Doubling of physical states in the quantum scalar field theory for a remote observer in the Schwarzschild space-time

Vadim Egorov, Mikhail Smolyakov, Igor Volobuev

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

Abstract

We discuss the problem of canonical quantization of a free real massive scalar field in the Schwarzschild space-time. It is shown that a consistent procedure of canonical quantization of the field can be carried out without taking into account the black hole interior, so that in the resulting theory the canonical commutation relations are satisfied exactly, and the Hamiltonian has the standard form. However, unlike some papers, in which the expansion of the quantum field in spherical harmonics was used, here we use an expansion in scattering states for energies larger than the mass of the field. This reveals a strange property of the resulting quantum field theory — doubling of the quantum states, which look as having the same fixed momentum to an observer located far away from the black hole. This purely topological effect poses a question about the existence of black holes with event horizons.

1 Introduction

The problem of quantization of fields in the presence of black holes is widely discussed in the scientific literature, starting from the pioneering papers [1, 2]. However, there still exist some unsolved problems within the standard treatment of quantum theory in the black hole background. For example, it is well known that in the Kruskal–Szekeres coordinates [3, 4], which describe the maximal analytic extension of the Schwarzschild space-time, there exists a second, so called "white hole". The latter reveals some problems with a physical interpretation of the resulting theory, in particular, there arises the well-known problem with locality, which is connected with the location of the white hole in our Universe or even in a parallel world. The amount of scientific literature on this topic is large, so we would like to highlight the latest papers by G. 't Hooft [5–7], in which mathematically rigorous attempts to solve this problem for the Schwarzschild solution were made. In papers [5, 6] a geometrical identification of some areas in the Kruskal–Szekeres space-time (which was called "antipodal identification") was proposed. Ideologically this identification is very similar to orbifolding in models with extra dimensions (see, for example, [8]). However, later it was shown [7] that such an identification leads to contradictions (namely, problems with CPT–invariance). Instead of the antipodal identification, the idea of "quantum cloning" of the black and white holes exteriors was proposed in [7]. An interesting property of the approach is that the interior regions of both holes do not play any role in the evolution and turn out to be mathematical artifacts that do not have a direct physical interpretation. However, the approach still has a drawback — there may emerge closed time-like curves [7].

According to the reasonings presented above (especially to the observation that at least some rigorous approaches to obtaining a consistent quantum theory in the black hole background lead to the needlessness of the black and white hole interiors), there arises a question about a possibility to build a consistent quantum field theory in the Schwarzschild space-time outside the horizon only. As we will see below, it is indeed possible even within the framework of canonical quantization, which is quite a surprising result. At least in the simplest case of a real massive scalar field, formally the resulting theory turns out to be complete and self-consistent.

Unlike some recent papers [9–11], in which the quantum scalar field is also considered only outside the event horizon of the Schwarzschild black hole, we use the field expansion in the scattering states, which are close to plane waves, if we go far away from the black hole. These states are very useful, because at large distances from the black hole they coincide with the states that are used in quantization of the scalar field in Minkowski space-time. This approach reveals an alarming feature of the resulting quantum field theory — doubling of the quantum states, which look as having the same fixed momentum to a distant observer. It is a purely topological effect that manifests itself even at such distances from the Schwarzschild black hole at which one may naively expect that the effects caused by the black hole can be neglected.

This paper is rather technical in the sense that it involves a detailed examination of the spectrum of states outside the Schwarzschild black hole, all canonical commutations relations are checked to be exactly satisfied, and the resulting Hamiltonian is obtained in the explicit form. All this is done in order to make sure that the resulting theory is indeed self-consistent and does not contain any contradictions.

2 Setup

Let us take a real massive scalar field $\phi(t,\vec{x})$ in a curved background described by the metric $g_{\mu\nu}$. The action of the theory is

$$S=\int\mathcal{L}\,d^4x=\int\sqrt{-g}\left(\frac{1}{2}\,g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi-\frac{M^2}{2}\phi^2\right)d^4x. \tag{1}$$

Suppose that the metric $g_{\mu\nu}$ is static, i.e., it does not depend on time. In such a case the equation of motion for the scalar field takes the form

$$\sqrt{-g}\,g^{00}\ddot{\phi}+\partial_i\left(\sqrt{-g}\,g^{ij}\partial_j\phi\right)+M^2\sqrt{-g}\,\phi=0, \tag{2}$$

where $\dot{\phi}=\partial_0\phi$. It is clear that the field $\phi(t,\vec{x})$ can be expanded in solutions of the form

$$e^{\pm iEt}F(E,\vec{x}), \tag{3}$$

where, without loss of generality, we can set $E\geq 0$. The latter representation leads to the equation

$$-E^2\sqrt{-g}\,g^{00}F+\partial_i\left(\sqrt{-g}\,g^{ij}\partial_jF\right)+M^2\sqrt{-g}\,F=0, \tag{4}$$

which, together with corresponding boundary conditions, defines an eigenvalue problem. In particular, Eq. (4) implies the following orthogonality conditions for the solutions with $E \neq E'$

$$\int \sqrt{-g}\, g^{00} F^*(E,\vec{x}) F(E',\vec{x}) d^3x = 0, \qquad \int \sqrt{-g}\, g^{00} F(E,\vec{x}) F(E',\vec{x}) d^3x = 0, \tag{5}$$

which can be easily obtained by multiplying the complex conjugate of Eq. (4) (or Eq. (4) as it is) by $F(E',\vec{x})$, integrating the result with respect to $\vec{x}$ and performing an integration by parts.

The component T_{00} of the energy-momentum tensor of a real massive scalar field has the standard form

$$T_{00} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2} g_{00} g^{ij} \partial_i \phi\, \partial_j \phi + \frac{M^2}{2} g_{00} \phi^2. \tag{6}$$

It is well known that in General Relativity the covariant conservation law is satisfied for any energy-momentum tensor and can be rewritten as [12]

$$\nabla_\mu T^\mu_\nu = \frac{1}{\sqrt{-g}} \frac{\partial\left(\sqrt{-g}\, T^\mu_\nu\right)}{\partial x^\mu} - \frac{1}{2} \frac{\partial g_{\mu\sigma}}{\partial x^\nu} T^{\mu\sigma} = 0. \tag{7}$$

Since we are interested in the cases, in which the metric is static (the Schwarzschild metric is exactly of this sort), it follows from (7) that for $\nu = 0$

$$\frac{\partial}{\partial x^0} \int \sqrt{-g}\, T^0_0 d^3x = 0. \tag{8}$$

Thus, we can define the energy of the system, which is conserved over time (i.e., the Hamiltonian of the system), as

$$H = \int \sqrt{-g}\, g^{00} T_{00} d^3x. \tag{9}$$

Substituting the explicit expression (6) into (9), performing an integration by parts and using equation of motion (2), we arrive at

$$H = \frac{1}{2} \int \sqrt{-g}\, g^{00} \left(\dot{\phi}^2 - \ddot{\phi}\, \phi\right) d^3x. \tag{10}$$

3 Solutions of the equation of motion

3.1 Properties of the spectrum

Let us start with a detailed examination of the spectrum of stationary states of Eq. (4). Let us consider the standard Schwarzschild metric and restrict ourselves to the domain $r > r_0$, where r_0 is the Schwarzschild radius. The field $F(E,\vec{x})$ can be expanded in spherical harmonics as

$$F(E,\vec{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_{lm}(E,r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta,\varphi) f_l(E,r), \tag{11}$$

where (we use the convention of [13])

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-|m|)!}{(l+|m|)!}}\, P_l^{|m|}(\cos\theta)\, e^{im\varphi}, \quad l = 0,1,2,..., \quad m = 0,\pm 1,\pm 2,..., \tag{12}$$

leading to the radial equation

$$E^2 \frac{r}{r-r_0} f_l(E,r) - M^2 f_l(E,r) + \frac{1}{r^2}\frac{d}{dr}\left(r(r-r_0)\frac{f_l(E,r)}{dr}\right) - \frac{l(l+1)}{r^2} f_l(E,r) = 0 \qquad (13)$$

for the functions $f_l(E,r)$. Without loss of generality we can take $f_l(E,r)$ to be real. The orthogonality condition for $f_l(E,r)$ is suggested by the form of Eq. (13)

$$\int\limits_{r_0}^{\infty} \frac{r^3}{r-r_0} f_l(E,r) f_l(E',r)\, dr = 0 \quad \text{for} \quad E \neq E', \qquad (14)$$

as well as the form of the norm

$$\int\limits_{r_0}^{\infty} \frac{r^3}{r-r_0} f_l^2(E,r)\, dr. \qquad (15)$$

Using Eq. (13), it is possible to show that there is no solution with $E = 0$. Indeed, for $r \to r_0$ a possible solution takes the form $f_l(0,r) \sim 1 + \left(r_0 M^2 + \frac{l(l+1)}{r_0}\right)(r - r_0)$, whereas for $r \to \infty$ it takes the form $f_l(0,r) \sim \frac{e^{-Mr}}{r}$. Let us multiply Eq. (13) by $r^2 f_l(r,0)$, integrate the result with respect to r from r_0 to ∞, and perform an integration by parts. We get

$$r(r-r_0)\frac{df_l(0,r)}{dr} f_l(0,r)\Big|_{r_0}^{\infty} - \int\limits_{r_0}^{\infty} \left(r(r-r_0)\left(\frac{df_l(0,r)}{dr}\right)^2 + \left(M^2 r^2 + l(l+1)\right) f_l^2(0,r)\right) dr = 0. \qquad (16)$$

Since the surface terms in (16) are equal to zero, we get

$$\int\limits_{r_0}^{\infty} \left(r(r-r_0)\left(\frac{df_l(r,0)}{dr}\right)^2 + \left(M^2 r^2 + l(l+1)\right) f_l^2(r,0)\right) dr = 0. \qquad (17)$$

The integrand in (17) is nonnegative for any r, which means that the only solution satisfying (17) is $f_l(0,r) \equiv 0$.

It should be noted that there is a controversy in the scientific literature concerning the properties of the spectrum of eigenfunctions of radial equation (13). For example, in the well-known paper [14] it is stated that the spectrum of states for $E < M$ is discrete (though each state has an infinite norm). In paper [15] it is shown that the spectrum is continuous and the radial solutions can be expressed in terms of the Heun functions.[1] In paper [18] it is stated that from a quantum mechanical point of view there exists the “fall to the center” regime [19,20] on the event horizon making the whole theory ill-behaved. To the best of our knowledge, the only paper in which the properties of the spectrum of the radial equation (13) are correctly described from a physical point of view without going into details (i.e., without obtaining explicit solutions like it was done in paper [15]) is [21]. So, below in this section we will reproduce the results of [21], though in more detail.

First, let us introduce the dimensionless variables

$$\rho = \frac{r}{r_0}, \qquad \mu = M r_0, \qquad \epsilon = E r_0, \qquad u_l(\epsilon,\rho) = r_0 f_l(E,r), \qquad (18)$$

[1] See also [16] for the Heun functions in the case of Regge-Wheeler equation [17].

where $\rho > 1$. In these variables Eq. (13) takes the form

$$-\frac{d}{d\rho}\left(\rho(\rho-1)\frac{du_l(\epsilon,\rho)}{d\rho}\right)+\left(\mu^2\rho^2+l(l+1)\right)u_l(\epsilon,\rho)=\epsilon^2\frac{\rho^3}{\rho-1}u_l(\epsilon,\rho). \tag{19}$$

Second, let us pass to the tortoise coordinate

$$z=\rho+\ln(\rho-1). \tag{20}$$

Then Eq. (19) takes the form

$$-\frac{d}{dz}\left(\rho^2(z)\frac{du_l(\epsilon,z)}{dz}\right)+\frac{\rho(z)-1}{\rho(z)}\left(\mu^2\rho^2(z)+l(l+1)\right)u_l(\epsilon,z)=\epsilon^2\rho^2(z)u_l(\epsilon,z), \tag{21}$$

where $\rho(z)$ is determined by (20). And third, using the substitution

$$u_l(\epsilon,z)=\frac{\psi_l(\epsilon,z)}{\rho(z)}, \tag{22}$$

Eq. (21) can be expressed in the form of a one-dimensional Schrödinger equation

$$-\frac{d^2\psi_l(\epsilon,z)}{dz^2}+V_l(z)\psi_l(\epsilon,z)=\epsilon^2\psi_l(\epsilon,z), \tag{23}$$

where the potential has the form [21]

$$V_l(z)=\frac{\rho(z)-1}{\rho(z)}\left(\mu^2+\frac{l(l+1)}{\rho^2(z)}+\frac{1}{\rho^3(z)}\right). \tag{24}$$

The potential $V_l(z)$ is such that $V_l(z)\to 0$ for $z\to-\infty$ and $V_l(z)\to\mu^2$ for $z\to\infty$. In Fig. 1 some examples of $V_l(z)$ are presented. Fig. 1 also supports the fact that $f_l(0,r)\equiv 0$. One can

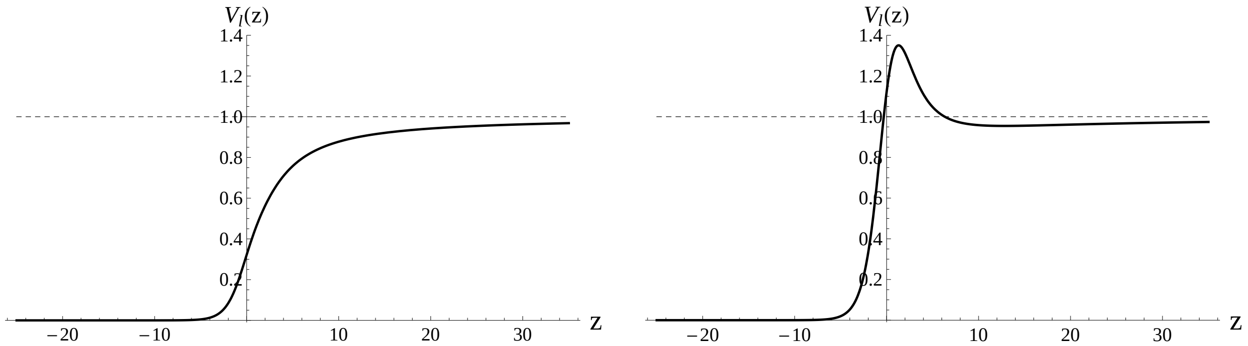

Figure 1: $V_l(z)$ for $\mu=1$: $l=0$ (left plot) and $l=2$ (right plot). Dashed lines stand for μ^2.

see that the asymptotic behavior of $V_l(z)$ corresponds to a step-like potential, though there can be rises and dips in the vicinity of $z=0$ depending on the value of l.[2] Such potentials imply the continuity of the spectrum for $\epsilon>0$.

For the initial norm (15) we get the result which is expected taking into account the form of Eq. (23):

$$\int\limits_1^\infty \frac{\rho^3}{\rho-1}u_l^2(\epsilon,\rho)d\rho=\int\limits_{-\infty}^{\infty}\rho^2(z)u_l^2(\epsilon,z)dz=\int\limits_{-\infty}^{\infty}\psi_l^2(\epsilon,z)dz. \tag{25}$$

[2] A detailed discussion of scattering by the square potential step can be found, for example, in [22].

Now let us turn to the discussion of the properties of the radial solutions $\psi_l(\epsilon, z)$. For $\epsilon < \mu$ and $z \to \infty$ formally there exist two asymptotics $\sim e^{\pm\sqrt{\mu^2-\epsilon^2}\, z}$ leading to a constant Wronskian. However, the only solution which can be properly normalized is the one with the asymptotic $\sim e^{-\sqrt{\mu^2-\epsilon^2}\, z}$ at $z \to \infty$, for $z \to -\infty$ the asymptotic of this solution is $\sim \cos(\epsilon z - \gamma_l)$, where γ_l is some phase. Thus, for a fixed ϵ and l there exists only one physically relevant solution. An important point is that the spectrum of radial states for $0 < \epsilon < \mu$ is continuous, which is provided by the phase γ_l [21].

For $\epsilon > \mu$ the situation is different. Since the asymptotics of solutions are finite at $z \to \pm\infty$, for fixed ϵ and l there exist two linearly independent solutions. In principle, these solutions can be connected with solutions corresponding to the waves moving towards $z \to \infty$ and towards $z \to -\infty$ (of course, one should take into account one-dimensional scattering by the potential $V_l(z)$ in these solutions) [10]. However, without loss of generality these linearly independent solutions can be chosen to be real, and we will denote them by $\psi_{lj}(\epsilon, z)$, where $j = 1, 2$.

It is necessary to note that at $z \to -\infty$ (i.e., at $r \to r_0$) the potential $V_l(z)$ vanishes, which means that the field becomes effectively massless in this area.

3.2 Orthogonality conditions and completeness relation for the eigenfunctions

It is clear that the functions $\psi_l(\epsilon, z)$ and $\psi_{lj}(\epsilon, z)$ can be normalized in such a way that the orthogonality conditions

$$\int_{-\infty}^{\infty} \psi_l(\epsilon, z)\psi_l(\epsilon', z)dz = \delta(\epsilon - \epsilon'), \tag{26}$$

$$\int_{-\infty}^{\infty} \psi_l(\epsilon, z)\psi_{lj}(\epsilon', z)dz = 0, \tag{27}$$

$$\int_{-\infty}^{\infty} \psi_{lj}(\epsilon, z)\psi_{lj'}(\epsilon', z)dz = \delta_{jj'}\delta(\epsilon - \epsilon') \tag{28}$$

hold. Then, using (18), (20) and (22), we get the orthogonality conditions for the initial radial functions $f_l(E, r)$ and $f_{lj}(E, r)$

$$\int_{r_0}^{\infty} f_l(E, r)f_l(E', r)\frac{r^3}{r - r_0}\, dr = \delta(E - E'), \tag{29}$$

$$\int_{r_0}^{\infty} f_l(E, r)f_{lj}(E', r)\frac{r^3}{r - r_0}\, dr = 0, \tag{30}$$

$$\int_{r_0}^{\infty} f_{lj}(E, r)f_{lj'}(E', r)\frac{r^3}{r - r_0}\, dr = \delta_{jj'}\delta(E - E'). \tag{31}$$

With (11), we get the resulting orthogonality conditions

$$\int_0^{2\pi}\int_0^{\pi}\int_{r_0}^{\infty} \phi^*_{lm}(E,r,\theta,\varphi)\phi_{l'm'}(E',r,\theta,\varphi)\frac{r^3}{r-r_0}\sin\theta\, drd\theta d\varphi = \delta_{ll'}\delta_{mm'}\delta(E-E'), \tag{32}$$

$$\int_0^{2\pi}\int_0^{\pi}\int_{r_0}^{\infty} \phi^*_{lm}(E,r,\theta,\varphi)\phi_{l'm'j}(E',r,\theta,\varphi)\frac{r^3}{r-r_0}\sin\theta\, drd\theta d\varphi = 0, \tag{33}$$

$$\int_0^{2\pi}\int_0^{\pi}\int_{r_0}^{\infty} \phi^*_{lmj}(E,r,\theta,\varphi)\phi_{l'm'j'}(E',r,\theta,\varphi)\frac{r^3}{r-r_0}\sin\theta\, drd\theta d\varphi = \delta_{jj'}\delta_{ll'}\delta_{mm'}\delta(E-E'), \tag{34}$$

where

$$\phi_{lmj}(E,r,\theta,\varphi) = Y_{lm}(\theta,\varphi)f_{lj}(E,r). \tag{35}$$

Since in Eq. (23) we have the standard Hermitian operator, the eigenfunctions of this equation form a complete set (see, for example, [13]). Taking into account the fact that the normalization on the "energy scale" (not the "energy scale" squared) was chosen in (26)–(28), the completeness relation for the radial functions $\psi_l(\epsilon,z)$ also has the standard form and looks like

$$\int_0^{\mu}\psi_l(\epsilon,z)\psi_l(\epsilon,z')\,d\epsilon + \sum_{j=1}^{2}\int_{\mu}^{\infty}\psi_{lj}(\epsilon,z)\psi_{lj}(\epsilon,z')\,d\epsilon = \delta(z-z'). \tag{36}$$

Consequently, the set of the corresponding solutions in the initial coordinate r also forms a complete set of eigenfunctions, the corresponding completeness relation

$$\int_0^{M} f_l(E,r)f_l(E,r')\,dE + \sum_{j=1}^{2}\int_{M}^{\infty} f_{lj}(E,r)f_{lj}(E,r')\,dE = \frac{r-r_0}{r^3}\,\delta(r-r') \tag{37}$$

can be easily obtained by substituting (18) and (22) into (36). Taking into account the angular parts of the eigenfunctions $Y_{lm}(\theta,\varphi)$, we can finally write

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\int_0^{M}\phi^*_{lm}(E,r,\theta,\varphi)\phi_{lm}(E,r',\theta',\varphi')\,dE + \sum_{j=1}^{2}\int_{M}^{\infty}\phi^*_{lmj}(E,r,\theta,\varphi)\phi_{lmj}(E,r',\theta',\varphi')\,dE\right)$$
$$= \frac{r-r_0}{r^3}\,\delta(r-r')\delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi'). \tag{38}$$

3.3 Scattering states

As we will see below, from a physical point of view, for the energies larger than the mass of the field, it is more convenient to use the scattering states instead of the states $\phi_{lmj}(E,r,\theta,\varphi)$ described above. At large r these states look like

$$\phi_j(\vec{k},\vec{x}) \sim e^{i\vec{k}\vec{x}} + A_j(\vec{k},\vec{n})\frac{e^{ikr}}{r}, \qquad j=1,2, \tag{39}$$

where $A_j(\vec{k},\vec{n})$ are the scattering amplitudes, $k=|\vec{k}|$, $r=|\vec{x}|$, and $\vec{n}=\frac{\vec{x}}{r}$. These solutions can be expressed through the real solutions $f_{lj}\left(\sqrt{k^2+M^2},r\right)$ of Eq. (13) as [23]

$$\phi_j(\vec{k},\vec{x})=\frac{1}{4\pi k}\sum_{l=0}^{\infty}(2l+1)e^{i\left(\frac{\pi l}{2}+\tilde{\delta}_{lj}(k)\right)}P_l\left(\frac{\vec{k}\vec{x}}{kr}\right)\left(\frac{\sqrt{k}}{(k^2+M^2)^{\frac{1}{4}}}\,f_{lj}\left(\sqrt{k^2+M^2},r\right)\right), \qquad (40)$$

where $P_l(...)$ are the Legendre polynomials and $\tilde{\delta}_{lj}(k)$ are phase shifts defined by the following representation of the radial solutions $f_{lj}(\sqrt{k^2+M^2},r)$ for $r\to\infty$ (like in the standard scattering theory [23]):

$$f_{lj}(E,r)\sim\frac{\sin\left(kr-\frac{\pi l}{2}+\tilde{\delta}_{lj}(k)\right)}{r}. \qquad (41)$$

The extra factor $\frac{\sqrt{k}}{(k^2+M^2)^{1/4}}$ in (40) is introduced in order to have

$$\int_{r_0}^{\infty}\left(\frac{\sqrt{k}\,f_{lj}\left(\sqrt{k^2+M^2},r\right)}{(k^2+M^2)^{\frac{1}{4}}}\right)\left(\frac{\sqrt{k'}\,f_{lj'}\left(\sqrt{k'^2+M^2},r\right)}{\left(k'^2+M^2\right)^{\frac{1}{4}}}\right)\frac{r^3}{r-r_0}\,dr=\delta_{jj'}\delta(k-k') \qquad (42)$$

and, consequently, to get a more physically reasonable normalization for the scattering states. It is evident that $\phi_j(\vec{k},\vec{x})$ defined by (40) is a solution of Eq. (4) with $E=\sqrt{k^2+M^2}$.

We would like to stress that, since for fixed l and k there exist two different solutions $f_{l1}\left(\sqrt{k^2+M^2},r\right)$ and $f_{l2}\left(\sqrt{k^2+M^2},r\right)$, for a fixed $\vec{k}$ we can build *two* scattering states of form (39) which differ in the scattering amplitudes.

It is clear that

$$\int_{r>r_0}\sqrt{-g}\,g^{00}\phi^*_{lm}(E,\vec{x})\phi_j(\vec{k},\vec{x})\,d^3x=0, \qquad (43)$$

because E in $\phi_{lm}(E,\vec{x})$ and $\sqrt{k^2+M^2}$ belong to different energy ranges ($\sqrt{k^2+M^2}>M$ and $E<M$). Thus, the orthogonality condition (43) follows directly from (5). In (43) the notation

$$\int_{r>r_0}\sqrt{-g}\,g^{00}d^3x=\int_{r_0}^{\infty}\frac{r^3}{r-r_0}\,dr\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi \qquad (44)$$

is used. The scattering states (40) are also orthogonal, the corresponding orthogonality condition takes the form

$$\int_{r>r_0}\sqrt{-g}\,g^{00}\phi^*_j(\vec{k},\vec{x})\phi_{j'}(\vec{k}',\vec{x})\,d^3x=\delta_{jj'}\delta^{(3)}(\vec{k}-\vec{k}'), \qquad (45)$$

a detailed proof can be found in Appendix A. The completeness relation involving the scattering states has the form

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^M\phi^*_{lm}(E,\vec{x})\phi_{lm}(E,\vec{y})\,dE+\sum_{j=1}^{2}\int\phi^*_j(\vec{k},\vec{x})\phi_j(\vec{k},\vec{y})\,d^3k=\frac{\delta^{(3)}(\vec{x}-\vec{y})}{\sqrt{-g}\,g^{00}}, \qquad (46)$$

a detailed proof can be found in Appendix B. In (46) the notation

$$\int d^3k = \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3 = \int_0^{\infty} k^2 dk \int_0^{\pi} \sin\theta_k d\theta_k \int_0^{2\pi} d\varphi_k, \tag{47}$$

where θ_k and φ_k are the angles in spherical coordinates in the momentum space, is used. The completeness relation (46) is necessary for performing a consistent procedure of quantization.[3]

4 Canonical quantization

4.1 Expansion of the quantum field

Usually, when a quantum theory outside the horizon of the Schwarzschild black hole is considered, the quantum scalar field $\phi(t,\vec{x})$ is expanded in spherical harmonics, i.e., an expansion of the form

$$\phi(t,r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^M \frac{dE}{\sqrt{2E}}\left(e^{-iEt}\phi_{lm}(E,r,\theta,\varphi)a_{lm}(E) + e^{iEt}\phi^*_{lm}(E,r,\theta,\varphi)a^\dagger_{lm}(E)\right)$$
$$+\sum_{j=1}^{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_M^{\infty} \frac{dE}{\sqrt{2E}}\left(e^{-iEt}\phi_{lmj}(E,r,\theta,\varphi)a_{lmj}(E) + e^{iEt}\phi^*_{lmj}(E,r,\theta,\varphi)a^\dagger_{lmj}(E)\right), \tag{48}$$

is used, see [9–11]. This expansion is quite natural for a spherically symmetric system like the one described by the Schwarzschild background. However, this expansion is not useful for examining the theory far away from the black hole, where we expect that the influence of the black hole can be neglected. Indeed, in Minkowski space-time we prefer to use the expansion in terms of plane waves, which provides a much more convenient description of quantum states. Of course, because of the Schwarzschild background we cannot use exactly the plane waves, however, we can use the scattering states described in the previous section for the expansion. These states behave like plane waves far away from what is considered as a "central potential", thus allowing one to describe particles in the usual manner in that area but rigorously take into account the effect produced by the "potential". Namely, let us take

$$\phi(t,\vec{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^M \frac{dE}{\sqrt{2E}}\left(e^{-iEt}\phi_{lm}(E,\vec{x})a_{lm}(E) + e^{iEt}\phi^*_{lm}(E,\vec{x})a^\dagger_{lm}(E)\right)$$
$$+\sum_{j=1}^{2}\int \frac{d^3k}{\sqrt{2\sqrt{k^2+M^2}}}\left(e^{-i\sqrt{k^2+M^2}\,t}\phi_j(\vec{k},\vec{x})a_j(\vec{k}) + e^{i\sqrt{k^2+M^2}\,t}\phi^*_j(\vec{k},\vec{x})a^\dagger_j(\vec{k})\right), \tag{49}$$

[3] Note that instead of the functions $\phi_j(\vec{k},\vec{x})$, we can take the set of functions $\tilde{\phi}_j(\vec{k},\vec{x}) = \phi^*_j(-\vec{k},\vec{x})$ which behave like $\tilde{\phi}_j(\vec{k},\vec{x}) \sim e^{i\vec{k}\vec{x}} + \tilde{A}_j(\vec{k},\vec{n})\frac{e^{-ikr}}{r}$ at large r [23]. Moreover, one can use, say, the set of functions $\phi_1(\vec{k},\vec{x})$ and $\tilde{\phi}_2(\vec{k},\vec{x})$ or vice versa. Together with $\phi_{lm}(E,\vec{x})$, they form a complete set of eigenfunctions. Since the results that will be obtained below do not depend on a particular choice of the set, in what follows we will use the functions $\phi_j(\vec{k},\vec{x})$.

where $\phi_{lm}(E,\vec{x}) = \phi_{lm}(E,r,\theta,\varphi)$ is defined by (11) with (12), $\phi_j(\vec{k},\vec{x})$ is defined by (40). We suppose that the creation and annihilation operators satisfy the standard commutation relations

$$[a_{lm}(E), a^{\dagger}_{l'm'}(E')] = \delta_{ll'}\delta_{mm'}\delta(E-E'), \tag{50}$$

$$[a_j(\vec{k}), a^{\dagger}_{j'}(\vec{k}')] = \delta_{jj'}\delta^{(3)}(\vec{k}-\vec{k}'), \tag{51}$$

all other commutators being equal to zero. As we will see below, this expansion is indeed more useful from a physical point of view.

4.2 Canonical commutation relations

A consistent procedure of canonical quantization demands that the canonical commutation relations are exactly satisfied. Let us check that it is indeed so for expansion (49). The canonical coordinate in a scalar field theory is $\phi(t,\vec{x})$, whereas the canonically conjugate momentum is

$$\pi(t,\vec{x}) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}(t,\vec{x})} = \sqrt{-g(\vec{x})}\, g^{00}(\vec{x})\dot{\phi}(t,\vec{x}). \tag{52}$$

The following canonical commutation relations should be satisfied:

$$[\phi(t,\vec{x}),\pi(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}), \qquad [\phi(t,\vec{x}),\phi(t,\vec{y})] = 0, \qquad [\pi(t,\vec{x}),\pi(t,\vec{y})] = 0. \tag{53}$$

Substituting (52) into (53), we get

$$[\phi(t,\vec{x}),\dot{\phi}(t,\vec{y})] = i\frac{\delta^{(3)}(\vec{x}-\vec{y})}{\sqrt{-g}\, g^{00}}, \tag{54}$$

$$[\phi(t,\vec{x}),\phi(t,\vec{y})] = 0, \tag{55}$$

$$[\dot{\phi}(t,\vec{x}),\dot{\phi}(t,\vec{y})] = 0. \tag{56}$$

Let us check that commutation relations (54)–(56) are satisfied for (49). Substituting (49) into the l.h.s. of (54) and using (50), (51), one gets

$$[\phi(t,\vec{x}),\dot{\phi}(t,\vec{y})] = \frac{1}{2}\left(\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^M\Big(\phi_{lm}(E,\vec{x})\phi^*_{lm}(E,\vec{y}) + \phi_{lm}(E,\vec{y})\phi^*_{lm}(E,\vec{x})\Big)dE \right.$$
$$\left. + \sum_{j=1}^{2}\int\Big(\phi_j(\vec{k},\vec{x})\phi^*_j(\vec{k},\vec{y}) + \phi_j(\vec{k},\vec{y})\phi^*_j(\vec{k},\vec{x})\Big)d^3k\right). \tag{57}$$

With the completeness relation (46), for (57) we get exactly (54).

Substituting (49) into the l.h.s. of (55), (56) and using (50), (51), one gets

$$[\phi(t,\vec{x}),\phi(t,\vec{y})] = \frac{1}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^M\Big(\phi_{lm}(E,\vec{x})\phi^*_{lm}(E,\vec{y}) - \phi_{lm}(E,\vec{y})\phi^*_{lm}(E,\vec{x})\Big)\frac{dE}{E}$$
$$+\frac{1}{2}\sum_{j=1}^{2}\int\Big(\phi_j(\vec{k},\vec{x})\phi^*_j(\vec{k},\vec{y}) - \phi_j(\vec{k},\vec{y})\phi^*_j(\vec{k},\vec{x})\Big)\frac{d^3k}{\sqrt{k^2+M^2}}, \tag{58}$$

$$[\dot\phi(t,\vec x),\dot\phi(t,\vec y)] = \frac{1}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int\limits_0^M \Big(\phi_{lm}(E,\vec x)\phi^*_{lm}(E,\vec y) - \phi_{lm}(E,\vec y)\phi^*_{lm}(E,\vec x)\Big)E\,dE$$
$$+\frac{1}{2}\sum_{j=1}^{2}\int \Big(\phi_j(\vec k,\vec x)\phi^*_j(\vec k,\vec y) - \phi_j(\vec k,\vec y)\phi^*_j(\vec k,\vec x)\Big)\sqrt{k^2+M^2}\,d^3k. \qquad (59)$$

Using formula (96) from Appendix B (see also (97)), the double sum in (58) can be represented as

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int\limits_0^M \Big(\phi_{lm}(E,\vec x)\phi^*_{lm}(E,\vec y) - \phi_{lm}(E,\vec y)\phi^*_{lm}(E,\vec x)\Big)\frac{dE}{E}$$
$$= \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\int\limits_0^M \Big(f_l(E,r)f_l(E,r') - f_l(E,r')f_l(E,r)\Big)\frac{dE}{E} = 0. \qquad (60)$$

Analogously, the double sum in (59) can be represented as

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int\limits_0^M \Big(\phi_{lm}(E,\vec x)\phi^*_{lm}(E,\vec y) - \phi_{lm}(E,\vec y)\phi^*_{lm}(E,\vec x)\Big)E\,dE$$
$$= \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\int\limits_0^M \Big(f_l(E,r)f_l(E,r') - f_l(E,r')f_l(E,r)\Big)E\,dE = 0. \qquad (61)$$

Next, using formulas (89)–(91) from Appendix B, the second line in (58) can be represented as

$$\sum_{j=1}^{2}\int \Big(\phi_j(\vec k,\vec x)\phi^*_j(\vec k,\vec y) - \phi_j(\vec k,\vec y)\phi^*_j(\vec k,\vec x)\Big)\frac{d^3k}{\sqrt{k^2+M^2}}$$
$$= \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\sum_{j=1}^{2}\int\limits_M^{\infty} \Big(f_{lj}\,(E,r)\,f_{lj}\,(E,r') - f_{lj}\,(E,r')\,f_{lj}\,(E,r)\Big)\frac{dE}{E} = 0, \qquad (62)$$

whereas the second line in (59) can be represented as

$$\sum_{j=1}^{2}\int \Big(\phi_j(\vec k,\vec x)\phi^*_j(\vec k,\vec y) - \phi_j(\vec k,\vec y)\phi^*_j(\vec k,\vec x)\Big)\sqrt{k^2+M^2}\,d^3k$$
$$= \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\sum_{j=1}^{2}\int\limits_M^{\infty} \Big(f_{lj}\,(E,r)\,f_{lj}\,(E,r') - f_{lj}\,(E,r')\,f_{lj}\,(E,r)\Big)E\,dE = 0. \qquad (63)$$

Substituting (60)–(63) into (58) and (59), we get exactly (55) and (56). Thus, all three canonical commutation relations are exactly satisfied for (49).

4.3 Hamiltonian

Now we turn to the calculation of the Hamiltonian of the system. Note that in addition to the orthogonality condition (43), the orthogonality conditions

$$\int\limits_{r>r_0} \sqrt{-g}g^{00}\phi_{lm}(E,\vec x)\phi_j(\vec k,\vec x)\,d^3x = 0, \qquad \int\limits_{r>r_0} \sqrt{-g}g^{00}\phi^*_{lm}(E,\vec x)\phi^*_j(\vec k,\vec x)\,d^3x = 0 \qquad (64)$$

hold as well. They are also a consequence of the fact that E in $\phi_{lm}(E,\vec{x})$ and $\sqrt{k^2+M^2}$ belong to different energy ranges ($\sqrt{k^2+M^2} > M$ and $E < M$) and follow directly from (5). Substituting (49) into (10) and using the orthogonality conditions (32), (43), (45) and (64), after straightforward calculations one gets

$$
\begin{aligned}
H = {} & \frac{1}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int\limits_{0}^{M} E\left(a_{lm}^{\dagger}(E)a_{lm}(E)+a_{lm}(E)a_{lm}^{\dagger}(E)\right)dE \\
& + \frac{1}{2}\sum_{j=1}^{2}\int \sqrt{k^2+M^2}\left(a_j^{\dagger}(\vec{k})a_j(\vec{k})+a_j(\vec{k})a_j^{\dagger}(\vec{k})\right)d^3k \\
& + \frac{1}{4}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'}\int\limits_{0}^{M}dE\int\limits_{0}^{M}dE'\left(\frac{E\sqrt{E}}{\sqrt{E'}}-\sqrt{EE'}\right) \\
& \times\left(e^{-i(E+E')t}a_{lm}(E)a_{lm}(E')\int\limits_{r>r_0}\sqrt{-g}g^{00}\phi_{lm}(E,\vec{x})\phi_{l'm'}(E',\vec{x})\,d^3x+\text{h.c.}\right) \\
& + \frac{1}{4}\sum_{j=1}^{2}\sum_{j'=1}^{2}\int d^3k\int d^3k'\left(\frac{(k^2+M^2)^{\frac{3}{4}}}{(k'^2+M^2)^{\frac{1}{4}}}-(k^2+M^2)^{\frac{1}{4}}(k'^2+M^2)^{\frac{1}{4}}\right) \\
& \times\left(e^{-i\left(\sqrt{k^2+M^2}+\sqrt{k'^2+M^2}\right)t}a_j(\vec{k})a_{j'}(\vec{k}')\int\limits_{r>r_0}\sqrt{-g}g^{00}\phi_j(\vec{k},\vec{x})\phi_{j'}(\vec{k}',\vec{x})\,d^3x+\text{h.c.}\right).
\end{aligned}
\tag{65}
$$

First, let us consider the term with $\phi_{lm}(E,\vec{x})$ and $\phi_{l'm'}(E',\vec{x})$. Using the explicit form of the functions $\phi_{lm}(E,\vec{x})$ and $\phi_{l'm'}(E',\vec{x})$ (see Appendix A for analogous calculations), it is not difficult to show that

$$
\int\limits_{r>r_0}\sqrt{-g}g^{00}\phi_{lm}(E,\vec{x})\phi_{l'm'}(E',\vec{x})\,d^3x = \delta_{ll'}\delta_{m,-m'}\delta(E-E'). \tag{66}
$$

Due to the presence of $\delta(E-E')$ in the latter relation, we have

$$
\left(\frac{E\sqrt{E}}{\sqrt{E'}}-\sqrt{EE'}\right)\delta(E-E') = (E-E)\delta(E-E') = 0, \tag{67}
$$

which means that the whole term with the coefficient $\frac{E\sqrt{E}}{\sqrt{E'}}-\sqrt{EE'}$ in (65) vanishes.

Second, let us consider the term with $\phi_j(\vec{k},\vec{x})$ and $\phi_{j'}(\vec{k}',\vec{x})$. It is not difficult to show that (again, see Appendix A for analogous calculations)

$$
\int\limits_{r>r_0}\sqrt{-g}g^{00}\phi_j(\vec{k},\vec{x})\phi_{j'}(\vec{k}',\vec{x})\,d^3x = \frac{1}{4\pi k^2}\sum_{l=0}^{\infty}(2l+1)e^{i(\pi l+2\tilde{\delta}_{lj}(k))}P_l(\cos\alpha)\delta_{jj'}\delta(k-k'), \tag{68}
$$

where α is the angle between the vectors $\vec{k}$ and $\vec{k}'$ (defined in spherical coordinates in the momentum space by k, θ_k, φ_k and k', θ'_k, φ'_k respectively), which looks like [13]

$$
\cos\alpha = \cos\theta_k\cos\theta'_k+\sin\theta_k\sin\theta'_k\cos(\varphi_k-\varphi'_k). \tag{69}
$$

Due to the presence of $\delta(k-k')$ in (68), we have

$$\left(\frac{(k^2+M^2)^{\frac{3}{4}}}{(k'^2+M^2)^{\frac{1}{4}}}-(k^2+M^2)^{\frac{1}{4}}(k'^2+M^2)^{\frac{1}{4}}\right)\delta(k-k')=\left(\sqrt{k^2+M^2}-\sqrt{k^2+M^2}\right)\delta(k-k')=0, \tag{70}$$

which means that the whole term with the coefficient $\frac{(k^2+M^2)^{\frac{3}{4}}}{(k'^2+M^2)^{\frac{1}{4}}}-(k^2+M^2)^{\frac{1}{4}}(k'^2+M^2)^{\frac{1}{4}}$ in (65) also vanishes. Thus, for (65) we get

$$\begin{aligned}H=&\frac{1}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int\limits_{0}^{M}E\left(a^{\dagger}_{lm}(E)a_{lm}(E)+a_{lm}(E)a^{\dagger}_{lm}(E)\right)dE\\&+\frac{1}{2}\sum_{j=1}^{2}\int\sqrt{k^2+M^2}\left(a^{\dagger}_{j}(\vec{k})a_{j}(\vec{k})+a_{j}(\vec{k})a^{\dagger}_{j}(\vec{k})\right)d^3k.\end{aligned} \tag{71}$$

Passing from $a_{lm}(E)a^{\dagger}_{lm}(E)$ to $a^{\dagger}_{lm}(E)a_{lm}(E)$ and from $a_{j}(\vec{k})a^{\dagger}_{j}(\vec{k})$ to $a^{\dagger}_{j}(\vec{k})a_{j}(\vec{k})$ in (71), and dropping the irrelevant c-number terms, finally we obtain

$$H=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int\limits_{0}^{M}E\,a^{\dagger}_{lm}(E)a_{lm}(E)\,dE+\int\sqrt{k^2+M^2}\left(a^{\dagger}_{1}(\vec{k})a_{1}(\vec{k})+a^{\dagger}_{2}(\vec{k})a_{2}(\vec{k})\right)d^3k. \tag{72}$$

We see that the resulting Hamiltonian has the standard form. However, this Hamiltonian implies that there is a degeneracy of states that are parameterized by the same $\vec{k}$. Of course, these states differ in the scattering amplitudes $A_1(\vec{k},\vec{n})$ and $A_2(\vec{k},\vec{n})$ in (39). Meanwhile, for an observer at large distances from the black hole, where the terms with the scattering amplitudes can be neglected because of the factor $\frac{1}{r}$, these different states look just as identical plane waves with the same momentum $\vec{k}$. The origin of this peculiarity will be briefly discussed in the next section.

4.4 Brief comparison with the case of Minkowski space-time

Let us consider a free real massive scalar field in Minkowski space-time. Formally, we can start the analysis from the solutions in spherical coordinates. In particular, the properly normalized solutions of the corresponding radial equation have the form [23]

$$R_l(k,r)=\sqrt{\frac{2}{\pi}}(-1)^l\left(\frac{r}{k}\right)^l\left(\frac{1}{r}\frac{d}{dr}\right)^l\frac{\sin kr}{r} \tag{73}$$

such that

$$\int\limits_{0}^{\infty}R_l(k,r)R_l(k',r)\,r^2dr=\delta(k-k'). \tag{74}$$

Using these solutions, we can also build the "scattering states" [23]

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\,e^{i\vec{k}\vec{x}}=\frac{1}{4\pi k}\sum_{l=0}^{\infty}(2l+1)\,e^{i\frac{\pi l}{2}}P_l\left(\frac{\vec{k}\vec{x}}{kr}\right)R_l(k,r), \tag{75}$$

which are just the exact plane waves. However, because there exits only *one* real radial solution for fixed k and l (this is the consequence of the existence of the dominant term $\frac{l(l+1)}{r^2}$ at $r \to 0$ in the corresponding equation), for a fixed $\vec{k}$ it is possible to build only one state that behaves as a plane wave for $r \to \infty$ (of course, in this particular case the state behaves as a plane wave everywhere in space). Thus, for the Hamiltonian we get the well-known exact result

$$H = \int \sqrt{k^2 + M^2}\, a^\dagger(\vec{k}) a(\vec{k})\, d^3k \tag{76}$$

without any degeneracy of states. Of course, the same reasonings are valid in the cases of standard spherically symmetric potentials in Minkowski space-time, the only difference with the case of the free field being a nonzero scattering amplitude. In contrast to this, in the case of the Schwarzschild metric there exist *two* real radial solution for fixed k and l, leading to two scattering states with the same $\vec{k}$. It is a purely topological effect, which cannot be eliminated by moving away from the black hole.

5 Discussion and conclusion

In the present paper, we have performed the procedure of canonical quantization of a real massive scalar field outside the horizon of an ideal Schwarzschild black hole. We have shown that the resulting theory turns out to be complete and self-consistent, i.e., the canonical commutation relation are satisfied exactly and the Hamiltonian has the standard form without any peculiarities. Although the scalar field theory we are working with is local in the sense that it is supposed to contain only local interactions, it relies on the existence of solutions of the corresponding equation of motion in the whole space. The latter means that in this sense the theory is "global". On the other hand, though the time coordinate t can be considered as proper time only at $r \to \infty$, it does not lead to any contradiction in performing the canonical quantization procedure. Moreover, since the initial theory is invariant under the translations in time t, it gives rise to Hamiltonian (10) that is conserved over time, which is essential for obtaining correct quantum field theory. Thus, time t can be considered as a global time in the resulting quantum theory.

It is clear that the results presented in this paper can be easily reduced to the massless case by setting $M = 0$. In this case, Eq. (23) with (24) takes the form of the Regge-Wheeler equation [17]. Of course, if $M = 0$, then the "localized" states (the first term with the integral with respect to E in (72)) are absent in the theory but all the other conclusion concerning the properties of the spectrum remain the same.

A feature of the resulting quantum theory is that the Schwarzschild black hole interior is not necessary for obtaining correct quantum field theory outside the black hole and does not affect it. In this sense this picture is similar to the one discussed in [7], the difference being that we do not need the white hole exterior as well.

Another feature is that in the vicinity of the horizon the scalar particles become effectively massless, which is a consequence of the fact that effective potential $V_l(z)$ (24) in the radial equation (23) is such that $V_l(z) \to 0$ as $z \to -\infty$ (i.e., as $r \to r_0$). It means that, presumably, bound states should decay as they approach the event horizon.

And one more feature, which is even more important, is connected with the spectrum of quantum states. This spectrum consists of two branches. The first branch is the continuous spectrum of states with energies less than the mass of the field, these states describe particles that are bound in the vicinity of the horizon. The second branch is the continuous spectrum of states with energies larger than the mass of the field. At large distances from the Schwarzschild black hole, the latter states look like the usual particles with definite momenta. However, it turns out that for a fixed momentum there exist *two* different states corresponding to such particles. Naively one would expect that if an observer is located far away from the Schwarzschild black hole, the effects caused by the black hole can be neglected. Nevertheless, it is not so — as it has been demonstrated above, because of the different topological structures of the Schwarzschild space-time and Minkowski space-time (the former is topologically $R^2 \times S^2$, whereas the latter is R^4), the structures of the spectra in both space-times are completely different, which manifests itself in the additional degeneracy in the case of the Schwarzschild space-time. This effect is a direct consequence of the fact that, as was mentioned above, the field theory is "global" in the sense that in order to have a consistent classical or quantum effective theory, it is necessary to have solutions of the corresponding equations of motion in the whole space. Of course, this conclusion was rigorously proven only for the simplest case of a real massive scalar field but we expect that such a degeneracy of states is inherent to fields of different types, i.e., to vector and spinor fields too. Moreover, it is quite possible that the existence of several black holes may lead to additional degeneracy of quantum states. It is clear that the cross-sections of various processes calculated in such a theory would differ considerably from those in the Standard Model, even in the case of a single black hole.

We would also like to note that, according to [24], a very compact object differs from a collapsing star in the existence of discrete energy levels in the former case. The same situation with the Schwarzschild black hole — the spectrum of "localized" states with energies less than the scalar field mass is also continuous. However, as was demonstrated above, there exists a more serious difference — we do not expect a degeneracy of states in the quantum theory for a very compact object, because such objects do not change the space-time topology. Meanwhile, black holes change the space-time topology, and even in the simplest case of the Schwarzschild black hole there emerges a degeneracy which leads to the consequences discussed above. In principle, the existence of such a degeneracy poses a question about the existence of black holes with horizons leading to substantial changes in the effective theory even far away from the black hole. However, the effects produced by the degeneracy, as well as finding possibilities to avoid it, call for a further analysis.

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Appendix A: Orthogonality condition for the scattering states

Let us consider (45) with $\phi_j(\vec{k},\vec{x})$ defined by (40). To prove this orthogonality condition, we will go along the lines of the method used in [23] for a similar proof. Let β be the angle between the vectors $\vec{k}$ and $\vec{x}$, β' is the angle between the vectors $\vec{k}'$ and $\vec{x}$, α is the angle between the vectors $\vec{k}$ and $\vec{k}'$, and $\tilde{\varphi}$ is the angle between the planes $(\vec{x},\vec{k})$ and $(\vec{k},\vec{k}')$. One can check that in such a case the relation

$$\cos\beta' = \cos\beta\cos\alpha + \sin\beta\sin\alpha\cos\tilde{\varphi} \tag{77}$$

between the angles holds, as well as the addition theorem for the Legendre polynomials [13]

$$P_{l'}(\cos\beta') = P_{l'}(\cos\beta)P_{l'}(\cos\alpha) + 2\sum_{m=1}^{l'}\frac{(l'-m)!}{(l'+m)!}P_{l'}^m(\cos\beta)P_{l'}^m(\cos\alpha)\cos(m\tilde{\varphi}). \tag{78}$$

Using these angles, the integral in the l.h.s. of (45) can be rewritten as[4]

$$\frac{1}{16\pi^2 kk'}\frac{\sqrt{k}}{(k^2+M^2)^{\frac{1}{4}}}\frac{\sqrt{k'}}{\left(k'^2+M^2\right)^{\frac{1}{4}}}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}(2l+1)(2l'+1)e^{i\left(\frac{\pi(l-l')}{2}+\tilde{\delta}_{lj}(k)-\tilde{\delta}_{l'j'}(k')\right)}$$
$$\times\int\limits_0^{2\pi}d\tilde{\varphi}\int\limits_0^{\pi}\sin\beta\, d\beta\int\limits_{r_0}^{\infty}dr\frac{r^3}{r-r_0}P_l(\cos\beta)P_{l'}(\cos\beta')f_{lj}\left(\sqrt{k^2+M^2},r\right)f_{l'j'}\left(\sqrt{k'^2+M^2},r\right). \tag{79}$$

Let us take only the angular part of the latter integral, which reads

$$\int\limits_0^{2\pi}d\tilde{\varphi}\int\limits_0^{\pi}\sin\beta\, d\beta P_l(\cos\beta)P_{l'}(\cos\beta'), \tag{80}$$

and substitute (78) into it. We get

$$\int\limits_0^{2\pi}d\tilde{\varphi}\int\limits_0^{\pi}\sin\beta\, d\beta P_l(\cos\beta)P_{l'}(\cos\beta') = 2\pi\int\limits_0^{\pi}\sin\beta\, d\beta P_l(\cos\beta)P_{l'}(\cos\beta)P_{l'}(\cos\alpha)$$
$$= \frac{4\pi}{2l+1}P_l(\cos\alpha)\delta_{ll'}. \tag{81}$$

Substituting (81) into (79), we get

$$\sum_{l=0}^{\infty}\frac{2l+1}{4\pi kk'}\frac{\sqrt{k}}{(k^2+M^2)^{\frac{1}{4}}}\frac{\sqrt{k'}}{\left(k'^2+M^2\right)^{\frac{1}{4}}}P_l(\cos\alpha)e^{i\left(\tilde{\delta}_{lj}(k)-\tilde{\delta}_{lj'}(k')\right)}$$
$$\times\int\limits_{r_0}^{\infty}dr\frac{r^3}{r-r_0}f_{lj}\left(\sqrt{k^2+M^2},r\right)f_{lj'}\left(\sqrt{k'^2+M^2},r\right). \tag{82}$$

[4] One can easily check that the integrations with respect to β and $\tilde{\varphi}$ in (79) go over the total solid angle of the vector $\vec{x}$.

With the orthogonality condition (31), formula (82) takes the form

$$\delta_{jj'}\frac{k}{\sqrt{k^2+M^2}}\,\delta\left(\sqrt{k^2+M^2}-\sqrt{k'^2+M^2}\right)\sum_{l=0}^{\infty}\frac{2l+1}{4\pi k^2}P_l(\cos\alpha)$$
$$=\delta_{jj'}\delta(k-k')\sum_{l=0}^{\infty}\frac{2l+1}{4\pi k^2}P_l(\cos\alpha). \tag{83}$$

Recall that [23]

$$\frac{1}{4}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\alpha)=\delta(1-\cos\alpha). \tag{84}$$

With the latter relation, formula (83) takes the form

$$\delta_{jj'}\frac{1}{\pi k^2}\,\delta(k-k')\delta(1-\cos\alpha). \tag{85}$$

It is clear that the delta functions in (85) select exactly $\vec{k}=\vec{k}'$, i.e., $\frac{1}{\pi k^2}\,\delta(k-k')\delta(1-\cos\alpha)$ should correspond to $\delta^{(3)}(\vec{k}-\vec{k}')$ in the initial variables. It is indeed so, because [23]

$$\int d^3k\left(\frac{1}{\pi k^2}\,\delta(k-k')\delta(1-\cos\alpha)\right)$$
$$=2\pi\int_0^{\infty}k^2dk\int_0^{\pi}\sin\alpha\,d\alpha\left(\frac{1}{\pi k^2}\,\delta(k-k')\delta(1-\cos\alpha)\right)=2\int_{-1}^{1}\delta(1-y)dy=1, \tag{86}$$

where the spherical coordinate system is chosen such that its "z-axis" coincides with the vector $\vec{k}'$ and the prescription $\int_{-1}^{1}\delta(1-y)dy=1/2$ is used. Relation (86) finalizes the proof of (45).

Appendix B: Completeness relation involving the scattering states

First, let us consider the second part in the l.h.s. of (46)

$$\sum_{j=1}^{2}\int\phi_j^*(\vec{k},\vec{x})\phi_j(\vec{k},\vec{y})\,d^3k. \tag{87}$$

Let β be the angle between the vectors $\vec{x}$ and $\vec{k}$, β' is the angle between the vectors $\vec{y}$ and $\vec{k}$, γ is the angle between the vectors $\vec{x}$ and $\vec{y}$, and $\tilde{\varphi}$ is the angle between the planes $(\vec{k},\vec{x})$ and $(\vec{x},\vec{y})$. One can check that in such a case the relation

$$\cos\beta'=\cos\beta\cos\gamma+\sin\beta\sin\gamma\cos\tilde{\varphi} \tag{88}$$

between the angles holds. Using these angles (contrary to the case discussed in Appendix A, here the angles β, β' and $\tilde{\varphi}$ parameterize the momentum space), the integral (87) can be rewritten as

$$\frac{1}{16\pi^2}\sum_{j=1}^{2}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}(2l+1)(2l'+1)e^{i\frac{\pi(l-l')}{2}}\int\limits_0^{2\pi}d\tilde{\varphi}\int\limits_0^{\pi}\sin\beta\,d\beta P_l(\cos\beta)P_{l'}(\cos\beta')$$
$$\times\int\limits_0^{\infty}dk\,e^{i\left(\tilde{\delta}_{lj}(k)-\tilde{\delta}_{l'j}(k)\right)}\frac{k}{\sqrt{k^2+M^2}}f_{lj}\left(\sqrt{k^2+M^2},r\right)f_{l'j}\left(\sqrt{k^2+M^2},r'\right), \tag{89}$$

where $r=|\vec{x}|$, $r'=|\vec{y}|$. Using the addition theorem for the Legendre polynomials [13]

$$P_{l'}(\cos\beta')=P_{l'}(\cos\beta)P_{l'}(\cos\gamma)+2\sum_{m=1}^{l'}\frac{(l'-m)!}{(l'+m)!}P_{l'}^m(\cos\beta)P_{l'}^m(\cos\gamma)\cos(m\tilde{\varphi}) \tag{90}$$

and (81), formula (89) can be brought to the form

$$\frac{1}{4\pi}\sum_{j=1}^{2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\int\limits_0^{\infty}dk\,\frac{k}{\sqrt{k^2+M^2}}f_{lj}\left(\sqrt{k^2+M^2},r\right)f_{lj}\left(\sqrt{k^2+M^2},r'\right)$$
$$=\frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\sum_{j=1}^{2}\int\limits_M^{\infty}f_{lj}\left(E,r\right)f_{lj}\left(E,r'\right)dE. \tag{91}$$

Now let us consider the first part in the l.h.s. of (46)

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int\limits_0^{M}\phi^*_{lm}(E,\vec{x})\phi_{lm}(E,\vec{y})dE. \tag{92}$$

Using the explicit form of spherical harmonics (12), we can get

$$\sum_{m=-l}^{l}Y^*_{lm}(\theta,\varphi)Y_{lm}(\theta',\varphi')$$
$$=\frac{2l+1}{4\pi}\left(P_l(\cos\theta)P_l(\cos\theta')+2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_l^m\left(\cos\theta\right)P_l^m\left(\cos\theta'\right)\cos(m(\varphi-\varphi'))\right). \tag{93}$$

It is known that for the standard spherical coordinates the relation

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi') \tag{94}$$

holds, where γ is the angle between the vectors $\vec{x}$ and $\vec{y}$ defined by r, θ, φ and r', θ', φ' respectively [13]. So, according to relation (94), the addition theorem for the Legendre polynomials takes the form

$$P_l(\cos\gamma)=P_l(\cos\theta)P_l(\cos\theta')+2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)P_l^m(\cos\theta')\cos(m(\varphi-\varphi')). \tag{95}$$

With (95), formula (93) can be rewritten as

$$\sum_{m=-l}^{l}Y^*_{lm}(\theta,\varphi)Y_{lm}(\theta',\varphi')=\frac{2l+1}{4\pi}P_l(\cos\gamma). \tag{96}$$

Thus, for (92) we obtain

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{0}^{M}\phi_{lm}^{*}(E,\vec{x})\phi_{lm}(E,\vec{y})dE=\frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\int_{0}^{M}f_l(E,r)f_l(E,r')dE. \qquad (97)$$

Combining (91) and (97), for the l.h.s. of (46) we get

$$\frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\left(\int_{0}^{M}f_l(E,r)f_l(E,r')dE+\sum_{j=1}^{2}\int_{M}^{\infty}f_{lj}\left(E,r\right)f_{lj}\left(E,r'\right)dE\right). \qquad (98)$$

With the completeness relation (37) and with (84), formula (98) can be brought to the form

$$\frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\frac{r-r_0}{r^3}\,\delta(r-r')=\frac{1}{\pi}\,\delta(1-\cos\gamma)\frac{r-r_0}{r^3}\,\delta(r-r'). \qquad (99)$$

Like in the case of (85), the delta functions in (99) select exactly $\vec{x}=\vec{y}$. The calculation (compare with (86))

$$\int_{r>r_0} d^3x\sqrt{-g}\,g^{00}\left(\frac{1}{\pi}\,\delta(1-\cos\gamma)\frac{r-r_0}{r^3}\,\delta(r-r')\right)$$

$$=2\pi\int_{r_0}^{\infty}dr\frac{r^3}{r-r_0}\int_{0}^{\pi}d\gamma\sin\gamma\left(\frac{1}{\pi}\,\delta(1-\cos\gamma)\frac{r-r_0}{r^3}\,\delta(r-r')\right)=1, \qquad (100)$$

in which the spherical coordinate system is chosen such that its "z-axis" coincides with the vector $\vec{y}$ and again the prescription $\int_{-1}^{1}\delta(1-y)dy=1/2$ is used, finalizes the proof of (46).

References

[1] D.G. Boulware, *"Quantum field theory in Schwarzschild and Rindler spaces"*, Phys. Rev. D **11** (1975) 1404.

[2] J.B. Hartle, S.W. Hawking, *"Path-integral derivation of black-hole radiance"*, Phys. Rev. D **13** (1976) 2188.

[3] M.D. Kruskal, *"Maximal extension of Schwarzschild metric"*, Phys. Rev. **119** (1960) 1743.

[4] G. Szekeres, *"On the singularities of a Riemannian manifold"*, Publ. Math. Debrecen **7** (1960) 285.

[5] G. 't Hooft, *"Virtual black holes and space-time structure"*, Found. Phys. **48** (2018) 1134.

[6] G. 't Hooft, *"The quantum black hole as a theoretical lab, a pedagogical treatment of a new approach"*, arXiv:1902.10469 [gr-qc].

[7] G. 't Hooft, *"Quantum clones inside black holes"*, arXiv:2206.04608 [gr-qc].

[8] R. Rattazzi, *"Cargese lectures on extra dimensions"*, arXiv:hep-ph/0607055 [hep-ph].

[9] E.T. Akhmedov, P.A. Anempodistov, K.V. Bazarov, D.V. Diakonov, U. Moschella, *"Heating up an environment around black holes and inside de Sitter space"*, Phys. Rev. D **103** (2021) 025023 [arXiv:2010.10877 [hep-th]].

[10] P.A. Anempodistov, *"Remarks on the thermofield double state in 4D black hole background"*, Phys. Rev. D **103** (2021) 105008 [arXiv:2012.03305 [hep-th]].

[11] K.V. Bazarov, *"Notes on peculiarities of quantum fields in space-times with horizons"*, arXiv:2112.02188 [hep-th].

[12] L.D. Landau, E.M. Lifshitz, *"The classical theory of fields"*, Butterworth-Heinemann (1987).

[13] G.A. Korn, T.M. Korn, *"Mathematical handbook for scientists and engineers"*, McGraw-Hill, Inc., New York (1968).

[14] N. Deruelle, R. Ruffini, *"Quantum and classical relativistic energy states in stationary geometries"*, Phys. Lett. B **52** (1974) 437.

[15] A. Zecca, *"Properties of radial equation of scalar field in Schwarzschild space-time"*, Il Nuovo Cim. B **124** (2009) 1251.

[16] P.P. Fiziev, *"Exact solutions of Regge-Wheeler equation and quasi-normal modes of compact objects"*, Class. Quant. Grav. **23** (2006) 2447 [arXiv:gr-qc/0509123 [gr-qc]].

[17] T. Regge, J. Wheeler, *"Stability of a Schwarzschild singularity"*, Phys. Rev. **108** (1957) 1063.

[18] M.V. Gorbatenko, V.P. Neznamov, *"Quantum mechanics of stationary states of particles in a space–time of classical black holes"*, Theor. Math. Phys. **205** (2020) 1492.

[19] K.M. Case, *"Singular potentials"*, Phys. Rev. **80** (1950) 797.

[20] A.M. Perelomov, V.S. Popov, *""Fall to the center" in quantum mechanics"*, Theor. Math. Phys. **4** (1970) 664.

[21] J. Barranco, A. Bernal, J.C. Degollado, A. Diez-Tejedor, M. Megevand, M. Alcubierre, D. Nunez, O. Sarbach, *"Are black holes a serious threat to scalar field dark matter models?"*, Phys. Rev. D **84** (2011) 083008 [arXiv:1108.0931 [gr-qc]].

[22] A. Messiah, *"Quantum mechanics. Volume I"*, North-Holland (1961).

[23] L.D. Landau, E.M. Lifshitz, *"Quantum mechanics. Non-relativistic theory", Second edition*, Pergamon press (1965).

[24] E.T. Akhmedov, D.A. Kalinov, F.K. Popov, *"Method for distinguishing very compact stellar objects from black holes"*, Phys. Rev. D **93** (2016) 064006 [arXiv:1601.03894 [gr-qc]].