

NONEXISTENCE OF CLOSED TIMELIKE GEODESICS IN KERR SPACETIMES

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Abstract

The Kerr-star spacetime is the extension over the horizons and in the negative radial region of the Kerr spacetime. Despite the presence of closed timelike curves below the inner horizon, we prove that the timelike geodesics cannot be closed in the Kerr-star spacetime. Since the existence of closed null geodesics was ruled out by the author in [G. Sanzeni, [16] (2024)], this result shows the absence of closed causal geodesics in the Kerr-star spacetime.

Keywords: closed timelike geodesics, closed timelike curves, Kerr-star spacetime, Kerr spacetime, elliptic integrals

1. Introduction

1.1. The Kerr solution and its chronology violations. The Kerr spacetime is a stationary, axisymmetric and asymptotically flat black hole solution of Einstein's vacuum field equations found by R. P. Kerr [12]. This spacetime depends on a *mass parameter* M and a *rotation parameter* a (angular momentum per unit mass). The static spherically symmetric Schwarzschild solution [17] is obtained from the Kerr solution in the limit case $a = 0$. The slowly rotating ($|a| < M$) Kerr spacetime have two horizons, an outer *event horizon* and an inner *causality horizon*. If the Kerr spacetime is analytically extended over the horizons and in the negative radial region [2, 15], from now on called the Kerr-star spacetime, through every point below the causality horizon there exists a closed timelike curve [5]. In this paper, we prove that despite chronology violations, the timelike geodesics cannot be closed in the Kerr-star spacetime. This work follows the strategy adopted in [16] in which we proved the absence of closed null geodesics. Therefore as the Gödel spacetime [10], the Kerr-star spacetime is not causal but it does not contain closed causal geodesics, see [13, 7, 14].

1.2. Result. Consider a spacetime $(\mathcal{M}, \mathbf{g})$, *i.e.* a time-oriented connected Lorentzian manifold, and a geodesic curve $\gamma : I = [a, b] \rightarrow \mathcal{M}$. γ is called *closed geodesic* if $\gamma(a) = \gamma(b)$ and $\gamma'(a) = \lambda \gamma'(b) \neq 0$, for some real number $\lambda \neq 0$. If γ is timelike, then $\lambda = 1$. The purpose of this paper is to prove the nonexistence of closed timelike geodesics in the Kerr-star extension of the slowly rotating ($|a| < M$) Kerr black hole, described in detail in §2.

Theorem 1.1. *Let K^* be the Kerr-star spacetime. Then there are no closed timelike geodesics in K^* .*

The nonexistence of closed null geodesics in the Kerr-star spacetime was proved in the following result.

Theorem 1.2 (Theorem 1.1, [16]). *Let K^* be the Kerr-star spacetime. Then there are no closed null geodesics in K^* .*

Corollary 1.3. *Let K^* be the Kerr-star spacetime. Then there are no closed causal geodesics in K^* .*

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1.3. Geodesic motion in Kerr spacetimes. The Kerr spacetimes are completely integrable systems. Indeed for any geodesic there exist four independent constants of motion: the *energy* (associated to a timelike Killing vector field), the *angular momentum* (associated to a spacelike Killing vector field), the *Lorentzian energy* (the causal character of the geodesic) and the *Carter constant* (associated to a Killing 2-tensor) [5]. Therefore one can study the geodesic motion solving a system of four coupled first-order differential equations [5, 15]. Geodesics restricted on submanifolds were firstly studied. Boyer and Price [3], then Boyer and Lindquist [2], and hence de Felice [9] considered geodesic motion in the equatorial hyperplane $E_q = \{\theta = \pi/2\}$. Geodesics in the axis of symmetry $A = \{\theta = 0, \pi\}$ of the black hole were analysed by Carter [4]. Wilkins instead studied trapped orbits, namely geodesics running over a finite radial interval [20]. The most exhaustive references about geodesic motion in Kerr spacetimes are the text-books by Chandrasekhar [6] and O'Neill [15]. In this paper, we first ruled out the existence of closed timelike geodesics strictly contained in $\{0 < r < r_-\}$ (Prop. 5.2), intersecting the horizons (Prop. 5.3) and tangent to the axis (Prop. 5.4). Starting from §5.3, the remaining timelike geodesics are analyzed. It turned out that the most difficult ones to investigate are those with non-vanishing energy and negative Carter constant. Firstly, we observed that if a geodesic of such kind is closed, it must have constant r -coordinate, so it must be a spherical geodesic (Prop. 4.4). Secondly, we obtained a lower bound on the negative constant r -coordinate (Prop. 5.7). Finally, arguing by contradiction we proved that the variation of the t -coordinate (see eq. (28)) on a full θ -oscillation must be positive (Prop. 5.17) for any spherical timelike geodesic with negative Carter constant. Therefore it is shown that the timelike geodesics cannot be closed in the Kerr-star spacetime.

1.4. Organization of the paper. In §2, we introduce the Kerr metric and discuss the definition and properties of the Kerr-star spacetime. In §3 we recall the set of first order differential equations satisfied by geodesic orbits. In §4, we study the properties of timelike geodesics required to prove the main theorem. In §5, we give the proof of Thm. 1.1 split into several cases. The overall structure of the proof is detailed in 5.1, 5.3 and Fig. 2.

2. The Kerr-star spacetime

Consider $\mathbb{R}^2 \times S^2$ with coordinates $(t, r) \in \mathbb{R}^2$ and $(\theta, \phi) \in S^2$. Fix two real numbers $a \in \mathbb{R} \setminus \{0\}$, $M \in \mathbb{R}_{>0}$ and define the functions

$$\rho(r, \theta) := \sqrt{r^2 + a^2 \cos^2 \theta}$$

and

$$\Delta(r) := r^2 - 2Mr + a^2.$$

We study the case $|a| < M$ called *slow Kerr*, for which $\Delta(r)$ has two positive roots

$$r_{\pm} = M \pm \sqrt{M^2 - a^2} > 0$$

and define two sets

- (1) the *horizons* $\mathcal{H} := \{\Delta(r) = 0\} = \{r = r_{\pm}\} := \mathcal{H}_- \sqcup \mathcal{H}_+$,
- (2) the *ring singularity* $\Sigma := \{\rho(r, \theta) = 0\} = \{r = 0, \theta = \pi/2\}$.

The *Kerr metric* [12] in *Boyer–Lindquist coordinates* is

$$\mathbf{g} = -dt \otimes dt + \frac{2Mr}{\rho^2(r, \theta)} (dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2(r, \theta)}{\Delta(r)} dr \otimes dr + a^2 \sin^4(\theta) d\phi \otimes d\phi + \rho^2(r, \theta) d\sigma^2, \quad (1)$$

where $d\sigma^2 = d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi$ is the 2-dimensional (Riemannian) metric of constant unit curvature on the unit sphere $S^2 \subset \mathbb{R}^3$ written in spherical coordinates.

Remark 2.1. The components of $\mathbf{g}$ in *Boyer–Lindquist coordinates* can be read off the common expression

$$\begin{aligned} \mathbf{g} = & - \left(1 - \frac{2Mr}{\rho^2(r, \theta)} \right) dt \otimes dt - \frac{4Mar \sin^2 \theta}{\rho^2(r, \theta)} dt \otimes d\phi + \\ & + \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2(r, \theta)} \right) \sin^2 \theta d\phi \otimes d\phi + \frac{\rho^2(r, \theta)}{\Delta(r)} dr \otimes dr + \rho^2(r, \theta) d\theta \otimes d\theta. \end{aligned} \quad (2)$$

Nevertheless this last expression does not cover the subsets $\{\theta = 0, \pi\}$.

Lemma 2.2. *The metric (1) is a Lorentzian metric on $\mathbb{R}^2 \times S^2 \setminus (\Sigma \cup \mathcal{H})$.*

The Boyer–Lindquist coordinates or the metric tensor fail on the sets $\mathcal{H}$ and Σ . In order to extend the metric tensor to the horizons, one has to introduce a new set of coordinates. No change of coordinates can be found in order to extend the metric across the ring singularity. For a detailed study of the nature of the ring singularity, see for instance [8].

Definition 2.3. *The subsets*

$$I := \{r > r_+\}, \quad II := \{r_- < r < r_+\}, \quad III := \{r < r_-\} \subset \{(t, r) \in \mathbb{R}^2, (\theta, \phi) \in S^2\} \setminus (\Sigma \cup \mathcal{H})$$

are called the Boyer–Lindquist (BL) blocks.

Remark 2.4. *The BL blocks I, II and III are the connected components of $\mathbb{R}^2 \times S^2 \setminus (\Sigma \cup \mathcal{H})$. Each block with the restriction of the metric tensor (1) is a connected Lorentzian 4-manifold. To get spacetimes, one has to choose a time orientation on each block.*

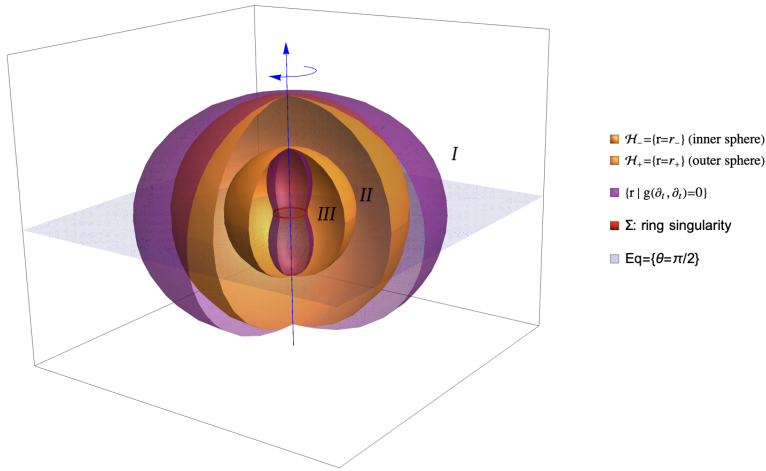


FIGURE 1. This picture shows a t -slice $\{t\} \times \mathbb{R} \times S^2$, with the radius drawn as e^r , so that $r = -\infty$ is at the center of the figure. The *Ergoregion* $\{g(\partial_t, \partial_t) > 0\}$ (at fixed time t) is the region between the purple ellipsoids in which ∂_t becomes spacelike.

2.1. Time orientation of BL blocks. We define a future time-orientation of block I using the gradient timelike vector field $-\nabla t$. Indeed, the hypersurfaces $\{t = \text{const}\}$ are spacelike in block I. Notice that the coordinate vector field ∂_t is timelike future-directed for $r \gg r_+$ on block I, since $g(-\nabla t, \partial_t) = -1$.

We define a time-orientation of block II by declaring the vector field $-\partial_r$, which is timelike in II, to be future-oriented.

We define a time-orientation of block III by declaring the vector field $V := (r^2 + a^2)\partial_t + a\partial_\phi$, which is timelike in III, to be future-oriented.

With this choice of time-orientations, each block is a *spacetime*, i.e. a connected time-oriented Lorentzian 4-manifold.

2.2. Kerr spacetimes.

Definition 2.5. *A Kerr spacetime is an analytic spacetime $(\mathcal{M}_{\text{Kerr}}, g)$ such that*

- (1) *there exists a family of open disjoint isometric embeddings $\Phi_i: \mathcal{B}_i \hookrightarrow \mathcal{M}_{\text{Kerr}}$ ($i \in \mathbb{N}$) of BL blocks $(\mathcal{B}_i, g|_{\mathcal{B}_i})$ (i.e. $g|_{\mathcal{B}_i} = \Phi_i^* g|_{\Phi_i(\mathcal{B}_i)}$) such that $\cup_{i \in \mathbb{N}} \Phi_i(\mathcal{B}_i)$ is dense in $\mathcal{M}_{\text{Kerr}}$;*
- (2) *there are analytic functions r and C on $\mathcal{M}_{\text{Kerr}}$ such that their restriction on each $\Phi_i(\mathcal{B}_i)$ of condition (1) is Φ_i -related to the Boyer–Lindquist functions r and $C = \cos \theta$ on $\mathcal{B}_i$;*

- (3) there is an isometry $\epsilon : \mathcal{M}_{Kerr} \rightarrow \mathcal{M}_{Kerr}$ called the equatorial isometry whose restrictions to each BL block sends θ to $\pi - \theta$, leaving the other coordinates unchanged;
- (4) there are Killing vector fields $\tilde{\partial}_t$ and $\tilde{\partial}_\phi$ on $\mathcal{M}_{Kerr}$ that restrict to the Boyer–Lindquist coordinate vector fields ∂_t and ∂_ϕ on each BL block.

Remark 2.6. With abuse of notation, we identify each block $\mathcal{B}_i$ with its image via the isometric embedding $\Phi_i(\mathcal{B}_i) \subset \mathcal{M}_{Kerr}$.

Lemma 2.7. Each time-oriented BL block is a Kerr spacetime.

Definition 2.8. In a Kerr spacetime $\mathcal{M}_{Kerr}$, on any BL block $\mathcal{B}_i$

- (1) the axis $A = \{\theta = 0, \pi\}$ is the set of zeroes of the Killing vector field $\tilde{\partial}_\phi$ as in (4) of Def. 2.5;
- (2) the equatorial hyperplane $Eq = \{\theta = \pi/2\}$ is the set of fixed points of the equatorial isometry ϵ as in (3) of Def. 2.5.

2.3. The Kerr-star spacetime.

Definition 2.9. On each BL block, we define the Kerr-star coordinate functions:

$$t^* := t + \mathcal{T}(r) \in \mathbb{R}, \quad \phi^* := \phi + \mathcal{A}(r) \in S^1, \quad (3)$$

with $d\mathcal{T}/dr := (r^2 + a^2)/\Delta(r)$ and $d\mathcal{A}/dr := a/\Delta(r)$.

Lemma 2.10 ([15], Lemma 2.5.1). For each BL block B , the map $\xi^* = (t^*, r, \theta, \phi^*) : B \setminus A \rightarrow \xi^*(B) \subseteq \mathbb{R}^4$ is a coordinate system on $B \setminus A$, where A is the axis. We call ξ^* a Kerr-star coordinate system.

Because the Kerr-star coordinate functions differ from BL coordinates only by additive functions of r , the coordinate vector fields $\partial_t, \partial_\theta, \partial_\phi$ are the same in the two systems, except that in K^* they extend over the horizons. However, the coordinate vector field associated to r does change its form, and we define $\partial_r^* := \partial_r - \Delta(r)^{-1}V$, where V is one of the canonical vector fields defined in Section 3. Note that if we use Kerr-star coordinates, we get $\mathbf{g}(\partial_r^*, \partial_r^*) = 0$, i.e. ∂_r^* is a null vector field of K^* , while in BL coordinates, $\mathbf{g}(\partial_r, \partial_r) = \rho^2(r, \theta)/\Delta(r)$, which is singular when $\Delta(r) = 0$.

Lemma 2.11. The Kerr metric, expressed in Kerr-star coordinates, takes the form

$$\begin{aligned} \mathbf{g} = & - \left(1 - \frac{2Mr}{\rho^2(r, \theta)} \right) dt^* \otimes dt^* - \frac{4Mar \sin^2 \theta}{\rho^2(r, \theta)} dt^* \otimes d\phi^* + \\ & + \left(r^2 + a^2 + \frac{2Mr a^2 \sin^2 \theta}{\rho^2(r, \theta)} \right) \sin^2 \theta d\phi^* \otimes d\phi^* + 2 dt^* \otimes dr + \\ & - 2a \sin^2 \theta d\phi^* \otimes dr + \rho^2(r, \theta) d\theta \otimes d\theta. \end{aligned} \quad (4)$$

Now all coefficients in $\mathbf{g}$ are well defined on the horizons $\mathcal{H} = \{\Delta(r) = 0\}$, hence it is a well defined Lorentzian metric on $\mathbb{R}^2 \times S^2 \setminus \Sigma$ and constitutes an analytic extension of (1) over $\mathcal{H}$.

Definition 2.12. The Kerr-star spacetime is a Kerr spacetime as defined in 2.5 given by the tuple $(K^*, \mathbf{g}, o)$ with $K^* = \{(t^*, r) \in \mathbb{R}^2, (\theta, \phi^*) \in S^2\} \setminus \Sigma$, $\mathbf{g}$ as in Lemma 2.11 (extended over the axis) and o is the future time-orientation induced by the null vector field $-\partial_r^*$.

Remark 2.13. Note that the time-orientations on individual BL blocks agree with the ones defined for the Kerr-star spacetime: $\mathbf{g}(-\partial_r^*, \partial_t) = -1 < 0$ on I , $\mathbf{g}(-\partial_r^*, -\partial_r) = \mathbf{g}(\partial_r, \partial_r) = \rho^2(r, \theta)/\Delta(r) < 0$ on II and $\mathbf{g}(-\partial_r^*, V) = \frac{1}{\Delta(r)}\mathbf{g}(V, V) = -\rho^2(r, \theta) < 0$ on III .

2.4. Totally geodesic submanifolds of the Kerr-star spacetime.

Lemma 2.14 (See p. 68 of [15]). Let K^* be the Kerr-star spacetime as in Def. 2.12. The axis A and the equatorial hyperplane Eq of K^* are closed totally geodesic submanifolds of K^* .

Proposition 2.15. [15] Let K^* be the Kerr-star spacetime. Then the horizon $\mathcal{H}$ is a closed totally geodesic null hypersurface, with future hemicone on the $-\partial_r^*$ side. Moreover, the restriction of $V := (r^2 + a^2)\partial_t + a\partial_\phi$ (called canonical vector field in §3) on $\mathcal{H}$ is the unique null vector field on $\mathcal{H}$ that is tangent to $\mathcal{H}$, hence also normal to $\mathcal{H}$. The integral curves of V in $\mathcal{H}$ are null pregeodesics.

2.5. Causal and vicious regions of the Kerr-star spacetime.

Proposition 2.16 ([15], Proposition 2.4.6). *The BL blocks I and II are causal.*

Corollary 2.17. *Let K^* be the Kerr-star spacetime. Then the region $I \cup II \cup \{r = r_{\pm}\} = \{t^* \in \mathbb{R}, r \in [r_-, +\infty), (\theta, \phi^*) \in S^2\} \setminus \Sigma \subset K^*$ is causal.*

Proof. Let γ be a future pointing curve. If γ is entirely contained either in I or in II, then by Prop. 2.16, γ cannot be closed. If γ is entirely contained in $\mathcal{H} = \{r = r_{\pm}\}$ (closed totally geodesic null hypersurface of K^* by Prop. 2.15), then by Lem. 1.5.11 of [15], except for restphotons, all other curves are spacelike, but restphotons are integral curves of $V|_{\mathcal{H}} = (r_{\pm}^2 + a^2)\partial_t + a\partial_{\phi}$, which cannot be closed. Since the time orientation $-\partial_r^*$ is null and transverse to the null hypersurface $\mathcal{H}$, the future directed curves always go in the direction of $-\partial_r^*$, if they hit $\mathcal{H}$ transversally. Henceforth, if γ starts in the BL block I (II), crosses $\mathcal{H}_+$ ($\mathcal{H}_-$) transversally, enters the block II (III), then γ cannot re-intersect $\mathcal{H}_+$ from II to I ($\mathcal{H}_-$ from III to II). The last possibility is the following: γ starts in I (II), becomes tangent to $\mathcal{H}_+$ ($\mathcal{H}_-$), hence either lies forever on $\mathcal{H}_+$ ($\mathcal{H}_-$) or leaves it at some point. In the first case, γ is obviously not closed, while in the second, it cannot be closed because it will necessarily have to enter the region $\{r < r_+\}$ ($\{r < r_-\}$), according to the time orientation. $\square$

Proposition 2.18 ([15], Proposition 2.4.7). *The BL block III in the Kerr-star spacetime is vicious, that is, given any two points $p, q \in III$ there exists a future directed timelike curve in III from p to q .*

Corollary 2.19. *Let p be a point in the BL block III of the Kerr-star spacetime. Then there exists a closed timelike curve through p .*

3. Geodesics in Kerr spacetimes

3.1. Constants of motion. Let $(\mathcal{M}_{Kerr}, \mathbf{g})$ be a Kerr spacetime as in Def. 2.5. Recall that there are two Killing vector fields $\tilde{\partial}_t$ and $\tilde{\partial}_{\phi}$ on $\mathcal{M}_{Kerr}$.

Definition 3.1 (Energy and angular momentum). *For a geodesic γ of $(\mathcal{M}_{Kerr}, \mathbf{g})$, the constants of motion*

$$E = E(\gamma) := -\mathbf{g}(\gamma', \tilde{\partial}_t)$$

and

$$L = L(\gamma) := \mathbf{g}(\gamma', \tilde{\partial}_{\phi})$$

are called its energy and its angular momentum (around the axis of rotation of the black hole), respectively.

Definition 3.2. *For every BL block $\mathcal{B}_i$ define the canonical vector fields*

$$V := (r^2 + a^2)\partial_t + a\partial_{\phi} \quad \text{and} \quad W := \partial_{\phi} + a\sin^2\theta\partial_t$$

via the isometry $\Phi_i: \mathcal{B}_i \hookrightarrow \mathcal{M}_{Kerr}$.

Remark 3.3. *V and W are not Killing vectors.*

Definition 3.4. *Let γ be a geodesic in $\mathcal{M}_{Kerr}$ with energy E and angular momentum L . Define the functions $\mathbb{P}$ and $\mathbb{D}$ along γ by*

$$\mathbb{P}(r) := -\mathbf{g}(\gamma', V) = (r^2 + a^2)E - La$$

and

$$\mathbb{D}(\theta) := \mathbf{g}(\gamma', W) = L - Ea\sin^2\theta.$$

A geodesic in a Kerr spacetime has two additional constants of motions. First, there is the Lorentian energy $q := \mathbf{g}(\gamma', \gamma')$, which is always constant along every geodesic in any pseudo-Riemannian manifold. The second one is K , which was first found by Carter in [5] using the separability of the Hamilton–Jacobi equation. K can be defined (see Ch. 7 in [6]) by

$$K := 2\rho^2(r, \theta)\mathbf{g}(l, \gamma')\mathbf{g}(n, \gamma') + r^2q,$$

where $l = \frac{1}{\Delta(r)}V + \partial_r$ and $n = \frac{1}{2\rho^2(r, \theta)}V - \frac{\Delta(r)}{2\rho^2(r, \theta)}\partial_r$. See also [19] for a definition using a Killing tensor for the Kerr metric.

Definition 3.5 (*Carter constant*). *On a Kerr spacetime, the constant of motion*

$$Q := K - (L - aE)^2 \quad \text{or} \quad \mathcal{Q} := Q/E^2 \quad \text{if} \quad E \neq 0$$

is called the Carter constant.

3.2. Equations of motion.

Proposition 3.6 ([15], Proposition 4.1.5, Theorem 4.2.2). *Let B be a BL block and γ be a geodesic with initial position in $B \subset \mathcal{M}_{\text{Kerr}}$ and constants of motion E, L, Q, q . Then the components of γ in the BL coordinates (t, r, θ, ϕ) satisfy the following set of first order differential equations*

$$\begin{cases} \rho^2(r, \theta)\phi' = \frac{\mathbb{D}(\theta)}{\sin^2 \theta} + a \frac{\mathbb{P}(r)}{\Delta(r)} \\ \rho^2(r, \theta)t' = a\mathbb{D}(\theta) + (r^2 + a^2) \frac{\mathbb{P}(r)}{\Delta(r)} \\ \rho^4(r, \theta)r'^2 = R(r) \\ \rho^4(r, \theta)\theta'^2 = \Theta(\theta) \end{cases} \quad (5)$$

where

$$\begin{aligned} R(r) &:= \Delta(r) [(qr^2 - K(E, L, Q)) + \mathbb{P}^2(r)] + \mathbb{P}^2(r) = \\ &= (E^2 + q)r^4 - 2Mqr^3 + \mathfrak{X}(E, L, Q)r^2 + 2MK(E, L, Q)r - a^2Q, \\ \Theta(\theta) &:= K(E, L, Q) + qa^2 \cos^2 \theta - \frac{\mathbb{D}(\theta)^2}{\sin^2 \theta} = \\ &= Q + \cos^2 \theta [a^2(E^2 + q) - L^2/\sin^2 \theta], \end{aligned}$$

with

$$\mathfrak{X}(E, L, Q) := a^2(E^2 + q) - L^2 - Q, \text{ and } K(E, L, Q) = Q + (L - aE)^2.$$

Remark 3.7. *Since in the third and in the fourth differential equations of Prop. 3.6 the left-hand sides are clearly non-negative, we see that the polynomials $R(r)$ and $\Theta(\theta)$ are non-negative along the geodesics. Hence the geodesic motion can only happen in the r, θ -region for which $R(r), \Theta(\theta) \geq 0$.*

In order to study geodesics that cross the horizons

$$\mathcal{H} = \{\Delta(r) = 0\} = \{r = r_{\pm}\},$$

it is necessary to introduce the Kerr-star coordinate system. Note however that since the change of coordinates modifies only the t and the ϕ coordinates and the r, θ -differential equations do not involve t and ϕ , the last two differential equations do extend over $\mathcal{H}$. Observe also that the r, θ -differential equations are not singular on $\mathcal{H}$, while the t, ϕ -differential equations are.

Notice that $\Theta(\theta)$ is also well-defined if the geodesic crosses $A = \{\theta = 0, \pi\}$. Indeed, $L = 0$ (because $\tilde{\partial}_\phi \equiv 0$ on A), hence $\mathbb{D}(\theta) = -Ea \sin^2 \theta$, and then

$$\begin{aligned} \Theta(\theta) &= K(E, 0, Q) + qa^2 \cos^2 \theta - (-Ea \sin^2 \theta)^2 / \sin^2 \theta = Q + a^2 E^2 + qa^2 \cos^2 \theta - a^2 E^2 \sin^2 \theta \\ &= Q + a^2(E^2 + q) \cos^2 \theta. \end{aligned}$$

Thus the r, θ -differential equations can be used to study geodesics on the whole Kerr-star spacetime.

Remark 3.8. *The system (5) is composed of first order differential equations, while the geodesic equation is second order. There exist solutions of (5), called singular, which do not correspond to geodesics. For example, if $r_0 \in \mathbb{R}$ is a multiplicity one zero of $r \mapsto R(r)$, then r_0 solves the radial equation in (5), since in this case $r'(s) = 0$ for all s , but we do not have a geodesic.*

3.3. Dynamics of geodesics. The non-negativity of $R(r)$ and $\Theta(\theta)$ in the first order differential equations of motion (5) can be used to study the dynamics of the r, θ -coordinates of the geodesics, together with the next proposition.

Proposition 3.9 ([15], Corollary 4.3.8). *Suppose $R(r_0) = 0$. Let γ be a geodesic whose r -coordinate satisfies the initial conditions $r(s_0) = r_0$ and $r'(s_0) = 0$.*

- (1) *If r_0 is a multiplicity one zero of $R(r)$, i.e. $R'(r_0) \neq 0$, then r_0 is an r -turning point, namely $r'(s)$ changes sign at s_0 .*

(2) If r_0 is a higher order zero of $R(r)$, i.e. at least $R'(r_0) = 0$, then γ has constant $r(s) = r_0$. Analogous results hold for r and $R(r)$ replaced by θ and $\Theta(\theta)$.

4. Properties of timelike geodesics in Kerr spacetimes

4.1. Principal geodesics. Since the vector fields $V, W, \partial_r, \partial_\theta$ are linearly independent, the tangent vector to a geodesic γ can be decomposed as $\gamma' = \gamma'_\Pi + \gamma'_\perp$ where $\Pi := \text{span}\{\partial_r, V\}$ (timelike plane) and $\Pi^\perp := \text{span}\{\partial_\theta, W\}$ (spacelike plane).

Definition 4.1. A Kerr geodesic γ is said to be *principal* if $\gamma' = \gamma'_\Pi$.

Proposition 4.2 ([15], Corollary 4.2.8(1)). If γ is a timelike geodesic, then $K \geq 0$, and $K = 0 \iff \gamma$ is a principal geodesic in the $Eq = \{\theta = \pi/2\}$.

4.2. Timelike geodesics with $Q < 0$.

Proposition 4.3. Let γ be a timelike ($q < 0$) geodesic with $Q < 0$. Then

- (1) γ does not intersect $Eq = \{\theta = \pi/2\}$;
- (2) $a^2(E^2 + q) > L^2$ and in particular $E \neq 0$ and $E^2 + q > 0$.

Proof. If $\gamma \cap A = \emptyset$, then from the θ -equation of (5) we have

$$\cos^2 \theta [L^2 / \sin^2 \theta - a^2(E^2 + q)] = Q - \rho^4(r, \theta) \theta'^2 < 0.$$

Hence $\cos^2 \theta \neq 0$ and $L^2 / \sin^2 \theta - a^2(E^2 + q) < 0$, hence $\gamma \cap Eq = \emptyset$ and $a^2(E^2 + q) > L^2$, so $E \neq 0$ and $E^2 + q > 0$.

If $\gamma \cap A \neq \emptyset$, then $L = 0$ since $\tilde{\partial}_\phi \equiv 0$ on A and

$$-a^2(E^2 + q) \cos^2 \theta = Q - \rho^4(r, \theta) \theta'^2 < 0.$$

Therefore $\cos^2 \theta \neq 0$ and $a^2(E^2 + q) > 0 = L^2$, $E \neq 0$ so $\gamma \cap Eq = \emptyset$. $\square$

Proposition 4.4 ([15], Corollaries 4.9.2, 4.9.3). For $Q < 0$ timelike geodesics, $R(r)$ is convex and has either zero or two negative roots, which may be coincident. Therefore the only possible bounded r -behaviour is $r(s) = \text{const} < 0$.

5. Proof of Theorem 1.1

5.1. Strategy of the proof. The following argument is similar to the one used to prove the nonexistence of closed null geodesics in [16]. Let $\gamma: I \rightarrow K^*$ be a closed timelike geodesic (CTG). Since the radius function $r: K^* \rightarrow \mathbb{R}$ is everywhere smooth the composition $r \circ \gamma$ has at least two critical points $s_0 < s_1$ in each period $[a, a+T)$, i.e. $(r \circ \gamma)'(s_0) = (r \circ \gamma)'(s_1) = 0$. Since $\rho: K^* \rightarrow \mathbb{R}$ does not vanish on K^* the differential equation for $r \circ \gamma$

$$(\rho \circ \gamma)^4 [(r \circ \gamma)']^2 = R(r \circ \gamma)$$

implies that $R(r \circ \gamma(s_{0,1})) = 0$. Because of the differential equation, the geodesic motion must happen in the r -region on which $R(r \circ \gamma) \geq 0$. Further since R is a polynomial in r we can distinguish two cases:

- (1) The zeros $r \circ \gamma(s_{0,1})$ of R are simple, i.e. $dR/dr \neq 0$ at these points. Then $r \circ \gamma(s_{0,1})$ are turning points of $r \circ \gamma$, i.e. $(r \circ \gamma)'$ changes its sign at s_0 and s_1 .
- (2) One of the zeros $r \circ \gamma(s_0)$ or $r \circ \gamma(s_1)$ is a higher order zero of R . Then $r \circ \gamma$ is constant.

Both the two facts follow from Proposition 3.9.

Most possible CTGs can be ruled out by comparing the location of the zeros of $R(r)$ with the following consequence of the causal structure of Kerr:

Lemma 5.1. Let $\gamma: I \rightarrow K^*$ be a closed timelike geodesic. Then $r \circ \gamma \subset \{r < r_-\}$.

Proof. The region

$$\{r \geq r_-\} = \{t^* \in \mathbb{R}, r \in [r_-, +\infty), (\theta, \phi^*) \in S^2\} \setminus \Sigma \subset K^*$$

is causal by Corollary 2.17 and closed timelike geodesics cannot intersect $\{r = r_-\}$ by Prop. 5.3. $\square$

Proposition 5.2. There are no closed timelike geodesics strictly contained in $\{0 < r < r_-\}$.

Proof. First we claim that the hypersurfaces $\mathcal{N}_t := \{t = \text{const}\} \cap \{0 < r < r_-\}$ are spacelike. Indeed, if $p \in \mathcal{N}_t \setminus A$, where $A = \{\theta = 0, \pi\}$, then $T_p \mathcal{N}_t$ is spanned by $\partial_r, \partial_\theta, \partial_\phi$ which are spacelike and orthogonal to each other. If $p \in A \subset \mathcal{N}_t$, then $p = (t, r, q)$ with $q = (0, 0, \pm 1) \in S^2 \subset \mathbb{R}^3$, and we may replace $\partial_\theta, \partial_\phi$ by any basis of $T_q S^2$. Suppose by contradiction that there exist a CTG γ in $\{0 < r < r_-\}$. Then $t \circ \gamma$ takes values in a closed t -interval, and so must attain maximum t_0 , say at parameter s_0 . Therefore $\gamma'(s_0) \in T_{\gamma(s_0)} \mathcal{N}_{t_0}$, so γ must be tangent $\mathcal{N}_{t_0}$. This is a contradiction since $\gamma'(s_0)$ is timelike. $\square$

5.2. Horizons and Axis cases. First we rule out CTGs entirely contained in the axis $A = \{\theta = 0, \pi\}$ and CTGs intersecting the horizon $\mathcal{H} = \{r = r_\pm\}$.

The case of the horizon $\mathcal{H} = \{r = r_\pm\}$.

Proposition 5.3. *There are no CTGs intersecting $\mathcal{H} = \{r = r_\pm\}$.*

Proof. First, notice that there are no timelike geodesics entirely contained in $\mathcal{H}$ by Prop. 2.15. Consider now a timelike geodesic intersecting $\mathcal{H}$ transversally. Since each connected component $\{r = r_\pm\}$ of $\mathcal{H}$ is an orientable hypersurface separating the orientable manifold K^* , every closed curve transversal to $\mathcal{H}$ has to intersect $\{r = r_\pm\}$ an even number of times. Further since K^* is time-oriented by $-\partial_r^*$, all tangent vectors to a timelike geodesic transversal to $\mathcal{H}$ have to lie on one side of $\mathcal{H}$. Therefore a timelike geodesic transversal to $\mathcal{H}$ can intersect each connected component $\{r = r_\pm\}$ only once. This shows that no timelike geodesic transversal to $\mathcal{H}$ can close. $\square$

The case of the axis $A = \{\theta = 0, \pi\}$.

Proposition 5.4. *There are no CTGs which are tangent at some point to $A = \{\theta = 0, \pi\}$. In particular, there are no CTGs entirely contained in A .*

Proof. First of all, $A = \{\theta = 0, \pi\}$ is a 2-dimensional closed totally geodesic submanifold by Lem. 2.14. Hence if a geodesic γ is tangent to A at some point, it will always lie on A . If $\gamma \in A$, then $L = 0$, since $\hat{\partial}_\phi \equiv 0$ on A . From the θ -equation of Prop. 3.6, we have $Q = -a^2(E^2 + q)$, hence $K = -a^2q$. Therefore we obtain

$$R(r) = \Delta(r)(qr^2 + a^2q) + (r^2 + a^2)^2 E^2 = (r^2 + a^2)[(r^2 + a^2)E^2 + q\Delta(r)].$$

Now distinguish the cases $E = 0$ and $E \neq 0$. If $E = 0$, $R(r) = q(r^2 + a^2)\Delta(r)$, so the only turning points must be on $\mathcal{H}$. Hence this polynomial cannot produce a CTG since the hypersurfaces $\mathcal{H} = \{r = r_\pm\}$ are closed totally geodesic submanifolds by Prop. 2.15 and a geodesic cannot have turning points on such hypersurfaces because it would be tangent to them there. If $E \neq 0$,

$$R(r) = (r^2 + a^2)[r^2(E^2 + q) - 2Mqr + a^2(E^2 + q)].$$

Let us consider the discriminant of the second factor: $\text{dis} = 4M^2q^2 - 4a^2(E^2 + q)^2$. If $\text{dis} < 0$, no bounded r -behaviour is possible. If $\text{dis} = 0$, the two coincident roots of $R(r)$ are $Mq/(E^2 + q) =: \bar{r}$. Since $\mathcal{H}$ is a null hypersurface by Prop. 2.15, there are no timelike geodesics in $\mathcal{H}$, hence we may assume $\bar{r} \neq r_\pm$. Therefore we can use the t -diff. equation of Prop. 3.6 to get

$$\rho^2(r, \theta)t' = (r^2 + a^2)^2 E / \Delta(r) \neq 0,$$

for every r . Hence $t(s)$ must be monotone and the geodesic cannot be closed.

If $\text{dis} > 0$, a bounded r -behaviour would require $E^2 + q < 0$. However the following root satisfies

$$\frac{Mq}{E^2 + q} + \sqrt{\frac{M^2q^2}{(E^2 + q)^2} - a^2} > r_+ = M + \sqrt{M^2 - a^2},$$

since $0 < \frac{q}{E^2 + q} = \left(1 + \frac{E^2}{q}\right)^{-1} > 1$, which contradicts Lemma 5.1. $\square$

5.3. Steps of the proof for other cases. The proof splits into two main cases $E = 0$ and $E \neq 0$.

If $E = 0$ (§5.4), by Prop. 4.2 we can analyse the only two possible subcases $K(0, L, Q) = 0$ (§5.4.1) and $K(0, L, Q) > 0$ (§5.4.2).

If $E \neq 0$ (§5.5), we analyse three subcases $Q = 0$ (§5.5.1), $Q > 0$ (§5.5.2) and $Q < 0$ (§5.5.3).

Remark 5.5. *The only case which requires a detailed analysis of the differential equations is the case $E \neq 0$ and $Q < 0$ (see 5.5.3).*

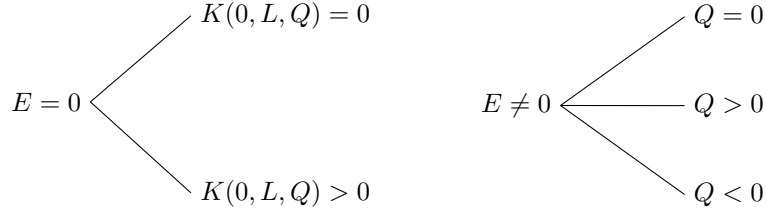


FIGURE 2. All the geodesic types which have to be studied.

5.4. Case $E = 0$. From Prop. 3.6, we have

$$R(r) = qr^4 - 2Mqr^3 + \mathfrak{X}(0, L, Q)r^2 + 2MK(0, L, Q)r - a^2Q \geq 0, \quad (6)$$

$$\Theta(\theta) = Q + \cos^2 \theta \left(a^2q - \frac{L^2}{\sin^2 \theta} \right) \geq 0, \quad (7)$$

with $\mathfrak{X}(0, L, Q) = a^2q - L^2 - Q$ and $K(0, L, Q) = L^2 + Q$. Notice that we must have $Q \geq 0$ by (7), hence $\mathfrak{X}(0, L, Q) < 0$.

5.4.1. Subcase $K(0, L, Q) = 0$. Then $Q = -L^2 \leq 0$. Since $Q \geq 0$, we hence must have $Q = L = 0$. Therefore

$$R(r) = qr^2\Delta(r). \quad (8)$$

Then by Lemma 5.1, the only possible r -behaviour for a CTG would be $r(s) = \text{const} = 0$. However, this geodesic would lie on the ring singularity by Prop. 4.2.

5.4.2. Subcase $K(0, L, Q) > 0$. The signs of the coefficients of $R(r)$ are $- + - + -$ if $Q > 0$ (respectively $- + - +$ if $Q = 0$), hence there are no roots in $r < 0$ and either four or two or zero positive roots (respectively either three or one positive roots) by the "Descartes' rule of signs". Therefore a CTG γ could have a bounded r -behaviour only if $r(s) \in [0, r_-)$ by Lemma 5.1. Now distinguish the cases $Q = 0$ and $Q > 0$. If $Q = 0$, then $\cos^2 \theta(s) = 0$ by (7), hence γ lies in Eq . Therefore $r(s) > 0$ because otherwise γ would hit the ring singularity. If $Q > 0$, then $R(0) < 0$, hence $r(s) > 0$. By Lemma 5.1, in both cases we have $0 < r(s) < r_-$. So γ cannot be closed by Prop. 5.2.

5.5. Case $E \neq 0$.

5.5.1. *Subcase $Q = 0$.* We have

$$R(r) = (E^2 + q)r^4 - 2Mqr^3 + \mathfrak{X}(E, L, 0)r^2 + 2MK(E, L, 0)r \geq 0, \quad (9)$$

$$\Theta(\theta) = \cos^2 \theta \left[a^2(E^2 + q) - \frac{L^2}{\sin^2 \theta} \right] \geq 0. \quad (10)$$

Proposition 5.6. *All the timelike geodesics with $E \neq 0$, $Q = 0$ which have bounded r -behaviour lie in $Eq = \{\theta = \pi/2\}$.*

Proof. Suppose there exists an r -bounded timelike geodesic with $E \neq 0$, $Q = 0$ for which $\theta(s) \neq \pi/2$ for some s . Then from (10), we get

$$a^2(E^2 + q) \geq \frac{L^2}{\sin^2 \theta} \geq L^2.$$

So $\mathfrak{X}(E, L, 0) = a^2(E^2 + q) - L^2 \geq 0$, hence $E^2 + q \geq 0$. Observe also that $K(E, L, 0) \geq 0$ by Prop. 4.2. Since the coefficients of $R(r)$ are all non-negative, $R(r)$ cannot have positive roots by the "Descartes' rule of signs". By Prop. 4.8.2 of [15] there are no timelike geodesics with bounded r -behaviour in $r < 0$. Therefore two possibilities are left, either $r(s) \in [\bar{r}, 0]$, with $\bar{r} < 0$ or $r(s) = \text{const} = 0$. In the first case we would have $R'(0) < 0$, which contradicts $K(E, L, 0) \geq 0$. In the second case we would have $R'(0) = 0$, hence $L = aE$, which contradicts $\mathfrak{X}(E, L, 0) \geq 0$. $\square$

By Prop. 5.6, we can suppose $\theta(s) = \pi/2$ for every s . Since the geodesics are constrained in Eq , $r = 0$ cannot be reached, hence the r -motion must be either confined in $\{r < 0\}$ or in $\{r > 0\}$. By Prop. 4.8.2 of [15] no bounded r -behaviour in $r < 0$ is allowed. By Lemma 5.1 the r -motion must then be constrained in the region $\{0 < r < r_-\}$. Such geodesic cannot be closed by Prop. 5.2.

5.5.2. *Subcase $Q > 0$.* We have $R(0) = -a^2Q < 0$. Hence a bounded r -behaviour is either confined in $\{r < 0\}$ or in $\{r > 0\}$. However if the timelike geodesic lies in $\{r > 0\}$, it must be constrained in $\{0 < r < r_-\}$ by Lemma 5.1 and it cannot be closed by Prop. 5.2. If it lies in $\{r < 0\}$, by Prop. 4.8.2 of [15] it must be a fly-by geodesic, i.e. $r \circ \gamma \subset [-\infty, r_{\text{turn}}]$, where at r_{turn} the geodesic reverse its r -motion.

5.5.3. *Subcase $Q < 0$.* This is the last remaining case and the most difficult one. By Prop. 4.4, the only possible bounded behaviour is $r(s) = \text{const} < 0$ (see Fig.3). Such geodesics are known in the literature as *spherical geodesics*, see e.g. [18].

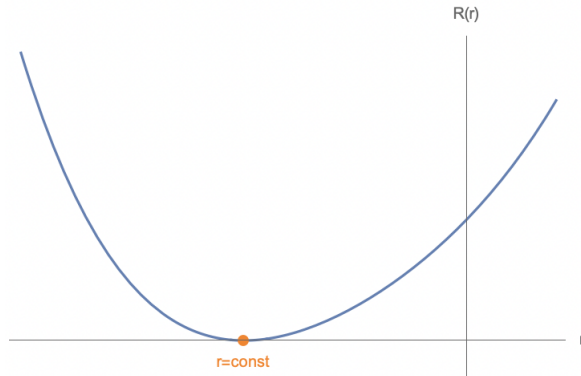


FIGURE 3. Plot of $R(r)$ with $a = 3$, $M = 5$, $q \approx -0.337$, $E \approx 1.156$, $L \approx 0.130$, $Q = -6$.

Proposition 5.7. *Consider a spherical timelike geodesic γ with $Q < 0$ at radius r . If $r \leq -M$, then γ cannot be closed.*

Proof. First observe that if $r \leq -M$ and $\theta \neq 0, \pi$, then ∂_ϕ is spacelike. Indeed the sign of $\mathbf{g}(\partial_\phi, \partial_\phi)$ is determined by the sign of the following function

$$\begin{aligned} r^4 + a^2(1 + \cos^2 \theta)r^2 + 2a^2 M \sin^2 \theta r + a^4 \cos^4 \theta &\geq r^4 + a^2 r^2 + 2a^2 M r \\ &\geq M^2 r^2 + 2a^2 M r + a^2 M^2 \\ &> 0, \end{aligned}$$

where we used respectively the bounds on the trigonometric functions, $r \leq -M < 0$, and the negativity of the discriminant of the r -polynomial since $|a| < M$.

Now we claim that the hypersurfaces $\mathcal{S}_t := \{t = \text{const}\} \cap \{r \leq -M\}$ are spacelike, hence $t \circ \gamma$ is monotonic, therefore γ cannot be closed. Indeed, if $p \in \mathcal{S}_t \setminus A$, where $A = \{\theta = 0, \pi\}$, then $T_p \mathcal{S}_t$ is spanned by $\partial_r, \partial_\theta, \partial_\phi$ which are spacelike and orthogonal to each other. If $p \in A \subset \mathcal{S}_t$, then $p = (t, r, q)$ with $q = (0, 0, \pm 1) \in S^2 \subset \mathbb{R}^3$, and we may replace $\partial_\theta, \partial_\phi$ by any basis of $T_q S^2$. $\square$

By Prop. 4.3, timelike geodesics with negative Carter constant do not meet $Eq = \{\theta = \pi/2\}$, hence $\cos^2 \theta \neq 0$. Then we may define $u := \cos^2 \theta \in (0, 1]$. Since $E^2 + q > 0$ by Prop. 4.3, we can re-write the θ -equation in (5) as

$$\left(\frac{\rho^2(r, u)}{\sqrt{E^2 + q}} \right)^2 \frac{(u')^2}{4u} = -a^2 u^2 + (a^2 - \hat{\Phi}^2 - \hat{\mathcal{Q}})u + \hat{\mathcal{Q}} =: \hat{\Theta}(u), \quad (11)$$

where $\hat{\Phi} := L/\sqrt{E^2 + q}$ and $\hat{\mathcal{Q}} := Q/(E^2 + q)$. Since we must have $\hat{\Theta}(u) \geq 0$ somewhere in $(0, 1]$, $w := a^2 - \hat{\Phi}^2 - \hat{\mathcal{Q}} > 0$ because $\hat{\mathcal{Q}} < 0$ and the coefficient of the second order term is negative. Therefore $\hat{\Theta}$ must have roots given by

$$u_{\pm} = \frac{w \pm \sqrt{\text{dis}}}{2a^2} \quad (12)$$

where $\text{dis} := w^2 + 4a^2 \hat{\mathcal{Q}}$, so that

$$\hat{\Theta}(u) = -a^2(u - u_+)(u - u_-). \quad (13)$$

Hence we have

$$0 < u_- \leq u_+.$$

Note that we must have $u_- \leq 1$, otherwise the downward parabola function $\hat{\Theta}(u)$ cannot be non-negative somewhere in $(0, 1]$. Since $\hat{\Theta}(u)$ is quadratic and $\hat{\Theta}(1) = -\hat{\Phi}^2 \leq 0$, we must either have $u_+ \leq 1$ or $u_- = 1$. However in the latter case $\theta \circ \gamma(s) = 0, \pi$ for all s , so the geodesic lies in A and it cannot be closed by Prop. 5.4. By hypothesis, we can hence assume

$$0 < u_- \leq u_+ \leq 1.$$

Proposition 5.8. *In the Kerr-star spacetime, consider a timelike geodesic γ with $\hat{\mathcal{Q}} < 0$ and $r = \text{const}$. If $\text{dis} = 0$, then $\theta = \text{const}$ and the geodesic cannot be closed.*

Proof. Since $\text{dis} = 0$, we have $u_- = u_+$. Then

$$\hat{\Theta}(u) = -a^2(u - u_+)^2 \geq 0.$$

Hence the last inequality is satisfied only if $u = u_+ = \text{const}$, therefore $\theta = \text{const}$. Then the geodesic γ cannot be closed. Indeed, there are two possibilities. First, if γ is entirely contained in A , then it cannot be closed by Prop. 5.4. Second, if γ is not entirely contained in A , by Prop. 3.6 a geodesic of the form $s \mapsto (t(s), r_0, \theta_0, \phi(s))$ has $t' \equiv \text{const}$ and $\phi' \equiv \text{const}$. It follows that $s \mapsto t(s)$ and $s \mapsto \phi(s)$ are affine functions. If the geodesic is bounded in K^* , then $t(s)$ must be constant. For $\gamma(s) = (t_0, r_0, \theta_0, b_0 s + b_1)$, $b_0, b_1 \in \mathbb{R}$, the geodesic equation can be written in BL coordinates as $\Gamma_{\phi\phi}^\alpha(\gamma(s))b_0^2 = 0$. We claim that the Christoffel symbol $\Gamma_{\phi\phi}^\theta$ cannot vanish on γ . Indeed,

$$\Gamma_{\phi\phi}^\theta = -\frac{\sin\theta\cos\theta}{\rho^6(r,\theta)} \left[\rho^4(r,\theta) \frac{\mathbf{g}(\partial_\phi, \partial_\phi)}{\sin^2\theta} + 2M(r^2 + a^2)a^2r\sin^2\theta \right] \neq 0$$

since $\theta \neq 0, \pi$ because we have already ruled out closed timelike geodesics in A , $\theta \neq \pi/2$ by Prop. 4.3, $\dot{\gamma} = \partial_\phi$ is timelike, and $r < 0$. Hence, $b_0 = 0$ and the geodesic degenerates to a point. $\square$

Remark 5.9. *Closed timelike curves exist in the Kerr-star spacetime: for instance, they are given by the integral curves of the vector field ∂_ϕ , whenever this last happens to be timelike for some negative r . Such curves cannot be geodesics by Prop. 5.8.*

We may now assume $\text{dis} > 0$. Therefore we have the following chain of inequalities

$$0 < u_- < u_+ \leq 1. \quad (14)$$

We hence define

$$\theta_1 := \arccos(\sqrt{u_+}), \theta_2 := \arccos(\sqrt{u_-}), \theta_3 := \arccos(-\sqrt{u_-}), \theta_4 := \arccos(-\sqrt{u_+}) \quad (15)$$

so that

$$0 \leq \theta_1 < \theta_2 < \frac{\pi}{2} < \theta_3 < \theta_4 \leq \pi. \quad (16)$$

Proposition 5.10. *In the Kerr-star spacetime, timelike geodesics with $\hat{Q} < 0$, $r = \text{const}$ and $\theta \neq \text{const}$ can have one of the following θ -behaviours:*

- if $0 < u_- < u_+ < 1$, then the θ -coordinate oscillates periodically in one of the following intervals $0 < \theta_1 \leq \theta \leq \theta_2 < \pi/2$ or $\pi/2 < \theta_3 \leq \theta \leq \theta_4 < \pi$;
- if $0 < u_- < u_+ = 1$, then $\hat{\Phi} (= L/\sqrt{E^2 + q}) = 0$ and the θ -coordinate oscillates periodically in one of the following intervals $0 = \theta_1 \leq \theta \leq \theta_2 < \pi/2$ or $\pi/2 < \theta_3 \leq \theta \leq \theta_4 = \pi$,

where the θ_i , $i = 1, 2, 3, 4$, are given by (15).

Proof. See the proof of Prop. 5.9 of [16]. $\square$

Consider the first order equations of motion (with the rescaled constants of motion $\hat{Q} := Q/(E^2 + q)$, $\hat{\Phi} := L/\sqrt{E^2 + q}$, $\Phi := L/E$), for a constant $r < 0$:

$$\frac{\rho^2(r,\theta)}{\sqrt{E^2 + q}} \frac{d\theta}{ds} = \pm \sqrt{\Theta(\theta)} = \pm \sqrt{\hat{Q} + a^2 \cos^2 \theta - \hat{\Phi}^2 \frac{\cos^2 \theta}{\sin^2 \theta}} \quad (17)$$

$$\frac{\rho^2(r,\theta)}{E} \frac{dt}{ds} = \frac{r^2 + a^2}{\Delta(r)} (r^2 + a^2 - a\Phi) + a(\Phi - a \sin^2 \theta), \quad (18)$$

where now the function $\Theta(\theta)$ is meant as the ratio of the Θ -function appearing in Prop. 3.6 and $(E^2 + q)$. Because of the θ -differential equation, we can restrict to an interval $\mathcal{U} \subset \theta^{-1}((\theta_1, \theta_2))$ on which $d\theta/ds$ is either everywhere positive or everywhere negative (depending on the initial condition). Due to the symmetry in (17) and the fact that $r = \text{const}$, $\theta(s)$ is periodic over twice the interval $\mathcal{U}$. For instance, set $\mathcal{U} = (0, T/2)$, starting from $\theta(0) = \theta_1$, hence $\theta'(s) = +\sqrt{\Theta(\theta)} > 0$ for $s \in (0, T/2)$, then $\theta'(s) = -\sqrt{\Theta(\theta)} < 0$ for $s \in (T/2, T)$, where $\theta'(T/2) = 0$, because $\theta(T/2) = \theta_2$ ($' \equiv d/ds$) and Prop. 3.9, which explains the change of sign of $\theta'(s)$ (using the fact that θ_1, θ_2 are multiplicity one zeroes of $\Theta(\theta)$). Hence every $\Delta s = T/2$, $\theta'(s)$ changes sign. See Figures 11, 12, 13, 14 of [16] for analogous θ -motions.

At parameters where $\Theta(\theta) \neq 0$ we can combine (18) and (17) to get

$$\frac{\sqrt{E^2 + q}}{E} \frac{dt}{d\theta} = \frac{r^2 \Delta(r) + 2Mr(r^2 + a^2 - a\Phi)}{\pm \Delta(r) \sqrt{\Theta(\theta)}} + a^2 \frac{\cos^2 \theta}{\pm \sqrt{\Theta(\theta)}} = B(r, a, \Phi) \frac{1}{\pm \sqrt{\Theta(\theta)}} + a^2 \frac{\cos^2 \theta}{\pm \sqrt{\Theta(\theta)}}, \quad (19)$$

with $B(r, a, \Phi) := \frac{r^2 \Delta(r) + 2Mr(r^2 + a^2 - a\Phi)}{\Delta(r)}$.

Lemma 5.11. *Consider a spherical timelike geodesic γ at radius r with negative Carter constant Q and angular momentum Φ . If $B(r, a, \Phi) \geq 0$, then γ cannot be closed.*

Proof. The equation (18) can be written as

$$\frac{\rho^2(r, \theta)}{E} \frac{dt}{ds} = B(r, a, \Phi) + a^2 \cos^2 \theta.$$

Then the t -coordinate is monotonic, hence it is non-periodic. $\square$

Proposition 5.12. *In a Kerr spacetime, a timelike geodesic with negative Carter constant $\mathcal{Q}$ and constant radial coordinate has one of the two following pairs $(\Phi_{\pm}, \mathcal{Q}_{\pm})$ of constants of motion given by*

$$\Phi_{\pm} = \Phi_{\pm}(r, q) = \frac{E^2 M(a^2 - r^2) \pm \sqrt{f(r, q)}}{aE^2(M - r)}, \quad (20)$$

$$\mathcal{Q}_{\pm} = \mathcal{Q}_{\pm}(r, q) = \frac{r^2}{a^2 E^2 (M - r)^2} \left\{ a^2 M [-Mq + (2E^2 + q)r] + r [4M^3 q - M^2(5E^2 + 8q)r + M(4E^2 + 5q)r^2 - (E^2 + q)r^3] \pm 2M\sqrt{f(r, q)} \right\},$$

where $f(r, q) := E^2 r [-Mq + (E^2 + q)r] \Delta(r)^2$ with $-Mq + (E^2 + q)r \leq 0$.

Proof. Since $E \neq 0$ by Prop. 4.3, we can divide the r -equation by E^2 to get

$$\begin{aligned} \left(\frac{\rho^2}{E} \right)^2 (r')^2 = & \left(1 + \frac{q}{E^2} \right) r^4 - 2M \frac{q}{E^2} r^3 + \left[a^2 \left(1 + \frac{q}{E^2} \right) - \Phi^2 - \mathcal{Q} \right] r^2 \\ & + 2M \left[\mathcal{Q} + (\Phi - a)^2 \right] r - a^2 \mathcal{Q} =: \mathcal{R}(r), \end{aligned}$$

where $\Phi := L/E$ and $\mathcal{Q} := Q/E^2$. A geodesic has constant radial behaviour if and only if $\mathcal{R}(r) = 0$ and $d\mathcal{R}(r)/dr = 0$. These two equations, quadratic in Φ , can be solved for $\mathcal{Q}$ and Φ to get the two pairs (20), as analogously done in Prop. 5.10 of [16]. (See also eq. (285) at p. 363 of [6].)

The condition $-Mq + (E^2 + q)r \leq 0$ is required for the reality of (20). $\square$

Lemma 5.13. *For the class $(\Phi_+, \mathcal{Q}_+)$ of spherical timelike geodesics with $Q < 0$ at radius $-M < r < 0$, the corresponding function $B(r, a, \Phi_+(r, q)) < 0$ if and only if $q < \frac{E^2}{4M^2}(r^2 + 3Mr)$.*

Proof. The inequality $B(r, a, \Phi_+(r, q)) < 0$ is equivalent to

$$\frac{-2Mr\sqrt{f(r, q)}}{E^2} < r^2(M + r)\Delta(r).$$

Since $\Delta(r) > 0$ in $r < 0$, $-M < r < 0$, the latter is equivalent to

$$E^2 r(r - M)[-4M^2 q + E^2(r^2 + 3Mr)] > 0,$$

or equivalently

$$q < \frac{E^2}{4M^2}(r^2 + 3Mr) < 0.$$

$\square$

Remark 5.14. *In the limit $q \rightarrow 0$, the class $(\Phi_-, \mathcal{Q}_-)$ of spherical timelike geodesics reduces to the admissible class of spherical null geodesics (21) of [16] while the class $(\Phi_+, \mathcal{Q}_+)$ reduces to the impossible class (23) of [16]. As seen at the beginning of 5.5.3, a necessary condition for the existence of the geodesic θ -behaviour is $w = a^2 - \Phi^2 - \mathcal{Q} > 0$. One can see that for the class $(\Phi_+, \mathcal{Q}_+)$, $w = a^2 - \Phi_+^2 - \mathcal{Q}_+ = a^2 - \Phi_+^2 - \frac{E^2}{E^2 + q} - \mathcal{Q}_+ = a^2 - \Phi_+^2 - \frac{E^2}{E^2 + q} - \frac{E^2}{4M^2}(r^2 + 3Mr)$ indeed can be positive for some negative $q < \frac{E^2}{4M^2}(r^2 + 3Mr)$ and some $-M < r < 0$ and it is negative when $q = 0$.*

We rule out closed spherical geodesics in the subcase $E \neq 0$, $Q < 0$. Consider a timelike geodesic $\gamma : I \rightarrow K^*$ with constants of motion $E \neq 0$, $Q < 0$, non-constant coordinate functions

$s \mapsto t(s), \theta(s), \phi(s)$ and constant negative radial coordinate $-M < r < 0$ such that $B(r, a, \Phi) < 0$. The differential equation (19) has the form

$$\frac{dt}{d\theta} = F(\theta),$$

for some function F . The variation of the t -coordinate on a full θ -oscillation is given by

$$\Delta t = 2 \int_{\theta_1}^{\theta_2} F(\theta) d\theta.$$

Remark 5.15. Notice the factor "2" in the last expression. On a full θ -oscillation, we have

$$\int_{\theta_1}^{\theta_2} F(\theta) d\theta + \int_{\theta_2}^{\theta_1} -F(\theta) d\theta = 2 \int_{\theta_1}^{\theta_2} F(\theta) d\theta.$$

Therefore the variation of the t -coordinate after n θ -oscillations is $n\Delta t$ because of the periodicity of the θ -coordinate. If the geodesic is closed, $\Delta t = 0$, otherwise the coordinate $t(s)$ cannot be periodic. Hence it suffices to study what happens on a single θ -oscillation.

Remark 5.16. A motion of the kind $\pi \geq \theta_4 \geq \theta \geq \theta_3 > \pi/2$ produces the same integrals since in this θ -interval $\cos \theta < 0$, hence with the substitution $u = \cos^2 \theta$ we have $d\theta = \frac{1}{2} \frac{du}{\sqrt{u}\sqrt{1-u}}$. Therefore

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{\Theta(\theta)}} = \int_{\theta_3}^{\theta_4} \frac{d\theta}{\sqrt{\Theta(\theta)}}, \quad \int_{\theta_1}^{\theta_2} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} = \int_{\theta_3}^{\theta_4} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}}.$$

Hence Δt is the same.

So without any loss of generality, we may consider a motion of the type $0 \leq \theta_1 \leq \theta \leq \theta_2 < \pi/2$. Then we can integrate (19) on a full oscillation to get

$$\frac{\sqrt{E^2 + q}}{E} \Delta t = 2B(r, a, \Phi) \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{\Theta(\theta)}} + 2a^2 \int_{\theta_1}^{\theta_2} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}}. \quad (21)$$

We now have to compute the following integrals

$$I_1 := \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{\Theta(\theta)}},$$

$$I_2 := \int_{\theta_1}^{\theta_2} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}}.$$

Let us start from the first integral:

$$I_1 = -\frac{1}{2} \int_{u_+}^{u_-} \frac{du}{\sqrt{u}\sqrt{\hat{\Theta}(u)}}, \quad (22)$$

where we have used the substitution $u := \cos^2 \theta$, hence $d\theta = -\frac{1}{2} \frac{du}{\sqrt{u}\sqrt{1-u}}$ since $\sin \theta \geq 0$ and $\cos \theta > 0$ if $\theta_1 \leq \theta \leq \theta_2$. Now we can use (11) and the substitution $u =: u_- + (u_+ - u_-)y^2$ adopted in [11] to get

$$\begin{aligned} I_1 &= \frac{1}{2} \int_{u_-}^{u_+} \frac{du}{\sqrt{u}\sqrt{a^2(u_+ - u)(u - u_-)}} \\ &= \frac{1}{2|a|} \int_0^1 \frac{2(u_+ - u_-)y dy}{\sqrt{u_- + (u_+ - u_-)y^2} \sqrt{(u_+ - u_- - (u_+ - u_-)y^2)(u_+ - u_-)y^2}} \\ &= \frac{1}{|a|} \int_0^1 \frac{dy}{\sqrt{u_- + (u_+ - u_-)y^2} \sqrt{1 - y^2}} \\ &= \frac{1}{|a|\sqrt{u_-}} \int_0^1 \frac{dy}{\sqrt{1 - y^2} \sqrt{1 - \left(1 - \frac{u_+}{u_-}\right)y^2}}. \end{aligned}$$

With the same substitutions, we also get

$$\begin{aligned}
I_2 &= -\frac{1}{2} \int_{u_+}^{u_-} \frac{udu}{\sqrt{u}\sqrt{\hat{\Theta}(u)}} \\
&= \frac{1}{2} \int_{u_-}^{u_+} \frac{udu}{\sqrt{u}\sqrt{a^2(u_+ - u)(u - u_-)}} \\
&= \frac{1}{|a|} \int_0^1 \frac{\sqrt{u_- + (u_+ - u_-)y^2}}{\sqrt{1 - y^2}} dy \\
&= \frac{\sqrt{u_-}}{|a|} \int_0^1 \frac{\sqrt{1 - (1 - \frac{u_+}{u_-})y^2}}{\sqrt{1 - y^2}} dy.
\end{aligned}$$

Then with the definition of the elliptic integrals in Appendix A we have

$$I_1 = \frac{1}{|a|\sqrt{u_-}} \mathcal{K}\left(1 - \frac{u_+}{u_-}\right), \quad (23)$$

$$I_2 = \frac{\sqrt{u_-}}{|a|} \mathcal{E}\left(1 - \frac{u_+}{u_-}\right). \quad (24)$$

Hence, we get

$$\Delta t = \frac{E}{\sqrt{E^2 + q}} \left[\frac{2B(r, a, \Phi)}{|a|\sqrt{u_-}} \mathcal{K}\left(1 - \frac{u_+}{u_-}\right) + 2|a|\sqrt{u_-} \mathcal{E}\left(1 - \frac{u_+}{u_-}\right) \right]. \quad (25)$$

Note that, since $u_+ > u_- > 0$, we have $1 - \frac{u_+}{u_-} < 0$, and hence $\mathcal{E}(1 - u_+/u_-) > \mathcal{K}(1 - u_+/u_-) > 0$ (see Appendix A). However, the prefactor of $\mathcal{E}$ does not dominate the opposite of the prefactor of $\mathcal{K}$ for every negative r , as one may check substituting $\hat{\Phi} = \Phi_{\pm}E/\sqrt{E^2 + q}$ and $\hat{Q} = Q_{\pm}E^2/(E^2 + q)$ from (20) into $u_{\pm}$ given by (12) and $\Phi = \Phi_{\pm}$ into $B(r, a, \Phi)$.

From now on set $x := 1 - u_+/u_-$. The elliptic integral $\mathcal{K}$ can be written as a hypergeometric function (see A.5):

$$\mathcal{K}(x) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; x\right).$$

Using the Pfaff transformation (see A.6)

$$F\left(\alpha, \beta; \gamma; x\right) = (1 - x)^{-\alpha} F\left(\alpha, \gamma - \beta; \gamma; \frac{x}{x - 1}\right), \quad (26)$$

we can decrease the modulus of the prefactor in front of the elliptic integral $\mathcal{K}$:

$$\mathcal{K}(x) = \frac{\sqrt{u_-}}{\sqrt{u_+}} \mathcal{K}\left(\frac{x}{x - 1}\right). \quad (27)$$

Hence we get

$$\Delta t = \frac{E}{\sqrt{E^2 + q}} \left[2|a|\sqrt{u_-} \mathcal{E}(x) + \frac{2B(r, a, \Phi)}{|a|\sqrt{u_+}} \mathcal{K}\left(\frac{x}{x - 1}\right) \right]. \quad (28)$$

Now we compare the elliptic integrals, after the Pfaff transformation. Since $x < 0$, we have

$$\mathcal{E}(x) > \mathcal{K}\left(\frac{x}{x - 1}\right) > 0, \quad (29)$$

by Rmk. A.2. Next we claim that the prefactors of the elliptic integrals in (28) satisfy

$$2|a|\sqrt{u_-} > -\frac{2B(r, a, \Phi)}{|a|\sqrt{u_+}}. \quad (30)$$

Indeed, both sides of the inequality are positive, so we can square them and use that $u_+u_- = -\dot{Q}/a^2 = -\frac{Q}{a^2} \frac{E^2}{E^2+q}$ by (11) to get an equivalent inequality

$$-a^2 Q \frac{E^2}{E^2+q} > B^2(r, a, \Phi). \quad (31)$$

Proposition 5.17. *For both classes (20) of $r = \text{const}$ timelike geodesics with $Q < 0$ the inequality (31) holds if $B(r, a, \Phi_{\pm}) < 0$ and $-M < r < 0$.*

Proof. Consider the class (Φ_-, Q_-) . Then (31) becomes

$$E^2 M (E^2 + q) (M - r)^2 r^2 \Delta(r) A(r, q) > 0, \quad (32)$$

with

$$A(r, q) := -\Delta(r) \left\{ a^2 E^2 [-Mq + (2E^2 + q)r] + r [-4M^2 q^2 + Mq(E^2 + 4q)r + E^2(6E^2 + 7q)r^2] \right. \\ \left. + 2a^2 E^2 \sqrt{f(r, q)} + 2r [3E^2 r + 2q(M + r)] \sqrt{f(r, q)} \right\}.$$

Since $E^2 + q > 0$ by Prop. 4.3, $\Delta(r) > 0$ if $r < 0$, (32) is equivalent to

$$A(r, q) > 0. \quad (33)$$

Since $-M < r < 0$, $q < 0$, we have

$$A(r, q) \geq -\Delta(r) \left\{ a^2 E^2 [-Mq + (2E^2 + q)r] + r [-4M^2 q^2 + Mq(E^2 + 4q)r + E^2(6E^2 + 7q)r^2] \right\} \\ =: G(r, q). \quad (34)$$

We claim that $G(r, q) > 0$, hence (33) is satisfied. Indeed, first $-Mq + (2E^2 + q)r < -Mq + (E^2 + q)r \leq 0$ by Prop. 5.12. Second we show that

$$-4M^2 q^2 + Mq(E^2 + 4q)r + E^2(6E^2 + 7q)r^2 \geq 0.$$

Since $-Mq + (E^2 + q)r \leq 0$, we can respectively use $q^2(r - M) \geq -E^2 qr$ and $r^2(E^2 + q) \geq Mqr$ in the following

$$\begin{aligned} & -4M^2 q^2 + Mq(E^2 + 4q)r + E^2(6E^2 + 7q)r^2 \\ &= ME^2 qr + 6E^4 r^2 + 4Mq^2(r - M) + 7E^2 qr^2 \\ &\geq ME^2 qr + 6E^4 r^2 - 4ME^2 qr + 7E^2 qr^2 \\ &= -3ME^2 qr + 6E^4 r^2 + 7E^2 qr^2 \\ &= -3ME^2 qr + 6E^2 r^2(E^2 + q) + E^2 qr^2 \\ &\geq -3ME^2 qr + 6ME^2 qr + E^2 qr^2 \\ &= qE^2 r(r + 3M) \\ &> 0, \end{aligned}$$

where in the last inequality we used that $q < 0$, $E \neq 0$ and $-M < r < 0$.

Consider now the class (Φ_+, Q_+) . From Lemma 5.13, $B(r, a, \Phi_+) < 0$ implies that $q < \frac{E^2}{4M^2}(r^2 + 3Mr)$. Prop. 5.12 implies that $q \geq -\frac{E^2 r}{r - M}$. Since $-M < r < 0$, we have

$$-\frac{E^2 r}{r - M} < \frac{E^2}{4M^2}(r^2 + 3Mr) < 0.$$

Therefore we must show that (31) holds for the class (Φ_+, Q_+) when $r \in (-M, 0)$ and

$$q \in \left[-\frac{E^2 r}{r-M}, \frac{E^2}{4M^2}(r^2 + 3Mr) \right).$$

For this class, (31) is equivalent to

$$E^2 M(E^2 + q)(M - r)^2 r^2 \Delta(r) \left\{ -G(r, q) + 2a^2 E^2 \sqrt{f(r, q)} + 2r[3E^2 r + 2q(M + r)] \sqrt{f(r, q)} \right\} < 0.$$

Since $E^2 + q > 0$ by Prop. 4.3, $\Delta(r) > 0$ if $r < 0$, $-M < r < 0$ the latter is equivalent to

$$-G(r, q) < -2a^2 E^2 \sqrt{f(r, q)} - 2r[3E^2 r + 2q(M + r)] \sqrt{f(r, q)} < 0.$$

Now we square the latter, reversing the sign of the inequality, to get

$$a^4 E^4 - 2a^2 E^2 r(-4Mq + E^2 r) + r^2[16M^2 q^2 + 8E^2 Mqr - E^2(15E^2 + 16q)r^2] > 0. \quad (35)$$

We claim that

$$\begin{aligned} -4Mq + E^2 r &> 0, \\ 16M^2 q^2 + 8E^2 Mqr - E^2(15E^2 + 16q)r^2 &> 0, \end{aligned}$$

hence (35) holds.

Indeed, using that $q < \frac{E^2}{4M^2}(r^2 + 3Mr)$ and $-M < r < 0$, we have

$$-4Mq + E^2 r > -4M \frac{E^2}{4M^2}(r^2 + 3Mr) + E^2 r = -E^2 r \left(\frac{r}{M} + 2 \right) > 0,$$

$$\begin{aligned} 16M^2 q^2 + 8E^2 Mqr - E^2(15E^2 + 16q)r^2 &> 16M^2 \left[\frac{E^2}{4M^2}(r^2 + 3Mr) \right]^2 + 8E^2 Mr \left[\frac{E^2}{4M^2}(r^2 + 3Mr) \right] \\ &\quad - E^2(15E^2 + 16q)r^2 \\ &= \frac{E^2 r^2}{M^2} (E^2 r^2 + 8E^2 Mr - 16M^2 q), \end{aligned}$$

which is positive if and only if

$$q < \frac{E^2 r^2 + 8E^2 Mr}{16M^2}.$$

The last indeed holds because $-M < r < 0$, hence

$$\frac{E^2 r^2 + 8E^2 Mr}{16M^2} > \frac{E^2}{4M^2}(r^2 + 3Mr).$$

□

Combining (28), (29) and Prop. 5.17 we conclude that $\Delta t > 0$ for all $-M < r < 0$ such that $B(r, a, \Phi) < 0$, which shows that the spherical timelike geodesics cannot be closed.

We have ruled out all the possibilities on Fig. 2, therefore there are no closed timelike geodesics in the Kerr-star spacetime.

6. CONCLUSION

We considered the Kerr-star spacetime, namely the analytical extension of the Kerr spacetime over the horizons and in the negative radial region. This spacetime contains closed timelike curves through every point below the inner horizon, in the BL block III. However we proved that the timelike geodesics cannot be closed. Using simple geometrical arguments and the first integral differential equations, we ruled out closed geodesics intersecting horizons, those in the axis and most of the remaining ones. It turned out that the most difficult geodesics to analyse were the spherical ones at negative radii with negative Carter constant. We first excluded those with constant θ -coordinate in Prop. 5.8. Then in Prop. 5.12 we found the two classes of spherical timelike

geodesics parametrized by the constant r -coordinate and the Lorentzian energy q . Hence we computed their variation Δt of the t -coordinate on a full θ -oscillation in eq. (25). Using the Pfaff transformation on the first elliptic integral in (25), we were able to compare the elliptic integrals in the resulting expression of Δt , eq. (28). Finally we proved Prop. 5.17 showing that Δt is in fact positive for any spherical geodesic with negative Carter constant.

In this article we proved the nonexistence of closed timelike geodesic in the Kerr-star spacetime (analytical extension of the slow Kerr spacetime over the horizons and in the negative radial region). Combining Thm. 1.1 with the result in [16], it follows that despite the existence of causality violations, the Kerr-star spacetime does not contain closed causal geodesics.

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APPENDIX A. ELLIPTIC INTEGRALS AND HYPERGEOMETRIC FUNCTIONS

Definition A.1. Let $\phi \in [-\pi/2, \pi/2]$. The elliptic integral of the first kind is

$$\mathcal{F}(\phi|k) := \int_0^{\sin \phi} \frac{ds}{\sqrt{(1-s^2)(1-ks^2)}}.$$

The complete ($\phi = \pi/2$) elliptic integral of the first kind is

$$\mathcal{K}(k) := \mathcal{F}(\pi/2|k) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-ks^2)}}.$$

The elliptic integral of the second kind is

$$\mathcal{E}(\phi|k) := \int_0^{\sin \phi} \sqrt{\frac{1-ks^2}{1-s^2}} ds.$$

The complete ($\phi = \pi/2$) elliptic integral of the second kind is

$$\mathcal{E}(k) := \mathcal{E}(\pi/2|k) = \int_0^1 \sqrt{\frac{1-ks^2}{1-s^2}} ds.$$

We define also

$$\mathcal{D}(k) := \int_0^1 \frac{s^2 ds}{\sqrt{(1-s^2)(1-ks^2)}} = \frac{\mathcal{K}(k) - \mathcal{E}(k)}{k} = -2 \frac{\partial \mathcal{E}(k)}{\partial k}.$$

Remark A.2. Let $0 < z, s < 1$, $x < 0$.

$$\sqrt{\frac{1-zs^2}{1-s^2}} > \frac{1}{\sqrt{1-s^2}\sqrt{1-xs^2}} \iff (1-zs^2)(1-xs^2) > 1 \implies \mathcal{E}(x) > \mathcal{K}(z).$$

If $z = x/(x-1)$, it satisfies $0 < z < 1$ and we have

$$(1-zs^2)(1-xs^2) > 1 \iff x+z < xzs^2 \iff 1 > s^2,$$

hence $\mathcal{E}(x) > \mathcal{K}(z)$.

Definition A.3 ([1]). The hypergeometric function $F(\alpha, \beta; \gamma; x)$ is defined by the series

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} x^n,$$

where $(\alpha)_n := \alpha(\alpha+1) \cdots (\alpha+n-1)$ for $n > 0$, $(\alpha)_0 \equiv 1$ (analogous for the others), for $|x| < 1$, and by continuation elsewhere.

Proposition A.4 (Euler's integral representation, see [1]). If $\operatorname{Re} \gamma > \operatorname{Re} \beta > 0$, then

$$F(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-xt)^{-\alpha} dt$$

in the complex x -plane cut along the real axis from 1 to $+\infty$, where $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$ is the Euler's gamma function.

Proposition A.5 ([1]). We can write the complete elliptic integral of the first kind as

$$\mathcal{K}(x) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; x\right).$$

Proof. Use the integral representation of the hypergeometric function given in Prop. A.4, the integral substitution $t = s^2$, with $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(1) = 1$. $\square$

Proposition A.6 ("Pfaff's formula", see Theorem 2.2.5 of [1]).

$$F(\alpha, \beta; \gamma; x) = (1-x)^{-\alpha} F\left(\alpha, \gamma-\beta; \gamma; \frac{x}{x-1}\right).$$

Proof. Use A.4 and the integral substitution $t = 1-s$. $\square$

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