

# CMB signature of a super-Hubble inhomogeneity in the gravitational field enclosing the present Hubble volume

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Repeated studies of the cosmic microwave background (CMB) based on data from the Wilkinson Microwave Anisotropy Probe have revealed an apparent asymmetry in the distribution of temperature fluctuations over the celestial sphere. The studies indicate that the amplitudes of temperature fluctuations are higher in one hemisphere than in the other. We consider whether this asymmetry could originate from a large scale inhomogeneity in the gravitational field enclosing the present Hubble volume. We examine what effect the presence of an inhomogeneity in the gravitational field of size larger than the present Hubble radius would have on the temperature distribution of the CMB and start eliciting its observational signature in the CMB power spectrum. The covariance function  $\langle a_{lm}^* a_{l'm'} \rangle$  contains, in addition to the diagonal  $C_l$  entries of the conventional CMB temperature anisotropy power spectrum, non-diagonal entries. We find that specific non-diagonal entries of the covariance function are sensitive to the strength of the inhomogeneity, while the diagonal  $C_l$  entries are not. These non-diagonal entries, which are not present in the case of a homogeneous background geometry, are observational signatures of a large-scale inhomogeneity in the background geometry of the universe. Furthermore, we find that an inhomogeneity in the gravitational potential of super-Hubble size would yield a power asymmetry in the CMB with maximal asymmetry at an angle of  $90^\circ$  to the CMB dipole axis. The axis of the CMB power asymmetry was recently estimated by Eriksen et. al. to be located at angles between  $83^\circ$  and  $96^\circ$  to the CMB dipole axis, which is consistent with the prediction of our model. This implies that the location of the observed power asymmetry in the CMB sky could be accounted for by a large-scale inhomogeneity in the gravitational field enclosing the present Hubble volume.

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## I. INTRODUCTION

Precision measurement and analysis of the cosmic microwave background (CMB) over the past two decades has exposed an amazingly detailed picture of the early universe, a picture in which not only the processes and structure of the early universe is imprinted, but also the signature of any subsequent gravitational processing of this structure. This makes the CMB an extraordinarily rich and powerful tool for probing and depicting physical processes in the early universe.

Hence, analysis of the CMB has uncovered evidence of how the observed structure in the universe has emerged: In the primordial universe, stochastic processes in the form of quantum fluctuations in primordial fields drove fluctuations in the gravitational field which subsequently gave rise to fluctuations in the matter and photon densities. These primordial density fluctuations are today imprinted in the temperature fluctuations in the CMB, and the properties of the primordial processes are encoded in the statistical properties of the CMB. In order to understand the primordial universe, it is therefore of utmost importance to understand the statistical properties of the CMB and decode its statistical information.

Full-sky analysis of data from the Wilkinson Microwave Anisotropy Probe (WMAP) are in good agreement with the predictions of simple inflationary models that prescribe a nearly scale free, Gaussian primordial power spectrum and a universe that is flat and homogeneous at large scales [1, 2]. The CMB spectrum expected in such a universe is statistically isotropic, implying that the expectation value of CMB observables are rotationally invariant. There is, however, an apparent asymmetry in the power of CMB fluctuations between two different hemispheres [3][4–6] of the CMB sky. Various explanations of the anomaly have been proposed. New inflationary models [7–9] may yield asymmetric power spectra, which may account for the CMB power asymmetry. Other authors have invoked conventional physics, such as non-linear gravitational effects of local inhomogeneities [10], or dust-filled voids in the local universe [11].

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Here, we are taking a simplistic phenomenological perspective where we consider whether the presence of a large scale inhomogeneity in the gravitational field enclosing the present Hubble volume could account for the power asymmetry[23]. At this stage, we are not concerned with how the inhomogeneity came about, but it is natural to assume that it has primordial origin. However, independent of its origin, a gravitational inhomogeneity in the form of a large scale perturbation to the gravitational potential could possibly manifest itself through a visible asymmetry in the CMB power spectrum. If successful, we may be able to capture the essentials of the physical phenomena under study in a rather model independent way, without excluding relevant models that may account for the phenomena.

If we were to detect large-scale asymmetries in the CMB temperature distribution, the determination of their origin, whether they originate from asymmetries in the primordial power spectrum or from large scale inhomogeneities in the gravitational field, would in any case provide us with valuable additional insight into the nature of the processes in the primordial universe.

Given an early inflationary universe, large scale fluctuations in the gravitational field will directly depict primordial processes, because as they leave the horizon early in the inflationary era, large scale scalar perturbations to the gravitational field freeze. Moreover, they would stay more or less constant until now, only decaying slightly throughout the history of the universe [12].

How could such large-scale perturbations to the gravitational field - of size larger than the observable universe - be observed? This is the first question that we would like to answer in this paper. Under the assumption that our observable universe is immersed in a scalar metric perturbation of super-Hubble size, the aim of this paper is to determine what effect the presence of such a perturbation would have on the CMB temperature and start to elicit its observational signature in the CMB spectrum.

Given the puzzling evidence of a hemispherical power asymmetry in the CMB [3], a natural follow-up question is: Could an hemispherical power asymmetry in the CMB be caused by a large-scale inhomogeneity in the gravitational field of super-Hubble size? Posing these questions, we make no assumptions about the origin of his perturbation or any physical mechanism that might have caused it. Our goal is simply to determine what effect the presence of such a perturbation to the gravitational potential would have on the observed CMB temperature and to elicit its observational signature in the CMB spectrum.

Naively, one could perhaps attempt to answer these questions by looking for an axis of maximal anisotropy in the CMB spectrum [3] and correlate the apparent CMB anisotropy with the expected anisotropy from a super-Hubble perturbation to the gravitational potential. There is, however, already such an axis of maximal anisotropy: the CMB dipole, which originates from our motion relative to the CMB rest frame and by far dominates the CMB anisotropy spectrum. We will therefore have to look for the signature of a secondary anisotropy axis in the residual temperature map after first removing the CMB dipole.

This paper starts by computing the expected temperature anisotropy from a super-Hubble perturbation to the gravitational potential, as seen from the CMB rest frame. Then, we transform this result to a moving frame of observation, before we factor out the best fit dipole. Treating the resulting residual as a lowest order perturbation to the temperature distribution expected from a standard, homogeneous background cosmology, we can therefore factor the residual into the temperature distribution of an arbitrary homogeneous background cosmology. By doing a standard multipole expansion of the temperature distribution, we are finally able to obtain the signature effect of the super-Hubble perturbation on the CMB spectrum.

## II. SUPER-HUBBLE PERTURBATIONS

Let us briefly review the large scale solutions to the first order perturbation equations for a  $\Lambda$ CDM cosmology. More details can be found in Appendix A. and in [12]. In conformal Newtonian gauge, the space-time metric takes the form

$$ds^2 = a^2(\eta) \left( - (1 + 2\Psi(\eta, x)) d\eta^2 + (1 + 2\Phi(\eta, x)) \delta_{ij} dx^i dx^j \right), \quad (1)$$

assuming implicit summation of spatial indices.  $\eta$  is the conformal time, while  $\Psi$  and  $\Phi$  are the two scalar metric perturbations. In this section, we will restrict our attention to the evolution of the gravitational potentials  $\Phi$  and  $\Psi$ , and will therefore discard any variable without significant contribution to their evolution.

Before we continue, let us define what we mean by a super-Hubble perturbation. As we primarily will work in the spatial domain, we need a definition of super-Hubble perturbations in the spatial domain that is compatible with the conventional definition in the frequency domain. In the frequency domain, a super-Hubble perturbation is a mode with wave length much larger than the comoving Hubble radius  $\mathcal{H}^{-1}$ , where  $\mathcal{H} \equiv \dot{a}/a$ . A super-Hubble mode with wave vector  $k_i$  will therefore have  $k^2 \ll \mathcal{H}^2$ . Similarly, for any perturbation variable  $X$ , we define a spatial domain super-Hubble perturbation to be any spatial perturbation with spatial derivatives that satisfy  $|\partial_i X/X| \ll \mathcal{H}$  and

$|\partial^2 X/X| \ll \mathcal{H}^2$ . By applying this definition to a Fourier mode  $\sim e^{ik \cdot x}$ , we see that the definitions of spatial-domain and frequency domain super-Hubble perturbations are consistent.

At super-Hubble scales, the temporal dependence of the perturbation variables can be separated from the spatial dependence, allowing us to write the potential as a product of a fixed spatial perturbation and a time-dependent factor:

$$\Phi \simeq -\Psi \simeq \Phi(\eta)\Phi_0(x) \quad (2)$$

where  $\Phi_0(x)$  is the primordial spatial perturbation. Now, since  $\Phi$  is a super-Hubble perturbation, we have  $\mathcal{H}^{-1}\partial_i\Phi_0 \ll \Phi_0$ . Consequently, within the present Hubble volume, higher-order derivatives of  $\Phi_0$  can be ignored, and  $\Phi_0$  may be approximated by its first order Taylor expansion around the position  $x_0$  of the observer:

$$\Phi_0(x) \simeq \Phi_0(x_0) + \partial_i\Phi_0(x_0)(x^i - x_0^i) \quad (3)$$

The spatial coordinates  $\{x^i\}$  are arbitrary, and we need to fix them in order to proceed. Let us choose cartesian spatial coordinates  $x, y, z$  that make the potential, and therefore the metric of eq. (1) manifestly axially symmetric to first order in  $\partial_i\Phi_0(x_0)$ . Align the  $z$  axis with the gradient  $\nabla\Phi_0(x_0)$  and let the origin be the observer's position. Then the linear expansion of the potential in eq. (3) takes the form

$$\Phi_0(z) \simeq -k(z - z_0) \quad (4)$$

where the constant  $k \equiv (\delta^{ij}\partial_i\Phi_0(x_0)\partial_j\Phi_0(x_0))^{1/2}$  and  $z_0$  is a constant. With this approximation, the metric of eq. (1) now takes an axially symmetric form:

$$ds^2 = a^2(\eta) \left( -(1 + 2\Psi(\eta, z))d\eta^2 + (1 + 2\Phi(\eta, z))(dx^2 + dy^2 + dz^2) \right) \quad (5)$$

In the following, we will use the space-time metric of eq. (5) as the background metric. This background can be treated as a perturbation to the homogeneous background  $ds^2 = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$ . To first order in the perturbation variables, this perturbation evolves independently of other perturbations, a result of the perturbation equations being linear. We are therefore able to treat the effect of a super-Hubble metric perturbation independently of other effects.

### III. THE EFFECT OF A SUPER-HUBBLE METRIC PERTURBATION ON THE ANGULAR CMB TEMPERATURE DISTRIBUTION

In order to compute the CMB temperature perturbation arising from a super-Hubble metric perturbation, we need to solve the equations of motion for photons arriving from an arbitrary direction  $(\theta, \varphi)$ . In the following, we will assume a space-time metric on the form

$$ds^2 = a^2(\eta) \left( -A(\eta, z)d\eta^2 + A(\eta, z)^{-1}(dx^2 + dy^2 + dz^2) \right), \quad (6)$$

where  $A(\eta, z) \equiv 1 + 2\Psi(\eta, z)$ , which is an idealization of the general case, but reduces to eq. (5) to first order in  $\Phi$  in the case  $\Psi = -\Phi$ .

Once we have computed the trajectories of free-streaming photons arriving from an arbitrary direction, we will be able to compute the gravitational potential at the time of emission. Assuming CMB photons were emitted simultaneously at a uniform temperature at recombination, the current CMB temperature will primarily depend on the gravitational potential at the time and place of emission as well as any subsequent development of the gravitational potential along the paths of the CMB photons.

#### A. Computing null geodesics

The first item on our agenda is to compute the trajectories of photons moving in the inhomogeneous background defined by eq. (6). These trajectories are null geodesics, determined by the geodesic equations. The geodesic equations are derived by minimizing the action  $S = \int d\lambda \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$  for a massless particle moving in the background of eq. (6). Null geodesics, as well as certain geometric quantities, like angles, are invariant under conformal transformations. We therefore choose to compute the null geodesics in a different conformal frame, using the conformally related metric

$$d\bar{s}^2 = -A(\eta, z)d\eta^2 + A(\eta, z)^{-1}(dx^2 + dy^2 + dz^2) \quad (7)$$

The reason for this is that the equations of motion simplify considerably in this metric. Notice that, even if the null geodesics and certain other quantities are invariant under conformal transformations, most physical properties of the photons depend on the conformal frame. We will therefore take care to use the physical spacetime metric of eq. (6) when computing conformally variant physical properties of the photons, such as the photon energy.

Two constants of motion,  $v_x$  and  $v_y$ , arise because coordinates  $x$  and  $y$  are cyclic coordinates of the action. The equations of motion for coordinates  $x$  and  $y$  are, using overdots to denote differentiation with respect to the affine parameter  $\lambda$ :

$$A^{-1}\dot{x} = v_x, A^{-1}\dot{y} = v_y \quad (8)$$

Furthermore, the equation for the time coordinate  $\eta$  is

$$\frac{d}{d\lambda}(A\dot{\eta}) - \frac{\partial A}{\partial \eta}\dot{\eta}^2 = 0 \quad (9)$$

Finally, the equation for the  $z$  coordinate becomes

$$\ddot{z} - \frac{1}{A}\frac{\partial A}{\partial \eta}\dot{z}\dot{\eta} + A\frac{\partial A}{\partial z}v_T^2 = 0 \quad (10)$$

where the constant  $v_T$  is defined by  $v_T^2 \equiv v_x^2 + v_y^2$ . These equations of motion are generally not solvable by analytical means. However, we will introduce an idealized model that allows analytical treatment. The model is motivated by the observation that, as shown in Appendix A, large scale perturbations to the gravitational potential stay more or less constant until the universe ceases to be matter dominated.

The idealized model that will be used from now on asserts a time-independent gravitational potential with  $\Phi(\eta) = \text{const} = 1$  in the metric of eq. (6). In this case, the metric is conformally static, with

$$A(z) = 1 + 2k(z - z_0) = A_0 + 2kz \quad (11)$$

Notice that this is an unrealistic approximation at all but the largest scales. At large, super-Hubble scales, still being a very crude approximation, it is nevertheless a useful idealization that grants us the luxury of analytical treatment and the ability of exploring important characteristics of super-Hubble perturbations and their effect on the CMB temperature anisotropy spectrum. As we will see later on in this section, the temperature perturbation arising from a super-Hubble metric perturbation is roughly proportional to the gain or loss in gravitational potential between the time of emission of a CMB photon and the time of its observation. This implies that, even if in a more realistic case, the super-Hubble perturbation to the gravitational potential decays slightly at late times, this decay is uniform at super-Hubble scales. Therefore, we would expect the directional distribution of the temperature perturbation to remain the same, modulo a time-dependent factor. Disregarding this time-dependent factor, which is what we are doing, makes parameter estimation imprecise, but does not change the qualitative signature of the super-Hubble perturbation in the CMB spectrum.

Given this approximation, the equations of motion for a massless particle now take the form

$$A\dot{\eta} = \mu = \text{const} \quad (12)$$

$$\ddot{z} + A(z)\frac{dA}{dz}v_T^2 = 0 \quad (13)$$

$$A^{-1}\dot{x} = v_x, A^{-1}\dot{y} = v_y \quad (14)$$

The four-velocity  $u^\mu \equiv \dot{x}^\mu$  is a null vector, which implies

$$A^{-2}\dot{z}^2 + v_x^2 + v_y^2 = \dot{\eta}^2 \quad (15)$$

Let us define the constant  $A_0 \equiv A(0) = 1 - 2kz_0$  and fix the integration constant  $\mu$  by setting  $\mu = A_0$ [24]. Furthermore, let us fix the affine parameter  $\lambda$  by demanding  $z(0) = 0$ . This gives  $\dot{\eta}(0) = 1$ . We notice that impact angles and photon velocity are conformally invariant quantities, and we are therefore free to compute them in any conformal frame. Therefore, the physical velocity of the photon in the frame of the observer is  $\hat{u}^i = A^{-1}\dot{x}^i/\dot{\eta} = \dot{x}^i/A_0$ , and eq. (15) simply states that  $|\hat{u}| = 1$ . We therefore find that the integration constants  $v_x$  and  $v_y$  are the velocities of the photon in the frame of the observer at the instance of observation. Defining  $\theta$  as the observed angle of the photon, as measured from the direction of positive  $z$ , we have:

$$v_T^2 = \sin^2\theta, \cos\theta = -\hat{u}^z = \dot{z}(0)/A_0 \quad (16)$$

The boundary conditions are now defined that allow us to find unique solutions to the equations of motion. Treating the case of vertical motion with  $\theta = 0$  or  $\theta = \pi$  separately, we obtain the following solutions to the equation of motion, expressed in terms of the observation angle  $\theta$ :

$$z(\lambda) = \begin{cases} \frac{A_0}{2k} \left( \frac{1}{\sin \theta} \sin(\theta - 2k \sin \theta \lambda) - 1 \right) & 0 < \theta < \pi \\ -\lambda \cos \theta & \theta = 0, \pi \end{cases} \quad (17)$$

$$\eta(\lambda) = \begin{cases} \eta_0(\theta) - \frac{1}{2k} \ln(\tan(\frac{1}{2}(\theta - 2k \sin \theta \lambda))) & 0 < \theta < \pi \\ -\frac{1}{2k \cos \theta} \ln \left( 1 - \frac{2k \cos \theta}{A_0} \lambda \right) & \theta = 0, \pi \end{cases} \quad (18)$$

where  $\eta_0(\theta) \equiv \frac{1}{2k} \ln(\tan(\theta/2))$ . For simplicity, we have set  $\eta = 0$  at the time of observation. We can now easily express the solution in terms of the conformal time  $\eta$ . We get

$$z(\theta, \eta) = \begin{cases} \frac{A_0}{2k} \left( \frac{1}{\sin \theta} \frac{1}{\cosh(2k(\eta_0 - \eta))} - 1 \right) & 0 < \theta < \pi \\ \frac{A_0}{2k} (e^{-2k\eta \cos \theta} - 1) & \theta = 0, \pi \end{cases} \quad (19)$$

Eq. (19) is parametrized by quantities that are identical in the two conformal frames, so it is a parametrization that also is valid in the physical frame. It allows us to compute the  $z$  coordinate of a photon observed at an angle  $\theta$  at any time  $\eta$  in its history. Next, we will use this to compute the effect of a super-Hubble metric perturbation on the temperature distribution of the CMB.

### B. CMB temperature distribution as measured by a static observer in the CMB rest frame

In the preceding section, we solved the null-geodesic equations in the background of a super-Hubble metric perturbation, asserting, for simplicity, that the perturbation is static in comoving coordinates:

$$ds^2 = a^2(\eta) \left( -A(z) d\eta^2 + A(z)^{-1} (dx^2 + dy^2 + dz^2) \right) \quad (20)$$

where  $A(z) \equiv 1 + 2k(z - z_0)$ . In this metric, the null geodesic equation for the time coordinate  $\eta$  takes the form  $\dot{p}_0 = 0$ , where  $p_0 \equiv \partial L / \partial \dot{\eta} = -a^2 A \dot{\eta}$ . This implies that  $p_0$  is a constant of motion, which we can relate to the energy of the photon. The photon energy as observed by a static observer with 4-velocity  $u^\mu = ((-g_{00})^{-1/2}, 0, 0, 0)$  is

$$E = -p_\mu u^\mu = -p_0 (-g_{00})^{-1/2} = -\frac{p_0}{a \sqrt{A(z)}}$$

For a photon with observed frequency  $\nu$ ,  $E = h\nu$ . We therefore have

$$a(\eta) \sqrt{A(z)} \nu = -p_0 / h = \text{const},$$

a generalization of both the gravitational redshift formula of a static universe and the cosmological redshift formula of a homogeneous, expanding universe. We will assume that CMB photons were emitted at time  $\eta_*$  at a uniform temperature  $T_*$ . Since the peak frequency of a black-body spectrum is proportional to the temperature of the radiation, we can use this relationship to relate the temperature at the time of emission to the observed temperature  $T(\theta)$  of CMB photons being observed at angle  $\theta$  by a static observer at the origin:

$$T(\theta) \sqrt{A_0} = T_* a(\eta_*) \sqrt{A(z(\theta, \eta_*))}, \quad (21)$$

where  $A(z(\theta, \eta_*))$  defines the gravitational field at the time and place of emission, and  $z(\theta, \eta_*)$  is given by eq. (19). Let  $T_0$  be the average CMB temperature at present, and let  $\Theta(\theta)$  to be the temperature perturbation defined by  $T(\theta) = T_0(1 + \Theta(\theta))$ . Since  $T_0/T_* = a(\eta_*)$ , the redshift relationship gives the following expression for the temperature perturbation  $\Theta(\theta)$  in terms of the gravitational potential at the time of emission for a photon observed at angle  $\theta$ :

$$\Theta(\theta) = \sqrt{\frac{A(z(\theta, \eta_*))}{A_0}} - 1 \quad (22)$$

With  $A(z) = 1 + 2k(z - z_0)$  and  $z(\theta, \eta)$  given by eq. (19), the expression for the temperature perturbation  $\Theta(\theta)$  simplifies to

$$\Theta(\theta) = \sqrt{\frac{2\chi_*}{\chi_*^2 + 1}} \frac{1}{\sqrt{1 + \frac{(\chi_*^2 - 1)}{(\chi_*^2 + 1)} \cos \theta}} - 1, \quad (23)$$

where the constant  $\chi_*$  is defined by  $2k\eta_* = \ln \chi_*$ .  $k$  is positive while  $\eta_* < 0$ .  $|\eta_*|$  is of the order of the  $H_0^{-1}$ , the inverse of the current Hubble parameter value.  $k$  is the spatial derivative of a super-Hubble perturbation to the potential  $\Phi$ . Therefore, using the definition of a super-Hubble perturbation, we must have  $|k\eta_*| \ll 1$ . Hence,  $\chi_*$  is slightly less than 1. It is natural then to introduce a small, positive, dimensionless parameter  $\Phi_* \equiv (1 - \chi_*)/2$  that will be useful for perturbative expansion of the temperature  $\Theta(\theta)$ . Expanding  $\Phi_*$  in terms of  $(k\eta_*)$ , we have, to lowest order in  $(k\eta_*)$ :

$$\Phi_* = -(k\eta_*) + \mathcal{O}((k\eta_*)^2),$$

$\Phi_*$  can be interpreted as the loss in gravitational potential since the time of emission for photons observed at angle  $\theta = 0$ . We will refer to  $\Phi_*$  as the *potential anisotropy*. Then, to second order in  $\Phi_*$ ,  $\Theta(\theta)$  is

$$\Theta(\theta) = \Phi_* \cos \theta - \Phi_*^2 \left( 1 - \left( 1 + \frac{3}{2} \cos \theta \right) \cos \theta \right) + \mathcal{O}(\Phi_*^3) \quad (24)$$

The first-order term  $\Phi_* \cos \theta$  equals the temperature perturbation expected from photons following the null geodesics of a homogeneous FLRW universe in eq. (21). Hence, higher-order terms stem from deviations from geodesics in a homogeneous space.

In this section, we assumed an observer at rest in the CMB rest frame. In reality, the frame of observation is moving relative to the CMB rest frame. Our motion relative to the CMB rest frame has been measured to about 370 km/s [13]. In the next section, we will transform the result of eq. (23) to a moving frame of observation.

### C. CMB temperature distribution as measured by a moving observer

Let  $\mathcal{O}$  and  $\mathcal{O}'$  be two inertial observers,  $\mathcal{O}$  at rest relative to the CMB, while  $\mathcal{O}'$  moves with constant velocity  $\mathbf{v} = v\hat{v}$ , where  $\hat{v}$  is the unit vector in the direction of movement and  $v$  is the velocity relative to the CMB rest frame. Given that  $\mathcal{O}$  observes a black-body distribution of CMB photons at temperature  $T(\hat{p})$ , with  $\hat{p}$  being the direction vector of the observed photon, observer  $\mathcal{O}'$  will observe a Doppler-shifted black-body spectrum with temperature [14]

$$T'(\hat{p}', \mathbf{v}) = T(\hat{p}) \frac{\sqrt{1 - v^2}}{(1 + v\hat{p}' \cdot \hat{v})} \quad (25)$$

$\hat{p}'$  is the direction vector of the photon as observed by observer  $\mathcal{O}'$ , and  $\hat{p}' \cdot \hat{v}$  is given by the aberration formula [14, 15]:

$$\hat{p}' \cdot \hat{v} = \frac{\hat{p} \cdot \hat{v} - v}{(1 - v\hat{p} \cdot \hat{v})} \quad (26)$$

Next, let us define two constants  $\gamma \equiv \sqrt{\frac{2\chi_*}{\chi_*^2 + 1}}$  and  $\beta \equiv \frac{(1 - \chi_*^2)}{(1 + \chi_*^2)}$ . This allows us to write the temperature perturbation of eq. (23) as  $\Theta(\theta) = \frac{\gamma}{\sqrt{1 - \beta \cos \theta}} - 1 = \frac{\gamma}{\sqrt{1 + \beta \hat{p} \cdot \hat{u}}}$ , where  $\hat{p}$  is the direction vector of the photon, and  $\hat{u}$  is the unit vector along the direction of the gradient  $\partial_i \Phi$  of the super-Hubble perturbation, which we hereafter will call the anisotropy axis.

Now, let us use eq. (25) to transform the temperature distribution of eq. (23) to the frame of observation:

$$T'(\hat{p}', \mathbf{v}) = \frac{\gamma}{\sqrt{1 + \beta \hat{p}' \cdot \hat{u}}} T_d(\hat{p}', \mathbf{v}), \quad (27)$$

Here, we introduced the dipole distribution  $T_d(\hat{p}', \mathbf{v}) \equiv T_0 \sqrt{1 - v^2} / (1 + v\hat{p}' \cdot \hat{v})$ . Let us first compute  $\hat{p} \cdot \hat{u}$  in the frame of observation. Expanding the unit vectors  $\hat{p}$  and  $\hat{u}$  in the CMB rest frame in spherical coordinates as  $\hat{p} \equiv -(\sin \theta_p \cos \phi_p, \sin \theta_p \sin \phi_p, \cos \theta_p)$  and  $\hat{u} \equiv (\sin \theta_u \cos \phi_u, \sin \theta_u \sin \phi_u, \cos \theta_u)$ , we get  $\hat{p} \cdot \hat{u} = -\sin \theta_p \sin \theta_u \cos(\phi_p - \phi_u) - \cos \theta_p \cos \theta_u$ . We can then transform to the frame of observation by using the aberration formula of eq. (26)

and the fact that longitudinal coordinates are invariant under a Lorentz boost along  $\hat{v}$ . We get, using  $\theta \equiv \theta'_p$ ,  $\phi \equiv \phi'_p$ ,  $\phi_u = \phi'_u = 0$  and  $\xi \equiv \cos \theta_u'$ :

$$f(\theta, \phi; \xi) \equiv \hat{p} \cdot \hat{u} = -\frac{1}{(1-v \cos \theta)(1-\xi v)} \left( (1-v^2) \sqrt{1-\xi^2} \sin \theta \cos \phi + (\xi-v)(\cos \theta - v) \right) \quad (28)$$

$(\theta, \phi)$  are now angular coordinates in the frame of observation, and  $\xi$  is the dot product between the anisotropy axis and the direction of motion in the frame of observation:  $\xi \equiv \hat{u}' \cdot \hat{v}$ .

If we assume a completely homogeneous temperature distribution in the CMB rest frame, the distribution observed in the moving frame of observation will, according to eq. (25), be the Doppler-shifted dipole distribution  $T_d(\hat{p}', \mathbf{v})$ . If we average this distribution over the sphere, we get the average temperature in the frame of observation:

$$T'_0 \equiv \frac{1}{4\pi} \int d\Omega' T_d(\hat{p}', \mathbf{v}) = \sqrt{1-v^2} \frac{\text{arctanh}(v)}{v} T_0 = \left( 1 - \frac{v^2}{6} + \mathcal{O}(v^3) \right) T_0$$

We notice that the average temperature in the frame of observation is slightly reduced compared to the static frame. The effect is small, though, and proceeding, we will use  $T'_0 = T_0$ , discarding terms higher than first order in velocity.

#### IV. THE EFFECT OF A SUPER-HUBBLE PERTURBATION ON THE CMB POWER SPECTRUM

Having computed the effect of a super-Hubble perturbation to the gravitational potential on the observed CMB temperature distribution, the next challenge is to elicit its effect on the CMB temperature anisotropy power spectrum. The standard way of computing this power spectrum is to first remove the dipole induced by the observatory's peculiar motion through the CMB sky [16], then decompose the residual temperature distribution into multipoles, and finally compute the covariance matrix of the multipole coefficients. With a homogeneous background geometry, this covariance matrix becomes diagonal. With an inhomogeneous background geometry, however, the covariance matrix obtains non-diagonal entries. Therefore, these non-diagonal entries in the temperature anisotropy covariance matrix represent the signature of the inhomogeneity of the background geometry.

##### A. Factoring out the Doppler dipole

Before computing the CMB power spectrum, the CMB dipole must be removed from the temperature data [16]. Thereafter, the power spectrum may be computed from the residual temperatures. Our moving observer  $\mathcal{O}'$  asserts a homogeneous background, so we will therefore fit the temperature distribution of eq. (27) to that of a Doppler-shifted uniform temperature distribution. The procedure is lengthy, but straight forward. We will therefore describe it briefly. Let us vary both the velocity and the average temperature by writing the fitted velocity as  $v'' \equiv v + \delta v$  and the fitted temperature average as  $T''_0 \equiv T_0(1 + \Delta T_0)$ . Defining the temperature residual as  $\delta T \equiv T_d(\hat{p}', \mathbf{v}'') - T'(\hat{p}', \mathbf{v})$ , we find the best fit dipole by minimizing the averaged squared error  $\int d\Omega' \delta T^2$  with respect to variations in velocity  $v''$  and temperature  $T''_0$ . The values of  $\delta v$  and  $\Delta T_0$  that provide the best fit dipole distribution are

$$\delta v = \Phi_*(1 + \Phi_*)\xi - v\Phi_*(1 - \xi^2), \Delta T_0 = -\frac{1}{2}\Phi_*^2 \quad (29)$$

Next, we will compute the temperature residual that remains after the Doppler dipole has been removed from the temperature distribution. The observed CMB temperature distribution of an assumed homogeneous background metric has two factors; the best fit dipole distribution and the observed residual distribution:

$$T''(\hat{p}', \mathbf{v}'') = T_d(\hat{p}', \mathbf{v}'') (1 + \Theta''(\hat{p}'))$$

Here,  $\Theta''$  is the residual temperature perturbation that remains after factoring out the best fit dipole.

Similarly, the expected temperature distribution of the inhomogeneous background of eq. (20) has three factors; the anisotropy factor given by eq. (23), the dipole distribution and some arbitrary residual temperature distribution  $T \equiv 1 + \Theta$  given a priori:

$$T'(\hat{p}', \mathbf{v}) = \frac{\gamma}{\sqrt{1 + \beta f(\hat{p}', \xi)}} T_d(\hat{p}', \mathbf{v}) (1 + \Theta(\hat{p}'))$$

Using eq. (27), we may then relate the observed temperature perturbation  $\Theta''$  to the given perturbation  $\Theta$ :

$$\Theta''(\hat{p}') = \frac{\gamma}{\sqrt{1 + \beta f(\hat{p}', \xi)}} \frac{T_d(\hat{p}', \mathbf{v})}{T_d(\hat{p}', \mathbf{v}'')} (1 + \Theta(\hat{p}')) - 1 \quad (30)$$

Eq. (30) will be used later in the final computation of the CMB anisotropy power spectrum.

## B. CMB anisotropy spectrum

The temperature distribution  $\Theta''(\hat{p})$  can be expanded in terms of spherical harmonics  $Y_l^m(\hat{p})$  and multipole coefficients  $a_{lm}$  as  $\Theta''(\hat{p}) = \sum_l \sum_{m=-l}^l a_{lm} Y_l^m(\hat{p})$ . This equation can be inverted, expressing the multipole coefficients  $a_{lm}$  in terms of the field  $\Theta''$ :

$$a_{lm} = \int d\Omega Y_l^m(\hat{p})^* \Theta''(\hat{p}) \quad (31)$$

Let us define the CMB covariance function as the expectation value of the square of the multipole coefficients:

$$\mathcal{C}_{l,k}^{m,n} \equiv \langle a_{kn} a_{lm}^* \rangle \quad (32)$$

For a homogeneous background geometry,  $\mathcal{C}_{l,k}^{m,n}$  can be expressed in terms of the  $C_l$  of the conventional CMB anisotropy power spectrum, in which case it takes the simple, diagonal form

$$\mathcal{C}_{l,k}^{m,n} = C_l \delta_{l,k} \delta_{m,n}$$

Now, let us compute  $\mathcal{C}_{l,k}^{m,n}$  for the inhomogeneous background of eq. (20). Using eq. (31), we get

$$\mathcal{C}_{l,k}^{m,n} = \int d\Omega' \int d\Omega Y_k^n(\hat{p}') Y_l^m(\hat{p})^* \langle \Theta''(\hat{p}') \Theta''(\hat{p}) \rangle$$

Now,  $\Theta''$  is given by eq. (30). Define  $\Delta\Theta$  as

$$\Delta\Theta \equiv \frac{\gamma}{\sqrt{1 + \beta f(\hat{p}', \xi)}} \frac{T_d(\hat{p}', \mathbf{v})}{T_d(\hat{p}', \mathbf{v}'')} - 1 \quad (33)$$

Now, using eq. (30), the observed temperature perturbation  $\Theta''$  can be expressed in terms of  $\Delta\Theta$  as follows:

$$\Theta''(\hat{p}') = (1 + \Delta\Theta) (1 + \Theta(\hat{p}')) - 1$$

This allows us to express  $\mathcal{C}_{l,k}^{m,n}$  in terms of  $\Delta\Theta$ :

$$\mathcal{C}_{l,k}^{m,n} = \int d\Omega' Y_k^n(\hat{p}') \Delta\Theta(\hat{p}')^* \int d\Omega Y_l^m(\hat{p})^* \Delta\Theta(\hat{p}) + \int d\Omega' Y_k^n(\hat{p}') (1 + \Delta\Theta(\hat{p}')) \int d\Omega Y_l^m(\hat{p})^* (1 + \Delta\Theta(\hat{p})) \langle \Theta(\hat{p}')^* \Theta(\hat{p}) \rangle \quad (34)$$

The expectation value  $\langle \Theta(\hat{p}')^* \Theta(\hat{p}) \rangle$  can be expressed in terms of the Fourier-transformed temperature distribution  $\Theta(k, \mathbf{k} \cdot \hat{p})$ , the matter overdensity  $\delta(k)$  and the matter power spectrum  $P(k)$  (see eq. (8.65) of [12]):

$$\langle \bar{\Theta}(\hat{p}')^* \bar{\Theta}(\hat{p}) \rangle = \int \frac{d^3 k}{(2\pi)^3} P(k) \frac{\Theta(k, \mathbf{k} \cdot \hat{p}')^*}{\delta(k)} \frac{\Theta(k, \mathbf{k} \cdot \hat{p})}{\delta(k)}$$

The first term on the right-hand side of eq. (34) is second order in the perturbation variables  $\Phi_*$  and  $v$ , so we will discard it in the following. This gives

$$\mathcal{C}_{l,k}^{m,n} = \int d\Omega' Y_k^n(\hat{p}') (1 + \Delta\Theta(\hat{p}')) \int d\Omega Y_l^m(\hat{p})^* (1 + \Delta\Theta(\hat{p})) \int \frac{d^3 k}{(2\pi)^3} P(k) \frac{\Theta(k, \mathbf{k} \cdot \hat{p}')^*}{\delta(k)} \frac{\Theta(k, \mathbf{k} \cdot \hat{p})}{\delta(k)}$$

First, we may expand  $\Theta(k, \mathbf{k} \cdot \hat{p})$  in terms of multipoles  $\Theta_l(k)$ :

$$\Theta(k, \mathbf{k} \cdot \hat{p}) = \sum_l (-i)^l (2l+1) \mathcal{P}_l(\hat{k} \cdot \hat{p}) \Theta_l(k)$$

Furthermore, the Legendre polynomials  $\mathcal{P}_l$  can be expanded in terms of spherical harmonics:

$$\mathcal{P}_l(\hat{k} \cdot \hat{p}) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_l^m(\hat{p}) Y_l^m(\hat{k})^*$$

Now, define the parameter  $\tau \equiv \Phi_* \sqrt{1 - \xi^2}$ . The covariance function  $\mathcal{C}_{l,k}^{m,n}$  then takes the form

$$\mathcal{C}_{l,k}^{m,n} = C_l \delta_{k,l} \delta_{m,n} + \tau (C_k + C_l) K_{l,k}^{m,n} \quad (35)$$



where  $C_l$  is defined by

$$C_l \equiv \frac{2}{\pi} \int dk k^2 \frac{P(k)}{\delta(k)^2} |\Theta_l(k)|^2 \quad (36)$$

and the coefficients  $K_{l,k}^{m,n}$  are defined by the integral

$$K_{l,k}^{m,n} \equiv \tau^{-1} \int d\Omega \Delta\Theta(\hat{p}) Y_l^m(\hat{p})^* Y_k^n(\hat{p}) \quad (37)$$

We see that  $C_l$  represents the variance of the multipole coefficients  $a_{lm}$ , and its definition in eq. (36) is indeed the conventional formula for computing the variance of the multipole coefficients  $a_{lm}$  for a homogeneous background, see p. 242 of [12]. Notice that when the expression for  $C_{l,k}^{m,n}$  in eq. (35) was computed, terms of order  $\tau^2$  were discarded.

Eq. (35) is the general form of the covariance function  $C_{l,k}^{m,n}$ . If the coefficients  $K_{l,k}^{m,n}$  are nonzero, the covariance function is nondiagonal. However, we have yet to compute the coefficients  $K_{l,k}^{m,n}$  and prove that there are indeed nondiagonal entries in the covariance function.

### C. Nondiagonal entries of the covariance function

Next, we will compute the nondiagonal coefficients  $K_{l,k}^{m,n}$  of the covariance function  $C_{l,k}^{m,n}$ . The coefficients are given by eq. (37). Before we are able to compute the integral, we must compute  $\Delta\Theta$  from its definition in eq. (33). To first order in the perturbation variables  $\Phi_*$  and  $v$ , we get

$$\Delta\Theta = \Phi_* \sqrt{1 - \xi^2} \sin\theta \cos\phi \quad (38)$$

Spherical harmonics may be written in terms of the associated Legendre functions  $P_l^m$  (see p. 670 of [17]):

$$Y_l^m(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta) e^{im\phi}$$

$K_{l,k}^{m,n}$  can then be expressed as follows, defining the angular variable  $\mu \equiv \cos\theta$ :

$$K_{l,k}^{m,n} = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} \sqrt{\frac{(2k+1)(k-n)!}{4\pi(k+n)!}} \int_{-1}^1 d\mu \sqrt{1-\mu^2} P_l^m(\mu) P_k^n(\mu) \int_0^{2\pi} d\phi e^{i(n-m)\phi} \cos\phi$$

Now, we may compute the integral over the longitudinal variable  $\phi$ :

$$\int_0^{2\pi} d\phi e^{i(n-m)\phi} \cos\phi = \pi(\delta_{m,n+1} + \delta_{m+1,n})$$

Furthermore, we may use the following recursion relationship for the associated Legendre functions [18, 19]:

$$\sqrt{1-\mu^2} P_l^{m-1}(\mu) = \frac{1}{(2l+1)} (P_{l+1}^m(\mu) - P_{l-1}^m(\mu))$$

as well as the orthogonality relationship (see p. 66 of [19]):

$$\int_{-1}^1 d\mu P_l^m(\mu) P_k^m(\mu) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{l,k}$$

By introducing the coefficients

$$c_l^m \equiv \begin{cases} \sqrt{\frac{l+m}{2l+1}} & -l \leq m \leq l \\ 0 & |m| > l \end{cases} \quad (39)$$

we obtain the following expression for the coefficients  $K_{l,k}^{m,n}$ :

$$K_{l,k}^{m,n} = \frac{1}{2} \left( (c_l^m c_k^m \delta_{m,n+1} - c_l^{-m} c_k^{-m} \delta_{m,n-1}) \delta_{l,k+1} + (c_l^n c_k^n \delta_{m+1,n} - c_l^{-n} c_k^{-n} \delta_{m-1,n}) \delta_{l+1,k} \right) \quad (40)$$

The coefficients  $K_{l,k}^{m,n}$  have the following symmetries:

$$K_{l,k}^{m,n} = K_{k,l}^{n,m} \quad (41)$$

and

$$K_{l,k}^{m,n} = -K_{l,k}^{-m,-n} \quad (42)$$

#### D. Estimating the potential anisotropy vector

The perturbation parameter  $\Phi_*$  measures the loss in gravitational potential for photons arriving along the anisotropy axis  $\hat{u}'$ . Therefore,  $\Phi_* \xi$  measures the gain or loss in gravitational potential for photons arriving along the dipole axis at  $\theta = 0$ , while  $\Phi_* \sqrt{1 - \xi^2}$  measures the gain or loss in gravitational potential for photons arriving from directions orthogonal to the dipole axis, at  $\theta = \pi/2$ .

We refer to the vector  $\Phi_* \hat{u}'$  as the *potential anisotropy vector*.  $\Phi_* \sqrt{1 - \xi^2}$  is then the transverse component of the potential anisotropy vector, which is the component orthogonal to the dipole axis, while  $\Phi_* \xi$  is the longitudinal component of the potential anisotropy vector, which is the component along the dipole axis. Define  $\lambda \equiv \Phi_* \xi$ . The parameter  $\tau \equiv \Phi_* \sqrt{1 - \xi^2}$  has already been introduced. We will refer to  $\tau$  as the *transverse potential anisotropy* and  $\lambda$  as the *longitudinal potential anisotropy*. In order to estimate the direction of the the anisotropy vector, we also need to estimate the longitude of the anisotropy axis in galactic coordinates. Alternatively, by noticing that the temperature anisotropy  $\Delta\Theta = \tau \sin\theta \cos\phi$  has extrema at  $\theta = \pi/2, \phi = \pm\pi$ , we could estimate the direction of the potential anisotropy based on an estimate of the direction of maximal temperature anisotropy perpendicular to the CMB dipole axis plus estimates of the parameters  $\tau$  and  $\lambda$ . If  $\hat{w}$  is the observed direction of maximal temperature asymmetry, we have

$$\Phi_* \hat{u}' = \tau \hat{w} + \lambda \hat{v} \quad (43)$$

We see from eq. (40) that to lowest order in the perturbation parameters, we are only able to estimate the transverse potential anisotropy  $\tau$ . In order to estimate the longitudinal potential anisotropy, the covariance function  $C_{l,k}^{m,n}$  must be evolved to higher order in the perturbation variables. In the scope of this paper, we will restrict ourselves to lowest order estimates, so we will therefore only be able to estimate the transverse potential anisotropy.

From eq. (40) we obtain four sets of estimators for  $\tau$ , one for each of the four terms. The first and third terms give the same estimator, and so do the second and fourth terms. The remaining two estimators can be identified by using the symmetry relation of eq. (42). The basic estimators can then be written

$$\hat{\tau} = \frac{2C_{l,l-1}^{m,m-1}}{(C_l + C_{l-1}) c_l^m c_{l-1}^{m-1}}, m = -(l-2), \dots, (l-1) \quad (44)$$

This basic estimator can be used for estimation of the transverse potential anisotropy  $\tau$  from multipole coefficients  $a_{lm}$  derived from CMB observations. For each multipole index  $l$ , we may sample  $\tau$   $2(l-1)$  times. If we choose a set of  $N$  multipoles ranging from  $l = 2$  to  $l = N + 1$ , we will therefore be able to retrieve  $\sum_{l=2}^{N+1} 2(l-1) = N(N+1)$  independent samples of  $\tau$ , thus reducing the uncertainty of the estimate considerably.

#### E. Hemispherical power asymmetry

Finally, let us have a glance at the hemispherical power asymmetry that would result from the temperature anisotropy  $\Delta\Theta = \tau \sin\theta \cos\phi$  of eq. (38). Designating  $(\theta = \frac{\pi}{2}, \phi = \pi)$  as the north pole, the northern hemisphere is parametrized by  $\pi/2 < \phi < 3\pi/2$ . We would therefore expect a maximal temperature difference between the two hemispheres to be  $\sim 2\tau\Theta$ , yielding a power difference between the two hemispheres of the order of  $\tau C_l$ . Furthermore, given that the north pole is at  $\theta = \pi/2$ , the axis of maximal power asymmetry should be perpendicular to the CMB dipole axis. Given a CMB dipole at  $(l, b) = (264^\circ, 48^\circ)$  [13] and a best fit power asymmetry axis at

TABLE I: Separation angle between the dipole axis and the axis of maximal power asymmetry for different results of Eriksen et. al. [6]. *a) small sky cut applied. b) large sky cut applied*

| Data             | (l,b)        | Separation angle |
|------------------|--------------|------------------|
| ILC <sup>a</sup> | (225°, -27°) | 83°              |
| ILC <sup>b</sup> | (208°, -27°) | 90°              |
| Q-band           | (222°, -35°) | 91°              |
| V-band           | (205°, -19°) | 85°              |
| W-band           | (204°, -31°) | 96°              |

$(l, b) = (225^\circ, -27^\circ)$ [6], the angle separating the two axes is  $83^\circ$ . Table I shows the angle of separation for each of the 5 different analyses made by Eriksen et. al [6]. In each case, the angle of separation between the CMB dipole axis and the axis of maximal power asymmetry lies between  $83^\circ$  and  $96^\circ$ , consistent with the prediction by our model.

We notice that eq. (43) can be applied to estimate the direction of the potential anisotropy from estimates of the power asymmetry axis  $\hat{w}$ , given estimates of parameters  $\tau$  and  $\lambda$ .

An interesting side note is that the anomalous CMB Cold Spot [20–22] is located in the same region of the sky as the north pole of the power anisotropy axis: The cold spot is located at  $(l, b) = (209^\circ, -57^\circ)$ . Given that there is a CMB power asymmetry with a minimal power in this particular region and assuming this power asymmetry can be explained by the presence of a super-Hubble inhomogeneity in the gravitational field, the cold spot will be less extreme, because the expectation value for the CMB temperature would be lower in this particular region. Consequently, the cold spot should be less anomalous in this case, as already noted by Eriksen et. al. [6].

## V. CONCLUSIONS

Prompted by the puzzling evidence of an hemispherical power asymmetry in the CMB [3], this paper started with two simple questions: 1): How could a large-scale inhomogeneity in the gravitational field of super-Hubble size - larger than the observable universe - be observed? And 2): Could an hemispherical power asymmetry in the CMB be caused by a super-Hubble scale inhomogeneity in the gravitational field enclosing the present Hubble volume? Posing these questions, we made no assumptions about the origin of this perturbation or any physical mechanism that might have caused it. Our goal was simply to determine what effect the presence of such a perturbation to the gravitational potential would have on the observed CMB temperature and to elicit its observational signature in the CMB spectrum.

In order to tackle these questions by analytical means, we introduced an idealized model in which large-scale perturbations to the gravitational potential stay constant in comoving coordinates. This model is an unrealistic approximation at all but the largest scales. At large, super-Hubble scales, still being a very crude approximation[25], it is nevertheless a useful idealization that grants us the luxury of analytical treatment and the ability of exploring important characteristics of super-Hubble perturbations and their effect on the CMB temperature anisotropy spectrum. We find that the temperature perturbation arising from a super-Hubble metric perturbation is roughly proportional to the loss in gravitational potential between the time of emission of a CMB photon and the time of its observation. Therefore, even if in a more realistic model, the super-Hubble perturbation to the gravitational potential decays slightly at late times, this decay is uniform at super-Hubble scales. Therefore, given a more realistic model, we still expect the directional distribution of the temperature perturbation to remain the same as in our idealized model, modulo a time-dependent factor. Disregarding this time-dependent factor, which is what we are doing, makes parameter estimation imprecise, but does not change the qualitative signature of the super-Hubble perturbation in the CMB spectrum.

Using our idealized model, we were able to solve the null-geodesic equations of motion and determine the resulting CMB temperature distribution in the CMB rest frame. Transforming the resulting temperature distribution to a moving frame of observation and removing the best fit Doppler dipole, we obtained a residual temperature distribution. By expanding the residual temperature distribution into multipoles, we obtained our main result: the covariance function of the multipole coefficients.

The covariance function contains, in addition to the  $C_l$  entries of the conventional CMB temperature anisotropy power spectrum, non-diagonal entries. We find that the  $C_l$  entries of the conventional anisotropy power spectrum are insensitive to the strength of the potential anisotropy. Thus, the non-diagonal entries, which are not present in the case of a homogeneous background geometry, constitute the main signature of a large-scale inhomogeneity in the background geometry of the universe. This answers the first of our two initial questions.

Regarding the second of our initial questions, an inhomogeneity in the gravitational potential of super-Hubble size would yield a power asymmetry in the CMB with maximal asymmetry at an angle of  $90^\circ$  to the CMB dipole axis. The power asymmetry that was observed appears to be at an angle that lies between  $83^\circ$  and  $96^\circ$  to the CMB dipole axis, which is consistent with the prediction of our model. This is suggestive of a simple explanation for the CMB power asymmetry, because it implies that the location of the observed power asymmetry in the CMB sky can be accounted for by a large-scale inhomogeneity in the gravitational field enclosing the present Hubble volume. At this point, we will not claim that our model can completely account for the CMB power asymmetry. More work remains. In particular, it still remains to be seen whether this model is also able to account for the power asymmetry in quantitative terms.

### Appendix A: Super-Hubble solution to the first order perturbation equations for $\Lambda$ CDM

Let us start by taking a look at the evolution equation for the temperature perturbation  $\Theta \equiv \Delta T/T$ . We use conformal Newtonian gauge and apply the notation and exposition of Dodelson (see ch. 7 of [12]).  $\Theta$  is a function, not only of conformal time  $\eta$  and position  $x$ , but also of photon direction,  $\hat{p}^i$ . The first order perturbation equation for  $\Theta(\eta, x, \hat{p})$  is

$$\frac{\partial \Theta}{\partial \eta} + \frac{\partial \Phi}{\partial \eta} + \hat{p}^i \partial_i \Theta + \hat{p}^i \partial_i \Psi = n_e \sigma_T a (\Theta_0 - \Theta + \hat{p} \cdot \vec{v}_b) \quad (\text{A1})$$

where  $n_e$  is the number density of free electrons,  $\vec{v}_b$  is the baryon velocity and  $\sigma_T$  is the Thomson cross section for electron-photon scattering.  $\Theta_0$  is the temperature monopole:  $\Theta_0(\eta, x) \equiv \frac{1}{4\pi} \int d\Omega \Theta(\eta, \hat{x}, p)$ . Seeking super-Hubble solutions to eq. (A1), we may discard the terms containing spatial derivatives. Integrating the equation by  $\frac{1}{4\pi} \int d\Omega$ , eq. (A1) simplifies, leaving an equation for the monopole  $\Theta_0$  and the potential  $\Phi$ :

$$\frac{\partial \Theta_0}{\partial \eta} + \frac{\partial \Phi}{\partial \eta} = 0 \quad (\text{A2})$$

Here, we have used that  $\int d\Omega \hat{p} \cdot \vec{u} = 0$  for any direction-independent vector  $\vec{u}$ , which implies that  $\int d\Omega \hat{p} \cdot \vec{v}_b = 0$ .

For super-Hubble perturbations we may therefore discard baryon and photon interactions, which occur at much smaller scales. At super-Hubble scales, perturbations to the dark matter and baryon distributions can be treated as a common distribution of matter. Similarly, at super-Hubble scales, perturbations to the distributions of photons and neutrinos can be treated the same, as a common distribution of relativistic radiation.

The two Einstein equations determining the evolution of the metric perturbations take the form

$$3\mathcal{H} \frac{\partial \Phi}{\partial \eta} - \partial^2 \Phi - 3\mathcal{H}^2 \Psi = 4\pi G a^2 (\rho_m^{(0)} \delta + 4\rho_r^{(0)} \Theta_0) \quad (\text{A3})$$

$$\partial^2 (\Phi + \Psi) = 0 \quad (\text{A4})$$

Here,  $\delta \equiv \delta\rho_m/\rho_m^{(0)}$  is the matter overdensity, while  $\Theta_0$  is the monopole moment of the radiation distribution.  $\rho_m^{(0)}$  is the zero-order matter density, which takes the form  $\rho_m^{(0)} = \rho_{\text{cr}} \Omega_m / a^3$ , where  $\rho_{\text{cr}}$  is the critical density today and  $\Omega_m$  is the present matter ratio.  $\rho_r^{(0)}$  is the zero-order radiation density, which takes the form  $\rho_r^{(0)} = \rho_{\text{cr}} \Omega_r / a^4$ .

The equation for the dark matter density perturbation is

$$\frac{\partial \delta}{\partial \eta} + 3 \frac{\partial \Phi}{\partial \eta} + \partial_i v^i = 0, \quad (\text{A5})$$

while the equation for the dark matter velocity  $v^i$  is

$$\frac{\partial v^i}{\partial \eta} + \mathcal{H} v^i + \partial_i \Psi = 0 \quad (\text{A6})$$

Eq. (A4) is easily solved by setting  $\Psi = -\Phi$ . Again, as we seek super-Hubble solutions to the equations, we discard terms with spatial derivatives. In that case, the velocity equations for matter decouple from the other equations, and we are left with the three equations for the three perturbation variables  $\Phi$ ,  $\Theta_0$  and  $\delta$ . Furthermore, as the equations now only contain time derivatives of these variables, we may separate each perturbation variable  $X(\eta, x)$  into a time-independent factor and a time-dependent factor. The time-independent factors can be determined by applying

initial conditions. We apply adiabatic initial conditions, which are  $\Phi(\eta_i, x) = \Phi_0(x)$ ,  $\Theta_0(\eta_i, x) = \frac{1}{2}\Phi_0(x)$  and  $\delta(\eta_i, x) = \frac{3}{2}\Phi_0(x)$ , where  $\eta_i$  is an arbitrarily chosen initial time.  $\Phi_0(x)$  is an arbitrary perturbation in the potential satisfying  $|\partial_i\Phi_0| \ll \mathcal{H}$  and  $\partial^2\Phi_0 \ll \mathcal{H}^2$ . Hence, the time-independent factor of each of the three perturbation variables are  $\Phi_0(x)$ . We will therefore rewrite the variables as follows:  $\Theta_0(\eta, x) \rightarrow \Theta_0(\eta)\Phi_0(x)$ ,  $\delta(\eta, x) \rightarrow \delta(\eta)\Phi_0(x)$  and  $\Phi(\eta, x) \rightarrow \Phi(\eta)\Phi_0(x)$ . This allows us to factor out the spatial dependence of the equations entirely, and we are left with the following ordinary differential equations, using overdots to denote differentiation with respect to  $\eta$ :

$$\dot{\Theta}_0 + \dot{\Phi} = 0 \quad (\text{A7})$$

$$3\mathcal{H}\dot{\Phi} + 3\mathcal{H}^2\Phi = 4\pi G a^2(\rho_m^{(0)}\delta + 4\rho_r^{(0)}\Theta_0) \quad (\text{A8})$$

$$\dot{\delta} + 3\dot{\Phi} = 0, \quad (\text{A9})$$

Applying adiabatic initial conditions  $\Phi(\eta_i) = 2\Theta_0(\eta_i)$  and  $\delta(\eta_i) = 3\Theta_0(\eta_i)$  at an initial time  $\eta_i$ , allows us to integrate eqs. (A7) and (A9). We get

$$\Theta_0(\eta) = \frac{\delta(\eta)}{3}, \delta(\eta) = \frac{9}{2} - 3\Phi(\eta)$$

Changing to a new temporal variable  $x \equiv \ln a$ , eq. (A8) can now be written

$$\frac{d\Phi}{dx} + \Phi = \frac{1}{2} \frac{(\Omega_m e^{-3x} + \frac{4}{3}\Omega_r e^{-4x})}{\Omega_\Lambda + \Omega_m e^{-3x} + \Omega_r e^{-4x}} \left( \frac{9}{2} - 3\Phi \right) \quad (\text{A10})$$

There are no known analytical solutions to eq. (A10), but we may find approximate solutions that cover the entire history of the universe. First, let us seek an approximate solution that covers the radiation and matter-dominated eras. In this case  $a^3 \ll 1$ . Introducing a new variable  $y = \frac{\Omega_m}{\Omega_r} e^x = \frac{\Omega_m}{\Omega_r} a$  allows us to rewrite eq. (A10) as follows:

$$y \frac{d\Phi}{dy} + \left( 1 + \frac{1}{2} \frac{3y+4}{y+1} \right) \Phi = \frac{3}{4} \frac{(3y+4)}{(y+1)} \quad (\text{A11})$$

Eq. (A11) has the following solution that satisfies the initial condition that requires  $\Phi(y) \rightarrow 1$  as  $y \rightarrow 0$ :

$$\Phi = \frac{1}{10y^3} \left( -16 - 8y + 2y^2 + 9y^3 + 16\sqrt{1+y} \right) \quad (\text{A12})$$

We see that  $\Phi$  has an almost constant value of 1 throughout the radiation era, then drops slightly to approach 9/10 in the matter-dominated era.

Next, let us seek an approximate solution to eq. (A11) that is valid in the matter-dominated era and later. In eq. (A11), we can now discard the radiation terms. Defining a new variable  $z \equiv \frac{\Omega_\Lambda}{\Omega_m} e^{3x}$ , we get the equation

$$z \frac{d\Phi}{dz} + \frac{2z+5}{6(z+1)} \Phi = \frac{3}{4} \frac{1}{z+1}$$

Its solution can be expressed in terms of the hypergeometric function  ${}_2F_1$ :

$$\Phi = \frac{3}{2} - \frac{3}{5} (1+z) {}_2F_1 \left( 1, \frac{4}{3}; \frac{11}{6}; -z \right) \quad (\text{A13})$$

This solution has the value  $\Phi(0) = 9/10$ . At present times,  $\Phi \simeq 0.7$  for  $\Omega_m = 0.3$ .

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- [23] It should be noted that, although the visible universe would be inhomogeneous at scales larger than the present Hubble volume (super-Hubble scales), a homogeneous universe at even larger scales is not precluded.
- [24] If we were making this computation in a physical spacetime, we would at this point relate  $\eta j$  to the energy of the photon. However, we must leave this aside, since we are computing the geodesic in an unphysical spacetime. We therefore choose the boundary conditions that are the most convenient.
- [25] We notice that in reality, large scale perturbations to the potential decay uniformly about 30% in a flat  $\Lambda$ CDM universe with  $\Omega_m = 0.3$