Mon. Not. R. astr. Soc. (1978) **182**, 443–455

The acceleration of cosmic rays in shock fronts – II

A. R. Bell* *Mullard Radio Astronomy Observatory, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE*

Received 1977 August 22; in original form 1977 July 28

Summary. The acceleration to relativistic energies of the high-energy tail of the particle distribution produced by a shock front is discussed. In order to be accelerated, particles need to be able to pass through the shock without being strongly deflected and it is argued, using the Earth's bow shock as an example, that a shock front produces large numbers of protons, and probably electrons also, which satisfy this condition. The resulting energy spectrum of these initially non-relativistic particles is calculated. It yields an estimate of the density of cosmic ray particles in a shocked gas and indicates that a large proportion of the random energy produced by a shock is given to accelerated particles. The synchrotron radio emission from the energetic electrons in a shocked gas is calculated, and the theoretical and observed flux densities of two supernova remnants (Tycho's SNR and Cas A) are compared and found to agree satisfactorily. The implications for the minimum energy of radio sources are considered, and the effect of multiple shocks on the cosmic ray density evaluated.

1 Introduction

In Paper I (Bell 1978) I described a mechanism by which particles can be accelerated in the region surrounding an astrophysical shock front, a prior condition being that the particles are sufficiently energetic to pass through the shock itself without significant deflection. This implies that they must have initial energies above the post-shock thermal energy, and the purpose of the present paper is firstly to determine what the initial energy must be for effective acceleration and how many particles are injected and accelerated to high energy, and secondly to consider the relevance of the acceleration mechanism to radio sources.

The energy necessary for injection depends on the structure of the shock front and especially on the shock thickness. Unfortunately, shocks of high Mach number are poorly understood (e.g. McKee 1974), and some aspects of the discussion here are correspondingly uncertain. The situation as regards the injection of protons is relatively clear, but that for electrons less clear and it is necessary to look for supporting evidence from the observed

*Present address: GEC-Marconi Ltd, Marconi Research Laboratories, Great Baddow, Chelmsford, Essex CM2 8HN.

densities of cosmic ray electrons. Much of the discussion is based upon the theoretical description of shock fronts by Tidman & Krall (1971) and the review of observations of the Earth's bow shock by Formisano (1974).

In Section 2, the conditions for the acceleration of a particle are considered and in Section 3 the energy spectrum is calculated for the energy range between barely supra-thermal and highly relativistic. The results of these sections are then combined in Section 4 to estimate the cosmic ray density produced by a shock front. This allows the radio emission from a shocked gas to be calculated, and the values for two supernova remnants are compared with observations. In Section 5, the implications for the energy requirements of radio sources are discussed. In Section 6, the cosmic ray density is calculated for a gas which has been shocked many times.

2 Conditions for acceleration

2.1 PROTONS

In Paper I it was assumed (i) that the particles being accelerated were sufficiently energetic to pass through the shock, i.e. the region in which the streaming energy is randomized, without significant deflection or change of kinetic energy, and (ii) that the particles were relativistic. This second assumption can be dropped without affecting the validity of the acceleration mechanism, although the energy spectrum is different for sub-relativistic particles. The first assumption, however, cannot be obviated since particles which are deflected through an angle of the order of π by the shock have their energy changed by an amount comparable to the energy gained in passing once from upstream to downstream and back, as described in Paper I. Nevertheless, it is probable that the acceleration mechanism of Paper I applies to particles which are deflected through smaller but not negligible angles.

Particles can be deflected in two ways:

(a) By the magnetic field if the gyroradius inside the shock is comparable to or less than the shock thickness. Since the magnetic field within the shock is comparable with the downstream time-averaged field, this condition corresponds to the shock thickness being greater than the downstream gyroradius.

(b) By the electric field resulting from plasma oscillations within the shock. The potentials within the shock are unlikely to exceed $\frac{1}{2}m_p v_s^2/e$ (McKee 1974), the potential required to stop a proton (with mass m_p) which is streaming into the shock with the shock velocity (v_s). Any particle which has an energy significantly above that of the incoming protons ($\frac{1}{2}m_p v_s^2$) will not be strongly deflected in crossing the shock front.

These statements concerning the magnitude of the magnetic field and electric potentials within the shock are in accordance both with the theoretical discussion by Tidman & Krall (1971) and with satellite measurements of the Earth's bow shock (Formisano 1974). The latter show that the thickness of the shock transition (a few hundred kilometres) is much less than the gyroradius (a few thousand kilometres) of a proton with energy $\frac{1}{2}m_p v_s^2$. Thus it is probable that a proton with a few times this energy will not be strongly deflected by the magnetic field as it crosses the shock, nor by the electric potentials as explained above. Such protons are then available for acceleration by the mechanism described in Paper I. Protons with a kinetic energy of $\frac{1}{2}m_p v_s^2$, the energy of the cold protons convecting into the shock, must be deflected since, by definition, a shock entails an abrupt change in the bulk velocity of the instreaming gas. The deflection is probably then due to the electric fields (b).

An indication of how many protons are accelerated by a shock front to an energy of a few times $\frac{1}{2}m_p v_s^2$ can be obtained from observations of the Earth's bow shock. The velocity of the incoming plasma in this case is about 400 km/s and its Mach number is about 8 (Volk 1970). Large numbers of protons are found to be accelerated to about $4(\frac{1}{2}m_p v_s^2)$ (Ashbridge, Bame & Strong 1968). The number density of these suprathermal protons upstream of the shock is sometimes 10 per cent of the number density of the solar wind protons convecting into the shock, but it is more often ~1 per cent or less. This implies that about one in a hundred of the solar wind protons convecting into the shock are accelerated to $4(\frac{1}{2}m_p v_s^2)$, although this proportion depends also on the degree to which these protons are trapped in the region near the shock. These protons are then sufficiently energetic for them to pass through the shock 'undeflected' if they encounter it subsequently.

It is probable that the thickness of shock fronts with much higher velocities and Mach numbers, such as occur in supernova remnants for example, is less than the gyroradius of the downstream thermal protons (Tidman & Krall 1971) and so protons with energies a few times $\frac{1}{2}m_p v_s^2$ will also pass through such shocks unhindered. It is not known whether the proportion of incoming protons accelerated to this energy is the same for these shocks as it is for the Earth's bow shock, but I here assume this to be the case and use the above measurements to estimate the proportion of protons overtaken by a shock which are injected into the acceleration process of Paper I. In Section 4 this estimate will be used to calculate the number of high-energy cosmic rays generated by a shock front.

2.2 ELECTRONS

The situation concerning the injection of electrons is even less clear than it is for protons since the gyroradius of an electron with a given energy is smaller by the factor $(m_e/m_p)^{1/2}$ than that of a proton with the same energy. The thickness of a shock must therefore be correspondingly smaller for an electron not to be strongly deflected in crossing it. In the case of the Earth's bow shock, the width of the magnetic transition is typically a few times the gyroradius (100 km) in a solar-wind magnetic field of 10^{-5} G of an electron with energy $\frac{1}{2}m_p v_s^2$, the characteristic thermal energy of the post-shock electrons. Thus electrons must have initial energies $\gg \frac{1}{2}m_p v_s^2$ if the acceleration mechanism of Paper I is to be effective. Shocks of very high Mach number, however, may be purely electrostatic (Volk 1970) with a thickness of the order of a Debye length (Tidman & Krall 1971), which is much less than the gyroradius of a post-shock thermal electron. If this is so then electrons with energies a few times $\frac{1}{2}m_p v_s^2$ will be susceptible to further acceleration, as are the protons.

Observations of the Earth's bow shock point to the production of electrons with energies of up to $100(\frac{1}{2}m_p v_s^2)$ (Anderson 1969). The proportion of electrons convected into the shock which are reflected upstream with energies greater than $\frac{1}{2}m_p v_s^2$ can be in excess of 1 in 30 (Scarf *et al.* 1971).

2.3 SCALE LENGTHS

In Paper I, an expression was derived for the scale length (x_0) for the occurrence of Alfvén waves and energetic particles upstream of the shock. If the radius of curvature of the shock is less than $L_{crit} = x_0(v/v_S)$ (where v is the particle velocity), then the probability of return of an upstream energetic particle is reduced and the efficiency of acceleration is also reduced. This is important in the case of the Earth's bow shock which has a radius of curvature of about 9×10^7 m. For the observed densities of suprathermal protons with energies a few times $\frac{1}{2}m_p v_s^2$, the value of L_{crit} is $\sim 10^7$ m. At higher proton energies the value of L_{crit} is much increased because of the lower number density and higher velocity. Acceleration will thus occur at the Earth's bow shock only for protons of the lowest energy. Electrons with similar energies have velocities which are $(m_p/m_e)^{1/2}$ larger than those of the protons, so the acceleration of even low-energy electrons is unlikely.

A similar problem occurs if the magnetic field changes direction on scales of the order of L_{crit} or less. In such cases the equations of Paper I do not apply. In particular, magnetic mirrors may convect through the shock, sweeping away energetic particles and thus preventing further acceleration.

3 Acceleration of non-relativistic particles and the resulting energy spectrum

It is interesting to calculate the proportion of particles injected with energies a few times $\frac{1}{2}m_p v_s^2$ which are accelerated to cosmic ray energies (in excess of 1 GeV). In Paper I, I derived the energy spectrum for relativistic particles, and the derivation is here extended to the non-relativistic regime in which the calculation is more complicated, owing to the energy dependence of the probability of return of a particle entering the downstream region, and the fractional energy increase of a particle as it crosses and recrosses the shock front. Low-energy protons have velocities which are not much larger than the shock velocity, thus making non-negligible the terms of order v_s/v which were neglected in Paper I. However, the inclusion of these terms would greatly add to the complexity of the derivation, and the errors resulting from their neglect are small compared with the uncertainties in the injection of particles for acceleration. Consequently they are neglected here also and this results in a small error in the energy spectrum for protons with velocities not much greater than v_s. The energy spectrum of electrons is much less affected because of their higher velocity. The notation used is that adopted in Paper I.

From equation (3) of Paper I, the fraction of particles which escape during one cycle of crossing and recrossing the shock is $\eta = 4u_2/v$. In calculating the energy gained by the particles, it is assumed that all particles with a particular kinetic energy $T = (\gamma - 1)\, mc^2$ have their energy increased by the average amount ΔT in one cycle of crossing and recrossing the shock. An argument equivalent to that used in Paper I – equations (4)–(7) – but without the approximation $v \simeq c$, yields

$$\Delta T = (T + E_R)\frac{4}{3}\frac{v(u_1 - u_2)}{c^2}, \tag{1}$$

where $E_R = mc^2$ is the particle's rest mass energy. This equation for ΔT neglects terms which are of order v_s/v. If $N_I(T)$ is the number of particles accelerated to a kinetic energy of least T (i.e. N_I is the integrated energy spectrum), then the number of particles (ΔN_I) which escape in one cycle is

$$\Delta N_I = N_I(T)\, 4u_2/v. \tag{2}$$

Equations (1) and (2) can now be combined in differential form by assuming that $\Delta T/T$ and $\Delta N_I/N_I$ are both small (which is true if $v \gg v_s$):

$$\frac{dN_I}{dT} = -\frac{N_I}{(T + E_R)}\frac{3c^2}{v^2}\frac{u_2}{u_1 - u_2}. \tag{3}$$

This can now be integrated to yield the integrated energy spectrum

$$N_I(T) = \left[\frac{T^2 + 2E_R T}{T_0^2 + 2E_R T_0}\right]^{-(\mu-1)/2}, \tag{4}$$

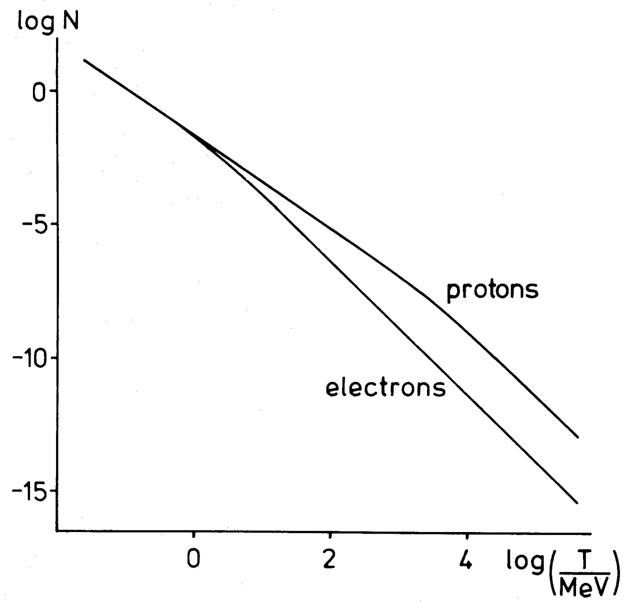

Figure 1. The energy spectra of protons and electrons injected at an energy $T_0 = 10\,\mathrm{keV}$.

where μ is defined as in Paper I, and T_0 is the kinetic energy of the injected particles. This expression is normalized so that $N_{\mathrm{I}}(T_0) = 1$, i.e. one particle is injected. The corresponding differential energy spectrum is

$$N(T) = (\mu - 1)(T_0^2 + 2E_{\mathrm{R}}T_0)^{(\mu-1)/2}(T + E_{\mathrm{R}})(T^2 + 2TE_{\mathrm{R}})^{-(\mu+1)/2}. \tag{5}$$

The energy spectra for protons and electrons are plotted in Fig. 1 for the case when the injection energy T_0 is 10 keV, which is equivalent to a shock velocity of 700 km/s if $T_0 = 4(\tfrac{1}{2}m_{\mathrm{p}}v_{\mathrm{s}}^2)$. At relativistic energies ($T \gg E_{\mathrm{R}}$) this reduces to

$$N(T) = (\mu - 1)(T_0^2 + 2E_{\mathrm{R}}T_0)^{(\mu-1)/2}\, T^{-\mu}, \tag{6}$$

an expression which applies closely to electrons, but is in error by a numerical factor of at most 3 for protons, on account of the approximations made at low energies. Inspection of Fig. 1 reveals the following:

(a) The energy spectrum is flatter for non-relativistic energies than for relativistic energies. In the non-relativistic regime $N(T) \propto T^{-(\mu+1)/2}$. If the relativistic spectral index (μ) is 2.5, then the non-relativistic spectral index is 1.75. The acceleration is thus more efficient for non-relativistic energies.

(b) As a result of the difference in their rest mass energies, the proton spectrum is flatter than the electron spectrum for kinetic energies in the range $m_{\mathrm{e}}c^2$ to $m_{\mathrm{p}}c^2$. This implies that, for equal numbers of electrons and protons injected at the same energy ($\ll 1\,\mathrm{GeV}$), the proton density at higher energies ($\gg 1\,\mathrm{GeV}$) is much greater than the electron density. The proton density exceeds the electron density by a factor

$$[(T_0 + 2m_{\mathrm{p}}c^2)/(T_0 + 2m_{\mathrm{e}}c^2)]^{(\mu-1)/2}.$$

For $T_0 \ll 2m_{\mathrm{e}}c^2$, this factor reduced to $(m_{\mathrm{p}}/m_{\mathrm{e}})^{(\mu-1)/2} = 280$ if $\mu = 2.5$. The condition $T_0 \ll 2m_{\mathrm{e}}c^2$ is equivalent to requiring that the shock velocity is much less than 7000 km/s if the injection energy is $4(\tfrac{1}{2}m_{\mathrm{p}}v_{\mathrm{s}}^2)$. The velocities occurring in supernova explosions are of the order of 10 000 km/s, and the injection of equal numbers of protons and electrons at the resulting shock front gives rise to a proton to electron ratio of about 100 at energies greatly

Table 1.

T_0 (keV)	v_s* (km/s)	U_p (MeV)	U_e (MeV)	U_p/U_e	N_p/N_e (at 100 GeV)	$\frac{10^{-2}\, U_p}{\frac{1}{2} m_p v_s^2}$
1	219	0.191	0.027	7.16	274	7.64
$10^{0.5}$	389	0.451	0.061	7.39	273	5.70
10^1	692	1.06	0.138	7.70	272	4.25
$10^{1.5}$	1 230	2.49	0.308	8.11	268	3.15
10^2	2 190	5.84	0.683	8.55	255	2.34
$10^{2.5}$	3 890	13.6	1.553	8.76	224	1.72
10^3	6 920	31.5	3.810	8.28	164	1.26
$10^{3.5}$	12 300	72.5	10.43	6.95	95	0.92
10^4	21 900	164.8	30.99	5.32	46	0.66

* Defined by $T_0 = 4(\frac{1}{2} m_p v_s^2)$.

in excess of 1 GeV. This ratio is close to that observed for cosmic rays arriving at the Earth (e.g. Meyer 1969).

(c) Despite the large proton to electron ratio at high energies, the total kinetic energy of the protons and electrons is more nearly equal since the major contribution comes from particles with low energies. The proton to electron ratio, and the average energies per electron (U_e) and proton (U_p) injected, are tabulated for different injection energies (i.e. different shock velocities) in Table 1.

4 High-energy cosmic rays

4.1 DENSITIES OF PROTONS AND ELECTRONS

If, in passing through a shock front, a fraction ϕ_p of the protons in a gas are pre-accelerated to an energy $\psi_p(\frac{1}{2} m_p v_s^2)$ at which they are injected into the acceleration mechanism of Paper I, then the density of high-energy protons in the post-shock gas is given (equation (6)) by

$$N_p(T) = \phi_p n(\mu - 1)(\psi_p m_p^2 v_s^2 c^2)^{(\mu-1)/2} \left(1 + \frac{\psi_p v_s^2}{4c^2}\right)^{(\mu-1)/2} T^{-\mu}, \tag{7}$$

where n is the post-shock hydrogen density. If $\mu = 2.5$, this may be expressed as

$$N_p(T) = 235 \left(\frac{\phi_p}{10^{-2}}\right)\left(\frac{n}{\text{cm}^{-3}}\right)\left(\frac{\psi_p}{4}\right)^{0.75}\left(\frac{v_s}{10^4\,\text{km/s}}\right)^{1.5}\left(\frac{T}{\text{GeV}}\right)^{-2.5} \text{m}^{-3}\,\text{GeV}^{-1}, \tag{8}$$

where the values 10^{-2} for ϕ_p and 4 for ψ_p are those found for the Earth's bow shock (Section 2), and where it is assumed that $\psi_p v_s^2/4c^2 \ll 1$. This may be compared with a typical density of cosmic rays in the interstellar medium of $3 \times 10^{-4} (T/\text{GeV})^{-2.5} m^{-3}\,\text{GeV}^{-1}$ (Wentzel 1974). For each proton overtaken by the shock front, an energy U_p is given to protons with an energy greater than $\psi_p(\frac{1}{2} m_p v_s^2)$. Hence a fraction $\phi_p U_p/\frac{1}{2} m_p v_s^2$ of the kinetic streaming energy of the gas flowing into the shock is given to high-energy protons. This fraction, listed in Table 1 for $\phi_p = 10^{-2}$ and $\psi_p = 4$, is found to be greater than unity, indicating that the injection rate must be less than 10^{-2}. It is clear, nevertheless, that this acceleration process can generate large cosmic ray densities.

As was discussed in Section 2, the energy of injection and the number injected are less easy to ascertain for electrons than for protons. However, since observations of the Earth's bow shock show that the numbers of electrons and protons generated at an energy of a few times $\frac{1}{2} m_p v_s^2$ are roughly comparable and, since the assumption of equal proton and electron injection rates leads to a proton to electron ratio at high energies close to that observed, I shall proceed on the assumption that their injection rates are indeed comparable and take ϕ_p and $\phi_e = 10^{-3}$ with injection energies of $4(\frac{1}{2}m_p v_s^2)$ ($\psi_p = \psi_e = 4$).

The electron energy spectrum at energies $\gg m_e c^2$ is

$$N_e(T) = \phi_e n(\mu - 1)(\psi_e m_p m_e v_s^2 c^2)^{(\mu-1)/2} \left(1 + \frac{\psi_e m_p v_s^2}{4 m_e c^2}\right)^{(\mu-1)/2} T^{-\mu}, \tag{9}$$

which can be recast in the form

$$N_e(T) = 0.0838(1.46 \times 10^{-3})^{\mu-2.5} \left(\frac{\phi_e}{10^{-3}}\right)\left(\frac{\psi_e}{4}\right)^{(\mu-1)/2} \left(\frac{n}{\mathrm{cm}^{-3}}\right)\left(\frac{\mu-1}{1.5}\right)\left(\frac{v_s}{10^4\,\mathrm{km/s}}\right)^{\mu-1}$$
$$\times \left[1 + \left(\frac{\psi_e}{4}\right)\left(\frac{v_s}{7000\,\mathrm{km/s}}\right)^2\right]^{(\mu-1)/2} \left(\frac{T}{\mathrm{GeV}}\right)^{-\mu} \mathrm{m}^{-3}\,\mathrm{GeV}^{-1}. \tag{10}$$

4.2 FLUX DENSITIES OF RADIO SOURCES

Energetic electrons gyrate under the influence of a magnetic field and produce synchrotron radio emission, the intensity of which can be calculated from the formulae given by Ginzburg & Syrovatskii (1965). The radio emissivity of a shocked gas is

$$\epsilon(\nu) = 2.94 \times 10^{-34}(1.435 \times 10^5)^{0.75-\alpha}\, \xi(2\alpha + 1) \left(\frac{\phi_e}{10^{-3}}\right)\left(\frac{n}{\mathrm{cm}^{-3}}\right)\left(\frac{\psi_e}{4}\right)^{2\alpha}\left(\frac{\alpha}{0.75}\right)$$
$$\times \left(\frac{v_s}{10^4\,\mathrm{km/s}}\right)^{4\alpha}\left(\frac{B}{10^{-4}\,\mathrm{G}}\right)^{\alpha+1}\left[1+\left(\frac{\psi_e}{4}\right)^{-1}\left(\frac{v_s}{7000\,\mathrm{km/s}}\right)^{-2}\right]^{\alpha}\left(\frac{\nu}{\mathrm{GHz}}\right)^{-\alpha} \mathrm{W\,Hz^{-1}\,m^{-3}}, \tag{11}$$

where B is the magnetic field in the emitting region, $\alpha = \frac{1}{2}(\mu - 1)$ is the spectral index of the radio emission, and $\xi(\mu)$ is a weak function of μ which is unity when $\mu = 2.5$. $\xi(\mu) = 11.7a(\mu)$ where $a(\mu)$ is the function tabulated by Ginzburg & Syrovatskii. The emissivity ($\propto v_s^{4\alpha}$ if $v_s \gtrsim 7000$ km/s) is strongly dependent on the shock velocity and the radio spectral index.

Supernova remnants are an example of a type of astrophysical object in which shock fronts are very important. Supernova explosions eject material into the interstellar medium at velocities of many thousand km/s (Woltjer 1970). Many cubic parsecs of interstellar gas are shocked at these high velocities and equation (11) can be used to estimate the flux density of a supernova remnant if the synchrotron-emitting electrons are accelerated in the shock which precedes the outflow of ejected material. The calculated flux densities for two young supernova remnants, Tycho's SNR (3C 10) and Cas A (3C 461), are given in Table 2, assuming that $\psi_e = 4$ and $\phi_e = 10^{-3}$. The expansion of both these remnants is decelerating (Gull 1973a; Bell 1977), but the flux density is calculated in the approximation of uniform expansion at the mean velocity deduced from their size and age. Since the number of hydrogen atoms is conserved when a gas passes through a shock, the value of n can be taken to be the unshocked interstellar value if the volume of the radio source is taken to be the total volume swept over by the shock, i.e. the volume of the sphere occupied by the supernova remnant. The magnetic field is taken to be 10^{-5} G, a value typical of an interstellar field compressed in a shock, although the medium in which a supernova explodes may be disturbed and contain an enhanced magnetic field. In view of the uncertainties in the distance to Tycho's SNR, its ambient density and the number of electrons injected into the acceleration mechanism, there is good agreement between the calculated and observed flux densities of Tycho's SNR. This suggests that the radio emission from Tycho's SNR can be explained in terms of electrons accelerated at the shock front moving into the interstellar gas.

Table 2.

SNR	Tycho		Cas A
n (cm^{-3})	0.2[1]		2[1,2]
α	0.6[6]		0.75[6]
D (kpc)	3[6]		3[5]
v_s (km/s)	7500		5900
Observed flux density at 1 GHz (Jy)	60[6]		2800[6]
B (G)	10^{-5}[3]	10^{-5}[3]	3×10^{-4}[4]
Calculated flux density at 1 GHz (Jy)	19	3.2	1200
Energy (J) electrons	1×10^{42}		8×10^{41}
Energy (J) electrons and protons	1×10^{43}		7×10^{42}
Energy (J) shocked gas	2×10^{43}		3×10^{43}

1. Gull (1973a).
2. McKee (1974).
3. This is a typical value for the compressed interstellar magnetic field.
4. The magnetic field assuming equipartition of energy between the magnetic field and relativistic electrons.
5. van den Bergh (1971).
6. Woltjer (1972).

In Cas A, however, the observed and calculated flux densities differ by three orders of magnitude if the magnetic field is 10^{-5} G. A possible reason for this is that the magnetic field in Cas A has been amplified by differential motions within the shell, resulting either from Rayleigh–Taylor instabilities (Gull 1973b), or from the passage of the fast-moving optical knots through the radio shell (Bell 1977). Indeed, the magnetic field in the compact radio peaks of Cas A must be much larger than 10^{-5} G or else their internal pressure due to synchrotron electrons is impossibly high (Bell 1977). If the magnetic field is instead set to the value (3×10^{-4} G) deduced by assuming equipartition between the energy in magnetic field and relativistic electrons, then the calculated flux density is 900 Jy, in better agreement with the observed value. The fast knots may also contribute to the radio emission by shocking the radio shell as they pass through it, thus accelerating more particles to high energies, and this may be related to the 'blobby' nature of the radio emission.

There is no evidence that the shell of Tycho's SNR is being intersected by fast-moving knots which could increase the radio emission, but it is probable that its shell was subject to the Rayleigh–Taylor instability at some time in the past, resulting in an increased magnetic field which may still persist. The calculated flux density in Table 2 is then too low.

Table 2 also gives the total energy of the accelerated electrons with energies greater than T_0, and the total energy of the electrons and protons under the assumption that equal numbers of each are injected. The energy of the accelerated particles is a substantial fraction of the total energy given to the shocked interstellar gas ($\frac{9}{32} m_p v_s^2$ per baryon in the case of compression by a factor 4 at the shock front) under the crude assumption of uniform expansion. The large proportion of this energy which is given to the accelerated particles is a consequence of the high value of ϕ_e (10^{-3}). The total energy of the particles is also a large fraction of the total energy released in a supernova explosion, 10^{44}–10^{45} J (e.g. Ginzburg & Syrovatskii 1964).

5 Calculations of the minimum energy in radio sources

A common calculation is that of the minimum energy of the magnetic field and relativistic electrons needed to account for the observed synchrotron emission from a radio source. This calculation usually takes account only of those electrons which radiate at frequencies higher than the low-frequency cut-off to the radio spectrum, neglecting both electrons of lower energy and also energetic protons. If, in reality, the energy spectrum of the electrons extends down to much lower energies as given by equation (5), then the minimum energy must be increased. In Cas A, these extra electrons increase the minimum energy by a factor of 2.5. If protons are injected into the acceleration process in equal numbers to the electrons, the minimum energy is increased by a further factor of 3.2, making 8.0 in total which is comparable with the factor of 13.9 by which the minimum energy is increased if the low-energy particles are neglected but protons are assumed to outnumber the electrons at all energies by 100 to 1. The large energy densities of the magnetic field and fast particles are an outstanding feature of non-thermal radio sources in general, and these considerations imply even higher energy densities.

6 Effect of a shock front on pre-existing cosmic rays

In previous sections the injection of particles into the acceleration mechanism has been considered as taking place at low energies, due to the existence of a 'high-energy' tail in the velocity distribution of the particles emerging from the shock front. An alternative source for the injection of particles is the cosmic ray population which already exists in the upstream gas. Since cosmic rays are thought to fill the Galaxy, any shock front moving into the interstellar gas will encounter these cosmic rays and accelerate them to even higher energies. The differential energy spectrum $R_1(T)$ which results can be obtained from the integral

$$R_1(T) = \int_0^\infty dZ\, P(Z)\, Q(T, Z), \tag{12}$$

where $P(Z)$ is the energy spectrum of the cosmic rays before they were shocked, and $Q(T, Z)$ is the energy spectrum resulting from the acceleration of particles which are injected monoenergetically with an energy Z. $Q(T, Z)$ is given by equation (5):

$$Q(T, Z) = \begin{cases} 0 & \text{when} \quad T < Z \\ (\mu - 1)(Z^2 + 2E_R Z)^{(\mu-1)/2}(T + E_R)(T^2 + 2TE_R)^{-(\mu+1)/2} & \text{when} \quad T > Z. \end{cases} \tag{13}$$

If the previously existing cosmic rays resulted from the acceleration at an earlier shock of particles injected at the low energy T_0, then $P(T) = N(T)$ (equation (5)) for $T > T_0$ and $P(T) = 0$ for $T < T_0$. The integral in equation (12) can now be evaluated. Indeed it is possible to derive a more general function $R_n(T)$ which is the energy spectrum of particles originally accelerated in a shock front, giving the energy spectrum $P(T)$, but which have subsequently passed through a further n shocks. $R_n(T)$ is thus the energy spectrum resulting from the passage of a gas through a total of $(n + 1)$ shocks. Only those particles injected at low energies in the first shock are considered, and those injected in the subsequent shocks are neglected (see (c) below). $R_n(T)$ is the energy probability distribution of one particle and is defined by

$$R_n(T) = \int_0^\infty dZ\, R_{n-1}(Z)\, Q(T, Z). \tag{14}$$

The general form of $R_n(T)$ in the range $T > T_0$ is

$$R_n(T) = \frac{(\mu - 1)^{n+1}}{2^n n!} \left[\ln\left(\frac{T^2 + 2TE_R}{T_0^2 + 2T_0 E_R}\right)\right]^n (T_0^2 + 2T_0 E_R)^{(\mu-1)/2}(T + E_R)(T^2 + 2TE_R)^{-(\mu+1)/2}. \tag{15}$$

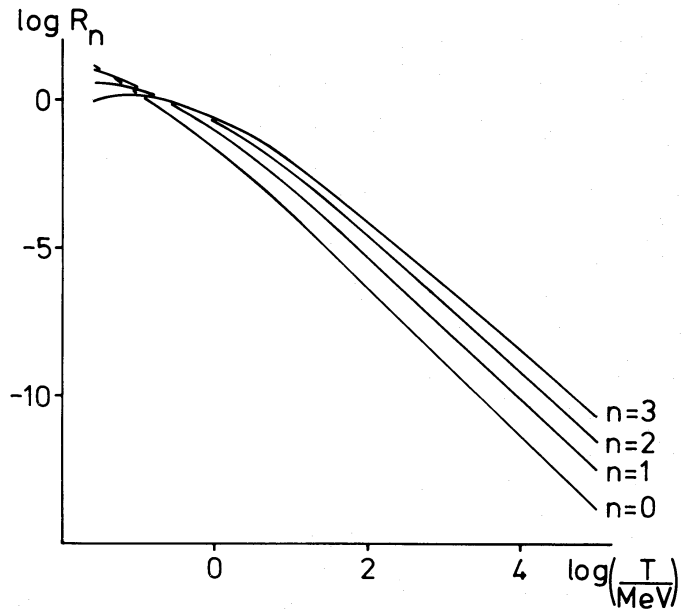

Figure 2. The energy spectra of electrons accelerated in an initial shock and which have passed through *n* subsequent shocks.

This function is plotted in Fig. 2 for electrons initially injected with an energy $T_0 = 10$ keV. The effect of multiple shocks on the number and average energy of protons and electrons is given in Table 3. The following points should be noted:

(a) At energies typical of electrons which emit radio waves, the energy spectrum flattens with succeeding shocks. Consequently the spectrum of radio emission also flattens.

(b) The expression for $R_n(T)$ is independent of the shock velocities, except for that of the initial shock, provided that the velocities of the subsequent shocks are less than that of the initial shock. The velocity of the initial shock determines T_0.

(c) For sufficiently high energies, $R_n(T)/R_{n-1}(T)$ is considerably greater than 2. This implies that particles injected at the initial shock and subsequently accelerated to higher energies outnumber the particles injected at low energies in each of the subsequent shocks.

The further acceleration of cosmic rays may be important in the case of low-velocity shock fronts. In such shocks the number of particles injected at low energies which are accelerated to high energies is small, on account of its strong dependence on the shock velocity – equations (7) and (9). This may be relevant to old supernova remnants such as the Cygnus Loop in which the shock velocity is only a few hundred kilometres per second (Woltjer 1972). Van der Laan (1962) suggested that the radio emission is due to the compression of the interstellar magnetic field and cosmic rays in the region of strong cooling which occurs behind a radiative shock front. The present discussion indicates that further acceleration of the interstellar cosmic rays takes place at the shock front, resulting in a cosmic ray density which is greater than that given merely by adiabatic compression. It is also possible that the shocked gas is subjected to further shocks while being compressed (Falle 1975), thus increasing the cosmic ray density yet again. The increase in the energy of the particles implies an increased pressure which will restrict the compression of the gas. The resulting energy spectrum can be expected to be flatter than that of the ambient interstellar cosmic rays.

A gas containing cosmic rays which passes through a shock front such as that of an old supernova remnant, and then at a later time re-expands adiabatically to its original volume, has the energy of its cosmic rays increased. A highly relativistic cosmic ray has its energy

Table 3.

T_0 (keV)	v_s* (km/s)	Number of shocks (n)	U_p (MeV)	U_e (MeV)	Number ratio of protons to electrons at 1 GeV	Increase in number of electrons $R_n(T)/R_0(T)$ $T = 0.1$ GeV	$T = 1$ GeV	$T = 10$ GeV
10^0	219	0	0.191	0.027	85.5	1.0	1.0	1.0
		1	2.10	0.162	58.8	12.1	15.5	19.0
		2	13.3	0.670	40.5	72.9	120.5	180.1
		3	62.2	2.33	27.8	293.6	623.6	1139.2
10^1	692	0	1.06	0.138	84.9	1.0	1.0	1.0
		1	9.99	0.666	55.2	10.3	13.8	17.2
		2	55.7	2.42	35.8	53.5	95.1	148.7
		3	237.5	7.85	23.3	184.5	437.2	854.7
10^2	2190	0	5.84	0.683	79.8	1.0	1.0	1.0
		1	46.0	2.63	48.1	8.6	12.0	15.5
		2	225.2	8.6	28.9	36.6	72.0	119.4
		3	873.4	26.7	17.4	104.3	288.1	615.2
10^3	6920	0	31.5	3.81	51.3	1.0	1.0	1.0
		1	202.9	12.3	28.7	6.4	9.8	13.3
		2	868.5	37.9	16.1	20.4	48.3	88.3
		3	3082.9	114.7	9.0	43.4	158.5	390.9

* Defined by $T_0 = 4(\frac{1}{2} m_p v_s^2)$.

increased on average by a factor of $3u_2/(4u_2 - u_1)$ when overtaken by a shock, and this is greater than the increase of $(u_1/u_2)^{1/3}$ which is obtained purely by adiabatic compression. Hence the average increase in energy is greater than the average decrease which takes place in the subsequent adiabatic expansion, and this is also true for energetic particles which are not highly relativistic. Consequently, energy is given to the cosmic ray gas. Some of the particle energy is lost to the Alfvén waves which confined the particles near the shock, and this results in a steeper energy spectrum of the accelerated particles which is entirely accounted for by the appearance of the terms $1/M_A$ in the expression for the spectral index of the differential energy spectrum given in equation (12) of Paper I. This energy loss is small for strong shocks.

7 Conclusions

Consideration of the theory of shock fronts and the results from satellite observations of the Earth's bow shock indicates that electrons and protons are produced by a shock at energies such that the particles can be accelerated to high energies by the process described in Paper I, thus producing large numbers of cosmic rays. The quantities of protons and electrons injected into the acceleration mechanism are estimated and the non-relativistic energy spectrum calculated, and these together give an estimate of the high-energy cosmic ray density in a shocked gas.

The following approximations and assumptions have been made:

(a) Protons and electrons with an energy of $4(\frac{1}{2}m_p v_s^2)$ are able to pass through a shock of very high Mach number without large deflection, and are therefore injected into the acceleration process (Section 2).

(b) The proportion of incoming protons (ϕ_p) and electrons (ϕ_e) which are injected with energy $4(\frac{1}{2}m_p v_s^2)$ is 10^{-3} in both cases (Section 4).

(c) The terms of order v_s/v are neglected in the calculation of non-relativistic energy spectrum and the approximation is made that $\Delta N_I/N_I$ and $\Delta T/T$ are small. This leads to an error in the proton energy spectrum but the error is small compared with the uncertainties in the number of protons injected (Section 3).

These assumptions and approximations yield expressions for the cosmic ray density in a shocked gas, and the cosmic ray electron density can be used to predict the intensity of radio emission from the gas. Application of the calculations to a crude model of Tycho's supernova remnant in which the magnetic field is taken to be the interstellar field compressed in a shock gives a predicted flux density consistent with the observed value. In the case of Cas A, agreement between the predicted and observed flux densities is obtained if the magnetic field is much greater than the compressed interstellar value as has already been implied by previous work on this object.

The further acceleration in a shock front of pre-existing cosmic rays may be important in relatively slow shock fronts such as occur in old supernova remnants.

Acknowledgments

I am grateful to Dr P. A. G. Scheuer for his help and encouragement in this work, and to Dr J. R. Shakeshaft, Dr S. F. Gull and the referee for helpful comments on the manuscript. I acknowledge the support of an SRC studentship.

References

Anderson, K. A., 1969. *J. geophys. Res.*, **74**, 95.
Ashbridge, J. R., Bame, S. J. & Strong, I. B., 1968. *J. geophys. Res.*, **73**, 5777.
Bell, A. R., 1977. *Mon. Not. R. astr. Soc.*, **179**, 573.
Bell, A. R., 1978. *Mon. Not. R. astr. Soc.*, **182**, 147.
Falle, S. A. E. G., 1975. *Mon. Not. R. astr. Soc.*, **172**, 55.
Formisano, V., 1974. *Correlated interplanetary and magnetospheric observations,* ed. Page, D. E., D. Reidel Publ. Co.
Ginzburg, V. L. & Syrovatskii, S. I., 1964. *The origin of cosmic rays,* Pergamon Press.
Ginzburg, V. L. & Syrovatskii, S. I., 1965. *A. Rev. Astr. Astrophys.*, 3, 297.
Gull, S. F., 1973a. *Mon. Not. R. astr. Soc.*, **162**, 135.
Gull, S. F., 1973b. *Mon. Not. R. astr. Soc.*, **161**, 47.
McKee, C. F., 1974. *Astrophys. J.*, **188**, 335.
Meyer, P., 1969. *A. Rev. Astr. Astrophys.*, **7**, 1.
Scarf, F. L., Fredericks, R. W., Frank, L. A. & Neugebauer, M., 1971. *J. geophys. Res.*, **76**, 5162.
Tidman, D. A. & Krall, N. A., 1971. *Shock waves in collisionless plasmas,* Wiley Interscience.
van den Bergh, S., 1971. *Astrophys. J.*, **165**, 457.
van der Laan, H., 1962. *Mon. Not. R. astr. Soc.*, **124**, 125.
Volk, H., 1970. Chapter *Collisionless shocks,* in *Physics of hot plasmas,* ed. Rye, B. J. & Taylor, J. C., Plenum Press.
Wentzel, D. G., 1974. *A. Rev. Astr. Astrophys.*, **12**, 71.
Woltjer, L., 1970. *IAU Symp. 39,* p. 229, D. Reidel Publ. Co.
Woltjer, L., 1972. *A. Rev. Astr. Astrophys.*, **10**, 33.