Observability of the total inflationary expansion

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I consider the question of possible observability of the total number of e-folds accumulated during the epoch of inflation. The total number of observable e-folds has been previously constrained by the de Sitter entropy after inflation, assuming that the null energy condition (NEC) holds. The NEC is violated by upward fluctuations of the local Hubble rate H, which occur with high probability in the fluctuation-dominated regime of inflation. These fluctuations lead at late times to the formation of black holes and thus limit the observability of inflationary evolution. I compute the average number $\langle \Delta N \rangle$ of e-folds accumulated during the last NEC-preserving fragment of the inflationary trajectory before reheating. This is the maximum number of inflationary e-folds that can be observed in principle through measurements of the CMB at arbitrarily late times (if the dark energy disappears). The calculation also provides a reasonably precise definition of the boundary of the fluctuation-dominated regime, with an uncertainty of a few percent. In simple models of single-field inflation compatible with current CMB observations, I find $\langle \Delta N \rangle$ of order 10^5. This upper bound on the observable e-folds, although model-dependent, is much smaller than the de Sitter entropy after inflation. The method of calculation can be used in other models of single-field inflation.

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I. INTRODUCTION AND SUMMARY

Inflation produces primordial metric fluctuations that may be observed indirectly through CMB measurements such as WMAP [1]. An observation of CMB at present corresponds to the measurement of the inflaton evolution about 60 e-folds before reheating [2]. Assuming that CMB measurements will be possible at indefinitely late times, one might hope to deduce information about arbitrarily early stages of inflation. Of course, late-time acceleration (persistent "dark energy") can make it impossible to observe CMB at very late times [3]. There is, however, another limit on our ability to see towards the past. This limit is caused by violations of the null energy condition (NEC) during inflation.

To make the following arguments more specific, let us consider a model of inflation driven by a canonical, minimally coupled scalar field ϕ such that the field evolves from a large initial value ϕ_{in} (perhaps near the Planck boundary ϕ_{Pl}) to the reheating point $\phi = \phi_*$. A typical model of this type has the inflaton action

$$\int d^4x\sqrt{-g}\left(\frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi)\right). \tag{1}$$

We assume that the inflaton potential $V(\phi)$ grows monotonically with ϕ and that the slow-roll approximation is valid. In models of chaotic type, e.g. $V(\phi) \propto \phi^{2n}$, we then expect that $\phi_* \ll \phi_{\text{in}} \lesssim \phi_{\text{Pl}}$, and that ϕ_{in} is deep in the fluctuation-dominated regime.

The evolution of ϕ during inflation can be pictured as a random walk superimposed on a deterministic drift towards $\phi = \phi_*$ [4–6]. The random walk can be modeled as "diffusion" in ϕ space with the diffusion coefficient

$$D(\phi) \equiv \frac{H^3}{8\pi^2}, \quad H(\phi) \equiv \frac{8\pi G}{3}V(\phi) \equiv \frac{8\pi}{3M_{\text{Pl}}^2}V(\phi), \tag{2}$$

while the mean drift velocity is the time derivative $\dot{\phi}$ of the slow-roll evolution,

$$\dot{\phi} = v(\phi) \equiv -\frac{H'}{4\pi G} = -\frac{H'}{4\pi}M_{\text{Pl}}^2. \tag{3}$$

During the last stages of inflation before reheating, the trajectory $\phi(t)$ is monotonic ($\dot{\phi} < 0$) with nearly unit probability, although there is always a small probability of an upward fluctuation ($\dot{\phi} > 0$). On the other hand, in the fluctuation-dominated regime an upward fluctuation of ϕ has around 50% probability because random fluctuations dominate over the slow-roll motion.

We now note that an upward fluctuation of ϕ corresponds to an upward fluctuation of the local Hubble rate H and thus to a local violation of the null energy condition (NEC) [7, 8]. A violation of the NEC due to an upward fluctuation of H on a distance scale $L \sim H^{-1}$ leads to the formation of a Hubble volume that looks like a black hole from the outside [9], and to a real black hole when the overdensity enters the local Hubble horizon at late times after inflation [10, 11]. This can be understood qualitatively by noting that a Hubble-size region of approximately de Sitter spacetime with local Hubble parameter H has exactly the energy density $\Lambda = \frac{3}{8\pi}M_{\text{Pl}}H^2$ that corresponds to a black hole with the Schwarzschild radius H^{-1}. An upward fluctuation of H therefore leads to an increase of the energy density beyond the Schwarzschild limit. In this way, an NEC violation in the far inflationary past will limit the lifetime of any future observers who may be trying to perform CMB observations at very late times.

The main focus of this paper is an investigation of this limiting effect of NEC violations. For the sake of this consideration, I will assume that dark energy eventually decays, so that the late-time universe is not expanding with acceleration and the local Hubble radius grows without limit, permitting (in principle) observations of the primordial density fluctuations on arbitrarily large scales.

It is interesting to determine the time range within which the NEC can be violated during inflation. For each random inflationary trajectory $\phi(t)$ there exists a well-defined time of the *last* NEC violation before reheating, i.e. a time t_{N} such that the NEC is violated around $t = t_{\mathrm{N}}$ but then is not violated any more. The evolution of $\phi(t)$ before $t = t_{\mathrm{N}}$ is thus not observable even in principle. On the other hand, the evolution of ϕ after $t = t_{\mathrm{N}}$ is in principle observable: If primordial fluctuations on a distance scale L are produced at $t > t_{\mathrm{N}}$, these fluctuations will be observed through the CMB fluctuations at sufficiently late times when the scale L reenters the Hubble horizon (we are assuming that the dark energy does not prevent such observations). Therefore, it is only the statistics of the last NEC-preserving[1] segment of the inflaton trajectory $\phi(t)$ that is — even in principle — accessible to observations.

In the following sections we will compute (within an adequate approximation) the mean duration of the last NEC-preserving portion of the trajectory $\phi(t)$ until reheating. When fluctuations are negligible, the field evolves according to the slow-roll equation (3), and so the time needed for evolving from $\phi = \phi_1$ to reheating (taking into account that $\phi_* < \phi_1$) is

$$\Delta t(\phi_1, \phi_*) = \int_{\phi_*}^{\phi_1} \frac{d\phi}{-v(\phi)}. \tag{4}$$

(We write $-v$ because the value of ϕ decreases with time, so $v(\phi) < 0$.) This formula, however, cannot be used directly to compute the duration of the last NEC-preserving portion of the trajectory, for two reasons: First, the NEC-preserving portion of the trajectory depends on chance and is not confined within a fixed interval, say $[\phi_*, \phi_1]$. Second, the initial stages of the trajectory $\phi(t)$ belong to the fluctuation-dominated regime where the evolution $\phi(t)$ is not well described by the deterministic slow-roll equation $\dot\phi = v(\phi)$.

Below we will compute the *average* duration $\langle \Delta t_{\mathrm{NEC}} \rangle$ of the last NEC-preserving portion of the trajectory $\phi(t)$, where the average is performed over the ensemble of all comoving trajectories.[2] Heuristically, we may attempt to determine a value $\phi = \phi_q$ such that the duration of the slow-roll trajectory between $\phi = \phi_q$ and $\phi = \phi_*$ is precisely equal to $\langle \Delta t_{\mathrm{NEC}} \rangle$, i.e. we first compute $\langle \Delta t_{\mathrm{NEC}} \rangle$ and then *define* ϕ_q such that

$$\Delta t(\phi_q, \phi_*) = \int_{\phi_*}^{\phi_q} \frac{d\phi}{-v(\phi)} = \langle \Delta t_{\mathrm{NEC}} \rangle . \tag{5}$$

This value ϕ_q can then be interpreted as the boundary between the fluctuation-dominated and the fluctuation-free regimes. One of the main results of this paper is a method of computing $\langle \Delta t_{\mathrm{NEC}} \rangle$; thus, the boundary ϕ_q between the fluctuation-dominated and the fluctuation-free regimes becomes a well-defined quantity. A precise definition of this boundary is relevant, e.g., for certain measure prescriptions for regulating eternal inflation [13, 14] as well as for attempts to count the observable degrees of freedom after inflation [15, 16].

The boundary between the fluctuation-dominated and the fluctuation-free regimes can be characterized in terms of the dimensionless ratio of $-v\delta t$ (the change of the field ϕ due to slow roll during one Hubble timestep $\delta t \equiv H^{-1}$) to $\sqrt{2D\delta t}$ (the typical fluctuation during the same time),

$$b(\phi) \equiv \frac{-v\delta t}{\sqrt{2D\delta t}} = \frac{-2\pi\dot\phi}{H^2} = \sqrt{\frac{3}{8}\varepsilon_1 \frac{M_{\mathrm{Pl}}^4}{V(\phi)}}, \tag{6}$$

where

$$\varepsilon_1 \equiv \frac{M_{\mathrm{Pl}}^2}{16\pi} \frac{V'^2}{V^2} \tag{7}$$

is the first slow-roll parameter. We note that $b^2(\phi)$ coincides, for inflationary models of the type (1), with the inverse magnitude of the power spectrum of the primordial scalar fluctuation mode that crossed the Hubble scale at that time,

$$P_S \approx \frac{1}{4\pi^2}\left(\frac{H^2}{\dot\phi}\right)^2 = \frac{8}{3\varepsilon_1}\frac{V(\phi)}{M_{\mathrm{Pl}}^4} = \frac{1}{b^2(\phi)}. \tag{8}$$

(In the equation above, we neglected the slow-roll corrections since our final result will only depend logarithmically on P_S.) The fluctuation-free regime is characterized by $b(\phi) \gg 1$ and the fluctuation-dominated regime by $b(\phi) \lesssim 1$ (i.e. by primordial fluctuations of order 1 or larger). However, this qualitative characterization cannot provide a sharply defined boundary value ϕ_q separating the two regimes.

We also note that the order parameter Ω, which was used in Ref. [17] to characterize the transition between the presence and the absence of eternal inflation, is related to b by

$$\Omega \equiv \frac{2\pi^2}{3}\frac{\dot\phi^2}{H^4} = \frac{\pi}{3}b^2. \tag{9}$$

The fluctuation-dominated regime was characterized by the condition $\Omega < 1$ in Ref. [17], which again corresponds qualitatively to $b \lesssim 1$.

It will be shown below that the rigorously computed value $\langle \Delta t_{\mathrm{NEC}} \rangle$ can be approximated as

$$\langle \Delta t_{\mathrm{NEC}} \rangle \approx \int_{\phi_*}^{\phi_{\mathrm{Pl}}} \frac{d\phi}{-v(\phi)} f(0;\phi), \tag{10}$$

[1] I talk about "NEC-preserving" rather than about "monotonically decreasing" trajectories $\phi(t)$ because in models with several fields $(\phi_1, ..., \phi_n)$, an NEC violation does not necessarily entail an upward fluctuation of a particular field $\phi_k(t)$.

[2] Thus we compute the "comoving" average rather than a "volume-weighted" average, which would require more complicated calculations left for future work. See, e.g., Ref. [12] for a review of comoving and volume-weighted averaging prescriptions.

where $f(0;\phi)$ is the probability of the event that an inflationary trajectory $\phi(t)$ starting at $t = 0$ with the given value of ϕ will *never* violate the NEC. It will be found that f is close to being a step function, $f(0;\phi) \approx \theta(\phi_q - \phi)$, effectively cutting the integration at $\phi = \phi_q$. The relevant value of ϕ_q will be determined from an explicit analytic approximation for $f(0;\phi)$.

If we use the number of e-foldings, $N \equiv \ln a$, as the time variable t, the same method yields the average number of e-foldings, $\langle \Delta N_{\rm NEC} \rangle$, during the last NEC-preserving portion of the trajectory before reheating. Below we will compute ϕ_q and $\langle \Delta N_{\rm NEC} \rangle$ explicitly for inflationary models with a power-law potential. In a specific example, we will show that the average number of NEC-preserving e-foldings in the model with $V(\phi) = \frac{1}{2}m^2\phi^2$ is of order 10^5, if one assumes the model parameters that fit the WMAP data. In this model, the value of ϕ_q turns out to be such that $b^2(\phi_q) \approx 14$. Thus, ϕ_q is well outside the fluctuation-dominated regime.

These results can be compared with the upper bound on the e-folds of inflation obtained in Ref. [15]. Assuming that the fluctuations are never dominant (equivalently, that the NEC always holds), it was found that the number of observable e-folds of inflation must be smaller than the entropy S_{dS} of the final de Sitter state *after* inflation. The latter is an extremely large number, of order 10^{120} (if we use the current value of the dark energy density). The present calculation shows that the limit on the number of observable e-folds is much more stringent. Therefore, the number of observable degrees of freedom, if it is expressed through the entropy of the final de Sitter state, is in any case not directly related to the total number of observable e-folds of inflation.

It must be noted that the value $\langle \Delta N_{\rm NEC} \rangle$ is highly model-dependent. In the present paper, we perform computations only for models with $V(\phi) \propto \phi^{2n}$ and derive a formula for $\langle \Delta N_{\rm NEC} \rangle$ [Eq. (90)] that shows a sensitive dependence on n. However, the method of calculation developed in this paper is sufficiently general so that the number of total observable e-folds can be computed in any other model of single-field slow-roll inflation.

II. THE TIME OF THE LAST NEC VIOLATION

The evolution of the inflaton $\phi(t)$ for a model of type (1) is a random process described by the Fokker-Planck (FP) equation (see e.g. [12, 18])

$$\partial_t P(\phi, t) = \hat{L}_\phi P(\phi, t), \tag{11}$$

$$\hat{L}_\phi P \equiv \partial_\phi \left(\partial_\phi (DP) - vP \right), \tag{12}$$

where the coefficients $D(\phi)$ and $v(\phi)$ were defined above. The FP equation is supplemented by appropriate initial and boundary conditions. The initial condition

$$P(\phi, t = 0) = \delta(\phi - \phi_{\rm in}) \tag{13}$$

reflects the initial value of the inflaton field at $t = 0$, while the boundary conditions are imposed at the Planck boundary $\phi = \phi_{\rm Pl}$ (we impose the reflecting boundary condition) and at the reheating boundary $\phi = \phi_*$:

$$\left[\partial_\phi (DP) - vP\right](\phi_{\rm Pl}, t) = 0; \quad \partial_\phi (DP)(\phi_*, t) = 0. \tag{14}$$

These equations describe the "comoving" evolution, i.e. $P(\phi, t)d\phi$ is the (infinitesimal) probability of having the value of the inflaton within the interval $[\phi, \phi + d\phi]$ at a fixed point in space and at time t. The "propagator" (i.e. Green's function) $P(\phi_1, \phi_2, t)$ for the FP equation describes the probability density of reaching the value $\phi = \phi_2$ at time t starting with $\phi = \phi_1$ at time $t = 0$. The propagator is the solution of Eq. (11) with respect to ϕ_2 with the initial condition

$$P(\phi_1, \phi_2, 0) = \delta(\phi_1 - \phi_2) \tag{15}$$

and the same boundary conditions as the FP equation, namely Eq. (14), with respect to ϕ_2. The propagator is a function only of the time interval t since the evolution is invariant under time translations. The probabilistic interpretation of the propagator is that $P(\phi_1, \phi_2, t)d\phi_2$ gives the probability of the field value $\phi(t)$ being within the interval $[\phi_2, \phi_2 + d\phi_2]$ at time t, while $\phi_1 \equiv \phi(0)$ and t are sharply fixed.

A technical complication for the present considerations is that an equation describing the statistics of NEC-preserving evolution cannot be formulated as another FP equation, e.g. with modified coefficients. This is so because the evolution strictly according to the FP equation will violate the NEC at *any* time. Mathematically, FP equations describe a Brownian motion superimposed onto a deterministic motion, while the Brownian motion admits large velocity fluctuations at short time scales: A typical fluctuation $\delta\phi \propto \sqrt{\delta t}$ over a time δt produces a velocity fluctuation $\propto (\delta t)^{-1/2}$, which is unbounded as $\delta t \to 0$. Thus, the description of the inflaton through Brownian motion effectively excludes the possibility that the trajectory $\phi(t)$ is strictly monotonic with $\dot{\phi} < 0$ for *any* finite duration of time. In reality, the mathematical picture of Brownian motion does not hold for $\phi(t)$ at arbitrarily small time scales. The FP equations may be used to describe the evolution of ϕ only on time scales of order $\delta t \sim H^{-1}$ or larger. A formulation of the stochastic evolution restricted to the subset of NEC-preserving trajectories requires an averaging over such time scales.

Therefore, one can formulate a statistical description of the subset of NEC-preserving trajectories only by using an equation that is nonlocal in time on time scales δt, or nonlocal in ϕ on some relevant scale $\delta\phi$. Below we will derive one such equation and obtain its approximate solution. For now, we focus on determining the time $t_{\rm N}$ of the last NEC violation, supposing that the statistical distribution of NEC-preserving trajectories is known.

We consider the ensemble of comoving worldlines with inflaton trajectories $\phi(t)$ starting at $t = 0$ with a fixed value $\phi(0) \equiv \phi_{\rm in}$. The time $t_{\rm N}$ is a random variable whose distribution can be computed as follows. We ask for the probability $\Pr(t_{\rm N} < T)$ of the event $t_{\rm N} < T$, where T is a fixed parameter. The event $t_{\rm N} < T$ means that the NEC holds for $\phi(t)$ after time T and until reheating but may be violated at any earlier time $t < T$. To compute $\Pr(t_{\rm N} < T)$, we split the random trajectory $\phi(t)$ into two stages: The first stage is the evolution from $\phi = \phi_{\rm in}$ at $t = 0$ to some intermediate value ϕ_T at time $t = T$; during this first stage, the NEC may be violated. The second stage is an NEC-preserving evolution from $\phi = \phi_T$ at time T until reheating at $\phi = \phi_*$ at some (random and not fixed) later time $t_* \geq T$. It is clear that the last NEC violation happens before $t = T$ for any trajectory consisting of these two stages, for any ϕ_T and t_*. On the other hand, trajectories with different values of ϕ_T or t_* are mutually exclusive random events. Therefore, we may simply integrate over all allowed values of ϕ_T and t_* in order to compute the probability $\Pr(t_{\rm N} < T)$.

We will now compute the probability of the event that the trajectory $\phi(t)$ has the two stages as just described. Since the first stage is not constrained with respect to the NEC, the evolution proceeds according to the FP equation. The probability of reaching an intermediate value ϕ_T at $t = T$ is thus given by the propagator $P(\phi_{\rm in}, \phi_T, T)d\phi_T$.

The evolution during the second stage needs to be NEC-preserving; in the single-field model we are considering, this is synonymous with the trajectory $\phi(t)$ being monotonic.

Since we are interested in the last NEC-preserving segment of the trajectory $\phi(t)$ before reheating at $\phi = \phi_*$, we need to compute the probability density of reaching a fixed value $\phi = \phi_*$ at an *unknown* time t_*, rather than of reaching an unknown value ϕ_2 at a *fixed* time t. Let us denote by $P_+(\phi_0; t_*)dt_*$ the probability of an NEC-preserving trajectory that starts at $\phi(t = 0) = \phi_0$ and reaches $\phi = \phi_*$ within a time interval $[t_*, t_* + dt_*]$. (In this section, we will treat P and P_+ as known; the necessary computations are postponed to the next sections.)

Now we may express $\Pr(t_{\rm N} < T)$ as the integral over t_* and ϕ_T of the probability density

$$\begin{aligned}
&\Pr(t_{\rm N} < T; t_*, \phi_T)dt_* d\phi_T \\
&\quad = P(\phi_{\rm in}, \phi_T, T)d\phi_T P_+(\phi_T, t_* - T)dt_*, \qquad (16)
\end{aligned}$$

namely,

$$\begin{aligned}
\Pr(t_{\rm N} < T) &= \int_T^\infty dt_* \int d\phi_T \Pr(t_{\rm N} < T; t_*, \phi_T) \\
&= \int_T^\infty dt_* \int d\phi_T \, P(\phi_{\rm in}, \phi_T, T)P_+(\phi_T; t_* - T). \qquad (17)
\end{aligned}$$

Here and below, the omitted range of integration over ϕ_T is from $\phi_{\rm Pl}$ to ϕ_*.

Once the probability $\Pr(t_{\rm N} < T)$ is known, the probability density $p(t_{\rm N})$ will be found from

$$p(T) = \frac{\partial}{\partial T}\Pr(t_{\rm N} < T). \qquad (18)$$

However, we will not proceed to compute $p(t_{\rm N})$ since our focus is on the duration $\Delta t \equiv t_* - t_{\rm N}$ of the last NEC-preserving segment of the trajectory $\phi(t)$. (As discussed above, the quantity Δt is observable in principle, while $t_{\rm N}$ is not observable.)

Let us denote by $p(\Delta t)$ the probability density for Δt; by definition $\Delta t \geq 0$. It is more convenient to compute the generating function

$$g(\lambda) \equiv \int_0^\infty d\tau \, e^{\lambda\tau} p(\tau). \qquad (19)$$

To assure the convergence of this integral, we will use $g(\lambda)$ only with $\lambda \leq 0$. Once this function is known, we can compute the moments of the distribution $p(\Delta t)$. For instance, the mean value of Δt and the dispersion $\sigma_{\Delta t}$ are given by

$$\langle \Delta t_{\rm NEC} \rangle = g'(0) \equiv \left.\frac{\partial g}{\partial \lambda}\right|_{\lambda=0}; \qquad (20)$$

$$\sigma_{\Delta t}^2 \equiv \langle \Delta t_{\rm NEC}^2 \rangle - \langle \Delta t_{\rm NEC} \rangle^2 = g''(0) - g'^2(0). \qquad (21)$$

The physical interpretation of $\langle \Delta t_{\rm NEC} \rangle$ is the mean time spent in the last NEC-preserving segment of the trajectory $\phi(t)$ before reheating, while $\sigma_{\Delta t}$ is the typical deviation from the mean among all the trajectories $\phi(t)$.

In order to compute $g(\lambda)$, we consider the joint probability density of the time of the last NEC violation with the parameters t_* and ϕ_T; this probability density is found using Eq. (16) as

$$\begin{aligned}
p(T; t_*, \phi_T) &\equiv \frac{\partial}{\partial T}\Pr(t_{\rm N} < T; t_*, \phi_T) \\
&= \frac{\partial}{\partial T}\left[P(\phi_{\rm in}, \phi_T, T)P_+(\phi_T, t_* - T)\right]. \qquad (22)
\end{aligned}$$

The value of $g(\lambda)$ equals the average of $e^{\lambda \Delta t} = e^{\lambda(t_* - T)}$ among all trajectories that reheat at $t = t_*$ and contain the last NEC violation at $t = T$. Hence

$$\begin{aligned}
g(\lambda) &= \int d\phi_T \int_0^\infty dT \int_T^\infty dt_* e^{\lambda(t_* - T)} p(T; t_*, \phi_T) \\
&= \int d\phi_T \int_0^\infty dt_* \int_0^{t_*} dT \times \\
&\quad \times e^{\lambda(t_* - T)} \frac{\partial}{\partial T}\left[P(\phi_{\rm in}, \phi_T, T)P_+(\phi_T, t_* - T)\right]. \qquad (23)
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
&\int_0^{t_*} dT e^{\lambda(t_* - T)} \frac{\partial}{\partial T}\left[P(\phi_{\rm in}, \phi_T, T)P_+(\phi_T, t_* - T)\right] \\
&= P(\phi_{\rm in}, \phi_T, t_*)P_+(\phi_T, 0) - e^{\lambda t_*} P(\phi_{\rm in}, \phi_T, 0)P_+(\phi_T, t_*) \\
&\quad + \lambda \int_0^{t_*} dT e^{\lambda(t_* - T)} P(\phi_{\rm in}, \phi_T, T)P_+(\phi_T, t_* - T) \\
&= P(\phi_{\rm in}, \phi_T, t_*)\delta(\phi_T - \phi_*) - e^{\lambda t_*}\delta(\phi_T - \phi_{\rm in})P_+(\phi_T, t_*) \\
&\quad + \lambda \int_0^{t_*} dT e^{\lambda(t_* - T)} P(\phi_{\rm in}, \phi_T, T)P_+(\phi_T, t_* - T).
\end{aligned}$$

Substituting this into Eq. (23) and simplifying, we find

$$\begin{aligned} g(\lambda) =& \int_0^\infty dt_* \big[P(\phi_{\rm in},\phi_*,t_*) - e^{\lambda t_*} P_+(\phi_{\rm in},t_*)\big] \\ &+ \lambda \int d\phi_T \int_0^\infty dT \int_0^\infty d(t_*-T) \times \\ &\times e^{\lambda(t_*-T)} P(\phi_{\rm in},\phi_T,T) P_+(\phi_T,t_*-T) \\ =& 1 - f(\lambda;\phi_{\rm in}) + \lambda \int d\phi_T \Psi(\phi_{\rm in},\phi_T) f(\lambda;\phi_T), \end{aligned} \tag{24}$$

where we defined the auxiliary functions

$$f(\lambda;\phi) \equiv \int_0^\infty d\tau\, e^{\lambda\tau} P_+(\phi,\tau); \tag{25}$$

$$\Psi(\phi_{\rm in},\phi) \equiv \int_0^\infty dT\, P(\phi_{\rm in},\phi,T). \tag{26}$$

So we will not actually need explicit expressions for the full distributions $P(\phi_{\rm in},\phi,T)$ and $P_+(\phi,\tau)$; it suffices to compute the functions f and Ψ.

As shown in Eq. (A11) in Appendix A below, the function $\Psi(\phi_{\rm in},\phi)$ can be approximated (up to slow-roll corrections) for $\phi<\phi_{\rm in}$ by

$$\Psi(\phi_{\rm in},\phi) \approx \frac{1}{-v(\phi)}. \tag{27}$$

The dependence on the value of $\phi_{\rm in}$ was omitted here because it is exponentially small as long as $\phi_{\rm in}$ is within the diffusion-dominated regime. Thus we will omit the dependence on $\phi_{\rm in}$ where appropriate.

The function $f(\lambda;\phi)$ will be computed in Sec. III as

$$f(\lambda;\phi_T) = \exp\left[-\int_{\phi_*}^{\phi_T} W(\lambda;\phi)d\phi\right], \tag{28}$$

where the auxiliary function $W(\lambda;\phi)$ is approximately found as the solution of Eq. (49) below. We can then rewrite Eq. (24) as

$$g(\lambda) = 1 - f(\lambda;\phi_{\rm in}) + \lambda\int_{\phi_*}^{\phi_{\rm Pl}} \frac{d\phi}{-v(\phi)} f(\lambda;\phi). \tag{29}$$

We note that $f(0;\phi_T)$ is interpreted physically as the total probability of never violating the NEC for a trajectory starting at $\phi=\phi_T$. If $\phi_{\rm in}$ is in the diffusion-dominated regime, the probability $f(0;\phi_{\rm in})$ is exponentially small and can be neglected in Eq. (24). It follows that

$$\langle \Delta t_{\rm NEC}\rangle = g'(0) \approx \int_{\phi_*}^{\phi_{\rm Pl}} \frac{d\phi}{-v(\phi)} f(0;\phi). \tag{30}$$

In the rest of the paper we will perform the calculations explicitly and show that the factor $f(0;\phi)$ effectively cuts off the integration at a model-dependent value $\phi=\phi_q$, which is in the regime where the diffusion is already small. A numerical calculation in a specific model of inflation is then given in Sec. IV.

III. DURATION OF NEC-PRESERVING TRAJECTORIES

It is necessary for our purposes to compute the function in Eq. (28), which we denoted by f:

$$f(\lambda;\phi_0) \equiv \int_0^\infty d\tau\, e^{\lambda\tau} P_+(\phi_0,\tau). \tag{31}$$

This is the generating function of the duration τ of NEC-preserving trajectories starting at a given value $\phi=\phi_0$ at time $t=0$ and finishing at $\phi=\phi_*$ at an unknown time $t=\tau$. For instance, the mean duration of time until reheating among all the NEC-*preserving* trajectories starting at $\phi=\phi_0$ is given by

$$\langle\tau\rangle = \frac{\partial}{\partial\lambda}\ln f(\lambda;\phi_0)\Big|_{\lambda=0}. \tag{32}$$

As discussed above, we expect that the function $f(\lambda;\phi_0)$ satisfies an equation nonlocal in ϕ_0. To derive this equation, we consider the change $\delta\phi=\phi(\delta t)-\phi(0)$ of the (spatially coarse-grained) value of ϕ after a single Hubble time step $\delta t\equiv H^{-1}$ at a given comoving point in space,

$$\delta\phi(\phi,\xi) \equiv v(\phi)\delta t + \xi\sqrt{2D(\phi)\delta t}, \tag{33}$$

where ξ is a normally distributed random variable. We denote for convenience by $p(\xi)$ the probability density of ξ,

$$p(\xi) \equiv \frac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{1}{2}\xi^2\right). \tag{34}$$

The NEC-preserving property at the presently considered Hubble time step is equivalent to the condition $\delta\phi<0$ or

$$\xi < b(\phi) \equiv \frac{-v(\phi)\delta t}{\sqrt{2D(\phi)\delta t}} = \frac{H' M_{\rm Pl}^2}{2H^2}. \tag{35}$$

(Note that the quantity b is always positive since $H'=dH/d\phi>0$ due to the assumption $dV/d\phi>0$.) Since the probability $P_+(\phi_0,\tau)$ includes only trajectories that preserve NEC throughout their evolution, we must include only values of ξ such that $\xi<b(\phi_0)$ when we describe the Hubble time step leading from ϕ_0 to $\phi_0+\delta\phi$. So we may express $P_+(\phi_0,\tau)$ through $P_+(\phi_0+\delta\phi,\tau-\delta t)$ as

$$P_+(\phi_0;\tau) = \int_{-\infty}^{b(\phi_0)} d\xi\, p(\xi)\, P_+(\phi_0+\delta\phi,\tau-\delta t). \tag{36}$$

Here $\delta\phi\equiv\delta\phi(\phi_0,\xi)$ under the integral is understood as a function of ξ. Using Eq. (31), we may now express the value $f(\lambda;\phi_0)$ through the values of f at the next Hubble step as follows. We first integrate Eq. (36) with $e^{\lambda\tau}d\tau$ from $\tau=\delta t$ to infinity and then exchange the order of integrals and shift the integration variable τ by δt:

$$\begin{aligned}
&\int_{\delta t}^{\infty} d\tau\, e^{\lambda\tau} P_+(\phi_0,\tau) \\
&=\int_{\delta t}^{\infty} d\tau\, e^{\lambda\tau}\int_{-\infty}^{b(\phi_0)} d\xi\, p(\xi)\, P_+(\phi_0+\delta\phi,\tau-\delta t) \\
&=\int_{-\infty}^{b(\phi_0)} d\xi\, p(\xi)\int_{0}^{\infty} d\tau\, e^{\lambda(\tau+\delta\tau)}\, P_+(\phi_0+\delta\phi,\tau) \\
&=\int_{-\infty}^{b(\phi_0)} d\xi\, p(\xi)\, e^{\lambda\delta t} f(\lambda;\phi_0+\delta\phi). \qquad (37)
\end{aligned}$$

Note that the top line in Eq. (37) is slightly different from the definition of $f(\lambda;\phi_0)$: The integration proceeds from $\tau=\delta t$ rather than from $\tau=0$ in order to allow the subtraction $\tau-\delta t$ in the argument of P_+. The difference,

$$\int_0^{\delta t} d\tau\, P_+(\phi_0,\tau) \approx \delta t\, P_+(\phi_0,\delta t), \qquad (38)$$

is negligible as long as ϕ_0 is at least a few e-foldings away from reheating. This is so because $P_+(\phi_0,\delta t)$ is equal to the (exponentially small) probability of jumping from $\phi=\phi_0$ directly to $\phi=\phi_*$ in one Hubble time δt. Therefore, we may replace the top line in Eq. (37) by $f(\lambda;\phi_0)$ and finally obtain the equation

$$f(\lambda;\phi_0)=\int_{-\infty}^{b(\phi_0)} d\xi\, p(\xi)\, e^{\lambda\delta t} f(\lambda;\phi_0+\delta\phi(\phi_0,\xi)). \quad (39)$$

This is the basic equation describing the function $f(\lambda;\phi_0)$; as expected, it is nonlocal in ϕ.

It is not possible to approximate Eq. (39) by a diffusion equation (as is the normal procedure while deriving FP equations) because the integration in Eq. (39) proceeds over a ϕ-dependent range. Rather than trying to solve Eq. (39) directly, we will approximate the solution of Eq. (39) by an adiabatic ansatz [Eq. (46) below].

Up to now we have been using the proper time as the variable t. If a different time parameterization is desired, such as

$$\tilde{t}=\int^t A(\phi(t))dt, \qquad (40)$$

where $A(\phi)$ is a known function, then the coefficients D, v, and δt must be modified as follows,

$$\tilde{v}=\frac{v}{A},\quad \tilde{D}=\frac{D}{A},\quad \delta\tilde{t}=A\delta t, \qquad (41)$$

while the dimensionless coefficient $b(\phi)$ is unchanged. For instance, passing to the e-folding time

$$N=\int^t H dt \qquad (42)$$

is implemented by choosing $A(\phi)=H(\phi)$. Below we will compute $\langle\Delta N_{\mathrm{NEC}}\rangle$ in a specific model of inflation by using this method.

At the end of the calculation, we will only need to evaluate $f(\lambda=0;\phi_0)$. As already mentioned above, $f(0;\phi_0)$ is the fraction of trajectories that never violate the NEC among all trajectories $\phi(t)$ starting at $\phi=\phi_0$. We note that for ϕ_0 in the fluctuation-dominated regime, the probability $f(0;\phi_0)$ rapidly decreases with growing ϕ_0 because there is a significant probability of violating the NEC at every Hubble time step at those ϕ_0. On the other hand, the probability of violating the NEC in the no-diffusion regime is exponentially small, and hence $f(0;\phi_0)$ is nearly constant and almost equal to 1 for ϕ_0 in that regime. Therefore, we expect that $f(\lambda;\phi_0)$ has exponentially strong dependence on ϕ_0. Moreover, $P_+(\phi_*,\tau)=\delta(\tau)$; this can be shown by considering

$$\int_0^\infty P_+(\phi_*,\tau)d\tau=1, \qquad (43)$$

$$P_+(\phi_*,\tau)=0 \text{ for } \tau>0, \qquad (44)$$

which holds because trajectories starting with $\phi=\phi_*$ immediately reheat and have zero duration. Therefore

$$f(\lambda;\phi_*)=\int_0^\infty e^{\lambda\tau}P_+(\phi_*,\tau)d\tau=1. \qquad (45)$$

Motivated by these considerations, we represent the exponential behavior of $f(\lambda;\phi_0)$ and the boundary condition (45) by the ansatz

$$f(\lambda;\phi_0)=\exp\left[-\int_{\phi_*}^{\phi_0} W(\lambda;\phi)d\phi\right], \qquad (46)$$

where $W(\lambda;\phi)$ is a new unknown function such that $W(0;\phi)>0$. We then divide Eq. (39) through by $f(\lambda;\phi_0)$ and expand $f(\lambda;\phi+\delta\phi)$ to first order in $\delta\phi$:

$$\begin{aligned}
1&=\int_{-\infty}^{b(\phi_0)} d\xi\, p(\xi)\, e^{\lambda\delta t}\exp\left[-\int_{\phi_0}^{\phi_0+\delta\phi} W(\lambda;\phi)d\phi\right] \\
&\approx\int_{-\infty}^{b(\phi_0)} d\xi\, p(\xi)\, e^{\lambda\delta t}\exp\left[-W(\lambda;\phi_0)\delta\phi(\phi_0,\xi)\right] \\
&=e^{(\lambda-vW)\delta t}\int_{-\infty}^{b}\frac{d\xi}{\sqrt{2\pi}}\exp\left[-\frac{\xi^2}{2}-\sqrt{2D\delta t}W\xi\right] \\
&=e^{\left(\lambda-vW+DW^2\right)\delta t}\left[\frac{1}{2}+\frac{1}{2}\mathrm{erf}\left(\frac{b+W\sqrt{2D\delta t}}{\sqrt{2}}\right)\right],
\end{aligned} \qquad (47)$$

where we suppressed the argument ϕ_0 in the last line, $b\equiv b(\phi)$ was defined in Eq. (35), while $\mathrm{erf}\,x$ is the standard error function

$$\mathrm{erf}\,x\equiv\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \qquad (48)$$

We now note that Eq. (47) does not contain derivatives of W; this means that we are using an adiabatic approximation where $W(\lambda;\phi)$ is assumed to vary slowly with ϕ.

Thus $W(\lambda;\phi)$ is the unique real root of the transcendental equation

$$\exp\left[\left(vW - DW^2 - \lambda\right)\delta t\right] = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left[\frac{b + W\sqrt{2D\delta t}}{\sqrt{2}}\right] \tag{49}$$

such that $W(\lambda;\phi) > 0$ for $\lambda = 0$.

The coefficient $b(\phi)$ measures the influence of quantum fluctuations on the evolution $\phi(t)$. It is possible to obtain approximate solutions of Eq. (49) in the cases $b \gg 1$ (a nearly fluctuation-free regime) and $b \ll 1$ (a fluctuation-dominated regime). To simplify calculations, we pass to a new dimensionless variable r by rewriting Eq. (49) as

$$\frac{r^2}{2} + \ln\left[\frac{1}{2} + \frac{1}{2}\mathrm{erf}\frac{r}{\sqrt{2}}\right] \equiv L(r) = \frac{b^2}{2} - \lambda\delta t, \tag{50}$$

where $r(\lambda;\phi)$ is related to $W(\lambda;\phi)$ by

$$W(\lambda;\phi) \equiv \frac{r(\lambda;\phi) - b}{\sqrt{2D\delta t}}. \tag{51}$$

We solve Eq. (50) by using the inverse function $L^{-1}(x)$,

$$r = L^{-1}(\tfrac{1}{2}b^2 - \lambda\delta t), \tag{52}$$

which then gives the solution $W(\lambda;\phi)$ through Eq. (51) as

$$W(\lambda;\phi) = \frac{L^{-1}(\frac{1}{2}b^2 - \lambda\delta t) - b}{\sqrt{2D\delta t}}. \tag{53}$$

Derivatives of $W(\lambda;\phi)$ with respect to λ can be expressed through $W(\lambda;\phi)$ by computing the derivative of L^{-1},

$$L'(r) = r + \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{1}{2}r^2}}{1 + \mathrm{erf}\frac{r}{\sqrt{2}}} = r + \frac{e^{-L(r)}}{\sqrt{2\pi}}, \tag{54}$$

$$\frac{\partial}{\partial x}\left[L^{-1}(x)\right] = \frac{1}{L'[L^{-1}(x)]} = \left[L^{-1}(x) + \frac{e^{-x}}{\sqrt{2\pi}}\right]^{-1}, \tag{55}$$

and using Eq. (53). For instance, we find

$$\begin{aligned}\frac{\partial}{\partial\lambda}W &= -\frac{\delta t}{\sqrt{2D\delta t}}\frac{1}{L'[L^{-1}(\frac{1}{2}b^2 - \lambda\delta t)]} \\ &= \frac{1}{v}\left[1 - \frac{2D}{v}W + \frac{e^{-\frac{1}{2}b^2 + \lambda\delta t}}{b\sqrt{2\pi}}\right]^{-1}.\end{aligned} \tag{56}$$

Further derivatives with respect to λ can be obtained similarly. Since we will ultimately compute the generating function $g(\lambda)$ and its derivatives only at $\lambda = 0$, it is sufficient to set $\lambda = 0$ in what follows.

We also note that the function $L(r)$ is "universal" in the sense that its definition does not depend on the inflaton potential $V(\phi)$. It is therefore useful to approximate the inverse function $L^{-1}(x)$ semi-numerically. We first consider the asymptotic behavior of $L^{-1}(x)$ for large x. Using the well-known asymptotic representation of the error function,

$$\mathrm{erf}\, x = 1 - \frac{1}{x\sqrt{\pi}}e^{-x^2}\left(1 + O(x^{-2})\right), \quad x \to +\infty, \tag{57}$$

we obtain

$$L(r) = \frac{r^2}{2} - \frac{\exp\left(-\frac{1}{2}r^2\right)}{r\sqrt{2\pi}}\left[1 + O(r^{-2})\right], \; r \to +\infty, \tag{58}$$

and hence

$$L^{-1}(x) = \sqrt{2x} + \frac{e^{-x}}{2x\sqrt{2\pi}}\left[1 + O(x^{-1})\right], \; x \to +\infty. \tag{59}$$

This asymptotic formula allows us to obtain the approximate solution $W(0;\phi)$ for the case $b \gg 1$ as

$$W(0;\phi) = \frac{L^{-1}(\frac{1}{2}b^2) - b}{\sqrt{2D\delta t}} \approx \frac{1 + O(b^{-2})}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}b^2}}{b^2\sqrt{2D\delta t}}. \tag{60}$$

The solution in the opposite regime $b \ll 1$ can be found by starting with the numerically obtained value

$$L^{-1}(0) \approx 0.7286 \equiv r_0 \tag{61}$$

and by expanding $L^{-1}(x)$ near $x = 0$,

$$\begin{aligned}L^{-1}(x) &= r_0 + x\left.\frac{\partial}{\partial x}\right|_{x=0}\left[L^{-1}\right] + O(x^2) \\ &= r_0 + x\left[r_0 + \frac{1}{\sqrt{2\pi}}\right]^{-1} + O(x^2), \; x \to 0.\end{aligned} \tag{62}$$

Hence for $b \ll 1$ we have

$$W(0;\phi) = \frac{r_0 - b + \frac{1}{2}b^2\left[r_0 + \frac{1}{\sqrt{2\pi}}\right]^{-1} + O(b^4)}{\sqrt{2D\delta t}}, \; b \to 0. \tag{63}$$

We have thus obtained the solution $W(0;\phi)$ in the two opposite regimes. An approximation that holds uniformly for all positive b can be obtained, if desired, by matching the asymptotic expressions near $b = 0$ and $b = \infty$, for instance, using the following interpolating function,

$$W(0;\phi) \approx \frac{e^{-\frac{1}{2}b^2}}{\sqrt{2D\delta t}}\frac{b + 0.7194}{b^3\sqrt{2\pi} + 1.803b^2 + 2.728b + 0.9874}. \tag{64}$$

Numerical verification shows that this function approximates $W(0,\phi)$ to within about 2.5% relative precision for all $b > 0$. (We note that W is model-independent only as a function of b and $\sqrt{2D\delta t}$, while $b(\phi)$, $D(\phi)$, and $\delta t \equiv H^{-1}(\phi)$ of course depend on the chosen model of inflation.)

However, it turns out that the approximation in Eq. (60), which holds in the fluctuation-free regime, is sufficient for our present purposes. Let us derive the corresponding approximation for the function $f(0;\phi_0)$,

$$f(0;\phi_0)=\exp\left[-\int_{\phi_*}^{\phi_0}W(0;\phi)d\phi\right], \tag{65}$$

assuming that ϕ_0 is such that $b^2(\phi_0)\gg 1$. Since $W(0;\phi)$ is quickly growing with ϕ, the integral under the exponential above is dominated by the upper limit, so we can use the asymptotic estimate

$$\int_{\phi_*}^{\phi_0}W(0;\phi)d\phi\approx\int_{\phi_*}^{\phi_0}\frac{2\pi}{H}d\phi\frac{e^{-\frac{1}{2}b^2}}{b^2\sqrt{2\pi}}\left[1+O(b^{-2})\right] \\ \approx\frac{\sqrt{2\pi}}{H}\frac{e^{-\frac{1}{2}b^2}}{b^3}\left[-\frac{\partial b}{\partial\phi}\right]^{-1}\Bigg|_{\phi=\phi_0}. \tag{66}$$

In deriving this estimate, we neglected terms of order b^{-2} as well as derivatives of H and b, since these are merely slow-roll corrections.

For ϕ_0 near reheating, we have $f(0;\phi_0)\approx 1$ with exponential precision. The value of ϕ_q at which $f(0;\phi_q)$ first drops to $\exp(-1)$ can then be found as the solution of the equation

$$\frac{\sqrt{2\pi}}{H(\phi_q)}\frac{e^{-\frac{1}{2}b_q^2}}{b_q^3}\left[-\frac{\partial b}{\partial\phi}\right]^{-1}_{\phi=\phi_q}=1, \tag{67}$$

where we need to substitute $b_q\equiv b(\phi_q)$. This can be interpreted as a closed-form equation for b_q if we express $\partial b/\partial\phi$ and $H(\phi_q)$ as functions of b_q. A numerical calculation needs to be performed to solve this equation for b_q in a particular inflationary model and to check that the resulting value of b_q satisfies $b_q^2\gg 1$, which is required for the validity of the approximation used to derive Eq. (67). (For instance, the calculations in the next section show that $b_q^2\approx 14\gg 1$ for the inflationary model with the potential $V(\phi)\propto\phi^2$.)

We have thus determined ϕ_q such that $f(0;\phi_q)=e^{-1}$. The function $f(0;\phi)$ has a sharp dependence on ϕ and interpolates from 1 to 0 within a narrow interval around $\phi=\phi_q$. To estimate the width of this interval, let us find the value $\phi_q^{(2)}$ such that $f(0;\phi_q^{(2)})=e^{-2}$. This value can be determined as a solution of

$$\frac{\sqrt{2\pi}}{H(\phi_q^{(2)})}\frac{e^{-\frac{1}{2}b^2}}{b^3}\left[-\frac{\partial b}{\partial\phi}\right]^{-1}_{\phi=\phi_q^{(2)}}=2. \tag{68}$$

Then the width of the interval in b_q can be estimated as $\delta b_q=b_q^{(2)}-b_q$. Since the exponential above is the fastest-varying function of b, to first approximation we have

$$b^2(\phi_q^{(2)})\approx b^2(\phi_q)-2\ln 2. \tag{69}$$

(Numerical calculations show that this is an overestimate of δb by about 20%.) This leads to a change in the value of ϕ_q that can be computed through

$$\frac{\delta\phi_q}{\phi_q}\approx\frac{\delta b}{b}\left[\frac{\partial\ln b}{\partial\ln\phi}\right]^{-1},\quad\frac{\delta b}{b}\approx -b^{-2}\ln 2. \tag{70}$$

Since $b^{-2}\ll 1$ while the logarithmic derivative $\partial\ln b/\partial\ln\phi$ is not large, we find that $\delta\phi_q/\phi_q\ll 1$. Hence, the function $f(0;\phi_0)$ has the effect of a cutoff near $\phi=\phi_q$ when integrated with a slowly-varying function of ϕ such as $1/v(\phi)$, as required for Eq. (30).

Below we will replace integrations with the factor $f(0;\phi)$ by integrations with the upper limit $\phi=\phi_q$. This approximation introduces a certain error; to estimate the effect of this error on the calculation of $\langle\Delta N_{\rm NEC}\rangle$, let us find the number of e-folds in the slow-roll trajectory between $\phi=\phi_q^{(2)}$ and $\phi=\phi_q$:

$$\delta N_q=\int_{\phi_q}^{\phi_q^{(2)}}\frac{H(\phi)d\phi}{-v(\phi)}\approx\frac{H(\phi_q)\delta\phi_q}{-v(\phi_q)} \\ \approx\frac{H\delta b}{-bv}\left[\frac{d\ln b}{d\phi}\right]^{-1}\Bigg|_{\phi=\phi_q}=\frac{H\ln 2}{b^2v}\left[\frac{d\ln b}{d\phi}\right]^{-1}\Bigg|_{\phi=\phi_q}. \tag{71}$$

To estimate $d\ln b/d\phi$, we use Eqs. (6) and find

$$\frac{d\ln b}{d\phi}=\frac{v'}{v}-\frac{1}{2}\frac{D'}{D}-\frac{1}{2}\frac{H'}{H}=-\frac{\varepsilon_2\sqrt{\pi}}{M_{\rm Pl}\sqrt{\varepsilon_1}}, \tag{72}$$

where ε_2 is the second slow-roll parameter,

$$\varepsilon_2\equiv\frac{M_{\rm Pl}^2}{4\pi}\left(\frac{V'^2}{V^2}-\frac{V''}{V}\right). \tag{73}$$

We also have

$$\frac{-v}{H}=\frac{\sqrt{\pi}}{2}M_{\rm Pl}\sqrt{\varepsilon_1}. \tag{74}$$

Therefore we obtain the estimate

$$\delta N_q=\frac{\ln 4}{\pi b^2\varepsilon_2}\Bigg|_{\phi=\phi_q}. \tag{75}$$

This estimate is important because it displays the error inherent in the definition of the boundary of the fluctuation-dominated regime. Below we will check that this error is acceptable when determining the average number of observable e-folds.

Let us summarize the calculations presented so far. We have derived an estimate of the mean time $\langle\Delta t_{\rm NEC}\rangle$ of the last NEC-preserving portion of the inflationary trajectory:

$$\langle\Delta t_{\rm NEC}\rangle\approx\int_{\phi_*}^{\phi_{\rm Pl}}\frac{d\phi}{-v(\phi)}f(0;\phi)\approx\int_{\phi_*}^{\phi_q}\frac{d\phi}{-v(\phi)}, \tag{76}$$

where the value of ϕ_q is determined from Eq. (67). We have introduced a simple approximation where the function $f(0;\phi)$ is replaced by a step cut-off at $\phi=\phi_q$; the error of this approximation is expected to be small. It is possible, in principle, to determine the function $f(0;\phi)$ numerically and thus to obtain a sharper estimate, as well as to compute the standard deviation $\sigma_{\Delta N}$ of the number of observable e-folds using Eq. (21). However, we expect that the standard deviation $\sigma_{\Delta N}$ will not be larger than the width of the function $f(0;\phi)$ around the point $\phi=\phi_q$, which is of order $\delta\phi_q$ as estimated above. Therefore, it will be sufficient for the present purposes to use the estimated width of the function $f(0;\phi)$ as the statistical uncertainty in $\langle\Delta t_{\mathrm{NEC}}\rangle$.

IV. EXAMPLE: INFLATION WITH A POWER-LAW POTENTIAL

We now perform specific calculations of ϕ_q and the average number of NEC-preserving e-folds, $\langle\Delta N_{\mathrm{NEC}}\rangle$, for a model of single-field inflation of type (1) with the potential

$$V(\phi)=\lambda M_{\mathrm{Pl}}^4\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{2n}. \tag{77}$$

This model can fit the current observations when $n=1$ or $n=2$ (see, e.g., [19]). We use the formalism developed in the previous sections for computing $\langle\Delta t_{\mathrm{NEC}}\rangle$, except that we divide δt, $v(\phi)$, and $D(\phi)$ in every formula by the factor $H(\phi)$ in order to pass from the proper time t to the e-folding time N.

For this model, we find in the slow-roll approximation

$$H(\phi)=\sqrt{\frac{8\pi}{3}}M_{\mathrm{Pl}}\sqrt{\lambda}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{n}, \tag{78}$$

$$v(\phi)=-\frac{M_{\mathrm{Pl}}^2}{4\pi}\sqrt{\frac{8\pi}{3}}n\sqrt{\lambda}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{n-1}, \tag{79}$$

$$b(\phi)=-\frac{2\pi v}{H^2}=\frac{n}{\sqrt{\lambda}}\sqrt{\frac{3}{32\pi}}\left(\frac{M_{\mathrm{Pl}}}{\phi}\right)^{n+1}. \tag{80}$$

The slow-roll parameters (computed as functions of ϕ through the potential V) are

$$\varepsilon_1\equiv\frac{M_{\mathrm{Pl}}^2}{16\pi}\frac{V'^2}{V^2}=\frac{n^2}{4\pi}\frac{M_{\mathrm{Pl}}^2}{\phi^2}; \tag{81}$$

$$\varepsilon_2\equiv\frac{M_{\mathrm{Pl}}^2}{4\pi}\left(\frac{V'^2}{V^2}-\frac{V''}{V}\right)=\frac{n}{2\pi}\frac{M_{\mathrm{Pl}}^2}{\phi^2}. \tag{82}$$

Reheating is assumed to happen at $\phi=\phi_*$ with $\varepsilon_1(\phi_*)=1$, which gives[3]

$$\phi_*=M_{\mathrm{Pl}}\frac{n}{\sqrt{4\pi}}. \tag{83}$$

[3] The value of ϕ_* is only an estimate because it is computed in the slow-roll approximation, which does not hold near reheating. However, our results are not sensitive to the precise value of ϕ_*.

The number N_e of inflationary e-foldings accumulated between some value $\phi=\phi_1$ until reheating is estimated (assuming $\phi_1\gg\phi_*$) as

$$N_e(\phi_1)=\int_{\phi_*}^{\phi_1}\frac{H(\phi)d\phi}{-v(\phi)}\approx\frac{2\pi\phi_1^2}{nM_{\mathrm{Pl}}^2}. \tag{84}$$

The squared amplitude of scalar primordial perturbations generated at $\phi=\phi_1$ is given by WMAP observations as $P_S\approx 2.3\cdot 10^{-9}$ (we use the data from Ref. [19]). Assuming that this amplitude is generated at N_e e-foldings before reheating (below we will set $N_e\approx 60$), we find

$$\phi_1=M_{\mathrm{Pl}}\sqrt{\frac{nN_e}{2\pi}}\gg\phi_*, \tag{85}$$

and then, using Eq. (8) with $\phi=\phi_1$, we get

$$\lambda=\frac{3}{32\pi}n^2P_S\left(\frac{2\pi}{nN_e}\right)^{n+1}\ll 1. \tag{86}$$

Substituting this value of λ and Eqs. (78)–(80) into Eq. (67), we can obtain an implicit equation for the value ϕ_q. However, it is more convenient to express ϕ_q through $b_q\equiv b(\phi_q)$ using Eqs. (80) and (86),

$$\frac{M_{\mathrm{Pl}}}{\phi_q}=P_S^{\frac{1}{2(n+1)}}\sqrt{\frac{2\pi}{nN_e}}b_q^{\frac{1}{n+1}}, \tag{87}$$

and to derive a closed-form equation for b_q^2,

$$b_q^2=\frac{2}{n+1}\ln\frac{1}{P_S}+2\ln N_e-2\ln\left[\pi\left(n+1\right)b_q^{\frac{3n+5}{n+1}}\right]. \tag{88}$$

This equation is in a form that can be solved numerically by direct iteration. We use the values $P_S=2.3\cdot 10^{-9}$ and $N_e=60$, while n can be 1 or 2 [19], and obtain

$$b_q^2\approx 13.9,\qquad n=1;$$
$$b_q^2\approx 8.9,\qquad n=2.$$

A rough order-of-magnitude analytic expression for b_q can be obtained by using only the first term in Eq. (88),

$$b_q^2\approx\frac{2}{n+1}\ln\frac{1}{P_S}\approx\frac{40}{n+1}. \tag{89}$$

However, this overestimates b_q^2 by about 50%.

Let us now compute the average number of e-foldings during the last NEC-preserving part of the trajectory, for potentials $V(\phi)\propto\phi^{2n}$. According to the results of the previous section, we need to integrate the e-foldings until the value ϕ_q. Using Eq. (87), we find

$$\langle\Delta N_{\mathrm{NEC}}\rangle\approx\int_{\phi_*}^{\phi_q}d\phi\frac{H(\phi)}{-v(\phi)}=\frac{2\pi}{n}\frac{\phi_q^2}{M_{\mathrm{Pl}}^2}$$
$$\approx P_S^{-\frac{1}{n+1}}N_eb_q^{-\frac{2}{n+1}} \tag{90}$$
$$=O(1)P_S^{-\frac{1}{n+1}}N_e\left(\frac{2\ln P_S^{-1}}{n+1}\right)^{-\frac{1}{n+1}}. \tag{91}$$

In the last line, we substituted for b_q the rough estimate (89) merely in order to obtain a simpler analytic expression for $\langle \Delta N_{\rm NEC}\rangle$ as an explicit function of the parameters. We use the more precise Eq. (90) for computing the numerical answers. With the value $n = 1$, which is preferred by observations, we find

$$\langle \Delta N_{\rm NEC}\rangle \approx 3.4 \cdot 10^5. \qquad (92)$$

With $n = 2$, we get $\langle \Delta N_{\rm NEC}\rangle \approx 2.2 \cdot 10^4$.

A curious coincidence is that Eq. (90) can be expressed through the slow-roll parameter ε_2 very simply as

$$\langle \Delta N_{\rm NEC}\rangle \approx \frac{1}{\varepsilon_2(\phi_q)}. \qquad (93)$$

Here we need to compute $\varepsilon_2(\phi)$ at the value ϕ_q determined through b_q as $b(\phi_q) \equiv b_q$.

The value of ϕ_q that corresponds to the obtained value of b_q can be expressed as

$$\frac{\phi_q}{M_{\rm Pl}} = P_S^{-\frac{1}{n+1}} \frac{nN_e}{2\pi} b_q^{-\frac{2}{n+1}} = \frac{n}{2\pi} \langle \Delta N_{\rm NEC}\rangle. \qquad (94)$$

Thus, for $n = 1$ we have $\phi_q \approx 5.4 \cdot 10^4 M_{\rm Pl}$, and for $n = 2$ we have $\phi_q \approx 7.0 \cdot 10^3 M_{\rm Pl}$.

We also need to check whether the intrinsic error of the present approximation, as given by Eq. (75), is small in comparison with the mean value $\langle \Delta N_{\rm NEC}\rangle$. We get

$$\delta N_q = \frac{\ln 4}{\pi b_q^2 \varepsilon_2(\phi_q)} = \frac{\ln 4}{\pi b_q^2} \langle \Delta N_{\rm NEC}\rangle. \qquad (95)$$

With the numerical values used above, we find $\delta N_q \sim 10^4$ for $n = 1$ and $\delta N_q \sim 10^3$ for $n = 2$. The relative error of the approximation is given by $\ln 4/(\pi b_q^2)$ and is about 3% for $n = 1$ and about 5% for $n = 2$, which is acceptable for the purpose of our estimates.

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Appendix A: Solving the stationary FP equation

The time integral of the propagator of the FP equation,

$$\Psi(\phi_{\rm in}, \phi_T) \equiv \int_0^\infty dT\, P(\phi_{\rm in}, \phi_T, T), \qquad (A1)$$

can be computed in closed form in one-field models [20]. One integrates Eq. (11) in time and finds

$$\begin{aligned}\hat{L}_\phi \int_0^\infty P(\phi_0, \phi, t) dt &= \int_0^\infty \partial_t P(\phi_0, \phi, t) dt \\ &= P(\phi, \infty) - P(\phi, 0) = -\delta(\phi - \phi_0)\end{aligned} \qquad (A2)$$

since $P(\phi, \infty) = 0$. Hence, the function $\Psi(\phi_0, \phi)$ satisfies the equation

$$\hat{L}_\phi \Psi = -\delta(\phi - \phi_0) \qquad (A3)$$

with the same boundary conditions in ϕ as the distribution $P(\phi_0, \phi, t)$. In other words, Ψ is the Green's function of the stationary FP equation. The function Ψ can be computed by integrating Eq. (A3). First, one obtains

$$\partial_\phi (D\Psi) - v\Psi + \theta(\phi - \phi_0) + C_1 = 0, \qquad (A4)$$

where the integration constant C_1 is expressed using the boundary condition (14) at $\phi = \phi_{\rm Pl}$ (assuming $\phi_0 > \phi_*$) as

$$C_1 = -\theta(\phi_{\rm Pl} - \phi_0) = -1. \qquad (A5)$$

Finally, Eq. (A4) can be integrated again; the general solution can be written as

$$\Psi(\phi_0, \phi) = \frac{1}{D(\phi)\mu(\phi)} \left[C_2 + \int_{\phi_*}^{\phi} \theta(\phi_0 - \phi_1)\mu(\phi_1) d\phi_1 \right], \qquad (A6)$$

where C_2 is an integration constant, and we introduced the auxiliary function

$$\begin{aligned}\mu(\phi) &\equiv \exp\left[-\int_{\phi_*}^{\phi} \frac{v}{D} d\phi \right] \\ &= \exp\left[\frac{3M_{\rm Pl}^4}{8} \left(\frac{1}{V(\phi_*)} - \frac{1}{V(\phi)} \right) \right].\end{aligned} \qquad (A7)$$

The value of C_2 is determined through the boundary condition at $\phi = \phi_*$. However, we note that $\mu(\phi)$ grows rapidly with ϕ; therefore the C_2 term is negligible for ϕ away from the reheating point. The closed-form expression for Ψ is thus

$$\begin{aligned}\Psi(\phi_0, \phi) &\approx \frac{1}{D(\phi)\mu(\phi)} \int_{\phi_*}^{\phi} \theta(\phi_0 - \phi_1)\mu(\phi_1) d\phi_1 \\ &= \frac{1}{D(\phi)} \int_{\phi_*}^{\min(\phi,\phi_0)} d\phi_1 \exp\left[-\int_{\phi}^{\phi_1} \frac{v(\phi_2)}{D(\phi_2)} d\phi_2 \right].\end{aligned} \qquad (A8)$$

We will now derive simplified forms of this expression in cases $\phi < \phi_0$ and $\phi > \phi_0$.

When $\phi < \phi_0$ (but for ϕ not too close to ϕ_*), we can simplify Eq. (A8) if we note that the outer integrand is dominated by the neighborhood of $\phi_1 = \phi$ where the exponent is close to 1. Then we can perform an asymptotic estimate of the integral in Eq. (A8). The easiest method is to integrate by parts repeatedly, which yields an asymptotic series:

$$\Psi(\phi_0,\phi) = \frac{1}{D(\phi)}\int_{\phi_*}^{\phi} d\left\{-\frac{D(\phi_1)}{v(\phi_1)}\exp\left[-\int_{\phi}^{\phi_1}\frac{v}{D}d\phi_2\right]\right\} + \frac{1}{D(\phi)}\int_{\phi_*}^{\phi}\exp\left[-\int_{\phi}^{\phi_1}\frac{v}{D}d\phi_2\right] d\left\{\frac{D(\phi_1)}{v(\phi_1)}\right\} \approx -\frac{1}{D(\phi)}\left\{\frac{D(\phi)}{v(\phi)}+\frac{D(\phi)}{v(\phi)}\left(\frac{D(\phi)}{v(\phi)}\right)'+...\right\}. \tag{A9}$$

Here we neglected the exponentially small terms of order

$$\exp\left[-\int_{\phi_*}^{\phi}\left|\frac{v}{D}\right|d\phi_2\right] = \frac{1}{\mu(\phi)} \ll 1. \tag{A10}$$

Thus we find for $\phi < \phi_0$ the required result,

$$\Psi(\phi_0,\phi) \approx -\frac{1}{v(\phi)}\left[1+\left(\frac{D(\phi)}{v(\phi)}\right)'+...\right] \approx -\frac{1}{v(\phi)}. \tag{A11}$$

In retaining only the first term of the asymptotic series, we neglect terms involving $H/M_{\rm Pl} \ll 1$ as well as terms proportional to the slow-roll parameters.

For completeness, we give the result also for $\phi > \phi_0$. In that case, the integral over ϕ_1 in Eq. (A8) becomes ϕ-independent, and we get

$$\Psi(\phi_0,\phi) = \frac{\mu(\phi_0)}{D(\phi)\mu(\phi)} = \frac{1}{D(\phi)}\exp\left[\frac{3M_{\rm Pl}^4}{8}\left(\frac{1}{V(\phi_0)}-\frac{1}{V(\phi)}\right)\right]. \tag{A12}$$

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