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Cosmic Variance in CMB Anisotropies: From 1° to COBE

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Abstract

Cosmic Microwave Background (CMB) anisotropies that result from quantum fluctuations during inflation are explored and the impact of their "cosmic variance" on the ability to use existing data to probe inflationary models is studied. We calculate the rms temperature fluctuation, and its cosmic variance, for a number of experiments and for models with primordial power spectra which range from $n = \frac{1}{2}$ to 1. We find: (1) cosmic variance obscures the information which can be extracted, so a comparison of the rms temperature fluctuation on small scales with the COBE result can fix n to only $\approx \pm 0.2$ at best; (2) measurements of the rms fluctuation on 1° scales may not allow one to unambiguously infer the tensor contribution to the COBE anisotropy; (3) comparison of this contribution with the predictions of inflation are ambiguous if the quadrupole anisotropy alone is utilized. We discuss means for minimizing the uncertainty due to cosmic variance in comparisons between experiments.

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1. Introduction

One of the central purposes of measuring CMB anisotropies is to gain information on the power spectrum of primordial density fluctuations, and from this, information on the physical processes in the early universe which may have produced such a spectrum. The well studied mechanism of inflation allows an *a priori* calculation of the spectrum of fluctuations. Both scalar and tensor fluctuations arise from calculable quantum fluctuations in elementary fields during an inflationary phase in the early universe, and any previously existing density fluctuations are inflated away. We explore here what constraints on the primordial power spectrum can be obtained, both in principle and in practice, from the COBE DMR observations in combination with observations of CMB anisotropies on scales as small as 1°. Comparison with large scale structure analyses (e.g. $\delta T/T|_{SW}$ as inferred from POTENT (Bertschinger & Dekel 1989; Bertschinger, Gorski & Dekel 1990)), could provide useful additional constraints although at present there are large theoretical and observational uncertainties.

When comparing theoretical predictions with observations, particular attention should be paid to the fact that CMB anisotropies resulting from inflation are stochastic. Because of the quantum nature of the process by which fluctuations are generated, only observations over an *ensemble* of universes can allow the unambiguous measurement of fundamental microphysical parameters, even in principle[2]. Constrained to observe only one universe, there remains an irremovable uncertainty in our ability to relate certain CMB measurements, no matter how precise, to predictions of inflationary models (such as the normalization and slope of the power spectrum). For example, inflation predicts a probability distribution for ratios of the moments of CMB temperature fluctuations (or functions thereof), and *not* the ratios themselves. This has important consequences for the normalization of the power spectrum. We term the induced uncertainty the "cosmic variance", not to be confused with other usages of this term. Among our results will be a description of the effect of this variance on the constraints which can be derived from present and future observations of the CMB.

2. The Primordial Power Spectrum and Inflation

The remarkable observations of primordial anisotropies in the CMB by the DMR instrument aboard COBE (Smoot et al. 1992) offered cosmologists the first "unprocessed" glimpse of what may be the initial conditions for the present observed Friedman-Robertson-Walker expansion of the universe. Inferring the primordial power spectrum of fluctuations based on observations of large scale structure had previously required detailed assumptions about both the equation of state of matter inside the horizon and the relationship between the observed structure and the underlying distribution of mass in the universe. Now model predictions can be normalized at COBE scales, which represent modes which had not crossed the horizon at recombination and hence could not have been processed by causal microphysical factors. This represents a significant theoretical advance.

[2] We shall discuss this issue in more detail later in the text.

Moreover, the scales observed by COBE in combination with small scale observations provide a large lever arm for exploring the scale dependence of the power spectrum.

A central quantity of interest has been the slope of the power spectrum of fluctuations. Defining the power spectrum, $P(k)$ in terms of the Fourier transform of the density fluctuation: $P(k) \equiv |\int(\delta\rho/\rho)\exp(ik\cdot x)d^3x|^2$, one finds that the rms density fluctuations on scales with comoving wavenumber, k can be written (e.g. Peebles 1980)

$$\left(\frac{\delta\rho}{\rho}\right)_k^2 = \frac{k^3}{2\pi^2}P(k) \tag{1}$$

One conventionally considers $P(k) \propto k^n$. This assumes a scale-free spectrum, namely that there is no preferred scale for fluctuations set by the underlying physics. Further, $n = 1$ represents a scale-invariant spectrum as predicted in standard inflation (Steinhardt & Turner 1984), up to logarithmic corrections. It has been stressed, coincident with COBE, that a number of inflationary models which attempt to circumvent the problems of new inflation also predict measurable deviations from a scale invariant spectrum, i.e they can predict $n < 1$ (Davis et al. 1992; Gelb et al. 1992; Liddle & Lyth 1992a,b; Lidsey & Coles 1992; Lucchin, Matarrese & Mollerach 1992; Salopek 1992). COBE reports a limit on this spectral index of $n = 1.1 \pm 0.5$, which is consistent with scale invariance as well as with significant deviations from scale invariance. Clearly to get an optimum handle on this spectrum one would like to compare large-scale and small-scale anisotropies in the CMB.

3. CMB anisotropies

It is useful to decompose the temperature fluctuations in spherical harmonics $\Delta T/T(\theta,\phi) \equiv \sum_{lm} a_{lm}Y_{lm}(\theta,\phi)$ and to define the rotationally symmetric quantity $a_l^2 \equiv \sum_m |a_{lm}|^2$. We choose to consider here the so-called rms fluctuation, or correlation function at "zero-lag", as our contact with experiment. This has a simple interpretation and is sensitive to a range of physics from both large and small scales. To be precise we consider

$$\left(\frac{\Delta T}{T}\right)^2 \equiv C(0) = \frac{1}{4\pi}\sum_{l=2}^{\infty}\left[(a_l^2)_S + (a_l^2)_T\right]W_l, \tag{2}$$

where the dipole has been subtracted and W_l is a "window function", which incorporates the experimental sensitivity to each moment. The window functions appropriate to the experiments that we will consider in this paper are described, and derived, in appendix A. As written, $\Delta T/T$ has contributions $(a_l)_S^2$ from scalar (density fluctuations) and $(a_l)_T^2$ from tensor (gravity wave) modes. For the COBE experiment, the window function is simply due to the gaussian smoothing, $W_l = \exp\left[-(l+1/2)^2\sigma^2\right]$ with $\sigma = 0.425 \times 10^\circ$ the smoothing scale (the factor 0.425 relates the gaussian width σ to the FWHM quoted by the COBE group). We will first concentrate on this "window", where $\Delta T/T$ then represents the rms temperature fluctuation over the observed sky. Understanding the cosmic variance of the COBE measurement is important if COBE is to be used to normalize the power spectrum to compare with other experiments.

Inflation predicts that the a_{lm} are independent gaussian random variables ($\Rightarrow a_l^2$ is χ^2-distributed with $2l+1$ d.o.f.) so the rms fluctuation as defined is itself a stochastic quantity (Abbott & Wise 1984; Krauss & White 1992; White 1992). It might be thought that measurements made over the entire sky afford the opportunity to make independent measurements of the quantities which have a cosmic variance, and so reduce it. However from the above, it is clear that this cannot be the case. No measurement of $\Delta T/T$ on our sky, no matter how accurate nor how many independent measurements it corresponds to, can reduce the cosmic variance for a given l. Determining each a_l^2 requires an all-sky average (we will not address the important issue of the effects of incomplete sky coverage in this paper as we are interested here in the *irremovable* cosmic variance. The issue of finite sky coverage is addressed in Scott et al. 1993.). The error in measuring a_l^2 will be reduced by more sky measurements, but not the uncertainty in going from a_l^2 to the average over an ensemble of universes $\langle a_l^2\rangle_{ens}$. (However, as we will see, window functions relevant to small scale measurements can give dominant contributions to $\Delta T/T$ from higher modes, which have more degrees of freedom and smaller cosmic variance, so that the effect of this uncertainty can in principle be lessened, provided one has adequate sky coverage.)

It is not possible in general to find a closed form analytic expression for the distribution of $\Delta T/T$. We outline a simple and accurate numerical procedure for obtaining the distribution in appendix B. It is straightforward, however, to calculate the first few moments of the distribution (since $\Delta T/T$ is a sum of *independent*, χ^2-distributed terms) which give us an estimate of the size of the variance. Write

$$\left(\frac{\Delta T}{T}\right)^2 = \sum_l \frac{a_l^2}{\langle a_l^2\rangle}(l+1/2)\Delta_l \qquad \text{with} \qquad \Delta_l = \frac{2}{2l+1}\frac{\langle a_l^2\rangle}{4\pi}\, W_l \tag{3}$$

so $\overline{(\Delta T/T)^2} \equiv \langle(\Delta T/T)^2\rangle = \sum_l (l+1/2)\Delta_l$. If we define $\Delta^{(n)} = \langle(((\Delta T/T)^2 - \overline{(\Delta T/T)^2})^n\rangle$ then we can use $\Delta^{(2)}$ as an estimate of the (cosmic) variance of $\Delta T/T$ and $\Delta^{(3)}$ as an indicator of the asymmetry of the distribution about the mean (and departures from normality). It is straightforward to compute[3]

$$\Delta^{(2)} = \sum_l (l+1/2)\Delta_l^2 \qquad\qquad \Delta^{(3)} = 2\sum_l (l+1/2)\Delta_l^3. \tag{4}$$

For $n=1$ Sachs-Wolfe fluctuations ($\langle a_l^2\rangle \sim (l+1/2)^{-1}$, $l \gg 1$), the rms fluctuation $\overline{(\Delta T/T(\sigma))^2} \sim$

[3]Our result for $\Delta^{(2)}$ differs from Bond & Efstathiou (1987) by a factor of 2.

$\log \sigma$ while the higher moments are all finite[4]. Thus the "cosmic variance" becomes less important at smaller scales. It is at these small angular scales (less than about 2°) however that the theoretical uncertainties due to cosmological model dependence and non-linear effects become progressively more severe.

To better illustrate the form of the distribution for $\Delta T/T$ we have compared two approximations to the exact distribution (which we checked by doing a Monte-Carlo analysis in which the a_l^2 were drawn at random from χ^2_{2l+1} distributions and $(\Delta T/T)^2$ computed using equation 2.). In the case of only a scalar component in fluctuations of the $n = 1$ Sachs-Wolfe form (specifically $\langle a_l^2 \rangle_S = (l + 1/2)^{-1}$ for $l \geq 2$), we show in Figure 1a the distribution of $\Delta T/T$ smoothed to 10° obtained from the method outlined in appendix B and the Monte-Carlo calculation (see also Table 1). These are in good agreement. Also shown are fits obtained from matching the mean and variance of the distribution to a χ^2 form ($P(x) \propto x^{c_1} e^{-c_2 x}$) and a gaussian. As can be seen, it is not well fit by a gaussian. The fit to a single χ^2 form gets progressively better for larger angles, or tilted spectra. This is because the sum is dominated by the lower multipoles in these cases and is therefore closest to being represented by a single χ^2 distribution. It is important to note that if one attempts to fix the normalization of the power spectra by fitting $\Delta T/T$ to the COBE measurement, there is an uncertainty due to cosmic variance of $\approx 20\%$ (figure 1a). This normalization uncertainty will affect any comparison with smaller scale experiments and reduce what can be inferred. In the future it may be better to normalize the spectrum not by the COBE $\Delta T/T$ but by something less sensitive to the low l multipoles. We will return to this point later.

For smoothing angles smaller than 10°, one can use the second moment of the distribution as an equivalent gaussian error for the purposes of computing confidence levels. We again stress that the distribution is *not gaussian* but for quantities depending only on the asymptotic form of the distribution (e.g. 90, 95, 99% confidence regions) such an approximation is adequate. Nevertheless this skewness should not be ignored when utilizing measurements of either $\Delta T/T$ or the correlation function $C(\theta)$ to place limits on microphysical parameters such as the scale of inflation.

4. The Anisotropy at 1° Scales

On degree scales, for the scalar fluctuations, one must consider in addition to the Sachs-Wolfe terms, other terms such as the Doppler contribution (from motions on the last scattering surface). These contributions depend on the cosmological model (e.g. the Hubble constant, Ω_B and the reionization history of the universe), which governs the growth of fluctuations inside the horizon, and generally require numerical solution of the Boltzmann equation for the evolving photon distribution for an accurate estimation. To calculate $\langle a_l^2 \rangle$ we have used the radiation "power spectrum" $k^3 W_T^2(k)$ of Vittorio & Silk (1992), which includes a "transfer function" for the evolution of modes which have

[4]In practice of course the rms fluctuation will not diverge as $\sigma \to 0$. Physics associated with the size and width of the last scattering surface damps the a_l^2 for high l. This can also be aided by spectral tilt.

crossed the horizon. As an illustrative example of the non-Sachs-Wolfe terms, we calculated the $\langle a_l^2 \rangle$ for a CDM model with $\Omega = 1, h = 0.5$ and $\Omega_B = 0.03, 0.1$ (consistent with the range favoured by nucleosynthesis: Smith et al. 1992, Krauss & Romanelli 1990, Walker et al. 1991). For $n \neq 1$ we multiplied the Sachs-Wolfe result by the ratio of the full calculation to Sachs-Wolfe for $n = 1$. This approximation is expected to be good to better than 10%, as can be seen from analytic work (Atrio-Barandela & Doroshkevich 1993). We normalized the a_l^2 to fit the COBE central value at 10°. The power spectra are shown in Figure 2 for $n = 1.0$ for the range of $k \equiv lH_0/2$ of relevance here. (An unprocessed Sachs-Wolfe power spectrum would go as k^{n-1}.) The $\langle a_l^2 \rangle$ are an integral of the "power spectrum" with measure $j_l^2(2k/H_0)dk/k$ — in effect an average of the "power spectrum" around $k \approx lH_0/2$. Hence the 'Doppler enhancement' of the power spectrum on degree scales translates to an increase in the $\langle a_l^2 \rangle$ for large l over the Sachs-Wolfe result. To obtain $\langle (\Delta T/T)^2 \rangle$, one performs a weighted sum of $\langle a_l^2 \rangle$ with weight W_l as in equation (2). Plotted in Figure 2 are the window functions for the COBE, MIT, UCSB South Pole (SP91) and MAX experiments to show the range of multipoles probed by each[5].

In Figure 3 we display the predicted value of $\Delta T/T$ (for $\Omega_B = 0.03, 0.10$) for SP91 vs n, for a spectrum which is normalized to fit the COBE measured $\Delta T/T$. (For a more detailed comparison of SP91 with COBE in the context of standard CDM see Gorski, Stompor & Juszkiewicz 1992). The solid lines represent the 90% confidence level predictions assuming a tensor component is present in the amount predicted by power-law inflation (Davis et al. 1993) and the dashed lines assume that the tensor component is absent (see section 5). Note that the entire uncertainty in the values shown comes from cosmic variance of the COBE result, with present observational uncertainty ignored and no uncertainty for SP91. This curve allows a determination of the constraints on n which may be possible from a comparison of SP91 and COBE. Concentrating for now on the solid curves, giving the full result (including tensor contributions to be described next), we see that even an exact measurement of $\Delta T/T$ by SP91 may only constrain n to within $\approx \pm 0.2$, due to the COBE cosmic variance. Including a SP91 measurement uncertainty might, depending on its value and the measured $\Delta T/T$, allow the whole range $n = 0.5 - 1$. The corresponding predictions for the MAX experiment are shown in Figure 4.

In principle one might reduce the uncertainty on the normalization of the $\langle a_l^2 \rangle$ inferred from COBE (and hence the theoretical uncertainty on smaller scales) by using not just $\Delta T/T$ but the whole correlation function $C(\theta)$, which contains more information. Because $C(\theta) \propto \sum_l a_l^2 P_l(\cos\theta) W_l$ probes a different linear combination of a_l^2 at each θ, one might hope for increased sensitivity to higher l modes which have smaller cosmic variance. Nevertheless, in practice the present sensitivity is such that most of the signal comes from small l, so that any treatment of the existing data cannot decrease our estimate of the errors to the point where cosmic variance is irrelevant. The relative

[5]Recall in our convention the window function is convolved with a_l^2 which fall with l in the Sachs-Wolfe region.

uncertainty and size of the $\langle a_l^2 \rangle$ are shown in Table 1 for reference.

As Figure 2 shows, the MIT experiment probes a larger range of l than COBE and the effects of cosmic variance on the normalization inferred from this experiment are less. We find that the normalization uncertainty is $\sim 13\%$ (thus the MIT experiment would be more useful than COBE for fixing the normalization of the spectrum if this experiment were to have extensive sky coverage and reduced experimental errors). As shown in Figure 1b $(\Delta T/T)^2$ still has significant skewness and is not well approximated by a gaussian of mean $\langle(\Delta T/T)^2\rangle$. Again the correlation function (assuming small errors at each angle) would contain more information than $\Delta T/T$, but would be more difficult to analyze. Even so it would be useful to have a measurement of the anisotropy on scales large enough that sensitivities to the doppler peak (Ω_B and amount of tensor component) and re-ionization are small, while being such that low l modes to not contribute significantly e.g. a "full sky MIT" correlation function or an experiment intermediate between Tenerife (Davies et al. 1987), which measures fluctuations on 8° scales, and SP91 on degree scales.

5. Tensor versus Scalar modes

It was pointed out in Krauss & White (1992) that gravitational wave (i.e tensor modes) could play an important role in interpreting the COBE anisotropy if it stems from inflation, so that these should not be ignored. As a result, several groups examined the ratio of scalar to tensor modes predicted to result from various inflationary models (Davis et al. 1992; Dolgov & Silk 1993; Krauss 1992; Salopek 1992). In Davis et al. (1992), a general connection was pointed out between the scalar to tensor ratio predicted by inflation and the spectral tilt as given by $n \neq 1$. For power law inflation this relation takes the approximate form $n \approx 1 - 1/7(T/S)$ (Davis et al. 1992) where (T/S) represents the ratio of the *expectation values* of the tensor and scalar quadrupole moments. As a result it appears that this ratio, if it were measurable, could provide important information about an inflationary epoch in the early universe (Davis et al. 1992). It is important to note however that while the ratio of the expectation values are fixed by the equation of state during inflation, the separate cosmic variance can alter the actual "observed" ratio of quadrupoles significantly from that predicted by T/S. For two χ^2-distributed random variables with $2l+1$ d.o.f., the ratio has the distribution

$$P(R)dR = \frac{\Gamma(2l+1)}{\Gamma^2(l+1/2)} \frac{R^{l-1/2}}{(1+R)^{2l+1}} \quad R = \frac{a_l^2/a_l'^2}{\langle a_l^2\rangle/\langle a_l'^2\rangle} \tag{5}$$

In Figure 5 we have plotted $P(R)$ and the integral $\int_0^R P(r)dr$ for the quadrupole ($l = 2$). Note that the distribution is peaked at a value below 1 but has a significant large R tail. With $P(R_1 < R < R_2) = 95\%$ we find that R can range from $0.2 - 5$! Hence fluctuations with $(a_2^2)_T/(a_2^2)_S$ 5 times (or 1/5) the ratio $\langle a_2^2\rangle_T/\langle a_2^2\rangle_S$ are not improbable. Thus we see that a direct measurement of the quadrupole ratio $(a_2^2)_T/(a_2^2)_S$ could not provide much information on the actual equation of state during inflation! One needs to look at quantities (like $\Delta T/T$) that have smaller cosmic variance such as the power spectrum normalized quadrupoles Q_{rms-PS} (Smoot et al. 1992). In this case the conversion to T/S is much less uncertain. The term quadrupole in this context should therefore be interpreted with care.

Nevertheless any confirmation of a tensor component to the CMB anisotropy on COBE scales or below would be of great significance for the interpretation of the COBE measurement, whether or not it pins down models. However, our results point out the difficulty in making such a confirmation based on comparing CMB anisotropies on $1°$ scales with the COBE result. The solid curve in Figure 3 includes the tensor contribution in the amount prescribed by power law inflation as a function of n. If the tensor contribution were absent for some reason (dashed curve), the predicted $\Delta T/T$ at small angles would be increased. This is because the $(a_l^2)_T$ begin to damp (to be conservative) at $l \simeq \sqrt{1+z_{dec}} \approx 30$ due to redshifting of the gravity waves (Starobinsky 1985; Turner et al. 1993). It is therefore smaller than the scalar for high l modes, so that its relative contribution to COBE is larger than it is to small angle experiments [Note that the spectral tilt required to have a large tensor component increases the contribution of low l modes to the sum, reducing the sensitivity to the Doppler peak and makes tensor and scalar contributions similar.] While this provides some sensitivity to a tensor component in principle, for SP91 the maximum variation is only slightly greater than the (90% confidence level) irremovable uncertainty due to cosmic variance. Including the uncertainty in Ω_B (and n) it seems that a comparison of $1°$ scale measurements with COBE can at best marginally probe the expected tensor contribution to $\Delta T/T$. The situation on still smaller angular scales is less pessimistic in this regard, but one should remember that these scales are most sensitive to non-linear effects and assumptions about the cosmological model, as shown in Figure 4.

The results of this section, and those of section 4, demonstrate the essential problem involved in comparing the COBE results to those of small angular scale experiments. If the window function of such experiments probes different l than COBE, the 20% COBE cosmic variance can blur the comparison. To overcome this normalization uncertainty, we suggest using the COBE correlation function or $\Delta T/T$ as measured by the MIT experiment to fix $\langle a_2^2 \rangle$. In the latter the window functions for small scale experiments can overlap the MIT window function significantly, and correlations between the variance affecting the normalization and the variance in the small angular scale predictions must be taken into account.

6. Cosmological Model Dependence

As alluded to earlier, while the COBE measurement probes scales outside the horizon today and is mostly insensitive to details of the cosmological model, on smaller scales the predictions are model-dependent. Apart from uncertainties in cosmological parameters (such as H_0 and Ω_B) which induce uncertainties in the prediction of $\Delta T/T$, the major unknown arises from the possiblity of early re-ionization and re-scattering of the primoridial CMB fluctuations. The unbiased CDM model, precisely that normalized to COBE, allows adequate non-linearity at early epochs that re-ionization and generation of a secondary "last scattering surface" (LSS) is not implausible (Tegmark & Silk 1993). The ionization of hydrogen imposes such modest energetic requirements that rare fluctuations of low mass can provide a plausible source of ionizing photons or particles (an effect that must have occurred by $z = 5$ to satisfy the Gunn-Peterson constraint). Such re-ionization would move the LSS in to lower redshift and reprocess fluctuations on scales smaller than the horizon size at the new LSS. Indeed for a range of (high) CDM normalizations with $n = 1$ one can reduce degree scale fluctuations by a factor $\sim 2-3$. However if the fluctuations are small (corresponding to biassing factors $b \approx 2$), the spectrum is tilted or is that of a mixed dark matter model, it appears that degree scale fluctuations remain largely unaffected.

Secondary fluctuations induced on very small (arc-minute) scales and their detection could eventually provide a useful constraint on the last scattering surface. Until then however primary fluctuations on degree scales are potentially more sensitive to the last scattering surface (re-ionization) than to the possible presence of tensor modes.

7. Conclusions

After the initial flurry of excitement over the COBE data and its potential ability — in combination with other CMB measurements — to probe the details of models for the generation of primordial density fluctuations, it is time to realistically assess the practical and theoretical limitations inherent in any such procedure. This is particularly important as new measurements of CMB anisotropies on 1° scales are expected. Our results indicate that the statistics of CMB anisotropies predicted by inflation play a fundamental and irremovable role in limiting the information we can gain from the data. Cosmic variance, even with an exact COBE result, still provides uncertainty in the extracted spectral index n. By incorporating this variance, we have been able to quantitatively display how limits on $\Delta T/T$ at small angles can translate to limits on n. Going to smaller scales, such as those comparable to that of observed large scale structures, would provide a longer lever arm and in principle tighter constraints on n. However the theoretical uncertainties due to causal effects and their relation to the growth of fluctuations are much larger in this case. Finally, while observation of a tensor component to the CMB anisotropy would be of great interest, we have shown that the predicted levels are not easily extractible from the data.

To reduce the effects of cosmic variance we suggest that the full correlation function of COBE or the MIT experiment would give a theoretically cleaner normalization of the power spectrum than the COBE $\Delta T/T$. This can then be used with the South Pole or MAX experiments to constrain models of structure formation or cosmology. The normalization uncertainty in this case can be reduced by perhaps a factor of 2. However this would still make a detection of a gravitational wave signal very difficult, coupled as it is with the uncertainties in n, Ω_B and re-ionization history.

Inflation is an attractive theory in part because it makes unambiguous predictions which can be tested. Similarly, measurements of the CMB anisotropy still provide the best hope we have of extracting information on the primordial power spectrum of density perturbations. As we have shown however, definitive answers will require careful attention to the theoretical uncertainties. Inflation predicts probability distributions for observed CMB anisotropies. It will be important to determine what CMB measurements can rule out conclusively. On the other hand, it is equally important not to overstate the case.

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Appendix A

The sensitivity of experiments to the spherical harmonic decomposition of the CMB temperature fluctuations is conventionally described in terms of window functions W_l as in equation (2). For completeness, and to fix our conventions, we describe in this appendix the window functions appropriate to the COBE, MIT, UCSB South Pole and MAX experiments. The functions are shown in Figure 2.

COBE and MIT

A simple way to view the window functions is in terms of a mapping $Y_{lm} \to \tau_{lm} Y_{lm}$ (no sum) describing the weighting the experiment gives to each mode. An example makes this clear. The most straightforward such mapping is that caused by a gaussian smoothing function of width $\sigma \ll 1$ (which could be due to antenna response or applied to the data "by hand")

$$Y_{lm}(\theta_0, \phi_0) \to \int \frac{\theta d\theta d\phi}{2\pi\sigma^2} \exp\left[\frac{-\theta^2}{2\sigma^2}\right] Y_{lm}(\Omega_0 + \Omega)$$

$$\approx \exp\left[\frac{-(l+1/2)^2\sigma^2}{2}\right] Y_{lm}(\theta_0, \phi_0)$$

(for details see White, 1992). If we substitute $\delta T/T = \sum_{lm} a_{lm} \tau_{lm} Y_{lm}$ into

$$\left(\frac{\Delta T}{T}\right)^2 \equiv \left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle_{sky} = \sum_l \frac{a_l^2}{4\pi} W_l$$

where $\langle \cdots \rangle$ indicates an average over the whole sky, and use the orthonormality of the Y_{lm} we obtain the window for the COBE experiment (Smoot et al. 1992) $W_l = \exp[-(l+1/2)^2\sigma^2]$ shown in Figure 2. In this case $\Delta T/T$ corresponds to an rms temperature fluctuation over the observed sky, smoothed on a scale $\sigma = 0.425 \times 10°$. (The COBE maps with no additional smoothing correspond to $\sigma = 0.425 \times 7°$, not shown in Figure 2). We find the same factor appearing in the other window functions due to the finite resolution of the antenna (which we assume to be gaussian). In what follows we will make the approximation of a "perfect" antenna, knowing we can put the antenna response in at the end.

The MIT experiment (Page et al. 1990) also calculates $\Delta T/T$ from a map of the sky and so the form of the window is identical to that of COBE. The smoothing scale is $\sigma = 0.425 \times 3.8°$.

South Pole and MAX

Now consider the UCSB South Pole experiment (Gaier et al. 1991) and focus on the measurement of the so-called $(\Delta T/T)_{rms}$ on one patch of the sky, centered at θ_0, ϕ_0. (Since it is our purpose here to show the effect of cosmic variance on inferences made from this experiment, and not to fit the data, we will not be concerned with multiple, correlated patches of the sky. For a discussion of the fitting of the data see Bond et al. (1991); Dodelson & Jubas (1992). Note that in Bond et al. (1991) a factor of $4\pi/(2l+1)$ is missing from equation (1).) The temperature difference at θ_0, ϕ_0 is measured by "chopping" the beam at fixed $\theta = \theta_0$ through $\phi(t) = \phi_0 + \alpha \sin(2\pi\nu t)$ and sampling $\delta T/T(\theta, \phi)$ with weight $\mathrm{sign}(\phi(t) - \phi_0)$. Recalling the temperature difference is twice the weighted average, the mapping we want is then (see Dodelson & Jubas 1992)

$$
\begin{aligned}
Y_{lm}(\theta_0, \phi_0) &\rightarrow 2\nu \int_0^{1/\nu} dt \; \mathrm{sign}(\phi(t) - \phi_0) Y_{lm}(\theta, \phi(t)) \\
&= 2i H_0(m\alpha) Y_{lm}(\theta_0, \phi_0)
\end{aligned}
$$

where in the last line we have written the integral dt as an integral $d(\phi(t) - \phi_0)$, extracted the $\exp[im\phi]$ dependence from Y_{lm} and used the identity (Gradshteyn & Ryzhik 1980)

$$
H_n(x) = \frac{2(x/2)^n}{\sqrt{\pi}\,\Gamma(n+1/2)} \int_0^1 dt \; (1-t^2)^{n-1/2} \sin(xt)
$$

for the Struve function. At this point we could proceed to define the sky rms fluctuation by averaging over θ_0, ϕ_0 as before (now with an m dependent τ_{lm}). However the "rms" $\Delta T/T$ quoted by the SP group does not come from this procedure. Instead we now argue that the small angular scales probed by this experiment make it sensitive only to high l multipoles. These multipoles have a small cosmic variance, thus our universe at θ_0, ϕ_0 should be similar to the ensemble average of all universes at the same point. We can then define $(\Delta T/T)^2 \equiv \langle (\delta T/T)^2 \rangle_{ens}$ and use (dropping the subscript ens)

$$
\langle a_{l'm'} a^*_{lm} \rangle = \frac{\langle a_l^2 \rangle}{(2l+1)} \delta_{l'l} \delta_{m'm}
$$

to write $(\Delta T/T)^2 = \sum_{lm} \langle a_l^2 \rangle |\tau_{lm} Y_{lm}|^2/(2l+1)$. Reinstating the finite beam width the window so defined is then

$$
W_l = \sum_{m=-l}^{l} W_{lm} = \frac{4\pi}{(2l+1)} \exp\left[-(l+1/2)^2\sigma^2\right] \sum_{m=-l}^{l} |Y_{lm}(\theta_0, \phi_0)|^2 \, 4H_0^2(m\alpha)
$$

For the configuration of (Gaier et al. 1991) $\theta_0 = 27.75°$, $\alpha \sin\theta_0 = 1.5°$ and $\sigma = 0.425 \times 1.65°$.

If the effects of sky rotation and smooth vs. step scanning are ignored the form of the window function for the MAX experiment is very similar to SP91. However the MAX experiment uses a phase lock-in which extracts the frequency component of the sky signal which matches the chop frequency ν. Extracting only this component corresponds to replacing the weight $\mathrm{sign}(\phi(t) - \phi_0)$ with $(\phi(t) - \phi_0)/\alpha$ in the Y_{lm} mapping defined above. The remaining integral over $d\phi$ is then not

H_0 but a Bessel function: $\pm(\pi/2)J_1(|m\alpha|)$. The window function is as above for SP91 but with $2H_0 \to \pi J_1$. [This explains why our window function and predictions for MAX differ from that of other authors.] The parameters (for the μ-Pegasii scan, Meinhold et al. 1992) are $\theta_0 = 90°, \alpha = 0.65°, \sigma = 0.425 \times 0.5°$.

We can test our assumption that the cosmic variance is "small". Recalling that the a_{lm} are gaussian independent complex variables it is straightforward to show that for an experiment covering the whole sky[6]:

$$\left\langle \left(\frac{\Delta T}{T}\right)^2 \right\rangle = \sum_{lm} \frac{\langle a_l^2 \rangle}{4\pi} W_{lm}$$

$$\left\langle \left[\left(\frac{\Delta T}{T}\right)^2\right]^2 \right\rangle - \left\langle \left(\frac{\Delta T}{T}\right)^2 \right\rangle^2 = 2\sum_{lm} \left(\frac{\langle a_l^2 \rangle}{4\pi} W_{lm}\right)^2$$

where the Y_{lm} factor is to be removed from W_{lm}. Notice that these reduce to equation (4) when W_{lm} is m-independent. Keeping in mind that the distribution of $\Delta T/T$ will not in general be gaussian, this allows us to derive an estimate for the size of the cosmic variance associated with each experiment. We find the uncertainty (at 1"σ") for an $n = 1$ spectrum to be 4 and 1% for the SP91 and MAX windows respectively. This would be larger for $n < 1$.

Appendix B

The probability distribution for the correlation function predicted by cosmic variance is not well fit by a gaussian or χ^2 form. In this appendix we derive a simple expression for the distribution which is amenable to numerical computation. We hope that this will facilitate comparisons of theory with data when they become available.

For now let us consider the correlation function predicted by a one component spectrum (either tensor or scalar). We will keep the θ dependence here for generality, and defer consideration of two component spectra till later. For COBE and MIT, by definition

$$C(\theta) \equiv \frac{1}{4\pi} \sum_{l=2}^{\infty} a_l^2 P_l(\cos\theta) W_l = \sum_{l=2}^{\infty} c_l y_l$$

where we have defined $y_l = a_l^2/\sigma_l^2$ with $\sigma_l^2 = \langle a_l^2 \rangle/(l + 1/2)$ and $c_l = (\sigma_l^2/4\pi)P_l(\cos\theta)W_l$. (Note c_l reduces to Δ_l of equation (3) for $\theta = 0$) For the Sachs-Wolfe part of the spectrum the $c_l \sim l^{-2} W_l$

[6] At the other extreme if the experiment looks at only one point on the sky the variance is twice the square of the mean, as expected for (the square of) a gaussian random variable. The intermediate case is discussed in Scott et al. 1993.

and so fall off rapidly with l. Each y_l is an independent random variable with distribution

$$P_\chi^{(l)}(y) = \frac{y^{l-1/2}e^{-y}}{\Gamma(l+1/2)}\theta(y)$$

with θ the Heaviside (or step) function. The correlation function is thus a random variable with distribution given by the convolution

$$P(C) = \lim_{N\to\infty}\left(\prod_{l=2}^{N} c_l^{-1}\right) P_2 \circ P_3 \circ \cdots \circ P_N(C)$$

where $P_l(x) \equiv P_\chi^{(l)}(x/c_l)$. To simplify this we take the Fourier transform (compare with Cayon et al. 1991)

$$\mathcal{F}[P] = \lim_{N\to\infty}\prod_{l=2}^{N}\frac{\mathcal{F}[P_l]}{c_l} = \prod_{l=2}^{\infty}(1-i\omega c_l)^{-(l+1/2)}$$

This can be easily inverted. Writing $\log(\mathcal{F}[P]) = -a_\omega + ib_\omega$ we need to calculate

$$P(C) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ e^{-a_\omega}\cos(b_\omega - \omega C)$$

where

$$a_\omega \equiv \sum_{l=2}^{\infty}(l+1/2)\log\sqrt{1+(\omega c_l)^2}$$

and

$$b_\omega \equiv \sum_{l=2}^{\infty}(l+1/2)\arctan(\omega c_l)$$

This expression is exact and the integral can be efficiently done numerically. One only needs to find a_ω and b_ω once for each set of c_l to evaluate the distribution for all C. The results shown in Figure 1 come from taking $\theta = 0$ and $\langle a_l^2\rangle = (l+1/2)^{-1}$. To include additional (independent) components of the spectrum one just has $a_\omega = \sum_i a_\omega^{(i)}$ and $b_\omega = \sum_i b_\omega^{(i)}$ where $a^{(i)}, b^{(i)}$ are computed from $c_l^{(i)}$ as above.

l	% Uncertainty	10°	7°	3.8°
2	63	1.00	1.00	1.00
3	53	0.69	0.70	0.71
4	47	0.51	0.53	0.55
5	43	0.40	0.43	0.45
6	39	0.32	0.35	0.37
7	37	0.25	0.29	0.32
8	34	0.20	0.25	0.28
9	32	0.17	0.21	0.25
10	31	0.13	0.18	0.22
11	29	0.11	0.15	0.20
12	28	0.09	0.13	0.18
13	27	0.07	0.12	0.16
14	26	0.06	0.10	0.15
15	25	0.04	0.09	0.13
$\sum_{l=2}^{\infty}$	—	4.19	5.01	6.48

Table 1: The relative uncertainty and expected size of the multipoles measured by the correlation function from COBE (smoothed on 10° and 7° scales) and from MIT (smoothed on 3.8° scales) for an $n = 1$ Sachs-Wolfe spectrum. For $n < 1$ the multipoles drop off faster with l. The quadrupoles are normalized to 1 and the sum $l = 2, \cdots, \infty$ is listed below each column. The uncertainty shown is the second moment of the χ^2_{2l+1}-distribution for a_l^2.

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Figure Captions

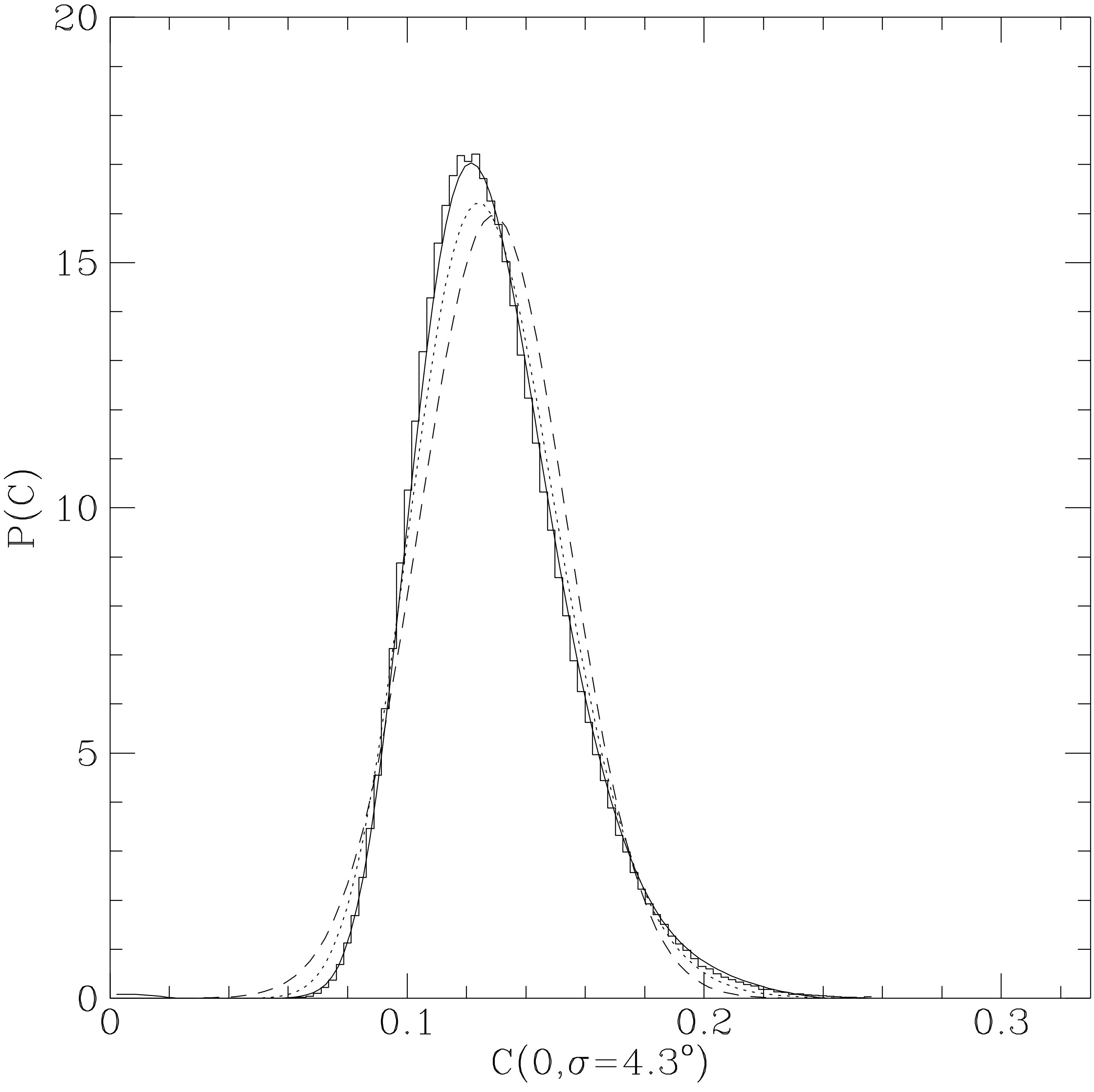
Figure 1 Results of a Monte-Carlo calculation of $\Delta T/T$ for the (a) COBE and (b) MIT experiments assuming $\langle a_l^2 \rangle_S = (l+1/2)^{-1}$ and $\langle a_l^2 \rangle_T = 0$. Also shown are the results obtained using the method of appendix B and fits to a χ^2 form (dotted line) and a gaussian (dashed line) with the same mean and variance as the Monte-Carlo distribution.

Figure 2 Power spectra $k^3 W_T^2(k)$ for a CDM model with $\Omega = 1, h = 0.5, n = 1$ and $\Omega_B = 0.03$ (solid line) and 0.10 (dotted line) as a function of $l \equiv 2k/H_0$. Also shown are the window functions W_l (as described in appendix A) versus multipole number l for COBE, MIT, SP91 and MAX.

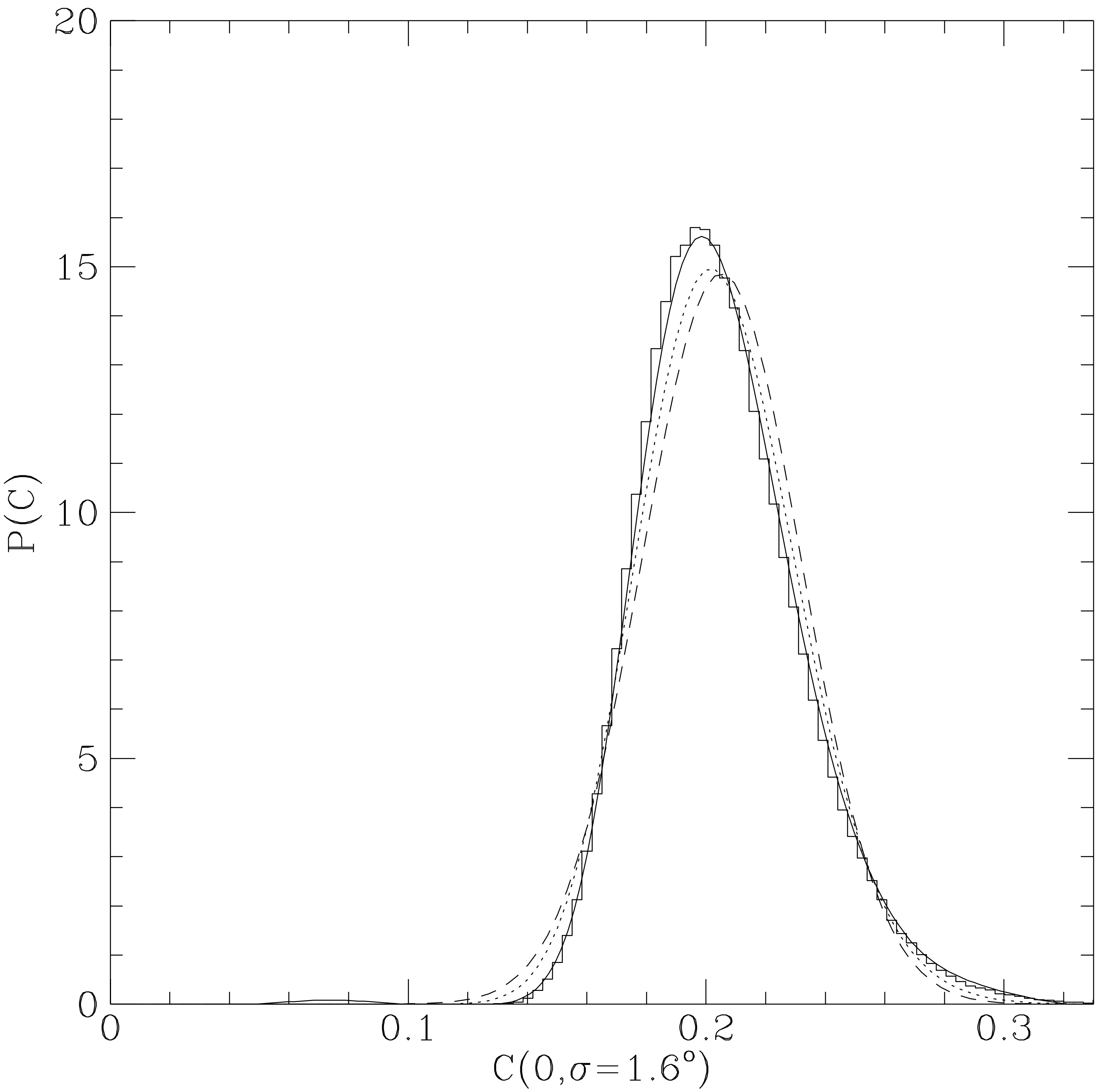
Figure 3 Predictions for SP91, assuming the COBE central value of $\Delta T/T$, as a function of n. Error bars are the COBE cosmic variance induced 90% confidence level uncertainties, neglecting experimental errors and the cosmic variance associated with SP91. Points are given for both scalar and tensor contributions (solid) and scalar only (dashed). Each group of points is at the same n, the offset is for ease of viewing only. Higher points represent $\Omega_B = 0.10$ and low points $\Omega_B = 0.03$.

Figure 4 Predictions for MAX, assuming the COBE central value of $\Delta T/T$, as a function of n. Error bars are the COBE cosmic variance induced 90% confidence level uncertainties, neglecting experimental errors and the cosmic variance associated with MAX. Points are given for both scalar and tensor contributions (solid) and scalar only (dashed). Each group of points is at the same n, the offset is for ease of viewing only. Higher points represent $\Omega_B = 0.10$ and low points $\Omega_B = 0.03$.

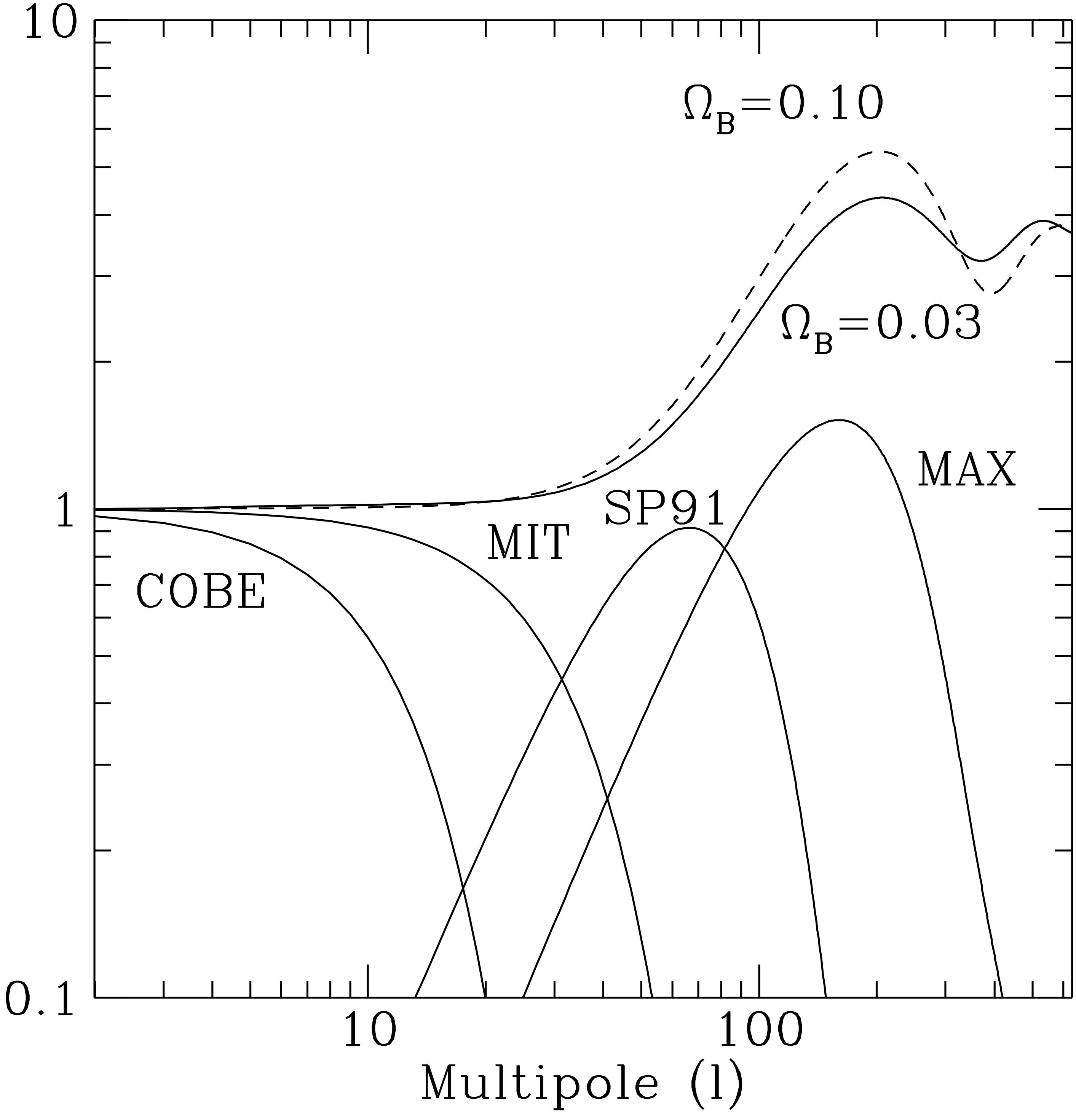
Figure 5 The probability distribution of the ratio of tensor to scalar contributions to the CMB quadrupole anisotropy, compared to the ratio of expectation values for these quantities. Also shown is the integrated probability distribution, useful for computing confidence limits.



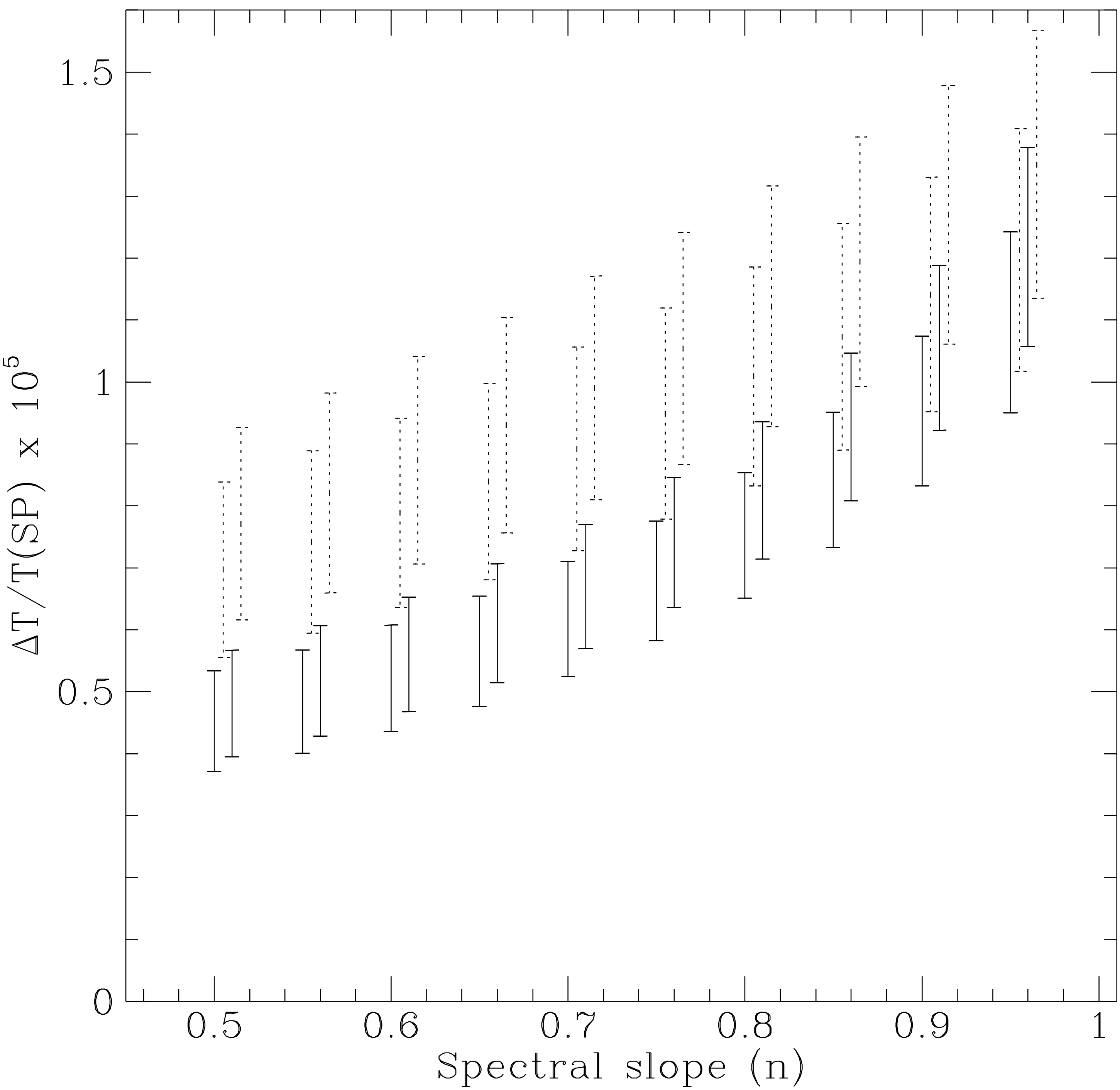
20
15
10
5
0
P(C)
0
0.1
0.2
0.3
C(0,σ=4.3°)



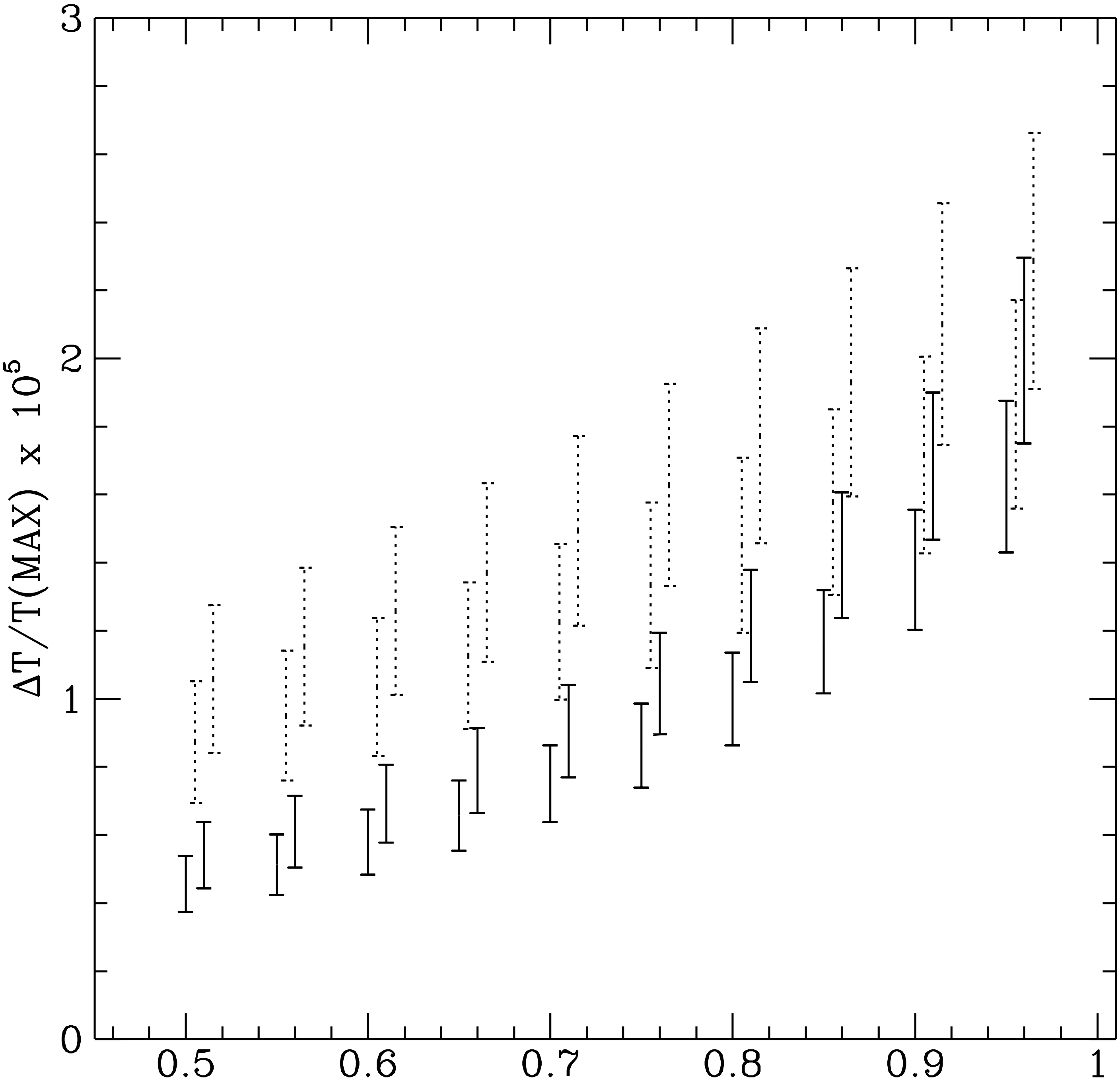
20
15
10
5
0
P(C)
0
0.1
0.2
0.3
C(0,σ=1.6°)



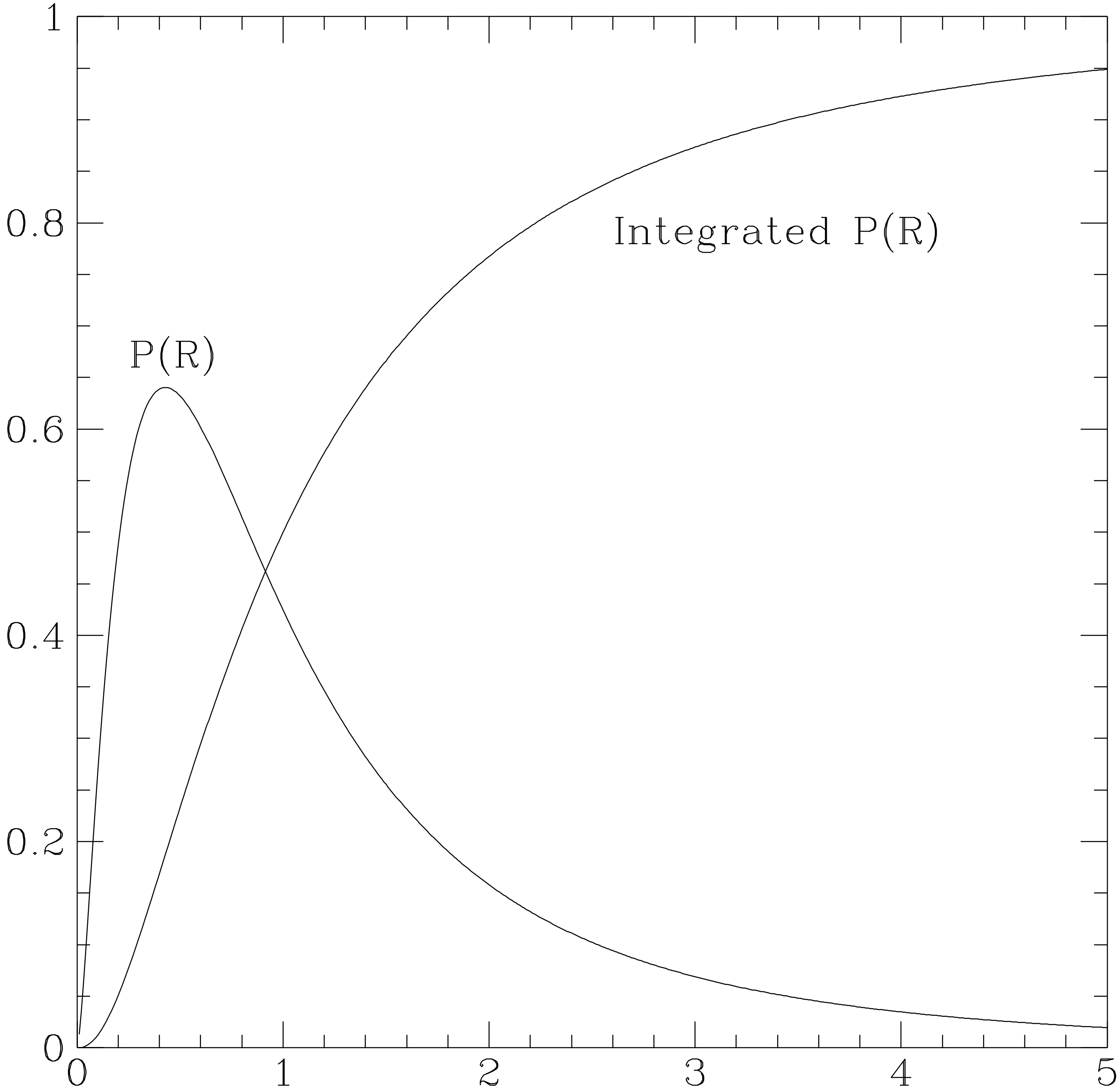
10
1
0.1
10
100
Multipole (l)
Ω_B=0.10
Ω_B=0.03
MAX
SP91
MIT
COBE



$\Delta T/T(SP) \times 10^5$
1.5
1
0.5
0
0.5
0.6
0.7
0.8
0.9
1
Spectral slope (n)



ΔT/T(MAX) x 10^5
0
1
2
3
0.5
0.6
0.7
0.8
0.9
1
Spectral slope (n)



Integrated P(R)
P(R)
0
0.2
0.4
0.6
0.8
1
0
1
2
3
4
5
R = (T/S) / (<T>/<S>)