The interior structure of astronomically realistic rotating black holes 1. Motivation and summary

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(Dated: October 8, 2010)

This paper motivates, summarizes, and discusses a new set of exact solutions for the interior structure of accreting, rotating black holes. The solutions are conformally stationary, axisymmetric, and separable. Hyper-relativistic counter-streaming between ingoing and outgoing streams leads to mass inflation at the inner horizon, followed by collapse. The papers solve a longstanding problem, providing for the first time a fully nonlinear solution for the interior structure of a rotating black hole.

PACS numbers: 04.20.-q

I. INTRODUCTION

Two companion technical papers [1, 2], hereafter Papers 2 and 3, present exact, conformally stationary, axisymmetric, separable solutions for the interior structure of a rotating black hole. Paper 2 deals with uncharged black holes, while Paper 3 generalizes to charged black holes. The claim is that these papers provide a solution to a problem that has remained outstanding since Kerr's 1963 [3] discovery of the exterior geometry of a rotating black hole. Consequently it is worthwhile to summarize the results of these papers, and to give an account of the reasoning underlying them.

The papers consider only classical general relativity, not alternate theories of gravity, nor speculative quantum processes that might for example destabilize the outer horizon.

The Penrose diagram of the analytically extended Kerr geometry, Figure 1, provides a good starting point for understanding where and how the interior Kerr geometry fails. A spherical charged (Reissner-Nordström) black hole has a similar interior structure, with essentially the same Penrose diagram, and much of the literature has focussed on this simpler case. The Kerr geometry, and more generally the Kerr-Newman geometry, has two inner horizons that are gateways to regions of unpredictability, signalled by the presence of timelike singularities. In 1968, Penrose [4] pointed out that an observer passing through the outgoing inner horizon (the Cauchy horizon) of a spherical charged black hole would see the outside Universe infinitely blueshifted, and he suggested that the infinite blueshift would destabilize the inner horizon. The infinite blueshift is plain from the Penrose diagram, Figure 1, which shows that a person passing through the outgoing inner horizon sees the entire future of the outside Universe go by in a finite time. Perturbation theory, much of it expounded in Chandrasekhar's (1983) monograph [5], confirmed that waves from the outside Universe would amplify to an infinite energy density on the outgoing inner horizon of both spherical charged and rotating black holes. The result was widely interpreted as indicating the instability of the inner horizon.

It was not until 1990 that the full nonlinear nature of the instability at the inner horizon was eventually clarified by Poisson & Israel [6]. Poisson & Israel showed that if ingoing (positive energy) and outgoing (negative energy) streams are simultaneously present just above the inner horizon of a spherical charged black hole, then cross-flow between the two streams would lead to an exponential growth of the interior mass. They called the instability "mass inflation." The inflationary instability in spherical charged black holes was confirmed analytically and numerically in several studies, as reviewed by [7].

The physical reason for the inflationary instability can be seen in the Penrose diagram, Figure 1. In the Black Hole region between the outer and inner horizons, the time coordinate (the one that expresses time translation symmetry) is spacelike, so that it is possible to go either forward or backward in time, and consequently to have either positive or negative energy. Inside the inner horizon, the time coordinate reverts to being timelike. Ingoing, positive energy particles want to fall into a region where the time coordinate is progressing forwards, while outgoing, negative energy particles want to fail into a region where the time coordinate is progressing backwards. The Penrose diagram, Figure 1, shows that indeed there are two distinct ingoing and outgoing inner horizons which disgorge on

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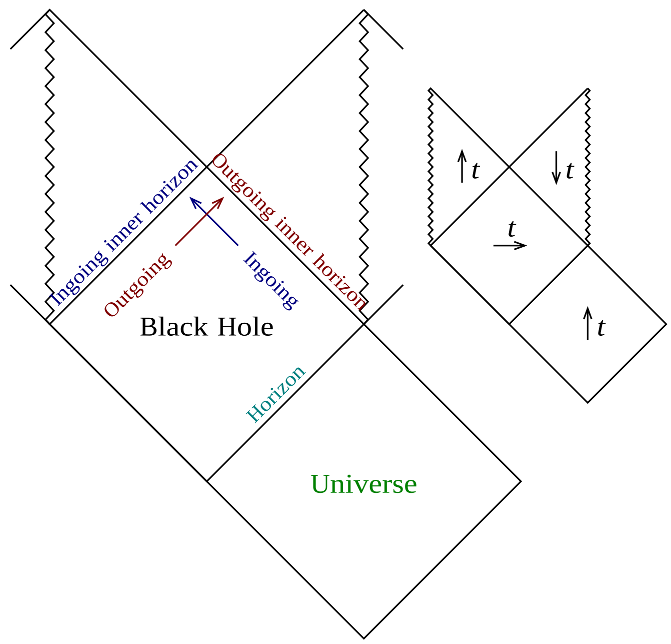

FIG. 1: Partial Penrose diagram illustrating why the Kerr geometry is subject to the inflationary instability. Ingoing and outgoing streams just outside the inner horizon must pass through separate ingoing and outgoing inner horizons into causally separated pieces of spacetime where the timelike Kerr time coordinate t goes in opposite directions. To accomplish this, the ingoing and outgoing streams must exceed the speed of light through each other, which physically they cannot do. In reality, hyper-relativistic counter-streaming between the ingoing and outgoing streams ignites and then drives the exponentially growing inflationary instability. The inset shows the direction of coordinate time t in the various regions. Proper time of course always increases upward in a Penrose diagram.

to two causally separated pieces of spacetime where the time coordinate is pointed in opposite directions. To achieve this causal separation, the ingoing and outgoing streams must exceed the speed of light relative to each other. This is Penrose's infinite blueshift. In reality, if ingoing and outgoing streams are present, then their attempt to exceed the speed of light relative to each other produces a counter-streaming energy and pressure that, however tiny the initial streams may be, inevitably grows to the point that it becomes a significant source of gravity. As expounded by [7], the counter-streaming pressure produces a gravitational force that is in opposite directions for ingoing and outgoing streams, accelerating the streams ever faster through each other, in turn increasing the counter-streaming pressure. The inflationary instability thus grows exponentially.

Almost the only prior work on the inflationary instability inside rotating black holes is that of Ori [8, 9] (see [10] for a full list of references). Ori considered the situation in which a black hole undergoes collapse, and inflation is driven by Price tails of ingoing and outgoing gravitational radiation generated by the collapse. The situation is more complicated than that considered in the present series of papers, and Ori was able to advance only qualitative arguments about the character of the instability.

II. APPROACH

The strategy adopted in the present papers is motivated by two key physical insights.

The first insight is that, as shown in §V of Paper 2 [1], collisionless ingoing and outgoing streams falling towards the inner horizon of the Kerr-Newman geometry become highly focussed into twin narrow, intense beams pointed along the ingoing and outgoing principal null directions. The focussing is along these two special directions regardless of the initial distributions of orbital parameters of the streams. This implies that the energy-momentum tensor of the ingoing and outgoing streams takes a simple and predictable form near the inner horizon.

As first shown by [11], the Kerr-Newman geometry (and some other electrovac geometries) is Hamilton-Jacobi separable in a tetrad aligned with the principal null directions. The fact that collisionless streams focus near the inner horizon along precisely these principal null directions suggests that the spacetime might continue to be separable in the presence of inflation.

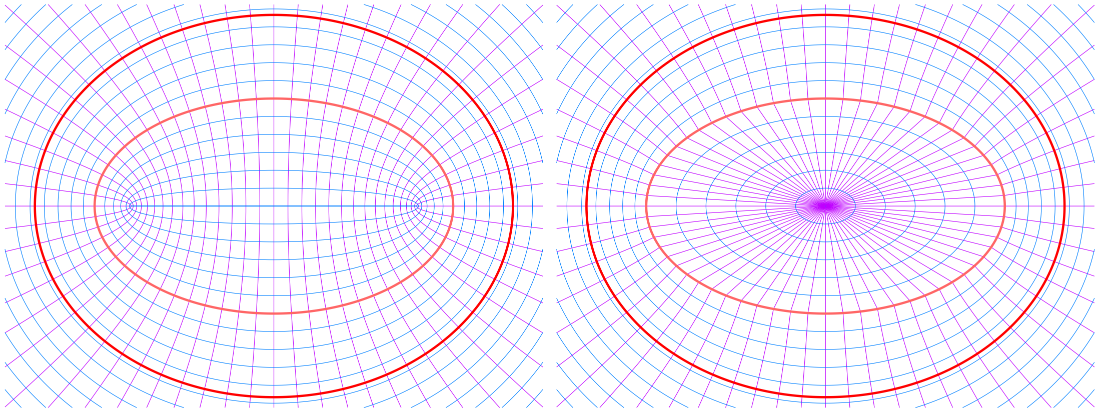

FIG. 2: Contours of constant radius and latitude in an uncharged black hole with spin parameter $a = 0.96 M_\bullet$. The thicker contours mark the outer and inner horizons. The left panel depicts a Kerr black hole. The right panel depicts a black hole of the kind considered in the present series of papers, which undergoes inflation just above the inner horizon, then collapses. In the Kerr geometry, surfaces of constant radius are confocal ellipsoids in Boyer-Lindquist coordinates, while surfaces of constant latitude are confocal hyperboloids, with a ring singularity at their focus. In the inflationary geometry, the streaming energy and density and Weyl curvature inflate to exponentially huge values at (just above) the inner horizon, which is destroyed. In the separable solutions presented here, the geometry then collapses radially to exponentially tiny size without changing shape.

The second insight is that the geometry of a spherical charged black hole undergoing inflation at (just above) its inner horizon has a step-function character. The spacetime is well-approximated by the electrovac (Reissner-Nordström) geometry down to just above the inner horizon. Then, in a tiny interval of radius and proper time near the inner horizon, the center-of-mass counter-streaming energy and pressure, the Weyl curvature, and the interior mass all inflate to exponentially huge values. Counter-intuitively, the smaller the incident ingoing and outgoing streams, the more rapidly quantities exponentiate [7]. In the limit of tiny accretion rate, the geometry tends to a step-function. This suggests that inflationary spacetimes might be found by looking for solutions with a step-like character. The strategy works, and provides a natural explanation for why relativists and mathematicians did not find these solutions before. The place where the step-function behaviour of inflation is entertained is §VI B of Paper 2 [1].

The step-like character of the inflationary solutions can be seen in the sharp turns at the inner horizon in the contours of constant radius and latitude in Figure 2, and the contours of constant conformal factor in Figure 3.

As an aside, it is worth commenting on the challenges and pitfalls of computing inflationary spacetimes numerically rather than analytically. One major numerical challenge arises from the fact that during inflation physical quantities inflate to exponentially huge values over tiny intervals of distance and time. One potential pitfall is that inflation requires ingoing and outgoing streams that can stream relativistically through each other. A code, or indeed analytic model, that treats the matter as a single fluid with a sound speed less than the speed of light artificially suppresses inflation by disallowing the relativistic counter-streaming that drives it.

III. SOLUTION SUMMARY

The solutions presented in this series of papers are conformally stationary, axisymmetric, and separable. Conformally stationarity combines the assumption of conformal time translation invariance (self-similarity) with an infinitesimal expansion rate. Let $x^\mu \equiv \{x, t, y, \phi\}$ be coordinates in which t is conformal time, ϕ is the azimuthal angle, and x and y are radial and angular coordinates. The line-element considered here is

$$ds^2 = \rho^2 \left[\frac{dx^2}{\Delta_x} - \frac{\Delta_x}{\sigma^4} (dt - \varpi \, d\phi)^2 + \frac{dy^2}{\Delta_y} + \frac{\Delta_y}{\sigma^4} (d\phi - \omega \, dt)^2 \right] , \tag{1}$$

where

$$\sigma \equiv \sqrt{1 - \omega \, \varpi} \ . \tag{2}$$

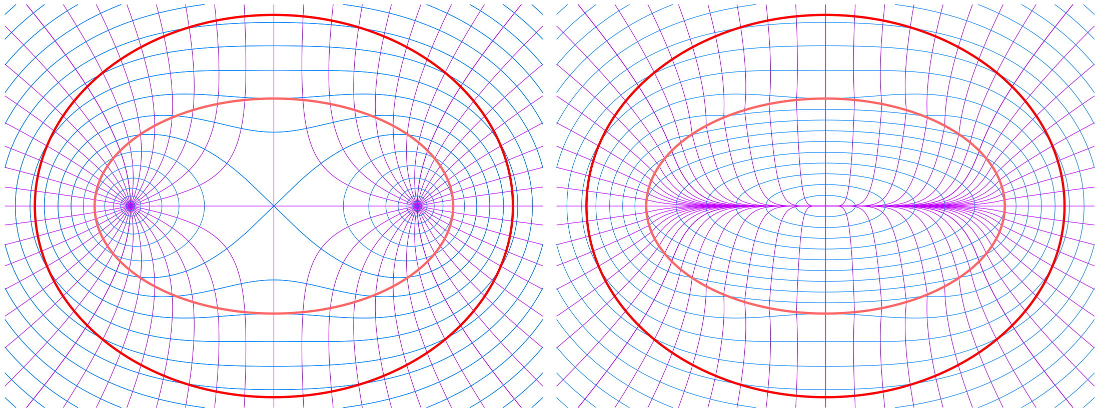

FIG. 3: Contours of constant conformal parameter ρ, and their normals, in an uncharged black hole with spin parameter $a = 0.96 M_\bullet$. The thicker contours mark the outer and inner horizons. The left panel depicts a Kerr black hole. The right panel depicts a black hole of the kind considered in the present series of papers.

The determinant of the 2×2 submatrix of t–ϕ metric coefficents defines the radial and angular horizon functions Δ_x and Δ_y:

$$g_{tt} g_{\phi\phi} - g_{t\phi}^2 = - \frac{\rho^4}{\sigma^4} \Delta_x \Delta_y \ . \tag{3}$$

Horizons occur when one or other of the horizon functions Δ_x and Δ_y vanish. The focus here is the region near the inner horizon where the radial horizon function Δ_x is negative and tending to zero. The line-element (1) defines a tetrad that is aligned with the principal null directions. Since the radial coordinate x is timelike near the inner horizon, it is convenient to take x as the time coordinate of the tetrad, and to choose the sign of x so that it increases inwards, the direction of advancing time.

The conformal factor ρ is a product of separable (electrovac) ρ_{sep}, time-dependent e^{vt}, and inflationary $e^{-\epsilon}$ factors,

$$\rho = \rho_{\text{sep}} e^{vt - \epsilon} \ . \tag{4}$$

Conformal time translation symmetry is expressed by the fact that the spacetime expands conformally (that is, without changing shape) by factor $\rho \to e^{v\Delta t} \rho$ when the conformal time increases by $t \to t + \Delta t$. Conformal stationarity means taking the limit of small expansion rate, or small accretion rate, *after* calculations are complete,

$$v \to 0 \ . \tag{5}$$

This is not the same as stationarity, which sets v to zero at the outset. A feature of inflation is that the smaller the accretion rate, the faster inflation exponentiates. Even in the limit of infinitesimal accretion rate, inflation drives the center-of-mass streaming density and pressure, and the Weyl curvature, to exponentially huge values. Mathematically, Einstein's and Maxwell's equations contain terms of order $\sim v/\Delta_x$ that grow large at the inner horizon $\Delta_x \to -0$ however small the accretion rate v may be.

The tetrad-frame covariant electromagnetic 4-potential A_k is

$$A_k \equiv \frac{1}{\rho} \left\{ \frac{\mathcal{A}_x}{\sqrt{-\Delta_x}}, \frac{\mathcal{A}_t}{\sqrt{-\Delta_x}}, \frac{\mathcal{A}_y}{\sqrt{\Delta_y}}, \frac{\mathcal{A}_\phi}{\sqrt{\Delta_y}} \right\} \ . \tag{6}$$

The parameters of the line element (1) and electromagnetic potential (6) satisfy the separability conditions

$$\begin{array}{ll} \omega \ , \ \ \Delta_x \ , \ \ e^{-vt} \mathcal{A}_x \ , \ \ e^{-vt} \mathcal{A}_t & \text{are functions of } x \text{ only} \ , \\ \varpi \ , \ \ \Delta_y \ , \ \ e^{-vt} \mathcal{A}_y \ , \ \ e^{-vt} \mathcal{A}_\phi & \text{are functions of } y \text{ only} \ . \end{array} \tag{7}$$

For boundary conditions provided by the Λ-Kerr-Newman geometry, the separable part ρ_{sep} of the conformal factor is

$$\rho_{\text{sep}} = \sqrt{r^2 + a^2 \cos^2\theta} \ , \quad r = a \cot(ax) \ , \quad \cos\theta = -y \ , \tag{8}$$

and ω, ϖ, and σ are

$$\omega = \frac{a}{R^2} \ , \quad \varpi = a \sin^2\theta \ , \quad \sigma \equiv \sqrt{1 - \omega\varpi} = \frac{\rho_{\rm sep}}{R} \ , \quad R \equiv \sqrt{r^2 + a^2} \ , \tag{9}$$

with a being the usual angular momentum parameter. During the electrovac phase leading up to inflation, the radial and angular horizon functions Δ_x and Δ_y take their usual Λ-Kerr-Newman forms

$$\Delta_x = \frac{1}{R^2}\left(1 - \frac{2M_\bullet r}{R^2} + \frac{Q_\bullet^2}{R^2} - \frac{\Lambda r^2}{3}\right) \ , \tag{10a}$$

$$\Delta_y = \sin^2\theta \left(1 + \frac{\Lambda a^2 \cos^2\theta}{3}\right) \ , \tag{10b}$$

where $M_\bullet$ is the black hole's mass, $Q_\bullet$ is its electric charge, and Λ is the cosmological constant.

Inflation begins as the radial horizon function approaches zero, $\Delta_x \to -0$, at the inner horizon of the electrovac geometry. During the electrovac and early inflationary phases, the inflationary exponent ϵ in the conformal factor remains negligibly small, while its first and second radial derivatives grow. During the later inflationary and collapse phases, the inflationary exponent ϵ becomes large, and continued satisfaction of the separability conditions (7) requires that

$$\epsilon \text{ is a function of } x \text{ only} \ . \tag{11}$$

During the inflationary and collapse phases, the inflationary exponent ϵ increases away from zero. The behaviour is conveniently characterized in terms of a positive parameter U, given in terms of the inflationary exponent ϵ by

$$U = \sqrt{\mathrm{v}^2 + (u^2 - \mathrm{v}^2) e^{4\epsilon}} \ , \tag{12}$$

with initial value

$$U = u \ . \tag{13}$$

The radial coordinate x is related to U by

$$dx = -\frac{\Delta_x \, dU}{2(U^2 - \mathrm{v}^2)} \ , \tag{14}$$

where the (negative) radial horizon function Δ_x during inflation and collapse is

$$\Delta_x = -\left(\frac{U^2 - \mathrm{v}^2}{u^2 - \mathrm{v}^2}\right)^{3/4} \left[\frac{(U + \mathrm{v})(u - \mathrm{v})}{(U - \mathrm{v})(u + \mathrm{v})}\right]^{\Delta_x'/(4\mathrm{v})} \ , \tag{15}$$

with $\Delta_x' \equiv d\Delta_x/dx|_{x_-}$ the (positive) derivative of the Λ-Kerr-Newman horizon function at the inner horizon $x = x_-$. The second factor in the expression (15) for the horizon function is initially a number slightly less than unity, taken to a large power $\Delta_x'/(4\mathrm{v})$. As U increases away from its initial value u, the horizon function (15) decreases in absolute value, becoming exponentially tiny,

$$\Delta_x \sim e^{-1/\mathrm{v}} \ . \tag{16}$$

As U continues to increase, the horizon function (15) goes through a minimum in absolute value, then grows exponentially large (negative) as the spacetime collapses to an exponentially tiny scale. During inflation and collapse, while U increases and the horizon function Δ_x varies exponentially, the radial x coordinate, equation (14), remains essentially frozen at its inner horizon value $x = x_-$. Finally, x starts to unfreeze, heralding the breakdown of separability, but this happens only deep into collapse, when the conformal factor has shrunk to an exponentially tiny scale.

Inflation is driven by the energy-momentum of counter-streaming ingoing and outgoing streams near the inner horizon. In this series of papers, the streams are taken to be a general, possibly charged, collisionless fluid. Einstein's equations are sourced by collisionless and electromagnetic energy-momenta and a cosmological constant,

$$G_{mn} = 8\pi T^{\rm c} + 8\pi T^{\rm e} - \Lambda \eta_{mn} \ . \tag{17}$$

The tetrad-frame electric current j_k and energy-momentum $T^{\rm c}_{mn}$ of the collisionless fluid, and the electromagnetic energy-momentum $T^{\rm e}_{mn}$, may be written

$$j_k = \frac{J_k}{\rho^2 \sqrt{|\Delta_k|}} \ , \tag{18a}$$

$$T^{\rm c}_{mn} = \frac{P_{mn}}{\rho^2 \sqrt{|\Delta_m \Delta_n|}} \ , \tag{18b}$$

$$T^{\rm e}_{mn} = \frac{Q^2}{8\pi\rho^4} \text{diag}(1,-1,1,1) + \frac{Q_{mn}}{\rho^2 \sqrt{|\Delta_m \Delta_n|}} \ , \tag{18c}$$

where $\Delta_m \equiv \Delta_x$ for $m = x, t$, and $\Delta_m \equiv \Delta_y$ for $m = y, \phi$. During inflation, the behaviour of the current and energy-momenta are dominated by the behaviour of the radial horizon function, which goes to zero near the inner horizon, $\Delta_x \to -0$, causing the x and t components of the current and energy-momenta to diverge. As the horizon function becomes exponentially tiny, the current and energy-momenta grow exponentially huge. By comparison, the current and energy momentum parameters J_k, P_{mn}, and Q_{mn} vary modestly. The parameters are sums of ingoing (positive energy) and outgoing (negative energy) parts

$$J_k = J^+_k + J^-_k \ , \quad P_{mn} = P^+_{mn} + P^-_{mn} \ , \quad Q_{mn} = Q^+_{mn} + Q^-_{mn} \ , \tag{19}$$

with non-vanishing components

$$J^\pm_x = \pm J^\pm_t = -\frac{j^\pm}{4\pi\rho_{\rm sep}} \left(\frac{U \mp {\rm v}}{u \mp {\rm v}} \right)^{1/2} \left(\frac{\omega''}{\omega'} + \frac{\varpi\omega'}{\sigma^2} \mp \frac{{\rm v}\varpi^2}{\Delta_y} \right) \ , \tag{20a}$$

$$P^\pm_{xx} + Q^\pm_{xx} = \pm \left(P^\pm_{xt} + Q^\pm_{xt} \right) = \frac{(U \mp {\rm v})}{16\pi} \left(\Delta'_x \pm {\rm v} \right) \ , \tag{20b}$$

$$P^\pm_{xy} + Q^\pm_{xy} = \pm \left(P^\pm_{ty} + Q^\pm_{ty} \right) = -\frac{(U \mp {\rm v})}{8\pi} \Delta_y \frac{\partial \ln \rho_{\rm sep}}{\partial y} \ , \tag{20c}$$

$$P^\pm_{x\phi} + Q^\pm_{x\phi} = \pm \left(P^\pm_{t\phi} + Q^\pm_{t\phi} \right) = \frac{(U \mp {\rm v})}{16\pi} \left(\pm \frac{\omega' \Delta_y}{\sigma^2} - 2{\rm v}\varpi \right) \ , \tag{20d}$$

$$Q^\pm_{xx} = \pm Q^\pm_{xt} = \frac{(j^\pm)^2 (U \mp {\rm v})}{4\pi\rho^2_{\rm sep}(u \mp {\rm v})} \frac{\varpi^2}{\Delta_y} \ , \tag{20e}$$

$$Q^\pm_{xy} = \pm Q^\pm_{ty} = \pm \frac{j^\pm (j^+ + j^-)(U \mp {\rm v})}{4\pi\rho^2_{\rm sep}{\rm v}(u \mp {\rm v})} \frac{\varpi}{\sigma^2} \frac{d\varpi}{dy} \ , \tag{20f}$$

$$Q^\pm_{x\phi} = \pm Q^\pm_{t\phi} = \frac{j^\pm (j^+ + j^-)(U \mp {\rm v})}{4\pi\rho^2_{\rm sep}{\rm v}(u \mp {\rm v})} \varpi \left(\frac{\omega''}{\omega'} + \frac{\varpi\omega'}{\sigma^2} \right) \ , \tag{20g}$$

where $\omega' \equiv d\omega/dx|_{x_-}$ and $\omega'' \equiv d^2\omega/dx^2\big|_{x_-}$ are the derivatives of ω at the inner horizon, $x = x_-$. Note that ω, ϖ, σ^2, and Δ_y retain their electrovac forms (9), (10b), during inflation and collapse; only the conformal factor ρ, equation (4), and the radial horizon function Δ_x, equation (15), are modified.

Equations (20) prescribe that the collisionless accretion flow incident on the inner horizon has a specific angular dependence, necessary to ensure that the separability conditions (7) continue to hold during inflation and collapse. The angular dependence of the collisionless parameters is frozen in during inflation and collapse, while they vary radially as $J^\pm_k \propto (U \mp {\rm v})^{1/2}$ and $P^\pm_{mn} \propto Q^\pm_{mn} \propto U \mp {\rm v}$. The radial dependences can be understood as consequences of current conservation and energy-momentum conservation in each of the ingoing and outgoing streams independently. Although particles can switch from ingoing to outgoing between the outer and inner horizons, once inflation begins there is no further switching: ingoing particles remain ingoing, and outgoing particles remain outgoing, so conservation laws apply separately to each of the ingoing and outgoing streams.

The solutions (20) are determined by four constants

$$u \ , \quad {\rm v} \ , \quad j^+ \ , \quad j^- \ , \tag{21}$$

fixed by accretion rates. All four constants are to be considered to be tiny in the conformally stationary limit, but not strictly zero, since the current and energy-momenta grow exponentially huge near the inner horizon $\Delta_x \to -0$ however small the accretion rates may be. The accretion rates of ingoing and outgoing streams incident on the inner horizon are proportional to $u - v$ and $u + v$ respectively. Positivity of both streams requires $u > |v|$, and the requirement that the black hole expand as it accretes imposes $v > 0$, so that

$$u > v > 0 \ . \tag{22}$$

The accretion rate parameter v, which appears as part of the conformal factor ρ, equation (4), measures the asymmetry between ingoing and outgoing streams at the inner horizon.

The accretion rates of charge in the ingoing and outgoing streams are proportional to j^+ and j^- respectively. It turns out that if both ingoing and outgoing streams are charged, then the counter-streaming current generates a diverging angular electromagnetic field that causes diagonal angular components of the electromagnetic energy-momentum to diverge during inflation, destroying separability. Thus separability during inflation and collapse requires that one or other of the ingoing or outgoing streams be neutral, so one or other of $j^\pm$ must vanish,

$$j^+ = 0 \quad \text{or} \quad j^- = 0 \ . \tag{23}$$

If both streams are charged, then separability holds during the early part of inflation, but fails as inflation develops.

The enclosed electric charge Q of the black hole is a sum of contributions from the ingoing ($+$) and outgoing ($-$) streams (the contribution being zero if the stream is neutral),

$$Q = Q^+ + Q^- \ , \quad Q^\pm = Q^\pm_\bullet \left[\frac{(u \pm v)(U \mp v)}{(u \mp v)(U \pm v)} \right]^{1/4} \ . \tag{24}$$

The charge $Q_\bullet = Q^+_\bullet + Q^-_\bullet$ of the black hole as seen by an outside observer is produced self-consistently by the accreted charge. The relation between the black hole charges $Q^\pm_\bullet$ and the charge accretion rates $j^\pm$ is

$$j^\pm = \frac{v e^{-vt} Q^\pm_\bullet \omega'}{2a} \ . \tag{25}$$

Whereas an electrovac black hole can possess a magnetic charge, the black holes in the present paper cannot, because the separability conditions (7) force Maxwell's equations to have zero magnetic current, so the black hole cannot accrete magnetic charge, so its cumulative magnetic charge is necessarily zero. This implies that the angular components of the electromagnetic potential vanish,

$$\mathcal{A}_y = \mathcal{A}_\phi = 0 \ . \tag{26}$$

The radial components of the electromagnetic potential $\mathcal{A}_k$ defined by equation (6) are related to the enclosed current Q, in both electrovac and inflationary/collapse phases, by

$$\mathcal{A}_k = \mathcal{A}^+_k + \mathcal{A}^-_k \ , \quad \mathcal{A}^\pm_t = \pm \mathcal{A}^\pm_x = -\frac{Q^\pm}{2a} \frac{d\omega}{dx} \ . \tag{27}$$

The Weyl tensor has only a complex spin-0 part, a sum of electrovac and inflationary parts

$$C = C_{\text{ev}} + C_{\text{inf}} \ , \tag{28}$$

$$C_{\text{ev}} = -\frac{\rho^2_{\text{sep}}}{\rho^2} \frac{1}{(r - ia\cos\theta)^3} \left(M_\bullet - \frac{Q^2_\bullet}{r + ia\cos\theta} \right) \ , \tag{29a}$$

$$C_{\text{inf}} = \frac{1}{\rho^2} \left[\frac{U^2 - v^2}{2\Delta_x} + \frac{1}{12\sigma^2} \left(\varpi\omega' - 3i\frac{d\varpi}{dy} \right) U \right] \ . \tag{29b}$$

IV. DISCUSSION

This paper has presented conformally stationary, axisymmetric, separable solutions for the interior structure of a rotating black hole that undergoes inflation just above its inner horizon, then collapses. It has long been known that the Kerr geometry is linearly unstable at its inner horizon [5], and it has been suspected that the instability would develop nonlinearly similarly to the inflationary instability [6] known to operate in spherical charged black holes [7]. The self-consistent nonlinear solutions found here confirm that the inflationary instability develops in rotating black holes as anticipated. The solutions in this paper are special, requiring that the accretion flow incident on the inner horizon take a certain precise form in order to maintain separability. Nevertheless, the generic and inevitable character of the inflationary instability suggests that the solutions here may be prototypical of what happens in real astronomical black holes.

It is quite remarkable that the solutions remain separable, in the sense of satisfying the separability conditions (7), not only during the electrovac and early inflation phases, but also through inflation deep into the subsequent collapse phase. Separability finally breaks down when the radial x coordinate, equation (14), which remains esssentially frozen at its inner horizon value during inflation and collapse, finally unfreezes, but that happens only after the conformal factor ρ has shrunk to an exponentially tiny value. We do not pursue what happens after the breakdown of separability, but further instabilities could perhaps ensue.

During the early part of inflation, the electrovac geometry remains unchanged (the inflationary exponent ϵ is negligible, and the horizon function Δ_x is unaffected), even while the energy and pressure of the counter-streaming ingoing and outgoing streams are growing large. The reason for the lack of back-reaction is the extremely short interval of proper time over which inflation develops. Inflation is like a bullet fired in the chamber of a gun: an explosion accelerates the bullet, and shortly after the bullet achieves high velocity, but still the bullet has hardly moved. Inflation does in due course back-react on the geometry, but in a predictable way: the conformal factor, having been accelerated to huge velocity in the inward radial direction, proceeds to shrink rapidly in that direction.

A limitation of the separable solutions is that the conditions on the ingoing and outgoing streams incident on the inner horizon required by separability cannot be accomplished by collisionless streams that fall freely from outside the outer horizon. Particles that fall from outside must be ingoing at the outer horizon, and this places constraints on parameters at the inner horizon that are incompatible with those imposed by separability. The problem is alleviated if the streams are charged, since electrical repulsion relaxes the constraints enough to achieve compatibility. However, separability also requires that only one of the ingoing or outgoing streams can be charged, and the problem remains for the neutral stream. The limitation does not invalidate the separable solutions, but it does mean that the streams must be considered to be delivered ad hoc to just above the inner horizon.

Acknowledgments

This work was supported by NSF award AST-0708607.

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