Conformal Superspace: the configuration space of general relativity

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It has long been considered that conformal superspace is the natural configuration space for canonical general relativity. However, this was never definitively demonstrated. We have found that the standard conformal method of solving the Einstein constraints has an unexpected extra symmetry. This allows us to complete the project. We show that given a point and a velocity in conformal superspace, the Einstein equations generate a unique curve in conformal superspace.

It was realised more than 50 years ago, due to the pioneering work of Dirac [1], and Arnowitt, Deser and Misner [2], that general relativity could be expressed as a dynamical theory, just like the other standard theories of physics such as particle mechanics or electromagentism. In GR one can specify initial data and then integrate forward in time. The initial data for gravity consists of a spacelike 3-slice, equipped with a Riemannian 3-metric, g_{ij} and a symmetric tensor K_{ij}, which is to be the extrinsic curvature of the slice. Thus it is the time derivative of the 3-metric. More precisely,

$$\mathcal{L}_{\hat{n}} g_{ij} = 2K_{ij}, \tag{1}$$

where $\mathcal{L}$ is the Lie derivative and $\hat{n}$ is the unit timelike normal to the 3-slice. This pair (g_{ij}, K_{ij}) can be regarded as the analogue of the position and momentum in mechanics. GR is like electromagnetism in that the 10 Einstein equations can be split into 6 evolution equations, which propogate the 3-metric, and 4 constraints on the initial data. The constraints, which prevent free specification of g_{ij} and K_{ij}, are the main focus of this letter. For an up-to-date and comprehensive account of the constraints, see [3], especially Chapter VII.

The constraints are

$$^{(3)}R - K^{ij}K_{ij} + K^2 = 0, \tag{2}$$

$$\nabla_j K_i^j - \nabla_i K = 0, \tag{3}$$

known as the Hamiltonian and momentum constraints respectively; $^{(3)}R$ is the scalar curvature of g_{ij} and $K = g_{ij}K^{ij}$ is the trace of the extrinsic curvature. To identify the 'true gravitational degrees of freedom' and the configuration space of general relativity, we need to 'factor' the metric and extrinsic curvature by the constraints.

We use three spaces: *Riem* is the space of Riemannian 3-metrics $g_{ij}(x^k)$, $(i,j) = 1,2,3$, on some 3-manifold $\mathcal{M}$ (closed without boundary). The six functions of x^k in $g_{ij}(x^k)$ define the geometry and the coordinates on it. A 3-geometry has three degrees of freedom at each point

and infinitely many cordinatizations. All explicit calculations will be done in Riem.

Superspace, the space of 3-geometries on $\mathcal{M}$, is the quotient of Riem by 3-diffeomorphisms, which eliminate redundant coordinate information. In its turn, a 3-geometry contains information of two kinds: the conformal 3-geometry (two degrees of freedom), which describes angle measurements, and the local scale factor.

Conformal Superspace (CS) is the quotient of superspace by 3-dimensional conformal transformations:

$$\bar{g}_{ij} = \phi^4 g_{ij}, \; \phi > 0. \tag{4}$$

The fourth power of ϕ is just for convenience: the transformed 3-scalar curvature R is then $\bar{R} = \phi^{-4}R - 8\phi^{-5}\nabla^2\phi$ without additional terms quadratic in $\nabla\phi$; the preferred power of ϕ is two in four dimensions.

The idea that conformal superspace was the natural configuration space for gravity goes back to Lichnerowicz [4], who showed that using a conformal transformation one could write the Hamiltonian constraint Eq.(2) as a nice elliptic equation for the conformal factor. Unfortunately, this identification was never made precise. We will do so here. This article should be viewed as a companion to and extension of the seminal work on the subject by James York [5].

A conformal 3-geometry $\mathcal{C}$ is an equivalence class of 3-geometries. It includes infinitely many distributions of the local scale, which can be seen as coded in $\sqrt{g}$, where g is the determinant of g_{ij}. We wish to treat $\sqrt{g}$ as gauge, just like the coordinates. The physical object is $\mathcal{C}$: it is a point in CS, or *shape* of the universe. If we take the trace of Eq.(1) we can show

$$\mathcal{L}_{\hat{n}} \sqrt{g} = K\sqrt{g}. \tag{5}$$

Therefore $(\sqrt{g}, K)$ are canonically conjugate variables. If we think of the volume element, $\sqrt{g}$, as gauge, we must think of K as one also.

It turns out that *transverse-traceless* (TT) tensors (a tensor h_{ij}^{TT} is TT if it is both *transverse* ($\nabla^j h_{ij}^{TT} = 0$) and *traceless* ($g^{ij}h_{ij}^{TT} = 0$)) are natural objects on CS.

First, TT tensors define the tangent space to CS. Consider two nearby metrics in Riem, g_{ij} and $g_{ij} + \epsilon h_{ij}$, with ϵ a small parameter. Any symmetric tensor, in particular h_{ij}, has a unique TT part with respect to a g_{ij} via [6]

$$h_{ij} = h_{ij}^{TT} + \nabla_i \lambda_j + \nabla_j \lambda_i + \frac{1}{3} h g_{ij}; \qquad (6)$$

$\nabla_i \lambda_j + \nabla_j \lambda_i$ is the change in g_{ij} due to an infinitesimal change of coordinates, and the final (scalar) term is an infinitesimal conformal transformation. These two are therefore the generators of the symmetries we factor out in going from Riem to CS, and merely change the representation of the $\mathcal{C}$ corresponding to the given g_{ij}. The perturbation of $\mathcal{C}$ is h_{ij}^{TT}. Because its magnitude can be changed by multiplying by a constant c without changing its essential nature (if h_{ij}^{TT} is TT, so is ch_{ij}^{TT}), it is a tangent vector.

Second, TT tensors are *conformally covariant* [6]. If h_{ij}^{TT} is TT wrt g_{ij} and ξ is any function, then $\xi^{-2} h_{ij}^{TT}$ is TT with respect to $\xi^4 g_{ij}$. This is just straightforward algebra. The exponent -2 is the power of the conformal factor for a down-down TT tensor, it is -6 for up-down and -10 for up-up. Thus $(\xi^4 g_{ij}, \xi^{-2} h_{ij}^{TT})$ represents the same $\mathcal{C}$ and tangent vector as g_{ij} and h_{ij}^{TT}.

Third, as first pointed out by Peter Higgs [7], if we have an action on superspace, the associated momentum must be divergence-free. If the action is defined on conformal superspace, the momentum must also be tracefree. Indeed, the variation of the action with respect to a change in the metric can be written

$$\delta L = \int \pi^{ij} \delta g_{ij} d^3 x,$$

where π^{ij} is the momentum conjugate to g_{ij}. If the change of the metric is just a conformal change, i.e., $\delta g_{ij} = (\delta f) g_{ij}$, and if the action is invariant, we get

$$\delta L = 0 = \int \pi^{ij} (\delta f) g_{ij} d^3 x.$$

If this vanishes for all δf, we must have $\pi^{ij} g_{ij} = 0$.

Fourth, asymptotically flat initial data have a well-defined total energy, the ADM energy. Brill and Deser, in [8], showed that if one made a perturbation expansion in the initial data about flat space, the first non-trivial contribution to the energy came at second order and had the form

$$16\pi(\frac{1}{2}\delta^2 E_{ADM}) = \int d^3 x [\frac{1}{4}(\delta g_{ij,k}^{TT})^2 + (\delta K_{ij}^{TT})^2].$$

This expression is very similar to the Poynting energy expression, $(E^2 + B^2)$, in electromagnetism, and shows that the TT terms are the true excitations of the gravitational field in the weak-field limit. The challenge is to extend this to the strong field/no boundary case.

The standard conformal method, [3], starts with the realization that if the extrinsic curvature has a constant trace then the momentum constraint, Eq.(3), implies that the tracefree part is transverse. This means that we can write

$$K_{ij} = K_{ij}^{TT} + \frac{1}{3} K g_{ij}. \qquad (7)$$

We start with freely specifiable $(\bar{g}_{ij}, \bar{K}_{ij})$ and make the decomposition (6), obtaining $(\bar{g}_{ij}, \bar{K}_{ij}^{TT})$. We call this pair *the initial data*, conceived as a point and tangent vector in CS but of necessity represented as a point and tangent vector in Riem. We adjoin a constant K, obtaining the triplet $(\bar{g}_{ij}, \bar{K}_{ij}^{TT}, K)$. We now seek a conformal factor ϕ which maps this triplet into a new triplet satisfying the constraints via $(g_{ij}, K_{ij}^{TT}, K) = (\phi^4 \bar{g}_{ij}, \phi^{-2} \bar{K}_{ij}^{TT}, K)$. Then K_{ij}, constructed as $K_{ij} = K_{ij}^{TT} + \frac{1}{3} K g_{ij}$, following from Eq.(7), satisfies the momentum constraint, while the Hamiltonian constraint transforms into the Lichnerowicz - York (L-Y) equation

$$8\bar{\nabla}^2 \phi - \bar{R}\phi + \bar{K}_{TT}^{ij} \bar{K}_{ij}^{TT} \phi^{-7} - \frac{2}{3} K^2 \phi^5 = 0; \qquad (8)$$

$\bar{R}$ is the 3-scalar curvature formed from $\bar{g}_{ij}$ (as is $\bar{\nabla}$). Equation (8) always has a unique positive solution $\phi > 0$ [9] as long as $K \neq 0$ and $\bar{K}^{TT} \neq 0$.

Further, we can transform the initial data with an arbitrary positive function ξ to $(\bar{g}'_{ij}, \bar{K}_{ij}^{'TT}, K') = (\xi^4 \bar{g}_{ij}, \xi^{-2} \bar{K}_{ij}^{TT}, K)$. The conformal covariance of the L-Y equation emerges via the fact that when these 'new' data are injected into the L-Y equation the 'new' conformal factor $\phi' = \phi/\xi$! This means that the data we construct to satisfy the constraints $(g'_{ij}, K_{ij}^{'TT}, K') = (\phi'^4 \bar{g}'_{ij}, \phi'^{-2} \bar{K}_{ij}^{'TT}, K')$ are identical to the set we got without the transformation with ξ. This is *almost* good enough: since making an arbitrary conformal transformation changes nothing, $\bar{g}_{ij}$ can be regarded as a point in CS and $\bar{K}_{ij}^{TT}$ can be regarded as a velocity in CS at that point. However, the need to specify K as an extra initial datum complicates things. In fact, this is unnecessary because the constraints turn out to have an extra, unexpected but simple, symmetry.

Pick a (positive or negative) constant A. Let (g_{ij}, K_{ij}) solve the constraints. Now transform them as follows: $(\bar{g}_{ij}, \bar{K}_{ij}) = (A^2 g_{ij}, A K_{ij})$. The new data will also satisfy the constraints. Each term in the Hamiltonian constraint picks up a factor of A^{-2} and each term in the momentum constraint is multiplied by A^{-1}.

This symmetry also commutes with the conformal method of constructing solutions to the constraints as follows: Let us take the specified initial data and transform them as follows: pick a (positive or negative) constant $A \neq 0$. Construct 'new' initial data (we think of these as 'rescaled' data, the terminology will become clear soon) $(\bar{g}'_{ij}, \bar{K}_{ij}^{'TT}, K') = (A^2 \bar{g}_{ij}, A\bar{K}_{ij}^{TT}, K/A)$. Substitute these data into the L-Y equation. One can see

that each term in the equation picks up a factor of A^{-2}. Therefore $\phi' = \phi$. Hence this rescaling commutes with the L-Y equation. We can rescale either before or after solving the L-Y equation. We get the same final (rescaled) data satisfying the constraints.

Why 'rescaling'? Take any initial data satisfying the constraints and propagate them. This gives a (patch of) spacetime with a spacetime 4-metric $g_{\mu\nu}$ satisfying the Einstein equations. If we use geometric units, so that the speed of light = 1, then we have only one dimensionful quantity (say 'meters'). Following Dicke [10] we choose to put the dimensions into the metric and consider the coordinates as pure numbers, labels of points. Let us decide to change our units from 'meters' to 'yards'. This is achieved by multiplying the spacetime metric by a spacetime constant A, i.e., $g_{\mu\nu} \to A^2 g_{\mu\nu}$. This new metric continues to satisfy the Einstein equations. The effect of this rescaling on the 3+1 data is $(g'_{ij}, K'_{ij}) = (A^2 g_{ij}, A K_{ij})$, or $(g'_{ij}, K'^{TT}_{ij}, K') = (A^2 g_{ij}, A K^{TT}_{ij}, K/A)$.

We should stress that this 'rescaling' transformation $(g'_{ij}, K'^{TT}_{ij}, K') = (A^2 g_{ij}, A K^{TT}_{ij}, K/A)$ is *not* a subset of the conformal transformations $(\bar{g}'_{ij}, \bar{K}'^{TT}_{ij}, K') = (\xi^4 \bar{g}_{ij}, \xi^{-2} \bar{K}^{TT}_{ij}, K)$ mentioned earlier. In one case the solution of the constraints that emerges is rescaled, in the other case the solution is unchanged. We now show that this new extra symmetry means that K does not correspond to an extra physical initial datum in CS but merely to a choice of units in spacetime.

We picked the initial data as a point in Riem (as Riemannian 3-metric $\hat{g}_{ij}$) and a TT tensor $\hat{K}^{TT}_{ij}$ on $\hat{g}_{ij}$, regarding these as a point and tangent in CS even though we have to work in Riem. Now we pick a constant K_1, which may be positive or negative, but not zero. From these we construct 'intermediate' data $(\bar{g}_{ij}, \bar{K}^{TT}_{ij}, \bar{K}) = (K_1^{-2} \hat{g}_{ij}, K_1^{-1} \hat{K}^{TT}_{ij}, K_1)$. These intermediate data are of the standard form, i.e., metric + TT tensor + constant, so we can substitute them into the L-Y equation, Eq.(8), find the solution ϕ_1 and construct data which satisfy the constraints $(g_{ij}, K^{TT}_{ij}, K) = (\phi_1^4 \bar{g}_{ij}, \phi_1^{-2} \bar{K}^{TT}_{ij}, \bar{K}_1)$.

Let us now go back and, leaving the initial data unchanged, pick a new constant, K_2, and repeat the construction. We find new intermediate data $(\bar{g}_{ij}, \bar{K}^{TT}_{ij}, \bar{K}) = (K_2^{-2} \hat{g}_{ij}, K_2^{-1} \hat{K}^{TT}_{ij}, K_2)$, a new solution ϕ_2 to the L-Y equation, and new solution data satisfying the constraints. What is the relationship between the two sets of solution data? If we look at the two sets of intermediate data we can see that the mapping between them is a rescaling transformation as introduced earlier. More precisely, $(K_2^{-2} \hat{g}_{ij}, K_2^{-1} \hat{K}^{TT}_{ij}, K_2) = (A^2 K_1^{-2} \hat{g}_{ij}, A K_1^{-1} \hat{K}^{TT}_{ij}, K_1/A)$ with $A = K_1/K_2$. This means that $\phi_1 = \phi_2$, and, one of the solutions of the constraints is just a rescaling of the other. Therefore, holding the initial data fixed, and changing the value of K generates solutions of the constraints that are related by rescaling.

We can see three routes to proceed from this point. The first, and for us least desirable, is to abandon the 3 + 1 viewpoint and return to a 4-dimensional picture. Then each set of initial data will generate a spacetime satisfying the vacuum Einstein equations, and each spacetime generated from a given set of initial data can be rescaled as above into another that satisfies the same equations.

The second is to maintain the 3 + 1 idea, but live with many-fingered time. From a given set of initial data, we know that the evolution equations generate an infinite family of curves through superspace, each corresponding to a different slicing of the spacetime. The family of curves arising from data set 1 is different from the curves from data set 2. However, when the families are mapped into conformal superspace, they coincide.

The third, and the one we favour, is to realise that we have constructed a CMC initial data slice, and that it is very natural to extend this into the spacetime as a CMC foliation. Look at Eq.(5), $\mathcal{L}_{\hat{n}} \sqrt{g} = K \sqrt{g}$. This tells us that, on a CMC slice, the fractional time rate of change of the local volume is a constant. Therefore these CMC slices are the natural 'Hubble time' slices of a cosmology. There always exists a (two-sided) CMC foliation around any given CMC slice. This, and only this, preserves the TT-ness of the extrinsic curvature. To maintain it, we solve the elliptic lapse-fixing equation

$$\nabla^2 N - K^{ij} K_{ij} N = C \qquad (9)$$

for the function N, the lapse function; C is some constant, conveniently taken to be $C = -1$. Equation (9) has a unique solution. In addition, if $C < 0$, then $N > 0$ and vice versa.

We now evolve (g_{ij}, K_{ij}) with respect to the time label t using

$$\frac{\partial g_{ij}}{\partial t} = 2 N K_{ij} + \nabla_i N_j + \nabla_j N_i. \qquad (10)$$

This is just rewriting Eq.(1) in 3 + 1 language. There is also a further equation for $\partial K_{ij}/\partial t$ that we can omit; in both we may set the freely specifiable N_j to zero, but must continuously update $N(t, x^k)$ using Eq. (9). This evolution system fixes $\partial K/\partial t = C$ and generates the desired CMC foliation.

Thus each choice of initial data with arbitrary, but constant, K, will generate different (rescaled) paths through superspace but will generate the *same* path through conformal superspace.

Let us point out that no natural 'arrow of time' emerges in this analysis. The key point is that the rescaling constant A can be either positive or negative. The metric remains Riemannian, with signature $(+, +, +)$, because it is multiplied by A^2, but the extrinsic curvature changes sign. This merely exchanges what we think of as 'future' with 'past'.

We choose the initial data to be a point and a velocity in conformal superspace, with no specification of the local

scale. This local scale emerges when we solve the L-Y equation. To find initial data in Riem or in superspace we also need to specify a 'unit length'. This is why we have to pick a K. However, the solution to the Einstein equations, regarded as a curve in conformal superspace, is independent of the choice of K.

We define *shape dynamics* as a theory in which a point $\mathcal{C}_0$ in CS and any one of $\mathcal{C}_0$'s tangent vectors together determine a unique curve in CS. The initial data are thus a conformal equivalence class $(\xi^4 g_{ij}, \xi^{-2} K_{ij}^{TT})$. We have shown that GR is such a theory. We emphasize that the analogue of (2) in [12], in which only *directions* in CS are dynamic, leads to an emergent local scale without 'expansion of the universe'. When K is in the algorithm, this allows tangent vectors to be dynamic, and global expansion emerges too. In [13] we construct an action which naturally generates G R in the CMC gauge.

We see a strong case for regarding local shape as the core of GR, and local scale as mere gauge. The conceptual basis of shape dynamics is minimal, matches what can actually be observed, and allows all the unquestionably physical solutions of Einstein's full theory while excluding many (such as those with closed timelike curves) that seem clearly unphysical. In fact, we considered only vacuum solutions here, but matter, both fundamental and phenomenological, can be included in York's method, as can open universes by specifying spatial boundary conditions [11]. At the same time, our mathematical framework is fully adequate to describe all cosmological observations, e.g., red shift determinations are *local* comparisons of galactic and laboratory wavelengths. Finally, we believe that the identification of shape dynamics as the core of GR removes the motivation for the more complicated conformal theories advocated in [15]. One of us has recently argued that the result developed here implements Mach's principle in general relativity [16]

We think shape dynamics could have consequences in cosmology but that it would be premature to try to identify them here. In contrast, there are clear possibilities in the quantum domain. The time at which wave-function collapse occurs is ill defined in relativity; however the simultaneity associated with the shape-dynamic CMC foliation could bring interpretational clarity. This is also true for the 'problem of time' [14], which arises from the ambiguity in the time evolution in superspace if foliation invariance (many-fingered time) is made inviolate. Moreover, insistence on this has hitherto made it impossible to identify the space on which the putative wave function of the universe should be defined. In shape dynamics, the space must be conformal superspace.

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