# Lectures on inflation and cosmological perturbations

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**Abstract**
The purpose of these lectures is to give a pedagogical introduction to inflation and the production of the primordial perturbations, as well as a review of some of the latest developments in this domain.
After a short introduction, we review the main principles of the Hot Big Bang model, as well as its limitations. These deficiencies provide the motivation for the study of a cosmological phase of accelerated expansion, called inflation, which can be induced by a slow-rolling scalar field. A few illustrative models are presented. We then turn to the analysis of cosmological perturbations, and explain how the vacuum quantum fluctuations are amplified during an inflationary phase. The next step consists in relating the perturbations generated during inflation to the perturbations of the cosmological fluid in the standard radiation dominated phase. One can thus confront the predictions of inflationary models with cosmological observations, such as the measurements of the Cosmic Microwave Background or the large-scale structure surveys. The present constraints on inflationary models are discussed.
The final part of these lectures gives a review of more general models of inflation, involving multiple fields or non standard kinetic terms. Although more complicated, these models are usually motivated by high energy physics and they can lead to specific signatures that are not expected in the simplest models of inflation. After introducing a very general formalism to describe perturbations in multi-field models with arbitrary kinetic terms, several interesting cases are presented. We also stress the role of entropy perturbations in the context of multi-field models. Finally, we discuss in detail the non-Gaussianities of the primordial perturbations and some models that could produce a detectable level of non-Gaussianities.

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## 1 Introduction

Inflation is today the main theoretical framework that describes the early Universe and that can account for the present observational data. In thirty years of existence, inflation has survived, in contrast with earlier competitors, the tremendous improvement of cosmological data. In particular, the fluctuations of the Cosmic Microwave Background (CMB) had not yet been measured when inflation was invented, whereas they give us today a remarkable picture of the cosmological perturbations in the early Universe. In the future, one can hope that more precise measurements of the primordial cosmological perturbations will allow us to go one step further in the confrontation of inflation models with data, and especially to discriminate between the many different possible realizations of inflation.

The purpose of these lectures is two-fold. The first goal is to explain, in a simple way and starting from first principles as much as possible, the conceptual basis of inflation and the elementary steps to calculate the cosmological perturbations predicted by the simplest models. The second objective of these lectures is to give an overview of the latest developments on inflation, in particular the study of more general models of inflation involving several scalar fields or non-standard kinetic terms. Although more complicated, these models can give very specific signatures in the primordial cosmological perturbations, in particular non-Gaussianities and isocurvature perturbations.

There is a huge literature on inflation and these lectures cover only a few topics, with a list of references that is far from exhaustive. More details and more references can be found in several textbooks (see e.g. [1, 2, 3]) and many reviews (including for instance [4, 5, 7, 6, 8]; more specialized reviews will also be mentioned in the text). A novel feature of these lectures is to introduce the latest methods used for the computation of perturbations. They have the advantage to be easily extendible to the study of non-linear perturbations, which has recently become an extremely active topic.

The outline of these lectures is the following. In the next section, we recall the basic elements of the Hot Big Bang model and discuss its limitations, which motivate inflation. Homogeneous inflation is introduced in Section 3. In Section 4, we turn to the theory of linear cosmological perturbations and explain how they are generated during an inflationary phase. The following section, Section 5, is devoted to the link between primordial perturbations and present cosmology, and thus to the confrontation of inflation models with the data. In Section 6, more general models of inflation are considered, with a discussion of several specific scenarios, which have attracted a lot of attention recently. Section 7 is devoted to the primordial non-Gaussianities and we conclude in the last section.

## 2 The hot Big Bang model

Modern cosmology is based on the theory of general relativity, according to which our Universe is described by a four-dimensional geometry $g_{\mu\nu}$ that satisfies Einstein's equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R \equiv g^{\mu\nu} R_{\mu\nu}$ the scalar curvature and $T_{\mu\nu}$ the energy-momentum tensor that describes the matter distribution.

### *2.1 The Friedmann equations*

One of the main assumptions of cosmology, which has been confirmed by observations so far, is to consider, as a first approximation, the universe as being homogeneous and isotropic. Note that these symmetries define implicitly a particular "slicing" of spacetime, in which the space-like hypersurfaces are homogeneous and isotropic. A different slicing of the *same* spacetime would give space-like hypersurfaces that are not homogeneous and isotropic.

Homogeneity and isotropy turn out to be very restrictive and the only geometries compatible with these requirements are the FLRW (Friedmann-Lemaître-Robertson-Walker) spacetimes, with metric

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-\kappa r^2} + r^2 \left( d\theta^2 + \sin^2\theta d\phi^2 \right) \right], \tag{2}$$

where $\kappa = 0, 1, -1$ determines the curvature of spatial hypersurfaces: respectively flat, elliptic or hyperbolic. Moreover, the matter content compatible with homogeneity and isotropy is necessarily characterized by an energy-momentum tensor of the form

$$T^{\mu}_{\nu} = \mathrm{Diag}\left[ -\rho(t), P(t), P(t), P(t) \right] \tag{3}$$

where $\rho$ corresponds to an energy density and $P$ to a pressure.

Substituting the metric (2) and the energy-momentum tensor (3) into Einstein's equations (1) gives the Friedmann equations,

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G \rho}{3} - \frac{\kappa}{a^2}, \tag{4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + 3P \right), \tag{5}$$

which govern the time evolution of the scale factor $a(t)$.

An immediate consequence of the two above equations is the *continuity equation*

$$\dot{\rho} + 3H \left( \rho + P \right) = 0, \tag{6}$$

where $H \equiv \dot{a}/a$ is the *Hubble parameter*. The continuity equation can also be obtained directly from the energy-momentum conservation

$$\nabla_\mu T^\mu_\nu = 0, \tag{7}$$

where $\nabla$ denotes the covariant derivative associated with the metric $g_{\mu\nu}$.

The cosmological evolution can be determined once the equation of state for the matter is specified. Let us assume here $P = w\rho$ with $w$ constant, which includes the two main types of matter that play an important rôle in cosmology, namely non relativistic matter ($w \simeq 0$) and a gas of relativistic particles ($w = 1/3$). The conservation equation (6) can be integrated to give

$$\rho \propto a^{-3(1+w)}. \tag{8}$$

Substituting into (4), one finds, for $\kappa = 0$,

$$a(t) \propto t^{\frac{2}{3(1+w)}}, \tag{9}$$

which thus gives the evolution $a(t) \propto t^{1/2}$ for relativistic matter and $a(t) \propto t^{2/3}$ for non-relativistic matter. Note that a different cosmological evolution, governed by modified Friedmann's equations, can be envisaged in the primordial Universe, as for example in the context of brane cosmology (see e.g. [9]), but this possibility will not be discussed in these notes.

The present cosmological observations seem to indicate that our Universe is currently accelerating. The simplest way to account for this acceleration is to assume the presence of a *cosmological constant* $\Lambda$ in Einstein's equations, i.e. an additional term $\Lambda g_{\mu\nu}$ on the left-hand side of (1). By moving this term on the right hand side of Einstein's equations it can also be interpreted as an energy-momentum tensor with equation of state $P = -\rho$, where $\rho$ is time-independent. This leads, for $\kappa = 0$ and without any other matter, to an exponential evolution of the scale factor

$$a(t) \propto \exp(Ht). \tag{10}$$

In our universe, several species with different equations of state coexist, and it has become customary to characterize their relative contributions by the dimensionless parameters

$$\Omega_{(i)} \equiv \frac{8\pi G \rho_0^{(i)}}{3H_0^2}, \tag{11}$$

where the $\rho_0^{(i)}$ denote the present energy densities of the various species, and $H_0$ is the present Hubble parameter. The first Friedmann equation (4), evaluated at the present time, implies

$$\Omega_0 = \sum_i \Omega_{(i)} = 1 + \frac{\kappa}{a_0^2 H_0^2}. \tag{12}$$

One can infer from present observations the following parameters: $\Omega_m \simeq 0.3$ for non-relativistic matter (which includes a small baryonic component $\Omega_b \simeq 0.05$),

$\Omega_\Lambda \simeq 0.7$ for a "dark energy" component (compatible with a cosmological constant), $\Omega_\gamma \simeq 5 \times 10^{-5}$ for the photons, and a total $\Omega_0$ close to 1, i.e. no detectable deviation from flatness.

### *2.2 The shortcomings of the standard Big Bang model*

The standard Big Bang model has encountered remarkable successes, in particular with primordial nucleosynthesis and the CMB, and it remains today a cornerstone in our understanding of the present and past universe. However, a few intriguing facts remain unexplained in the strict scenario of the Hot Big Bang model and seem to necessitate a larger framework. We review now the main problems:

- Homogeneity problem
  A first question is why the approximation of homogeneity and isotropy turns out to be so good. Indeed, inhomogeneities are unstable, because of gravitation, and they tend to grow with time. It can be verified for instance with the CMB that inhomogeneities were much smaller at the last scattering epoch than today. One thus expects that these homogeneities were still smaller further back in time. How to explain a universe so smooth in its past ?

- Flatness problem
  Another puzzle lies in the (spatial) flatness of our universe. Indeed, the first Friedmann equation, Eq. (4), implies
  $$\Omega - 1 \equiv \frac{8\pi G\rho}{3H^2} - 1 = \frac{\kappa}{a^2 H^2}. \tag{13}$$
  In standard cosmology, the scale factor behaves like $a \sim t^p$ with $p < 1$ ($p = 1/2$ for radiation and $p = 2/3$ for non-relativistic matter). As a consequence, $(aH)^{-2}$ grows with time and $|\Omega - 1|$ must thus diverge with time. Therefore, in the context of the standard model, the quasi-flatness observed today requires an extreme fine-tuning of $\Omega$ near 1 in the early universe.

- Horizon problem
  One of the most fundamental problems in standard cosmology is certainly the *horizon problem*. The (particle) *horizon* is the maximal distance that can be covered by a light ray. For a light-like radial trajectory, $dr = a(t)dt$ and the horizon is thus given by
  $$d_H(t) = a(t)\int_{t_i}^{t} \frac{dt'}{a(t')} = a(t)\frac{t^{1-q} - t_i^{1-q}}{1-q}, \tag{14}$$
  where the last equality is obtained by assuming $a(t) \sim t^q$ and $t_i$ is some initial time.

In standard cosmology ($q < 1$), the integral converges in the limit $t_i = 0$ and the horizon has a finite size, of the order of the so-called Hubble radius $H^{-1}$:

$$d_H(t) = \frac{q}{1-q} H^{-1}. \quad (15)$$

It also useful to consider the *comoving Hubble radius*, $(aH)^{-1}$, which represents the fraction of comoving space in causal contact. One finds that it *grows* with time, which means that the *fraction of the universe in causal contact increases with time* in the context of standard cosmology. But the CMB tells us that the Universe was quasi-homogeneous at the time of last scattering on a scale encompassing many regions a priori causally independent. How to explain this ?

A solution to the horizon problem and to the other puzzles is provided by the inflationary scenario, which we will examine in the next section. The basic idea is to "decouple" the causal size from the Hubble radius, so that the real size of the horizon region in the standard radiation dominated era is much larger than the Hubble radius. Such a situation occurs if the comoving Hubble radius *decreases* sufficiently in the very early universe. The corresponding condition is

$$\ddot{a} > 0, \quad (16)$$

i.e. the Universe undergoes a *phase of acceleration*.

## 3 Inflation

The broadest definition of inflation is that it corresponds to a phase of acceleration of the universe,

$$\ddot{a} > 0. \quad (17)$$

In this sense, the current cosmological observations, if correctly interpreted, mean that our present universe is undergoing an inflationary phase. It is worth noting that many of the models suggested for inflation have been adapted to account for the present acceleration. We are however interested here in an inflationary phase taking place in the *early* universe, thus characterized by very different energy scales. Another difference is that inflation in the early universe *must end* to leave room to the standard radiation dominated cosmological phase.

Cosmological acceleration requires, according to the second Friedmann equation, Eq. (5), an equation of state satisfying

$$P < -\frac{1}{3}\rho, \quad (18)$$

condition which looks at first view rather exotic.

A very simple example giving such an equation of state is a cosmological constant, corresponding to a cosmological fluid with the equation of state

$$P = -\rho. \tag{19}$$

However, a strict cosmological constant leads to exponential inflation *forever* which cannot be followed by a radiation era. Another possibility is a scalar field, which we now discuss in some details.

### 3.1 Cosmological scalar fields

The dynamics of a scalar field minimally coupled to gravity is governed by the action

$$S_\phi = \int d^4x\sqrt{-g}\left(-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right), \tag{20}$$

where $g \equiv \det(g_{\mu\nu})$ and $V(\phi)$ is the potential of the scalar field. The corresponding energy-momentum tensor, obtained by varying the action (20) with respect to the metric, is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi + V(\phi)\right). \tag{21}$$

In the homogeneous and isotropic geometry (2), the energy-momentum tensor is of the perfect fluid form, with the energy density

$$\rho = -T_0^0 = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{22}$$

where one recognizes the sum of a kinetic energy and of a potential energy, and the pressure

$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{23}$$

The equation of motion for the scalar field is the Klein-Gordon equation, obtained by taking the variation of the above action (20) with respect to the scalar field,

$$\nabla^\mu\nabla_\mu\phi = \frac{dV}{d\phi}, \tag{24}$$

which reduces to

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{25}$$

in a homogeneous and isotropic universe.

The system of equations governing the dynamics of the scalar field and of the cosmological geometry is thus given by

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{26}$$

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0, \tag{27}$$

$$\dot{H} = -4\pi G\dot{\phi}^2. \tag{28}$$

The last equation can be derived from the first two and is therefore redundant.

### 3.2 The slow-roll regime

The dynamical system (26-28) does not always give an accelerated expansion but it does so in the so-called *slow-roll regime* when the potential energy of the scalar field dominates over its kinetic energy.

More specifically, the slow-roll approximation consists in neglecting the kinetic energy of the scalar field, $\dot{\phi}^2$, in (26) and its acceleration, $\ddot{\phi}$, in the Klein-Gordon equation (27). One then gets the simplified system

$$H^2 \simeq \frac{8\pi G}{3}V, \tag{29}$$

$$3H\dot{\phi} + V' \simeq 0. \tag{30}$$

Let us now examine in which regime this approximation is valid. From (30), the velocity of the scalar field is given by

$$\dot{\phi} \simeq -\frac{V'}{3H}. \tag{31}$$

Substituting this relation into the condition $\dot{\phi}^2/2 \ll V$ yields the requirement

$$\varepsilon_V \equiv \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 \ll 1, \tag{32}$$

where we have introduced the *reduced Planck mass*

$$M_P \equiv \frac{1}{\sqrt{8\pi G}}. \tag{33}$$

Alternatively, one can use the parameter

$$\varepsilon \equiv -\frac{\dot{H}}{H^2}, \tag{34}$$

which coincides with $\varepsilon_V$ at leading order in slow-roll, since $\varepsilon = \dot{\phi}^2/(2M_P^2H^2)$.

Similarly, $\ddot{\phi} \ll V'$ implies, after using the time derivative of (31) and (29), the condition

$$\eta_V \equiv M_P^2 \frac{V''}{V} \ll 1. \tag{35}$$

In summary, the slow-roll approximation is valid when the conditions $\varepsilon_V, \eta_V \ll 1$ are satisfied by the potential, which means that the slope and the curvature of the potential, in Planck units, must be sufficiently small.

### *3.3 Number of e-folds*

Inflation must last long enough, in order to solve the problems of the Hot Big Bang model. To investigate this question, one usally introduces the *number of e-folds before the end of inflation*, denoted $N$, and simply defined by

$$N = \ln \frac{a_{end}}{a}, \tag{36}$$

where $a_{end}$ is the value of the scale factor at the end of inflation and $a$ is a fiducial value for the scale factor during inflation. By definition, $N$ *decreases* during the inflationary phase and reaches zero at its end.

In the slow-roll approximation, it is possible to express $N$ as a function of the scalar field. Since $dN = -d\ln a = -Hdt = -(H/\dot{\phi})d\phi$, one easily finds, using (31) and (29), that

$$N(\phi) \simeq \int_{\phi}^{\phi_{end}} \frac{V}{M_P^2 V'} d\phi. \tag{37}$$

Given an explicit potential $V(\phi)$, one can in principle integrate the above expression to obtain $N$ in terms of $\phi$. This will be illustrated below for some specific models.

Let us now discuss the link between $N$ and the present cosmological scales. If one considers a given scale characterized by its comoving wavenumber $k = 2\pi/\lambda$, this scale crossed out the Hubble radius, during inflation, at an instant $t_*(k)$ defined by

$$k = a(t_*)H(t_*). \tag{38}$$

To get a rough estimate of the number of e-foldings of inflation that are needed to solve the horizon problem, let us first ignore the transition from a radiation era to a matter era and assume for simplicity that the inflationary phase was followed instantaneously by a radiation phase that has lasted until now. During the radiation phase, the comoving Hubble radius $(aH)^{-1}$ increases like $a$. In order to solve the horizon problem, the increase of the comoving Hubble radius during the standard evolution must be compensated by *at least* a decrease of the same amount during inflation. Since the comoving Hubble radius roughly scales like $a^{-1}$ during inflation, the minimum amount of inflation is simply given by the number of e-folds between the end of inflation and today

$$\ln(a_0/a_{end}) = \ln(T_{end}/T_0) \sim \ln(10^{29}(T_{end}/10^{16}\mathrm{GeV})), \tag{39}$$

i.e. around 60 e-folds for a temperature $T \sim 10^{16}$Gev at the beginning of the radiation era. As we will see later, this energy scale is typical of inflation in the simplest models.

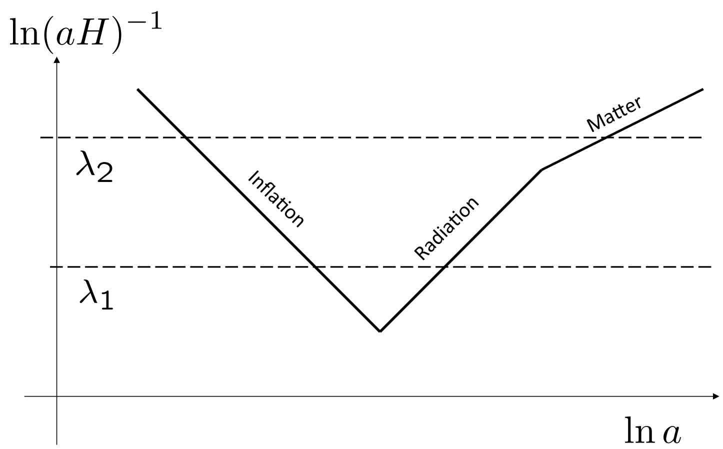


**Fig. 1** Evolution of the comoving Hubble radius $\lambda_H = (aH)^{-1}$, during inflation, radiation dominated era and matter dominated era. The horizontal dashed lines correspond to two different comoving lengthscales: the larger scales cross out the Hubble radius *earlier* during inflation and reenter the Hubble radius *later* in the standard cosmological era.

This determines roughly the number of e-folds $N(k_0)$ between the moment when the scale corresponding to our present Hubble radius $k_0 = a_0 H_0$ exited the Hubble radius during inflation and the end of inflation. The other lengthscales of cosmological interest are *smaller* than $k_0^{-1}$ and therefore exited the Hubble radius during inflation *after* the scale $k_0$, whereas they entered the Hubble radius during the standard cosmological phase (either in the radiation era for the smaller scales or in the matter era for the larger scales) *before* the scale $k_0$ (see Fig. 1).

A more detailed calculation, which distinguishes between the energy scales at the end of inflation and after the reheating, gives for the number of e-folds between the exit of the mode $k$ and the end of inflation (see e.g. [2, 10])

$$N(k) \simeq 62 - \ln \frac{k}{a_0 H_0} + \ln \frac{V_k^{1/4}}{10^{16}\mathrm{GeV}} + \ln \frac{V_k^{1/4}}{V_{\mathrm{end}}^{1/4}} + \frac{1}{3} \ln \frac{\rho_{\mathrm{reh}}^{1/4}}{V_{\mathrm{end}}^{1/4}}. \tag{40}$$

Since the smallest scale of cosmological relevance is of the order of 1 Mpc, the range of cosmological scales covers about 9 e-folds.

The above number of e-folds is altered if one changes the thermal history of the universe between inflation and the present time by including for instance a period of so-called thermal inflation.

## *3.4 A few examples*

It is now time to illustrate all the points discussed above with some specific potentials.

### 3.4.1 Power-law potential

We consider first the case of power-law monomial potentials, of the form

$$V(\phi) = \lambda \phi^p, \tag{41}$$

which have been abundantly studied in the literature. In particular, the above potentials include the case of a free massive scalar field, $V(\phi) = m^2\phi/2$.

The slow-roll parameters are given by

$$\varepsilon = \frac{p^2 M_P}{2\phi^2}, \qquad \eta = p(p-1)\frac{M_P^2}{\phi^2}. \tag{42}$$

The slow-roll conditions $\varepsilon \ll 1$ and $\eta \ll 1$ thus imply

$$\phi \gg p\, M_P, \tag{43}$$

which means that the scalar field amplitude must be above the Planck mass during inflation.

After substituting the potential (41) into the slow-roll equations of motion (29-30), one can integrate them explicitly to get

$$\phi^{2-\frac{p}{2}} - \phi_i^{2-\frac{p}{2}} = -\frac{2p}{4-p}\sqrt{\frac{\lambda}{3}}\, M_P\,(t-t_i) \tag{44}$$

for $p \neq 4$ and

$$\phi = \phi_i \exp\left[-4\sqrt{\frac{\lambda}{3}}\, M_P(t-t_i)\right] \tag{45}$$

for $p = 4$.

One can also express the scale factor as a function of the scalar field [and thus as a function of time by substituting the above expression for $\phi(t)$] by using $d\ln a/d\phi =$

$H/\dot{\phi} \simeq -\phi/(pM_P^2)$. One finds

$$a = a_{end} \exp\left[-\frac{\left(\phi^2 - \phi_{end}^2\right)}{2pM_P^2}\right]. \tag{46}$$

Defining the end of inflation by $\varepsilon = 1$, one gets $\phi_{end} = pM_P/\sqrt{2}$ and the number of e-folds is thus given by

$$N(\phi) \simeq \frac{\phi^2}{2pM_P^2} - \frac{p}{4}. \tag{47}$$

This can be inverted, so that

$$\phi(N) \simeq \sqrt{2Np}\, M_P, \tag{48}$$

where we have ignored the second term of the right hand side of (47), consistently with the condition (43).

### 3.4.2 Exponential potential

Cosmological scalar fields with a potential of the form

$$V = V_0 \exp\left(-\sqrt{\frac{2}{q}}\frac{\phi}{M_P}\right), \tag{49}$$

admit an *exact* solution (i.e. valid beyond the slow-roll approximation) of the system (26-28), with a power-law scale factor, i.e.

$$a(t) \propto t^q. \tag{50}$$

The evolution of the scalar field is given by the expression

$$\phi(t) = \sqrt{2q}\, M_P \ln\left[\sqrt{\frac{V_0}{q(3q-1)}}\frac{t}{M_P}\right]. \tag{51}$$

Note that one recovers the slow-roll approximation in the limit $q \gg 1$, since the slow-roll parameters are given by

$$\varepsilon_V = \frac{1}{q} \qquad \eta_V = \frac{2}{q}. \tag{52}$$

### 3.4.3 Hybrid inflation

In this type of model, the potential contains a constant piece in addition to a power-law potential, the simplest example being

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2. \tag{53}$$

In fact, the full model relies on the presence of two scalar fields, where one plays the traditional rôle of the inflaton, while the other is necessary to end inflation. In the original model of hybrid inflation [11], one starts from the potential

$$V(\phi,\psi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\lambda'\psi^2\phi^2 + \frac{1}{4}\lambda\left(M^2-\psi^2\right)^2. \tag{54}$$

For values of the field $\phi$ larger than the critical value $\phi_c = \lambda M^2/\lambda'$, the potential for $\psi$ has its minimum at $\psi = 0$. This is the case during inflation: $\psi$ is thus trapped in this minimum $\psi = 0$, so that the effective potential for the scalar field $\phi$, which plays the rôle of the inflaton, is given by (53) with $V_0 = \lambda M^4/4$. During the inflationary phase, the field $\phi$ slow-rolls until it reaches the critical value $\phi_c$. The shape of the potential for $\psi$ is then modified and new minima appear in $\psi = \pm M$. $\psi$ will thus roll down into one of these new minima and, as a consequence, inflation will end.

During the inflationary phase, the slow-roll parameters are given by

$$\varepsilon = \frac{m^2 M_P^2 \tilde{\phi}^2}{V_0(1+\tilde{\phi}^2)^2}, \qquad \eta = \frac{m^2 M_P^2}{V_0(1+\tilde{\phi}^2)}, \tag{55}$$

where we have introduced the rescaled scalar field $\tilde{\phi}$, which is dimensionless and defined so that $V = V_0(1+\tilde{\phi}^2)$. Note that there are two limiting regimes: if $\tilde{\phi} \gg 1$, the constant term is negligible and one recovers a power-law potential with $p = 2$; if $\tilde{\phi} \ll 1$, $V_0$ dominates and the potential is extremely flat with $\varepsilon \ll \eta$.

## *3.5 The inflationary "zoology"*

### 3.5.1 Historical perspective

The first model of inflation is usually traced back to Alan Guth [12] in 1981, although one can see the model of Alexei Starobinsky [13] as a precursor. Guth's model, which is named today *old inflation* is based on a first-order phase transition, from a false vacuum with non zero energy, which generates an exponential inflationary phase, into a true vacuum with zero energy density. The true vacuum phase appears in the shape of bubbles via quantum tunneling. The problem with this inflationary model is that, in order to get sufficient inflation to solve the problems of the standard model mentioned earlier, the nucleation rate must be sufficiently small; but, then, the bubbles never coalesce because the space that separates the bubbles undergoes inflation and expands too rapidly. Therefore, the first model of inflation is not phenomenologically viable.

After this first and unsuccessful attempt, a new generation of inflationary models appeared, usually denoted *new inflation* models [14]. They rely on a second order phase transition, based on thermal corrections of the effective potential and thus assume that the scalar field is in thermal equilibrium.

This hypothesis of thermal equilibrium was given up in the third generation of models, initiated by Andrei Linde, and whose generic name is *chaotic inflation* [15]. This allows to use extremely simple potentials, quadratic or quartic, which lead to inflationary phases when the scalar field is displaced from the origin with values of the order of several Planck masses.

In the last few years, there has been an intense activity in building inflationary models based on high energy theories, in particular in the context of supersymmetry and string theory. Details can be found in several recent reviews [16, 17, 18, 19, 20].

### 3.5.2 Classification

There exist a huge number of models of inflation. As far as *single-field* models are concerned[1], it is convenient to regroup them into three broad categories:

- Large field models ($0 < \eta \leq \varepsilon$)
  The scalar field is displaced from its stable minimum by $\Delta\phi \sim M_P$. This includes the chaotic models with monomial potentials
$$V(\phi) = \Lambda^4 \left(\frac{\phi}{\mu}\right)^p , \tag{56}$$
  or the exponential potential
$$V(\phi) = \Lambda^4 \exp(\phi/\mu) , \tag{57}$$
  which have already been discussed.

- Small field models ($\eta < 0 < \varepsilon$)
  In this type of models, the scalar field is rolling away from an unstable maximum of the potential. This is a characteristic feature of spontaneous symmetry breaking. A typical potential is
$$V(\phi) = \Lambda^4 \left[1 - \left(\frac{\phi}{\mu}\right)^p\right] , \tag{58}$$
  which can be interpreted as the lowest-order term in a Taylor expansion about the origin. Historically, this potential shape appeared in the so-called 'new inflation' scenario.
  A particular feature of these models is that tensor modes are much more suppressed with respect to scalar modes than in the large-field models, as it will be

[1] or at least *effectively* single field during inflation (the hybrid models require a second field to *end* inflation as discussed earlier).

shown later.

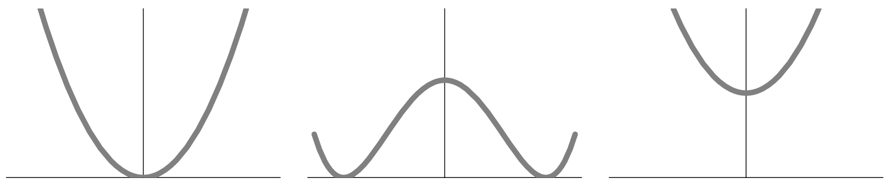

**Fig. 2** Schematic potential for the three main categories of inflationary models: large-field models, small-field models, hybrid models.

- Hybrid models ($0 < \varepsilon < \eta$)
  Although a second scalar field is needed to end inflation, hybrid models correspond effectively to single-field models with a potential characterized by $V''(\phi) > 0$ and $0 < \varepsilon < \eta$. A typical potential is

$$V(\phi) = \Lambda^4 \left[1 + \left(\frac{\phi}{\mu}\right)^p\right]. \tag{59}$$

  Once more, this potential can be seen as the lowest order in a Taylor expansion about the origin.
  In the case of hybrid models, the value $\phi_N$ of the scalar field as a function of the number of e-folds before the end of inflation is not determined by the above potential and, therefore, $(\phi_N/\mu)$ can be considered as a freely adjustable parameter.

## 4 Quantum fluctuations and "birth" of cosmological perturbations

So far, we have concentrated our attention on strictly homogeneous and isotropic aspects of cosmology. Of course, this idealized version, although extremely useful, is not sufficient to account for real cosmology and it is now time to turn to the study of deviations from homogeneity and isotropy.

In cosmology, inhomogeneities grow because of the attractive nature of gravity, which implies that inhomogeneities were much smaller in the past. As a consequence, for most of their evolution, inhomogeneities can be treated as *linear perturbations*. The linear treatment ceases to be valid on small scales in our recent past, hence the difficulty to reconstruct the primordial inhomogeneities from large-scale structure, but it is quite adequate to describe the fluctuations of the CMB at the time of last scattering. This is the reason why the CMB is currently the best observational probe of primordial inhomogeneities.

In this section, we concentrate on the perturbations of the inflaton and show how the accelerated expansion during inflation converts its *initial vacuum quantum fluctuations* into "macroscopic" cosmological perturbations (see [21, 22, 23, 24, 25, 26] for some of the historical works). In this sense, inflation provides us with "natural" initial conditions. We will also see how the perturbations of the inflaton can be translated into perturbations of the geometry.

### *4.1 Massless scalar field in de Sitter*

As a warming-up, it is instructive to discuss the case of a massless scalar field in a de Sitter universe, described by a cosmological metric with exponential expansion,

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \quad a(t) = e^{Ht}. \tag{60}$$

It turns out it is more convenient to use, instead of the cosmic time $t$, a conformal time $\tau$, defined by

$$\tau = \int \frac{dt}{a(t)}, \tag{61}$$

so that the metric takes the particularly simple form

$$ds^2 = a^2(\tau)\left[-d\tau^2 + d\mathbf{x}^2\right]. \tag{62}$$

In the de Sitter case, the conformal time is given by

$$\tau = -\frac{e^{-Ht}}{H} = -\frac{1}{aH}, \tag{63}$$

so that the scale factor in terms of $\tau$ is simply

$$a(\tau) = -\frac{1}{H\tau}. \tag{64}$$

The conformal time is here negative (so that the scale factor is positive) and goes from $-\infty$ to 0.

The action for a massless scalar field is given by

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right) = \int d\tau\, d^3x\, a^4\left[\frac{1}{2a^2}\phi'^2 - \frac{1}{2a^2}\nabla\phi^2\right], \tag{65}$$

where we have substituted in the action the cosmological metric (62) and where a prime denotes a derivative with respect to the conformal time $\tau$. Note that, whereas we still allow for spatial variations of the scalar field, i.e. inhomogeneities, we will assume here, somewhat inconsistently, that the geometry is completely fixed as homogeneous. We will deal later with the question of the metric perturbations.

It is possible to eliminate the factor $a^2$ in front of the kinetic term $\phi'^2$ by introducing the new function

$$u = a\,\phi. \tag{66}$$

This will generate a term proportional to $uu'$ but one can get rid of it by an integration by parts. The action (65) can then be rewritten as

$$S = \frac{1}{2}\int d\tau\, d^3x \left[u'^2 - \nabla u^2 + \frac{a''}{a}u^2\right]. \tag{67}$$

The first two terms are familiar since they are the same as in the action for a free massless scalar field in Minkowski spacetime. The fact that our scalar field here lives in de Sitter spacetime rather than Minkowski has been reexpressed as a *time-dependent effective mass*

$$m^2_{eff} = -\frac{a''}{a} = -\frac{2}{\tau^2}. \tag{68}$$

Let us now proceed to the quantization of the scalar field $u$ by using the standard procedure of quantum field theory. One first turns $u$ into a quantum field denoted $\hat{u}$, which we expand in Fourier space as

$$\hat{u}(\tau,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int d^3k \left\{\hat{a}_{\mathbf{k}} u_k(\tau) e^{i\mathbf{k}.\mathbf{x}} + \hat{a}^\dagger_{\mathbf{k}} u^*_k(\tau) e^{-i\mathbf{k}.\mathbf{x}}\right\}, \tag{69}$$

where the $\hat{a}^\dagger$ and $\hat{a}$ are creation and annihilation operators, satisfying the usual commutation rules

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = \left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\right] = 0, \quad \left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\right] = \delta(\mathbf{k}-\mathbf{k}'). \tag{70}$$

The function $u_k(\tau)$ is a complex time-dependent function that must satisfy the *classical* equation of motion in Fourier space, namely

$$u''_k + \left(k^2 - \frac{a''}{a}\right) u_k = 0, \tag{71}$$

which is simply the equation of motion for an oscillator with a time-dependent mass. In the case of a massless scalar field in Minkowski spacetime, this effective mass is zero ($a''/a = 0$) and one usually takes

$$u_k = \sqrt{\frac{\hbar}{2k}} e^{-ik\tau} \quad \text{(Minkowski)}, \tag{72}$$

where the choice for the normalization factor will be clear below. In the case of de Sitter, one can solve explicitly the above equation with $a''/a = 2/\tau^2$ and the general solution is given by

$$u_k = \alpha e^{-ik\tau}\left(1 - \frac{i}{k\tau}\right) + \beta e^{ik\tau}\left(1 + \frac{i}{k\tau}\right). \tag{73}$$

Canonical quantization consists in imposing the following commutation rules on the $\tau$ =constant hypersurfaces:

$$\left[\hat{u}(\tau,\mathbf{x}),\hat{u}(\tau,\mathbf{x}')\right]=\left[\hat{\pi}_u(\tau,\mathbf{x}),\hat{\pi}_u(\tau,\mathbf{x}')\right]=0 \tag{74}$$

and

$$\left[\hat{u}(\tau,\mathbf{x}),\hat{\pi}_u(\tau,\mathbf{x}')\right]=i\hbar\delta(\mathbf{x}-\mathbf{x}'), \tag{75}$$

where $\pi_u \equiv \delta S/\delta u'$ is the conjugate momentum of $u$. In the present case, $\pi_u = u'$ since the kinetic term is canonical.

Subtituting the expansion (69) in the commutator (75), and using the commutation rules for the creation and annihilation operators (70), one obtains the relation

$$u_k u_k'^* - u_k^* u_k' = i\hbar, \tag{76}$$

which determines the normalization of the Wronskian.

The choice of a specific function $u_k(\tau)$ corresponds to a particular prescription for the physical vacuum $|0\rangle$, defined by

$$\hat{a}_{\mathbf{k}}|0\rangle = 0. \tag{77}$$

A different choice for $u_k(\tau)$ is associated to a different decomposition into creation and annihilitation modes and thus to a different vacuum.

Let us now note that the wavelength associated with a given mode $k$ can always be found *within* the Hubble radius provided one goes sufficiently far backwards in time, since the comoving Hubble radius is shrinking during inflation. In other words, for $|\tau|$ sufficiently big, one gets $k|\tau| \gg 1$. Moreover, for a wavelength smaller than the Hubble radius, one can neglect the influence of the curvature of spacetime and the mode behaves as in a Minkowski spacetime, as can also be checked explicitly with the equation of motion (71) (the effective mass is negligible for $k|\tau| \gg 1$). Therefore, the most natural physical prescription is to take the particular solution that corresponds to the usual Minkowski vacuum, i.e. $u_k \sim \exp(-ik\tau)$, in the limit $k|\tau| \gg 1$. In view of (73), this corresponds to the choice

$$u_k = \sqrt{\frac{\hbar}{2k}}e^{-ik\tau}\left(1-\frac{i}{k\tau}\right), \tag{78}$$

where the coefficient has been determined by the normalisation condition (76). This choice, in the jargon of quantum field theory on curved spacetimes, corresponds to the *Bunch-Davies vacuum*.

Finally, one can compute the *correlation function* for the scalar field $\phi$ in the vacuum state defined above. When Fourier transformed, the correlation function defines the *power spectrum* $\mathscr{P}_\phi(k)$:

$$\langle 0|\hat{\phi}(\mathbf{x}_1)\hat{\phi}(\mathbf{x}_2)|0\rangle = \int d^3k\, e^{i\mathbf{k}.(\mathbf{x}_1-\mathbf{x}_2)}\frac{\mathscr{P}_\phi(k)}{4\pi k^3}. \tag{79}$$

Note that the homogeneity and isotropy of the quantum field is used implicitly in the definition of the power spectrum, which is "diagonal" in Fourier space (homogeneity) and depends only on the norm of **k** (isotropy). In our case, we find

$$2\pi^2 k^{-3} \mathscr{P}_\phi = \frac{|u_k|^2}{a^2}, \tag{80}$$

which gives in the limit $k|\tau| \ll 1$, i.e. when the wavelength is *larger than the Hubble radius*,

$$\mathscr{P}_\phi(k) \simeq \hbar \left(\frac{H}{2\pi}\right)^2 \qquad (k \ll aH)\,. \tag{81}$$

Note that, in the opposite limit $k|\tau| \gg 1$, one recovers the usual vacuum fluctuations in Minkowski, with $\mathscr{P}_\phi(k) = \hbar(k/2\pi a)^2$.

We have used a quantum description of the scalar field. But cosmological perturbations in the standard cosmological eras are usually described by *classical random fields*. Roughly speaking, the transition between the quantum and classical (although stochastic) descriptions makes sense when the perturbations exit the Hubble radius. Indeed each of the terms in the Wronskian (76) is roughly of the order $\hbar/2(k\tau)^3$ in the super-Hubble limit and the non-commutativity can then be neglected. In this sense, one can see the exit outside the Hubble radius as a quantum-classical transition, although some refinement is required to make this statement more precise (see e.g. [27]).

## *4.2 Quantum fluctuations with metric perturbations*

Let us now move to the more realistic case of a perturbed inflaton field living in a *perturbed* cosmological geometry. In fact, Einstein's equations imply that scalar field fluctuations must necessarily coexist with *metric fluctuations*. A correct treatment, either classical or quantum, must therefore involve both the scalar field perturbations and metric perturbations. We thus need to resort to the theory of relativistic cosmological perturbations, which we briefly present below (more details can be found in e.g. [28, 29, 30, 7, 31]).

### 4.2.1 Linear perturbations of the metric

The most general linear perturbation about the homogenous metric can be expressed as

$$ds^2 = a^2 \left\{ -(1+2A)d\tau^2 + 2B_i dx^i d\tau + (\delta_{ij} + h_{ij})\,dx^i dx^j \right\}, \tag{82}$$

where we have assumed, for simplicity, a spatially flat background metric[2]. We have introduced a time plus space decomposition of the perturbations. The indices $i$, $j$ stand for *spatial* indices and the perturbed quantities defined in (82) can be seen as three-dimensional tensors, for which the indices can be lowered (or raised) by the spatial metric $\delta_{ij}$ (or its inverse).

It is very convenient to separate the perturbations into three categories, the so-called "scalar", "vector" and "tensor" modes. For example, a spatial vector field $B^i$ can be decomposed uniquely into a longitudinal part and a transverse part,

$$B_i = \partial_i B + \bar{B}_i, \quad \partial_i \bar{B}^i = 0, \tag{83}$$

where the longitudinal part is curl-free and can thus be expressed as a gradient, and the transverse part is divergenceless. This yields one "scalar" mode, $B$ , and two "vector" modes $\bar{B}^i$ (the index $i$ takes three values but the divergenceless condition implies that only two components are independent).

A similar procedure applies to the symmetric tensor $h_{ij}$, which can be decomposed as

$$h_{ij} = 2C\,\delta_{ij} + 2\partial_i\partial_j E + 2\partial_{(i}E_{j)} + \overline{E}_{ij}, \tag{84}$$

with $\overline{E}^{ij}$ transverse and traceless (TT), i.e. $\partial_i\overline{E}^{ij} = 0$ (transverse) and $\overline{E}^{ij}\delta_{ij} = 0$ (traceless), and $E_i$ transverse. The parentheses around the indices denote symmetrization, namely $2\partial_{(i}E_{j)} \equiv \partial_i E_j + \partial_j E_i$. We have thus defined two scalar modes, $C$ and $E$, two vector modes, $E_i$, and two tensor modes, $\bar{E}_{ij}$.

### 4.2.2 Coordinate transformations

The metric perturbations, introduced in (82), are modified in a coordinate transformation of the form

$$x^\alpha \to x^\alpha + \xi^\alpha, \qquad \xi^\alpha = (\xi^0, \xi^i). \tag{85}$$

It can be shown that the change of the metric components can be expressed as

$$\delta g_{\mu\nu} \to \delta g_{\mu\nu} - 2\nabla_{(\mu}\xi_{\nu)}\,, \tag{86}$$

using the symbol $\nabla$ for the four-dimensional covariant derivative, where the variation due the coordinate transformation is defined for the *same* old and new coordinates (and thus at different physical points).

The above variation can be decomposed into individual variations for the various components of the metric defined earlier. One finds

$$A \to A - \xi^{0\prime} - \mathscr{H}\xi^0 \tag{87}$$

$$B_i \to B_i + \partial_i\xi^0 - \xi_i' \tag{88}$$

[2] This is all the more justified given that the metric in the early Universe was closer to a spatially flat metric than our present metric, which is itself indistiguishible from a flat geometry, according to observations.

$$h_{ij} \to h_{ij} - 2\left(\partial_{(i}\xi_{j)} - \mathscr{H}\xi^0\delta_{ij}\right), \tag{89}$$

where $\mathscr{H} \equiv a'/a$.

The effect of a coordinate transformation can also be decomposed along the scalar, vector and tensor sectors introduced earlier. The generator $\xi^\alpha$ of the coordinate transformation can indeed be written as

$$\xi^\alpha = (\xi^0, \partial^i\xi + \overline{\xi}^i), \tag{90}$$

with $\overline{\xi}^i$ transverse, which shows explicitly that $\xi^\alpha$ contains two scalar components, $\xi^0$ and $\xi$, and two vector components, $\overline{\xi}^i$. The transformations (88-89) are then decomposed into :

$$\begin{aligned}
B &\to B + \xi^0 - \xi' \\
C &\to C - \mathscr{H}\xi^0 \\
E &\to E - \xi \\
\overline{B}^i &\to \overline{B}^i - \overline{\xi}^{i\prime} \\
E^i &\to E^i - \overline{\xi}^i.
\end{aligned} \tag{91}$$

The tensor perturbations remain unchanged since $\xi^\alpha$ does not contain any tensor component.

To summarize, the whole system scalar field plus gravitation is described by eleven perturbations. They can be decomposed into five scalar quantities: $A, B, C$ and $E$ from the metric and $\delta\phi$; four vector quantities $\bar{B}^i$ and $\bar{E}^i$; two tensor quantities: the two polarizations of $E_{ij}^{TT}$. However, these quantities are physically redundant since the same physical situation can be described by different sets of values of these perturbations, provided they are related by the coordinate transformations described above.

One would thus like to identify the *true* degrees of freedom, i.e. the physically independent quantities characterizing the system. One can reduce the effective number of degrees of freedom by using the four coordinate transformations, which consist of two scalar transformations and two vector transformations as we saw earlier. Moreover, Einstein's equations contain nondynamical equations, i.e. constraints, which are also the consequence of the invariance by coordinate transformations. They can be decomposed into two scalar constraints and two vector constraints. The situation for the scalar, vector and tensor sectors, respectively, is summarized in Table 1. By taking into account the coordinate changes and the constraints, one finds three true degrees of freedom: two polarizations of the gravitational waves and one scalar degree of freedom. If matter was composed of $N$ scalar fields, one would get $N$ scalar degrees of freedom in addition to the two tensor modes.

In a coordinate transformation, the scalar field perturbation is also modified, according to

$$\delta\phi \to \delta\phi - \phi'\xi^0. \tag{92}$$

| | Metric | Scalar field | Gauge choice | Constraints | True d.o.f. |
|---|---|---|---|---|---|
| S | 4 | 1 | -2 | -2 | 1 |
| V | 4 | 0 | -2 | -2 | 0 |
| T | 2 | 0 | 0 | 0 | 2 |

**Table 1** Counting of the degrees of freedom in the scalar, vector and tensor sectors.

In single-field inflation, there are thus two natural choices of gauge to describe the scalar perturbation. The first is to work with hypersurfaces that are flat, i.e. $C = 0$, in which case we will denote the scalar field perturbation by $Q$, i.e.

$$Q = \delta\phi_{C=0}\,. \tag{93}$$

The other choice is to work with hypersurfaces where the scalar field is uniform, i.e. $\delta\phi = 0$, in which case the scalar degree of freedom is embodied by the metric perturbation $C_{\delta\phi=0}$. In other words, the true scalar degree of freedom can be represented either as a pure matter perturbation or a pure metric perturbation. In the general case, we have

$$Q = \delta\phi - \frac{\phi'}{\mathscr{H}}C\,, \tag{94}$$

which is a gauge-invariant combination (often called the Mukhanov-Sasaki variable [25, 32]).

## 4.3 Quantizing the scalar degree of freedom

In order to quantize the true scalar degree of freedom, one needs the action that governs its dynamics. Let us first note that the *linearized* equations of motion for the coupled system {gravity + scalar field} are obtained from the expansion of the full action at *second-order* in the perturbations. Indeed the equations for the linear perturbations correspond to the Euler-Lagrange equations derived from a quadratic Lagrangian. In our case, the difficulty is that there are several scalar perturbations that are not independent. In order to quantize this coupled system, one can work directly with the second-order Lagrangian [30], or resort to a Hamiltonian approach [33, 34].

The modern approach, introduced by Maldacena [35] to study perturbations beyond linear order, is based on the Arnowitt-Deser-Misner (ADM) formalism [36]. In the ADM approach, the metric is written in the form

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{95}$$

where $N$ is called the lapse function and $N^i$ the shift vector. The full action for the scalar field and gravity

$$S=\int d^4x\sqrt{-g}\left(-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi-V(\phi)\right)+\frac{M_P^2}{2}\int d^4x\sqrt{-g}R \tag{96}$$

becomes, after substitution of (95),

$$S=\int dtd^3x\sqrt{h}N\left[\frac{\mathscr{V}^2}{2N^2}-\frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi-V(\phi)\right]+\frac{M_P^2}{2}\int dtd^3x\frac{\sqrt{h}}{N}(E_{ij}E^{ij}-E^2)\,, \tag{97}$$

where $h=\det(h_{ij})$,

$$\mathscr{V}\equiv\dot{\phi}-N^j\partial_j\phi\,. \tag{98}$$

and the symmetric tensor $E_{ij}$, defined by

$$E_{ij}\equiv\frac{1}{2}\dot{h}_{ij}-N_{(i|j)}\,, \tag{99}$$

(the symbol $|$ denotes the spatial covariant derivative associated with the spatial metric $h_{ij}$) is proportional to the extrinsic curvature of the spatial slices.

The variation of the action with respect to $N$ yields the energy constraint,

$$\frac{\mathscr{V}^2}{2N^2}+\frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi+V(\phi)+\frac{M_P^2}{2N^2}(E_{ij}E^{ij}-E^2)=0\,, \tag{100}$$

while the variation of the action with respect to the shift $N^i$ gives the momentum constraint,

$$M_P^2\left(\frac{1}{N}(E_i^j-E\delta_i^j)\right)_{|j}=\frac{\mathscr{V}}{N}\partial_i\phi. \tag{101}$$

In order to study the linear perturbations about the FLRW background, we now restrict ourselves to the flat gauge, which corresponds to the choice

$$h_{ij}=a^2(t)\delta_{ij}. \tag{102}$$

The scalar fields on the corresponding flat hypersurfaces can be decomposed as

$$\phi=\bar{\phi}+Q \tag{103}$$

where $\bar{\phi}$ is the spatially homogeneous background value of the scalar field and $Q$ represents its perturbation (on flat hypersurfaces). In the following, we will often omit the bar and simply write the homogeneous value as $\phi$, unless this generates ambiguities.

We can also write the (scalarly) perturbed lapse and shift as

$$N=1+\alpha,\qquad N_i=\beta_{,i}\,, \tag{104}$$

where the linear perturbations $\alpha$ and $\beta$ are determined in terms of the scalar field perturbation $Q$ by solving the linearized constraints. At first-order, the momentum constraint implies

$$\alpha = \frac{\dot{\phi}}{2M_P^2 H} Q, \tag{105}$$

while the energy constraint gives $\partial^2\beta$ in terms of $Q$ and $\dot{Q}$.

### 4.4 Second order action

We now expand the action, up to quadratic order, in terms of the linear perturbations. This action can be written solely in terms of the physical degree of freedom $Q$ by substituting the expression (105) for $\alpha$ (it turns out that $\beta$ disappears of the second order action, after an integration by parts). The second order action can be written in the rather simple form

$$S_{(2)} = \frac{1}{2}\int dt\, \mathrm{d}^3x\, a^3 \left[\dot{Q}^2 - \frac{1}{a^2}\partial_i Q \partial^i Q - \mathscr{M}^2 Q^2\right], \tag{106}$$

with the effective (squared) mass

$$\mathscr{M}^2 = V'' - \frac{1}{a^3}\frac{d}{dt}\left(\frac{a^3}{H}\dot{\phi}^2\right). \tag{107}$$

As we did earlier, it is convenient to use the conformal time $\tau$ and to introduce the canonical degree of freedom

$$v = a\,Q \tag{108}$$

which leads to the action

$$S_v = \frac{1}{2}\int d\tau\, d^3x \left[v'^2 + \partial_i v \partial^i v + \frac{z''}{z} v^2\right], \tag{109}$$

with

$$z = a\,\frac{\phi'}{\mathscr{H}}. \tag{110}$$

This action is analogous to that of a scalar field in Minkowski spacetime with a time-dependent mass. The situation is quite similar to what we obtained previously, with the notable difference that the effective time-dependent mass is now $z''/z$, instead of $a''/a$.

The quantity we will be eventually interested in is the comoving curvature perturbation $\mathscr{R}$, which is related to the canonical variable $v$ by the relation

$$v = z\mathscr{R}. \tag{111}$$

Since, by analogy with (80), the power spectrum for $v$ is given by

$$2\pi^2 k^{-3}\mathscr{P}_v(k) = |v_k|^2, \tag{112}$$

the corresponding power spectrum for $\mathscr{R}$ is found to be

$$2\pi^2 k^{-3}\mathscr{P}_{\mathscr{R}}(k) = \frac{|v_k|^2}{z^2}. \qquad (113)$$

In the case of an inflationary phase in the *slow-roll* approximation, the evolution of $\phi$ and of $H$ is much slower than that of the scale factor $a$. Consequently, one gets approximately

$$\frac{z''}{z} \simeq \frac{a''}{a}, \qquad (\mathrm{slow-roll}) \qquad (114)$$

and all results obtained previously for $u$ apply directly to our variable $v$ in the slow-roll approximation. This implies that the properly normalized function corresponding to the Bunch-Davies vacuum is approximately given by

$$v_k \simeq \sqrt{\frac{\hbar}{2k}} e^{-ik\tau}\left(1 - \frac{i}{k\tau}\right). \qquad (115)$$

In the super-Hubble limit $k|\tau| \ll 1$ the function $v_k$ behaves like

$$v_k \simeq -\sqrt{\frac{\hbar}{2k}}\frac{i}{k\tau} \simeq i\sqrt{\frac{\hbar}{2k}}\frac{aH}{k}, \qquad (116)$$

where we have used $a \simeq -1/(H\tau)$.

Consequently, combining (113), (110) and (115) and reintroducing the cosmic time gives the power spectrum for $\mathscr{R}$, on scales larger than the Hubble radius,

$$\mathscr{P}_{\mathscr{R}} \simeq \frac{\hbar}{4\pi^2}\left(\frac{H^4}{\dot{\phi}^2}\right)_{k=aH} = \frac{\hbar}{2M_P^2\varepsilon_*}\left(\frac{H_*}{2\pi}\right)^2 \qquad (117)$$

where we have used $\varepsilon \equiv -\dot{H}/H^2$ in the second equality, and the subscript $*$ means that the quantity is evaluated at Hubble crossing ($k = aH$). This is the main result for the spectrum of scalar cosmological perturbations generated from vacuum fluctuations during a slow-roll inflation phase.

## *4.5 Gravitational waves*

We have focused so far our attention on scalar perturbations, which are the most important in cosmology. Tensor perturbations, or primordial gravitational waves, if ever detected in the future, would be a remarkable additional probe of the early universe. In the inflationary scenario, like scalar perturbations, primordial gravitational waves are generated from vacuum quantum fluctuations [37]. Let us now explain briefly how.

The action expanded at second order in the perturbations contains a tensor part, given by

$$S_g^{(2)} = \frac{M_P^2}{8}\int d\tau\, d^3x\, a^2 \eta^{\mu\nu}\partial_\mu \bar{E}^i_j \partial_\nu \bar{E}^j_i, \tag{118}$$

where $\eta^{\mu\nu}$ denotes the Minkoswki metric. Apart from the tensorial nature of $E^i_j$, this action is quite similar to that of a scalar field in a FLRW universe (65), up to a renormalization factor $M_P/2$. The decomposition

$$a\bar{E}^i_j = \sum_{\lambda=+,\times}\int \frac{d^3k}{(2\pi)^{3/2}} v_{k,\lambda}(\tau)\varepsilon^i_j(\mathbf{k};\lambda)e^{i\mathbf{k}.\mathbf{x}} \tag{119}$$

where the $\varepsilon^i_j(\mathbf{k};\lambda)$ are the polarization tensors, shows that the gravitational waves are essentially equivalent to two massless scalar fields (for each polarization) $\phi_\lambda = M_P\bar{E}_\lambda/2$.

The total power spectrum is thus immediately deduced from (80) and reads

$$\mathscr{P}_T = 2\times\frac{4}{M_P^2}\times\hbar\left(\frac{H}{2\pi}\right)^2, \tag{120}$$

where the first factor comes from the two polarizations, the second from the renormalization with respect to a canonical scalar field, the last term being the power spectrum for a scalar field derived earlier. In summary, the tensor power spectrum is

$$\mathscr{P}_T = \frac{8\hbar}{M_P^2}\left(\frac{H_*}{2\pi}\right)^2, \tag{121}$$

where the subscript $*$ recalls that the Hubble parameter, which can be slowly evolving during inflation, must be evaluated when the relevant scale exited the Hubble radius during inflation.

A measurement of the tensor amplitude (121) gives direct access, in this context, to the energy scale $H_*$ during inflation, in contrast with the scalar amplitude (117) which depends on the slow-roll parameter $\varepsilon_*$ as well. The tensor to scalar ratio,

$$r \equiv \frac{\mathscr{P}_T}{\mathscr{P}_\mathscr{R}} = 16\,\varepsilon_*, \tag{122}$$

is proportional to the slow-roll parameter.

## 5 From inflation to the standard era

Once the perturbations have been computed during inflation, one must relate them to perturbations in the standard radiation dominated era, where they will be used as “initial conditions”. A priori, one could think that it is necessary to follow the details of how the inflaton is converted into ordinary particles in order to establish this relation. In fact, all these details turn out to be irrelevant, at least in the case of single field inflation, because there exists a conservation law for scales larger than the Hubble radius, which is the case for all relevant scales at the end of inflation.

## 5.1 Covariant approach

Instead of the traditional metric-based approach, we use here a more geometrical approach to cosmological perturbations [38], which will enable us to recover easily and intuitively the main useful results, not only for linear perturbations but also for non-linear perturbations.

Let us consider a spacetime with metric $g_{ab}$ and some perfect fluid characterized by its energy density $\rho$, its pressure $P$ and its four-velocity $u^a$. The corresponding energy momentum-tensor is given by

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b). \tag{123}$$

Let us also introduce the expansion along the fluid worldlines,

$$\Theta = \nabla_a u^a, \tag{124}$$

and the integrated expansion

$$\alpha = \frac{1}{3}\int d\tau_p \, \Theta \,, \tag{125}$$

where $\tau_p$ is the proper time defined along the fluid worldlines. In a FLRW spacetime, one would find $\Theta = 3H$. Therefore, in the general case, one can interpret $\Theta/3$ as a local Hubble parameter and $S = \exp(\alpha)$ as a local scale factor, while $\alpha$ represents the local number of e-folds.

As shown in [39, 40], the conservation law for the energy-momentum tensor,

$$\nabla_a T^a_b = 0, \tag{126}$$

implies that the covector

$$\zeta_a \equiv \nabla_a \alpha - \frac{\dot{\alpha}}{\dot{\rho}} \nabla_a \rho \tag{127}$$

satisfies the relation

$$\dot{\zeta}_a \equiv \mathscr{L}_u \zeta_a = -\frac{\Theta}{3(\rho + p)} \left( \nabla_a p - \frac{\dot{p}}{\dot{\rho}} \nabla_a \rho \right) \,, \tag{128}$$

where a dot denotes the time derivative defined as the Lie derivative along $u^a$[3]. This result is valid for any spacetime geometry and does not depend on Einstein's equations.

The covector $\zeta_a$ can be defined for the global cosmological fluid or for any of the individual cosmological fluids (the case of interacting fluids is discussed in [41]). Using the non-linear conservation equation

$$\dot{\rho} = -3\dot{\alpha}(\rho + P)\,, \tag{129}$$

which follows from $u^b \nabla_a T^a_b = 0$, one can re-express $\zeta_a$ in the form

$$\zeta_a = \nabla_a \alpha + \frac{\nabla_a \rho}{3(\rho + P)}\,. \tag{130}$$

If $w \equiv P/\rho$ is constant, the above covector is a total gradient and can be written as

$$\zeta_a = \nabla_a \left[\alpha + \frac{1}{3(1+w)} \ln \rho\right]. \tag{131}$$

On scales larger than the Hubble radius, our definition agrees with the non-linear curvature perturbation on uniform density hypersurfaces as defined in [42] (see also [43])

$$\zeta = \delta N - \int_{\bar{\rho}}^{\rho} H \frac{d\tilde{\rho}}{\dot{\tilde{\rho}}} = \delta N + \frac{1}{3} \int_{\bar{\rho}}^{\rho} \frac{d\tilde{\rho}}{(1+w)\tilde{\rho}}\,, \tag{132}$$

where $N = \alpha$. The above equation is simply the integrated version of (127), or of (130).

## 5.2 Linear conserved quantities

Let us now introduce a coordinate system, in which the metric (with only scalar perturbations) reads

$$ds^2 = a^2 \left\{-(1+2A)d\tau^2 + 2\partial_i B\, dx^i d\tau + [(1+2C)\delta_{ij} + 2\partial_i \partial_j E]\, dx^i dx^j\right\}. \tag{133}$$

One can decompose the fluid four-velocity as

$$u^\mu = \bar{u}^\mu + \delta u^\mu, \quad \delta u^\mu = \left\{-A/a, v^i/a\right\}, \quad v_i = \partial_i v + \bar{v}_i\,, \tag{134}$$

where $\bar{v}_i$ is transverse.

At linear order, the spatial components of $\zeta_a$ are simply

[3] For scalar quantities, this is equivalent to an ordinary derivative along $u^a$ (e.g. $\dot{\rho} \equiv u^a \nabla_a \rho$), but for $\zeta_a$, one has $\dot{\zeta}_a \equiv u^b \nabla_b \zeta_a + \zeta_b \nabla_a u^b$.

$$\zeta_i^{(1)} = \partial_i \zeta^{(1)}, \qquad \zeta^{(1)} \equiv \delta\alpha - \frac{\bar{\alpha}'}{\bar{\rho}'}\delta\rho\,, \tag{135}$$

where a prime denotes a derivative with respect to $\tau$. Linearizing (128) implies that the curvature perturbation on *uniform-energy-density hypersurfaces*, defined by

$$\zeta = C - \mathscr{H}\frac{\delta\rho}{\rho'} = C + \frac{\delta\rho}{3(\rho+P)} \tag{136}$$

and originally introduced in [44], obeys the evolution equation (see also [45])

$$\zeta' = -\frac{\mathscr{H}}{\rho+P}\delta P_{nad} - \frac{1}{3}\nabla^2(E'+v), \tag{137}$$

where $\delta P_{nad}$ is the non-adiabatic part of the pressure perturbation, defined by

$$\delta P_{nad} = \delta P - c_s^2 \delta\rho. \tag{138}$$

Note that $\zeta^{(1)}$ differs from $\zeta$ but they coincide when the spatial gradients can be neglected, for instance on large scales. The expression (137) shows that $\zeta$ is conserved *on super-Hubble scales* in the case of *adiabatic* perturbations.

Another convenient quantity, which is sometimes used in the literature instead of $\zeta$, is the *curvature perturbation on comoving hypersurfaces*, which can be written in any gauge as

$$\mathscr{R} = -C - \frac{\mathscr{H}}{\rho+P}\delta q, \qquad \partial_i \delta q \equiv \delta_{(S)} T_i^0, \tag{139}$$

where the subscript $(S)$ denotes the perturbations of scalar type. For a perfect fluid, $\delta q = (\rho+P)v$, where $v$ has been defined in (134).

One can relate the two quantities $\zeta$ and $\mathscr{R}$ by using the energy and momentum constraints, which were derived earlier in the ADM formalism. Linearizing (100) and (101) yields, respectively,

$$3\mathscr{H}^2\delta N + a\mathscr{H}\partial^2\beta = -\frac{a^3}{2M_P^2}\delta\rho \tag{140}$$

$$\mathscr{H}\delta N = -\frac{a^3}{2M_P^2}\delta q\,. \tag{141}$$

Combining these two equations yields the relativistic analog of the Poisson equation, namely

$$\partial^2\Psi = \frac{a^2}{2M_P^2}(\delta\rho - 3\mathscr{H}\delta q) \equiv \frac{a^2}{2M_P^2}\delta\rho_c\,, \tag{142}$$

where we have replaced $\beta$ by the Bardeen potential $\Psi \equiv -C - \mathscr{H}(B-E') = -\mathscr{H}\beta$ and introduced the comoving energy density $\delta\rho_c \equiv \delta\rho - 3\mathscr{H}\delta q$. Since

$$\zeta = -\mathscr{R} + \frac{\delta\rho_c}{\rho+P} = -\mathscr{R} - \frac{2\rho}{3(\rho+P)}\left(\frac{k}{aH}\right)^2\Psi, \tag{143}$$

one finds that $\zeta$ and $\mathscr{R}$ coincide in the super-Hubble limit $k \ll aH$.

### 5.3 *"Initial" conditions for standard cosmology*

In standard cosmology, the "initial" conditions for the perturbations are usually defined in the radiation dominated era around the time of nucleosynthesis, when the main cosmological components are restricted to the usual photons, baryons, neutrinos and cold dark matter (CDM) particles. The scales that are cosmologically relevant today correspond to lengthscales much larger than the Hubble radius at that time.

Before the invention of inflation, "initial" conditions were put "by hand", with the restriction that their late time consequences should be compatible with observations. Inflation now provides a precise prescription to determine these "initial conditions"[4].

Since several species are present, one needs to specify the density perturbation of each species. A simplification arises in the case of *single field* inflation, since exactly the same cosmological history, i.e. inflation followed by the decay of the inflaton into the usual species, occurs in all parts of our Universe, up to a small time shift depending on the perturbation of the inflaton in each region. As a consequence, even if the number densities of the various species vary from point to point, their ratio should be fixed, i.e.

$$\delta\left(n_A/n_B\right) = 0\,, \tag{144}$$

for any pair of species denoted $A$ and $B$ (see e.g. [47] for a more detailed discussion). This is not necessarily true in *multi-field inflation*, as the perturbations in the radiation era may depend on *different* combinations of the scalar field perturbations.

The variation in the relative particle number densities between two species can be quantified by the quantity

$$S_{A,B} \equiv \frac{\delta n_A}{n_A} - \frac{\delta n_B}{n_B}\,, \tag{145}$$

which is usually called the *entropy* perturbation between $A$ and $B$. When the equation of state for a given species is such that $w \equiv P/\rho = const$, one can reexpress the entropy perturbation in terms of the density contrast, in the form

$$S_{A,B} \equiv \frac{\delta_A}{1+w_A} - \frac{\delta_B}{1+w_B}. \tag{146}$$

It is convenient to choose a species of reference, for instance the photons, and to define the entropy perturbations of the other species relative to it. The quantities

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[4] although one must be aware that present cosmological scales can correspond to scales smaller than the Planck scale during inflation, suggesting the possibility of trans-Planckian effects (see e.g. [46]).

$$S_b \equiv \delta_b - \tfrac{3}{4}\delta_\gamma, \tag{147}$$

$$S_c \equiv \delta_c - \tfrac{3}{4}\delta_\gamma, \tag{148}$$

$$S_\nu \equiv \tfrac{3}{4}\delta_\nu - \tfrac{3}{4}\delta_\gamma, \tag{149}$$

thus define respectively the baryon, CDM and neutrino entropy perturbations.

For single field inflation, all these entropy perturbations vanish, $S_b = S_c = S_\nu = 0$, and the primordial perturbations are said to be *adiabatic*. An adiabatic primordial perturbation is thus characterized by

$$\frac{1}{4}\delta_\gamma = \frac{1}{4}\delta_\nu = \frac{1}{3}\delta_b = \frac{1}{3}\delta_c\,. \tag{150}$$

Only one density constrast needs to be specified. However, since it is a gauge-dependent quantity, it is better to use the gauge-invariant quantity $\zeta$, i.e. the uniform density curvature perturbation, which is also equivalent to $-\mathscr{R}$, since we are on super-Hubble scales here.

Note that the *adiabatic mode*, which is directly related to the curvature perturbation, is also called *curvature* mode. By contrast, the entropy perturbations can be non-zero even if the curvature is zero, and the corresponding modes are called *isocurvature* modes.

## 5.4 Inflation and cosmological data

Let us now discuss the confrontation of single-field inflation models with the current cosmological data. The main idea is that one can predict precisely the statistics of the CMB perturbations, once the amplitude of the primordial perturbation as a function of scale, $\mathscr{R}(k)$, is given, provided some choice for the cosmological parameters $\Omega_i$. In other words, the measurements of the CMB, together with other cosmological data, allow us to constrain both the cosmological parameters, which are numbers, and the primordial spectrum, which is a function (see e.g. [48, 49] for details on the CMB physics). From the present data, one finds that the primordial spectrum is nearly (although not quite) scale-invariant , with an amplitude

$$\mathscr{P}_{\mathscr{R}}^{1/2} \simeq 5\times 10^{-5}. \tag{151}$$

In order to derive some constraints on the inflation models, it is useful to reexpress the scalar and tensor power spectra, respectively given in (117) and (121), in terms of the scalar field potential. This can be done by using the slow-roll equations (29-30). One finds for the scalar spectrum

$$\mathscr{P}_{\mathscr{R}}(k) = \frac{1}{12\pi^2}\left(\frac{V^3}{M_P^6 V'^2}\right)_{k=aH}, \tag{152}$$

the subscript meaning that the term on the right hand side must be evaluated at *Hubble radius exit* for the scale of interest. The scalar spectrum can also be written in terms of the first slow-roll parameter defined in (32), in which case it reads

$$\mathscr{P}_{\mathscr{R}}(k) = \frac{1}{24\pi^2}\left(\frac{V}{M_P^4 \varepsilon_V}\right)_{k=aH}. \tag{153}$$

If $\varepsilon_V$ is not much smaller than 1, as in chaotic models, the observed amplitude (152) implies that the typical energy scale during inflation is

$$V^{1/4} \sim 10^{-3} M_P \sim 10^{15}\mathrm{GeV}. \tag{154}$$

As for the tensor power spectrum, it is given in terms of the scalar field potential by

$$\mathscr{P}_T = \frac{2}{3\pi^2}\left(\frac{V}{M_P^4}\right)_{k=aH}. \tag{155}$$

The scalar and tensor spectra are almost scale invariant but not quite since the scalar field slowly evolves during the inflationary phase. In order to evaluate quantitatively this variation, it is convenient to introduce a scalar *spectral index* as well as a tensor one, defined respectively by

$$n_S(k) - 1 = \frac{d\ln \mathscr{P}_{\mathscr{R}}(k)}{d\ln k}, \quad n_T(k) = \frac{d\ln \mathscr{P}_T(k)}{d\ln k}. \tag{156}$$

One can express the spectral indices in terms of the slow-roll parameters. For this purpose, let us note that, in the slow-roll approximation, $d\ln k = d\ln(aH) \simeq d\ln a$, so that

$$\frac{d\phi}{d\ln a} = \frac{\dot\phi}{H} \simeq -\frac{V'}{3H^2} \simeq -M_P^2 \frac{V'}{V}, \tag{157}$$

where the slow-roll equations (29-30) have been used. Therefore, one gets

$$n_s(k) - 1 = 2\eta_V - 6\varepsilon_V, \tag{158}$$

where $\varepsilon_V$ and $\eta_V$ are the two slow-roll parameters defined in (32) and (35). Similarly, one finds for the tensor spectral index

$$n_T(k) = -2\varepsilon_V. \tag{159}$$

Comparing with Eq. (122) yields the relation

$$r = -8 n_T, \tag{160}$$

the so-called *consistency relation* which relates purely *observable* quantities (at least in principle). This means that if one was able to observe the primordial gravitational waves and measure the amplitude and spectral index of their spectrum, a rather formidable task, then one would be able to test directly the paradigm of single field slow-roll inflation.

Let us finally go back to the particular models which we have already considered, in order to establish the predictions of these models in the $(n_s, r)$ plane, where they can be easily compared with the observational constraints. For the power-law potentials (41), one finds, using (42),

$$n_s - 1 = -6\varepsilon + 2\eta = -2\frac{p+2}{p}\varepsilon \tag{161}$$

and

$$r = 16\varepsilon = \frac{8p}{p+2}(1-n_s)\,. \tag{162}$$

Moreover,

$$\varepsilon = \frac{p}{4N}, \tag{163}$$

where $N$ is the number of e-folds before the end of inflation when the scales of cosmological interest crossed out the Hubble radius. Therefore, the observational prediction for a model with a power-law potential lie on a line in the $(n_s, r)$ plane, the precise point depending on the number of e-folds when the perturbations were generated.

For an exponential potential (49), one finds, using (52),

$$n_s - 1 = -\frac{2}{q}, \qquad r = \frac{16}{q}. \tag{164}$$

The prediction in the $(n_s, r)$ plane thus depends only on the parameter in the exponential of the potential, but not on the number of e-folds as in the previous case.

For potentials (53) like in hybrid inflation, one finds

$$\eta = \frac{1+\tilde{\phi}^2}{\tilde{\phi}^2}\varepsilon, \tag{165}$$

and

$$r = 8\frac{\tilde{\phi}^2}{2\tilde{\phi}^2 - 1}(1-n_s). \tag{166}$$

One can proceed in a similar way for any model of inflation and thus be able to confront it with observational data. In general, it is worth noticing that a significant amount of gravitational waves, and thus a detectable $r$, requires a variation of the inflaton of the order of the Planck mass during inflation [50].

# 6 More general inflationary scenarios

So far, the simplest models of inflation are compatible with the available data but it is instructive to study more refined models for at least two reasons. First, models inspired by high energy physics are usually more complicated than the simplest phenomenological inflationary models. Second, exploring more general models of inflation and identifying their specific observational features is a healthy procedure to prepare the interpretation of the future data.

At present, two types of extensions of the simplest scenarios have been mainly studied:

- models with non standard kinetic terms;
- models with multiple scalar fields.

Of course, the two aspects can be combined and there exist scenarios involving several scalar fields with non-standard kinetic terms, as we will see later.

Among the scenarios involving several scalar fields, it is useful to distinguish three subclasses. The first, and oldest, category consists of models with *multiple inflatons*, i.e. models where several scalar fields play a dynamical role in the homogenous cosmological evolution during inflation. In the second category, one finds the *curvaton* scenarios. These models assume the existence, in addition to the inflaton, of a scalar field, which is light during inflation but does not participate to inflation *per se*. Its energy density, which decreases less quickly than radiation, becomes significant only after inflation. Its decay produces a second reheating, and its fluctuations are then imprinted in the curvature perturbation.

The final subclass regroups what we will name the *modulaton* scenarios. Like in the curvaton models, one assumes the presence of a light scalar field, the modulaton, which is subdominant during inflation but acquires some fluctuations. The fluctuations of the modulaton are transferred to the curvature perturbation because the cosmological evolution is governed by some parameter that depends on the modulaton. This parameter can be for instance the value of the inflaton at the end of inflation, or the coupling of the inflaton to other particles during the reheating. Of course, one can envisage even more complicated scenarios which combine several of these mechanisms.

## 6.1 Generalized Lagrangians

We now consider multi-field models, which can be described by an action of the form

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{16\pi G}+P(X^{IJ},\phi^K)\right] \tag{167}$$

where $P$ is an arbitrary function of $N$ scalar fields and of the kinetic term

$$X^{IJ} = -\frac{1}{2}\nabla_\mu\phi^I\nabla^\mu\phi^J. \tag{168}$$

The relations obtained earlier for the single field model can then be generalized, as we now show. The energy-momentum tensor, derived from (167), is of the form

$$T^{\mu\nu} = Pg^{\mu\nu} + P_{<IJ>}\partial^\mu\phi^I\partial^\nu\phi^J\,, \tag{169}$$

where $P_{<IJ>}$ denotes the partial derivative of $P$ with respect to $X^{IJ}$ (symmetrized with respect to the indices $I$ and $J$). The equations of motion for the scalar fields, which can be seen as generalized Klein-Gordon equations, are obtained from the variation of the action with respect to $\phi^I$ and read

$$\nabla_\mu\left(P_{<IJ>}\nabla^\mu\phi^J\right) + P_{,I} = 0\,. \tag{170}$$

where $P_{,I}$ denotes the partial derivative of $P$ with respect to $\phi^I$.

In a homogeneous spacetime, $X^{IJ} = \dot\phi^I\dot\phi^J/2$, and the energy-momentum tensor reduces to that of a perfect fluid with pressure $P$ and energy density

$$\rho = 2P_{<IJ>}X^{IJ} - P\,. \tag{171}$$

The evolution of the scale factor $a(t)$ is governed by the Friedmann equations, which can be written in the form

$$H^2 = \frac{1}{3M_P^2}\left(2P_{<IJ>}X^{IJ} - P\right)\,, \qquad \dot H = -X^{IJ}P_{<IJ>}/M_P^2\,. \tag{172}$$

The equations of motion for the scalar fields reduce to

$$\left(P_{<IJ>} + P_{<IL>,<JK>}\dot\phi^L\dot\phi^K\right)\ddot\phi^J + \left(3HP_{<IJ>} + P_{<IJ>,K}\dot\phi^K\right)\dot\phi^J - P_{,I} = 0\,, \tag{173}$$

where $P_{<IL>,<JK>}$ denotes the (symmetrized) second derivative of $P$ with respect to $X^{IL}$ and $X^{JK}$.

The expansion of the action (167) up to second order in the perturbations is useful to obtain the classical equations of motion for the linear perturbations and to calculate the spectra of the primordial perturbations generated during inflation, as we have seen earlier in the case of a single scalar field.

Working for convenience with the scalar field perturbations $Q^I$ defined in the spatially flat gauge, the second order action can be written in the compact form [51]

$$\begin{aligned} S_{(2)} = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^3x\,a^3\,\Big[&\left(P_{<IJ>} + 2P_{<MJ>,<IK>}X^{MK}\right)\dot Q^I\dot Q^J \\ &-P_{<IJ>}h^{ij}\partial_i Q^I\partial_j Q^J - \mathscr{M}_{KL}Q^KQ^L + 2\,\Omega_{KI}Q^K\dot Q^I\Big]\,, \end{aligned} \tag{174}$$

where the mass matrix is

$$\mathscr{M}_{KL} = -P_{,KL} + 3X^{MN}P_{<NK>}P_{<ML>} + \frac{1}{H}P_{<NL>}\dot\phi^N\left[2P_{<IJ>,K}X^{IJ} - P_{,K}\right]$$

$$- \frac{1}{H^2} X^{MN} P_{<NK>} P_{<ML>} \left[ X^{IJ} P_{<IJ>} + 2P_{<IJ>,<AB>} X^{IJ} X^{AB} \right]$$
$$- \frac{1}{a^3} \frac{\mathrm{d}}{\mathrm{dt}} \left( \frac{a^3}{H} P_{<AK>} P_{<LJ>} X^{AJ} \right) \tag{175}$$

and the mixing matrix is

$$\Omega_{KI} = \dot{\phi}^J P_{<IJ>,K} - \frac{2}{H} P_{<LK>} P_{<MJ>,<NI>} X^{LN} X^{MJ} . \tag{176}$$

This formalism is very general and, in the following, we will consider two particular cases, which have often been studied in the literature.

## 6.2 *Simple multi-inflaton scenarios*

A more restrictive, although still very large, class of models consists of multi-field scenarios governed by a Lagrangian of the form

$$P = G_{IJ} X^{IJ} - V = -\frac{1}{2} G_{IJ}(\phi)\, \partial^\mu \phi^I \partial_\mu \phi^J - V(\phi) , \tag{177}$$

where the field metric $G_{IJ}$ can be non trivial (also studied in e.g. [52, 53, 54]). It can then be shown that the second-order action can be rewritten in the form

$$S_{(2)} = \frac{1}{2} \int dt\, \mathrm{d}^3 x\, a^3 \left[ G_{IJ} \mathscr{D}_t Q^I \mathscr{D}_t Q^J - \frac{1}{a^2} G_{IJ} \partial_i Q^I \partial^i Q^J - \tilde{M}_{IJ} Q^I Q^J \right] , \tag{178}$$

with

$$\tilde{M}_{IJ} = \mathscr{D}_I \mathscr{D}_J V - R_{IKLJ} \dot{\phi}^K \dot{\phi}^L - \frac{1}{a^3} \mathscr{D}_t \left[ \frac{a^3}{H} \dot{\phi}_I \dot{\phi}_J \right] , \tag{179}$$

and where $\mathscr{D}_I$ denotes the covariant derivative with respect to the field space metric $G_{IJ}$ (so that $\mathscr{D}_I \mathscr{D}_J V = V_{,IJ} - \Gamma^K_{IJ} V_{,K}$ where the $\Gamma^K_{IJ}$ denote the Christoffel symbols of the metric $G_{IJ}$), while $R_{IJKL}$ is the corresponding Riemann tensor and $\mathscr{D}_t Q^I \equiv \dot{Q}^I + \Gamma^I_{JK} \dot{\phi}^I Q^K$.

It is now convenient, following the approach of [55], to introduce a particular direction in field space, which we will call the *instantaneous adiabatic* direction, defined by the unit vector tangent to the inflationary trajectory in field space,

$$e^I_\sigma = \frac{\dot{\phi}^I}{\sqrt{2X}} = \frac{\dot{\phi}^I}{\dot{\sigma}} , \tag{180}$$

where we have introduced the notation $X \equiv G_{IJ} X^{IJ}$ and $\dot{\sigma} \equiv \sqrt{2X}$. The *instantaneous entropic* directions, which are orthogonal to $e^I_\sigma$, span an hyperplane in field space.

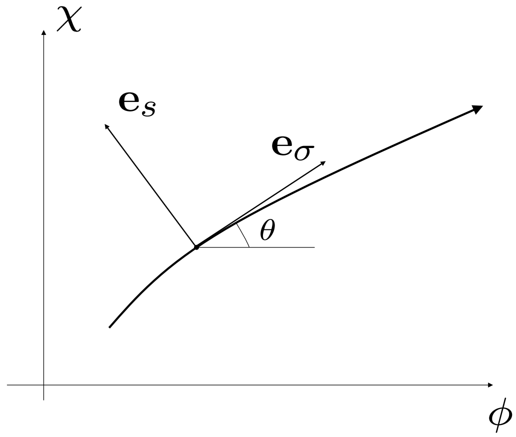


**Fig. 3** Inflationary trajectory in a two-field model. The (instantaneous) adiabatic vector $e_\sigma$ is tangent to the trajectory while the (instantaneous) entropic vector $e_s$ is orthogonal to it.

For simplicity, let us now concentrate on two-field scenarios, where there is as single entropic degree of freedom. Defining the entropy vector $e_s^I$ as the unit vector orthogonal to the adiabatic vector $e_\sigma^I$, i.e.

$$G_{IJ}e_s^I e_s^J = 1, \qquad G_{IJ}e_s^I e_\sigma^J = 0, \tag{181}$$

the scalar field perturbations can be uniquely decomposed into (instantaneous) adiabatic and entropic modes,

$$Q^I = Q_\sigma e_\sigma^I + Q_s e_s^I. \tag{182}$$

One can then derive the equations of motion for the quantities $Q_\sigma$ and $Q_s$ from the second-order action. One finds [54]

$$\ddot{Q}_\sigma + 3H\dot{Q}_\sigma + \left(\frac{k^2}{a^2} + \mu_\sigma^2\right) Q_\sigma = (\Xi Q_s)^\cdot - \left(\frac{\dot{H}}{H} + \frac{V_{,\sigma}}{\dot{\sigma}}\right) \Xi Q_s, \tag{183}$$

with

$$\Xi \equiv -\frac{2}{\dot{\sigma}} V_{,s}, \qquad \mu_\sigma^2 \equiv -\frac{(\dot{\sigma}/H)^{\cdot\cdot}}{\dot{\sigma}/H} - \left(3H + \frac{(\dot{\sigma}/H)^\cdot}{\dot{\sigma}/H}\right) \frac{(\dot{\sigma}/H)^\cdot}{\dot{\sigma}/H}, \tag{184}$$

where $V_{,\sigma} \equiv e_\sigma^I V_{,I}$ and $V_{,s} \equiv e_s^I V_{,I}$. The equation of motion for the entropy part is given by

$$\ddot{Q}_s + 3H\dot{Q}_s + \left(\frac{k^2}{a^2} + \mu_s^2\right) Q_s = -\Xi \left[\dot{Q}_\sigma - H\left(\frac{\dot{\sigma}^2}{2H^2} + \frac{\ddot{\sigma}}{H\dot{\sigma}}\right) Q_\sigma\right], \tag{185}$$

with

$$\mu_s^2 \equiv V_{ss} + \frac{1}{2}\dot{\sigma}^2 R - \frac{V_{,s}^2}{2X}\,, \tag{186}$$

where $R$ is the trace of the Ricci tensor on field space, i.e. the scalar curvature.

The adiabatic perturbation is naturally related to the comoving curvature perturbation (139). Indeed, using the energy-momentum tensor (169), with the property $\rho + P = 2X$, which follows from (171), one finds that the comoving perturbation (139) is given by

$$\mathscr{R} = \frac{H}{2X}\dot{\phi}_I Q^I = \frac{H}{\sqrt{2X}} Q_\sigma\,. \tag{187}$$

Taking the time derivative of this expression and using the analog of (142),

$$-2\frac{k^2}{a^2}\Psi = \delta\rho_c = \sqrt{2X}\left[\dot{Q}_\sigma + \left(\frac{\dot{H}}{H} - \frac{\dot{X}}{2X}\right) Q_\sigma\right] + 2V_{,s}Q_s \tag{188}$$

one finds

$$\dot{\mathscr{R}} = \frac{H}{\dot{H}}\frac{k^2}{a^2}\Psi - 2\frac{H}{\dot{\sigma}^2}V_{,s}Q_s\,. \tag{189}$$

By noting that the right hand side of Eq. (185) is proportional to $\dot{\mathscr{R}}$, one can rewrite the entropic equation of motion as

$$\ddot{Q}_s + 3H\dot{Q}_s + \left(\frac{k^2}{a^2} + \mu_s^2 + \Xi^2\right) Q_s = -\frac{\dot{\sigma}}{\dot{H}}\,\Xi\,\frac{k^2}{a^2}\Psi\,. \tag{190}$$

When the spatial gradients can be neglected on large scales, the above equation shows that the entropy perturbation $Q_s$ evolves independently of the adiabatic mode. In the same limit, the adiabatic mode is governed by a *first-order* equation

$$\dot{\mathscr{R}} \approx \frac{H}{\dot{\sigma}}\,\Xi\,Q_s \quad \text{or} \quad \dot{Q}_\sigma + \left(\frac{\dot{H}}{H} - \frac{\ddot{\sigma}}{\dot{\sigma}}\right) Q_\sigma - \Xi\,Q_s \approx 0\,, \tag{191}$$

This implies that, in contrast with the entropy mode, the adiabatic mode is affected by the entropy on large scales, as soon as the *mixing parameter* $\Xi = -2V_{,s}/\dot{\sigma}$ is non zero. When the field metric is flat, $G_{IJ} = \delta_{IJ}$, one can introduce the rotation angle $\theta$ between the initial basis and the adiabatic/entropy basis, which gives $\Xi = 2\dot{\theta}$. In the case of a field metric of the form

$$G_{IJ}\,d\phi^I\,d\phi^J = d\phi^2 + e^{2b(\phi)}d\chi^2\,, \tag{192}$$

investigated in [56, 57], the coupling is now given by $\Xi = 2\dot{\theta} + b'\dot{\sigma}\sin\theta$, where the additional term simply comes from the non-trivial covariant derivative. Note that non-linear extensions of the adiabatic and entropic equations have been obtained in [58, 59, 60] (see also [61, 62] for other works on non-linear perturbations in multi-field inflation).

The above results show that a generic feature of multi-inflaton scenarios is that the curvature perturbation is not frozen after horizon crossing, like in single-field

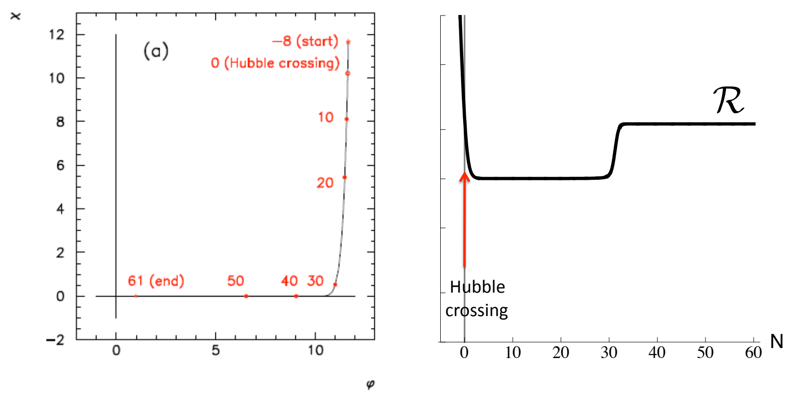


**Fig. 4** In a double inflation model, with two different masses for the scalar fields, the inflationary trajectory is bent (left). This induces an evolution of the curvature perturbation, *after* Hubble crossing (right). Other examples can be found in [57].

inflation, but can, instead, evolve on large scales as a consequence of the *transfer* of entropy perturbations into adiabatic perturbations, as illustrated on Fig. 4. This property was pointed out originally in [63] in the context of generalized gravity theories. As a consequence, it is crucial, when working with a model involving several scalar fields during inflation, to identify all the light directions in field space and to evolve the curvature perturbation until any transfer from entropy into adiabatic modes has completely ceased (the transfer can even occur long after inflation, as is the case in the curvaton scenario, which we will discuss later).

As we have just seen, the instantaneous entropy perturbations can affect the evolution of the curvature perturbation during inflation, on large scales, but they could also survive the end of inflation and the reheating phase and therefore, cause the existence of "initial" isocurvature perturbations, for instance between the CDM and photon fluids, in the radiation era. Moreover, these isocurvature perturbations could be correlated with the "initial" adiabatic perturbations [64], since part of the adiabatic perturbation can originate from an (instantaneous) entropy perturbation during inflation. We will discuss later the observational constraints on this possibility.

### 6.3 K-inflation

Let us now consider *single-field* inflation, but with a generalized Lagrangian

$$L = P(X,\phi), \qquad X \equiv -\partial_\mu\phi\partial^\mu\phi/2\,. \tag{193}$$

This class of models, studied in [65], has been called K-inflation because inflation can arise from the presence of non-standard kinetic terms, and not necessarily from a quasi-flat potential as in standard inflation.

Linear perturbations have been investigated in [66]. Here, they can simply be obtained from the single-field limit of (174)

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^3x\,a^3\left[(P_X + 2P_{XX}X)\,\dot{Q}^2 - P_X h^{ij}\partial_i Q\partial_j Q \right. \\ \left. -\mathscr{M}Q^2 + 2\,\Omega Q\dot{Q}\right]\,, \tag{194}$$

where $P_X \equiv \frac{\partial P}{\partial X}$ and $P_{XX} \equiv \frac{\partial^2 P}{\partial X^2}$. The first line of the above action shows that the perturbations of the scalar field propagate with an effective sound speed given by

$$c_s^2 = \frac{P_X}{P_X + 2XP_{XX}}\,, \tag{195}$$

which, in some models, can be much smaller than the usual speed of light.

Introducing the conformal time $\tau$ and the canonically normalized field

$$v = \frac{a\sqrt{P_X}}{c_s}Q \tag{196}$$

yields the action

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^3x\left[v'^2 - c_s^2(\partial v)^2 + \frac{z''}{z}v^2\right]\,, \tag{197}$$

with

$$z = \frac{a\dot{\phi}\sqrt{P_X}}{c_s H}\,. \tag{198}$$

In Fourier space, this leads to the equation of motion

$$v'' + \left(k^2c_s^2 - \frac{z''}{z}\right)v = 0, \tag{199}$$

where one notes the presence of $c_s^2$ multiplying $k^2$. As a consequence, the fluctuations are amplified at *sound horizon* crossing, i.e. when $kc_s \sim aH$, and not at Hubble radius crossing as in the standard case (the two of course coincide for $c_s \simeq c$).

Assuming a slow variation of the Hubble parameter $H$ and of the sound speed $c_s$, one can use the approximation $z''/z \simeq 2/\tau^2$ and the solution corresponding to the vacuum on small scales is given by

$$v = \frac{1}{\sqrt{2kc_s}} e^{-ikc_s\tau} \left(1 - \frac{i}{kc_s\tau}\right). \tag{200}$$

This expression differs from (78) only by the presence of $c_s$.

One can then proceed exactly as in the standard case to obtain the power spectrum of the scalar field fluctuations

$$\mathscr{P}_Q \simeq \frac{H^2}{4\pi^2 c_s P_X} \tag{201}$$

and the power spectrum of the curvature perturbation

$$\mathscr{P}_{\mathscr{R}_*} = \frac{k^3}{2\pi^2} \frac{|v_{\sigma k}|^2}{z^2} \simeq \frac{H^4}{4\pi^2 \dot{\sigma}^2} = \frac{H^2}{8\pi^2 \varepsilon c_s}, \tag{202}$$

where $\varepsilon = -\dot{H}/H^2$.

### 6.4 A specific example: multi-field DBI inflation

The two previous subsections have illustrated *separately* the consequences of multiple inflatons, on the one hand, and of non-standard kinetic terms, on the other hand. Here, these two aspects will be naturally combined in a category of models motivated by string theory, where inflation is due to the motion of a $D3$-brane in an internal six-dimensional compact space.

The dynamics of the brane, with tension $T_3$, is governed by the Dirac-Born-Infeld Lagrangian (we ignore here the dilaton and the various form fields, but they can be included, as in [67])

$$L_{\rm DBI} = -T_3 \sqrt{-\det \gamma_{\mu\nu}} \tag{203}$$

which depends on the determinant of the induced metric on the 3-brane,

$$\gamma_{\mu\nu} = H_{AB}\, \partial_\mu Y^A_{(\rm b)} \partial_\nu Y^B_{(\rm b)}, \tag{204}$$

where $H_{AB}$ is metric of the compactified 10-dimensional spacetime, assumed to be of the form

$$H_{AB}\, dY^A\, dY^B = h^{-1/2}(y^K)\, g_{\mu\nu} dx^\mu dx^\nu + h^{1/2}(y^K)\, G_{IJ}(y^K)\, dy^I dy^J, \tag{205}$$

and $Y^A_{(\rm b)}(x^\mu) = (x^\mu, \psi^I(x^\mu))$, with $\mu = 1\ldots3$ and $I = 1\ldots6$, defines the brane embedding.

After using the rescalings $\phi^I \equiv \sqrt{T_3} Y^I$ and $f = h/T_3$, one ends up with a Lagrangian of the form

$$P = -\frac{1}{f(\phi^I)} \left(\sqrt{\mathscr{D}} - 1\right) - V(\phi^I) \tag{206}$$

with

$$\mathscr{D} \equiv \det(\delta^\mu_\nu + f\, G_{IJ}\partial^\mu\phi^I\partial_\nu\phi^J)$$
$$= 1 - 2fG_{IJ}X^{IJ} + 4f^2X_I^{[I}X_J^{J]} - 8f^3X_I^{[I}X_J^{J}X_K^{K]} + 16f^4X_I^{[I}X_J^{J}X_K^{K}X_L^{L]} \quad (207)$$

where the field indices are lowered by the field metric $G_{IJ}$, i.e. the metric of the internal compact space, and the brackets denote antisymmetrization of the indices. A potential term, which arises from the brane's interactions with bulk fields or other branes, is also included.

Assuming that the brane is moving in a conical geometry, many works have concentrated on the purely radial dynamics of the brane, while ignoring the angular directions. The effective action then reduces to

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{f}\left(\sqrt{1+f\,\partial_\mu\phi\partial^\mu\phi}-1\right)-V(\phi)\right]. \quad (208)$$

If $f\dot{\phi}^2 \ll 1$, one can expand the square root in the Lagrangian and one recovers the usual kinetic term familiar to slow-roll inflation.

This Lagrangian also leads to another type of inflation, called DBI inflation [68, 69], in the "relativistic" limit

$$1 - f\dot{\phi}^2 \ll 1 \quad \Leftrightarrow \quad |\dot{\phi}| \simeq 1/\sqrt{f}. \quad (209)$$

Indeed, using (172), one can check that it is possible to obtain $\varepsilon \equiv -\dot{H}/H^2 \ll 1$ in this limit, provided $V \gg 1/fc_s$. An interesting property of DBI inflation is that the potential can be rather steep, in contrast with standard slow-roll inflation.

The Lagrangian in (208) is of the form $P(X,\phi)$, discussed in the previous subsection, with

$$P(X,\phi) = -\frac{1}{f(\phi)}\left(\sqrt{1-2fX}-1\right) - V(\phi), \quad (210)$$

and therefore, using (195),

$$c_s = \sqrt{1-2fX} = \frac{1}{P_X}. \quad (211)$$

If the angular directions are relevant, the above single field simplification is not valid and one must work in a multi-field framework with the Lagrangian (206). The perturbations generated by such a scenario have been studied in detail in [70] and we now summarize the main results.

After decomposing the perturbations into adiabatic and entropy modes, one finds that the single field results apply to the adiabatic mode, so that its spectrum at *sound horizon crossing* is given by

$$\mathscr{P}_{Q\sigma*} \simeq \frac{H^2}{4\pi^2} \quad (212)$$

(the subscript $*$ here indicates that the corresponding quantity is evaluated at sound horizon crossing $kc_s = aH$).

As for the (canonically normalized) entropy mode, $v_s \equiv (a/\sqrt{c_s})Q_s$, its evolution is governed by the equation

$$v_s'' + \xi v_\sigma' + \left(k^2 c_s^2 - \frac{\alpha''}{\alpha}\right) v_s = 0\,, \qquad \alpha \equiv \frac{a}{\sqrt{c_s}} \tag{213}$$

where we have neglected a possible coupling with the adiabatic mode and assumed that the effective mass of the entropy mode is small with respect to $H$. $v_s$ has thus the same spectrum as $v_\sigma$, but since the normalization coefficients in front of the adiabatic and entropy modes differ, one finds that the spectrum for the fluctuations along the entropy direction in field space, is given by

$$\mathscr{P}_{Q_s*} \simeq \frac{H^2}{4\pi^2 c_s^2}, \tag{214}$$

which shows that, for small $c_s$, the entropic modes are *amplified* with respect to the adiabatic modes:

$$Q_{s*} \simeq \frac{Q_{\sigma*}}{c_s}. \tag{215}$$

Since we are in a multi-field scenario, the curvature perturbation can be modified, after sound horizon crossing, if there is a transfer from the entropic to the adiabatic modes, as we saw earlier. The final curvature perturbation can be formally written as

$$\mathscr{R} = \mathscr{R}_* + T_{\mathscr{R}\mathscr{S}}\mathscr{S}_*\,, \tag{216}$$

where, for convenience, we have introduced the *rescaled* entropy perturbation

$$\mathscr{S} = c_s \frac{H}{\dot\sigma} Q_s\,, \tag{217}$$

defined such that its power spectrum at sound horizon crossing is the same as that of the curvature perturbation, i.e. $\mathscr{P}_{\mathscr{S}_*} = \mathscr{P}_{\mathscr{R}_*}$. The final curvature power-spectrum is thus given by

$$\mathscr{P}_{\mathscr{R}} = (1+T_{\mathscr{R}\mathscr{S}}^2)\mathscr{P}_{\mathscr{R}_*} = \frac{\mathscr{P}_{\mathscr{R}_*}}{\cos^2\Theta}\,, \tag{218}$$

where we have introduced the “transfer angle” $\Theta$ defined by

$$\sin\Theta = \frac{T_{\mathscr{R}\mathscr{S}}}{\sqrt{1+T_{\mathscr{R}\mathscr{S}}^2}} \tag{219}$$

(so that $\Theta = 0$ if there is no transfer and $|\Theta| = \pi/2$ if the final curvature perturbation is mostly of entropic origin).

The power spectrum for the tensor modes is still governed by the transition at *Hubble radius* and its amplitude, given by (121), is unchanged. The tensor to scalar ratio is thus

$$r \equiv \frac{\mathscr{P}_T}{\mathscr{P}_{\mathscr{R}}} = 16\,\varepsilon\, c_s \cos^2\Theta\,. \tag{220}$$

Interestingly this expression combines the result of $k$-inflation [66], where the ratio is suppressed by a small sound speed $c_s$, and that of multi-field inflation with standard kinetic terms [71], where the ratio is suppressed by a large transfer from entropy to adiabatic modes.

### 6.5 The curvaton scenario

The transfer from entropy into adiabatic perturbations can occur during inflation, as we have seen in scenarios with multiple inflatons, but it can also take place long after the end of inflation. A much studied example of this possibility is the curvaton scenario [72, 73, 74] (see also [75]).

The curvaton is a weakly coupled scalar field, $\sigma$, which is light relative to the Hubble rate during inflation, and hence acquires perturbations during inflation, with an almost scale-invariant power spectrum

$$\mathscr{P}_{\delta\sigma} = \left(\frac{H}{2\pi}\right)^2, \tag{221}$$

where the curvaton perturbation is defined here in the flat gauge, i.e. $\delta\sigma = Q_\sigma$.

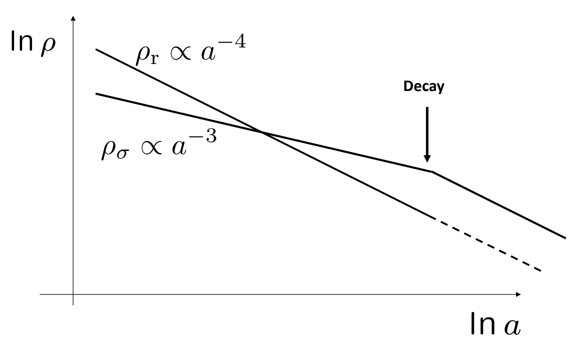


**Fig. 5** Evolution of the energy density of the radiation, $\rho_r$, produced by the inflaton and of the energy density of the curvaton, $\rho_\sigma$, before and after the curvaton decay.

After inflation, the Hubble rate drops and, eventually, the curvaton becomes non-relativistic so that its energy density grows relatively to that of radiation, until it represents a significant fraction of the total energy density, $\Omega_\sigma \equiv \bar{\rho}_\sigma/\bar{\rho}$, before it finally decays (see Fig. 5). Hence the initial curvaton field perturbations on large scales can give rise to a primordial density perturbation after the decay of the curvaton.

Before it decays, the non-relativistic curvaton (with mass $m \gg H$) behaves effectively as a pressureless, non-interacting fluid with energy density

$$\rho_\sigma = m^2\sigma^2 , \tag{222}$$

where $\sigma$ is the rms amplitude of the curvaton field. The corresponding perturbations are characterized, using (136) and (221), by

$$\zeta_\sigma = \left(\frac{\delta\rho_\sigma}{3\rho_\sigma}\right)_{\rm flat} = \frac{2}{3}\frac{\delta\sigma}{\sigma} \quad \Rightarrow \mathscr{P}_{\zeta_\sigma} \simeq \frac{H^2}{9\pi^2\sigma^2} . \tag{223}$$

When the curvaton decays into radiation, its perturbations are converted into perturbations of the resulting radiation fluid. The subsequent perturbation is described by

$$\zeta = r_\sigma\zeta_\sigma + (1-r_\sigma)\zeta_{\rm inf} , \quad r_\sigma \equiv \frac{3\Omega_{\sigma,\rm decay}}{4-\Omega_{\sigma,\rm decay}} . \tag{224}$$

This implies that the power spectrum for the primordial adiabatic perturbation $\zeta_{\rm r}$ can be expressed as

$$\mathscr{P}_\zeta = \mathscr{P}_{\zeta_{\rm inf}} + r_\sigma^2 \mathscr{P}_{\zeta_\sigma} . \tag{225}$$

where $\mathscr{P}_{\zeta_{\rm inf}}$ is the spectrum of perturbations generated directly by the inflaton fluctuations.

In the case of single field inflation, $\mathscr{P}_{\zeta_{\rm inf}}$ is given in (117) and one can rewrite the total power spectrum as we have

$$\mathscr{P}_\zeta = (1+\lambda)\mathscr{P}_{\zeta_{\rm inf}} , \qquad \lambda \equiv \frac{8}{9} r_\sigma^2 \varepsilon_* \left(\frac{\sigma_*}{M_P}\right)^{-2} \tag{226}$$

The limit $\lambda \gg 1$ corresponds to the original curvaton scenario where the inflaton perturbations are negligible: since $r_\sigma$ and $\varepsilon_*$ are bounded by 1, this requires $\sigma_* \ll M_P$.

A value of $\lambda$ of order 1 or smaller is possible if $r_\sigma$ or $\varepsilon_*$ are sufficiently small and/or $\sigma_*$ is of the order of $M_P$. In the latter case the curvaton starts to oscillate at about the same time as it decays and cannot be described as a dust field. A more refined treatment [76] shows that the curvature perturbation due to the inflaton and curvaton perturbations is given

$$\mathscr{R} = -\frac{V}{M_P^2 V'}\delta\phi - \frac{3}{2} f(\sigma_*)\frac{\delta\sigma_*}{M_P}, \tag{227}$$

where the function $f(\sigma_*)$ interpolates between the limiting situations of a pure curvaton and of a secondary inflaton,

$$f(\sigma_*) \simeq \begin{cases} \frac{4}{9}\frac{M_P}{\sigma_*}, & \sigma_* \ll M_P \\ \\ \frac{\sigma_*}{3M_P}, & \sigma_* \gg M_P. \end{cases} \tag{228}$$

Interestingly, the curvaton scenario can also produce entropic, or isocurvature, perturbations [77]. It can produce a CDM isocurvature perturbation if the CDM is created before the curvaton decay and thus inherits the perturbations of the inflaton so that $S_{\rm cdm} = 3(\zeta_{\rm inf} - \zeta_r)$; or, on the contrary, if the CDM is created by the curvaton decay, in which case $S_{\rm cdm} = 3(\zeta_\sigma - \zeta_r)$. Similarly, baryon isocurvature perturbations can be generated if the baryon asymmetry exists before the curvaton decay.

## *6.6 Modulaton*

In the curvaton scenario, the curvaton dominates the energy density of the Universe at some epoch in order to give the main contribution to the primordial perturbations. Alternatively, one can also envisage scenarios where the primordial perturbations are due to the perturbations of a scalar field, which has never dominated the matter content of the universe but has played a crucial rôle during some cosmological transition. We will name this field a *modulaton*.

The best example is the *modulated reheating* scenario [78, 79] where the decay rate of the inflaton, $\Gamma$, depends on a modulaton $\sigma$, which has acquired classical fluctuations during inflation. The decay rate is thus slightly different from one super-Hubble patch to another, which generates a curvature perturbation.

A simple way to quantify this effect is to compute the number of e-folds between some initial time $t_i$ during inflation, when the scale of interest crossed out the Hubble radius, and some final time $t_f$. The curvature perturbation is then directly related to the fluctuations of the number of e-folds, as we discussed at the beginning of Section 5.

For simplicity, we will assume that, just after the end of inflation at time $t_e$, the inflaton behaves like pressureless matter (as is the case for a quadratic potential) until it decays instantaneously at the time $t_d$ characterized by $H_d = \Gamma$. At the decay, the energy density is thus $\rho_d = \rho_e \exp[-3(N_d - N_e)]$ and is transferred into radiation, so that, at time $t_f$, one gets

$$\rho_f = \rho_d e^{-4(N_f - N_d)} = \rho_e e^{-3(N_f - N_e) - (N_f - N_d)}. \tag{229}$$

Using the relation $\Gamma = H_d = H_f \exp[2(N_f - N_d)]$ to eliminate $(N_f - N_d)$ in (229), we finally obtain

$$N_f = N_e - \frac{1}{3}\ln\frac{\rho_f}{\rho_e} - \frac{1}{6}\ln\frac{\Gamma}{H_f}\,. \tag{230}$$

If one ignores the inflaton fluctuations, the final curvature perturbation is therefore

$$\zeta = N_{,\sigma}\delta\sigma_* = -\frac{1}{6}\frac{\Gamma_{,\sigma}}{\Gamma}\delta\sigma_*, \tag{231}$$

which yields the curvature power spectrum

$$\mathscr{P}_\zeta = \frac{1}{36}\left(\frac{\Gamma_{,\sigma}}{\Gamma}\right)^2\left(\frac{H_*}{2\pi}\right)^2. \tag{232}$$

The dependence on the modulaton can alternatively show up in the mass of the particles created by the decay of the inflaton [80, 81].

The modulaton can also affect the cosmological evolution *during* inflation, as in the *modulated trapping* scenario [82], which is based on the resonant production of particles during inflation [83] (see also [84, 85] for other recent scenarios based on particle production). If the inflaton is coupled to some particles, whose effective mass becomes zero for a critical value of the inflaton, then there will be a burst of production of these particles when the inflaton crosses the critical value. These particles will be quickly diluted but they will slow down the inflaton. This effect, which increases the number of e-folds until the end of inflation, can depend on a modulaton, for example via the coupling between the inflaton and the particles, and a significant curvature perturbation might be generated (see [82] for details).

### 6.7 "Initial" adiabatic and entropic perturbations

In contrast with single field inflation, multi-field inflation can generate isocurvature "initial" perturbations in the radiation era. Note that this is only a possibility but not a necessity: purely adiabatic initial conditions are perfectly compatible with multi-field scenarios.

The CMB is a powerful way to study isocurvature perturbations because (primordial) adiabatic and isocurvature perturbations produce very distinctive features in the CMB anisotropies. On large angular scales, one can show for instance that [64]

$$\frac{\delta T}{T} \simeq \frac{1}{5}\left(\mathscr{R} - 2S\right). \tag{233}$$

On smaller angular scales, an adiabatic initial perturbation generates a cosine oscillatory mode in the photon-baryon fluid, leading to an acoustic peak at $\ell \simeq 220$ (for a flat universe), whereas a pure isocurvature initial perturbation generates a sine oscillatory mode resulting in a first peak at $\ell \simeq 330$. The unambiguous observation of the first peak at $\ell \simeq 220$ has eliminated the possibility of a dominant isocurvature perturbation. The recent observation by WMAP of the CMB polarization has also confirmed that the initial perturbation is mainly an adiabatic mode. But this does not exclude the presence of a subdominant isocurvature contribution, which could be detected in future high-precision experiments such as Planck.

The combined impact of adiabatic and entropic perturbations crucially depends on their correlation [64, 86]

$$\beta = \frac{\mathscr{P}_{S\mathscr{R}}}{\sqrt{\mathscr{P}_S \mathscr{P}_{\mathscr{R}}}}. \tag{234}$$

Parametrizing the relative amplitude between the two types of perturbations by a coefficient $\alpha$,

$$\frac{\mathscr{P}_S}{\mathscr{P}_{\mathscr{R}}} \equiv \frac{\alpha}{1-\alpha}, \tag{235}$$

the WMAP5 data [87] yield the following constraints on the entropy contribution

$$\beta = 0: \ \alpha_0 < 0.067 \ (95\% \,\mathrm{C.L.}) \qquad \beta = -1: \ \alpha_{-1} < 0.0037 \ (95\% \,\mathrm{C.L.}) \tag{236}$$

in the uncorrelated case ($\beta = 0$) and in the totally anti-correlated case ($\beta = -1$), respectively.

## 7 Primordial non-Gaussianities

One of the most promising probes of the early Universe, which has been investigately very actively in the last few years, is the non-Gaussianity of the primordial perturbations (see [88] for a review, but the field has grown considerably in the last few years). Whereas the simplest models of inflation, based on a slow-rolling single field with standard kinetic term, generate undetectable levels of non-Gaussianity [35, 89], a significant amount of non-Gaussianity can be produced in scenarios with i) non-standard kinetic terms; ii) multiple fields; iii) a non standard vacuum; iv) a non slow-roll evolution. We will discuss in this section the first two possibilities.

### *7.1 Higher order correlation functions*

The most used estimate of non-Gaussianity is the bispectrum defined, in Fourier space, by

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle \equiv (2\pi)^3 \delta^{(3)}(\sum_i \mathbf{k}_i) B_\zeta(k_1, k_2, k_3), \tag{237}$$

where the Fourier modes are defined by

$$\zeta_{\mathbf{k}} = \int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, \zeta(\mathbf{x}). \tag{238}$$

Equivalently, one often uses the so-called $f_{\mathrm{NL}}$ parameter, which can be defined in general by

$$B_\zeta(k_1,k_2,k_3) \equiv \frac{6}{5} f_{\mathrm{NL}}(k_1,k_2,k_3) \left[ P_\zeta(k_1) P_\zeta(k_2) + P_\zeta(k_2) P_\zeta(k_3) + P_\zeta(k_3) P_\zeta(k_1) \right], \tag{239}$$

where $P_\zeta$ is the power spectrum[5] defined by

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) P(k_1)\,. \tag{240}$$

The $f_{\rm NL}$ parameter was initially introduced in [90] for a very specific type of non-Gaussianity characterized by

$$\zeta(\mathbf{x}) = \zeta_G(\mathbf{x}) + \frac{3}{5} f_{\rm NL} \zeta_G^2(\mathbf{x})\,, \tag{241}$$

in the physical space, where $\zeta_G$ is Gaussian and the factor $3/5$ appears because $f_{\rm NL}$ was originally defined with respect to the gravitational potential $\Phi = (3/5)\zeta$, instead of $\zeta$. In this particular case, $f_{\rm NL}$, as defined in (239), is independent of the vectors $\mathbf{k}_i$. In general, $f_{\rm NL}$ is a function of the norm of the three vectors $\mathbf{k}_i$ (which define a triangle in Fourier space since they are constrained by $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$ as a consequence of homogeneity), and the “shape” of the three-point function is an important characterization of how non-Gaussianity was generated [91].

In the context of multi-field inflation, the so-called $\delta N$-formalism [92, 52] is particularly useful to evaluate the primordial non-Gaussianity generated on large scales [93]. The idea is to describe, on scales larger than the Hubble radius, the non-linear evolution of perturbations generated during inflation in terms of the perturbed expansion from an initial flat hypersurface (usually taken at Hubble crossing during inflation) up to a final uniform-density hypersurface (usually during the radiation-dominated era). Using the Taylor expansion of the number of e-folds given as a function of the initial values of the scalar fields,

$$\zeta \simeq \sum_I N_{,I} \delta\varphi_*^I + \frac{1}{2} \sum_{IJ} N_{,IJ} \delta\varphi_*^I \delta\varphi_*^J \tag{242}$$

one finds [93, 94], in Fourier space,

$$\begin{aligned} \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = & \sum_{IJK} N_{,I} N_{,J} N_{,K} \langle \delta\varphi^I_{\mathbf{k}_1} \delta\varphi^J_{\mathbf{k}_2} \delta\varphi^K_{\mathbf{k}_3} \rangle + \\ & \frac{1}{2} \sum_{IJKL} N_{,I} N_{,J} N_{,KL} \langle \delta\varphi^I_{\mathbf{k}_1} \delta\varphi^J_{\mathbf{k}_2} (\delta\varphi^K \star \delta\varphi^L)_{\mathbf{k}_3} \rangle + \text{perms.} \end{aligned} \tag{243}$$

The first term on the right hand side corresponds to non-Gaussianities arising from nonvanishing three-point function(s) of the scalar field(s). This is the case for models with non-standard kinetic terms [95, 96, 97], leading to a specific shape of non-Gaussianities, usually called *equilateral*, where the dominant contribution comes from configurations with three wavevectors of similar length $k_1 \sim k_2 \sim k_3$.

[5] In this section on non-Gaussianities, we have followed the recent literature and adopted the definition (238) for the Fourier modes, which differs slightly from our convention (69) of the previous chapters. This changes the expression of the power spectrum, but the quantiy $\mathscr{P}(k)$ is the same in the two conventions.

The terms appearing in second line of (243) can also lead to sizable non-Gaussianities. Indeed, substituting

$$\langle \delta\varphi^I_{\mathbf{k}_1} \delta\varphi^J_{\mathbf{k}_2} \rangle = (2\pi)^3 \delta_{IJ} \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \frac{2\pi^2}{k_1^3} \mathscr{P}_*(k_1), \qquad \mathscr{P}_*(k) \equiv \frac{H_*^2}{4\pi^2}, \tag{244}$$

in (243), one gets

$$\frac{6}{5} f_{\rm NL} = \frac{N_{,I} N_{,J} N^{,IJ}}{(N_{,K} N^{,K})^2}, \tag{245}$$

where we use Einstein's summation convention for the field indices, which are raised with $\delta^{IJ}$. This corresponds to another type of non-Gaussianity, usually called *local* or *squeezed*, for which the dominant contribution comes from configurations where the three wavevectors form a squeezed triangle.

The present observational constraints [87] are

$$-9 < f_{NL}^{(\rm local)} < 111 \quad (95\%\,{\rm CL}), \qquad -151 < f_{NL}^{(\rm equil)} < 253 \quad (95\%\,{\rm CL}), \tag{246}$$

for the local non-linear coupling parameter and the equilateral non-linear coupling parameter, respectively.

Extending the Taylor expansion (242) up to third order, one can compute in a similar way the trispectrum [98], i.e. the Fourier transform of the connected four-point function defined by

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \zeta_{\mathbf{k}_4} \rangle_c \equiv (2\pi)^3 \delta^{(3)}(\sum_i \mathbf{k}_i) T_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4). \tag{247}$$

Assuming the scalar field perturbations to be quasi-Gaussian, the trispectrum can be written in the form [99]

$$T_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \tau_{\rm NL} \left[ P(k_{13}) P(k_3) P(k_4) + 11 \text{ perms} \right] \tag{248}$$

$$+ \frac{54}{25} g_{\rm NL} \left[ P(k_2) P(k_3) P(k_4) + 3 \text{ perms} \right], \tag{249}$$

with

$$\tau_{\rm NL} = \frac{N_{IJ} N^{IK} N^J N_K}{(N_L N^L)^3}, \qquad g_{\rm NL} = \frac{25}{54} \frac{N_{IJK} N^I N^J N^K}{(N_L N^L)^3} \tag{250}$$

and where $k_{13} \equiv |\mathbf{k_1} + \mathbf{k_3}|$.

## 7.2 A few examples

It is not always easy to obtain significant non-Gaussianities even in inflationary models with several inflatons (see e.g. [100, 101, 94, 102, 103, 104, 105]). Local non-Gaussianity can also be generated at the end of inflation [106, 107, 108]. Below, we discuss in more details the non-Gaussianities generated, first, in the curvaton scenario and, then, in multi-field DBI inflation.

### 7.2.1 Curvaton

In scenarios with a curvaton (or a modulaton), the total number of e-folds can be written as the sum of two contributions: one from the inflaton field $\phi$ and the other from the curvaton/modulaton $\sigma$. In the case of standard slow-roll inflation, the second derivatives with respect to $\phi$ are negligible and (245) reduces to

$$\frac{6}{5} f_{\rm NL} = \frac{N_\sigma^2 N_{\sigma\sigma}}{(N_\phi^2 + N_\sigma^2)^2} = \frac{N_{\sigma\sigma}}{N_\sigma^2 (1+\lambda^{-1})^2}, \tag{251}$$

where we have introduced the parameter $\lambda \equiv N_\sigma^2/N_\phi^2$, which represents the ratio of the contribution of $\sigma$ with that of the inflaton in the power spectrum (see (226) for the curvaton).

For the curvaton, Eq. (224) tell us that $N_\sigma = 2r_\sigma/3\sigma$ and the extension of this equation to second order yields

$$N_{\sigma\sigma} = \frac{4r_\sigma}{9\sigma^2}\left(\frac{3}{2} - 2r_\sigma - r_\sigma^2\right), \tag{252}$$

which leads to a local non-Gaussianity characterized by

$$\frac{6}{5} f_{\rm NL} = \frac{1}{r_\sigma} \frac{\left(\frac{3}{2} - 2r_\sigma - r_\sigma^2\right)}{(1+\lambda^{-1})^2}. \tag{253}$$

Non-Gaussianities are thus significant when the curvaton decays well before it dominates, $r_\sigma \ll 1$. When $\lambda \gg 1$ and the perturbations from inflation are negligible, one recovers the standard curvaton result [77].

Note however that $f_{\rm NL}$ does not grow indefinitely as $r_\sigma$ becomes small because both $r_\sigma$ and $\lambda$ depend on the curvaton expectation value $\sigma_*$. Indeed, substituting $r_\sigma \sim (\sigma_*/M_P)^2/\sqrt{\Gamma_\sigma/m_\sigma}$ (valid in the limit $r \ll 1$), where $\Gamma_\sigma$ is the decay rate of the curvaton, into the definition (226), one sees that $\lambda$ is proportional to $\sigma_*^2$, like $r_\sigma$. One thus finds [109] that the non-linearity parameter reaches its maximal value $f_{\rm NL}(\max) \sim \varepsilon_*/\sqrt{\Gamma_\sigma/m_\sigma}$ for $\lambda \sim 1$, i.e., for $\sigma_* \sim \sqrt{\Gamma_\sigma/(m_\sigma \varepsilon_*)} M_P$. A significant non-Gaussianity is thus possible if $\varepsilon_* \gg \sqrt{\Gamma_\sigma/m_\sigma}$. It is easy to extend the above procedure for the computation of the trispectrum [110].

Moreover, in the curvaton scenarios, isocurvature perturbations can be present. Even if their contribution to the power spectrum is constrained to be small, they could contribute significantly to non-Gaussianities. It is thus interesting to study the non-Gaussianities of isocurvature perturbations as well (see e.g. [111, 112, 113, 114]). Non-Gaussianities in modulaton scenarios have also been investigated (see e.g. [115, 116, 82]).

### 7.2.2 Multi-field DBI inflation

Multi-field DBI inflation is another example where non-Gaussianities have been investigated. In this case, the three-point correlation functions of the scalar fields are not negligible and they can be computed from the third order action, which is given, in the small sound speed limit, by [70, 51]

$$S_{(3)} = \int \mathrm{d}t\, \mathrm{d}^3x \left\{ \frac{a^3}{2c_s^5\dot{\sigma}} \left[ (\dot{Q}_\sigma)^3 + c_s^2 \dot{Q}_\sigma (\dot{Q}_s)^2 \right] - \frac{a}{2c_s^3\dot{\sigma}} \left[ \dot{Q}_\sigma (\partial Q_\sigma)^2 - c_s^2 \dot{Q}_\sigma (\partial Q_s)^2 + 2c_s^2 \dot{Q}_s \partial Q_\sigma \partial Q_s) \right] \right\} \tag{254}$$

in terms of the instantaneous adiabatic and entropic perturbations. The contribution from the scalar field three-point functions to the coefficient $f_{\rm NL}$ is

$$f_{NL}^{(3)} = -\frac{35}{108} \frac{1}{c_s^2} \frac{1}{1+T_{\mathscr{RS}}^2} = -\frac{35}{108} \frac{1}{c_s^2} \cos^2\Theta \tag{255}$$

which is similar to the single-field DBI result [69, 117], but with a suppression due to the transfer between the entropic and adiavatic modes.

In the trispectrum, multi-field effects induce a shape of non-Gaussianities that differs from the single-field case [118]. Moreover, multi-field DBI inflation could also produce a local non-Gaussianity in addition to the equilateral one (see [119] for an explicit illustration).

## 8 Conclusions

As these notes have tried to emphasize, inflation provides an attractive framework to describe the very early Universe and to account for the "initial" seeds of the cosmological perturbations, which we are able to observe today with increasing precision. In particular, the idea that the present structures in the Universe arose from the gravitational amplification of quantum vacuum fluctuations is especially appealing.

At present, inflation is more a general framework than a specific theory and there exists a plethora of models, based on various types of motivation, which can all satisfy the present observational data. The simplest models, based on a slow-rolling single field, produce only adiabatic perturbations, with negligible non-Gaussianities, but with a possibly detectable amount of gravitational waves for the large-field subclass.

More sophisticated models, involving multiple scalar fields or non-standard kinetic terms, can lead to a much richer spectrum of possibilities: isocurvature perturbations that could be correlated with the adiabatic ones, or a detectable level of non-Gaussianities.

Any clear evidence in the future of one or several of these additional features (gravitional waves, isocurvature perturbations and/or primordial non-Gaussianities) would allow us to discriminate between the main species of inflationary models and would thus have a huge impact on our understanding of the early Universe.

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