A physical derivation of the Kerr–Newman black hole solution

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Abstract According to the no-hair theorem, the Kerr–Newman black hole solution represents the most general asymptotically flat, stationary (electro-) vacuum black hole solution in general relativity. The procedure described here shows how this solution can indeed be constructed as the unique solution to the corresponding boundary value problem of the axially symmetric Einstein–Maxwell equations in a straightforward manner.

1 Introduction: From Schwarzschild to Kerr–Newman

The Schwarzschild solution, depending on a single parameter (mass M), represents the *general* spherically symmetric vacuum solution to the Einstein equations. Similarly, the Reissner–Nordström solution, depending on two parameters (M and electric charge Q), is the *general* spherically symmetric (electro-) vacuum solution to the Einstein–Maxwell equations. In contrast, the Kerr–Newman solution, depending on three parameters (M, Q and angular momentum J), is only a *particular* stationary and axially symmetric electro-vacuum solution to the Einstein–Maxwell equations. However, one can show under quite general conditions that the Kerr–Newman solution represents the most general asymptotically flat, stationary electro-vacuum *black hole* solution ("no-hair theorem"). Important contributions to the subject of black hole uniqueness were made by Israel, Carter, Hawking, Robinson and Mazur (1967–1982), for details see the recent review [3].

Assuming stationarity and axial symmetry, it is indeed possible to derive the Kerr–Newman black hole solution in straightforward manner, by solving the corresponding boundary value problem of the Einstein–Maxwell equations [7]. In the following sections, an outline of this work will be given. The method is a generaliza-

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tion of the technique developed for solving a boundary value problem of the vacuum Einstein equations leading to the global solution describing a uniformly rotating disc of dust in terms of ultraelliptic functions [12, 13], see also [9]. It is based on the "integrability" of the stationary and axisymmetric vacuum Einstein and electro-vacuum Einstein–Maxwell equations via the "inverse scattering method", see [1]. In the pure vacuum case, the method was also used to derive the Kerr black hole solution [10, 13, 9].

2 Einstein–Maxwell equations and related Linear Problem

The stationary and axisymmetric, electro-vacuum Einstein–Maxwell equations are equivalent to the Ernst equations [4]

$$f\,\Delta\mathscr{E} = (\nabla\mathscr{E} + 2\bar{\Phi}\nabla\Phi)\cdot\nabla\mathscr{E}, \quad f\,\Delta\Phi = (\nabla\mathscr{E} + 2\bar{\Phi}\nabla\Phi)\cdot\nabla\Phi \tag{1}$$

$$\text{with}\quad f \equiv \Re\mathscr{E} + |\Phi|^2, \quad \Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\zeta^2}, \quad \nabla = \left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\zeta}\right). \tag{2}$$

The line element reads

$$\mathrm{d}s^2 = f^{-1}[h(\mathrm{d}\rho^2 + \mathrm{d}\zeta^2) + \rho^2\mathrm{d}\phi^2] - f(\mathrm{d}t + A\,\mathrm{d}\phi)^2, \tag{3}$$

where the coordinates t and ϕ are adapted to the Killing vectors corresponding to stationarity and axial symmetry:

$$\boldsymbol{\xi} = \frac{\partial}{\partial t}, \quad \boldsymbol{\eta} = \frac{\partial}{\partial\phi}. \tag{4}$$

We assume an asymptotic behaviour as $r \to \infty$ $(\rho = r\sin\theta, \zeta = r\cos\theta)$ given by

$$\Re\mathscr{E} = 1 - \frac{2M}{r} + \mathscr{O}(r^{-2}), \quad \Im\mathscr{E} = -\frac{2J\cos\theta}{r^2} + \mathscr{O}(r^{-3}), \quad \Phi = \frac{Q}{r} + \mathscr{O}(r^{-2}) \tag{5}$$

corresponding to asymptotic flatness and the absence of a magnetic monopole term (Q real). The metric functions h and A can be calculated from the complex Ernst potentials $\mathscr{E}(\rho,\zeta)$ and $\Phi(\rho,\zeta)$ according to

$$(\ln h)_{,z} = \frac{\rho}{f^2}(\mathscr{E}_{,z} + 2\bar{\Phi}\Phi_{,z})(\bar{\mathscr{E}}_{,z} + 2\Phi\bar{\Phi}_{,z}) - \frac{4\rho}{f}\Phi_{,z}\bar{\Phi}_{,z}, \tag{6}$$

$$A_{,z} = \frac{\mathrm{i}\rho}{f^2}[(\Im\mathscr{E})_{,z} - \mathrm{i}\bar{\Phi}\Phi_{,z} + \mathrm{i}\Phi\bar{\Phi}_{,z}] \qquad (r \to \infty: \ h \to 1, A \to 0). \tag{7}$$

Here complex variables

$$z = \rho + \mathrm{i}\zeta, \quad \bar{z} = \rho - \mathrm{i}\zeta \tag{8}$$

have been used instead of ρ and ζ. Note that f has already been given in (2). The electromagnetic field tensor

$$F_{ik} = A_{k,i} - A_{i,k}, \quad A_i \, dx^i = A_\phi \, d\phi + A_t \, dt \tag{9}$$

can also be obtained from the Ernst potentials:

$$A_t = -\Re\,\Phi, \qquad A_{\phi,z} = A A_{t,z} - \frac{\mathrm{i}\rho}{f}(\Im\,\Phi)_{,z} \qquad (r \to \infty: \ A_\phi \to 0). \tag{10}$$

The Ernst equations (1) can be formulated as the integrability condition of a related Linear Problem (LP). We use the LP of [11] in a slightly modified form, which is advantageous in the presence of ergospheres:

$$\mathbf{Y}_{,z} = \left[\begin{pmatrix} b_1 & 0 & c_1 \\ 0 & a_1 & 0 \\ d_1 & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & b_1 & 0 \\ a_1 & 0 & -c_1 \\ 0 & d_1 & 0 \end{pmatrix} \right] \mathbf{Y}, \tag{11}$$

$$\mathbf{Y}_{,\bar{z}} = \left[\begin{pmatrix} b_2 & 0 & c_2 \\ 0 & a_2 & 0 \\ d_2 & 0 & 0 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & b_2 & 0 \\ a_2 & 0 & -c_2 \\ 0 & d_2 & 0 \end{pmatrix} \right] \mathbf{Y} \tag{12}$$

with

$$\lambda = \sqrt{\frac{K - \mathrm{i}\bar{z}}{K + \mathrm{i}z}}, \tag{13}$$

$$a_1 = \bar{b}_2 = \frac{\mathscr{E}_{,z} + 2\bar{\Phi}\Phi_{,z}}{2f}, \quad a_2 = \bar{b}_1 = \frac{\mathscr{E}_{,\bar{z}} + 2\bar{\Phi}\Phi_{,\bar{z}}}{2f}, \tag{14}$$

$$c_1 = f\bar{d}_2 = \Phi_{,z}, \quad c_2 = f\bar{d}_1 = \Phi_{,\bar{z}}. \tag{15}$$

The integrability condition

$$\mathbf{Y}_{,z\bar{z}} = \mathbf{Y}_{,\bar{z}z} \tag{16}$$

is equivalent to the Ernst equations. The following points are relevant for the application of soliton theoretic solution methods:

- The 3×3 matrix $\mathbf{Y}$ depends not only on the coordinates ρ and ζ (or z and $\bar{z}$), but also on the additional complex "spectral parameter" K.
- Since $\bar{K}$ does not appear, we can assume without loss of generality that the elements of $\mathbf{Y}$ are holomorphic functions of K defined on the two-sheeted Riemann surface associated with (13), except from the locations of possible singularities.
- Each column of $\mathbf{Y}$ is itself a solution to the LP. We assume that these three solutions are linearly independent.
- For a given solution $\mathscr{E}$, Φ to the Einstein–Maxwell equations, the solution to the LP can be fixed (normalized) by prescribing $\mathbf{Y}$ at some point ρ_0, ζ_0 of the ρ-ζ plane as a (matrix) function of K in one of the two sheets of the Riemann surface.
- $\mathbf{Y}$ can be discussed in general as a unique function of ρ, ζ and λ.

Three interesting relations result directly from the structure of the LP (11, 12):

$$[f(\rho,\zeta)]^{-1}\det \mathbf{Y}(\rho,\zeta,\lambda) = C_0(K), \tag{17}$$

$$\mathbf{Y}(\rho,\zeta,-\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{Y}(\rho,\zeta,\lambda)\mathbf{C}_1(K), \tag{18}$$

$$\left[\mathbf{Y}(\rho,\zeta,1/\bar{\lambda})\right]^{\dagger} \begin{pmatrix} [f(\rho,\zeta)]^{-1} & 0 & 0 \\ 0 & -[f(\rho,\zeta)]^{-1} & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{Y}(\rho,\zeta,\lambda) = \mathbf{C}_2(K), \tag{19}$$

where $C_0(K)$ as well as the matrices $\mathbf{C}_1(K)$ and $\mathbf{C}_2(K)$ do not depend on ρ and ζ.

3 Solving the black hole boundary value problem

After formulating the black hole boundary value problem, we will use the LP to find its solution. The most important part comprises deriving the Ernst potentials on the axis of symmetry [7]. It is well known that these "axis data" uniquely determine the solution everywhere, see [5, 14]. A straightforward method for obtaining the full solution from the axis data is based on the analytical properties of $\mathbf{Y}$ as a function of λ [8].

3.1 Boundary conditions

The event horizon $\mathcal{H}$ of a stationary and axisymmetric black hole is characterized by the conditions

$$\mathcal{H}: \quad \chi^i\chi_i = 0, \quad \chi^i\eta_i = 0, \tag{20}$$

where $\chi^i \equiv \xi^i + \Omega\eta^i$ and the constant Ω is the "angular velocity of the horizon" [6, 2]. Because of

$$\rho^2 = (\xi^i\eta_i)^2 - \xi^i\xi_i\eta^k\eta_k = (\chi^i\eta_i)^2 - \chi^i\chi_i\eta^k\eta_k \tag{21}$$

the horizon must be located on the ζ-axis of our Weyl coordinate system:

$$\mathcal{H}: \quad \rho = 0. \tag{22}$$

This results in two possibilities for a connected horizon[1]: (i) a finite interval on the ζ-axis and (ii) a point on the ζ-axis, see Fig. 1. Note that the two parts of the symmetry axis, $\mathscr{A}^+$ and $\mathscr{A}^-$, where the Killing vector $\boldsymbol{\eta}$ vanishes, are also characterized by $\rho = 0$. The black hole boundary value problem consists of finding a solution that is regular everywhere outside the horizon and satisfies (20) and (5).

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[1] A connected horizon means a single black hole. We are not interested here in the problem of multi-black-hole equilibrium states.
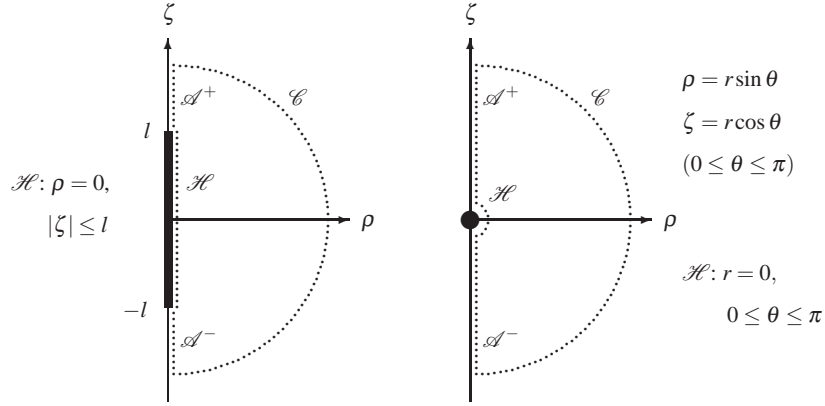
Fig. 1 In Weyl ccordinates, the horizon is either a finite interval or a point on the ζ-axis (adapted from [7])

3.2 Axis data

At $\rho = 0$, the branch points $K = i\bar{z}$ and $K = -iz$ of (13) merge to $K = \zeta$ and for $K \neq \zeta$ holds $\lambda = \pm 1$. Consequently, the solution to the LP, for $\lambda = +1$, is of the form

$$\mathscr{A}^{\pm}: \quad \mathbf{Y}_{\pm} = \begin{pmatrix} \bar{\mathscr{E}} + 2|\Phi|^2 & 1 & \Phi \\ \mathscr{E} & -1 & -\Phi \\ 2\bar{\Phi} & 0 & 1 \end{pmatrix} \mathbf{C}_{\pm}, \tag{23}$$

$$\mathscr{H}: \quad \mathbf{Y}_{\mathrm{h}} = \begin{pmatrix} \bar{\mathscr{E}} + 2|\Phi|^2 & 1 & \Phi \\ \mathscr{E} & -1 & -\Phi \\ 2\bar{\Phi} & 0 & 1 \end{pmatrix} \mathbf{C}_{\mathrm{h}}. \tag{24}$$

We fix $\mathbf{C}_+(K)$ by the normalization condition

$$\lim_{K \to \zeta} \mathbf{Y}_+(\zeta, K) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \mathbf{C}_+ = \begin{pmatrix} F & 0 & 0 \\ G & 1 & L \\ H & 0 & 1 \end{pmatrix} \tag{25}$$

and the functions $F(K)$, $G(K)$, $H(K)$ and $L(K)$, for $K = \zeta$, are given by the potentials $\mathscr{E} = \mathscr{E}_+$, $\Phi = \Phi_+$ on $\mathscr{A}^+$:

$$F(\zeta) = [f_+(\zeta)]^{-1}, \quad G(\zeta) = \left[|\Phi_+(\zeta)|^2 + i\Im\,\mathscr{E}_+(\zeta)\right][f_+(\zeta)]^{-1}, \tag{26}$$

$$H(\zeta) = -2\bar{\Phi}_+(\zeta)[f_+(\zeta)]^{-1}, \quad L(\zeta) = -\Phi_+(\zeta) \tag{27}$$

and, vice versa,

$$\mathscr{E}_+(\zeta) = \frac{1 - \bar{G}(\zeta)}{F(\zeta)}, \quad \Phi_+(\zeta) = -\frac{\bar{H}(\zeta)}{2F(\zeta)}. \tag{28}$$

We can calculate $C_0(K)$, $\mathbf{C}_1(K)$ and $\mathbf{C}_2(K)$ of relations (17–19) for our normalization:

$$C_0 = -2F, \quad \mathbf{C}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{C}_2 = \begin{pmatrix} 0 & 2F & 0 \\ 2F & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{29}$$

On $\mathscr{A}^+$, (19) reads

$$[\mathbf{C}_+(\bar{K})]^\dagger \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{C}_+(K) = \begin{pmatrix} 0 & 2F & 0 \\ 2F & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{30}$$

From continuity conditions at the "poles" of the horizon ($\rho = 0$, $\zeta = \pm l$ or $r = 0$, $\theta = 0, \pi$; see Fig. 1) and using the boundary conditions, one can calculate $\mathbf{C}_h(K)$ and $\mathbf{C}_-(K)$ in terms of $\mathbf{C}_+(K)$, for details I refer to [7]. Closing the path of integration via infinity (curve $\mathscr{C}$: $\rho = R \sin \theta$, $\zeta = R \cos \theta$ with $0 \leq \theta \leq \pi$, $R \to \infty$), where $\mathbf{Y}$ is constant because of the LP and (5), but λ changes from ± 1 at $\theta = 0$ to ∓ 1 at $\theta = \pi$, we obtain with (18) and (29) an explicit expression for $\mathbf{C}_+(K)$ in terms of the parameters Ω, l (with $l = 0$ for a horizon at $r = 0$) and the values of the Ernst potentials at the poles. Using (28), we can calculate $\mathscr{E}_+$ and Φ_+. The number of free real parameters is reduced to four as a consequence of the constraint (30) and to three if no magnetic monopole is allowed. The final result is

$$F(K) = \frac{(K - L_1)(K - L_2)}{(K - K_1)(K - K_2)}, \quad G(K) = \frac{Q^2 - 2iJ}{(K - K_1)(K - K_2)}, \tag{31}$$

$$H(K) = -\frac{2Q(K - L_1)}{(K - K_1)(K - K_2)}, \quad L(K) = -\frac{Q}{K - L_1} \tag{32}$$

$$\text{with} \quad L_{1/2} = -M \pm i\frac{J}{M}, \quad K_{1/2} = \pm\sqrt{M^2 - Q^2 - \frac{J^2}{M^2}} \tag{33}$$

and, correspondingly,

$$\mathscr{E}_+(\zeta) = 1 - \frac{2M}{\zeta + M - iJ/M}, \quad \Phi_+(\zeta) = \frac{Q}{\zeta + M - iJ/M} \tag{34}$$

together with the parameter relations

$$\frac{l^2}{M^2} + \frac{Q^2}{M^2} + \frac{J^2}{M^4} = 1 \quad \text{and} \quad \Omega M = \frac{J/M^2}{(1 + l/M)^2 + J^2/M^4}. \tag{35}$$

3.3 Solution everywhere outside the horizon

Relation (18) together with the expression for $\mathbf{C}_1(K)$ in (29) is equivalent to the following structure of $\mathbf{Y}$:

$$\mathbf{Y}(\rho,\zeta,\lambda) = \begin{pmatrix} \psi(\rho,\zeta,\lambda) & \psi(\rho,\zeta,-\lambda) & \alpha(\rho,\zeta,\lambda) \\ \chi(\rho,\zeta,\lambda) & -\chi(\rho,\zeta,-\lambda) & \beta(\rho,\zeta,\lambda) \\ \varphi(\rho,\zeta,\lambda) & \varphi(\rho,\zeta,-\lambda) & \gamma(\rho,\zeta,\lambda) \end{pmatrix}, \tag{36}$$

where $\alpha(\rho,\zeta,\lambda) = \alpha(\rho,\zeta,-\lambda)$, $\beta(\rho,\zeta,\lambda) = -\beta(\rho,\zeta,-\lambda)$ and $\gamma(\rho,\zeta,\lambda) = \gamma(\rho,\zeta,-\lambda)$. The general solution of the LP for $K \to \infty$ and $\lambda = +1$ reads

$$\mathbf{Y}(\rho,\zeta,1) = \begin{pmatrix} \bar{\mathscr{E}} + 2|\Phi|^2 & 1 & \Phi \\ \mathscr{E} & -1 & -\Phi \\ 2\bar{\Phi} & 0 & 1 \end{pmatrix} \mathbf{C}, \tag{37}$$

where $\mathbf{C}$ is a constant matrix. Eqs. (23, 25, 31, 32) imply $\mathbf{C} = \mathbf{1}$ and lead to the ansatz

$$\psi = 1 + k_1 \left(\frac{1}{\kappa_1 - \lambda} - \frac{1}{\kappa_1 + 1} \right) + k_2 \left(\frac{1}{\kappa_2 - \lambda} - \frac{1}{\kappa_2 + 1} \right), \tag{38}$$

$$\chi = 1 + l_1 \left(\frac{1}{\kappa_1 - \lambda} - \frac{1}{\kappa_1 + 1} \right) + l_2 \left(\frac{1}{\kappa_2 - \lambda} - \frac{1}{\kappa_2 + 1} \right), \tag{39}$$

$$\varphi = m_1 \left(\frac{1}{\kappa_1 - \lambda} - \frac{1}{\kappa_1 + 1} \right) + m_2 \left(\frac{1}{\kappa_2 - \lambda} - \frac{1}{\kappa_2 + 1} \right), \tag{40}$$

$$\alpha = \Phi + \frac{\alpha_0}{K - L_1}, \quad \beta = -\Phi \frac{\lambda(K + \mathrm{i}z)}{K - L_1}, \quad \gamma = 1 + \frac{\gamma_0}{K - L_1}, \tag{41}$$

where

$$\kappa_\mu = \sqrt{\frac{K_\mu - \mathrm{i}\bar{z}}{K_\mu + \mathrm{i}z}} \quad (\mathscr{A}^+: \quad \kappa_\mu = +1). \tag{42}$$

According to the LP, $\mathbf{Y}_{,z}\mathbf{Y}^{-1}$ and $\mathbf{Y}_{,\bar{z}}\mathbf{Y}^{-1}$ must be holomorphic functions of λ for all $\lambda \neq 0, \infty$. The regularity at $\lambda = \pm\kappa_\mu$ ($\mu = 1,2$), the poles of the first two columns of $\mathbf{Y}$, is automatically guarantied, whereas regularity at $\lambda = \pm\lambda_\mu$ with $\lambda_\mu = \sqrt{(L_\mu - \mathrm{i}\bar{z})/(L_\mu + \mathrm{i}z)}$ ($\mathscr{A}^+$: $\lambda_\mu = +1$), where poles of the third column ($\mu = 1$) and zeros of $\det\mathbf{Y}$ ($\mu = 1,2$) occur, see (17, 29, 31), is equivalent to a set of linear algebraic equations, which together with (23, 25, 31, 32) uniquely determine the unknowns $k_\mu(\rho,\zeta)$, $l_\mu(\rho,\zeta)$, $m_\mu(\rho,\zeta)$, $\alpha_0(\rho,\zeta)$, $\gamma_0(\rho,\zeta)$ and $\Phi(\rho,\zeta)$. With $\mathscr{E}(\rho,\zeta) = \chi(\rho,\zeta,1)$, see (37), this leads to the result

$$\mathscr{E} = 1 - \frac{2M}{\tilde{r} - \mathrm{i}(J/M)\cos\tilde{\theta}}, \quad \Phi = \frac{Q}{\tilde{r} - \mathrm{i}(J/M)\cos\tilde{\theta}} \tag{43}$$

$$\text{with} \quad \rho = \sqrt{\tilde{r}^2 - 2M\tilde{r} + J^2/M^2 + Q^2}\,\sin\tilde{\theta}, \quad \zeta = (\tilde{r} - M)\cos\tilde{\theta}. \tag{44}$$

The "domain of outer communication" (the region outside the event horizon $\mathscr{H}$) is given by $\tilde{r} > \tilde{r}_h = M + \sqrt{M^2 - J^2/M^2 - Q^2}$. The horizon itself is characterized by $\tilde{r} = \tilde{r}_h$, and the axis of symmetry is located at $\tilde{\theta} = 0$ ($\mathscr{A}^+$) and $\tilde{\theta} = \pi$ ($\mathscr{A}^-$). Note that (35) implies $Q^2 + J^2/M^2 \leq M^2$. The equality sign, corresponding to $l = 0$, is valid for the extremal Kerr–Newman black hole.

3.4 Full metric and electromagnetic field

Using Eqs. (2, 6, 7, 10) we can calculate the full metric and the electromagnetic four-potential:

$$ds^2 = \frac{\Sigma}{\Delta}\,d\tilde{r}^2 + \Sigma\,d\tilde{\theta}^2 + \left(\tilde{r}^2 + a^2 + \frac{(2M\tilde{r} - Q^2)a^2\sin^2\tilde{\theta}}{\Sigma}\right)\sin^2\tilde{\theta}\,d\phi^2 \quad (45)$$

$$- \frac{(2M\tilde{r} - Q^2)2a\sin^2\tilde{\theta}}{\Sigma}\,d\phi\,dt - \left(1 - \frac{2M\tilde{r} - Q^2}{\Sigma}\right)dt^2 \quad (46)$$

$$\text{with} \quad \Delta = \tilde{r}^2 - 2M\tilde{r} + a^2 + Q^2, \quad \Sigma = \tilde{r}^2 + a^2\cos^2\tilde{\theta}, \quad a \equiv J/M \quad (47)$$

and

$$A_i\,dx^i = \frac{Q\tilde{r}}{\Sigma}(a\sin^2\tilde{\theta}\,d\phi - dt). \quad (48)$$

This is the well-known Kerr–Newman solution in Boyer–Lindquist coordinates $\tilde{r}$ and $\tilde{\theta}$. For $Q = 0$ it reduces to the Kerr solution, $J = 0$ gives the Reissner–Nordström solution and $Q = J = 0$ leads back to the Schwarzschild solution.

References

1. Belinski, V., Verdaguer, E.: Gravitational Solitons. Cambridge University Press, Cambridge (2001)
2. Carter, B.: Black hole equilibrium states. In: DeWitt, C., DeWitt, B.S. (eds.) Black Holes, pp. 57–214. Gordon and Breach Science Publishers, New York (1973)
3. Chruściel, P.T., Costa, J.L., Heusler, M.: Stationary black holes: uniqueness and beyond. Living Rev. Relativity **15**, 7 (2012)
4. Ernst, F.J.: New formulation of the axially symmetric gravitational field problem. II. Phys. Rev. **168**, 1415–1417 (1968)
5. Hauser, I., Ernst, F.J.: A homogeneous Hilbert problem for the Kinnersley–Chitre transformations of electrovac space-times. J. Math. Phys. **21**, 1418–1422 (1980)
6. Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Space-Time. Cambridge University Press, Cambridge (1973)
7. Meinel, R.: Constructive proof of the Kerr–Newman black hole uniqueness including the extreme case. Class. Quant. Grav. **29**, 035004 (2012)
8. Meinel, R., Richter, R.: Constructive proof of the Kerr–Newman black hole uniqueness: derivation of the full solution from scratch. arXiv:1208.0294 [gr-qc] (2012)
9. Meinel, R., Ansorg, M., Kleinwächter, A., Neugebauer, G., Petroff, D.: Relativistic Figures of Equilibrium. Cambridge University Press, Cambridge (2008)
10. Neugebauer, G.: Rotating bodies as boundary value problems. Ann. Phys. (Leipzig) **9**, 342–354 (2000)
11. Neugebauer, G., Kramer, D.: Einstein–Maxwell solitons. J. Phys. A **16**, 1927–1936 (1983)
12. Neugebauer, G., Meinel, R.: General relativistic gravitational field of a rigidly rotating disk of dust: solution in terms of ultraelliptic functions. Phys. Rev. Lett. **75**, 3046–3047 (1995)
13. Neugebauer, G., Meinel, R.: Progress in relativistic gravitational theory using the inverse scattering method. J. Math. Phys. **44**, 3407–3429 (2003)
14. Sibgatullin, N.R.: Oscillations and Waves in Strong Gravitational and Electromagnetic Fields. Springer, Berlin (1991)