*Astronomy & Astrophysics* manuscript no. kinetic7aa
January 17, 2025


# Cosmic voids and the kinetic analysis. IV. Hubble tension and the cosmological constant

V.G.Gurzadyan[1,2], N.N.Fimin[3], V.M.Chechetkin[3,4]

[1] Center for Cosmology and Astrophysics, Alikhanian National Laboratory, Alikhanian Brothers str.2, 0036, and Yerevan State University, Manukian str.1, 0025 Yerevan, Armenia
[2] SIA, Sapienza Universita di Roma, Via Salaria 851/881, 00191 Rome, Italy
[3] Keldysh Institute of Applied Mathematics of RAS, Miusskaya Sq. 4, 125047 Moscow, Russia
[4] Institute of Computer Aided Design of RAS, 2nd Brestskaya st., 123056 Moscow, Russia

Submitted: XXX; Accepted: XXX



**ABSTRACT**

The formation of the cosmic structures in the late Universe is considered using Vlasov kinetic approach. The crucial point is the use of the gravitational potential with repulsive term of the cosmological constant which provides a solution to the Hubble tension, that is the Hubble parameter for the late Universe has to differ from its global cosmological value. This also provides a mechanism of formation of stationary semi-periodic gravitating structures of voids and walls, so that the cosmological constant has a role of the scaling and hence can be compared with the observational data for given regions. The considered mechanism of the structure formation in late cosmological epoch then is succeeding the epoch described by the evolution of primordial density fluctuations.

**Key words.** Cosmology: theory

## 1. Introduction

The Hubble tension (Riess 2020; Riess et al 2024a,b; Di Valentino et al 2021; Dainotti et al 2023; Scolnik et al 2024a,b) has attracted the attention to the possible genuine differences in the descriptions of the late and early Universe, including of the structure formation and the large scale flows e.g. (Capozziello and Lambiase 2022; Bouche et al 2022; Bajardi et al 2022; Capozziello and D'Agostino 2024). Given the still open problems of the nature of the dark matter and dark energy, diverse approaches and models are developed to deal with the entire scope of observational challenges, see (Capozziello et al 2024; de Pedreira et al 2024) and references therein.

Evolution of primordial density fluctuations as the origin of the cosmic structure formation (Peebles 1993) at large scales is described by the hydrodynamical pancake theory of Zeldovich (Zeldovich 1970; Shandarin and Zeldovich 1989), predicting the formation of the cosmic web and the filaments. At smaller scales, those of clusters and groups of galaxies, the role of local gravitational interaction becomes dominating. Namely, while the hydrodynamic approach assumes little influence of the intrinsic gravity of the involved particles as compared to the influence on the geometry of inhomogeneities (2-dimensional caustics) (Arnold, Shandarin, Zeldovich 1982; Arnold 1982) of the Friedmannian flow, the kinetic-field approach is assumed to be suitable for relatively fast non-equilibrium phenomena in an external quasi-ordered medium, when the wave fronts overturning and intersection of three-dimensional matter flows channels for kinetic processes are formed (Gurzadyan, Fimin, Chechetkin 2022, 2023a,b).

The possible tension of the late (local) and early (global) Universe outlines the mentioned differences in the mechanisms of structure formation and of the flows at large and small scales. Namely, the explanation of the Hubble tension as a result of two flows, local and global one, with non-identical Hubble constant, is provided in (Gurzadyan and Stepanian 2021a,b), based on the theorem proved in (Gurzadyan 1985). That theorem states the general function fulfilling the equivalency of sphere and of a point mass in the following form for the force

$$F = -\frac{GMm}{r^2} + \frac{\Lambda c^2 mr}{3}. \tag{1}$$

This function ensures non-force-free gravitational field inside a shell and the second term in l.h.s involves the cosmological constant in weak-field General Relativity and McCrea-Milne non-relativistic cosmology (McCrea and Milne 1934; Zeldovich 1981). Importantly, it is shown (Gurzadyan and Stepanian 2018; Gurzadyan 2019; Gurzadyan and Stepanian 2019, 2020) that Eq.(1) enables to describe the observational data on the dynamics of groups and clusters of galaxies. Also note, that there are observational data supporting the non-force-free field inside a shell as predicted by Eq.(1), i.e. on the influence of the galactic halos on the properties of the spiral galaxies (Kravtsov 2013).

Then, the late Hubble flow is described by the equation (Gurzadyan and Stepanian 2021a,b)

$$H_0^2 = \frac{8\pi G\rho_0}{3} + \frac{\Lambda c^2}{3}, \tag{2}$$

*Send offprint requests to*: V.G. Gurzadyan, e-mail: `gurzadyan@yerphi.am`

where $H_0$ is the local Hubble parameter, and $\rho_0$ is the local mean density of matter. This equation which follows from Eq.(1) has the structure of the standard Friedmannian equation, while it has a different content within non-relativistic McCrea-Milne model with a cosmological constant.

Below, we continue the Vlasov kinetic analysis (Gurzadyan, Fimin, Chechetkin 2022, 2023a,b) of the structure formation in the small scale Universe. Essential point in the analysis is the role of the cosmological constant and hence that the evolutionary paths of the filament formation are result of self-consistent gravitational interaction of the particles along with the repulsive cosmological term. Namely, it appears that zones of dominating influence of ascending and descending branches of the modified gravitational potential of Eq.(1) form a topologically different multi-connected matter structures in the Universe. We analyse the of quasi-static processes occurring in the system defined by Vlasov-Poisson equations near its state of relative equilibrium. In this case, as shown, it is reasonable to reduce the problem to the analysis of the integral equation of Hammerstein type. For the latter, the corresponding boundary problem is again based on the theorem in (Gurzadyan 1985). This is due to the fact that the structure of the of solutions of that integral equation and of the possible singularities can be expressed using known formalizable representations. We investigate the properties of the spectrum of the nonlinear integral operator and its relation to the spectrum of the Fredholm operator for the linear potential which enables to reveal the deterministic way of formation of coherent filaments in the local regions in the Universe.

## 2. The kinetic approach

The starting point for obtaining hydrodynamic models with singularities–caustics and kinetic models without taking into account the influence of gravity is the introduction of the “Friedmann flow” characterized by a field of velocities whose values for each pair of points in space will be proportional to the distance between these points $\mathbf{v} = H\Delta\mathbf{r}$. The Friedmann flow in the above-mentioned models was considered as some analog of an equilibrium state, which is perturbed either by inhomogeneity in the initial data set, or random fluctuations of the density leads to the formation of geometrical “small” inhomogeneity (adiabatically stable) of the spatial distribution of particles of the system on the background of this flow; the “smallness” is caused by random deviations, which are assumed a priori to be essentially limited in norm in comparison with the basis values of mean parameters.

The system of Vlasov–Poisson equations for describing of dynamics in a system of $N$ cosmological objects (with masses $m_{i=1,\dots,N} = m \equiv 1$) may be represented as (see (Gurzadyan, Fimin, Chechetkin 2022) for details)

$$\frac{\partial F(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathrm{div}_{\mathbf{x}}(\mathbf{v}F) + \widehat{G}(F;F) = 0, \quad \widehat{G}(F;F) \equiv$$

$$-\big(\nabla_{\mathbf{v}}F\big)\big(\nabla_{\mathbf{x}}(\Phi[F(\mathbf{x})]\big), \tag{3}$$

$$\Delta^{(3)}_{\mathbf{x}}\Phi[F(\mathbf{x})]\big|_{t=t_0} = AS_3G\int F(\mathbf{x},\mathbf{v},t_0)\,d\mathbf{v} - \frac{c^2\Lambda}{2}, \tag{4}$$

$$S_3 \equiv \mathrm{meas}\,\mathcal{S}^2 = 4\pi, \quad \mathcal{S}^d = \{\mathbf{x}\in\mathbb{R}^d, |\mathbf{x}=1|\},$$

where $F(\mathbf{x},\mathbf{v},t)$ is the distribution function of gravitationally interacting particles, $A$ is a normalization factor for particle density, $t_0$ is a fixed moment of time, $G$ is the gravitational constant. The system of objects/particles is considered within finite domain of configurational space $\Omega \subseteq \mathbb{R}^3$ ($\mathrm{diam}\,\Omega \le \infty$), with a $C^2$-smooth boundary $\partial\Omega$.

Equation (4) is the nonlinear Poisson equation, with account the cosmological term of Eq.(1). The third term on the right hand side of the kinetic equation (3) may be represented as

$$\widehat{G}(F;F) = \mathbf{G}(F)\frac{\partial F}{\partial \mathbf{v}}, \tag{5}$$

$$\mathbf{G}(F) = -\nabla_{\mathbf{x}}\Phi[F(\mathbf{x})],$$

$$\Phi[F(\mathbf{x})] = AS_3G\int\int \mathfrak{K}_3(\mathbf{x}-\mathbf{x}')F(\mathbf{x}',\mathbf{v}',t_*)\,d\mathbf{x}'d\mathbf{v}' +$$

$$\frac{\Lambda c^2}{12}|\mathbf{x}|^2 + \widehat{\mathfrak{B}}_3(\mathbf{x},\mathbf{x}'),$$

where: $\mathfrak{K}_3(\mathbf{x}-\mathbf{x}') = -|\mathbf{x}-\mathbf{x}'|^{-1}$, $\widehat{\mathfrak{B}}_3(\mathbf{x},\mathbf{x}')$ is an operator term that takes into account the influence of the boundary conditions. Classical Newtonian potential $\Phi_N(r) = -Gm/r$ increases monotonically on the interval $r \in (0,+\infty)$ ($\Phi_N \in (-\infty,0)$), while the generalized (with cosmological term) Newton gravity potential

$$\Phi_{GN}(r) \equiv -Gm/r - \frac{1}{2}c^2\Lambda r^2,$$

has a maximum

$$\Phi^{(max)}_{GN}(r_c) = -\frac{1}{2}G(3mc^{2/3})\Lambda^{1/3},$$

where $r_c = \big(Gm/(3\Lambda c^2)\big)^{1/3}$ (it increases on the interval $r \in (0; r_c]$ and decreases on the interval $r \in (r_c; \infty)$).

We will consider the stationary case of dynamics: $F = F(\mathbf{x},\mathbf{v})$. However, it should be pointed out that further analysis will mainly concern the second equation of the system, which is the Poisson equation relative to the potential, and no explicit time dependence is observed in it. That is why, when varying, the Hilbert–Einstein–Maxwell action (Vedenyapin, Fimin, Chechetkin 2021) can be separate variation over fields (for a fixed particle distribution) and variation over distribution functions (with fixed fields); thus, approach considered in this paper, is applicable for adiabatic processes at a quasi-equilibrium (weakly varying) particle distribution functions. In this case, we can use the energy substitution for unique variable of the distribution function Vedenyapin et al (2011): $F(\mathbf{x},\mathbf{v}) = f(\varepsilon) \in C^1_+(\mathbb{R}^1)$, where $\varepsilon = m\mathbf{v}^2/2 + \Phi(\mathbf{x})$. Thus, the particle density in the right side of the Poisson equation can be expressed in terms of the equilibrium solution of Vlasov equations. This solution is identical to Maxwell–Boltzmann distributions $f = f_0(\varepsilon) = AN\exp(-\varepsilon/\theta)$. However the physical meaning of the equilibrium solution of the Vlasov equation is essentially different from that of the Boltzmann equation. This solution must meet the following requirements: 1) the maximal possible statistical independence, 2) isotropy of velocity distribution, 3) stationarity of distribution in the form $F(\mathbf{x},\mathbf{v}) = \rho(\mathbf{x})\prod_{i=1,2,3}\mathfrak{f}(v_i^2)$. The substitution expression into the Vlasov equation gives

$$\sum_i \left( v_i \frac{\partial \ln(\rho)}{\partial x_i} - \frac{\partial \Phi}{m\partial x_i}\frac{\partial \mathfrak{f}(v_i^2)}{\mathfrak{f}(v_i^2)\,\partial v_i}\right) F = 0, \tag{6}$$

and we get system of ODEs

$$\frac{\partial(\ln\,\rho)/\partial x_i}{-\partial\Phi/\partial x_i} = \frac{\partial \ln\big(\mathfrak{f}(v_i^2)/\partial v_i\big)}{m v_i} = -\theta^{-1}, \tag{7}$$

where $\theta$ is a constant of separation of variables, its physical meaning is *kinetic temperature* in the system of interacting collisionless particles (in accordance of Vlasov's definition (Vlasov 1961, 1978) thermodynamic/collisional equilibrium is globally absent in this system).

Equation (4) for gravitational potential can be written as

$$\Delta\Phi(\mathbf{x}) = ANGS_3^2\bigg(\int_{y\in[0,\infty]} \exp\big(-y^2/(2\theta)\big)y^2\,dy\bigg)\cdot$$

$$\exp(-\Phi/\theta) - \frac{c^2\Lambda}{2}, \quad A,\theta,R_\Omega \in \mathbb{R}^1, \tag{8}$$

where: $R_\Omega$ — radius of region $\Omega$ is accepted in the form of a ball in configurational 3–space (as the simplest physically realizable case).

So, the Poisson equation (4) takes the form of an inhomogeneous equation of Liouville–Gelfand (LG) (Dupaigne 2011) with local (generalized) temperature changing sign depending on the value of the derivative of the potential at a given point: as mentioned above, for two-particle problem (in particular, for a formal pair in the form of a center coalescence of the main part of the particles and the conditional "extremely distant" particle) can dominate the repulsive force due to the presence of a quadratic term $\sim |\mathbf{x}|^2$; generalized to the indefinite thermodynamics of a system of gravitating particles, it becomes similar to that for the Onsager vortices in the classical hydrodynamics (Fimin and Chechetkin 2020), and the existence of solutions of the LG equation for large system sizes ensures the existence of solutions to the Vlasov equation (3). This can be shown using the parametric Young's inequality (Pokhozhaev 2010). It was be shown (Gurzadyan, Fimin, Chechetkin 2022), for the conditions $c^2\Lambda \gtrless 3\pi\lambda^\dagger$ ($\theta \gtrless 0$), the system of Vlasov–Poisson equations has solutions of the type of distribution functions that admit the energy substitution, and potential of gravitational field, which have the property of convexity (in the general case, for an arbitrary $R_\Omega \leq \infty$, in contrast to the case of the attraction potential, for which there is a limitation $R_\Omega < \big(C_0\theta^2/\big(\lambda^\dagger/S_3^2\big)^2\big)^{1/4}$); we used notation $\lambda^\dagger \equiv ANGS_3^2\mathcal{J}(\theta)$, $\mathcal{J}(\theta) \equiv \int_0^{v_{max}} \exp\big(-v^2/(2\theta)\big)v^2dv$.

As already noted in (Gurzadyan, Fimin, Chechetkin 2022), in the formulation of the Dirichlet problem for the Poisson equation (4) (or (8)) with a constant right-hand side on the boundary of the $\Omega$ region according to McCrea-Milne averaging gravitational field outside the compact subdomain $\Omega_0$ containing system of particles (which is situated in the region $\Omega$ (meas $\Omega_0 \ll$ meas $\Omega$)), we can assume the boundary condition on the $\partial\Omega$ as given in accordance with the theorem in (Gurzadyan 1985).

The solution of Dirichlet problem may be obtained with the help of integral representation of the equation for gravitational (double–layer) potential (Nowakowski 2001). Equation for the potential with Maxwell–Boltzmann particle density, corresponding to internal Dirichlet problem in a bounded domain $\Omega$ (under boundary conditions corresponding to the Milne–McCrea model) has the following form:

$$\Phi(\mathbf{x}) = \lambda_I \int_{\Omega'} \mathcal{K}(\mathbf{x},\mathbf{x}')\exp\big(-\Phi(\mathbf{x}')/\theta\big)d\mathbf{x}' - \frac{c^2\Lambda}{12}\mathbf{x}^2 + C_0', \tag{9}$$

$$\mathcal{G}(\mathbf{x},\mathbf{x}') \equiv 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{Y^*_{\ell m}(\vartheta',\varphi')Y_{\ell m}(\vartheta,\varphi)}{2\ell+1}\frac{x_<^\ell x_>^\ell}{R_\Omega^{2\ell+1}},$$

$$x_< = \min(|\mathbf{x}|,|\mathbf{x}'|), \quad x_> = \max(|\mathbf{x}|,|\mathbf{x}'|),$$

$$\mathcal{K}(|\mathbf{x}-\mathbf{x}'|) \equiv \mathcal{G}(\mathbf{x},\mathbf{x}') - \frac{1}{|\mathbf{x}-\mathbf{x}'|}, \quad C_0' = -\frac{GNm}{R_\Omega} - \frac{c^2\Lambda R_\Omega{}^2}{12},$$

$$\lambda_I = \lambda^\dagger/S_3.$$

In essence, the above is the explicit form of the equation for the potential introduced in expression (5), where the Green's function $\mathcal{G}(\mathbf{x},\mathbf{x}')$ for the inner boundary value problem in the domain $\Omega$ (in this case, due to symmetry of the latter we have $\int_{\Omega'}\mathcal{G}(\mathbf{x},\mathbf{x}')\rho(|\mathbf{x}'|)d\mathbf{x}' \to C_1 =$ const($\propto 1/R_\Omega$)). Let's introduce a new variable $U(\mathbf{x}) \equiv (\Phi(\mathbf{x}) - C_0' - C_1)/\theta + \alpha|\mathbf{x}|^2$, $\alpha \equiv c^2\Lambda/(12\theta)$, the above equation can be written as the uniform Hammerstein integral equation

$$U(\mathbf{x}) = \lambda_\theta\widehat{\mathfrak{G}}(U), \quad \widehat{\mathfrak{G}}(U) \equiv \int_{\Omega'}\mathcal{K}(|\mathbf{x}-\mathbf{y}|)\Psi\big(\mathbf{y},U(\mathbf{y})\big)\,d\mathbf{y}, \tag{10}$$

$$\lambda_\theta \equiv \frac{\lambda^\dagger}{\theta S_3}\exp\big((-C_0 - C_1)/\theta\big),$$

$$\mathcal{K}(|\mathbf{x}-\mathbf{y}|) = -|\mathbf{x}-\mathbf{y}|^{-1}, \quad \Psi\big(\mathbf{y},U(\mathbf{y})\big) \equiv -\exp\big(-\alpha\mathbf{y}^2 - U(\mathbf{y})\big).$$

For $\theta > 0$ we get $\lambda_\theta > 0$, the mapping $\widehat{\mathfrak{G}}(U)$ is compact (in $L_2(\Omega)$) nonlinear operator, since for it the conditions of the Nemytskij–Vainberg theorem (Krasnosel'sky 1964) are satisfied ($\int_\Omega\int_\Omega \mathcal{K}^2(|\mathbf{x}-\mathbf{y}|)\,d\mathbf{x}d\mathbf{y} = \mathcal{K}_\Omega < \infty$, $\Psi\big(\mathbf{y},U(\mathbf{y})\big) \in C(\Omega\otimes\mathbb{R})$ and $|\Psi\big(\mathbf{y},U(\mathbf{y})\big)| \leq g(\mathbf{y}) + C_\Psi\cdot|U|$, $g \in L_2(\Omega)$, $g(\mathbf{y}), C_\Psi > 0$).

Consider the expression $\Psi^\dagger \equiv \int_0^U \Psi\big(\mathbf{y},U\big)dU = \exp(-\alpha\mathbf{y}^2)\cdot(\exp(-U) - 1)$; it is obviously, $\Psi^\dagger \leq \tau_1|U| + \tau_2$ ($\mathbf{x}\in\Omega$), where $\tau_{1,2} > 0$, $\tau_1 < 1/\lambda_{\mathcal{K}}$, $\lambda_{\mathcal{K}}$ is a maximal eigenvalue of integral equation kernel $\mathcal{K}$. Then, in accordance with Theorem 2.8 in (Zabreiko et al 1975), the Hammerstein equation has at least one solution $U_0(\mathbf{x})$. We will deal with the question of the uniqueness of a solution or the presence of many solutions in the next paragraph.

## 3. Solutions and cosmological consequences

We will assume that for $\lambda = (\lambda_\theta)_0$ the equation (10) has nontrivial solution $U = U_0$. Consider the Fredholm's determinant $\mathcal{D}(\lambda)$ (Zemyan 2012) for integral kernel of linearized (in the vicinity $O(U_0)$ of basic solution $U_0$) Hammerstein equation $\widetilde{\mathcal{K}}(\mathbf{x},\mathbf{y}) \equiv \mathcal{K}(|\mathbf{x}-\mathbf{y}|)\cdot\partial\Psi\big(\mathbf{y},U_0(\mathbf{y})\big)/\partial U_0(\mathbf{y})$. After linearization (an application of Frechet derivative) we obtained linear Fredholm self-adjoint compact operator with discrete spectrum (on real axis) (Krasnosel'sky 1964). If $\mathcal{D}\big((\lambda_\theta)_0\big) \neq 0$ (i. e. $(\lambda_\theta)_0$ isn't characteristic value of kernel $\widetilde{\mathcal{K}}(\mathbf{x},\mathbf{y})$), then in the vicinity $O\big((\lambda_\theta)_0\big)$ the equation (10) has unique analytic (by powers of $\big(\lambda_\theta - (\lambda_\theta)_0\big)^j$, $j = 1, 2, ...$) solution $U(\mathbf{x}|\lambda_\theta)$ (Corduneanu 1991), for which $\lim_{\lambda_\theta \to (\lambda_\theta)_0} U(\mathbf{x}|\lambda_\theta) = U_0(\mathbf{x})$ $(\forall \mathbf{x} \in \Omega)$.

Let us consider the solution of (10) in the vicinity $O(U_0) \times O(\lambda_\theta)_0$, and introduce the perturbed characteristic value $\lambda_\theta = (\lambda_\theta)_0 + \xi$ and perturbed solution $U = U_0(\mathbf{x}) + \zeta(\mathbf{x})$. Following the methodology in (Akhmedov 1957), we substitute these expressions in the equation (10) written in the form

$$-\zeta(\mathbf{x}) = (\lambda_\theta)_0 \int_\Omega \frac{\big(\omega_1(\mathbf{y})\zeta + \omega_2(\mathbf{y})\zeta^2 + ...\big)d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} +$$

$$\xi \int_\Omega \frac{\big(\omega_0 + \omega_1(\mathbf{y})\zeta + \omega_2(\mathbf{y})\zeta^2 + ...\big)d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}, \tag{11}$$

and taking into account the Taylor expansion $\Psi\big(\mathbf{x}, U\big) = \sum_{j=0,1,...} \omega_j(\mathbf{x})\zeta^j(\mathbf{x})$ (here $\zeta(\mathbf{x}) = \sum_{k=1} \xi^k \zeta_k(\mathbf{x})$),

$$\omega_0(\mathbf{x}) = \Psi\big(\mathbf{x}, U_0\big) = -\exp\big(-\alpha\mathbf{x}^2 - U_0(\mathbf{x})\big),$$

$$\omega_1(\mathbf{x}) = \frac{\partial\Psi\big(\mathbf{x},U\big)}{1!\partial U}\bigg|_{U=U_0} = \exp\big(-\alpha\mathbf{x}^2 - U_0(\mathbf{x})\big), ..., \tag{12}$$

we obtain a system of recurrent linear and multi-linear equations for variables $\zeta_i(\mathbf{x})$

$$-\zeta_1(\mathbf{x}) = (\lambda_\theta)_0 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\omega_1(\mathbf{y})\zeta_1(\mathbf{y})\, d\mathbf{y} +$$

$$(\lambda_\theta)_0 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\omega_0(\mathbf{y})\, d\mathbf{y}, \tag{13}$$

$$-\zeta_2(\mathbf{x}) = (\lambda_\theta)_0 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\omega_1(\mathbf{y})\zeta_2(\mathbf{y})\, d\mathbf{y} +$$

$$\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big((\lambda_\theta)_0\omega_2(\mathbf{y})\zeta_1^2(\mathbf{y}) + \omega_1(\mathbf{y})\zeta_1(\mathbf{y})\big)\, d\mathbf{y}, ... \tag{14}$$

Since, by assumption, $\mathcal{D}(\lambda_0) \neq 0$ then for linear non-uniform Fredholm $II$ type equation (13) there exists a resolvent $R(\mathbf{x},\mathbf{y};\lambda_0)$ and we can write

$$\zeta_k(\mathbf{x}) = -(\lambda_\theta)_0 \int_\Omega R(\mathbf{x},\mathbf{y};\lambda_0)\mathcal{H}_k\big(\zeta_1(\mathbf{y}), \zeta_1(\mathbf{y}), ...,$$

$$\zeta_{k-1}(\mathbf{y})\big)\, d\mathbf{y} + \mathcal{H}_k\big(\zeta_1(\mathbf{x}), \zeta_1(\mathbf{x}), ..., \zeta_{k-1}(\mathbf{x})\big) \tag{15}$$

(here operator–function $\mathcal{H}_k$ is a sum of all integrals including $\zeta_1, \zeta_2, ...$ up to $(k-1)$–th power: $\zeta_k = -(\lambda_\theta)_0 \int |\mathbf{x}-\mathbf{y}|^{-1}\zeta_1(\mathbf{y})d\mathbf{y} + \mathcal{H}_k(\mathbf{x})$).

Thus, we can consistently and uniquely define all functions $\zeta_k$. Consequently, we can construct a power series for $\zeta(\mathbf{x}) = \zeta\big(\xi^i, \zeta_i(\mathbf{x})\big) \equiv \xi^i\zeta_i$ (summation over repeating indices), and it only remains to prove that the constructed series converge (for all values from some convergence circle) for the deviation parameter $\xi$ (this will completely establish the equivalence of the expansion $\zeta\big(\xi^i, \zeta_i(\mathbf{x})\big)$ and the function $\zeta(\mathbf{x})$).

Based on the properties of the Lyapunov–Schmidt integral operator with a weakly polar kernel (Lemma 8.1 in (Krasnosel'sky 1964)), and explicit forms of $\omega_k \propto \exp(-\alpha\mathbf{x}^2 - U)/k!$, we can state $|\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\omega_k(\mathbf{y})d\mathbf{y}| < Z_1(=\text{const})$ $(k = 0, 1, ..., \forall\mathbf{x} \in \Omega)$. Next, by definition, $|R(\mathbf{x},\mathbf{y};\lambda_0)| < Z_2(=\text{const})$ $(\forall\mathbf{x},\mathbf{y} \in \Omega)$. Then, we can write the majorant series for series $\zeta\big(\xi^i, \zeta_i(\mathbf{x})\big)$. Let us introduce an algebraic equation $\zeta^\dagger = Z_3\big(\xi + \xi\zeta^\dagger + (\xi + (\lambda_\theta)_0)\cdot((\zeta^\dagger)^2 + (\zeta^\dagger)^3 + ...)\big)$, $Z_3 \equiv Z_1(1 + Z_2|(\lambda_\theta)_0|)$. We substitute $\zeta(\mathbf{x}) = \xi^j\varkappa_j|_{j=1,2,...}$ into the last equation and then compare the coefficients at different powers of deviation variable $\xi$:

$$\varkappa_1 = Z_3, \quad \varkappa_2 = Z_3(\varkappa_1 + |(\lambda_\theta)_0|\cdot\varkappa_1^2), \tag{16}$$

$$\varkappa_3 = Z_3\big(\varkappa_2 + \varkappa_1^2 + 2|(\lambda_\theta)_0|\cdot\varkappa_1\varkappa_2 + |(\lambda_\theta)_0|\varkappa_1^3\big), ...$$

Consequently, $|\zeta_k(\mathbf{x})| < \varkappa_k$ $(\mathbf{x} \in \Omega)$, and the convergence region of the series $\zeta = Z_3\big(\xi + \xi\zeta + ...\big)$ is equivalent to the convergence region of the series $\zeta(\mathbf{x}) = \xi^j\zeta_j(\mathbf{x})|_{j=1,2,...}$. To define the convergence region of $\zeta = \xi^i\varkappa_i$ we ought to investigate the implicit function $\zeta = Z_3\big(\xi + \xi\zeta + ...\big)$ as function $\zeta = \widetilde{\zeta}(\xi)$

$$\widetilde{\zeta}(\xi) = Z_3\xi + (Z_3\xi - 1)\zeta^\dagger + Z_3(\xi + |(\lambda_\theta)_0|)\cdot((\zeta^\dagger)^2 + (\zeta^\dagger)^3 + ...) = 0. \tag{17}$$

By Implicit Function Theorem (Deimling 2010), since $\partial\widetilde{\zeta}/\partial\zeta^\dagger = -1$ for $\xi = \zeta^\dagger = 0$, then there exists a convergence circle with positive radius for series $\zeta^\dagger = Z_3(\xi + \xi\zeta^\dagger + (\xi + |(\lambda_\theta)_0|)((\zeta^\dagger)^2 + (\zeta^\dagger)^3 + ...)$. Consequently, there exists a function $\zeta(\mathbf{x})$, and the function

$$U(\mathbf{x}) = U_0(\mathbf{x}) + \zeta(\mathbf{x}) = U_0(\mathbf{x}) +$$

$$\big(\lambda_\theta - (\lambda_\theta)_0\big)\zeta_1(\mathbf{x}) + \big(\lambda_\theta - (\lambda_\theta)_0\big)^2\zeta_2(\mathbf{x}) + ..., \tag{18}$$

which is a holomorphic solution of the Hammerstein equation (10) in the vicinity $O\big((\lambda_\theta)_0\big)$ (herewith $\lim_{(\lambda_\theta)\to(\lambda_\theta)_0} U = U_0$).

We recall some of the properties of linear compact operator $\widehat{L}\phi = \lambda\int_\Omega \mathcal{K}(|\mathbf{x}-\mathbf{y}|)\phi(\mathbf{y})d\mathbf{y}$. The kernel of this operator is symmetric, of weak polar type; consequently, the operator $\widehat{L}$ belongs to Hilbert–Schmidt type of operators. The existence of at least one characteristic function is known since Kellogg (Kellogg 1922). Moreover, there exists a sequence of characteristic values and corresponding eigenfunctions of the investigated kernel of the linear integral operator (Theorem 146 in (Titchmarsh 1957)). In accordance with (Kalmenov and Suragan 2011) we can consider

$$\lambda_{\ell,j} = R_\Omega^{-2}\cdot\big(\varphi_j^{(\ell+1/2)}\big)^2, \quad \ell \geq 0, \quad j \geq 1, \tag{19}$$

where $\varphi_j^{(\ell+1/2)}$ are the roots of the transcendental equation

$$(2\ell+1)J_{\ell+1/2}\big(\varphi_j^{(\ell+1/2)}\big) + \frac{\varphi_j^{(\ell+1/2)}}{2}\big(J_{\ell-1/2}\big(\varphi_j^{(\ell+1/2)}\big) -$$

$$J_{\ell+3/2}\big(\varphi_j^{(\ell+1/2)}\big)\big) = 0, \tag{20}$$

where $J_\nu(...)|_{\nu\in\mathbb{R}}$ refers to the Bessel function of fractional order. The eigenfunctions corresponding to each eigenvalue

$\lambda_{\ell,j}$ can be represented, in spherical coordinates, in the form $\phi_{\ell,j,m}(r,\theta,\chi) = J_{\ell+1/2}(\sqrt{\lambda_{\ell,j}}r)Y_\ell^m(\theta,\chi)$ ($|m| \le \ell$), $Y_\ell^m(\theta,\chi) = P_\ell^m(\cos(\theta))\cos(m\chi)$. It should be noted, that we can consider the Hammerstein equation (10) for an arbitrarily shaped region $\Omega$, but for spherical region we assume $\ell = m = 0$ ($Y_0^0 = \sqrt{1/(4\pi)}$). The kernel $\widetilde{\mathcal{K}}(\mathbf{x},\mathbf{y}) = -|\mathbf{x}-\mathbf{y}|^{-1}\omega_1(\mathbf{y},U_0(\mathbf{y}))$ of the Fredholm equation for potential belongs to Schmidt class, and can be transformed to symmetrical form

$$\widetilde{\mathcal{K}}(\mathbf{x},\mathbf{y}) = -\sqrt{\omega_1(\mathbf{y})/\omega_1(\mathbf{x})}|\mathbf{x}-\mathbf{y}|^{-1}\sqrt{\omega_1(\mathbf{y})\omega_1(\mathbf{x})}. \quad (21)$$

In this case the resolvent for kernel $\widetilde{\mathcal{K}}(\mathbf{x},\mathbf{y})$ is equal to $-\sqrt{\omega_1(\mathbf{y})/\omega_1(\mathbf{x})}R_1(\mathbf{x},\mathbf{y};\lambda)$, where $R_1$ is a resolvent for symmetric kernel $-|\mathbf{x}-\mathbf{y}|^{-1}\sqrt{\omega_1(\mathbf{y})\omega_1(\mathbf{x})}$. In fact, $\omega_1(\mathbf{y})$ is a weight for the Newtonian kernel. We denote $\lambda_1 \equiv (\lambda_\theta)_0 = \lambda_{0,1}(\widetilde{\mathcal{K}})$ as the characteristic number to which the eigenfunction corresponds $\phi_1 \sim J_{1/2}(\sqrt{\lambda_1}r)$. For simplification of further calculations we will assume $\omega_1 \equiv 1$ (this can always be accomplished by multiplying of left-hand and right-hand sides of (11) by $\sqrt{\omega_1}$ and redefining of $\sqrt{\omega_1}\zeta \to \zeta'$, $\omega_0/\sqrt{\omega_1} \to \omega_0'$ etc.). We will be looking for solutions of (11) in the form of a series $\zeta = \xi^i\zeta_i|_{i=1,2,...}$. After substituting the last series we obtain a system of recurrent integral equations

$$-\zeta_1(\mathbf{x}) = \lambda_1\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\zeta_1(\mathbf{y})d\mathbf{y} + \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\omega_0 d\mathbf{y}, \quad (22)$$

$$-\zeta_2(\mathbf{x}) = \lambda_1\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\zeta_2(\mathbf{y})d\mathbf{y} + \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big(\lambda_1\omega_2\zeta_2^2(\mathbf{y}) + \zeta_1(\mathbf{y})\big)d\mathbf{y}, ...$$

According to the Fredholm alternative, the condition of existence of the solution of the first equation of the system will be the orthogonality of characteristic function $\phi_1$ and of the second term in the right-hand side of (19): $\widehat{\mathfrak{H}}(\phi_1) \equiv \int_\Omega \omega_0(\mathbf{y})\phi_1(\mathbf{y})d\mathbf{y} = 0$. This case is physically unrealizable (for $\phi_1, \phi_2, ...$), which can be checked directly; this means the absence (in the neighborhood of the characteristic value $\lambda_k$, $k \ge 1$) of the analytic solution of the nonlinear equation for gravitational potential. Therefore, we turn to the case $\widehat{\mathfrak{H}}(\phi_1) \ne 0$. Then, the equation (14) is unsolvable (condition of Fredholm theorem is absent), and consequently, the analytic series $\zeta = \xi^i\zeta_i$ (for integer indices and powers) do not exist. However, we can consider the representation $\zeta(\mathbf{x})$ as a Puiseux series: $\zeta = \xi^{k/2}\zeta_k|_{k=1,2,...}$. Let's denote $\xi^{1/2} \equiv \nu$, then equation (11) takes the form

$$-\zeta(\mathbf{x}) = \nu^2\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big(\omega_0 + \zeta(\mathbf{y}) + \omega_2\zeta^2(\mathbf{y}) + ...\big)d\mathbf{y} + \lambda_1\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big(\zeta(\mathbf{y}) + \omega_2\zeta^2(\mathbf{y}) + ...\big)d\mathbf{y}, \quad (23)$$

and above–mentioned, the Puiseux series takes the form $\zeta = \nu^k\zeta_k|_{k=1,2,...}$. Substituting this series into equation (23), we obtain an infinite system of equations

$$-\zeta_1(\mathbf{x}) = \lambda_1\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\zeta_1(\mathbf{y})\,d\mathbf{y}, \quad (24)$$

$$-\zeta_2(\mathbf{x}) = \lambda_1\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\zeta_2(\mathbf{y})\,d\mathbf{y} + \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big(\lambda_1\omega_2\zeta_1^2(\mathbf{y}) + \zeta_1(\mathbf{y})\big)d\mathbf{y}, \; ..., \quad (25)$$

$$-\zeta_n(\mathbf{x}) = \lambda_1\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\zeta_n(\mathbf{y})\,d\mathbf{y} + \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big(2\lambda_1\omega_2\zeta_1(\mathbf{y})\zeta_{n-1}(\mathbf{y}) + \mathfrak{Q}(\zeta_1,...,\zeta_{n-1})\big)d\mathbf{y}, \quad (26)$$

Then we obtain $\zeta_1(\mathbf{x}) = \mathfrak{E}_1\cdot\phi_1(\mathbf{x})$, $\mathfrak{E}_1 = \mathrm{const}$. The condition (by Fredholm alternative) of existence of solution of equation (25) can be written as

$$\int_\Omega\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big(\lambda_1\omega_2\zeta_1^2(\mathbf{y}) + \omega_0\big)\phi_1(\mathbf{x})d\mathbf{x}d\mathbf{y} = 0, \quad (27)$$

or, after integration by the variable $\mathbf{x}$: $\int\big(\lambda_1\omega_2\zeta_1^2(\mathbf{y}) + \omega_0\big)\phi_1(\mathbf{y})d\mathbf{y} = 0$. If we substitute in this formula the obtained above expression $\zeta_1 = \mathfrak{E}_1\cdot\phi_1$, then for constant $\mathfrak{E}_1$ we have an explicit expression

$$\mathfrak{E}_1 = \pm\sqrt{-\frac{\int_\Omega \omega_0\phi_1(\mathbf{y})d\mathbf{y}}{\int_\Omega \lambda_1\omega_2\zeta_1^3(\mathbf{y})d\mathbf{y}}} \quad (28)$$

The equation (25) may be written as

$$-\zeta_2(\mathbf{x}) = \mathcal{P}_2(\mathbf{x}) + \mathfrak{E}_1\phi_1(\mathbf{x}), \quad \mathcal{P}_2(\mathbf{x}) \equiv \sum_{j=2,...}\frac{\phi_j(\mathbf{x})}{\lambda_j-\lambda_1}\int_\Omega\big(\lambda_1\omega_2\zeta_1^2(\mathbf{y}) + \omega_0\big)\phi_j(\mathbf{y})d\mathbf{y}. \quad (29)$$

In general case $n \ge 3$ we obtain

$$-\zeta_3(\mathbf{x}) = \mathcal{P}_2(\mathbf{x}) + \mathfrak{E}_3\phi_1(\mathbf{x}), \quad (30)$$

$$\mathcal{P}_2(\mathbf{x}) \equiv \sum_{j=2,...}\frac{\phi_j}{\lambda_j-\lambda_1}\int_\Omega\big(2\lambda_1\omega_2\zeta_1(\mathbf{y})\zeta_2(\mathbf{y}) + \mathfrak{Q}(\zeta_1,\zeta_2)\big)\phi_j(\mathbf{y})d\mathbf{y},$$

$$-\zeta_n(\mathbf{x}) = \mathcal{P}_n(\mathbf{x}) + \mathfrak{E}_n\phi_1(\mathbf{x}), \quad (31)$$

$$\mathcal{P}_n(\mathbf{x}) \equiv \sum_{j=2,...}\frac{\phi_j}{\lambda_j-\lambda_1}\int_\Omega\big(2\lambda_1\omega_2\zeta_1(\mathbf{y})\zeta_{n-1}(\mathbf{y}) + \mathfrak{Q}(\zeta_1,\zeta_2,...,\zeta_{n-1})\big)\phi_j(\mathbf{y})d\mathbf{y},$$

$$\frac{\mathfrak{E}_n}{\mathfrak{E}_1}\int_\Omega \omega_0\phi_1(\mathbf{y})d\mathbf{y} = \int_\Omega\big(2\lambda_1\omega_2\zeta_1(\mathbf{y})\mathcal{P}_n(\mathbf{y}) + \mathfrak{Q}(\zeta_1,\zeta_2,...,\zeta_{n+1})\big)\phi_1(\mathbf{y})d\mathbf{y}.$$

Here $\mathcal{P}_n(\mathbf{x})$ can be associated to Ruston pseudoresolvent (Ruston 1948). Consequently, we can write the series $\zeta = \xi^{k/2}\zeta_k|_{k=1,2,...}$, this series formally satisfies the equation (11); we ought to demonstrate the convergence of this series in the neighborhood $O(\xi)$. Let us introduce: 1) constant $\mathfrak{X}_0$, defined by condition $|\mathfrak{E}_1\phi_1(\mathbf{x})| = |\zeta_1(\mathbf{x})| < \mathfrak{X}_0$ ($\forall\mathbf{x} \in \Omega$); 2) function $S(z) = (|\lambda_1| + \nu^2)\mathfrak{M}z^2/(1 - z/\rho^\ddagger) + \nu^2(\omega_0^{(m)} + z)$, where $|\omega_0| < \omega_0^{(m)}$, $\rho^\ddagger \in (0, \rho^\ddagger_{max})$, $\rho^\ddagger_{max}$ is a convergence radius of the series $\omega_2(\mathbf{x}) + \omega_3(\mathbf{x})z + \omega_4(\mathbf{x})z^2 + ...$, $|\omega_2(\mathbf{x})| < \mathfrak{M}$. We substitute in the definition $S(z)$ the decomposition $z = \nu\mathfrak{X}_0 + \nu^2(\mathfrak{X}_1 + \mathfrak{Y}_1) + \nu^3(\mathfrak{X}_1 + \mathfrak{Y}_1) + ...$ and formally obtain series $S(z) = \nu^2S_2 + \nu^3S_3 + ....$

The value $S_n(z)$ is a majorant function for expression $2\lambda_1\omega_2\zeta_1(\mathbf{x})\zeta_{n-1}(\mathbf{x})+\mathfrak{Q}(\zeta_1,\zeta_2,...,\zeta_n)$ for condition: $\mathfrak{X}_n+\mathfrak{Y}_n$ is a majorant function for $\zeta_{n+1}(\mathbf{x})$, $n=1,2,...$

We denote: 1) $(1/\mathfrak{E}_1)\int_\Omega \omega_0\phi_1(\mathbf{x})d\mathbf{x} = \mathfrak{N}(=\text{ const})$; 2) $\mathfrak{Y}_n = S_{n+1}\rho^\ddagger_{max}$, 3) $(\mathfrak{N}+2|\lambda_1|\mathfrak{M}\mathfrak{X}_0a^2)\mathfrak{X}_n = S_{n+1}a^2$ $(a>|\phi_1(\mathbf{x})|)$. Consequently, $|\mathcal{P}_n(\mathbf{x})|<\mathfrak{Y}_n$, $\mathfrak{E}_{n+1}<\mathfrak{X}_n$. Let introduce functions $\mathfrak{X}\equiv\nu^2\mathfrak{X}_1+\nu^3\mathfrak{X}_2+...$, $\mathfrak{Y}\equiv\nu^2\mathfrak{Y}_1+\nu^3\mathfrak{Y}_2+...$, then above given definition of variables $\mathfrak{X}_1,...,\mathfrak{X}_n,...$ and $\mathfrak{Y}_1,...,\mathfrak{Y}_n,...$ is equivalent to solving of the system of equations

$$\mathfrak{Y}=\rho^\ddagger_{max}\big((|\lambda_1|+\nu^2)\frac{\mathfrak{M}(\nu\mathfrak{X}_0+\mathfrak{X}+\mathfrak{Y})^2}{1-(\nu\mathfrak{X}_0+\mathfrak{X}+\mathfrak{Y})/\rho^\ddagger}+$$

$$\nu^2(\omega_0^{(m)}+\nu\mathfrak{X}_0+\mathfrak{Y}+\mathfrak{X})\big), \tag{32}$$

$$(|\mathfrak{N}|+2|\lambda_1|\mathfrak{M}\mathfrak{X}_0a^2)\nu\mathfrak{X}=a^2\Bigg((\nu^2+|\lambda_1|)\frac{\mathfrak{M}(\nu\mathfrak{X}_0+\mathfrak{X}+\mathfrak{Y})^2}{1-(\nu\mathfrak{X}_0+\mathfrak{X}+\mathfrak{Y})/\rho^\ddagger} \tag{33}$$

$$+\nu^2\big(\rho^\ddagger_{max}+\nu\mathfrak{X}_0+\mathfrak{X}+\mathfrak{Y}\big)-\nu^2\big(\rho^\ddagger_{max}+\mathfrak{M}|\lambda_1|\mathfrak{X}_0^2\big)\Bigg).$$

Let us replace the variables: $\mathfrak{X}=\nu\mathfrak{X}^\dagger$, $\mathfrak{Y}=\nu\mathfrak{Y}^\dagger$. Then the system (32)–(33) takes the form

$$\Theta_1=\mathfrak{Y}^\dagger-\nu\mathfrak{Y}^\dagger\frac{\mathfrak{X}_0+\mathfrak{Y}^\dagger+\mathfrak{X}^\dagger}{\rho^\ddagger}-\rho^\ddagger_{max}\Bigg((\nu^2+|\lambda_1|)$$

$$\mathfrak{M}\nu(\mathfrak{X}_0+\mathfrak{Y}^\dagger+\mathfrak{X}^\dagger)+ \tag{34}$$

$$+\nu\big(\rho^\ddagger_{max}+\nu(\mathfrak{X}_0+\mathfrak{Y}^\dagger+\mathfrak{X}^\dagger)\big)\big(1-\nu\frac{\mathfrak{X}_0+\mathfrak{Y}^\dagger+\mathfrak{X}^\dagger}{\rho^\ddagger})\Bigg)=0,$$

$$\Theta_2=(|\mathfrak{N}|+2|\lambda_1|\mathfrak{X}_0\mathfrak{M}a^2)\mathfrak{X}^\dagger-(|\mathfrak{N}|+$$

$$2|\lambda_1|\mathfrak{X}_0\mathfrak{M}a^2)\mathfrak{X}^\dagger\nu\frac{\mathfrak{X}_0+\mathfrak{Y}^\dagger+\mathfrak{X}^\dagger}{\rho^\ddagger}- \tag{35}$$

$$-(|\lambda_1|+\nu^2)\mathfrak{M}(\mathfrak{X}_0+\mathfrak{Y}^\dagger+\mathfrak{X}^\dagger)^2a^2-$$

$$-\big(\nu(\mathfrak{X}_0+\mathfrak{Y}^\dagger+\mathfrak{X}^\dagger)-\mathfrak{M}|\lambda_1|\mathfrak{X}_0^2\big)\big(1-\nu\frac{\mathfrak{X}_0+\mathfrak{Y}^\dagger+\mathfrak{X}^\dagger}{\rho^\ddagger}\big)=0.$$

The Jacobi determinant of the latter system of equations: $\Delta=D(\Theta_1,\Theta_2)/D(\mathfrak{X}^\dagger,\mathfrak{Y}^\dagger)=-|\mathfrak{N}|<0$ for $\mathfrak{X}^\dagger=\mathfrak{Y}^\dagger=\nu=0$. Consequently, series defined variables $\mathfrak{X}$ and $\mathfrak{Y}$, are converging in the vicinity of the point $\nu=0$. The series $\nu\mathfrak{X}_0+\nu^2(\mathfrak{X}_1+\mathfrak{Y}_1)+\nu^3(\mathfrak{X}_2+\mathfrak{Y}_2)+...$ has convergence circle (with center in the point $\nu=0$). This indicates that the Puiseux series $\zeta=\xi^{k/2}\zeta_k|_{k=1,2,...}$ converge in the vicinity of the point $\xi=0$.

We can conclude that the Hammerstein equation for gravitational potential in the vicinity $O(\lambda_1)\ni\lambda$ (where $\lambda_1$ is one of the characteristic values of linear Fredholm equation for Newton potential) have two nonholomorphic solutions of the form

$$U(\mathbf{x})=U_0(\mathbf{x})+(\lambda_\theta-(\lambda_\theta)_0)^{1/2}\zeta_1(\mathbf{x})+$$

$$(\lambda_\theta-(\lambda_\theta)_0)\zeta_2(\mathbf{x})+(\lambda_\theta-(\lambda_\theta)_0)^{3/2}\zeta_3(\mathbf{x})+... \tag{36}$$

Now we return to the equation (10) and assumption: $\big((\lambda_\theta)_0,U_0\big)$ is a characteristic value and eigenvalue of Hammerstein operator $\widehat{\mathfrak{G}}(U)$. The above analysis of the structure of the solutions of the Hammerstein equation utilized the fact that the kinetic temperature in the system was assumed to be positive $(\theta>0)$. For the case of negative temperatures $(\theta<0)$, the properties of the solutions of the equation for the potential change slightly, although the methodology of their constructive continuation on the parameter plane remains the same. Since the presence of negative temperatures in the system is due to the effect of antigravity (due to the presence of the cosmological term in the integral equation), we can assume the Newtonian kernel to be continuous and bounded function (although we can also use the Hilbert-Schmidt theory), and, on the basis of Theorem 2 in (Jingxian and Bendong 1997), conclude that the equation (10) has at least two analytic solutions outside the points of the spectrum of the Fréchet derivative of the Hammerstein operator $(\lambda_{\theta<0})_k(\widehat{\mathfrak{G}}')$. In the vicinity of the characteristic points corresponding to the solutions of the nonlinear equation and simultaneously coinciding with the special points of the resolvent of the linearized equation, the solutions are continued algebraically and can be written out in the form of Puiseux series on half-integer degrees of deviation from the solution points.

In physical terms the expression (36) can be written as

$$\Phi(\mathbf{x})=(C_1^\ddagger+c^2\Lambda/12)|\mathbf{x}|^2+U_0(\mathbf{x})\sum_i\big(ANG\mathcal{J}(\theta)\exp(C_1^\ddagger/\theta)-$$

$$(\lambda_\theta)_0\big)^{i/2}, \tag{37}$$

$$\mathcal{J}(\theta>0)=-\exp(-v_{max}^2/(2\theta))v_{max}\theta+$$

$$\theta^{3/2}\sqrt{\pi/2}\,\mathrm{erf}\big(v_{max}/\sqrt{2\theta}\big),$$

$$\mathcal{J}(\theta<0)=\exp(v_{max}^2/(2\theta))v_{max}\theta-$$

$$\theta\sqrt{-\pi\theta/2}\,\mathrm{erf}\big(v_{max}/\sqrt{-2\theta}\big),$$

where $U_0(\mathbf{x})$ is a solution of the Hammerstein equation for the potential (existing due to the properties of the corresponding integral operator), $\lambda_1(=\lambda_\theta)_0)$ is the characteristic value of kernel $\omega_1/|\mathbf{x}-\mathbf{y}|$ (for Fredholm equation) in the vicinity of the point $\lambda_\theta\in((\lambda_\theta)_0-\epsilon,(\lambda_\theta)_0+\epsilon)$. If $sgn\big(\int_\Omega\omega_0\phi_1(\mathbf{y})d\mathbf{y}\big)\neq sgn\big(\int_\Omega\lambda_1\omega_2\phi_1^3(\mathbf{y})d\mathbf{y}\big)$, then for $\lambda>\lambda_1$ there exist two solutions, and for $\lambda<\lambda_1$ there aren't solutions. For $sgn\big(\int_\Omega\omega_0\phi_1(\mathbf{y})d\mathbf{y}\big)=sgn\big(\int_\Omega\lambda_1\omega_2\phi_1^3(\mathbf{y})d\mathbf{y}\big)$, then for $\lambda<\lambda_1$ there exist two solutions, and for $\lambda>\lambda_1$ there aren't solutions.

Thus, it has been shown that the solution of the system of Vlasov-Poisson equations in the case of using the energy substitution into the stationary Vlasov equation and choosing the quasi-Maxwell distribution (depending on the kinetic temperature) as a basis solution of the kinetic equation, leads to the nonlinear Hammerstein integral equation for the potential including repulsion due to the presence of the cosmological term in Eq.(1). The Hammerstein equation in essence is an equation for the self-consistent field in the system of massive many-particles on cosmological scales. This leads to the simultaneous realization in the cosmological system with self-consistent field of two types of ordering — a significant increase of the of matter density and its decrease (at one-dimensional motion of matter in the channel there appear walls with high value of gravitational potential, and voids — practically empty spaces between walls). Oscillations of eigenfunctions even in the simplest case of spherical symmetry of the Green's function can form analogs of interference patterns, which leads to the production of structure formation in a homogeneous medium far from the line connecting the massive objects. Since the eigenfunctions of the linearized version of the equation for the potential are proportional to the Bessel function (with a rapidly decreasing weight function), the loss of periodicity of solutions in physical space becomes obvious. Since for nonlinear integral operators it is possible to establish the the continuum character of the spectrum, one can observe the effect of secondary solutions (on the hyperplane of parameters near the basic solution of the equation) coexisting with the initial ones, which leads to the construction of the second type of periodicity - translation of the solution isotropically along all directions. Thus, the composition of solutions with a potential of two types in the presence of additional translational transgression due to the structure of the spectrum, results in a distribution of field and matter in space peculiar to the cosmic web. The scaling of the semi-periodic structures corresponds to the scale of the involved constant, i.e. the cosmological constant in agreement with observational data (Gurzadyan, Fimin, Chechetkin 2023a,b). Incidentally, the change of the kinetic temperature at the zero-point transition can lead to dipole type structures involving a repeller.

The study performed above of the properties of solutions of the nonlinear inhomogeneous integral Hammerstein equation obtained from the Dirichlet boundary value problem allows, without the need to invoke the assumption of the existence of a correlation of fluctuations, in a completely deterministic manner, to describe the existence in the vicinity of equilibrium points of many-particle systems (at different scales) defined by secondary solutions with significantly different properties. In the above analysis it was also proven that such solutions exist nonlocally. The presence of analytical and non-holomorphic branches of solutions of the equation for the potential given by Eq.(10) from a physical point of view in distinguished directions with sharp gradients of the density parameter of the medium has to be associated with the walls as the internal spaces between the voids. It should be noted that, the presence of the cosmological term in the Hammerstein equation leads to the necessity of introducing of the concept of a modified kinetic temperature as defined by Vlasov (Vlasov 1961, 1978). In contrast to the linear analysis based on the properties of the Fredholm equations of the second kind, which, as was previously established by the authors (Gurzadyan, Fimin, Chechetkin 2022, 2023a,b), should lead to periodic and mixed-periodic cosmological structures, the methodology proposed in this study is a more refined one, suitable for numerical modeling and can be used in certain cases as an alternative to statistical computations.

## 4. Conclusions

We had performed an analysis of filament formation in the late Universe based on the Vlasov kinetic technique and using the modified law of gravitational interaction with a cosmological constant, Eq.(1). While the primordial density fluctuations are considered to lead to the early structure formation via Zeldovich pancake theory i.e. via stochastic dynamics, the Vlasov self-consistent field mechanism can provide the structure features at late stages of the evolution.

We continued the rigorous analysis of the Vlasov-Poisson equations of (Gurzadyan, Fimin, Chechetkin 2022, 2023a,b). The equations reduced to the Hammerstein integral equation are shown to predict the existence of semi-periodic structures i.e. those peculiar to the cosmic web. The role of the repulsive term, the cosmological constant is a key one, as it is also defining the scaling of the formed semi-periodic structures, i.e. of the voids. In that context the considered mechanism of the formation of the structures in the local Universe is complementing the role of the cosmological constant term in Eq.(1) in the appearance of two galactic flows, the local one determined by the in McCrea-Milne non-relativistic model, and the global one determined by the Friedmannian equations, while the discrepancy of the Hubble constants of the flows thus naturally explains the Hubble tension.

Thus, the Hubble tension (Riess et al 2024a,b; Scolnik et al 2024a,b) and other observational indications on the possible genuine difference in the properties of the early and late Universe can serve as tests for the discussed above mechanism of the structure formation in the local scale. Particularly, the measurement by the Dark Energy Spectroscopic Instrument (DESI) collaboration (Said 2024) of the local value of the Hubble constant and the distance to the Coma cluster using the fundamental plane of galaxies confirms that tension. An additional informative window to trace the cosmological evolution is due to the DESI released first year data (Adame et al 2024a,b,c) on the baryon acoustic oscillation using several tracers and which revealed additional tensions with the $\Lambda CDM$ model. These observational data do support the approach of the present study, i.e. the need of development of different, more refined descriptions of the properties of the late Universe. Particularly, this approach with the scaling radius $r_c = \left(Gm/(3\Lambda c^2)\right)^{1/3}$ defining the role of the cosmological constant term in Eq.(1), enables to fit the dynamics of galactic flow in the vicinity e.g. of the Virgo supercluster, the Laniakea supercluster (Gurzadyan and Stepanian 2021a,b).

The revealing of the dominating role of self-consistent gravitational field in the formation of the cosmic structures in local cosmological scales can be considered as the main result of the performed rigorous analysis. The refined comparison of the predictions of the considered mechanism with the observational surveys on the voids and of the filaments, e.g. to study the distributions of their scales vs the redshift, etc. can therefore, be of particular interest.

*Acknowledgements.* We are thankful to the referee for valuable comments. VMC is acknowledging the Russian Science Foundation grant 20-11-20165.

## References

Adame A.G. et al, 2024a, arXiv:2404.03000
Adame A.G. et al, 2024b, arXiv:2404.03001
Adame A.G. et al, 2024c, arXiv:2404.03002
Akhmedov K,T., 1957, Uspekhi Mat.Nauk, 12, 135
Arnold V.I., 1982, Comm. Petrovskij seminar, 8, 21
Arnold V.I., Shandarin S.F., Zeldovich Ya.B., 1982, Geophys. Astrophys. Fluid Dynamics, 20, 111

Bajardi, F., D'Agostino, R., Benetti, M., De Falco V. and Capozziello S., 2022, Eur. Phys. J. Plus 137, 1239
Bateman H., Erdelyi A., 1955, *Higher Transcendental Functions, V.3*, McGraw-Hill Book Company, (New York – Toronto – London).
Bouche F., Capozziello S., Salzano V., Umetsu K., 2022, Eur. Phys. J. C, 82, 652
Capozziello, S., Lambiase, G., 2022, Eur. Phys. J. Plus 137, 735
Capozziello, D'Agostino R., 2023, Phys. Dark Universe, 42, 101346
Capozziello S., Sarracino G., De Somma G., 2024, Universe, 10, 140
Corduneanu C., 1991, *Integral Equations and Applications*, (Cambridge University Press, Cambridge–N.Y.–Port Chester–Melbourne–Sydney)
Dainotti M. et al, 2023, arXiv:2301.10572
Deimling K., 2010, *Nonlinear Functional Analysis*, (Dover Publications, Inc., New York)
Di Valentino E., et al, 2021 Class. Quant. Grav., 38, 153001
Dupaigne L., 2011, *Stable solutions of elliptic partial differential equations*, Boca Raton (FL): Chapman & Hall/CRC
Fimin N.N., Chechetkin V.M., 2020, *The Physical Foundations of Hydrodynamic Processes: Macroscopic and Kinetic Approaches*, World Scientific
Gurzadyan V.G., 1985, Observatory, 105, 42
Gurzadyan V.G., 2019, Eur. Phys. J. Plus, 134, 14
Gurzadyan V.G., Stepanian A., 2018, Eur. Phys. J. C, 78, 632
Gurzadyan V.G., Stepanian A., 2019, Eur. Phys. J. C, 79, 169
Gurzadyan V.G., Stepanian A., 2020, Eur. Phys. J. C, 80, 24
Gurzadyan V.G., Stepanian A., 2021a, Eur. Phys. J. Plus, 136, 235
Gurzadyan V.G., Stepanian A., 2021b, A&A, 653, A145
Gurzadyan V.G., Fimin N.N., Chechetkin V.M., 2022, A&A, 666, A149
Gurzadyan V.G., Fimin N.N., Chechetkin V.M., 2023a, A&A, 672, A95
Gurzadyan V.G., Fimin N.N., Chechetkin V.M., 2023b, A&A, 677, A161
Hewitt E. and Savage L. J., 1955, Trans. Am. Math. Soc., 80, 470
Jingxian S. and Bendong L., 1997, Nonlin. Anal.Theor. Meth. Appl., 29, 1277
Kalmenov T.Sh., Suragan D., 2011, *A Boundary Condition and Spectral Problems for the Newton Potential, in: Modern Aspects of the Theory of Partial Differential Equations* (Operator Theory: Advances and Applications, V. 216, eds. M. Ruzhansky and J. Wirth, Birkhauser, p.187)
Keller H.B., Langford W.F., 1972, Arch. Rat. Mech. Anal. , 48, 83
Kellogg O.D., 1922, Math. Annalen, 86, 14
Kravtsov A.V., 2013, ApJ Lett, 764, L31
Krasnosel'sky M. A., 1964, *Topological Methods in the Theory of Nonlinear Integral Equations*, (Macmillan Co.)
McCrea W.H., Milne E.A., 1934, Q. J. Math. 5, 73
Nowakowski M., 2001, Int. J. Mod. Phys. D, 10, 649
O'Regan D., Meehan M., 1998, *Existence Theory for Nonlinear Integral and Integrodifferential Equations*, Kluwer Academic Publishers
de Pedreira I., et al, 2024, Phys. Rev. D 109, 103525
Peebles P.J.E., 1993, *Principles of Physical Cosmology*, Princeton University Press, Princeton
Pokhozhaev S.I., 2010, Diff. Equations, 46, 530
Riess A.G., 2020, Nature Review Physics, 2, 10
Riess A.G. et al, 2024a, ApJ, 962, 17
Riess A.G. et al, 2024b, ApJ, 977, 120
Ruston A.F., 1948, Proc. London Math. Soc., 2, 53
Said, K., et al., 2024, arXiv:2408.13842
Scolnic D. et al, arXiv:2409.14546
Scolnic D. et al, arXiv:2412.08449
Shandarin S.F., Zeldovich Ya.B., 1989, Rev. Mod. Phys., 61, 185
Titchmarsh E.C., 1957, *Introduction to the Theory of Fourier Integrals*, (Oxford University Press, Oxford)
Vedenyapin V., Sinitsyn A., Dulov E., 2011, *Kinetic Boltzmann, Vlasov and Related Equations*, Elsevier Insights.
Vedenyapin V.V., Fimin N.N., Chechetkin V.M., 2021, Eur. Phys. J. Plus, 136, 670
Vlasov A.A., 1961, *Many Particle Theory and its Application to Plasma*, Gordon and Breach
Vlasov A.A., 1978, *Nonlocal Statistical Mechanics*, Nauka, Moscow, (in Russian)
Zabreiko P.P., Koshelev A.I., Krasnoselsky M.A., Mikhlin S.G., Rakovshchik L.S., Stetsenko V.Ya., 1975, *Integral Equations*, (Noordhoff International Pub., Leyden, 443p.)
Zeldovich Ya.B., 1970, A&A, 5, 84
Zeldovich Ya.B., 1981, *Non-relativistic cosmology*, 181, Editor's appendix 1 to Russian edition of S. Weinberg, *The First Three Minutes*, Energoizdat, Moscow (in Russian)
Zemyan S.M., 2012, *The Classical Theory of Integral Equations: A Concise Treatment,* (Springer Science+Business Media)