The Theory of Inflation

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Summary. — This article contains a concise review of the theory of inflation. We discuss its main theoretical aspects as well as its observational predictions. We also explain how the most recent astrophysical observations constrain the inflationary scenario.

1. – Introduction

The theory of inflation was invented at the end of the 70's and beginning of the 80's in order to improve the hot Big Bang model [1, 2, 3, 4, 5, 6]. It consists in a phase of accelerated expansion taking place in the early Universe, at very high energy scales, possibly as high as 10^{15}GeV. Not only inflation solves the puzzles of the standard model but it also provides a convincing mechanism for structure formation [7, 8, 9, 10, 11, 12] (for reviews, see *e.g.* Refs. [13, 14]) which, interestingly enough, is based on General Relativity (GR) and Quantum Mechanics (QM), two theories notoriously difficult to combine.

On the observational front, the progresses have also been enormous, culminating

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recently with the publication of the high accuracy measurement of the Cosmic Microwave Background (CMB) anisotropies by the European Space Agency (ESA) satellite Planck [15, 16, 17, 18]. For the first time, this satellite has been able to show that the spectral index of the scalar power spectrum is close to one (exact scale invariance) but not exactly one, the deviation from one being detected at a statistically significant level, namely at more than 5σ. This is a crucial landmark because this was a prediction of inflation (and not a post-diction). This is the reason why inflation is now viewed as the front-runner candidate for describing the physical conditions that prevailed in the early Universe [19].

The aim of these lectures is to give a brief introduction to the theory of inflation. It is organized as follows. In the next section, Section **2**, we discuss the motivations for inflation. In Section **2**'1, we first present the standard model of Cosmology, the hot Big Bang phase, as it was prior to the invention of inflation. Then, in Section **2**'2, we discuss the puzzles of the hot Big Bang phase and why a phase of accelerated expansion can solve them. In Section **2**'3, we discuss how inflation can be realized in practice and how it comes to an end (the theory of reheating). In Section **3**, we discuss the theory of inflationary cosmological perturbations of quantum-mechanical origin. We first show that the quantum state of the perturbations at the end of inflation is peculiar (a two-mode squeezed state) and then we calculate the power spectrum in the slow-roll approximation. In Section **4**, we briefly describe more complicated ways to realize inflation, in particular multiple field inflation. In Section **5**, we discuss the observational status of inflation. We argue that the simplest class of scenarios is the preferred one and present observational constraints on the shape of the potential and on the reheating phase. Finally, in Section **6**, we recap the main points and briefly discuss the future of inflation.

2. – Why Inflation?

2'1. *The pre-inflationary standard model*. – Among the four fundamental interactions that have been identified in Nature, gravity is the important one when it comes to Cosmology. Indeed, the Universe being neutral, this is the only force left with an infinite range and, therefore, the only one which can shape the Universe on astrophysical scales. The gravitational interaction being described by GR, any attempt to construct a model of the cosmos must be based on this theory. In addition, the standard model of cosmology, the so-called hot Big Bang model, is based on a second fundamental assumption, namely the cosmological principle which states that, on large scales, the Universe in homogeneous and isotropic. This means that the general relativistic metric describing our Universe can be taken to be the Friedman-Lemaitre-Robertson-Walker (FLRW) one, namely

$$(1) \qquad \mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right),$$

where $a(t)$ is the scale factor and $\mathcal{K}$ is a constant related to the curvature radius of space $r_{\mathrm{curv}} = a(t)/\sqrt{|\mathcal{K}|}$. Assuming that matter is described by perfect fluids, the

corresponding Einstein equations read

$$(2) \qquad \frac{\dot{a}^2}{a^2} + \frac{\mathcal{K}}{a^2} = \frac{1}{3M_{\mathrm{Pl}}^2} \sum_{i=1}^{N} \rho_i + \frac{\Lambda_{\mathrm{B}}}{3} \,,$$

$$(3) \qquad -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\mathcal{K}}{a^2} \right) = \frac{1}{M_{\mathrm{Pl}}^2} \sum_{i=1}^{N} p_i - \Lambda_{\mathrm{B}} \,,$$

where M_{Pl} is the Planck mass and Λ_{B} is the bared cosmological constant. The quantities ρ_i and p_i are respectively the energy density and pressure of the fluid "i". In the hot Big Bang model, one has five species, photons, neutrinos (which form radiation) and cold dark matter (cdm) and baryons (which form pressure-less matter) plus dark energy (given by the cosmological constant). Photons and neutrinos have a constant equation of state equals to 1/3, which means that $p_\gamma = \rho_\gamma/3$ and $p_\nu = \rho_\nu/3$. As already mentioned cdm and baryons have vanishing pressure. Finally dark energy (de) has a vacuum equation of state, meaning that $p_{\mathrm{de}} = -\rho_{\mathrm{de}}$. The standard model is also such that the spatial curvature vanishes, $\mathcal{K} = 0$. The free parameters are $H_0 \equiv \dot{a}/a|_{\mathrm{now}}$ (a dot denotes a derivative with respect to cosmic time), Λ_{B}, ρ_γ, ρ_ν, ρ_{cdm}, ρ_{b} and τ the optical depth that describes how the universe re-ionizes. We also have two extra parameters describing the perturbations, A_{s} and n_{s} that will be introduced later on. This means a total of nine parameters. However, introducing the critical energy density $\rho_{\mathrm{cri}} = 3H^2 M_{\mathrm{Pl}}^2$ and defining $\Omega_i \equiv \rho_i/\rho_{\mathrm{cri}}$, the fact that $\mathcal{K} = 0$ means that the Friedman equation (2) can be rewritten as a constraint, $\Omega_\gamma + \Omega_\nu + \Omega_{\mathrm{cdm}} + \Omega_{\mathrm{b}} + \Omega_{\mathrm{de}} \equiv \Omega_{\mathrm{tot}} = 1$. So, in fact, we have eight free parameters (often, ρ_γ and ρ_ν are not viewed as free parameters because they are precisely determined by the CMB measurement and the number of neutrinos family; in that case we have a six parameter model). These free parameters have now been measured with good precision (at the percentage level) [15, 17]. For the expansion rate, one has $H_0 = 100h\,\mathrm{km} \times \mathrm{s}^{-1} \times \mathrm{Mpc}^{-1}$ with $h \simeq 0.67$, and for the matter content in the present day Universe, $\Omega_\gamma h^2 \simeq 2.47 \times 10^{-5}$, $\Omega_\nu h^2 \simeq 1.68 \times 10^{-5}$ (assuming three families of neutrinos), $\Omega_{\mathrm{cdm}} h^2 \simeq 0.1198$, $\Omega_{\mathrm{b}} h^2 \simeq 0.02255$ and $\Omega_{\mathrm{de}} h^2 \simeq 0.306$.

Knowing the matter content, by integrating the Einstein equations, we can infer the history of the Universe. The early Universe was dominated by radiation, with a scale factor given by $a(t) \propto t^{1/2}$ from the initial singularity until a redshift $z_{\mathrm{eq}} \simeq 3400$. Then, pressure-less matter took over with a scale factor $a(t) \propto t^{2/3}$ until a redshift of order one. Then, dark energy started to dominate and we still live in this epoch. The history of the Universe is thus made of three successive eras.

This simple model, except for the presence of dark energy, was already known before the 80's (although, at that time, the parameters were not measured with today accuracy) and has a great explanatory power. As mentioned before, it is known as the hot Big Bang model or the ΛCDM model in its most modern incarnation and is considered as the most convincing model for cosmology. Why, then, the simple version presented above is nevertheless considered as not fully satisfactory thus motivating the introduction of inflation? We now turn to this question.

2˙2. *The puzzles of the standard model.* – With a few parameters, the pre-inflationary standard model of Cosmology was (is) able to explain a very large number of observational facts. Therefore it may seem strange to view it as not totally satisfactory. In fact, the difficulties of the hot Big Bang model are all related to the initial conditions. For instance, it is difficult to understand why spatial curvature is so small today. Indeed, the expansion during the hot Big Bang phase is decelerated and this means that $\Omega_{\mathrm{tot}} - 1$ is growing. Therefore, since $\Omega_{\mathrm{tot}} - 1$ is, today, very close to zero, this implies that it was in fact incredibly small in the early Universe (say, at BBN). Of course, it is always possible to postulate that the initial conditions were just such that it was the case. However, there is another explanation which consists in assuming that there was an accelerated phase of expansion, $\ddot{a} > 0$, prior to the hot Big Bang epoch. This new phase of accelerated expansion is called "inflation". Then, the initial conditions at the beginning of the hot Big Bang epoch are now viewed as the "final conditions" at the end of inflation. Moreover, during a phase of accelerated expansion $\Omega_{\mathrm{tot}} - 1$ is decreasing. Therefore, if $\Omega_{\mathrm{tot}} - 1$ sufficiently decreases during inflation, it can entirely compensate the subsequent growth during the hot Big Bang phase and we understand why it is still small today. One can show that the compensation occurs if we have more than 60 e-folds of inflation. In some sense, inflation is a physical mechanism which puts the hot Big Bang phase on the "right tracks" by automatically single outing the right initial conditions.

Quite remarkably, one can show that all the puzzles of the standard model can be solved by the same mechanism [3]. For instance, this is the case of the so-called horizon problem. According to the hot Big Bang model, the angular scale of the horizon on the last scattering surface (where the CMB radiation was emitted) is $\simeq 1°$. This means that we should expect the temperature to be strongly inhomogeneous on this scale all over the sky. As is well-known, this is not the case since the CMB is, on the contrary, extremely homogeneous and isotropic. However, if one has 60 e-folds of inflation before the hot Big Bang phase, then the horizon at the last scattering surface covers the entire celestial sphere today and the problem is gone. We stress again that the number of e-folds needed to solve the problem turns out to be the same as for the flatness problem, namely 60.

Of course, postulating a phase of accelerating is not sufficient. One must also identify a physical mechanism that could be responsible for it. In the next section, we discuss this question.

2˙3. *Basics of inflation.* – We have seen before that, if there is a phase of accelerated expansion in the early Universe, then the puzzles of the hot Big Bang model can be explained. As long as the gravitational field is described by GR and the cosmological principle valid, the acceleration of the scale factor can be expressed as

$$(4) \qquad \frac{\ddot{a}}{a} = -\frac{1}{6M_{\mathrm{Pl}}^2} \sum_{i=1}^{N} (\rho_i + 3p_i) + \frac{1}{3}\Lambda_{\mathrm{B}}.$$

Assuming that the cosmological constant does not play a role in the early Universe (given its present day value), the condition for having $\ddot{a} > 0$ reads

$$(5) \qquad\qquad\qquad \rho_{\text{T}} + 3p_{\text{T}} < 0 \,,$$

where $\rho_{\text{T}} = \sum_{i=1}^{N} \rho_i$ and $p_{\text{T}} = \sum_{i=1}^{N} p_i$ denote the total energy density and pressure. Given that the energy density must be positive, we are left with the condition that the pressure must be negative.

In usual situations, the pressure of a fluid is positive. This is for instance the case of radiation. However, inflation is supposed to take place in the very early Universe, at extremely high redshifts, and at those energies, hydrodynamics is clearly not the appropriate framework to describe matter. We should rather use field theory. The simplest type of field, compatible with the cosmological principle and the FLRW symmetries is a scalar field. We therefore assume that the matter content of the early Universe was dominated by a homogeneous scalar field $\phi(t)$ called, for obvious reasons, the "inflaton". The corresponding action is given by

$$(6) \qquad\qquad \mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + \mathcal{L}_{\text{int}}(\phi, A_\mu, \Psi),$$

where $V(\phi)$ is the inflaton potential and $\mathcal{L}_{\text{int}}$ describes the interaction of the inflaton field with the other fields present such as gauge bosons A_μ or fermions Ψ. Then, by varying this action with respect to the metric tensor, one can calculate the energy momentum tensor and, therefore, the energy density and the pressure of the system. Ignoring for the moment the interaction term, this leads to

$$(7) \qquad\qquad \rho = \frac{\dot{\phi}^2}{2} + V(\phi), \quad p = \frac{\dot{\phi}^2}{2} - V(\phi).$$

We see that energy density is positive definite as it should [of course, $V(\phi) > 0$] but this is not the case of pressure. If the potential energy dominates over the kinetic energy, then $p < 0$. This will be the case if the kinetic energy is small or, in other words, if the inflaton moves slowly along its potential. And this will happen if the potential is nearly flat. We conclude that, if the inflaton dominates the energy budget at early times and if its potential is almost flat, then a phase of inflation can occur. This is the basics idea that underlies the theory of inflation.

At the technical level, the evolution of the system is controlled by the Friedmann and Klein-Gordon equations, namely

$$(8) \qquad\qquad H^2 = \frac{1}{3M_{\text{Pl}}^2} \left[\frac{\dot{\phi}^2}{2} + V(\phi) \right], \quad \ddot{\phi} + 3H\dot{\phi} + V_\phi = 0,$$

where a subscript ϕ means a derivative with respect to the inflaton field. Unfortunately, this system of equations cannot be solved analytically unless the potential has a very

specific form [for instance, $V(\phi) \propto e^{-\alpha\phi}$, a model called power law inflation]. Therefore, we have to use either numerical calculations or a perturbative method. In general, a perturbative method is based on an expansion of the relevant physical quantities in terms of a small parameter (or several) naturally present in the problem (for instance a coupling constant in field theory). Here, one can use the fact that the potential is nearly flat. If it is exactly flat, then the scalar field acts as a cosmological constant and the corresponding solution is de Sitter. One can then expand the solution of the system (8) around de Sitter. Since the de Sitter solution corresponds to a constant Hubble parameter, one can define small parameters by considering the derivatives of H and, then, expand the solution in these parameters. They are called horizon flow parameters or slow-roll parameters and are defined by [20, 21]

$$(9) \qquad \epsilon_{n+1} \equiv \frac{\mathrm{d}\ln|\epsilon_n|}{\mathrm{d}N}, \quad n \geq 0,$$

where $\epsilon_0 \equiv H_{\mathrm{ini}}/H$ stands at the top of the hierarchy and $N \equiv \ln(a/a_{\mathrm{ini}})$ is the number of e-folds. The first Hubble flow parameter can be expressed as

$$(10) \qquad \epsilon_1 = -\frac{\dot{H}}{H^2} = 1 - \frac{\ddot{a}}{aH^2} = \frac{3\dot{\phi}^2}{2}\frac{1}{\dot{\phi}^2/2 + V(\phi)}.$$

As mentioned above, it is related to the first derivative of the Hubble parameter. The second Hubble flow parameter, ϵ_2, would be related to $\ddot{H}$ and so on. We also see on the second expression of ϵ_1 that $\epsilon_1 < 1$ when $\ddot{a} > 0$, that is to say when inflation occurs. Of course, $\epsilon_1 \ll 1$ when the inflationary expansion is close to that of de Sitter. Finally, the third expression of ϵ_1 makes clear that it is a very small quantity when the kinetic energy is small compared to the total energy and, therefore, compared to the potential energy. In fact, there is yet another way to express the Hubble flow parameters. If one assumes that $\epsilon_n \ll 1$ [the following expressions are therefore approximate contrary to Eqs. (10) which are exact], then the first three Hubble flow parameters can be written as as [22]

$$(11) \qquad \epsilon_1 \simeq \frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{V_\phi}{V}\right)^2,$$

$$(12) \qquad \epsilon_2 \simeq 2M_{\mathrm{Pl}}^2\left[\left(\frac{V_\phi}{V}\right)^2 - \frac{V_{\phi\phi}}{V}\right],$$

$$(13) \qquad \epsilon_2\epsilon_3 \simeq 2M_{\mathrm{Pl}}^4\left[\frac{V_{\phi\phi\phi}V_\phi}{V^2} - 3\frac{V_{\phi\phi}}{V}\left(\frac{V_\phi}{V}\right)^2 + 2\left(\frac{V_\phi}{V}\right)^4\right].$$

It is then clear that, when the inflaton potential is nearly flat, one has $\epsilon_n \ll 1$. The Hubble flow parameters are in fact nothing but a measure of the flatness of the inflaton potential.

   Having identified the small parameters of the problem, one can now use them and design a method of approximation based on an expansion in terms of the ϵ_n's. This is

called the slow-roll approximation. The first step consists in re-writing the Friedman and Klein-Gordon equations (8) in terms of the ϵ_n's. This leads to

$$(14) \qquad\qquad H^2 = \frac{V}{M_{\mathrm{Pl}}^2(3 - \epsilon_1)},$$

$$(15) \qquad\qquad \left(1 + \frac{\epsilon_2}{6 - 2\epsilon_1}\right)\frac{\mathrm{d}\phi}{\mathrm{d}N} = -M_{\mathrm{Pl}}^2\frac{\mathrm{d}\ln V}{\mathrm{d}\phi}\,.$$

At this stage, these expressions are exact. Then, we expand them at leading order in the Hubble flow parameters. This gives

$$(16) \qquad\qquad H^2 \simeq \frac{V}{3M_{\mathrm{Pl}}^2}, \quad \frac{\mathrm{d}\phi}{\mathrm{d}N} \simeq -M_{\mathrm{Pl}}^2\frac{\mathrm{d}\ln V}{\mathrm{d}\phi}.$$

Unsurprisingly, we now see that the expansion rate of the Universe is solely controlled by the potential energy. One great advantage of the above equations is that they can be integrated exactly. The solution reads

$$(17) \qquad\qquad N - N_{\mathrm{ini}} = -\frac{1}{M_{\mathrm{Pl}}^2}\int_{\phi_{\mathrm{ini}}}^{\phi}\frac{V(\chi)}{V_\chi(\chi)}\,\mathrm{d}\chi\,,$$

ϕ_{ini} being the initial value of the inflaton. If the above integral can be performed, then one obtains $N = N(\phi)$ and by inverting it, one arrives at the trajectory, $\phi = \phi(N)$. If one assumes a potential $V(\phi)$, this solution can be compared with the exact solution obtained by a numerical integration. In practice, as long as $\epsilon_n \ll 1$, Eq. (17) turns out to be an excellent approximation.

We now turn to another crucial question of the inflationary scenario, namely how it comes to an end [23, 24, 25, 26]. At this stage, let us recall that inflation is not an alternative to the ΛCDM model but just an additional ingredient. A phase of inflation is supposed to take place in the early Universe for the reasons explained in Sec. 2˙2 but, then, it must be smoothly connected to the standard ΛCDM phase. On a more practical side, it is known that the expansion of the Universe was radiation dominated during the Big Bang Nucleosynthesis (BBN) (otherwise the production of light elements, which is known to be in good agreement with the data, would be drastically modified) and, therefore, inflation must have stopped by that time.

There exists different mechanisms to stop inflation but the simplest one is just that, at some point, the potential is no longer flat enough to support inflation. Usually this happens in the vicinity of the minimum of the potential. Technically, this means that the slow-roll approximation is no longer valid. In fact, from Eq. (10), one sees that the expansion is no longer accelerated when $\epsilon_1 = 1$ which, therefore, defines the time at which inflation comes to an end. Then, the field starts oscillating at the bottom of its potential. If $m^2 = \mathrm{d}^2V/\mathrm{d}\phi^2$ is the mass around the local minimum, the field behaves

as [23]

$$\phi(t) = \phi_{\mathrm{end}} \left(\frac{a_{\mathrm{end}}}{a} \right)^{3/2} \sin\left(mt\right), \tag{18}$$

namely the field oscillates with a frequency given by its mass. Of course, in this regime, the kinetic energy is no longer sub-dominant compared to the potential energy. In fact, there is now equipartition between them which means that $\langle p \rangle_t = 0$. This implies that the averaged energy density behaves as dust as also revealed by the fact that the overall amplitude of the inflaton is proportional to $a^{-3/2}$.

The above behavior is valid if one neglects the interaction of the inflaton with the other fields or, in other words, for times much smaller than the inflaton life time Γ^{-1}, where Γ is the total inflaton decay rate. If this is taken into account, then Eq. (18) becomes

$$\phi(t) = \phi_{\mathrm{end}} e^{-\Gamma t} \left(\frac{a_{\mathrm{end}}}{a} \right)^{3/2} \sin\left(mt\right), \tag{19}$$

which shows that the total energy density stored in the inflaton field quickly goes to zero. This energy is transferred to the inflaton decay products. Then, these decay products thermalize and the radiation dominated epoch starts at a temperature which is known as the reheating temperature T_{rh}. This is the first time that a temperature can be defined in the history of the Universe. Equivalently, this also determines the reheating energy density, ρ_{reh}, that is to say the energy density at which one starts the ΛCDM model. It is given by

$$\rho_{\mathrm{reh}} = g_* \frac{\pi^2}{30} T_{\mathrm{reh}}^4, \tag{20}$$

where g_* encodes the number of relativistic degrees of freedom.

It is also interesting to study the evolution of the equation of state during the reheating. We know it must transit between -1 and $1/3$. In fact, observationally speaking, the mean equation of state is easier to probe. It is defined by [27, 28, 29, 30]

$$\overline{w}_{\mathrm{reh}} \equiv \frac{1}{\Delta N} \int_{N_{\mathrm{end}}}^{N_{\mathrm{reh}}} w_{\mathrm{reh}}(n)\mathrm{d}n, \tag{21}$$

where $\Delta N \equiv N_{\mathrm{reh}} - N_{\mathrm{end}}$ is the total number of e-folds during reheating and $w_{\mathrm{reh}} \equiv p_{\mathrm{T}}/\rho_{\mathrm{T}}$ is the instantaneous equation of state. The quantity $\overline{w}_{\mathrm{reh}}$ controls the evolution of the total energy density since one has

$$\rho_{\mathrm{reh}} = \rho_{\mathrm{end}}\, e^{-3(1+\overline{w}_{\mathrm{reh}})\Delta N}, \tag{22}$$

where ρ_{end} is the energy density at the end of inflation, namely when $\epsilon_1 = 1$. If one is given a model of inflation, then this quantity can be easily calculated.

It is also relevant to introduce the reheating parameter which is a quantity depending on ρ_{reh} and $\overline{w}_{\mathrm{reh}}$. Explicitly, it reads [27, 28, 29, 30]

$$(23) \qquad R_{\mathrm{rad}} \equiv \left(\frac{\rho_{\mathrm{reh}}}{\rho_{\mathrm{end}}}\right)^{(1-3\overline{w}_{\mathrm{reh}})/(12+12\overline{w}_{\mathrm{reh}})}.$$

The reason why this parameter is important can be found in Refs. [27, 28, 29, 19, 30]. It turns out that, when one tries to constrain reheating with the CMB, we end up constraining this parameter. As simple check allows us to understand why. Observationally speaking there should not be any difference between a model where reheating proceeds instantaneously and a model where reheating proceeds with an equation of state $1/3$. If R_{rad} is the only combination of parameters we can access to, it should therefore have the same value for those two situations. And, indeed, it is easy to check that $R_{\mathrm{rad}} = 1$ if $\rho_{\mathrm{reh}} = \rho_{\mathrm{end}}$ (instantaneous reheating) or $\overline{w}_{\mathrm{reh}} = 1/3$ (radiative reheating).

Let us now illustrate the previous considerations on a simple example. Supposed the inflaton potential is given by $V(\phi) = m^2\phi^2/2$. Then, it is easy to perform the integral in Eq. (17) and the corresponding trajectory reads

$$(24) \qquad \phi(N) = \sqrt{\phi_{\mathrm{ini}}^2 - 4M_{\mathrm{Pl}}^2(N - N_{\mathrm{ini}})}$$

As explained before, inflation stops when $\epsilon_1 = 1$ which, in this case, means $\phi_{\mathrm{end}} = \sqrt{2}M_{\mathrm{Pl}}$. From this result, one can also compute the total number of e-folds. One finds

$$(25) \qquad N_{\mathrm{T}} \equiv N_{\mathrm{end}} - N_{\mathrm{ini}} = \frac{1}{4}\frac{\phi_{\mathrm{ini}}^2}{M_{\mathrm{Pl}}^2} - \frac{1}{2}.$$

This relation means that, in order to have more than 60 e-folds, one should start from $\phi_{\mathrm{ini}} \gtrsim 15M_{\mathrm{Pl}}$. Finally, the reheating will be completed when $H \simeq \Gamma$, namely $g_*\pi^2 T_{\mathrm{reh}}^4/30 \simeq M_{\mathrm{Pl}}^2\Gamma^2$ or

$$(26) \qquad T_{\mathrm{reh}} \simeq \left(\frac{30}{g_*\pi^2}\right)^{1/4} M_{\mathrm{Pl}}^{1/2}\Gamma^{1/2}.$$

We see that the reheating temperature scales as the square root of the decay rate.

3. – Inflationary Cosmological Perturbations

We now turn to the theory of cosmological perturbations of quantum mechanical origin. This part of the inflationary scenario makes use of GR and QM and as such is particularly interesting. Moreover, it allows us to build a bridge between theoretical considerations and actual astrophysical measurements. Therefore, it plays a crucial role in our attempts to observationally probe inflation.

So far, we have considered that the Universe was homogeneous and isotropic. Clearly, in the real world, this is not the case. Going beyond the cosmological principle is a priori

technically challenging since this means solving Einstein equations in an inhomogeneous and anisotropic situation. Fortunately, we know that the amplitude of these inhomogeneities were small in the early Universe as revealed by the fact that $\delta T/T \simeq 10^{-5}$ on the last scattering surface located at a redshift of $z_{\text{lss}} \simeq 1100$. Since the amplification of the fluctuations proceeds by gravitational collapse, the amplitude of the inhomogeneities were even smaller during inflation. As a consequence, one can study their behavior perturbatively. Moreover, restricting ourselves to linear perturbations (leading order) is sufficient. Based on the previous considerations, we can then write [31]

$$(27) \qquad g_{\mu\nu}(\eta, \boldsymbol{x}) = g_{\mu\nu}^{\text{FLRW}}(t) + \delta g_{\mu\nu}(\eta, \boldsymbol{x}) + \cdots$$

with the assumption that $|\delta g_{\mu\nu}(\eta, \boldsymbol{x})| \ll |g_{\mu\nu}^{\text{FLRW}}(t)|$. The tensor $\delta g_{\mu\nu}$ can be decomposed in three types of fluctuations, scalar, vector and tensor or gravitational waves. The study of scalar perturbations can be reduced to the study of a single quantity, the curvature perturbation $\zeta(\eta, \boldsymbol{x})$ and the primordial gravitational waves can be described by a transverse and traceless two rank tensor $h_{ij}(\eta, \boldsymbol{x})$, $h^i{}_i = \partial_i h^i{}_j = 0$. Vector perturbations do not play a role during inflation. As was already mentioned, the evolution of the Universe is controlled by the Einstein equations, $G_{\mu\nu} = T_{\mu\nu}$. Since we expand the metric tensor in terms of the perturbations, one must do the same for the Einstein tensor, $G_{\mu\nu} = G_{\mu\nu}^{\text{FLRW}} + \delta G_{\mu\nu}$ and for the stress energy tensor, $T_{\mu\nu} = T_{\mu\nu}^{\text{FLRW}} + \delta T_{\mu\nu}$. Then, the equations describing the behavior of the perturbations are

$$(28) \qquad \delta G_{\mu\nu} = \delta T_{\mu\nu}.$$

Of course, these equations are now partial differential equations since the perturbations are supposed to describe the early inhomogeneous and anisotropic Universe. But since these equations are linear, they can be solved by going to Fourier space.

Then, the idea is to quantize the system. The motivation is that this will provide a source for the cosmological perturbations (in other words, this will fix the initial conditions). This source will be the unavoidable quantum fluctuations of the inflaton and gravitational fields at the beginning of inflation. On the technical front, this means that $\delta g_{\mu\nu}$ will be promoted to a quantum operator, $\delta g_{\mu\nu} \to \delta \hat{g}_{\mu\nu}$. As consequence, curvature perturbations and gravitational waves also become quantum operators, $\hat{\zeta}$ and $\hat{h}_{ij}$.

One fundamental assumption of inflation is that, initially, the quantum perturbations are placed in the vacuum state. Then, this state will evolve as the Universe expands. At the end of inflation, the system will be placed into a strongly two-mode squeezed state. This state is a very peculiar state and is defined as follows (here, we follow the presentation of Ref. [32]). Let us consider a one-dimensional quantum oscillator. As is well-known, its vacuum state is a Gaussian state whose wavefunction is given by

$$(29) \qquad \Psi_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2},$$

where x is the position of the oscillator. This state, written in the momentum basis, reads

$$(30) \qquad \tilde{\Psi}_0(p) = \frac{1}{\pi^{1/4}} e^{-p^2/2},$$

where p is the conjugate momentum of x. An interesting feature of the vacuum state is that the dispersion in position and momentum are equal, namely

$$(31) \qquad \langle \Delta \hat{x}^2 \rangle = \langle \Delta \hat{p}^2 \rangle = \frac{1}{2}$$

and saturates the Heisenberg inequality $\langle \Delta \hat{x}^2 \rangle \langle \Delta \hat{p}^2 \rangle = \frac{1}{4}$. A one-mode squeezed state is a also a Gaussian state but, in position basis and momentum basis, its wave function is given by

$$(32) \qquad \Psi_R(p) = \frac{\sqrt{R}}{\pi^{1/4}} e^{-R^2 x^2/2}, \quad \tilde{\Psi}_R(p) = \frac{1}{\pi^{1/4}\sqrt{R}} e^{-p^2/(2R^2)}.$$

We see that the wavefunction now depends on an additional parameter, R. As a consequence, the dispersion in position and momentum are no longer equal,

$$(33) \qquad \langle \Delta \hat{x}^2 \rangle = \frac{1}{2R^2}, \quad \langle \Delta \hat{p}^2 \rangle = \frac{R^2}{2}$$

although they still saturates the Heisenberg inequality. If $R > 1$, then the dispersion in position is smaller than that of the vacuum. We say that the state is squeezed in position, hence its name. Of course, since one has to satisfy the Heisenberg inequality, the price to pay is that the dispersion on momentum is larger. If $R < 1$, we have the opposite situation and the state is squeezed in momentum.

Then, let us consider two oscillators. The vacuum state of this system in position basis (namely the position of the first oscillator also referred to as the position of Alice and the position of the second oscillator also referred as to the position of Bob) can be written as

$$(34) \qquad \Psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} e^{-x_1^2/2 - x_2^2/2} = \frac{1}{\sqrt{\pi}} e^{-(x_1-x_2)^2/4} e^{-(x_1+x_2)^2/4}.$$

We see that the position of Alice and Bob are uncorrelated. From this expression, we are now in a position to introduce the two-mode squeezed state which is given by

$$(35) \qquad \Psi_R(x_1, x_2) = \frac{1}{\sqrt{\pi}} e^{-R^2(x_1-x_2)^2/4} e^{-(x_1+x_2)^2/(4R^2)},$$

where the squeezing factor R appears again and is related to the squeezing parameter r by $R = \ln r$. We see that the position of Alice and Bob are now correlated. It is also

interesting to notice that the two-mode squeezed state does not imply squeezing for Alice or Bob. Indeed, it is easy to check that

$$(36) \qquad \langle \Delta \hat{x}_1^2 \rangle = \langle \Delta \hat{x}_2^2 \rangle = \frac{1 + R^4}{4R^2}.$$

These dispersions are always larger than those one would obtain from the vacuum state. This is related to the fact that, if one traces out, say, Alice's degree of freedom, the obtained state of Bob is not a one-mode squeezed state but a thermal state.

The quantum-mechanical properties of inflation discussed above are clearly fascinating. Based on this aspect of the theory, one can wonder whether it would be possible to exhibit quantum effects in the sky. This was first discussed in Refs. [33, 34] and, more recently, in Refs. [35, 36, 37, 38, 39, 40, 41].

Let us now turn to a quantitative characterization of the cosmological fluctuations originating from inflation. As usual this will be done by computing the various correlation functions of scalar and tensor perturbations (in the following, we mainly focus on the scalar sector). The simplest correlation function is evidently the two-point correlation function which is given by

$$(37) \qquad \langle \zeta^2(\eta, \boldsymbol{x}) \rangle = \int_0^{+\infty} \frac{\mathrm{d}k}{k} \mathcal{P}_\zeta(k),$$

where brackets mean quantum averages in the two mode squeezed state described above and where $\mathcal{P}_\zeta(k) = k^3 |\zeta_{\boldsymbol{k}}|^2 / (2\pi^2)$ is, by definition, the power spectrum of scalar perturbations. This scalar power spectrum is a very important quantity because it can be probed observationally by measuring the CMB anisotropies or by measuring the distributions of galaxies across our Universe. Using the slow-roll approximation introduced above, it can also be calculated for an arbitrary potential $V(\phi)$ and the result reads

$$(38) \qquad \mathcal{P}_\zeta(k) = \mathcal{P}_{\zeta 0}(k_{\mathrm{P}}) \left[a_0^{(\mathrm{S})} + a_1^{(\mathrm{S})} \ln \left(\frac{k}{k_{\mathrm{P}}} \right) + \frac{a_2^{(\mathrm{S})}}{2} \ln^2 \left(\frac{k}{k_{\mathrm{P}}} \right) + \cdots \right],$$

where k_{P} is a pivot scale and the global amplitude can be expressed as

$$(39) \qquad \mathcal{P}_{\zeta 0} = \frac{H_*^2}{8\pi^2 \epsilon_{1*} M_{\mathrm{Pl}}^2}.$$

In the above formula, a star means that the corresponding quantity has been calculated at the time at which the pivot scale crossed out the Hubble radius during inflation. We notice that the amplitude of the correlation function depends on the square of the Hubble rate during inflation (measured in Planck units) and is inversely proportional to the first slow-roll parameter. All these quantities are scale independent and so is the global amplitude. This result is viewed as one of the most important success of inflation. Indeed, before the invention of inflation, it was already known that a scale invariant power spectrum (or Harrisson-Zeldovitch power spectrum) is a good fit to the data. But

its origin was mysterious and there was no convincing physical mechanism to produce it. Inflation, on the contrary, naturally implies this property. In fact, generically, exact scale invariance is not a prediction of inflation because, as can be seen in Eq. (38), the overall amplitude receives small, scale dependent, logarithmic corrections. The amplitudes of those corrections is determined by the Hubble flow parameters, namely [20, 42, 43, 44, 45, 46, 21, 47, 48],

$$a_0^{(\mathrm{S})} = 1 - 2\left(C + 1\right)\epsilon_{1*} - C\epsilon_{2*} + \left(2C^2 + 2C + \frac{\pi^2}{2} - 5\right)\epsilon_{1*}^2$$

$$+ \left(C^2 - C + \frac{7\pi^2}{12} - 7\right)\epsilon_{1*}\epsilon_{2*} + \left(\frac{1}{2}C^2 + \frac{\pi^2}{8} - 1\right)\epsilon_{2*}^2$$

$$(40) \qquad + \left(-\frac{1}{2}C^2 + \frac{\pi^2}{24}\right)\epsilon_{2*}\epsilon_{3*} + \cdots ,$$

$$(41) \quad a_1^{(\mathrm{S})} = -2\epsilon_{1*} - \epsilon_{2*} + 2(2C + 1)\epsilon_{1*}^2 + (2C - 1)\epsilon_{1*}\epsilon_{2*} + C\epsilon_{2*}^2 - C\epsilon_{2*}\epsilon_{3*} + \cdots ,$$

$$(42) \quad a_2^{(\mathrm{S})} = 4\epsilon_{1*}^2 + 2\epsilon_{1*}\epsilon_{2*} + \epsilon_{2*}^2 - \epsilon_{2*}\epsilon_{3*} + \cdots ,$$

$$(43) \quad a_3^{(\mathrm{S})} = \mathcal{O}(\epsilon_{n*}^3) ,$$

where $C \equiv \gamma_{\mathrm{E}} + \ln 2 - 2 \approx -0.7296$, γ_{E} being the Euler constant. Therefore, the exact prediction of inflation (really a prediction since it was made before it was checked) is that the power spectrum should be almost scale invariant but not exactly scale invariant. This prediction has been recently confirmed for the first time by the Planck data. Technically, one defines the spectral index, which is the logarithmic derivative of $\ln \mathcal{P}_\zeta(k)$, namely

$$(44) \qquad n_{\mathrm{S}} = 1 - 2\epsilon_{1*} - \epsilon_{2*},$$

where $n_{\mathrm{S}} = 1$ corresponds to exact scale invariance. As will be discussed in more details in the following, Planck has measured $n_{\mathrm{S}} \simeq 0.96$ and $n_{\mathrm{S}} = 1$ is now excluded at more than 5σ. We also see that the spectral index depends on the two first Hubble flow parameters. As a consequence, a measurement of n_{S} is also a measurements of ϵ_{1*} and ϵ_{2*}, that is to say of the first and second derivative of the inflaton potential. This explains how astrophysical measurements can constrain the theory of inflation.

The treatment of tensor modes (primordial gravitational waves) proceeds in the very same way. One can compute the two-point correlation and the power spectrum using the slow-roll approximation. One then arrives at the following expression

$$(45) \qquad \mathcal{P}_h(k) = \mathcal{P}_{h0}(k_{\mathrm{P}})\left[a_0^{(\mathrm{T})} + a_1^{(\mathrm{T})}\ln\left(\frac{k}{k_{\mathrm{P}}}\right) + \frac{a_2^{(\mathrm{T})}}{2}\ln^2\left(\frac{k}{k_{\mathrm{P}}}\right) + \cdots\right] ,$$

where the amplitude $\mathcal{P}_{h0}(k_{\mathrm{P}})$ is given by

$$(46) \qquad \mathcal{P}_{h_0} = \frac{2H_*^2}{\pi^2 M_{\mathrm{Pl}}^2} .$$

As it was the case for scalar perturbations, the overall amplitude is also given by the square of the expansion rate during inflation measured in Planck units. Of course the big difference is that the first slow-roll parameter ϵ_{1*} is now absent. This means that a measurement of the tensor modes would immediately provide the energy scale of inflation. Notice that $\mathcal{P}_{h0}(k_{\mathrm{P}})$ is also scale independent and, at leading order, the tensor power spectrum is therefore scale invariant. However, as it was also the case for scalar modes, this scale invariant amplitude receives small, scale dependent, logarithmic corrections the amplitude of which can be expressed as [21]

$$a_0^{(\mathrm{T})} = 1 - 2\left(C+1\right)\epsilon_{1*} + \left(2C^2 + 2C + \frac{\pi^2}{2} - 5\right)\epsilon_{1*}^2$$

$$(47) \qquad\qquad + \left(-C^2 - 2C + \frac{\pi^2}{12} - 2\right)\epsilon_{1*}\epsilon_{2*} + \cdots,$$

$$(48) \qquad a_1^{(\mathrm{T})} = -2\epsilon_{1*} + 2(2C+1)\epsilon_{1*}^2 - 2(C+1)\epsilon_{1*}\epsilon_{2*} + \cdots,$$

$$(49) \qquad a_2^{(\mathrm{T})} = 4\epsilon_{1*}^2 - 2\epsilon_{1*}\epsilon_{2*} + \cdots,$$

$$(50) \qquad a_3^{(\mathrm{T})} = \mathcal{O}(\epsilon_{n*}^3).$$

From the coefficient $a_1^{(\mathrm{T})}$, one can read the tensor spectral index (at first order in slow-roll). One obtains

$$(51) \qquad\qquad n_{\mathrm{T}} = -2\epsilon_1.$$

Exact scale invariance corresponds to $n_{\mathrm{T}} = 0$ (for historical reasons, the convention differs from that of scalars). Another difference is that n_{T} depends on ϵ_{1*} only while n_{S} depends on ϵ_{1*} and ϵ_{2*}. Given that ϵ_{1*} is always positive, this implies that n_{T} is always negative (or red).

Finally, one can also calculate the tensor amplitude to scalar amplitude r. Using the previous expressions, one obtains

$$(52) \qquad\qquad r \equiv \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} = 16\epsilon_{1*}.$$

Since, by definition, $\epsilon_{1*} \ll 1$, this means that gravitational waves are sub-dominant (which explains why they have not yet been detected [49, 50]). Notice that there is a priori no lower bound on r. Therefore, if r turns out to be very small, primordial gravitational waves will probably never been detected but this would be in no way in contradiction with the predictions of inflation. At the time of writing, it is believed that the next generations of telescope and satellites will be able to reach the level $r \sim 10^{-3}$ maybe a bit smaller. Let us hope that Nature has produced a r larger than this limit!

To conclude this section, let us mention Non-Gaussianities (NG). So far, we have restricted our considerations to two-point correlation functions. Of course, higher correlation functions are also of great interest. Usually, the three-point correlation function (bispectrum) and the four-point correlation function (trispectrum) are considered. For

the models described previously, NG are very small (of the order of the slow-roll parameters) [51, 52, 53, 54]. The reason is easy to understand. We have started from a Gaussian state and the evolution of the perturbations is linear. As a consequence, the appearance of any NG is necessarily related to non linearities, which are very small.

4. – Extensions

So far, we have described the simplest way to realize inflation. However, since the invention of inflation in the 80's, more complicated scenarios have been imagined. In this section, we say a few words about them.

The most generic extension is probably to consider models where, instead of having one scalar field, one has several ones playing an active role during inflation [55]. This appears to be a natural approach given that inflation can occur at energy scales as high as 10^{15}GeV. At those scales, it is believed that particle physics is no longer described by the standard model but by its extensions (SUSY, SUGRA, string theory, etc . . .). And, usually, in these alternative frameworks, there are plethora of scalar fields.

Clearly, multiple field inflation scenarios are more complicated and it is more difficult to make generic predictions. However, one can list three main modifications. Firstly, there is the possibility of having non adiabatic perturbations, which is impossible for single field models. The reason is that, if several scalar fields are present during inflation, then the corresponding decay products can have different origin resulting in the possible presence of non adiabatic perturbations. Secondly, non adiabatic perturbations can source the evolution of curvature perturbations. As a result, if they are present during inflation and reheating, $\zeta(\eta, \boldsymbol{x})$ on large scales is no longer a conserved quantity. This has drastic consequences, especially for reheating, which then becomes potentially dependent on the details of physical processes going on on scales smaller than the Hubble radius. Thirdly, it is possible to produce non negligible NG. As already mentioned, these modifications are not mandatory and must be analyzed on a model by model basis.

Yet other extensions are also possible such that having a non canonical kinetic term for the scalar field. They are called K-inflation models [56, 57] (for the observational status of this class of models, see Refs. [47, 58, 48]). It is also possible to have models with features [59, 60]. This means a model of inflation where, in some limited region, the potential is not flat. This usually causes a transitory violation of the slow-roll approximation which can result in oscillations in the power spectrum and non negligible NG [61, 62, 63]. More complicated models are possible, for instance by combining the various ingredients discussed above [64], but we will not discuss them here. We now turn to another question, namely how the observations can discriminate among these various possibilities.

5. – Inflation and CMB Observations

The Planck satellite has recently measured the CMB temperature, see Fig. 1, and polarization, see Figs. 2 and 3, anisotropies with unprecedented accuracy. These new
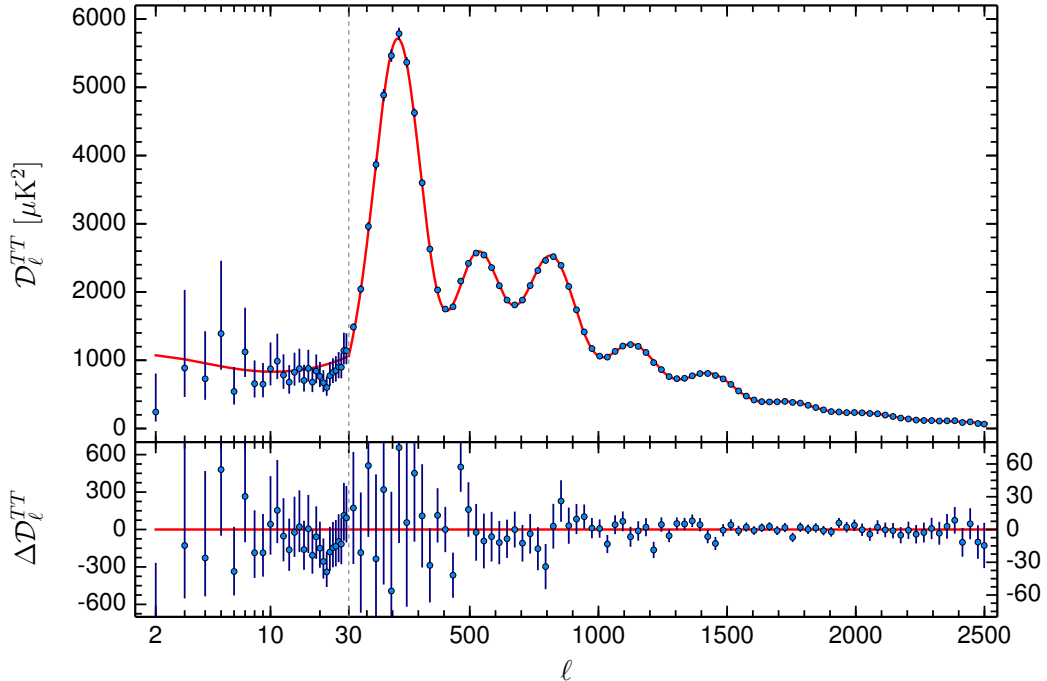
Fig. 1. – Multipole moments versus angular scale from Planck 2015 data. The multipole moments are obtained from the CMB map by Fourier transforming it according to: $\langle \delta T/T(\boldsymbol{e}_1) \delta T/T(\boldsymbol{e}_2) \rangle = (4\pi)^{-1} \sum_\ell (2\ell+1) C_\ell P_\ell(\cos\theta)$ where θ is the angle between two directions $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ and P_ℓ is a Legendre polynomial. The multipole moments C_ℓ are interpreted as the power of the signal at a given angle θ. Notice that $\mathcal{D}_\ell$ is related to C_ℓ by $\mathcal{D}_\ell = \ell(\ell+1)C_\ell/(2\pi)$. The red curve corresponds to the best fit and is consistent with the predictions of single field, slow-roll, inflation. Figure taken from Ref. [17].

data allow us to constrain inflation and to learn which was version of inflation realized in the early Universe.

In brief, Planck has shown that the Universe is spatially flat, that the perturbations are adiabatic and Gaussian [18]. These results are all consistent with single field (with minimal kinetic term), slow-roll, inflation which, therefore, appears to be the preferred class of models. This does not mean that the more complicated versions discussed in Section **4** are ruled out but just that, at the moment, they are not needed in order to explain the data.

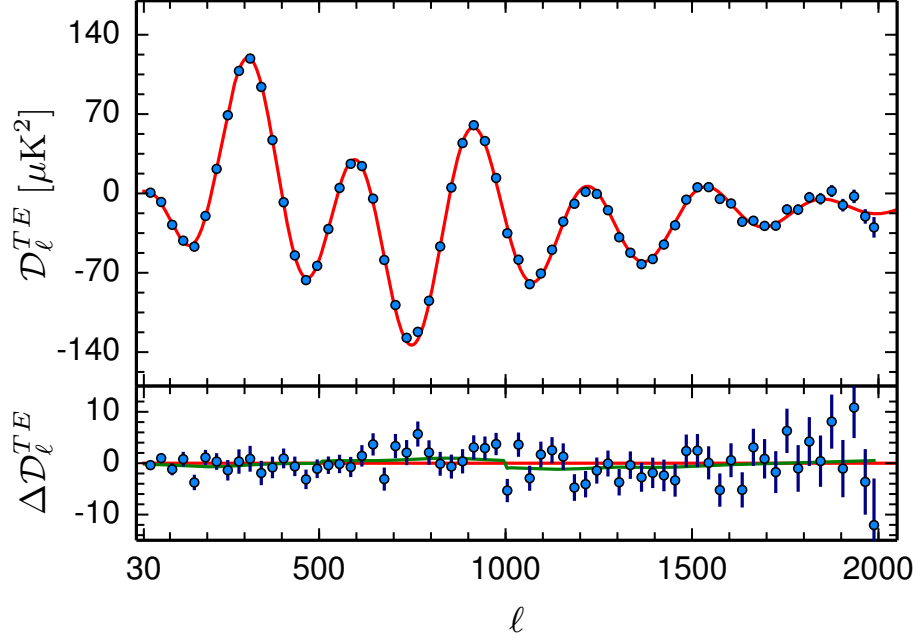With regards to inflation, probably the most important discovery made by the Planck

Fig. 2. – Multipole moments corresponding to the correlation between temperature and E-mode polarization anisotropies. The red solid line is obtained from temperature measurements only, see Fig. 1. The lower panel shows the residual with respect to this best fit. Figure taken from Ref. [17].

satellite is the measurement of the scalar spectral index [18]

$$(53) \qquad\qquad\qquad n_{\mathrm{s}} = 0.969 \pm 0.005.$$

For the first time, the value $n_{\mathrm{s}} = 1$ is excluded at more than 5σ. As was already discussed above, the fact that the power spectrum must be scale invariant (the so called Harrisson-Zeldovitch power spectrum) was known long ago (before the invention of inflation). But the non trivial prediction of inflation was that n_{s} should be close to one but not exactly one. And this is exactly what has been observed for the first time by the Planck satellite.

Another important of piece of information is that, unfortunately, so far, no gravitational waves has been detected. This means the following upper bound on the tensor to scalar ratio r [49]

$$(54) \qquad\qquad\qquad r \lesssim 0.08.$$

From the measurements of those quantities, one can also infer constraints on the
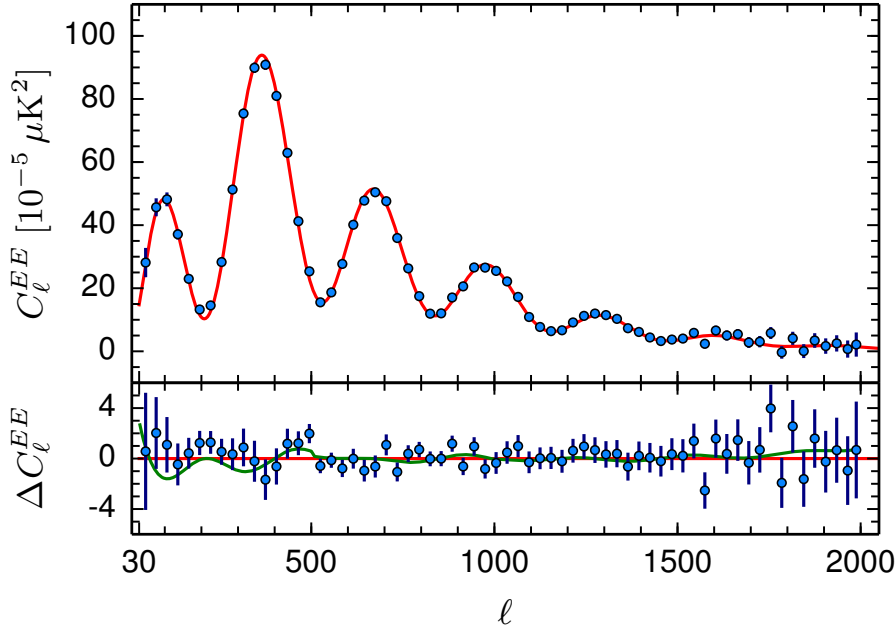
Fig. 3. – Same as in Fig. 2 but for the *E*-mode power spectrum obtained from Planck 2015. Figure taken from Ref. [17].

Hubble flow parameters, see Fig. 4 and Refs. [65, 66, 19]. We see that $P_* \equiv \mathcal{P}_{\zeta 0} a_0^{(\mathrm{S})}$ and ϵ_{2*} are constrained while there only exists an upper bound on ϵ_{1*}. Of course, P_* is determined because one knows the amplitude of CMB fluctuations (namely $\delta T/T \simeq 10^{-5}$). On the other hand, the upper bound on ϵ_{1*} originates from Eq. (52) and the fact that we only have an upper bound on r. Given that $H_*^2/M_{\mathrm{Pl}}^2 \simeq 8\pi^2 \epsilon_{1*} P_*$, this means that we only have an upper bound on the energy scale of inflation, namely

(55) $$H_* \lesssim 1.2 \times 10^{14} \mathrm{GeV},$$

or $\rho_*^{1/4} \lesssim 2.2 \times 10^{16} \mathrm{GeV}$. Finally, the third slow-roll parameter, ϵ_{3*} is not well constrained which means that we do not have yet a detection of a running.

We have seen before that the slow-roll parameters carry information about the shape of the inflaton potential. Since we have obtained constraints on these parameters, we must be able to say something about the shape of the inflaton potential itself [65, 66, 19]. In order to answer this question, one can calculate the Bayesian evidence of the various models of inflation. The Bayesian evidence is the integral of the likelihood function over the prior space. It characterizes the performance of a model and its ability to fit the data [67]. The larger the evidence, the better the model. In Refs. [65, 66, 19],
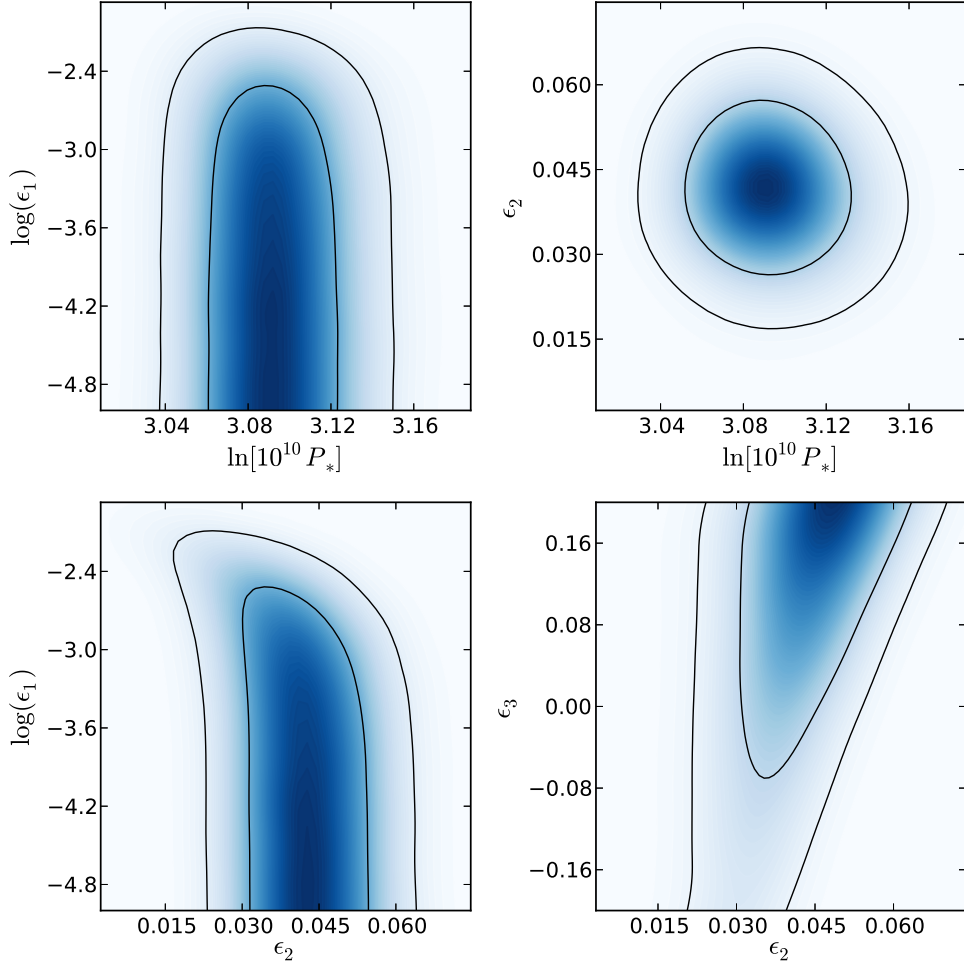
Fig. 4. – Posterior distributions of the parameters ϵ_{1*}, ϵ_{2*}, ϵ_{3*} and $P_* \equiv a_0^{(S)} \mathcal{P}_{\zeta 0}$. The posteriors are taken to be Jeffreys's priors for P_* and ϵ_{1*} and flat priors for ϵ_{2*} and ϵ_{3*}.

the Bayesian evidence of nearly two hundred models were computed. The result of this computation is displayed in Figs. 5 where the number of unconstrained parameters is also indicated. A detailed analysis of those results has been published in Refs. [65, 66, 19], but the bottom line is that plateau inflationary models are the "best" models according to the Planck data. A plateau potential is a potential which flattens out at infinity. The prototype of this class of models is the so-called Starobinsky model given by

$$(56) \qquad V(\phi) = M^4 \left(1 - e^{-\sqrt{2/3}\phi/M_{\mathrm{Pl}}}\right)^2 .$$
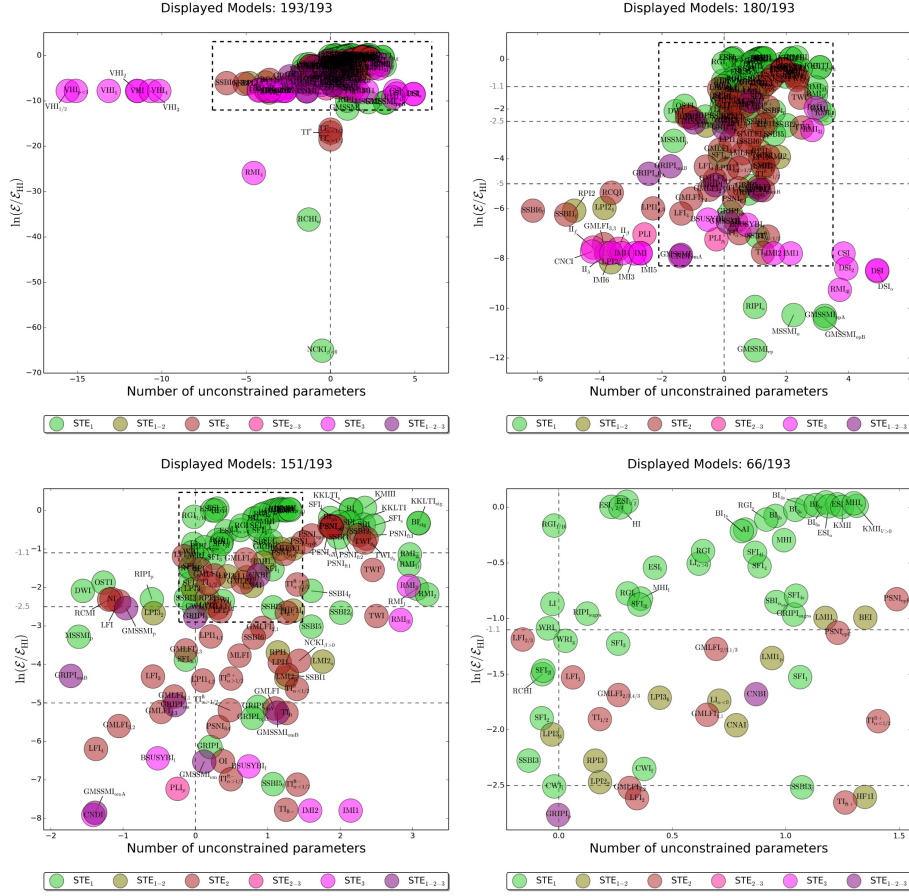
Fig. 5. – Bayesian evidence versus number of unconstrained parameters for different models of inflation. Each circle represents a given inflationary scenario (the size of the circle has no meaning). The upper right panel is a zoom on the "best" region (the square delimited by the dashed black line) of the upper left panel. In the same way, the bottom left plot is a zoom on the "best" region of the upper right. Finally, the bottom right is a zoom on the "best" region of the bottom left figure.

This conclusion is non trivial since models that were historically considered as leading candidates, such as $V(\phi) = m^2\phi^2/2$, are now strongly disfavored compared to plateau models.

Let us also notice another interesting point. The prediction of plateau models for r is, roughly speaking, $r \simeq 10^{-3}$. As indicated before, this value is in principle reachable by the next generation of instruments. This means that there is maybe a good chance to detect primordial gravitational in a non too distance future (say, a decade).
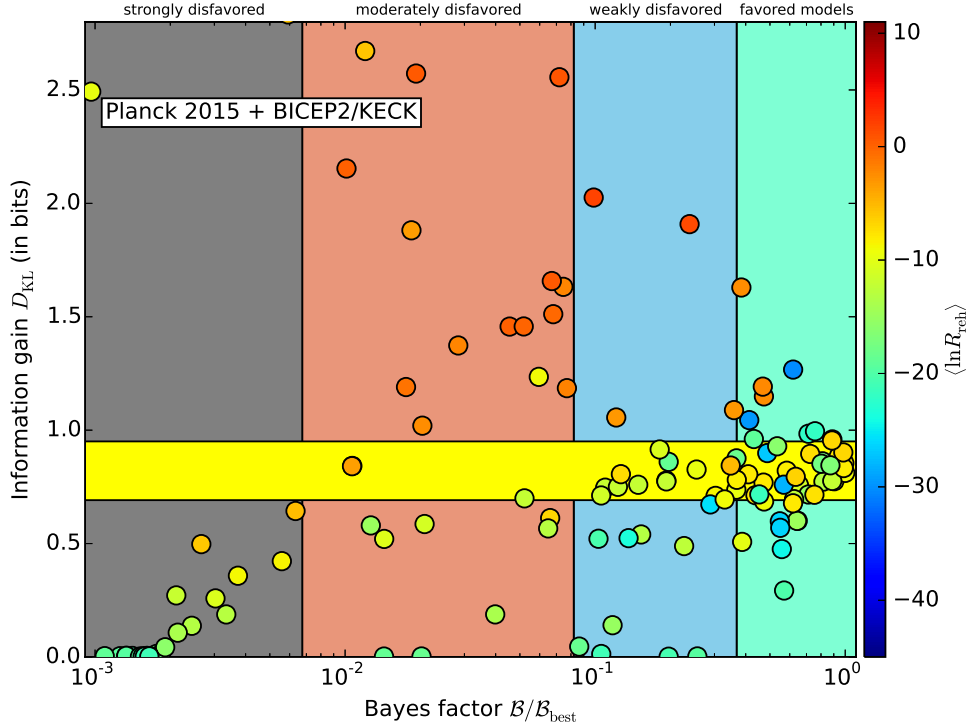
Fig. 6. – Kullback-Leibler D_{KL} divergence versus Bayesian evidence for various models of inflation. The mean value is given by $\langle D_{\mathrm{KL}} \rangle = 0.82 \pm 0.13$ and the yellow band represents the one-sigma deviation around this mean value.

Finally, let us discuss what the Planck data imply for reheating. As was discussed before, constraints on reheating are expressed through constraints on the reheating parameter R_{rad} defined in Eq. (23). In Refs. [27, 28, 29, 30], the posterior distributions was derived for the nearly two hundred models already considered before for the calculation of the Bayesian evidence. The situation is summarized in Fig. 6. It represents the Kullback-Leibler divergence between the prior distribution and the posterior versus the Bayesian evidence for different models of inflation (represented by circles). The Kullback-Leibler divergence is defined by

$$(57) \qquad D_{\mathrm{KL}} = \int P\left(\ln R_{\mathrm{reh}} | D\right) \ln \left[\frac{P\left(\ln R_{\mathrm{reh}} | D\right)}{\pi\left(\ln R_{\mathrm{reh}}\right)} \right] \mathrm{d} \ln R_{\mathrm{reh}},$$

where R_{reh} is given by $\ln R_{\mathrm{reh}} = \ln R_{\mathrm{rad}} + \ln(\rho_{\mathrm{end}}/M_{\mathrm{Pl}}^4)/4$ and is therefore, for a given model of inflation, in a one-to-one correspondence with R_{rad}. The quantity π represents the prior on R_{reh} and P the posterior. The Kullback-Leibler divergence measures the

"distance" between the prior and the posterior and, as a consequence, also represents the amount of information provided by the data D (of course, here, the Planck data) about $\ln R_{\rm reh}$. The constraints are model dependent and one has a posterior distribution per model of inflation, an amount of information which, given the number of scenarios analyzed, is difficult to deal with. The value of $D_{\rm KL}$ is one way to summarize the information about reheating for a given model to one number. In this sense, Fig. 6 completely describes what, for each known model of inflation, the Planck data implies with regards to the ability to fit the data and to reheating. Let us also notice that one can calculate the mean value of $D_{\rm KL}$. One finds $\langle D_{\rm KL} \rangle = 0.82 \pm 0.13$, which expresses the fact that reheating is globally constrained by the Planck data.

6. – Conclusions

In this short review, we have discussed the theory of inflation. Over the years, the inflationary scenario has become a crucial ingredient in our understanding of Cosmology. It is important to stress that inflation is not an alternative to the standard model of Cosmology, it is rather a new part of it.

Invented in the 80's, inflation has recently witnessed new developments with the publication of the high accuracy Planck data. Clearly, these data have boosted our confidence in inflation. In particular, the measurement of the spectral index to be close but not equal to one is an important confirmation of an inflationary prediction. Admittedly, it is probably not the final proof that inflation actually occurred in the early Universe but it nevertheless represents a very strong argument in its favor. From the Planck data, we have also learned that inflation is probably realized in its simplest version (single field, slow-roll, with minimal kinetic term) and that the best scenario is a plateau model for which the potential flattens out at very large values of the field.

What is then the next step? Clearly, the detection of primordial gravitational waves will play a crucial role. It is an unambiguous prediction of inflation that has not yet been confirmed. Future missions will be able to reach $r \sim 10^{-3}$. Unfortunately, inflation, as a paradigm, does not predict the value of r even if r is predicted if a precise scenario is given. However, the best model of inflation, the Starobinsky model, predicts a value of r which, in principle, could be detected in the future.

Let us also add that the detection of NG will also certainly play an important role in the future. Given that we deal with the simplest class of models, the expected signal is very small and its detection will be challenging (if possible). But, obviously, this would be of crucial importance.

Of course, inflation is not a perfect scenario and some of its aspects remain unclear. But, as an effective model of the early Universe, it scores pretty well. Let us see whether its performances remain so efficient in the future.

* * *

School of Physics Enrico Fermi (from 26th June to 19th July 2017) at the beautiful Villa Monastero located in Varenna, Lake of Como (Italy).

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