Cosmological perturbations

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A R T I C L E   I N F O

A B S T R A C T

We review the study of inhomogeneous perturbations about a homogeneous and isotropic background cosmology. We adopt a coordinate based approach, but give geometrical interpretations of metric perturbations in terms of the expansion, shear and curvature of constant-time hypersurfaces and the orthogonal time-like vector field. We give the gauge transformation rules for metric and matter variables at first and second orders. We show how gauge-invariant variables are constructed by identifying geometric or matter variables in physically-defined coordinate systems, and give the relations between many commonly used gauge-invariant variables. In particular we show how the Einstein equations or energy–momentum conservation can be used to obtain simple evolution equations at linear order, and discuss extensions to non-linear order. We present evolution equations for systems with multiple interacting fluids and scalar fields, identifying adiabatic and entropy perturbations. As an application we consider the origin of primordial curvature and isocurvature perturbations from field perturbations during inflation in the very early universe.

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1. Introduction

The standard model of hot big bang cosmology is based upon the spatially homogeneous and isotropic Friedmann–Robertson–Walker (FRW) model. This successfully describes the average expansion of the universe on large scales according to Einstein's theory of general relativity, and the evolution from a hot, dense initial state dominated by radiation to the cool, low density state dominated by non-relativistic matter and, apparently, vacuum energy at the present day. This standard model is described by just a handful of numbers specifying the expansion rate, the temperature of the present microwave background radiation, the density of visible matter, dark matter and dark vacuum energy.

But a homogeneous model cannot describe the complexity of the actual distribution of matter and energy in our observed universe where stars and galaxies form clusters and superclusters of galaxies across a wide range of scales. For this we need to be able to describe spatial inhomogeneity and anisotropy. But there are few exact solutions of general relativity that incorporate spatially inhomogeneous and anisotropic matter and hence geometry. Therefore we use a perturbative approach starting from the spatially homogeneous and isotropic FRW model as a background solution with simple properties within which we can study the increasing complexity of inhomogeneous perturbations order by order.

The introduction of a homogeneous background model to describe the inhomogeneous universe leads to an ambiguity in the choice of coordinates. In the FRW model the homogeneous three-dimensional hypersurfaces provide a natural time slicing of four-dimensional spacetime. For instance, hypersurfaces of a uniform density coincide with hypersurfaces with uniform spatial curvature. However in the real, inhomogeneous universe spatial hypersurfaces of uniform density do not in general have uniform spatial curvature, and hypersurfaces of uniform curvature do not have uniform density. In general relativity there is, a priori, no preferred choice of coordinates. Choosing a set of coordinates in the inhomogeneous universe which will then be described by an FRW model plus perturbations amounts to assigning a mapping between spacetime points in the inhomogeneous universe and the homogeneous background model. The freedom in this choice is the gauge freedom, or gauge problem, in general relativistic perturbation theory. It can lead to apparently different descriptions of the same physical solution simply due to the choice of coordinates. This freedom can be a powerful tool as it allows one to work in terms of variables best suited to the problem in hand, however it is important to work with gauge-invariant definitions of the perturbations variables if the results are to be easily assimilated and cross-referenced to the work of others.

In this review we will focus on how one can construct a variety of gauge-invariant variables to deal with perturbations in different cosmological models at first order and beyond. We will emphasise the geometrical meaning of metric and matter perturbations and their gauge-invariant definitions.

Most work to date has been done only to linear order where the perturbations obey linear field equations. Even then we must consistently solve the linear evolution equations, subject to the constraint equations of general relativity which relate the matter variables to the geometry. Beyond first order, the non-linearity of Einstein's equations becomes evident, making progress much more difficult. But in limiting cases, notably the large-scale limit, it is possible to extend some of the simple results of linear theory to higher orders.

Previous reviews on the topic of linear perturbation theory in cosmology include, for example, Refs. [1–6], and the relevant chapters in Refs. [7–10]. For second-order perturbation theory see Refs. [11–13]. For reviews on inflation in particular see, e.g., Refs. [14–17,10,18–20] and for cosmic microwave background anisotropies see, e.g., Refs. [21–24]. Finally, for reviews on perturbation theory in the context of the higher-dimensional brane-world scenario see, e.g., Refs. [25–29] and for perturbation theory in the context of the low-energy string effective action see, e.g., Refs. [30,31].

For simplicity we work with a flat background spatial metric which is compatible with current observations. For generalisation to spatially hyperbolic or spherical FRW models see other reviews, e.g., Ref. [1]. We use a prime to denote derivatives with respect to conformal time, and we use a comma to denote partial derivatives with respect to comoving spatial coordinates, i.e.,

$$T_{,i} \equiv \frac{\partial T}{\partial x^i}. \tag{1.1}$$

For further definitions and notation see Appendix A.

2. Perturbations in cosmology

Throughout this review we will assume that our observable Universe can approximately be described by a homogeneous and isotropic Friedmann–Robertson–Walker (FRW) spacetime. Thus we assume that the physical quantities can usefully be decomposed into a homogeneous background, where quantities depend solely on cosmic time, and inhomogeneous perturbations. The perturbations thus "live" on the background spacetime and it is this background spacetime which is used to split four-dimensional spacetime into spatial three-hypersurfaces, using a $(3 + 1)$ decomposition. In addition we work with a spatially flat FRW background, though results can be readily extended to non-zero spatial curvature.

We start this section by defining arbitrary perturbations of tensorial quantities and then proceed by decomposing vectors and tensors into "time" and "space" parts on the spatial hypersurfaces.

2.1. Defining perturbations

Any tensorial quantities can then be split into a homogeneous background and an inhomogeneous perturbation

$$\mathbf{T}(\eta, x^i) = \mathbf{T}_0(\eta) + \delta\mathbf{T}(\eta, x^i). \tag{2.1}$$

The background part is a time-dependent quantity, $\mathbf{T}_0 \equiv \mathbf{T}_0(\eta)$, whereas the perturbations depend on time and space coordinates $x^\mu = [\eta, x^i]$. The perturbation can be further expanded as a power series,

$$\delta\mathbf{T}(\eta, x^i) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \delta\mathbf{T}_n(\eta, x^i), \tag{2.2}$$

where the subscript n denotes the order of the perturbations, and we explicitly include here the small parameter ϵ. In linear perturbation theory, for example, we only consider first-order terms, ϵ^1, and can neglect terms resulting from the product of two perturbations, which would necessarily be of order ϵ^2 or higher, which considerably simplifies the resulting equations. In the following sections we shall omit the small parameter ϵ whenever possible, as is usually done to avoid the equations getting too cluttered.

2.2. Decomposing tensorial quantities

It is convenient to slice the spacetime manifold into a one parameter family of spatial hypersurfaces of constant time, which is the standard $(3 + 1)$ split of spacetime. This foliation was introduced by Darmois already in 1927 (see Ref. [32]) and popularised by Arnowitt, Deser and Misner [33] (for conditions on the existence of the foliation see e.g. Ref. [34]). We refer to the foliation of spacetime by spatial hypersurfaces of given conformal time as the time slicing, and the identification of spatial coordinates on each hypersurface as the threading.

Note that the $(3 + 1)$ split of spacetime precedes the decomposition of three-dimensional quantities into scalars and vectors, or scalar, vectors, and tensors below.

2.2.1. Vectors

We can split any 4-vector V^μ into a temporal and spatial part,

$$\mathcal{U}^\mu = \left[\mathcal{U}^0, \mathcal{U}^i\right]. \tag{2.3}$$

Note that $\mathcal{U}^0$ is a *scalar* on spatial hypersurfaces. The spatial part $\mathcal{U}^i$ can then be further decomposed into a further *scalar* part $\mathcal{U}$ and a *vector* part $\mathcal{U}^i_{\text{vec}}$,

$$\mathcal{U}^i \equiv \delta^{ij} \mathcal{U}_{,j} + \mathcal{U}^i_{\text{vec}}, \tag{2.4}$$

where $\partial\mathcal{U}^i_{\text{vec}}/\partial x^i = 0$. The denominations *scalar* and *vector* parts go back to Bardeen [35] and are due to the transformation behaviour of $\mathcal{U}$ and $\mathcal{U}^i_{\text{vec}}$ on spatial hypersurfaces. The decomposition of a vector field into potential (or curl-free) part and a divergence-free part in Euclidean space is known as Helmholtz's theorem. The curl-free and divergence-free parts are also called longitudinal and solenoidal parts, respectively.

In our isotropic (FRW) background, there can be no spatial vector part at zeroth order (as this would correspond to a preferred direction), but there can be a non-zero temporal part,

$$\mathcal{U}^0_{(0)} \neq 0, \qquad \mathcal{U}^i_{(0)} = 0. \tag{2.5}$$

A non-trivial vector, with a non-zero spatial part, can appear only at first order. We give examples of vector fields in a FRW background including perturbations in Section 3.1, where we discuss the unit vector field normal to constant-time hypersurfaces and the fluid 4-velocity.

Note that "divergence-free", etc, is defined with respect to the flat-space metric, rather than using covariant derivatives, since perturbations are defined with respect to the spatially flat background.

2.2.2. Tensors

As for vector fields, we can decompose a rank-2 tensor into a time part and spatial part, but now have also mixed time–space parts.

Consider the metric tensor $g_{\mu\nu}$ which we require to be symmetric,

$$g_{\mu\nu} \equiv g_{\nu\mu}. \tag{2.6}$$

The metric tensor has therefore only 10 independent components in 4 dimensions. We first split the metric tensor into a background and a perturbed part, using Eq. (2.1). It then turns out to be useful to split the metric perturbation into different parts labelled *scalar*, *vector* or *tensor* according to their transformation properties on spatial hypersurfaces [35,36], which are themselves expanded into first- and higher-order parts using Eq. (2.2).

Our background spacetime is described by a spatially flat FRW background metric

$$ds^2 = a^2 \left[-d\eta^2 + \delta_{ij} dx^i dx^j \right],$$ (2.7)

where η is conformal time and $a = a(\eta)$ is the scale factor. The cosmic time, measured by observers at fixed comoving spatial coordinates, x^i, is given by $t = \int a(\eta) \, d\eta$.

The perturbed part of the metric tensor can be written as

$$\delta g_{00} = -2a^2 \phi,$$ (2.8)

$$\delta g_{0i} = a^2 B_i,$$ (2.9)

$$\delta g_{ij} = 2a^2 C_{ij}.$$ (2.10)

The 0–i and the i–j components of the metric tensor can be further decomposed into scalar, vector and tensor parts

$$B_i = B_{,i} - S_i,$$ (2.11)

$$C_{ij} = -\psi \delta_{ij} + E_{,ij} + F_{(i,j)} + \frac{1}{2} h_{ij}$$ (2.12)

where ϕ, B, ψ and E are *scalar* metric perturbations, S_i and F_i are *vector* metric perturbations, and h_{ij} is a *tensor* metric perturbation, which we will now define.

Scalar perturbations can always be constructed from a scalar, or its derivatives, and background quantities. Any 3-vector, such as $B_{,i}$, constructed from a scalar is necessarily curl-free, i.e., $B_{,[ij]} = 0$.

Vector perturbations are divergence-free. For instance one can distinguish an intrinsically *vector* part of the metric perturbation δg_{0i}, which we denote by $-S_i$, which gives a non-vanishing contribution to $\delta g_{0[i,j]}$. Similarly we define the vector contribution to δg_{ij} constructed from the (symmetric) derivative of a vector $F_{(i,j)}$.

Finally there is a *tensor* contribution to $\delta g_{ij} = a^2 h_{ij}$ which is both transverse, $h_{ij,}{}^j = 0$ (i.e., divergence-free), and trace-free ($h^i_i = 0$) which therefore cannot be constructed from inhomogeneous scalar or vector perturbations.

Note that when raising and lowering spatial indices of vector and tensor perturbations we use the comoving background spatial metric, δ_{ij}, so that, for instance, $h^j_i \equiv \delta^{jk} h_{ik}$.

The reason for splitting the metric perturbation into these three types is that the governing equations decouple at linear order, and hence we can solve each perturbation type separately. At higher order this is no longer true: we find for example at second order that although the "true" second-order perturbations, $\delta g_{2\mu\nu}$, still decouple, their governing equations have source terms quadratic in the first-order variables, $\delta g_{1\mu\nu}$, mixing the different types [37]. Indeed at all higher orders, $n > 1$, the different types of perturbations of order n decouple, but are sourced by terms comprising perturbations of lower order.

We have introduced four scalar functions, two spatial vector valued functions with three components each, and a symmetric spatial tensor with six components. But these functions are subject to several constraints: h_{ij} is transverse and traceless, which contributes four constraints, F_i and S_i are divergence-free, one constraint each. We are therefore finally left 10 new degrees of freedom, the same number as the independent components of the perturbed metric.

The choice of variables to describe the perturbed metric is not unique. Already at first order there are different conventions for the split of the spatial part of the metric. We follow the notation of Mukhanov et al. [2] so that the metric perturbation, ψ, can be identified directly with the intrinsic scalar curvature of spatial hypersurfaces at first order, see later. Sometimes it is useful to work in terms of the trace of the perturbed spatial metric

$$C = C^i_i = -3\psi + \nabla^2 E.$$ (2.13)

At first order this coincides with the perturbation of the determinant of the spatial metric. Including terms up to second order we have

$$\det \left(\delta_{ij} + 2C_{ij} \right) = 1 + 2C + 2 \left(C^2 - C_{ij} C^{ij} \right)$$

$$= 1 - 6\psi + 2\nabla^2 E + 12\psi^2 - 8\psi \nabla^2 E + 2(\nabla^2 E)^2 - 2E_{,ij} E^{,ij} - 2F_{i,j} F^{i,j}_{,}$$

$$- \frac{1}{2} h_{ij} h^{ij} - 2E_{,ij} h^{ij} - 2F_{i,j} h^{ij}$$ (2.14)

where we have used the general result $\det(e^\gamma) = e^{\text{Tr}(\gamma)}$. There are further choices for the way the spatial metric is split into the different perturbation variables at second (and higher) order in the perturbations.

Note that our metric perturbations in Eqs. (2.8)–(2.12) include all orders. If we write out the complete metric tensor, up to and including second order in the perturbations we have

$$g_{00} = -a^2 \left(1 + 2\phi_1 + \phi_2 \right),$$

$$g_{0i} = a^2 \left(B_{1i} + \frac{1}{2} B_{2i} \right),$$

$$g_{ij} = a^2 \left[\delta_{ij} + 2C_{1ij} + C_{2ij} \right],$$ (2.15)

where the first and second-order perturbations B_{1i} and C_{1ij}, and B_{2i} and C_{2ij}, can be further split according to Eqs. (2.11) and (2.12).

The contravariant metric tensor follows from the constraint (to the required order),

$$g_{\mu\nu} \, g^{\nu\lambda} = \delta_{\mu}^{\ \lambda},$$
(2.16)

which up to second order gives

$$g^{00} = -a^{-2} \left[1 - 2\phi_1 - \phi_2 + 4\phi_1^{\ 2} - B_{1k}B_1^{\ k} \right],$$

$$g^{0i} = a^{-2} \left[B_1^i + \frac{1}{2}B_2^{\ i} - 2\phi_1 B_1^i - 2B_{1k}C_1^{\ ki} \right],$$

$$g^{ij} = a^{-2} \left[\delta^{ij} - 2C_1^{\ ij} - C_2^{\ ij} + 4C_1^{\ ik}C_{1k}^{\ \ j} - B_1^{\ i}B_1^{\ j} \right].$$
(2.17)

3. Geometry of spatial hypersurfaces

3.1. Time-like vector fields

The perturbed metric given in Section 2.2.2 implicitly defines a unit time-like vector field orthogonal to constant-η hypersurfaces,

$$n_\mu \propto \frac{\partial \eta}{\partial x^\mu},$$
(3.1)

subject to the constraint

$$n^\mu n_\mu = -1.$$
(3.2)

In the FRW background this coincides with the 4-velocity of matter and the expansion of the velocity field $\theta = 3H$, where H is the Hubble expansion rate. We define the conformal Hubble parameter

$$\mathcal{H} \equiv aH.$$
(3.3)

In this section we calculate corresponding geometrical quantities for n^ν, and thus the spacetime, defined by the perturbed metric tensor. Note however that in the perturbed spacetime the vector field n^μ, need no longer coincide with the 4-velocity of matter fields at first order and beyond.

Up to and including second order, the covariant vector field is

$$n_\mu = -a \left[1 + \phi_1 + \frac{1}{2}\phi_2 + \frac{1}{2}\left(B_{1k}B_1^k - \phi_1^2 \right), \mathbf{0} \right],$$
(3.4)

and the contravariant vector field is

$$n^0 = \frac{1}{a} \left[1 - \phi_1 - \frac{1}{2}\phi_2 + \frac{3}{2}\phi_1^2 - \frac{1}{2}B_{1k}B_1^k \right],$$

$$n^i = \frac{1}{a} \left[-\left(B_1^i + \frac{1}{2}B_2^{\ i} \right) + 2B_{1k}C_1^{\ ki} + \phi_1 B_1^i \right].$$
(3.5)

Observers moving along the hypersurface orthogonal vector field, n^μ, have a vanishing 3-velocity with respect to the spatial coordinates x^i when the shift vector B^i is zero. We will refer to these as orthogonal coordinate systems.

3.2. Geometrical quantities

The covariant derivative of any time-like unit vector field n_μ can be decomposed uniquely as follows [34]:

$$n_{\mu;\nu} = \frac{1}{3}\theta \, \mathcal{P}_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - a_\mu n_\nu,$$
(3.6)

where the spatial projection tensor $\mathcal{P}_{\mu\nu}$, orthogonal to n^μ, is given by

$$\mathcal{P}_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu.$$
(3.7)

The overall expansion rate is given by

$$\theta = n^\mu_{\ ;\mu},$$
(3.8)

the (trace-free and symmetric) shear is

$$\sigma_{\mu\nu} = \frac{1}{2}\mathcal{P}_\mu^{\ \alpha}\mathcal{P}_\nu^{\ \beta} \left(n_{\alpha;\beta} + n_{\beta;\alpha} \right) - \frac{1}{3}\theta \, \mathcal{P}_{\mu\nu},$$
(3.9)

the (antisymmetric) vorticity is

$$\omega_{\mu\nu} = \frac{1}{2} \mathcal{P}_\mu{}^\alpha \mathcal{P}_\nu{}^\beta \left(n_{\alpha;\beta} - n_{\beta;\alpha} \right),$$ (3.10)

and the acceleration is

$$a_\mu = n_{\mu;\nu} n^\nu.$$ (3.11)

On spatial hypersurfaces the expansion, shear, vorticity and acceleration coincide with their Newtonian counterparts in fluid dynamics [38,39]. In this subsection we focus on the unit normal vector field n^μ, but the expansion, shear, vorticity and acceleration defined in this way can readily be applied to any other 4-vector field, such as the 4-velocity u^μ. One can easily verify that the vorticity (3.10) is automatically zero for the hypersurface orthogonal vector field, n_μ defined in Eq. (3.4). Note however that the perturbed fluid velocity can have vorticity and this is described by the vector (divergence-free) part of the fluid 3-velocity which we will define in Section 4.

The projection tensor $\mathcal{P}_{\mu\nu}$ is the induced 3-metric on the spatial hypersurfaces, and the Lie derivative of $\mathcal{P}_{\mu\nu}$ along the vector field n^μ is the extrinsic curvature of the hypersurface embedded in the higher-dimensional spacetime [34,40]. The extrinsic curvature of the spatial hypersurfaces defined by n_μ is thus given by

$$K_{\mu\nu} \equiv \frac{1}{2} \pounds_n \mathcal{P}_{\mu\nu} = \mathcal{P}_\nu{}^\lambda n_{\mu;\lambda} = \frac{1}{3} \theta \, \mathcal{P}_{\mu\nu} + \sigma_{\mu\nu}.$$ (3.12)

At first order we can easily identify the metric perturbations with geometrical perturbations of the spatial hypersurfaces or the associated vector field, n_μ.

The intrinsic curvature of spatial hypersurfaces up to first order is given by

$$^{(3)}R_1 = \frac{4}{a^2} \nabla^2 \psi_1.$$ (3.13)

The scalar part of the shear (3.9) up to first order is given by

$$\sigma_{1ij} = \left(\partial_i \partial_j - \frac{1}{3} \nabla^2 \delta_{ij} \right) a\sigma_1,$$ (3.14)

where we define the shear potential

$$\sigma_1 \equiv E_1' - B_1.$$ (3.15)

The vector and tensor parts are given by, respectively,

$$\sigma_{1Vij} = a \left(F_{1(i,j)}' - B_{1(i,j)} \right), \qquad \sigma_{1Tij} = \frac{a}{2} h_{1ij}'.$$ (3.16)

The acceleration up to first order is

$$a_i = \phi_{,i}.$$ (3.17)

The expansion rate up to first order is given by

$$\theta = \frac{3}{a} \left[\mathcal{H} - \mathcal{H}\phi - \psi' + \frac{1}{3} \nabla^2 \sigma \right].$$ (3.18)

The intrinsic spatial curvature, shear and acceleration of n_μ are given up to second order in Appendix C.
The overall expansion, up to second order is given by

$$\theta = \frac{1}{a} \left[3\frac{a'}{a} - 3\frac{a'}{a}\phi_1 + C_{1k}{}^{k'} - B_{1k,}{}^k - \frac{3}{2}\frac{a'}{a}\left(\phi_2 - 3\phi_1{}^2\right) + \frac{1}{2}\left(C_{2k}{}^{k'} - B_{2k,}{}^k\right) + \phi_1\left(B_{1k,}{}^k - C_{1k}{}^{k'}\right) \right.$$
$$\left. - \frac{3}{2}\frac{a'}{a}B_{1k}B_1^k - 2C_1^{kl}C_{1kl}' + 2C_1^{kl}B_{1l,k} + 2B_1^l C_{1lk,}{}^k - B_1^k C_1{}^l{}_{l,k} \right].$$ (3.19)

Focusing for the moment only on scalar perturbations (neglecting first-order vectors and tensors) on large scales (neglecting spatial derivatives) the perturbed part of the expansion simplifies to

$$\delta\theta_1 \simeq -\frac{3}{a}\left(\frac{a'}{a}\phi_1 + \psi_1'\right),$$ (3.20)

$$\delta\theta_2 \simeq -\frac{3}{a}\left[\frac{a'}{a}\phi_2 + \psi_2' + 2\psi_1'\left(2\psi_1 - \phi_1\right) - 3\frac{a'}{a}\phi_1^2\right].$$ (3.21)

Note that the expansion rate, θ in Eq. (3.19), is defined with respect to proper time (comoving with n^μ). The expansion rate with respect to conformal time on large scales is

$$\theta_{\text{coord}} \simeq 3 \left[\frac{a'}{a} - \psi_1' - \frac{1}{2}\psi_2' - 2\psi_1'\psi_1 \right]. \tag{3.22}$$

We see that on large scales (and considering only scalar perturbations) spatially flat hypersurfaces ($\psi = 0$) are also uniform coordinate expansion hypersurfaces (on which the perturbed expansion vanishes).

In the following we will often refer to the perturbed logarithmic expansion, or "number of e-foldings"

$$N \equiv \frac{1}{3} \int \theta \, d\tau = \frac{1}{3} \int \theta_{\text{coord}} \, d\eta = \ln a - \psi_1 - \frac{1}{2}\psi_2 - \psi_1^2. \tag{3.23}$$

This quantity, and its perturbation δN becomes a particularly useful quantity to describe the primordial scalar perturbation beyond linear order, as we will discuss in Section 11.

4. Energy–momentum tensor for fluids

We are interested in how the spacetime geometry, described by the metric tensor, is affected by the perturbed matter content, described by the energy–momentum tensor.

The four-velocity of matter, u^μ, is defined by

$$u^\mu = \frac{dx^\mu}{d\tau}, \tag{4.1}$$

where τ is the proper time comoving with the fluid, subject to the constraint

$$u_\mu u^\mu = -1. \tag{4.2}$$

The components of the 4-velocity up to second order are then given by

$$u_0 = -a \left[1 + \phi_1 + \frac{1}{2}\phi_2 - \frac{1}{2}\phi_1^2 + \frac{1}{2}v_{1k}v_1^k \right],$$

$$u_i = a \left[v_{1i} + B_{1i} + \frac{1}{2}(v_{2i} + B_{2i}) - \phi_1 B_{1i} + 2C_{1ik}v_1^k \right],$$

$$u^0 = a^{-1} \left[1 - \phi_1 - \frac{1}{2}\phi_2 + \frac{3}{2}\phi_1^2 + \frac{1}{2}v_{1k}v_1^k + v_{1k}B_1^k \right], \tag{4.3}$$

$$u^i = a^{-1} \left(v_1^i + \frac{1}{2}v_2^i \right). \tag{4.4}$$

The spatial part of the velocity can be split into a scalar (potential) part and a vector (solenoidal) part, order by order, following Eq. (2.4) as

$$v^i \equiv \delta^{ij} v_{,j} + v_{\text{vec}}^i, \tag{4.5}$$

for each order n where we refer to $v_{(n)}$ as the scalar velocity potential, and to $v_{(n)\text{vec}}^i$ as the vector part.

Note that v^i is the 3-velocity of matter defined with respect to the spatial coordinates, x^i, and so is not the velocity with respect to the hypersurface orthogonal vector field n^i, except in orthogonal coordinate systems for which $B^i = 0$. In comoving orthogonal coordinates, which we will discuss later, $v^i = 0$ and $B^i = 0$.

4.1. Single fluid

The energy–momentum tensor of a fluid with density ρ, isotropic pressure P and 4-velocity u^μ, defined above in Eq. (4.3), is defined as [1,39,41,7]

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu} + \pi_{\mu\nu}. \tag{4.6}$$

The anisotropic stress tensor $\pi_{\mu\nu}$ is split into first- and second-order parts in the usual way,

$$\pi_{\mu\nu} \equiv \pi_{(1)\mu\nu} + \frac{1}{2}\pi_{(2)\mu\nu}, \tag{4.7}$$

and is subject to the constraints

$$\pi_{\mu\nu} u^\nu = 0, \qquad \pi^\mu_\mu = 0. \tag{4.8}$$

The anisotropic stress vanishes for a perfect fluid or minimally coupled scalar fields, but may be non-zero in the presence of free-streaming neutrinos or a non-minimally coupled scalar field.

We follow Kodama and Sasaki [1] in defining the proper energy density as the eigenvalue of the energy–momentum tensor, and the four-velocity u^μ as the corresponding eigenvector

$$T^\mu_\nu u^\nu = -\rho u^\mu. \tag{4.9}$$

The anisotropic stress tensor decomposes into a trace-free scalar part, Π, a vector part, Π_i, and a tensor part, Π_{ij}, at each order according to

$$\pi_{ij} = a^2 \left[\Pi_{,ij} - \frac{1}{3}\nabla^2 \Pi \delta_{ij} + \frac{1}{2}\left(\Pi_{i,j} + \Pi_{j,i}\right) + \Pi_{ij} \right]. \tag{4.10}$$

We get for the components of the stress energy tensor in the background

$$T^0_0 = -\rho_0, \qquad T^0_i = 0, \qquad T^i_j = \delta^i_j P_0, \tag{4.11}$$

at first order,

$$^{(1)}\delta T^0_0 = -\delta\rho_1, \tag{4.12}$$

$$^{(1)}\delta T^0_i = (\rho_0 + P_0)(v_{1i} + B_{1i}), \tag{4.13}$$

$$^{(1)}\delta T^i_j = \delta P_1 \delta^i_j + a^{-2}\pi_{(1)}{}^i_j, \tag{4.14}$$

and at second order

$$^{(2)}\delta T^0_0 = -\delta\rho_2 - 2(\rho_0 + P_0)v_{1k}\left(v_1^k + B_1^k\right), \tag{4.15}$$

$$^{(2)}\delta T^0_i = (\rho_0 + P_0)\left[v_{2i} + B_{2i} + 4C_{1ik}v_1^k - 2\phi_1(v_{1i} + 2B_{1i})\right] + 2(\delta\rho_1 + \delta P_1)(v_{1i} + B_{1i}) + \frac{2}{a^2}\left(B_1^k + v_1^k\right)\pi_{1ik}, \tag{4.16}$$

$$^{(2)}\delta T^i_j = \delta P_2 \delta^i_j + a^{-2}\pi_2{}^i_j - \frac{4}{a^2}C_1^{ik}\pi_{1jk} + 2(\rho_0 + P_0)v_1^i(v_{1j} + B_{1j}). \tag{4.17}$$

Note, that for compactness of presentation we have not split perturbations into their constituent scalar, vector and tensor parts in the above expressions. The decompositions are given for v_i and π_{ij} in Eqs. (4.5) and (4.10), and given for B_i and C_{ij} in Section 2.2.2, in Eqs. (2.11) and (2.12).

Note that contracting the $i - j$ part of the energy–momentum tensor, Eq. (4.17), including the constraints for the anisotropic stress, Eq. (4.8), guarantees that the anisotropic stress cancels on the trace,

$$^{(1)}\delta T^k_k = 3\delta P_1, \tag{4.18}$$

$$^{(2)}\delta T^k_k = 3\delta P_2 + 2(\rho_0 + P_0)v_1^k(v_{1k} + B_{1k}). \tag{4.19}$$

This cancellation is true at all orders.

Coordinate transformations affect the split between spatial and temporal components of the matter fields and so quantities like the density, pressure and 3-velocity are gauge dependent, as described in Section 6. Density and pressure are 4-scalar quantities which transform as given in Eq. (6.18) in the following section, but the 4-velocity is a 4-vector which transforms as described in Section 6.3. The anisotropic stress is gauge-invariant at first order, but becomes gauge dependent at second order.

4.2. Multiple fluids

In the multiple fluid case the total energy–momentum tensor is the sum of the energy–momentum tensors of the individual fluids

$$T^{\mu\nu} = \sum_\alpha T^{\mu\nu}_{(\alpha)}. \tag{4.20}$$

For each fluid we define the local energy–momentum transfer 4-vector $Q^\nu_{(\alpha)}$ through the relation

$$\nabla_\mu T^{\mu\nu}_{(\alpha)} = Q^\nu_{(\alpha)}, \tag{4.21}$$

where the energy–momentum tensor, $T^{\mu\nu}_{(\alpha)}$, is locally conserved only for non-interacting fluids, for which $Q^\nu_{(\alpha)} = 0$. Eqs. (8.2) and (4.21) imply the constraint

$$\sum_\alpha Q^\nu_{(\alpha)} = 0. \tag{4.22}$$

Following Refs. [1,42] we split the energy–momentum transfer 4-vector using the total fluid velocity u^μ as

$$Q^\mu_{(\alpha)} = Q_{(\alpha)} u^\mu + f^\mu_{(\alpha)}, \tag{4.23}$$

where $Q_{(\alpha)}$ is the energy transfer rate and $f^\mu_{(\alpha)}$ is the momentum transfer rate, subject to the constraint

$$u_\mu f^\mu_{(\alpha)} = 0. \tag{4.24}$$

From Eq. (4.24) we find for the temporal component of the momentum transfer rate vector $f^\mu_{(\alpha)}$

$$f^0_{1(\alpha)} = 0, \qquad f^0_{2(\alpha)} = 2 f^k_{1(\alpha)} (v_{1k} + B_{1k}). \tag{4.25}$$

We then find for the temporal components of the energy transfer 4-vector to be at zeroth, first and second orders, respectively

$$Q^0_{(\alpha)} = \frac{1}{a} Q_{0\alpha}, \tag{4.26}$$

$$Q^0_{(\alpha)} = \frac{1}{a} (\delta Q_{1\alpha} - \phi_1 Q_{0\alpha}), \tag{4.27}$$

$$Q^0_{(\alpha)} = \frac{1}{2a}\left[\delta Q_{2\alpha} + Q_{0\alpha} \left(3{\phi_1}^2 - \phi_2 \right) - 2\phi_1 \delta Q_{1\alpha} + (v_{1k} + B_{1k}) \left(\frac{2}{a} f^k_{1(\alpha)} + Q_{0\alpha} v^k_1 \right) \right], \tag{4.28}$$

where $Q_{0\alpha}, \delta Q_{1\alpha}$, and $\delta Q_{2\alpha}$ are the energy transfer to the α-fluid in the background, at first and at second orders, respectively.

For the spatial components of the energy transfer 4-vector, the momentum part, we get at first and second orders, respectively

$$Q^i_{(\alpha)} = \frac{1}{a} Q_{0\alpha} v^i_1 + \frac{1}{a^2} f^i_{1(\alpha)}, \tag{4.29}$$

$$Q^i_{(\alpha)} = \frac{1}{2a}\left[\frac{1}{a} f^i_{2(\alpha)} + \delta Q_{1\alpha} v^i_1 + Q_{0\alpha} \left(v^i_2 + 2\phi_1 B^i_1 - 4 C^i_{1k} v^k_1 \right) \right], \tag{4.30}$$

where $f^i_{1(\alpha)}$ and $f^i_{2(\alpha)}$ are the spatial parts of the momentum transfer rates at first and second orders.

Note that the homogeneous and isotropic FRW background excludes a zeroth-order momentum transfer.

Using Eq. (2.4) the spatial momentum transfer vector of order n, $f^i_{n(\alpha)}$, can be further decomposed into a scalar and a vector part, according to

$$f^i_{n(\alpha)} \equiv \delta^{ij} f_{n(\alpha),j} + \hat{f}^i_{n(\alpha)}. \tag{4.31}$$

5. Energy–momentum tensor for scalar fields

5.1. Single field

A minimally coupled scalar field is specified by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} - U(\varphi), \tag{5.1}$$

where the scalar field kinetic energy is then non-negative for our choice of metric signature.

The energy–momentum tensor is defined as

$$T_{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}, \tag{5.2}$$

and we therefore get for a scalar field φ

$$T^\mu_{\ \nu} = g^{\mu\alpha} \varphi_{,\alpha} \varphi_{,\nu} - \delta^\mu_\nu \left(U(\varphi) + \frac{1}{2} g^{\kappa\lambda} \varphi_{,\kappa} \varphi_{,\lambda} \right). \tag{5.3}$$

Comparing to the energy–momentum tensor for a perfect fluid, Eq. (4.6), we can identify the non-linear 4-velocity, density and pressure of the scalar field [43]

$$u_\mu = \frac{\varphi_{,\mu}}{|g^{\lambda\kappa} \varphi_{,\lambda} \varphi_{,\kappa}|}, \qquad \rho = -g^{\lambda\kappa} \varphi_{,\lambda} \varphi_{,\kappa} + U,$$

$$P = -g^{\lambda\kappa} \varphi_{,\lambda} \varphi_{,\kappa} - U. \tag{5.4}$$

Note that the anisotropic stress, $\pi_{\mu\nu}$, is identically zero for minimally coupled scalar fields. In fact we can subdivide the energy–momentum tensor for a single field into that of a stiff kinetic fluid with $\rho_{(\varphi)} = P_{(\varphi)} = -g^{\lambda\kappa}\varphi_{,\lambda}\varphi_{,\kappa}$ and a vacuum energy $\rho_{(U)} = -P_{(U)} = U(\varphi)$, which exchange energy–momentum $Q^{\mu}_{(\varphi)} = -Q^{\mu}_{(U)} = (dU/d\varphi)\nabla^{\mu}\varphi$.

Splitting the scalar field into a homogeneous background field and a perturbation,

$$\varphi(\eta, x^i) = \varphi_0(\eta) + \delta\varphi_1(\eta, x^i), \tag{5.5}$$

and using the definitions above we find for the components of the energy–momentum tensor of a perturbed scalar field at linear order without specifying a gauge yet

$$T^0_0 = -\frac{1}{2}a^{-2}\varphi_0'^2 - U_0 + a^{-2}\varphi_0'\left(\phi_1\varphi_0' - \delta\varphi_1'\right) - U_{,\varphi}\delta\varphi_1, \tag{5.6}$$

$$T^0_i = -a^{-2}\left(\varphi_0'\delta\varphi_{1,i}\right), \tag{5.7}$$

$$T^i_j = \left[\frac{1}{2}a^{-2}\varphi_0'^2 - U_0 - U_{,\varphi}\delta\varphi_1 + a^{-2}\varphi_0'\left(\delta\varphi_1' - \phi_1\varphi_0'\right)\right]\delta^i_j, \tag{5.8}$$

where $U_{,\varphi} \equiv dU/d\varphi$ and $U_0 = U(\varphi_0)$. By comparing Eq. (5.8) with Eq. (4.14), we see that scalar fields do not support vector or tensor perturbations to first order.

5.2. Multiple fields

For N minimally coupled scalar fields the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}\sum_I\left(g^{\mu\alpha}\varphi_{I,\alpha}\varphi_{I,\mu}\right) - U(\varphi_1, \ldots, \varphi_N). \tag{5.9}$$

The energy–momentum tensor is

$$T_{\mu\nu} = \sum_{K=1}^N\left[\varphi_{K,\mu}\varphi_{K,\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\varphi_{K,\alpha}\varphi_{K,\beta}\right] - g_{\mu\nu}U, \tag{5.10}$$

where φ_K is the Kth scalar field and U is the scalar field potential and $\varphi_{K,\mu} \equiv \partial\varphi_K/\partial x^\mu$.

Analogous to the energy–momentum tensor for a single field, we can identify the non-linear 4-velocity, density and pressure of each of the scalar fields [42]

$$u_{(I)\mu} = \frac{\varphi_{I,\mu}}{|g^{\lambda\kappa}\varphi_{I,\lambda}\varphi_{I,\kappa}|},$$

$$\rho_{(I)} = P_{(I)} = -g^{\lambda\kappa}\varphi_{I,\lambda}\varphi_{I,\kappa}. \tag{5.11}$$

The energy–momentum transfer to each fluid is $Q_{(I)\mu} = (\partial U/\partial\varphi_I)\varphi_{I,\mu}$.

The total energy–momentum tensor (5.10) is the sum over N stiff fluids plus the vacuum energy

$$T_{\mu\nu} = \sum_{I=1}^N\rho_{(I)}\left[2u_{(I)\mu}u_{(I)\nu} + g_{\mu\nu}\right] - Ug_{\mu\nu}. \tag{5.12}$$

The anisotropic stress, $\pi_{\mu\nu}$, is identically zero for any number of minimally coupled scalar fields. The total energy–momentum tensor is only equivalent to that of a single scalar field in the special case where all the 4-velocities of the fields, $u_{(I)\mu}$, are identical. This is true in the homogeneous FRW cosmology, but in general breaks down when inhomogeneous perturbations are considered.

We split the scalar fields φ_I into a background and perturbations up to and including second order according to Eqs. (2.1) and (2.2),

$$\varphi_I(x^\mu) = \varphi_{0I}(\eta) + \delta\varphi_{1I}(x^\mu) + \frac{1}{2}\delta\varphi_{2I}(x^\mu). \tag{5.13}$$

The potential $U \equiv U(\varphi_I)$ can be split similarly according to

$$U(\varphi_I) = U_0 + \delta U_1 + \frac{1}{2}\delta U_2, \tag{5.14}$$

where

$$\delta U_1 = \sum_K U_K\delta\varphi_{1K}, \tag{5.15}$$

$$\delta U_2 = \sum_{K,L} U_{KL}\delta\varphi_{1K}\delta\varphi_{1L} + \sum_K U_K\delta\varphi_{2K} \tag{5.16}$$

and we use the shorthand $U_K \equiv \partial U/\partial\varphi_K$. The energy–momentum tensor, Eq. (5.10), expanded up to second order in the perturbations for the metric tensor equation (2.15) is given in Appendix B.2.

6. Gauge transformations

We now review how tensorial quantities change under coordinate transformations [35,36,44–46] (see Ref. [47,7,48] for earlier work on this subject). While the order of the perturbation is indicated by a subscript, we also keep the small parameter ϵ in the following equations whenever appropriate.

A problem which arises in cosmological perturbation theory is the presence of spurious coordinate artefacts or gauge modes in the calculations. Although the gauge modes had been dealt with on a "case by case" basis before, the gauge issue was resolved in a systematic way by Bardeen [35]. The gauge issue will arise in any approach to GR that splits quantities into a background and a perturbation. Although GR is covariant, i.e. manifestly coordinate choice independent, splitting variables into a background part and a perturbation is not a covariant procedure, and therefore introduces a coordinate or gauge dependence. By construction this only affects the perturbations; the background quantities remain the same in the different coordinate systems. Here we assume that the difference between the coordinate systems is small, of $O(\epsilon)$, however the gauge problem would persist also for finite transformations. Note that the "covariant approach" [49] also corresponds to a choice of gauge, namely the comoving one, which is made explicit by the inclusion of the velocity field [50,51].

In order to restore covariance as far as possible, we usually wish to eliminate the gauge degrees of freedom. We will show in Section 7 how, by constructing variables corresponding to perturbations in physically defined coordinate systems, the gauge dependencies can be made to cancel out (the quantities so constructed will not change under a gauge transformation).

6.1. Active and passive approaches to gauge transformations

There are two approaches to calculate how perturbations change under a small coordinate or gauge transformation. For the *active view* we study how perturbations change under a mapping, where the map directly induces the transformation of the perturbed quantities. In the *passive view* the relation between two coordinate systems is specified, and we calculate how the perturbations are changed under this coordinate transformation. In the passive approach the transformation is taken at the same physical point, whereas in the active approach the transformation of the perturbed quantities is evaluated at the same coordinate point. We will discuss both approaches briefly in the following, but shall use the active approach to calculate the transformation behaviour of the first- and second-order variables. For a mathematically more rigorous discussion see e.g. Ref. [46].

6.1.1. Active approach

The starting point in the active approach is the exponential map, that allows us to immediately write down how a tensor **T** transforms up to second order, once the generator of the gauge transformation, ξ^{μ}, has been specified. The exponential map is

$$\widetilde{\mathbf{T}} = e^{\pounds_{\xi}} \mathbf{T}, \tag{6.1}$$

where $\pounds_{\xi}$ denotes the Lie derivative with respect to ξ^{λ}. The vector field generating the transformation, ξ^{λ}, is up to second order

$$\xi^{\mu} \equiv \epsilon \xi_1^{\mu} + \frac{1}{2} \epsilon^2 \xi_2^{\mu} + O(\epsilon^3). \tag{6.2}$$

The exponential map can be readily expanded

$$\exp(\pounds_{\xi}) = 1 + \epsilon \pounds_{\xi_1} + \frac{1}{2} \epsilon^2 \pounds_{\xi_1}^2 + \frac{1}{2} \epsilon^2 \pounds_{\xi_2} + \cdots \tag{6.3}$$

where we kept terms up to $O(\epsilon^2)$. Splitting the tensor **T** up to second order, as given in Eq. (2.2), and collecting terms of like order in ϵ we find that tensorial quantities transform at zeroth, first and second orders, respectively, as [44,45]

$$\widetilde{\mathbf{T}}_0 = \mathbf{T}_0, \tag{6.4}$$

$$\epsilon \widetilde{\delta \mathbf{T}}_1 = \epsilon \delta \mathbf{T}_1 + \epsilon \pounds_{\xi_1} \mathbf{T}_0, \tag{6.5}$$

$$\epsilon^2 \widetilde{\delta \mathbf{T}}_2 = \epsilon^2 \left(\delta \mathbf{T}_2 + \pounds_{\xi_2} \mathbf{T}_0 + \pounds_{\xi_1}^2 \mathbf{T}_0 + 2\pounds_{\xi_1} \delta \mathbf{T}_1 \right). \tag{6.6}$$

Note that the background quantities are not affected by the mapping. We will apply Eqs. (6.5) and (6.6) to scalars, vectors, and tensors after discussing the passive approach next.

Applying the map (6.1) to the coordinate functions x^{μ} we get a relation for the coordinates of a point q in and a point p as

$$x^{\mu}(q) = e^{\xi^{\lambda} \frac{\partial}{\partial x^{\lambda}} \big|_p} x^{\mu}(p), \tag{6.7}$$

where we have used the fact that when acting on scalars $\pounds_{\xi} = \xi^{\mu}(\partial/\partial x^{\mu})$ and the partial derivatives are evaluated at p. The left-hand side and the right-hand side of Eq. (6.7) are evaluated at different points. Eq. (6.7) can then be expanded up to second order as

$$x^{\mu}(q) = x^{\mu}(p) + \epsilon \xi_1^{\mu}(p) + \frac{1}{2} \epsilon^2 \left(\xi_{1,\nu}^{\mu}(p) \xi_1^{\nu}(p) + \xi_2^{\mu}(p) \right). \tag{6.8}$$

Note that we do not need Eq. (6.8) to calculate how perturbations change under a gauge transformation in the active approach, it simply tells us how the coordinates of the points p and q are related in this approach.

6.1.2. Passive approach

In the passive approach we specify the relation between two coordinate systems directly, and then calculate the change in the metric and matter variables when changing from one system to the other. As long as the two coordinate systems are related through a small perturbation, the functional form relating them is quite arbitrary.

However, in order to make contact with the active approach, discussed above, we take Eq. (6.8) as our starting point. Note, that all quantities in the passive approach are evaluated at the same physical point. Eq. (6.8) can be rewritten to give a relation between the "old" (untilded) and the "new" (tilde) coordinate systems [45,46],

$$\widetilde{x^\mu}(q) = x^\mu(q) - \epsilon \xi_1^\mu(q) + \epsilon^2 \frac{1}{2} \left(\xi_{1,\nu}^\mu(q) \xi_1^\nu(q) - \xi_2^\mu(q) \right), \tag{6.9}$$

evaluated at the same physical point q.

The passive point of view is very popular at first order, see e.g. the original paper by Bardeen [35] and the widely used reviews by Kodama and Sasaki [1], and Mukhanov, Feldman, and Brandenberger [2].

The starting point in the passive approach is to identify an invariant quantity, that allows us to relate quantities to be evaluated in the two coordinate systems. We denote the two coordinate systems by $\tilde{x}^\mu$ and x^μ systems, and their relation is given by Eq. (6.9). We choose as an example the energy density, ρ, which as a four scalar will not change under a coordinate transformation. However, once it has been split into the background quantity and perturbation at different orders, these variables will change.

The two coordinate systems are related by Eq. (6.9), which we can use to expand the right-hand side of Eq. (6.11) in a Taylor expansion up to second order. At first order, the two coordinate systems are simply related, using the linear part of Eq. (6.8), by

$$\tilde{x}^\mu = x^\mu - \xi_1^\mu. \tag{6.10}$$

We get the transformation behaviour of the density perturbation, $\delta\rho$, from the requirement that the total density, $\rho = \rho_0 + \delta\rho$, has to be invariant under a change of coordinate system and therefore has to be the same in the $\tilde{x}^\mu$ and the x^μ system, that is

$$\tilde{\rho}(\tilde{x}^\mu) = \rho(x^\mu). \tag{6.11}$$

Expanding both sides of Eq. (6.11) using Eq. (2.1), we get

$$\rho(x^\mu) = \rho_0(x^0) + \epsilon \delta\rho_1(x^\mu) + O(\epsilon^2), \tag{6.12}$$

$$\begin{aligned}
\tilde{\rho}(\tilde{x}^\mu) &= \rho_0\left(\widetilde{x^0}\right) + \epsilon \widetilde{\delta\rho_1}\left(\widetilde{x^\mu}\right) + O(\epsilon^2), \\
&= \rho_0(x^0) + \epsilon \left(-\rho_0'(x^0)\xi_1^0(x^\mu) + \widetilde{\delta\rho_1}(x^\mu) \right) + O(\epsilon^2).
\end{aligned} \tag{6.13}$$

Note that we use the same background solution $\rho_0(\eta)$ in both expressions. Thus we obtain the transformation rule at first order

$$\widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\xi_1^0. \tag{6.14}$$

Another invariant is the line element ds^2, which allows us to deduce the transformation properties of the metric tensor by exploiting the invariance of ds^2, i.e.,

$$ds^2 = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = g_{\mu\nu} dx^\mu dx^\nu. \tag{6.15}$$

Here we will not follow this approach further, but see e.g. [1,52].

6.2. Four-scalars

We now return to the active approach by studying the simplest tensorial quantity, the four-scalar. Examples of four-scalar are the energy density, ρ, and the scalar field φ, and we shall use the former below.

From Eqs. (2.1) and (2.2) we immediately get the perturbed four-scalar up to second order

$$\rho = \rho_0 + \delta\rho_1 + \frac{1}{2}\delta\rho_2, \tag{6.16}$$

using the energy density as an example.

6.2.1. First order

Before we can study the transformation behaviour of the perturbations at first order, we split the generating vector ξ_1^μ into a scalar temporal part α_1 and a spatial scalar and vector part, β_1 and γ_1^i, according to

$$\xi_1^\mu = \left(\alpha_1, \beta_{1,}^i + \gamma_1^i \right), \tag{6.17}$$

where the vector part is divergence-free $\partial_k \gamma_1^k = 0$.

Under a first-order transformation a four scalar, here the energy density, ρ, then transforms from Eqs. (6.5) and (A.8) as,

$$\widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\alpha_1. \tag{6.18}$$

The first-order density perturbation is fully specified by prescribing the first-order temporal gauge or time slicing, α_1.

6.2.2. Second order

At second order, as at first order, we split the generating vector ξ_2^μ into a scalar time and scalar and vector spatial parts, similarly as at first order, as

$$\xi_2^\mu = \left(\alpha_2, \beta_{2,}{}^i + \gamma_2{}^i\right), \tag{6.19}$$

where the vector part is divergence-free $\partial_k\gamma_2{}^k = 0$. We then find from Eqs. (6.6) and (A.8) that a four scalar transforms as

$$\widetilde{\delta\rho_2} = \delta\rho_2 + \rho_0'\alpha_2 + \alpha_1\left(\rho_0''\alpha_1 + \rho_0'\alpha_1' + 2\delta\rho_1'\right) + \left(2\delta\rho_1 + \rho_0'\alpha_1\right)_{,k}\left(\beta_{1,}{}^k + \gamma_1{}^k\right). \tag{6.20}$$

We see here already the coupling between vector and scalar perturbations in the last term through the gradient and $\gamma_1{}^i$. The gauge is only specified once the scalar temporal gauge perturbations at first and second order, α_1 and α_2, and the first-order spatial gauge perturbations, β_1 and $\gamma_1{}^i$, are specified.

6.3. Four-vectors

We now turn to four-vectors and their transformation properties. Of particular interest in cosmology is the unit four-velocity u^μ, which we defined in Section 4 above.[1]

6.3.1. First order

A four-vector transforms at first order, using Eqs. (6.5) and (A.9), as

$$\widetilde{\delta\mathcal{U}_{1\mu}} = \delta\mathcal{U}_{1\mu} + \mathcal{U}'_{(0)\mu}\alpha_1 + \mathcal{U}_{(0)\lambda}\xi_{1,\mu}^\lambda, \tag{6.21}$$

where we used the fact that in a FRW spacetime background quantities are time dependent only.

For the specific example of the four-velocity, defined in Eq. (4.3), we find,

$$\widetilde{v_{1i} + B_{1i}} = v_{1i} + B_{1i} - \alpha_{1,i}. \tag{6.22}$$

Using the transformation of the metric perturbation B_{1i}, given in Eq. (6.30), and using the decompositions given in Section 2.2.1, we get the transformations for the scalar and vector parts, respectively, at first order

$$\widetilde{v_1} = v_1 - \beta_1', \tag{6.23}$$

$$\widetilde{v_{\text{vec}(1)}^i} = v_{\text{vec}(1)}^i - \gamma_1{}^{i\prime}, \tag{6.24}$$

and the perturbed temporal part of u^μ does indeed transform as a scalar.

6.3.2. Second order

At second order we find that a four-vector transforms, using Eqs. (6.6) and (A.9), as

$$\widetilde{\delta\mathcal{U}_{2\mu}} = \delta\mathcal{U}_{2\mu} + \mathcal{U}'_{(0)\mu}\alpha_2 + \mathcal{U}_{(0)0}\alpha_{2,\mu} + \mathcal{U}''_{(0)\mu}\alpha_1^2 + \mathcal{U}'_{(0)\mu}\alpha_{1,\lambda}\xi_1^\lambda$$
$$+ 2\mathcal{U}'_{(0)0}\alpha_1\alpha_{1,\mu} + \mathcal{U}_{(0)0}\left(\xi_1^\lambda\alpha_{1,\mu\lambda} + \alpha_{1,\lambda}\xi_{1,\mu}^\lambda\right) + 2\left(\delta\mathcal{U}_{1\mu,\lambda}\xi_1^\lambda + \delta\mathcal{U}_{1\lambda}\xi_{1,\mu}^\lambda\right), \tag{6.25}$$

where as before we used for the background $\mathcal{U}_{(0)\mu} \equiv \mathcal{U}_{(0)\mu}(\eta)$ and $\mathcal{U}_{(0)i} = \mathbf{0}$.

Focusing again on the four-velocity, Eq. (4.3), and following a similar procedure as at first order, we find that the second order combined scalar and vector spatial part transforms as

$$\widetilde{v_{2i}} = v_{2i} - \xi_{2i}' + \mathcal{X}_{\text{v}i}, \tag{6.26}$$

where $\mathcal{X}_{\text{v}i}$ contains the terms quadratic in the first-order perturbations and is given by

$$\mathcal{X}_{\text{v}i} \equiv \xi_{1i}'\left(2\phi_1 + \alpha_1' + 2\mathcal{H}\alpha_1\right) - \alpha_1\xi_{1i}'' - \xi_1^k\xi_{1i,k}' + \xi_1^{k\prime}\xi_{1i,k} - 2\alpha_1\left(v_{1i}' + \mathcal{H}v_{1i}\right) + 2v_{1i,k}\xi_1^k - 2v_1^k\xi_{1i,k}, \tag{6.27}$$

and we already substituted for the transformation of the metric perturbation B_{2i}, given in Eq. (6.48). Then decomposing the second-order velocity transformation, Eq. (6.26), into scalar and vector parts, we get the transformations as

$$\widetilde{v_2} = v_2 - \beta_2' + \nabla^{-2}\mathcal{X}_{\text{v},k}^k, \tag{6.28}$$

$$\widetilde{v_{\text{vec}(2)}^i} = v_{\text{vec}(2)}^i - \gamma_2{}^{i\prime} + \mathcal{X}_{\text{v}i} - \nabla^{-2}\mathcal{X}_{\text{v},ki}^k. \tag{6.29}$$

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[1] Note that under a gauge transformation the vector field normal to the constant-η hypersurface, n^μ defined in Section 3.1, will be replaced by a new vector field normal to the constant-$\tilde{\eta}$ hypersurfaces, therefore it is not particularly helpful to study the transformation of n^μ. We will study the gauge transformation of the metric tensor in Section 6.4, from which we can derive the transformation rules for the hypersurface orthogonal field and its expansion, shear, etc.

6.4. Tensors

6.4.1. First-order coordinate transformation

We can now calculate how the first-order metric perturbations change under a gauge transformation. We get the change of the $\delta^{(1)}g_{00}$ and hence the lapse function ϕ_1 immediately from Eqs. (6.5) and (A.10), since this component of the metric is scalar in nature. The change of the $\delta^{(1)}g_{0i}$ is slightly more involved, since this component contains scalar and vector perturbations. We therefore have to compute the overall transformation of this metric component using Eqs. (6.5) and (A.10), and then split the result for B_{1i} into its scalar part, B_1, and its divergence-free part $-S_i$. We therefore get for the combined part B_{1i},

$$\widetilde{B}_{1i} = B_{1i} + \xi'_{1i} - \alpha_{1,i}, \tag{6.30}$$

and taking the divergence gives for the scalar part,

$$\nabla^2 \widetilde{B}_1 = \nabla^2 B_1 + \nabla^2 \beta'_1 - \nabla^2 \alpha_1, \tag{6.31}$$

which, after "removal" of the Laplacian gives the transformation behaviour of B_1. We can then subtract the scalar part from Eq. (6.31) and are left with the vector part. The results are given below.

To get the change of the metric functions in the spatial part of the metric under a gauge transformation, we again first use Eqs. (6.5) and (A.10) to get transformation of the spatial part of the metric $\delta^{(1)}g_{ij}$, or C_{1ij},

$$2\widetilde{C}_{1ij} = 2C_{1ij} + 2\mathcal{H}\alpha_1\delta_{ij} + \xi_{1i,j} + \xi_{1j,i}, \tag{6.32}$$

where we reproduce Eq. (2.12) above for convenience,

$$2C_{1ij} = -2\psi_1\delta_{ij} + 2E_{1,ij} + 2F_{1(i,j)} + h_{1ij}.$$

Taking the trace of Eq. (6.32) we get

$$-3\widetilde{\psi}_1 + \nabla^2 \widetilde{E}_1 = -3\psi_1 + \nabla^2 E_1 + 3\mathcal{H}\alpha_1 + \nabla^2 \beta_1. \tag{6.33}$$

Now applying the operator $\partial^i\partial^j$ to Eq. (6.32) we get a second equation relating the scalar perturbations ψ_1 and E_1,

$$-3\widetilde{\nabla}^2\psi_1 + \nabla^2\nabla^2\widetilde{E}_1 = -3\nabla^2\psi_1 + \nabla^2\nabla^2 E_1 + 3\mathcal{H}\nabla^2\alpha_1 + \nabla^2\nabla^2\beta_1. \tag{6.34}$$

Taking the divergence of Eq. (6.32) we get

$$2\widetilde{C}_{1ij,}{}^j = 2C_{1ij,}{}^j + 2\mathcal{H}\alpha_{1,i} + \nabla^2\xi_{1i} + \nabla^2\beta_{1,i}. \tag{6.35}$$

Substituting in our results for $\widetilde{\psi}_1$ and $\widetilde{E}_1$ we arrive at

$$\nabla^2\widetilde{F}_{1i} = \nabla^2 F_{1i} + \nabla^2\gamma_1{}^i. \tag{6.36}$$

We can sum up the transformations of the first-order metric perturbations we have from the above, first for the scalars as

$$\widetilde{\phi}_1 = \phi_1 + \mathcal{H}\alpha_1 + \alpha'_1, \tag{6.37}$$

$$\widetilde{\psi}_1 = \psi_1 - \mathcal{H}\alpha_1, \tag{6.38}$$

$$\widetilde{B}_1 = B_1 - \alpha_1 + \beta'_1, \tag{6.39}$$

$$\widetilde{E}_1 = E_1 + \beta_1, \tag{6.40}$$

and for the vector perturbations as

$$\widetilde{S}_1{}^i = S_1{}^i - \gamma_1{}^{i'}, \tag{6.41}$$

$$\widetilde{F}_1{}^i = F_1{}^i + \gamma_1{}^i. \tag{6.42}$$

The first-order tensor perturbation is found to be gauge-invariant,

$$\widetilde{h}_{1ij} = h_{1ij} \tag{6.43}$$

by substituting Eqs. (6.37)–(6.42) into Eq. (6.32).

For later use, we note that the scalar shear potential, $\sigma_1 = E'_1 - B_1$, defined in Eq. (3.15), and the combination $v_1 + B_1$ corresponding to the momentum scalar, transform as

$$\widetilde{\sigma}_1 = \sigma_1 + \alpha_1, \tag{6.44}$$

$$\widetilde{v}_1 + \widetilde{B}_1 = v_1 + B_1 - \alpha_1. \tag{6.45}$$

6.4.2. Second-order gauge transformations

The metric tensor transforms at second order, from Eqs. (6.6) and (A.10), as

$$
\widetilde{\delta g_{\mu\nu}^{(2)}} = \delta g_{\mu\nu}^{(2)} + g_{\mu\nu,\lambda}^{(0)} \xi_2^\lambda + g_{\mu\lambda}^{(0)} \xi_{2\,,\nu}^\lambda + g_{\lambda\nu}^{(0)} \xi_{2\,,\mu}^\lambda + 2\Big[\delta g_{\mu\nu,\lambda}^{(1)} \xi_1^\lambda + \delta g_{\mu\lambda}^{(1)} \xi_{1\,,\nu}^\lambda + \delta g_{\lambda\nu}^{(1)} \xi_{1\,,\mu}^\lambda\Big]
$$
$$
+ g_{\mu\nu,\lambda\alpha}^{(0)} \xi_1^\lambda \xi_1^\alpha + g_{\mu\nu,\lambda}^{(0)} \xi_{1\,,\alpha}^\lambda \xi_1^\alpha + 2\Big[g_{\mu\lambda,\alpha}^{(0)} \xi_1^\alpha \xi_{1\,,\nu}^\lambda + g_{\lambda\nu,\alpha}^{(0)} \xi_1^\alpha \xi_{1\,,\mu}^\lambda + g_{\lambda\alpha}^{(0)} \xi_{1\,,\mu}^\lambda \xi_{1\,,\nu}^\alpha\Big]
$$
$$
+ g_{\mu\lambda}^{(0)} \left(\xi_{1\,,\nu\alpha}^\lambda \xi_1^\alpha + \xi_{1\,,\alpha}^\lambda \xi_{1\,,\nu}^\alpha\right) + g_{\lambda\nu}^{(0)} \left(\xi_{1\,,\mu\alpha}^\lambda \xi_1^\alpha + \xi_{1\,,\alpha}^\lambda \xi_{1\,,\mu}^\alpha\right). \tag{6.46}
$$

As at first order, in the previous subsection, we get the transformation behaviour for the second-order lapse function ϕ_2 straight from the 0–0 component of Eq. (6.46), which gives

$$
\widetilde{\phi_2} = \phi_2 + \mathcal{H}\alpha_2 + \alpha_2' + \alpha_1\left[\alpha_1'' + 5\mathcal{H}\alpha_1' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\alpha_1 + 4\mathcal{H}\phi_1 + 2\phi_1'\right]
$$
$$
+ 2\alpha_1'\left(\alpha_1' + 2\phi_1\right) + \xi_{1k}\left(\alpha_1' + \mathcal{H}\alpha_1 + 2\phi_1\right)_{,}^{\ k} + \xi_{1k}'\left[\alpha_{1,}^{\ k} - 2B_{1k} - \xi_1^{k'}\right]. \tag{6.47}
$$

The combined scalar and vector 0–i metric part transforms from Eqs. (6.6) and (A.10) as

$$
\widetilde{B}_{2i} = B_{2i} + \xi_{2i}' - \alpha_{2,i} + \mathcal{X}_{Bi}, \tag{6.48}
$$

where B_{2i} is similarly vector and scalar combined, and we defined $\mathcal{X}_{Bi}$ to contain the terms quadratic in the first-order perturbations, as

$$
\mathcal{X}_{Bi} \equiv 2\Big[\left(2\mathcal{H}B_{1i} + B_{1i}'\right)\alpha_1 + B_{1i,k}\xi_1^k - 2\phi_1\alpha_{1,i} + B_{1k}\xi_{1,\ i}^k + B_{1i}\alpha_1' + 2C_{1ik}\xi_1^{k'}\Big] + 4\mathcal{H}\alpha_1\left(\xi_{1i}' - \alpha_{1,i}\right)
$$
$$
+ \alpha_1'\left(\xi_{1i}' - 3\alpha_{1,i}\right) + \alpha_1\left(\xi_{1i}'' - \alpha_{1,i}'\right) + \xi_1^{k'}\left(\xi_{1i,k} + 2\xi_{1k,i}\right) + \xi_1^k\left(\xi_{1i,k}' - \alpha_{1,ik}\right) - \alpha_{1,k}\xi_{1,\ i}^k. \tag{6.49}
$$

To get the transformation behaviour of the vector and the scalar parts separately, we take the divergence of Eq. (6.48) and find after applying the inverse Laplacian, the transformation scalar part $\widetilde{B}_2$,

$$
\widetilde{B}_2 = B_2 - \alpha_2 + \beta_2' + \nabla^{-2}\mathcal{X}_{B\ ,k}^{\ k}, \tag{6.50}
$$

or explicitly

$$
\widetilde{B}_2 = B_2 - \alpha_2 + \beta_2' + \nabla^{-2}\bigg\{2\Big[\nabla^2\left(2\mathcal{H}B_1 + B_1'\right)\alpha_1 + \left(2\mathcal{H}B_{1k} + B_{1k}'\right)\alpha_{1,}^{\ k} + \nabla^2 B_{1,k}\xi_1^k + B_{1\ ,k}^{\ l}\xi_{1,l}^k
$$
$$
- 2\phi_1\nabla^2\alpha_1 - 2\phi_{1,k}\alpha_{1,}^{\ k} + B_{1k}\nabla^2\xi_1^k + B_{1k,l}^{\ \ l}\xi_{1,}^{k} \xi_{1,l}^k \nabla^2 B_1\alpha_1' + B_{1k}\alpha_{1,}^{\ k'} + 2C_{1\ k}^l\xi_{1,\ l}^{k\ '} + 2C_{1\ k,l}^l\xi_1^{k'}\Big]
$$
$$
+ 4\mathcal{H}\Big[\alpha_1\nabla^2\left(\beta_1' - \alpha_1\right) + \alpha_{1,k}\left(\xi_1^{k'} - \alpha_{1,}^{\ k}\right)\Big] + \alpha_1'\nabla^2\left(\beta_1' - 3\alpha_1\right) + \alpha_{1,k}\left(\xi_1^{k'} - 3\alpha_{1,}^{\ k}\right)
$$
$$
+ \alpha_1\nabla^2\left(\beta_1'' - \alpha_1'\right) + \alpha_{1,k}\left(\xi_1^{k''} - \alpha_{1,}^{\ k'}\right) + \xi_1^{k'}\nabla^2\left(\beta_{1,k} + 2\xi_{1k}\right) + \xi_1^{k\ '}_{\ l}\left(\xi_{1\ k}^l + 2\xi_{1k,}^l\right)
$$
$$
+ \xi_1^k\nabla^2\left(\beta_{1,k}' - \alpha_{1,k}\right) + \xi_{1\ l}^k\left(\xi_{1\ k}^{l\ '} - \alpha_{1,\ k}^{\ l}\right) - \alpha_{1,k}\nabla^2\xi_1^k - \alpha_{1,k}^{\ l}\xi_{1,l}^k\bigg\}. \tag{6.51}
$$

The vector part is then simply found by subtracting the scalar part from Eq. (6.48), and is given by

$$
\widetilde{S}_{2i} = S_{2i} - \gamma_2^{\ i'} - \mathcal{X}_{Bi} + \nabla^{-2}\mathcal{X}_{B\ ,ki}^{\ k}. \tag{6.52}
$$

We now turn to the transformation behaviour of the perturbations in the spatial part of the metric tensor. We can follow here along similar lines as in the linear case. However, the task is made more complicated not only by the size of the expressions but more importantly by the fact that now we will have to let inverse gradients operate on products of first-order quantities.

The perturbed spatial part of the metric, C_{2ij}, transforms at second order as

$$
2\widetilde{C}_{2ij} = 2C_{2ij} + 2\mathcal{H}\alpha_2\delta_{ij} + \xi_{2i,j} + \xi_{2j,i} + \mathcal{X}_{ij}, \tag{6.53}
$$

where we defined $\mathcal{X}_{ij}$ to contain the terms quadratic in the first-order perturbations as

$$
\mathcal{X}_{ij} \equiv 2\left[\left(\mathcal{H}^2 + \frac{a''}{a}\right)\alpha_1^2 + \mathcal{H}\left(\alpha_1\alpha_1' + \alpha_{1,k}\xi_1^k\right)\right]\delta_{ij} + 4\Big[\alpha_1\left(C_{1ij}' + 2\mathcal{H}C_{1ij}\right) + C_{1ij,k}\xi_1^k + C_{1ik}\xi_{1,\ j}^k + C_{1kj}\xi_{1,\ i}^k\Big]
$$
$$
+ 2\left(B_{1i}\alpha_{1,j} + B_{1j}\alpha_{1,i}\right) + 4\mathcal{H}\alpha_1\left(\xi_{1i,j} + \xi_{1j,i}\right) - 2\alpha_{1,i}\alpha_{1,j} + 2\xi_{1k,i}\xi_{1,\ j}^k + \alpha_1\left(\xi_{1i,j}' + \xi_{1j,i}'\right)
$$
$$
+ \left(\xi_{1i,jk} + \xi_{1j,ik}\right)\xi_1^k + \xi_{1i,k}\xi_{1,\ j}^k + \xi_{1j,k}\xi_{1,\ i}^k + \xi_{1i}'\alpha_{1,j} + \xi_{1j}'\alpha_{1,i}. \tag{6.54}
$$

The perturbed spatial part of the metric, C_{2ij}, is decomposed in Eq. (2.12) above into scalar, vector, and tensor parts, which we reproduce here at second order,

$$2C_{2ij} = -2\psi_2\delta_{ij} + 2E_{2,ij} + 2F_{2(i,j)} + h_{2ij}.$$

Taking the trace of Eq. (6.53) we get

$$-3\widetilde{\psi_2} + \nabla^2\widetilde{E_2} = -3\psi_2 + \nabla^2 E_2 + 3\mathcal{H}\alpha_2 + \nabla^2\beta_2 + \frac{1}{2}\mathcal{X}^k{}_k, \tag{6.55}$$

where we find $\mathcal{X}^k{}_k$ to be

$$\frac{1}{2}\mathcal{X}^k{}_k = 3\left(\mathcal{H}^2 + \frac{a''}{a}\right)\alpha_1^2 + 3\mathcal{H}\left(\alpha_1\alpha_1' + \alpha_{1,k}\xi_1^k\right) + 2\left[\alpha_1\left(C_1{}^k{}_k{}' + 2\mathcal{H}C_1{}^k{}_k\right) + C_1{}^k{}_{k,l}\xi_1^l + 2C_1{}^{kl}\xi_{1l,k}\right]$$

$$+ 2B_{1k}\alpha_{1,}{}^k - \alpha_{1,k}\alpha_{1,}{}^k + \xi_{1k,}{}^l\xi_{1,}{}^k{}_l + \xi_{1k,}{}^l\xi_{1l,}{}^k + \alpha_1\nabla^2\left(\beta_1' + 4\mathcal{H}\beta_1\right) + \nabla^2\beta_{1,k}\xi_1^k + \xi_{1k}'\alpha_{1,}{}^k. \tag{6.56}$$

Now applying the operator $\partial^i\partial^j$ to Eq. (6.53) we get a second equation relating the scalar perturbations ψ_2 and E_2,

$$-\widetilde{\nabla}^2\psi_2 + \nabla^2\nabla^2\widetilde{E_2} = -\nabla^2\psi_2 + \nabla^2\nabla^2 E_2 + \mathcal{H}\nabla^2\alpha_2 + \nabla^2\nabla^2\beta_2 + \frac{1}{2}\mathcal{X}^{ij}{}_{,ij}. \tag{6.57}$$

This gives the second-order scalar metric perturbations

$$\widetilde{\psi_2} = \psi_2 - \mathcal{H}\alpha_2 - \frac{1}{4}\mathcal{X}^k{}_k + \frac{1}{4}\nabla^{-2}\mathcal{X}^{ij}{}_{,ij}, \tag{6.58}$$

and

$$\widetilde{E_2} = E_2 + \beta_2 + \frac{3}{4}\nabla^{-2}\nabla^{-2}\mathcal{X}^{ij}{}_{,ij} - \frac{1}{4}\nabla^{-2}\mathcal{X}^k{}_k. \tag{6.59}$$

Taking the divergence of Eq. (6.53) we get

$$2\widetilde{C}_{2ij,}{}^j = 2C_{2ij,}{}^j + 2\mathcal{H}\alpha_{2,i} + \nabla^2\xi_{2i} + \nabla^2\beta_{2,i} + \mathcal{X}_{ik,}{}^k. \tag{6.60}$$

Substituting in our results for $\widetilde{\psi_2}$ and $\widetilde{E_2}$ we then arrive at

$$\nabla^2\widetilde{F}_{2i} = \nabla^2 F_{2i} + \nabla^2\gamma_{2i} + \mathcal{X}_{ik,}{}^k - \nabla^{-2}\mathcal{X}^{kl}{}_{,kli}. \tag{6.61}$$

Finally we obtain the second-order vector metric perturbation

$$\widetilde{F}_{2i} = F_{2i} + \gamma_{2i} + \nabla^{-2}\mathcal{X}_{ik,}{}^k - \nabla^{-2}\nabla^{-2}\mathcal{X}^{kl}{}_{,kli}. \tag{6.62}$$

We can now turn to the tensor perturbation at second order. Substituting our previous results for ψ_2, E_2, and F_{2i} into Eq. (6.53) we get

$$\widetilde{h}_{2ij} = h_{2ij} + \mathcal{X}_{ij} + \frac{1}{2}\left(\nabla^{-2}\mathcal{X}^{kl}{}_{,kl} - \mathcal{X}^k{}_k\right)\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}\mathcal{X}^{kl}{}_{,klij} + \frac{1}{2}\nabla^{-2}\mathcal{X}^k{}_{k,ij} - \nabla^{-2}\left(\mathcal{X}_{ik,}{}^k{}_j + \mathcal{X}_{jk,}{}^k{}_i\right). \tag{6.63}$$

Although the second-order tensor transformation h_{2ij} is not dependent on the second-order part of the gauge transformation, ξ_2^μ, it does depend on first-order quantities quadratically. The tensor metric perturbations are no longer gauge-invariant at second and higher orders.

6.4.3. The large scale or small k limit

From Eqs. (6.20), (6.58) and (6.47) we see that on large scales, where gradient terms can be neglected, the definition of the second-order perturbations in the "new" coordinate is independent of the spatial coordinate choice (the "threading") at second order in the gradients. It is therefore sufficient on large scales (at $O(k^2)$) to specify the time slicing by prescribing α_1 and α_2, in order to define gauge-invariant variables [53–55]. The procedure to neglect the gradient terms, is explained in detail in Ref. [54]. For the approximation to hold one assumes that each quantity can be treated as smooth on some sufficiently large scale. Formally one multiplies each spatial gradient ∂_i by a fictitious parameter k, and expands the exact equations as a power series in k, keeping only the zero- and first-order terms, finally setting $k = 1$.

7. Gauge-invariant variables

The notion of invariance under coordinate reparametrisation is central to Einstein's theory of general relativity. This is both a blessing and a curse in the study of cosmological perturbations. We are free to pick coordinate systems best adapted to the problem at hand, but we also obtain apparently different results depending upon this arbitrary choice of coordinates. This is the gauge problem.

Ultimately physical observables are not dependent on the choice of coordinate, though observables may be different for different observers. All one can do is to specify quantities unambiguously, such that they have a *gauge-invariant* definition. This is not the same as *gauge independence*. A quantity like the tensor metric perturbation, h_{1ij}, is truly gauge independent at first order in that the tensor part of the metric perturbation is the same in all gauges. The scalar curvature perturbation, ψ_1, on the other hand is intrinsically gauge dependent. It is different under different time slicings. (Indeed in the spatially flat slicing the curvature perturbation is zero by construction.) One can construct gauge-invariant combinations, which may be referred to as the gauge-invariant curvature perturbation, but they only correspond with the curvature perturbation in one particular gauge. As a result one can find in the literature many different gauge-invariant curvature perturbations corresponding to the many different choices of gauge, such as Ψ, ζ and $\mathcal{R}$, corresponding to the curvature perturbations in the longitudinal, the uniform density, and the comoving gauge, respectively, to name just three.

In this section we shall show how different gauge-invariant combinations of otherwise gauge-dependent quantities can be constructed by fixing the otherwise arbitrary coordinate transformations at first order and beyond, yielding gauge-invariant definitions of the physical perturbations in specified gauges. Residual gauge degrees of freedom only remain in cases where the coordinate choice is not unambiguously fixed, as in the synchronous gauge.

At first order the tensor metric perturbation, h_{1ij}, is not affected by the mapping, or by the change of coordinate system, and hence is gauge-invariant. Thus we only have to construct gauge-invariant scalar and vector perturbations at first order. However at second order the tensor part of the metric perturbation also becomes gauge dependent.

At first order we can define scalar and vector type gauge-invariant variables independently of each other, but matters are more complicated at second order. Whereas we can still specify the "proper" second-order scalar and vector slicings and threadings independently, we now also have to specify the first-order gauge functions of both types simultaneously.

We could specify different gauges at first and second order, but we would be losing the physical interpretation of the quantity thus constructed. We therefore choose the same gauge at first and second order, and at first order the same physical gauge condition for the scalar and vector gauge functions. It is however sometimes necessary to combine different temporal and spatial gauge conditions. For example imposing the uniform density condition only specifies the slicing, and we are free to combine it with a flat threading.

7.1. Longitudinal gauge

7.1.1. First order

The gauge dependence of the metric perturbations led Bardeen [35] to propose that only quantities that are explicitly invariant under gauge transformations should be considered. By studying the transformation equations (6.37)–(6.40), Bardeen[2] constructed two such quantities [35]

$$\Phi \equiv \phi_1 + \mathcal{H}(B_1 - E_1') + (B_1 - E_1')', \tag{7.1}$$

$$\Psi \equiv \psi_1 - \mathcal{H}\left(B_1 - E_1'\right). \tag{7.2}$$

These turn out to coincide with the scalar metric perturbations in a particular gauge, called variously the orthogonal zero-shear [35,1], conformal Newtonian [4,56] or longitudinal gauge [2]. It may therefore appear that this gauge is somehow preferred over other gauge choices. However any unambiguous choice of time slicing and threading can be used to define explicitly gauge-invariant perturbations. The longitudinal gauge of Ref. [2] provides but one example.

The two scalar gauge functions, α and β defined in Eq. (6.17), which represent different choices of time slicings (choice of spatial hypersurfaces) and threading (choice of spatial coordinates on these hypersurfaces) respectively, allow two of the scalar metric perturbations to be eliminated, implying that the two remaining gauge-invariant combinations should then be gauge-invariant. If we choose to work on spatial hypersurfaces with vanishing shear, we find from Eqs. (6.39), (6.40) and (3.15) that the shear scalar transforms as $\widetilde{\sigma}_1 = \sigma_1 + \alpha_1$ and this implies that to obtain perturbations in the longitudinal gauge starting from arbitrary coordinates we should perform a transformation

$$\alpha_{1\ell} = -\sigma_1 = B_1 - E_1'. \tag{7.3}$$

This is sufficient to determine the geometrical perturbations ϕ_1, ψ_1, σ_1 or other scalar quantities on these hypersurfaces. In addition, the longitudinal gauge is completely determined by the spatial gauge choice $\widetilde{E}_{1\ell} = 0$ [and hence from Eq. (7.3) $\widetilde{B}_{1\ell} = 0$] which requires from Eq. (6.40)

$$\beta_{1\ell} = -E_1. \tag{7.4}$$

The remaining scalar metric perturbations, ϕ_1 and ψ_1, are given from Eqs. (6.37) and (6.38) as

$$\widetilde{\phi_{1\ell}} = \phi_1 + \mathcal{H}(B_1 - E_1') + (B_1 - E_1')', \tag{7.5}$$

$$\widetilde{\psi_{1\ell}} = \psi_1 - \mathcal{H}\left(B_1 - E_1'\right). \tag{7.6}$$

Note that $\widetilde{\phi_{1\ell}}$ and $\widetilde{\psi_{1\ell}}$ are then identical to Φ and Ψ defined in Eqs. (7.1) and (7.2).

---

[2] In Bardeen's notation these gauge-invariant perturbations are given as $\Phi \equiv \Phi_A Q^{(0)}$ and $\Psi \equiv -\Phi_H Q^{(0)}$.

The fluid density perturbation, $\delta\rho_1$, and scalar velocity, v_1, are given from (6.18) and (6.23)

$$\widetilde{\delta\rho_{1\ell}} = \delta\rho_1 + \rho_0' \left(B_1 - E_1'\right), \tag{7.7}$$

$$\widetilde{v_{1\ell}} = v_1 + E_1'. \tag{7.8}$$

These gauge-invariant quantities are simply a gauge-invariant definition of the perturbations in the longitudinal gauge.

This gauge is widely used, for example, throughout Ref. [2]. It has also proven useful for calculations on small scales, since it gives evolution equations closest to the Newtonian ones, e.g. Ref. [57]. Recently it has also become popular in backreaction studies, e.g. [58–60]. After imposing the gauge conditions the metric tensor is diagonal, which simplifies many calculations, for example the derivation of the governing equations of the Boltzmann-hierarchy. Moreover we shall show in Section 8 that in many cases of physical interest (in the absence of anisotropic stress) one finds $\Phi = \Psi$ and there is only one variable required to describe all scalar metric perturbations.

However, it can be difficult to define quantities in this gauge in the super-horizon limit, since in the super-horizon limit the shear vanishes and hence numerical instabilities can occur on large scales in the longitudinal gauge, see e.g. Refs. [56, 61,22].

The extension to include vector and tensor metric perturbations is called the Poisson gauge [4,45]. Tensor metric perturbations are automatically gauge independent at first order (and hence gauge-invariant). Eliminating the spatial part of the contravariant vector field n^μ in Eq. (3.5) requires both $\widetilde{B_\ell}{}_{,}^i$ and $\widetilde{S_\ell}{}^i = 0$ which from Eq. (6.41) fixes the vector part of the spatial gauge transformation (6.17)

$$\gamma_{1\ell}^i = \int S_1^i \, \mathrm{d}\eta + \hat{c}_1^i(x^j), \tag{7.9}$$

up to an arbitrary constant 3-vector $\hat{c}_1^i$ which depends on the choice of spatial coordinates on an initial hypersurface. The remaining vector metric perturbation is

$$\widetilde{F_{1\ell}}{}^i = F_1^{i\prime} + \int S_1^i \, \mathrm{d}\eta + \hat{c}_1^i(x^j). \tag{7.10}$$

7.1.2. Second order

It is possible to extend the longitudinal, or Poisson, gauge to higher orders. The principle for constructing gauge-invariant variables remains the same. We use a physical choice of gauge to specify the vector field ξ^μ generating the transformation (6.7) from an arbitrary gauge [53,62]. Requiring first that $\widetilde{E}_{2\ell} = 0$ fixes the scalar part of the spatial gauge using Eq. (6.59), which gives

$$\beta_{2\ell} = -E_2 - \frac{3}{4}\nabla^{-2}\nabla^{-2}\mathcal{X}^{ij}{}_{,ij} + \frac{1}{4}\nabla^{-2}\mathcal{X}^k{}_k. \tag{7.11}$$

Note that having already imposed the Poisson gauge at first order, α_1, β_1 and γ_{1i} are fixed by Eqs. (7.3), (7.4) and (7.9), and thus so is $\mathcal{X}_{ij}$, given in Eq. (6.54).

Requiring that the scalar part of the perturbed shift function vanishes, $\widetilde{B}_{2\ell} = 0$, then sets the temporal gauge, $\alpha_{2\ell}$ using Eq. (6.50), while requiring that the vector part vanishes, $F_2^i = \mathbf{0}$, can be used along with Eq. (6.62) to fix the vector part of the spatial gauge, $\gamma_{2\ell}^i$, up to a constant of integration as at first order, Eq. (7.9).

We then obtain gauge-invariant definitions of Φ, Ψ and other perturbations at second order by substituting these specific gauge transformations into Eqs. (6.47) and (6.58) to obtain

$$\widetilde{\phi_{2\ell}} = \phi_2 + \mathcal{H}\alpha_{2\ell} + \alpha_{2\ell}' + \alpha_{1\ell}\left[\alpha_{1\ell}'' + 5\mathcal{H}\alpha_{1\ell}' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\alpha_{1\ell} + 4\mathcal{H}\phi_1 + 2\phi_1'\right]$$
$$+ 2\alpha_{1\ell}'\left(\alpha_{1\ell}' + 2\phi_1\right) + \xi_{1\ell k}\left(\alpha_{1\ell}' + \mathcal{H}\alpha_{1\ell} + 2\phi_1\right){}_{,}^k + \xi_{1\ell k}'\left[\alpha_{1\ell,}^k - 2B_{1k} - \xi_{1\ell}^{k\,\prime}\right]. \tag{7.12}$$

$$\widetilde{\psi_{2\ell}} = \psi_2 - \mathcal{H}\alpha_{2\ell} - \frac{1}{4}\mathcal{X}_{\ell k}^k + \frac{1}{4}\nabla^{-2}\mathcal{X}_\ell{}^{ij}{}_{,ij}, \tag{7.13}$$

where $\mathcal{X}_{\ell ij}$ denotes the quadratic first-order terms in Eq. (6.54) using the longitudinal gauge transforms $\alpha_{1\ell}$ and $\xi_{\ell i}$.

The tensor (transverse, trace-free) part of the metric perturbation at second order, h_{ij} in Eq. (6.63), is not affected by the second-order gauge transformations α_2 and ξ_{2i}, but it does depend on the choice of gauge at first order, α_1 and ξ_{1i}. Thus we need to include the corresponding first-order gauge definition to obtain a gauge-invariant definition of the tensor metric perturbations at second order.

In particular, recent work on the generation of gravitational waves at second order [63,13,64–66] has calculated the resulting tensor mode in the Poisson gauge. To give a gauge-invariant definition of the tensor perturbation in the Poisson gauge one needs to explicitly include the transverse and trace-free (tensor) part of the second-order gauge transformation

from an arbitrary gauge. From Eq. (6.63) we obtain the gauge-invariant definition of the tensor metric perturbation in the Poisson gauge:

$$\widetilde{h}_{2\ell ij} = h_{2ij} + \mathcal{X}_{\ell ij} + \frac{1}{2}\left(\nabla^{-2}\mathcal{X}^{\ell kl}_{,kl} - \mathcal{X}^{k}_{\ell k}\right)\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}\mathcal{X}^{\ell kl}_{,klij} + \frac{1}{2}\nabla^{-2}\mathcal{X}^{k}_{\ell k,ij} - \nabla^{-2}\left(\mathcal{X}_{\ell ik,\ j}^{\ \ \ k} + \mathcal{X}_{\ell jk,\ i}^{\ \ \ k}\right). \quad (7.14)$$

7.2. Spatially flat gauge

7.2.1. First order

An alternative gauge choice, defined purely by local metric quantities is the spatially flat or uniform curvature gauge [1,67–70], also called the off-diagonal gauge [71]. In this gauge one selects spatial hypersurfaces on which the induced 3-metric on spatial hypersurfaces is left unperturbed by scalar or vector perturbations, which requires $\widetilde{\psi}_1 = \widetilde{E}_1 = 0$ and $\widetilde{F}_{1i} = \mathbf{0}$. Using Eqs. (6.38), (6.40) and (6.42) this corresponds to a gauge transformation (6.17) where

$$\alpha_{1\text{flat}} = \frac{\psi_1}{\mathcal{H}}, \qquad \beta_{1\text{flat}} = -E_1, \qquad \gamma^i_{1\text{flat}} = -F^i_1. \quad (7.15)$$

The gauge-invariant definitions of the remaining scalar metric degrees of freedom are then from Eq. (6.37) and (6.39)

$$\widetilde{\phi_{1\text{flat}}} = \phi_1 + \psi_1 + \left(\frac{\psi_1}{\mathcal{H}}\right)', \quad (7.16)$$

$$\widetilde{B_{1\text{flat}}} = B_1 - E'_1 - \frac{\psi_1}{\mathcal{H}}. \quad (7.17)$$

These gauge-invariant combinations were denoted $\mathcal{A}$ and $\mathcal{B}$ by Kodama and Sasaki [1]. The gauge-invariant definition of the remaining vector metric perturbation is the time derivative of the vector metric perturbation in the Poisson gauge (7.10):

$$\widetilde{S^i_{1\text{flat}}} = S^i_1 + F^{i'}_1 = \widetilde{F^{i'}_{1\ell}}. \quad (7.18)$$

Perturbations of scalar quantities in this gauge, such as the density perturbation, have gauge-invariant definitions from Eq. (6.18):

$$\widetilde{\delta\rho_{1\text{flat}}} = \delta\rho_1 + \rho'_0\frac{\psi_1}{\mathcal{H}}, \quad (7.19)$$

and the velocity potential (6.23) is given from Eq. (6.23):

$$\widetilde{v_{1\text{flat}}} = v_1 + E'_1. \quad (7.20)$$

The shear perturbation in the spatially flat gauge is given by $\widetilde{\sigma_{1\text{flat}}} = -\widetilde{B_{1\text{flat}}}$. This is closely related to the curvature perturbation in the zero-shear (longitudinal) gauge, $\widetilde{\psi}_\ell = \Psi$, given in Eq. (7.2) or (7.6),

$$\widetilde{B_{1\text{flat}}} = -\frac{\widetilde{\psi_{1\ell}}}{\mathcal{H}} = -\frac{\Psi}{\mathcal{H}}. \quad (7.21)$$

Gauge-invariant quantities, such as $\widetilde{B_{1\text{flat}}}$ or $\widetilde{\psi_{1\ell}}$ are proportional to the displacement between two different choices of spatial hypersurface,

$$\widetilde{B_{1\text{flat}}} = -\frac{\widetilde{\psi_{1\ell}}}{\mathcal{H}} = \alpha_{1\text{flat}} - \alpha_{1\ell}, \quad (7.22)$$

which would vanish for a homogeneous cosmology.

In some circumstances it is more convenient to use the spatially flat gauge-invariant variables instead of those in the longitudinal gauge. For instance, when calculating the evolution of perturbations during a collapsing "pre big bang" era the perturbations $\widetilde{\phi}_{1\text{flat}}$ and $\widetilde{B}_{1\text{flat}}$ may remain small even when Φ and Ψ become large [71,72]. On the other hand the metric perturbation $\widetilde{B}_{1\text{flat}}$ grows on large scales in radiation or matter dominated eras, $\widetilde{B}_{1\text{flat}} \propto \mathcal{H}^{-1} \propto \eta$, when the longitudinal gauge metric perturbation Ψ remains constant.

Note that the scalar field perturbation on spatially flat hypersurfaces,

$$\widetilde{\delta\varphi_{1\text{flat}}} \equiv \delta\varphi_1 + \varphi'_0\frac{\psi_1}{\mathcal{H}}, \quad (7.23)$$

is the gauge-invariant Sasaki–Mukhanov variable [73,74], often denoted $\mathcal{Q}$.

7.2.2. Second order

At second order we get from the gauge condition $\widetilde{\psi}_2 = 0$ using Eq. (6.58)

$$\alpha_{2\text{flat}} = \frac{\psi_2}{\mathcal{H}} + \frac{1}{4\mathcal{H}}\left[\nabla^{-2}\mathcal{X}^{ij}_{\text{flat},ij} - \mathcal{X}^k_{\text{flat}k}\right], \quad (7.24)$$

where we get $\mathcal{X}_{\text{flat}ij}$ from Eq. (6.54) using the first-order gauge generators given above, as

$$
\begin{aligned}
\mathcal{X}_{\text{flat}ij} = {} & 2\left[\psi_1\left(\frac{\psi_1'}{\mathcal{H}} + 2\psi_1\right) + \psi_{1,k}\xi_{1\text{flat}}^k\right]\delta_{ij} + \frac{4}{\mathcal{H}}\psi_1\left(C_{1ij}' + 2\mathcal{H}C_{1ij}\right) \\
& + 4C_{1ij,k}\xi_{1\text{flat}}^k + \left(4C_{1ik} + \xi_{1\text{flat}i,k}\right)\xi_{1\text{flat},j}^k + \left(4C_{1jk} + \xi_{1\text{flat}j,k}\right)\xi_{1\text{flat},i}^k \\
& + \frac{1}{\mathcal{H}}\left[\psi_{1,i}\left(2B_{1j} + \xi_{1\text{flat}j}'\right) + \psi_{1,j}\left(2B_{1i} + \xi_{1\text{flat}i}'\right)\right] - \frac{2}{\mathcal{H}^2}\psi_{1,i}\psi_{1,j} \\
& + \frac{2}{\mathcal{H}}\psi_1\left(\xi_{1\text{flat}(i,j)}' + 4\mathcal{H}\xi_{1\text{flat}(i,j)}\right) + 2\xi_{1\text{flat}}^k\xi_{1\text{flat}(i,j)k} + 2\xi_{1\text{flat}k,i}\xi_{1\text{flat},j}^k,
\end{aligned} \tag{7.25}
$$

where we define

$$
\xi_{1\text{flat}i} = -\left(E_{1,i} + F_{1i}\right). \tag{7.26}
$$

The trace of Eq. (7.25) is then

$$
\begin{aligned}
\mathcal{X}_{\text{flat}k}^k = {} & 6\left[\psi_1\left(\frac{\psi_1'}{\mathcal{H}} + 2\psi_1\right) + \psi_{1,k}\xi_{1\text{flat}}^k\right] + \frac{4}{\mathcal{H}}\psi_1\left(C_{1\ k}^{k'} + 2\mathcal{H}C_{1\ k}^k\right) \\
& + 4C_{1\ k,l}^k\xi_{1\text{flat}}^l + 4\left(2C_1^{kl} + \xi_{1\text{flat},}^{k\ l}\right)\xi_{1\text{flat}(k,l)} - 2\nabla^2 E_{1,k}\xi_{1\text{flat}}^k \\
& + \frac{2}{\mathcal{H}}\left(2B_{1k} + \xi_{1\text{flat}k}' - \frac{1}{\mathcal{H}}\psi_{1,k}\right)\psi_{1,}^k - \frac{2}{\mathcal{H}}\left(\psi_1\nabla^2 E_1' + 4\mathcal{H}\nabla^2 E_1\right).
\end{aligned} \tag{7.27}
$$

As an example of a second-order scalar metric perturbation we give the lapse function in the flat gauge

$$
\begin{aligned}
\widetilde{\phi_{2\text{flat}}} = {} & \phi_2 + \frac{1}{\mathcal{H}}\left[\psi_2' + \left(\mathcal{H} - \frac{\mathcal{H}'}{\mathcal{H}}\right)\psi_2\right] + \frac{1}{4\mathcal{H}}\left[\nabla^{-2}\mathcal{X}_{\text{flat},kl}^{kl'} - \mathcal{X}_{\text{flat}k}^{k'} + \left(\mathcal{H} - \frac{\mathcal{H}'}{\mathcal{H}}\right)\left(\nabla^{-2}\mathcal{X}_{\text{flat},kl}^{kl} - \mathcal{X}_{\text{flat}k}^k\right)\right] \\
& + \frac{1}{\mathcal{H}^2}\left(\psi_1''\psi_1 + 2\psi_1'^2\right) + \left(2 - \frac{\mathcal{H}''}{\mathcal{H}^3}\right)\psi_1^2 + \frac{1}{\mathcal{H}}\left(5 - 6\frac{\mathcal{H}'}{\mathcal{H}^2}\right)\psi_1\psi_1' + \frac{2}{\mathcal{H}}\phi_1'\psi_1 \\
& + \frac{4}{\mathcal{H}}\phi_1\left[\psi_1' + \left(\mathcal{H} - \frac{\mathcal{H}'}{\mathcal{H}}\right)\psi_1\right] + \frac{1}{\mathcal{H}}\left[\psi_1' + \left(\mathcal{H} - \frac{\mathcal{H}'}{\mathcal{H}}\right)\psi_1 + 2\mathcal{H}\phi_1\right]_{,k}\xi_{1\text{flat}}^k \\
& + \frac{1}{\mathcal{H}}\left[\left(\psi_1' + \left(\mathcal{H} - \frac{\mathcal{H}'}{\mathcal{H}}\right)\psi_1\right)_{,k} - 2\mathcal{H}B_{ik}\right]\xi_{1\text{flat}}^{k'}.
\end{aligned} \tag{7.28}
$$

The second-order tensor perturbation is in the flat gauge

$$
\begin{aligned}
\widetilde{h}_{2\text{flat}ij} = {} & h_{2ij} + \mathcal{X}_{\text{flat}ij} + \frac{1}{2}\left(\nabla^{-2}\mathcal{X}_{\text{flat},kl}^{kl} - \mathcal{X}_{\text{flat}k}^k\right)\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}\mathcal{X}_{\text{flat},klij}^{kl} \\
& + \frac{1}{2}\nabla^{-2}\mathcal{X}_{\text{flat}k,ij}^k - \nabla^{-2}\left(\mathcal{X}_{\text{flat}ik,j}^{\ \ k} + \mathcal{X}_{\text{flat}jk,i}^{\ \ k}\right).
\end{aligned} \tag{7.29}
$$

As an example of a matter variable we choose the energy density, which in the flat gauge is

$$
\begin{aligned}
\widetilde{\delta\rho_{2\text{flat}}} = {} & \delta\rho_2 + \frac{\rho_0'}{\mathcal{H}}\psi_2 + \frac{\rho_0'}{4\mathcal{H}}\left(\nabla^{-2}\mathcal{X}_{\text{flat},ij}^{ij} - \mathcal{X}_{\text{flat}k}^k\right) \\
& + \frac{\psi_1}{\mathcal{H}^2}\left[\rho_0''\psi_1 + \rho_0'\left(\psi_1' - \frac{\mathcal{H}'}{\mathcal{H}}\psi_1\right) + 2\mathcal{H}\delta\rho_1'\right] + \left(2\delta\rho_1 + \frac{\rho_0'}{\mathcal{H}}\psi_1\right)_{,k}\xi_{1\text{flat}}^k.
\end{aligned} \tag{7.30}
$$

To show the relation between gauge-invariant perturbations defined in the flat gauge and those previously defined in the longitudinal gauge we note that in the longitudinal gauge we have

$$
2\widetilde{C}_{1\ell ij} = -2\Psi\delta_{ij} + 2\widetilde{F}_{1\ell(i,j)} + h_{1ij}, \tag{7.31}
$$

and the first-order gauge shifts to the flat gauge, Eq. (7.15), become

$$
\alpha_{1\text{flat}}|_\ell = \frac{\Psi}{\mathcal{H}}, \qquad \beta_{1\text{flat}}|_\ell = 0, \qquad \gamma_{1\text{flat}}^i = -\widetilde{F}_{1\ell}^{\ i}. \tag{7.32}
$$

Substituting these into Eq. (7.25) we obtain

$$
\begin{aligned}
\mathcal{X}_{\text{flat}ij}|_\ell = {} & \frac{2}{\mathcal{H}}\left[-\Psi\left(\Psi' + 2\mathcal{H}\Psi\right) - \Psi_{,k}\widetilde{F}_{1\ell}^{\ k}\right]\delta_{ij} + \frac{2}{\mathcal{H}}\Psi\left(h_{1ij}' + 2\mathcal{H}h_{1ij}\right) \\
& + 2\left(2\Psi_{,k}\delta_{ij} - 2\widetilde{F}_{1\ell(i,j)k} - h_{ij,k}\right)\widetilde{F}_{1\ell}^{\ k} - 4h_{k(i}\widetilde{F}_{1\ell,j)}^{\ k} - \frac{2}{\mathcal{H}}\Psi_{,(i}\widetilde{F}_{1\ell j)} - \frac{2}{\mathcal{H}^2}\Psi_{,i}\Psi_{,j} \\
& - \frac{2}{\mathcal{H}}\Psi\widetilde{F}_{1\ell(i,j)}' + 2\widetilde{F}_{1\ell}^{\ k}\widetilde{F}_{1\ell(i,j)k} + 2\widetilde{F}_{1\ell k,i}\widetilde{F}_{1\ell,j}^{\ k}.
\end{aligned} \tag{7.33}
$$

If we can neglect the first-order vector and tensor perturbations, $\widetilde{F}_{1\ell i}$ and h_{1ij}, then we have

$$\mathcal{X}_{\text{flat}ij}|_\ell = -\frac{2}{\mathcal{H}}\Psi\left(\Psi' + 2\mathcal{H}\Psi\right)\delta_{ij} - \frac{2}{\mathcal{H}^2}\Psi_{,i}\Psi_{,j}. \tag{7.34}$$

Similarly, neglecting first vector and tensor perturbations, i.e. setting $F_{1i} = S_{1i} = h_{1ij} = 0$ and hence only considering scalar perturbations, the energy density at second order, Eq. (7.30), simplifies to

$$\widetilde{\delta\rho_{2\text{flat}}} = \delta\rho_2 + \frac{\rho_0'}{\mathcal{H}}\psi_2 + \frac{\psi_1}{\mathcal{H}^2}\left[\left(\rho_0'' + 2\mathcal{H}\rho_0'\right)\psi_1 + \rho_0'\left(2\psi_1' - \frac{\mathcal{H}'}{\mathcal{H}}\psi_1\right) + 2\mathcal{H}\delta\rho_1'\right] - 2\left(\delta\rho_1 + \frac{\rho_0'}{\mathcal{H}}\psi_1\right)_{,k}E_{1,}^{k}$$

$$+ \frac{\rho_0'}{2\mathcal{H}}\left\{E_{1,kl}E_{1,}^{kl} + \nabla^2 E_{1,k}E_{1,}^{k} - 2\psi_1\nabla^2\left(\frac{E_1'}{\mathcal{H}} + 2E_1\right) - \frac{\psi_{1,k}}{\mathcal{H}}\left(2B_1 + E_1' - \frac{\psi_1}{\mathcal{H}}\right)_{,k}\right.$$

$$\left. + \nabla^{-2}\left[2\psi_1\left(\frac{E_1'^{ij}}{\mathcal{H}} + 2E_1^{ij}\right) - E_1^{ijk}E_{1,k} - E_1^{ik}E_{1,k}^{j} + \frac{\psi_{1,}^{(i}}{\mathcal{H}}\left(2B_1 + E_1' - \frac{\psi_1}{\mathcal{H}}\right)_{,}^{j)}\right]_{,ij}\right\}. \tag{7.35}$$

This expression was first derived in Ref. [53], however with a different, incorrect sub-horizon part [75]. On super-horizon scales, where gradient terms can be neglected, we recover the expressions given in Ref. [76].

7.3. Synchronous gauge

The synchronous gauge is defined by $\widetilde{\phi} = \widetilde{B}_i = 0$, so that the proper time for observers at fixed spatial coordinates coincides with cosmic time in the FRW background, i.e., $d\tau = a d\eta$. This simplifies dynamical equations as the time derivatives can be directly related to proper time derivatives. This gauge is very popular for numerical studies, and used in many Boltzmann solvers such as CMBFAST [77]. It is also popular in the older literature [7,8].

The gauge condition at first order is $\widetilde{\phi}_1 = B_{1i} = 0$, which from Eqs. (6.37) and (6.39) gives

$$\alpha_{1\text{syn}} = -\frac{1}{a}\left(\int a\phi_1 d\eta - \mathcal{C}_1(x^i)\right), \tag{7.36}$$

$$\beta_{1\text{syn}} = \int\left(\alpha_{1\text{syn}} - B_1\right)d\eta + \hat{c}_1(x^i), \tag{7.37}$$

$$\gamma_{1\text{syn}}^i = \int S_1^i d\eta + \hat{c}_1^i(x^i). \tag{7.38}$$

This does not determine the time slicing unambiguously and we are left with two arbitrary scalar functions of the spatial coordinates, $\mathcal{C}_1$ and $\hat{c}_1$. Note that $\hat{c}_{1,i} + \hat{c}_{1i}$ affects only the labelling of the coordinates on the initial spatial hypersurface, but $\mathcal{C}_1$ affects scalar perturbations on spatial hypersurfaces. We are left with two non-zero geometrical scalar perturbations,

$$\widetilde{\psi_{1\text{syn}}} = \psi_1 + \frac{\mathcal{H}}{a}\left(\int a\phi_1 d\eta - \mathcal{C}(x^i)\right), \tag{7.39}$$

$$\widetilde{\sigma}_{1\text{syn}} = \sigma_1 + \alpha_{1\text{syn}} - B_1, \tag{7.40}$$

and the matter variables are

$$\widetilde{\delta\rho_{1\text{syn}}} = \delta\rho_1 - \frac{\rho_0'}{a}\left(\int a\phi_1 d\eta - \mathcal{C}(x^i)\right), \tag{7.41}$$

$$\widetilde{v_{1\text{syn}}} = v_1 + B_1 - \alpha_{1\text{syn}}. \tag{7.42}$$

Thus it is not possible to define gauge-invariant quantities in general using this gauge condition [78]. This gauge was originally used by Lifshitz in his pioneering work on perturbations in a FRW spacetime [79] (see also Ref. [9]). He dealt with the residual gauge freedom by eliminating the unphysical gauge modes through symmetry arguments.

To remove the ambiguity, we can follow Ref. [80] and choose the initial velocity of cold dark matter to be zero, $\widetilde{v_{1\text{cdm}}} \equiv 0$, which fixes the residual gauge freedom

$$\mathcal{C}_1(x) = a(v_{1\text{cdm}} + B_1). \tag{7.43}$$

Note that for pressureless matter, momentum conservation equation ensures that $a(v_{1\text{cdm}} + B_1)$ is a constant (see Section 8).

7.4. Comoving orthogonal gauge

The comoving gauge is defined by choosing spatial coordinates such that the 3-velocity of the fluid vanishes, $\widetilde{v}_i = 0$. Orthogonality of the constant-η hypersurfaces to the 4-velocity, u^μ, then requires $\widetilde{v}_i + B_i = 0$, which shows that the momentum vanishes as well. From Eq. (6.39) and (6.23) this implies

$$\alpha_{1com} = v_1 + B_1,$$

$$\beta_{1com} = \int v_1 \mathrm{d}\eta + \hat{c}(x^i), \tag{7.44}$$

where $\hat{c}(x^i)$ represents a residual gauge freedom, corresponding to a constant shift of the spatial coordinates. All the 3-scalars like curvature, expansion, acceleration and shear are independent of $\hat{c}(x^i)$. Applying the above transformation from arbitrary coordinates, the scalar perturbations in the comoving orthogonal gauge can be written as

$$\widetilde{\phi_{1com}} = \phi_1 + \mathcal{H}(v_1 + B_1) + \left(v_1' + B_1'\right), \tag{7.45}$$

$$\mathcal{R} \equiv \widetilde{\psi_{1com}} = \psi_1 - \mathcal{H}(v_1 + B_1), \tag{7.46}$$

$$\widetilde{\sigma_{1com}} = v_1 + E_1'. \tag{7.47}$$

Defined in this way, these combinations are gauge-invariant under transformations of their component parts in exactly the same way as, for instance, Φ and Ψ defined in Eqs. (7.1) and (7.2).

Note that the curvature perturbation in the comoving gauge given above, Eq. (7.46) was used (with a constant pre-factor) by Lukash [81]. It was later employed by Lyth and denoted $\mathcal{R}$ in his seminal paper, [82], and in many subsequent works, e.g. [83,16].

The density perturbation on the comoving orthogonal hypersurfaces is given by Eqs. (6.18) and (7.44) in gauge-invariant form as

$$\widetilde{\delta\rho_{1com}} = \delta\rho_1 + \rho_0'(v_1 + B_1), \tag{7.48}$$

and corresponds to the gauge-invariant density perturbation $\epsilon_m E_0 Q^{(0)}$ in the notation of Bardeen [35]. The gauge-invariant scalar density perturbation Δ introduced in Refs. [84,49] corresponds to $\delta\widetilde{\rho}_{1com,i}^{\;\;i}/\rho_0$.

If we wish to write these gauge-invariant quantities in terms of the metric perturbations rather than the velocity potential then we can use the Einstein equations, presented in Section 8, to obtain

$$v_1 + B_1 = \frac{\mathcal{H}\phi_1 + \psi_1'}{\mathcal{H}' - \mathcal{H}^2}. \tag{7.49}$$

In particular we note that we can write the comoving curvature perturbation, given in Eq. (7.46), in terms of the longitudinal gauge-invariant quantities as

$$\mathcal{R} = \Psi - \frac{\mathcal{H}(\mathcal{H}\Phi + \Psi')}{\mathcal{H}' - \mathcal{H}^2}, \tag{7.50}$$

which coincides with the quantity denoted ζ by Mukhanov, Feldman and Brandenberger in Ref. [2].

By comparing the definitions of the energy–momentum tensor for a single fluid and a single scalar field in Sections 4.1 and 5.1 we can relate the $v_1 + B_1$ to $\delta\varphi_1$, which allows one to rewrite the definition of the comoving curvature perturbations, Eq. (7.46), as

$$\mathcal{R} = \psi_1 + \frac{\mathcal{H}}{\varphi_0'}\delta\varphi_1. \tag{7.51}$$

From the definition above we immediately see that the comoving curvature perturbation is related to the field fluctuation on flat slices, defined in Eq. (7.23), by

$$\widetilde{\delta\varphi_{1flat}} = \frac{\varphi_0'}{\mathcal{H}}\mathcal{R}. \tag{7.52}$$

For extensions to the multi-field case see Section 8.2.4.

7.5. Total matter gauge

This gauge is also known as the velocity orthogonal isotropic gauge [1] but here we follow the terminology of Ref. [10]. It is closely related to comoving orthogonal and longitudinal gauges.

To fix the temporal gauge we require the total momentum potential on spatial hypersurfaces to vanish

$$\widetilde{v}_1 + \widetilde{B}_1 = 0. \tag{7.53}$$

In addition we require $\widetilde{E}_1 = 0$ and $\widetilde{F}_{1i} = 0$, which fixes the spatial gauge. These require

$$\alpha_{1\text{tom}} = v_1 + B_1, \qquad \beta_{1\text{tom}} = -E_1, \qquad \gamma_{1\text{tom}}^i = -F_1^i. \tag{7.54}$$

We therefore get the metric perturbations in the total matter gauge related to the comoving orthogonal and longitudinal gauge perturbations:

$$\widetilde{\phi_{1\text{tom}}} = \widetilde{\phi_{1\text{com}}}, \tag{7.55}$$

$$\widetilde{\psi_{1\text{tom}}} = \widetilde{\psi_{1\text{com}}}, \tag{7.56}$$

$$\widetilde{B_{1\text{tom}}} = -\widetilde{v_{1\ell}}, \tag{7.57}$$

and for the matter quantities we get in the total matter gauge

$$\widetilde{\delta\rho_{1\text{tom}}} = \widetilde{\delta\rho_{1\text{com}}}, \tag{7.58}$$

$$\widetilde{v_{1\text{tom}}} = \widetilde{v_{1\ell}}. \tag{7.59}$$

Note that in the total matter gauge velocity potential is not identically zero (unlike in the comoving orthogonal gauge), but equal to the shear potential, $\widetilde{v_{1\text{tom}}} = \widetilde{\sigma_{1\text{tom}}} = -\widetilde{B_{1\text{tom}}}$, which also coincides with velocity potential in longitudinal gauge.

7.6. Uniform density gauge

Alternatively we can use the matter to pick out a foliation of uniform density hypersurfaces on which to define perturbed quantities.

7.6.1. First order

Using Eq. (6.18) we see that $\widetilde{\delta\rho_1} = 0$ implies a temporal gauge transformation

$$\alpha_{1\delta\rho} = -\frac{\delta\rho_1}{\rho_0'}. \tag{7.60}$$

On these hypersurfaces the gauge-invariant curvature perturbation is [40,78]

$$-\zeta_1 \equiv \widetilde{\psi_{1\delta\rho}} = \psi_1 + \mathcal{H}\frac{\delta\rho_1}{\rho_0'}. \tag{7.61}$$

The sign is chosen to coincide with ζ defined in Refs. [85,86].[3] There is still the freedom to choose the spatial gauge. In particular we can choose either $\tilde{B}$, $\tilde{E}$ or $\tilde{v}$ to be zero and thus fix β.

Note that ζ_1 is simply related to the density perturbation in the flat gauge, Eq. (7.19) by

$$-\zeta_1 = \frac{\mathcal{H}}{\rho_0'}\delta\rho_{1\text{flat}}. \tag{7.62}$$

The curvature perturbation in the uniform density gauge is also closely related to the comoving curvature perturbation (7.46). At first order we have

$$\zeta_1 = -\mathcal{R}_1 - \frac{\mathcal{H}}{\rho_0'}\delta\rho_{1\text{com}}, \tag{7.63}$$

where $\delta\rho_{1\text{com}}$ is the comoving density perturbation (7.48). In Section 8 we shall use the Einstein equations to show that ζ_1 and $\mathcal{R}_1$ differ only by an overall minus sign in the large-scale limit where the comoving density perturbation vanishes.

7.6.2. Second order

The transformation behaviour of scalars at second order, Eq. (6.20), allows us to define the temporal gauge corresponding to uniform density hypersurfaces as

$$\alpha_{2\delta\rho} = -\frac{\delta\rho_2}{\rho_0'} - \frac{\alpha_1}{\rho_0'}\left(\rho_0''\alpha_1 + \rho_0'\alpha_1' + 2\delta\rho_1'\right) - \frac{1}{\rho_0'}\left(2\delta\rho_1 + \rho_0'\alpha_1\right)_{,k}\left(\beta_{1,}^{\ k} + \gamma_1^{\ k}\right). \tag{7.64}$$

---

[3] Note, that ζ_1 defined in Eq. (7.61) is related to the curvature perturbation ζ_{SBB} defined in Ref. [87] by $\zeta_{\text{SBB}} \equiv 3\zeta_1$.

Then using the definition of uniform density hypersurfaces at first order, Eq. (7.60), and choosing a spatially flat threading by using Eq. (7.15), we finally get

$$\alpha_{2\delta\rho} = \frac{1}{\rho_0'}\left[-\delta\rho_2 + \frac{\delta\rho_1'}{\rho_0'}\delta\rho_1 + \left(E_{1,}^{\ k} + F_1^{\ k}\right)\delta\rho_{1,k}\right]. \tag{7.65}$$

Using Eq. (6.54) we find

$$\begin{aligned}
\mathcal{X}_{ij\delta\rho} &\equiv -2\frac{\mathcal{H}}{\rho_0'}\Big[\mathcal{H}\left(1+3c_s^2\right)\left(\frac{\delta\rho_1^2}{\rho_0'}\right) - \frac{\delta\rho_1'}{\rho_0'}\delta\rho_1 + \delta\rho_{1,k}\xi_1^{\ k}\Big]\delta_{ij} \\
&\quad + 4\Big[-\frac{\delta\rho_1}{\rho_0'}\left(C_{1ij}' + 2\mathcal{H}C_{1ij}\right) + C_{1ij,k}\xi_1^{\ k}\Big] + \left(4C_{1ik} + \xi_{1\mathrm{flat}i,k}\right)\xi_{1\mathrm{flat},j}^{\ k} \\
&\quad + \left(4C_{1jk} + \xi_{1\mathrm{flat}j,k}\right)\xi_{1\mathrm{flat},i}^{\ k} - \frac{1}{\rho_0'}\Big[\delta\rho_{1,i}\left(2B_{1j} + \xi_{1\mathrm{flat}j}'\right) + \delta\rho_{1,j}\left(2B_{1i} + \xi_{1\mathrm{flat}i}'\right)\Big] \\
&\quad - 2\frac{\delta\rho_1}{\rho_0'}\left(\xi_{1\mathrm{flat}(i,j)}' + 4\mathcal{H}\xi_{1\mathrm{flat}(i,j)}\right) - \frac{2}{\rho_0'^2}\delta\rho_{1,i}\delta\rho_{1,j} + 2\xi_{1\mathrm{flat}}^{\ k}\xi_{1\mathrm{flat}(i,j)k} + 2\xi_{1\mathrm{flat}k,i}\xi_{1\mathrm{flat},j}^{\ k},
\end{aligned} \tag{7.66}$$

where we choose a flat threading by defining,

$$\xi_{1\mathrm{flat}}^i \equiv -\left(E_{1,}^{\ i} + F_1^i\right). \tag{7.67}$$

From Eq. (7.66) we get for the trace in the uniform density gauge

$$\begin{aligned}
\mathcal{X}_{k\delta\rho}^k &= -6\frac{\mathcal{H}}{\rho_0'}\Big[\mathcal{H}\left(1+3c_s^2\right)\left(\frac{\delta\rho_1^2}{\rho_0'}\right) - \frac{\delta\rho_1'}{\rho_0'}\delta\rho_1 + \delta\rho_{1,k}\xi_1^{\ k}\Big] + 4\Big[C_{1\ k,l}^{k}\xi_{1\mathrm{flat}}^l - \frac{\delta\rho_1}{\rho_0'}\left(C_{1\ k}^{k'} + 2\mathcal{H}C_{1\ k}^k\right)\Big] \\
&\quad + 4\left(2C_1^{kl} + \xi_{1\mathrm{flat},}^{k\ l}\right)\xi_{1\mathrm{flat}(k,l)} - \frac{2}{\rho_0'}\left(2B_1^k + \xi_{1\mathrm{flat}k}'\right)\delta\rho_{1,k} \\
&\quad + 2\frac{\delta\rho_1}{\rho_0'}\nabla^2\left(E_1' + 4\mathcal{H}\nabla^2 E_1\right) - \frac{2}{\rho_0'^2}\delta\rho_{1,}\delta\rho_1^k - 2\nabla^2 E_{1,k}\xi_{1\mathrm{flat}}^k.
\end{aligned} \tag{7.68}$$

Finally, from Eq. (6.58) we get for ζ_2, the curvature perturbation on uniform density hypersurfaces,

$$-\zeta_2 = \psi_2 + \frac{\mathcal{H}}{\rho_0'}\left[\delta\rho_2 - \frac{\delta\rho_1'}{\rho_0'}\delta\rho_1 + \xi_{1\mathrm{flat}}^k\delta\rho_{1,k}\right] - \frac{1}{4}\mathcal{X}_{\delta\rho\ k}^{\ k} + \frac{1}{4}\nabla^{-2}\mathcal{X}_{\delta\rho\ ,ij}^{ij}. \tag{7.69}$$

The second-order tensor perturbation is in the uniform density gauge

$$\begin{aligned}
\widetilde{h}_{2\delta\rho ij} &= h_{2ij} + \mathcal{X}_{\delta\rho ij} + \frac{1}{2}\left(\nabla^{-2}\mathcal{X}_{\delta\rho,kl}^{kl} - \mathcal{X}_{\delta\rho k}^k\right)\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}\mathcal{X}_{\delta\rho,klij}^{kl} \\
&\quad + \frac{1}{2}\nabla^{-2}\mathcal{X}_{\delta\rho k,ij}^k - \nabla^{-2}\left(\mathcal{X}_{\delta\rho ik,\ j}^{\ k} + \mathcal{X}_{\delta\rho jk,\ i}^{\ k}\right).
\end{aligned} \tag{7.70}$$

The curvature perturbation on uniform density hypersurfaces, defined in Eq. (7.69), simplifies considerably if we neglect first-order vector and tensor perturbations, i.e. setting $F_{1i} = S_{1i} = h_{1ij} = 0$ and hence only considering scalar perturbations,

$$\begin{aligned}
-\zeta_2 &= \psi_2 + \frac{\mathcal{H}}{\rho_0'}\delta\rho_2 - \frac{\mathcal{H}}{\rho_0'}\left[2\frac{\delta\rho_1'}{\rho_0'}\delta\rho_1 - \mathcal{H}\left(1+3c_s^2\right)\frac{\delta\rho_1^2}{\rho_0'} + 2\delta\rho_1\left(\frac{\psi_1'}{\mathcal{H}} + 2\psi_1\right)\right] \\
&\quad - \left(\frac{5}{2}\frac{\mathcal{H}}{\rho_0'}\delta\rho_1 + \psi_1\right)_{,k}E_{1,}^{\ k} + \psi_1\nabla^2 E_1 - E_{1,kl}E_1^{kl} - \frac{1}{2}\left[\frac{\delta\rho_1}{\rho_0'}\nabla^2 E_1' + \nabla^2 E_{1,k}E_1^k + \frac{\delta\rho_1^k}{\rho_0'}\left(2B_1 - E_1'\frac{\delta\rho_1}{\rho_0'}\right)_{,k}\right] \\
&\quad + \nabla^{-2}\left\{2E_{1,}^{\ ij}\left(4\psi_1 - \frac{\delta\rho_1}{\rho_0'}\right) - 2E_{1,}^{\ ijk}E_{1,k} - 4E_{1,}^{\ ik}E_{1,k}^{\ j} - \frac{2\delta\rho_{1,}^{(i}}{\rho_0'}\left(2B_1 - E_1'\frac{\delta\rho_1}{\rho_0'}\right)_{1,}^{\ j)}\right\}_{,ij}.
\end{aligned} \tag{7.71}$$

This expression was first derived in Ref. [53], however with a different, incorrect sub-horizon part [75]. On super-horizon scales, where gradient terms can be neglected, we recover the expressions given in Ref. [76].

8. Dynamics

In this section we give the Einstein equations governing the evolution of the FRW background and perturbations in general relativity. This will allow us to derive some key properties of the perturbation variables, such as the conservation of the curvature perturbation ζ on super-horizon scales in the adiabatic case.

In general relativity the Einstein equations relate the local spacetime curvature to the local energy–momentum:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{8.1}$$

In more general theories of gravity we can still equate the local spacetime curvature, $G_{\mu\nu}$, with an effective energy–momentum, though this may not be simply related to the energy–momentum tensor derived, say, from the matter Lagrangian. Moreover, many modified gravity theories, including Brans–Dicke gravity or higher-order theories, may be rewritten in terms of general relativity plus non-minimally coupled matter fields through a conformal rescaling of coordinates [88,89]. In this review we will restrict our analysis to general relativity.

We can project the tensor equation (8.1) into components tangent to and orthogonal to the time-like 4-vector field, n^μ defined in Eq. (3.4), which defines the coordinate system (see Section 3). This gives two constraint equations for the metric perturbations, which we will refer to as the energy and momentum constraint equations. We also have two evolution equations driven by the trace and trace-free parts of the pressure. Through the Bianchi identities, $\nabla_\mu G^\mu_\nu = 0$, the field equation (8.1) imply the local conservation the total energy and momentum,

$$\nabla_\mu T^{\mu\nu} = 0 \tag{8.2}$$

which can similarly be split into energy and momentum conservation equations with respect to a given coordinate system.

In the case of multiple matter components the total energy–momentum tensor is the sum of the energy–momentum tensors of the individual fluids, $T^{\mu\nu}_{(\alpha)}$, given in Eq. (4.20). For each fluid the local energy–momentum "conservation" equation (4.21), has an energy–momentum transfer 4-vector, $Q^\nu_{(\alpha)}$ on the right-hand side, which is zero only for non-interacting fluids. However local conservation of the total energy–momentum imposes the constraint equation (4.22).

We also have at our disposal the equations of motion for specific matter fields, such as the Klein–Gordon equation for canonical scalar fields, φ_I, with interaction potential energy U:

$$\Box \varphi_I = \frac{dU}{d\varphi_I}. \tag{8.3}$$

In the following we equate terms order by order in a perturbative expansion about a homogeneous background spacetime.

8.1. Background

The Einstein equation (8.1) give the Friedmann constraint and evolution equation for the background FRW universe

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \rho, \tag{8.4}$$

$$\mathcal{H}' = -\frac{4\pi G}{3} a^2 (\rho + 3P), \tag{8.5}$$

and energy–momentum conservation, Eq. (8.2), gives the continuity equation

$$\rho' = -3\mathcal{H}(\rho + P), \tag{8.6}$$

where ρ and P are the total energy density and the total pressure, a prime denotes a derivative with respect to conformal time, η, the scale factor is a, and $\mathcal{H} \equiv a'/a$ is the conformal Hubble parameter.

The total density and the total pressure are related to the density and pressure of the component fluids by

$$\sum_\alpha \rho_\alpha = \rho, \qquad \sum_\alpha P_\alpha = P. \tag{8.7}$$

The continuity equation (4.21) for each individual fluid in the background is [1]

$$\rho'_\alpha = -3\mathcal{H}(\rho_\alpha + P_\alpha) + aQ_\alpha, \tag{8.8}$$

where the energy transfer to the α-fluid is given by the time component of the energy–momentum transfer vector

$$Q_\alpha \equiv -u_\mu Q^\mu_{(\alpha)}. \tag{8.9}$$

Eq. (4.22) implies that the energy transfer obeys the constraint

$$\sum_\alpha Q_\alpha = 0. \tag{8.10}$$

Homogeneous scalar fields in the FRW metric obey the Klein–Gordon equation

$$\varphi_I'' + 2\mathcal{H}\varphi_I' + a^2 \frac{\mathrm{d}U}{\mathrm{d}\varphi_I} = 0. \tag{8.11}$$

It is sometimes useful to identify the kinetic energy density and (isotropic) pressure of each field as

$$\rho_I = P_I = \frac{1}{2}a^{-2}\varphi_I'^2. \tag{8.12}$$

The Klein–Gordon equation (8.11) then implies an energy transfer of the form given by Eq. (8.8)

$$aQ_I = -\varphi' \frac{\mathrm{d}U}{\mathrm{d}\varphi_I} \tag{8.13}$$

where this energy is transferred to the potential energy

$$\rho_U = -P_U = U, \tag{8.14}$$

and overall energy conservation (8.10) implies

$$Q_U = -\sum_I Q_I = a^{-1}U'. \tag{8.15}$$

8.2. First-order scalar perturbations

In the following we discuss the linear constraint and evolution equations for inhomogeneous perturbations at first order. We omit the subscript "1" denoting the order of the perturbations to avoid unnecessary clutter.

8.2.1. Einstein equations

The scalar metric perturbations in an arbitrary gauge are related to matter perturbations via the first-order energy and momentum constraints [1,2]

$$3\mathcal{H}\left(\psi' + \mathcal{H}\phi\right) - \nabla^2\left[\psi + \mathcal{H}\sigma\right] = -4\pi Ga^2\delta\rho, \tag{8.16}$$

$$\psi' + \mathcal{H}\phi = -4\pi Ga^2(\rho + P)V \tag{8.17}$$

where the total covariant velocity perturbation is given by

$$V \equiv v + B, \tag{8.18}$$

and v is the total scalar velocity potential (4.5).

In a specific gauge, such as the spatially flat gauge these can be written in terms of the corresponding gauge-invariant quantities. For instance, in the spatially flat gauge we have

$$3\mathcal{H}^2\phi_{\text{flat}} - \mathcal{H}\nabla^2\sigma_{\text{flat}} = -4\pi Ga^2\delta\rho_{\text{flat}}, \tag{8.19}$$

$$\mathcal{H}\phi_{\text{flat}} = -4\pi Ga^2(\rho + P)V_{\text{flat}}, \tag{8.20}$$

which makes it straightforward to eliminate the metric variables ϕ_{flat} and $\sigma_{\text{flat}} = -B_{\text{flat}}$ in favour of the energy and momentum in the flat gauge.

Alternatively in the longitudinal gauge the shear terms are absent and we obtain first-order differential equations for the curvature perturbation

$$3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) - \nabla^2\Psi = -4\pi Ga^2\delta\rho_\ell, \tag{8.21}$$

$$\Psi' + \mathcal{H}\Phi = -4\pi Ga^2(\rho + P)v_\ell. \tag{8.22}$$

Typically one then uses these equations to eliminate the density and velocity perturbations, $\delta\rho_\ell$ and v_ℓ, in terms of the metric perturbations in the longitudinal gauge.

The same energy and momentum constraints can be rewritten in terms of gauge-invariant variables to give expressions for the curvature perturbation in the uniform density gauge (7.61) and the comoving curvature perturbation (7.46), respectively, in terms of the longitudinal gauge metric perturbations (7.1) and (7.2):

$$\Psi' + \mathcal{H}\Phi - \frac{\mathcal{H}' - \mathcal{H}^2}{\mathcal{H}}\Psi - \frac{1}{3\mathcal{H}}\nabla^2\Psi = \frac{\mathcal{H}' - \mathcal{H}^2}{\mathcal{H}}\zeta, \tag{8.23}$$

$$\Psi' + \mathcal{H}\Phi - \frac{\mathcal{H}' - \mathcal{H}^2}{\mathcal{H}}\Psi = -\frac{\mathcal{H}' - \mathcal{H}^2}{\mathcal{H}}\mathcal{R}. \tag{8.24}$$

These can be combined to give the gauge-invariant generalisation of the Newtonian Poisson equation

$$\nabla^2 \Psi = -3 \left(\mathcal{H}' - \mathcal{H}^2 \right) (\zeta + \mathcal{R}) = 4\pi G a^2 \delta \rho_{\text{com}}, \tag{8.25}$$

relating the longitudinal gauge curvature perturbation (7.2) to the comoving density perturbation (7.48). We see that the comoving density perturbation is suppressed relative to the metric perturbation Ψ on large scales, and that ζ and $-\mathcal{R}$ coincide in the large-scale limit as long as Ψ is finite in this limit.[4]

The perturbed Einstein equations at first order also yield two evolution equations for the scalar metric perturbations

$$\psi'' + 2\mathcal{H}\psi' + \mathcal{H}\phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right) \phi = 4\pi G a^2 \left(\delta P + \frac{2}{3}\nabla^2 \Pi \right), \tag{8.27}$$

$$\sigma' + 2\mathcal{H}\sigma + \psi - \phi = 8\pi G a^2 \Pi, \tag{8.28}$$

where Π is the scalar part of the (trace-free) anisotropic stress, defined in Eq. (4.10).

Eq. (8.28) in a general gauge can be interpreted as the evolution equation for the scalar shear, but in the longitudinal gauge it becomes a constraint equation for the gauge-invariant perturbations Φ and Ψ, defined in Eq. (7.1) and (7.2),

$$\Psi - \Phi = 8\pi G a^2 \Pi, \tag{8.29}$$

and hence we have $\Psi = \Phi$ in the absence of anisotropic stresses.

Eq. (8.27) then provides a second-order evolution equation for the metric perturbation in the longitudinal gauge driven by isotropic pressure:

$$\Psi'' + 3\mathcal{H}\Psi' + \left(2\mathcal{H}' + \mathcal{H}^2\right) \Psi = 4\pi G a^2 \delta P. \tag{8.30}$$

For adiabatic perturbations we can relate the pressure to the density, $\delta P = c_s^2 \delta \rho$ where c_s^2 is the adiabatic sound speed, in which case (8.16) and (8.30) yield a closed second-order differential equation [2]

$$\Psi'' + 3(1 + c_s^2)\mathcal{H}\Psi' + [2\mathcal{H}' + (1 + 3c_s^2)\mathcal{H}^2 - c_s^2\nabla^2]\Psi = 0. \tag{8.31}$$

8.2.2. Energy and momentum conservation

Energy–momentum conservation gives evolution equations for the perturbed energy and momentum

$$\delta\rho' + 3\mathcal{H} \left(\delta\rho + \delta P\right) - 3\left(\rho + P\right)\psi' + (\rho + P)\nabla^2 (V + \sigma) = 0, \tag{8.32}$$

$$V' + (1 - 3c_s^2)\mathcal{H}V + \phi + \frac{1}{\rho + P} \left(\delta P + \frac{2}{3}\nabla^2 \Pi \right) = 0, \tag{8.33}$$

where c_s^2 is the adiabatic speed of sound, defined as

$$c_s^2 \equiv \frac{P'}{\rho'}. \tag{8.34}$$

From the momentum conservation equation in the total matter gauge, such that $V = 0$, we see that the acceleration is proportional to the pressure perturbation: $(\rho + P)\phi = \delta P + (2/3)\nabla^2 \Pi$. Alternatively, for pressureless, non-interacting dust we have $(aV)' + a\phi = 0$ and hence the scalar velocity potential redshifts as $V \propto 1/a$ in a synchronous gauge.

Rewriting the energy conservation equation (8.32) in terms of the curvature perturbation on uniform density hypersurfaces, ζ in (7.61), we obtain the important result

$$\zeta' = -\mathcal{H}\frac{\delta P_{\text{nad}}}{\rho + P} - \Sigma_V, \tag{8.35}$$

where δP_{nad} is the non-adiabatic pressure perturbation, defined in (9.5), and Σ describes the divergence of the velocity in the longitudinal gauge, Eq. (7.8) or, equivalently, the scalar shear along comoving worldlines [90]

$$\Sigma_V \equiv \frac{1}{3}\nabla^2 (V + \sigma) = \frac{1}{3}\nabla^2 \widetilde{v}_\ell. \tag{8.36}$$

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[4] Note that Eq. (8.24) shows that the variable denoted ζ in the review by Mukhanov, Feldman and Brandenberger:

$$\zeta_{\text{MFB}} \equiv \Phi - \frac{\mathcal{H}(\Phi' + \mathcal{H}\Phi)}{\mathcal{H}' - \mathcal{H}^2}, \tag{8.26}$$

coincides with the comoving curvature perturbation when $\Psi = \Phi$.

Thus the curvature perturbation on uniform density hypersurfaces is constant for adiabatic perturbations on large scales when the shear of comoving worldlines becomes negligible. This follows directly from local energy conservation and holds independently of the gravitational field equations [91–94].

Using the definition of ζ and Ψ in Eqs. (7.61) and (7.2) and the constraint equation for the comoving density perturbation (8.25), we have

$$\frac{\Sigma_V}{\mathcal{H}} = \frac{\nabla^2}{3\mathcal{H}^2}(\zeta + \Psi) + \frac{2\rho}{3(\rho + P)}\left(\frac{\nabla^2}{\mathcal{H}^2}\right)^2 \Psi. \tag{8.37}$$

Thus we see that ζ is constant for adiabatic perturbations ($\delta P_{\mathrm{nad}} = 0$) on super-Hubble scales ($k/\mathcal{H} \ll 1$), as long as Ψ remains finite. This makes ζ a convenient variable to characterise the primordial density perturbation on super-Hubble scales, either during a period of inflation in the very early universe, or in the subsequent radiation dominated era. This is an excellent approximation throughout reheating at the end of inflation and the subsequent radiation era on scales relevant for observations of degree-scale anisotropies in the cosmic microwave background and large-scale galaxy surveys [10].

Conversely, local variations in the pressure leading to a non-adiabatic pressure perturbation, δP_{nad}, will lead to a change in the curvature perturbation ζ on super-Hubble scales [95–97]. This mechanism is at the heart of the curvaton scenario for the origin of large-scale structure in the universe [98–100].

8.2.3. Multiple fluids

The perturbations in the total energy–momentum can be related to the perturbations of individual fluids by

$$\sum_\alpha \delta \rho_\alpha = \delta \rho, \qquad \sum_\alpha \delta P_\alpha = \delta P, \qquad \sum_\alpha \Pi_\alpha = \Pi, \tag{8.38}$$

and

$$V = \sum_\gamma \frac{\rho_\gamma + P_\gamma}{\rho + P} V_\gamma, \tag{8.39}$$

where $\delta \rho_\alpha$ and δP_α are the perturbed energy density and the perturbed pressure of the α-fluid, respectively, and V_α is the covariant velocity perturbation of the α-fluid defined as

$$V_\alpha \equiv v_\alpha + B, \tag{8.40}$$

where v_α is the scalar velocity potential of the α-fluid.

The perturbed energy transfer 4-vector, Eq. (4.21), for individual fluids including terms up to first order, is written as [1,101]

$$Q_{(\alpha)0} = -aQ_\alpha(1 + \phi) - a\delta Q_\alpha,$$
$$Q_{(\alpha)i} = (f_\alpha + aQ_\alpha V)_{,i}, \tag{8.41}$$

and Eq. (4.22) implies that the perturbed energy and momentum transfer obey the constraints

$$\sum_\alpha \delta Q_\alpha = 0, \qquad \sum_\alpha f_\alpha = 0. \tag{8.42}$$

Note that the momentum transfer, f_α, is by convention [1,42] defined with respect to the total momentum, V, so is non-zero only if the momentum transfer vanishes in the total matter frame ($V = 0$).

The perturbed energy conservation equation for a particular fluid, including energy transfer, is then obtained by the first-order part of the time component of the perturbed continuity Eq. (4.21) to give [1,52]

$$\delta \rho_\alpha' + 3\mathcal{H}(\delta \rho_\alpha + \delta P_\alpha) - 3(\rho_\alpha + P_\alpha)\psi' + (\rho_\alpha + P_\alpha)\nabla^2(V_\alpha + \sigma) = aQ_\alpha \phi + a\delta Q_\alpha. \tag{8.43}$$

The momentum conservation equation of the α-fluid is

$$V_\alpha' + \left[\frac{aQ_\alpha}{\rho_\alpha + P_\alpha}(1 + c_\alpha^2) + (1 - 3c_\alpha^2)\mathcal{H}\right]V_\alpha + \phi + \frac{1}{\rho_\alpha + P_\alpha}\left[\delta P_\alpha + \frac{2}{3}\nabla^2 \Pi_\alpha - aQ_\alpha V - f_\alpha\right] = 0, \tag{8.44}$$

where $c_\alpha^2 \equiv P_\alpha'/\rho_\alpha'$ is the adiabatic sound speed of the α-fluid and $a^2[\Pi_{\alpha,ij} - (1/3)\delta_{ij}\nabla^2 \Pi_\alpha]$ is the scalar anisotropic stress of that fluid. The total adiabatic sound speed, Eq. (8.34), is the weighted sum of the adiabatic sound speeds of the individual fluids,

$$c_s^2 = \sum_\alpha \frac{\rho_\alpha'}{\rho'}c_\alpha^2. \tag{8.45}$$

We recover the evolution equation for the total density perturbation (8.32) from Eq. (8.43) by summing over all fluids, using Eq. (8.38) and the constraint (8.42).

Analogous to the curvature perturbation on uniform total density hypersurfaces, ζ defined in Eq. (7.61), we can define a gauge-invariant perturbation on the uniform α density hypersurfaces

$$\zeta_\alpha \equiv -\psi - \mathcal{H} \frac{\delta \rho_\alpha}{\rho_\alpha'}. \tag{8.46}$$

The perturbed energy conservation equation for each fluid can then be written as

$$\zeta_\alpha' = 3 \frac{\mathcal{H}^2}{\rho_\alpha'} \delta P_{\mathrm{intr},\alpha} - \Sigma_\alpha - \frac{\nabla^2}{3\mathcal{H}} \left[\frac{aQ_\alpha}{\rho_\alpha'} \mathcal{R}_\alpha \right] - \left(\frac{\mathcal{H}}{a} \right)' \frac{aQ_\alpha}{\rho_\alpha'} \left(\frac{\delta \rho_\alpha}{\rho_\alpha'} - \frac{\delta \rho}{\rho'} \right) - \frac{\mathcal{H}}{\rho_\alpha'} \left(\delta Q_\alpha - \frac{Q_\alpha'}{\rho_\alpha'} \delta \rho_\alpha \right), \tag{8.47}$$

where the intrinsic non-adiabatic pressure perturbation of each fluid is given by

$$\delta P_{\mathrm{intr},\alpha} \equiv \delta P_\alpha - c_\alpha^2 \delta \rho_\alpha, \tag{8.48}$$

the scalar shear along worldlines comoving with the α-fluid is

$$\Sigma_\alpha \equiv \frac{1}{3} \nabla^2 (\sigma + V_\alpha), \tag{8.49}$$

and, extending Eq. (7.46), the curvature perturbation comoving with the α-fluid is

$$\mathcal{R}_\alpha \equiv \psi + \mathcal{H}(v_\alpha + B). \tag{8.50}$$

Thus we see that ζ_α is constant on large scales for adiabatic perturbations of any perfect fluid, with $\delta P_{\mathrm{intr},\alpha} = 0$, whose energy is conserved, $Q_\alpha = 0$ [91]. In fact we shall show later that ζ_α is constant even in the presence of energy transfer, $Q_\alpha \neq 0$, as long as that energy transfer is adiabatic [42].

8.2.4. Multiple fields

If we consider N scalar fields with Lagrangian density

$$\mathcal{L} = -U(\varphi_1, \ldots, \varphi_N) - \frac{1}{2} \sum_{I=1}^{N} g^{\mu\nu} \varphi_{I,\mu} \varphi_{I,\nu}, \tag{8.51}$$

and minimal coupling to gravity, then the total energy, pressure and momentum perturbations are given by

$$\delta \rho = \sum_I \left[a^{-2} \varphi_I' \left(\delta \varphi_I' - \varphi_I' \phi_1 \right) + U_I \delta \varphi_I \right], \tag{8.52}$$

$$\delta P = \sum_I \left[a^{-2} \varphi_I' \left(\delta \varphi_I' - \varphi_I' \phi_1 \right) - U_I \delta \varphi_I \right], \tag{8.53}$$

$$(\rho + P)(v + B)_{,i} = -\sum_I a^{-1} \varphi_I' \delta \varphi_{I,i}, \tag{8.54}$$

where $U_I \equiv \partial U / \partial \varphi_I$. These then give the gauge-invariant comoving density perturbation (7.48)

$$\delta \rho_m = a^{-2} \sum_I \left[\varphi_I' \left(\delta \varphi_I' - \varphi_I' \phi_1 \right) - (\varphi_I'' - \mathcal{H} \varphi_I') \delta \varphi_I \right]. \tag{8.55}$$

The comoving density is sometimes used to represent the total matter perturbation, but for a single scalar field it is proportional to the non-adiabatic pressure (9.5):

$$\delta P_{\mathrm{nad}} = -\frac{2a^2 U_{,\varphi}}{3\mathcal{H}\varphi'} \delta \rho_{\mathrm{com}}. \tag{8.56}$$

From the Einstein constraint equation (8.25) this will vanish on large scales ($k/aH \to 0$) if Ψ remains finite, and hence single scalar field perturbations become adiabatic in this large-scale limit, even without assuming slow-roll [102,52].

The anisotropic stress, π_{ij}, vanishes to linear order for any number of scalar fields minimally coupled to gravity.

The first-order scalar field perturbations obey the wave equation

$$\delta \varphi_I'' + 2\mathcal{H} \delta \varphi_I' - \nabla^2 \delta \varphi_I + a^2 \sum_J U_{IJ} \delta \varphi_J = -2a^2 U_I \phi_1 + \varphi_I' \left[\phi_1' + 3\psi' - \nabla^2 \sigma \right] \tag{8.57}$$

where the terms on the right-hand side represent the effect of metric perturbations at first order (sometimes called the gravitational backreaction).

The scalar field wave equation in a perturbed FRW cosmology can most easily be written in closed form in terms of the field perturbations in the spatially flat gauge, defined in Eq. (7.23), which in the multi-field case have the gauge-invariant definitions

$$\delta\varphi_{\text{flat}I} \equiv \delta\varphi_I + \frac{\varphi_I'}{\mathcal{H}}\psi.$$

(8.58)

Note that since the scalar field can be thought of as a potential for the 4-velocity, this variable is a rescaling of the curvature perturbation on the comoving orthogonal hypersurface for each field

$$\delta\varphi_{\text{flat}I} = \frac{\varphi_I'}{\mathcal{H}}\mathcal{R}_I.$$

(8.59)

Using the Einstein equations to eliminate the metric perturbations on the right-hand side of Eq. (8.57) yields [103–106]

$$\delta\varphi_{\text{flat}I}'' + 2\mathcal{H}\delta\varphi_{\text{flat}I}' - \nabla^2\delta\varphi_{\text{flat}I} + a^2\left[\sum_J U_{IJ} - \frac{8\pi G}{a^2}\left(\frac{a^2\varphi_I'\varphi_J'}{\mathcal{H}}\right)'\right]\delta\varphi_{\text{flat}J} = 0.$$

(8.60)

The effect of gravitational coupling is now evident due to the terms proportional to Newton's gravitational constant. It is also evident that this gravitational coupling vanishes at first order for fields whose time derivative vanishes in the background solution, which is why at lowest order in a slow-roll approximation during inflation one can neglect the gravitational coupling.

In the next section we will discuss how the coupled equations for multiple fields may be partially decoupled by identifying the adiabatic and isocurvature field perturbations on large scales.

8.3. First-order vector perturbations

The divergence-free part of the 3-momentum [see Eqs. (2.11), (4.5) and (4.13)]

$$\delta q_i = (\rho + P)(v_{\text{vec}i} - S_i),$$

(8.61)

obeys the momentum conservation equation

$$\delta q_i' + 4\mathcal{H}\delta q_i = -\nabla^2\Pi_i,$$

(8.62)

where the vector part of the anisotropic stress, Eq. (4.10), is given by $a^2\partial_{(i}\Pi_{j)}$. The gauge-invariant vector metric perturbation is then directly related to the divergence-free part of the momentum via the constraint equation

$$\nabla^2\left(F_i' + S_i\right) = -16\pi Ga^2\delta q_i.$$

(8.63)

Thus the Einstein equations constrain the gauge-invariant vector metric perturbation, $F_i' + S_i$, to vanish in the presence of only scalar fields, for which the divergence-free momentum necessarily vanishes.

Eq. (8.63) shows that vector metric perturbations can be supported only by divergence-free momenta, but even then Eq. (8.62) shows that the vector perturbations are redshifted away by the Hubble expansion on large scales unless they are driven by an anisotropic stress.

8.4. First-order tensor perturbations

There is no constraint equation for the tensor perturbations as these are the free gravitational degrees of freedom (gravitational waves). The spatial part of the Einstein equations yields a wave equation

$$h_{ij}'' + 2\mathcal{H}h_{ij}' - \nabla^2 h_{ij} = 8\pi Ga^2\Pi_{ij},$$

(8.64)

where $\Pi_{ij}^{(TT)}$ is the transverse and trace-free part of the anisotropic stress (4.10).

We can decompose arbitrary tensor perturbations into eigenmodes of the spatial Laplacian, $\nabla^2 e_{ij} = -(k^2/a^2)e_{ij}$, with comoving wavenumber k, and scalar amplitude $h(t)$:

$$h_{ij} = h(t)e_{ij}^{(+,\times)}(x),$$

(8.65)

with two possible polarisation states, $+$ and $\times$. In the absence of any such anisotropic stress, e.g., in the presence of scalar fields and perfect fluids, the amplitude, defined in Eq. (8.65), of the tensor metric perturbation with comoving wavenumber, k, obeys the wave equation for a massless scalar field (8.57) in an unperturbed FRW metric.

$$h'' + 2\mathcal{H}h' + k^2 h = 0.$$

(8.66)

9. Adiabatic and entropy perturbations

9.1. Multiple fluids

We will refer to primordial perturbations as the perturbations at the epoch of primordial nucleosynthesis. The abundances of the light elements provide constraints on the matter content and expansion rate of the universe at this epoch, so we will assume that the material content of the universe is known (photons, neutrinos, baryonic matter and cold dark matter) and the gravitational laws are described by general relativity. This is expected to be some time after an early inflationary epoch when the perturbations originated as vacuum fluctuations on much smaller scales. By the time of primordial nucleosynthesis, the scales responsible for the large-scale structure of our observable universe today were far outside the Hubble scale and well described by the large-scale limit.

In the standard hot big bang the entropy density of the universe is dominated by the number of relativistic particles and for most of the history of the universe it is proportional to the number of photons, $s = 1.8 g_s n_\gamma$, where g_s is the effective number of light species [15]. In particular the perturbed baryon–entropy ratio n_B/s (assuming that g_s remains constant) is given by [95]

$$S_B \equiv \frac{\delta(n_B/n_\gamma)}{n_B/n_\gamma} = \frac{\delta n_B}{n_B} - \frac{\delta n_\gamma}{n_\gamma}. \tag{9.1}$$

Written in terms of the energy density of photons, $\rho_\gamma \propto n_\gamma^{4/3}$, and baryons, $\rho_B \propto n_B$, at the time of primordial nucleosynthesis this becomes

$$S_B = \frac{\delta\rho_B}{\rho_B} - \frac{3}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}. \tag{9.2}$$

More generally, density perturbations in an n-component system can be decomposed into an overall density perturbation and $n - 1$ relative density perturbations between the different components. The overall density perturbation is naturally gauge dependent, however the gauge transformation rule for the linear density perturbation (6.18) suggests a natural gauge-invariant definition of the relative density perturbation at first order,

$$\delta\rho_{IJ} \propto \delta\rho_I - \frac{\rho_I'}{\rho_J'}\delta\rho_J, \tag{9.3}$$

corresponding to the density perturbation of fluid I on surfaces of uniform density of the fluid J. Comparing this expression with the conventional definition of the primordial baryon–entropy perturbation (9.2) suggests a gauge-invariant definition of the relative perturbation between *any* two fluids [101,42]

$$S_{IJ} \equiv 3\mathcal{H}\left(\frac{\delta\rho_J}{\rho_J'} - \frac{\delta\rho_I}{\rho_I'}\right) \equiv 3\left(\zeta_I - \zeta_J\right), \tag{9.4}$$

which reduces to baryon–entropy perturbation (9.2) for $S_B \equiv S_{B\gamma}$. Hence we refer to S_{IJ} as the relative entropy perturbation for two fluids. It is the correct generalisation of the entropy perturbation defined in Ref. [1] to the case of non-interacting fluids.

The non-adiabatic pressure perturbation is given by

$$\delta P_{\text{nad}} \equiv \delta P - \frac{P'}{\rho'}\delta\rho. \tag{9.5}$$

For a detailed recent discussion of the non-adiabatic pressure see Ref. [107].

In a system of more than one fluid the total non-adiabatic pressure perturbation, δP_{nad}, may be further split into two parts [1],

$$\delta P_{\text{nad}} \equiv \delta P_{\text{intr}} + \delta P_{\text{rel}}. \tag{9.6}$$

The first part is due to the intrinsic entropy perturbation of each fluid

$$\delta P_{\text{intr}} = \sum_\alpha \delta P_{nad,\alpha}, \tag{9.7}$$

where the intrinsic non-adiabatic pressure perturbation of each fluid was given in Eq. (8.48). The second part of the non-adiabatic pressure perturbation (9.6) is due to the *relative entropy perturbation $S_{\alpha\beta}$* between different fluids (9.4)

$$\delta P_{\text{rel}} \equiv -\frac{1}{6\mathcal{H}\rho'}\sum_{\alpha,\beta}\rho_\alpha'\rho_\beta'\left(c_\alpha^2 - c_\beta^2\right)S_{\alpha\beta}. \tag{9.8}$$

In analogy with the non-adiabatic pressure perturbation for each fluid (8.48), we can identify an intrinsic non-adiabatic part of the energy transfer perturbation [101,42] that appears in the perturbed energy conservation equation for each fluid (8.43)

$$\delta Q_{\text{intr},\alpha} \equiv \delta Q_\alpha - \frac{Q'_\alpha}{\rho'_\alpha} \delta \rho_\alpha. \tag{9.9}$$

This is automatically zero if the local energy transfer Q_α is a function of the local density ρ_α so that $\delta Q_\alpha = (dQ_\alpha/d\rho_\alpha)\delta\rho_\alpha$, just as the intrinsic non-adiabatic pressure perturbation (8.48) vanishes when $\delta P_\alpha = (dP_\alpha/d\rho_\alpha)\delta\rho_\alpha$.

We can also identify in Eq. (8.43) a relative non-adiabatic energy transfer

$$\delta Q_{\text{rel},\alpha} = Q_\alpha \frac{\mathcal{H}' - \mathcal{H}^2}{\mathcal{H}^2} \left(\frac{\delta\rho_\alpha}{\rho'_\alpha} - \frac{\delta\rho}{\rho'} \right)$$

$$= -\frac{Q_\alpha}{6\mathcal{H}\rho} \sum_\beta \rho'_\beta S_{\alpha\beta}, \tag{9.10}$$

due to the presence of relative entropy perturbations whenever the background energy transfer is non-zero, $Q_\alpha \neq 0$.

The perturbed energy conservation Eq. (8.47) for each fluid can then be written as

$$\zeta'_\alpha = 3\frac{\mathcal{H}^2}{\rho'_\alpha} \delta P_{\text{intr},\alpha} - \frac{\mathcal{H}}{\rho'_\alpha} \left(\delta Q_{\text{intr},\alpha} + \delta Q_{\text{rel},\alpha} \right) - \Sigma_\alpha - \frac{\nabla^2}{3\mathcal{H}} \left[\frac{aQ_\alpha}{\rho'_\alpha} \mathcal{R}_\alpha \right] \tag{9.11}$$

where the non-adiabatic pressure perturbation of each fluid is given by Eq. (8.48) and the non-adiabatic energy transfer is given by Eqs. (9.9) and (9.10). We thus see that ζ_α is constant for adiabatic perturbations in the large-scale limit where the shear of comoving worldlines, Σ_α defined in Eq. (8.49), vanishes [42].

More generally [90] one finds a conserved perturbation whenever there is a local conservation equation of the form $dy/d\tau = -\theta f(y)$, where θ is the local expansion rate and τ is the proper time along comoving worldlines. When one integrates this conservation equation one finds the local logarithmic expansion as a function of y:

$$\tilde{N} = \int \theta d\tau = \int \frac{dy}{f(y)} \equiv F(y). \tag{9.12}$$

Thus the difference $\delta F = F(y_B) - F(y_A)$ evaluated along different worldlines remains a fixed constant of integration if one evaluates δF on constant-time hypersurfaces separated by a fixed expansion, and spatially flat hypersurfaces provide a suitable time slicing on large scales. The classic example of such a conserved quantity is the curvature perturbation on uniform density hypersurfaces, ζ defined in Eq. (7.61), which is conserved on large scales when $P = P(\rho)$ and thus the perturbations are adiabatic. But as we have seen the same result holds for the curvature perturbation on uniform density hypersurfaces for any fluid whose pressure perturbation and energy transfer is adiabatic.

Thus the matter isocurvature perturbation (9.2) is constant on large scales if there is negligible energy transfer between non-relativistic matter and radiation. However at even higher energies we can still define a conserved perturbation associated with conserved baryon number density [108]

$$\tilde{\zeta}_B = \frac{\delta n_B}{n_B} - \psi \tag{9.13}$$

as long as we have a conserved quantum number associated with baryon number. The observed stability of the proton implies that baryon number is conserved up to very high energy, possibly the GUT scale, and thus it should be possible to relate any primordial baryon isocurvature perturbation to physics at very high energies.

9.2. Multiple fields

In the background FRW cosmology driven by multiple scalar fields it is possible to identify an adiabatic direction along the background trajectory in field space

$$\hat{r}_I = \frac{\varphi'_I}{\sqrt{\varphi'^2_I}}. \tag{9.14}$$

The background solution, even in the presence of multiple fields, can then be described in terms of an effective single field, r, obeying the usual Klein–Gordon equation

$$r'' + 2\mathcal{H}r' + U_r = 0 \tag{9.15}$$

where $r' = \sum_I \hat{r}_I \varphi'_I$ and $U_r = \sum_I \hat{r}_I U_I$. However field perturbations need not follow this background trajectory and we encounter qualitatively different behaviour from that in the single field case when we consider inhomogeneous perturbations about the background trajectory.
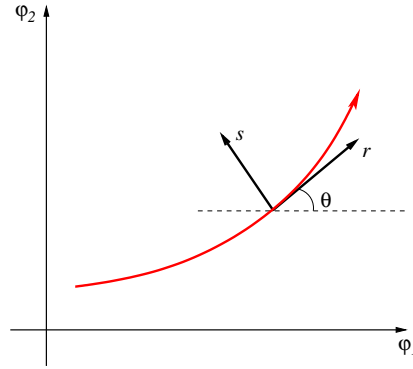
Fig. 1. r and s are used to denote the instantaneous adiabatic and entropy fields respectively, along and orthogonal to the curved background solution in field space.

In analogy with our treatment of fluid perturbations one can identify adiabatic and isocurvature field perturbations in a cosmology with more than one scalar field. Indeed even a single scalar field, φ, can support a non-adiabatic pressure perturbation, given in Eq. (8.56). We refer to this as the intrinsic pressure perturbation for the field. However this intrinsic non-adiabatic perturbation for a single field is proportional to the comoving density perturbation and thus vanishes in a scalar field dominated universe on large scales according to Eq. (8.25), leaving effectively only adiabatic perturbations in this large-scale limit.

For multiple fields we can perform a local rotation in field space to identify the adiabatic part of arbitrary perturbations along the background trajectory [109,102]

$$\delta r \equiv \sum_I \hat{r}_I \delta \varphi_I. \tag{9.16}$$

The generalisation to non-canonical fields with arbitrary metric in field space is given in Refs. [110,111].

Field perturbations orthogonal to the adiabatic field are isocurvature field perturbations, or entropy perturbations in analogy with the fluid density perturbations (9.2),

$$\delta s_{IJ} \propto \frac{\delta \varphi_I}{\varphi_I'} - \frac{\delta \varphi_J}{\varphi_J'}. \tag{9.17}$$

Note that the adiabatic field perturbation is naturally gauge dependent, whereas the isocurvature field perturbations are gauge independent at first order. For simplicity we will consider the case of two scalar fields where the direction in field space is given by the angle θ, see Fig. 1, so that $\hat{r} = (\cos\theta, \sin\theta)$. We then have

$$\delta r = \cos\theta \, \delta \varphi_1 + \sin\theta \, \delta \varphi_2, \tag{9.18}$$
$$\delta s = -\sin\theta \, \delta \varphi_1 + \cos\theta \, \delta \varphi_2 \tag{9.19}$$

and we will work with the adiabatic field perturbation in the spatially flat gauge

$$\delta r_{\text{flat}} = \delta r + \frac{r'}{\mathcal{H}} \psi. \tag{9.20}$$

Note that δr_{flat} is thus proportional to the total comoving curvature perturbation

$$\mathcal{R} = \psi + \frac{\mathcal{H}}{r'} \delta r = \frac{\mathcal{H}}{r'} \delta r_{\text{flat}} \tag{9.21}$$

since the adiabatic field, r, is the potential for the total velocity, $u_\mu \propto \mathrm{d}r/\mathrm{d}x^\mu$.

The adiabatic and isocurvature field perturbations obey the coupled evolution equations [102]

$$\delta r_{\text{flat}}'' + 2\mathcal{H}\delta r_{\text{flat}}' + \left[k^2 + a^2 U_{rr} - \theta'^2 - \frac{8\pi G}{a^2} \left(\frac{a^2 r'^2}{\mathcal{H}} \right)' \right] \delta r_{\text{flat}} = 2(\theta'\delta s)' - 2 \left(\frac{a^2 U_r}{r'} + \frac{\mathcal{H}'}{\mathcal{H}} \right) \theta'\delta s, \tag{9.22}$$

$$\delta s'' + 2\mathcal{H}\delta s' + \left(k^2 + a^2 U_{ss} + 3\theta'^2 \right) \delta s = \frac{\theta'}{r'} \frac{k^2}{2\pi G} \Psi, \tag{9.23}$$

where

$$U_{rr} \equiv (\cos^2\theta) U_{11} + (\sin 2\theta) U_{12} + (\sin^2\theta) U_{22}, \tag{9.24}$$
$$U_{ss} \equiv (\sin^2\theta) U_{11} - (\sin 2\theta) U_{12} + (\cos^2\theta) U_{22}. \tag{9.25}$$

We can identify a purely adiabatic mode where $\delta s = 0$ and remains zero on large scales. However a non-zero isocurvature perturbation appears as a source term in the perturbed inflaton equation (9.22) whenever $\theta' \neq 0$ and the inflaton trajectory is curved in field space. Note that θ' is given by [102]

$$\theta' = -\frac{a^2 U_s}{r'},\tag{9.26}$$

where $U_s = (\cos\theta)U_2 - (\sin\theta)U_1$ is the potential gradient orthogonal to the inflaton trajectory in field space. In the slow-roll approximation the background field always follows the potential gradient so the adiabatic–isocurvature coupling is suppressed in this slow-roll limit, the integrated effect of isocurvature field perturbations on the adiabatic field perturbation cannot in general be neglected.

Eq. (9.22) shows that the isocurvature perturbation δs works as a source term for the adiabatic curvature perturbation. This is in fact clearly seen if we take the time derivative of the comoving curvature perturbation (9.21):

$$\mathcal{R}' = \frac{\mathcal{H}}{\mathcal{H}' - \mathcal{H}^2}k^2\Psi + \frac{2\mathcal{H}}{r'}\theta'\delta s.\tag{9.27}$$

Therefore $\mathcal{R}$ (or equivalently ζ) is not conserved even in the large-scale limit in the presence of an isocurvature field perturbation, δs, with a non-straight trajectory in field space ($\theta' \neq 0$).

By contrast, the solution for the isocurvature field perturbation is independent of any initial adiabatic perturbation on large scales. The adiabatic perturbation provides a source term for the isocurvature field only through the spatial gradient of the longitudinal gauge metric potential, Ψ, which rapidly becomes negligible on super-Hubble scales during slow-roll inflation.

10. Perturbations from inflation

The standard hot big bang model of cosmology has a major shortcoming in that there is no causal explanation for the existence of primordial density perturbations on super-Hubble scales during the radiation dominated era. The CMB, and in particular the acoustic peaks in the temperature and polarisation anisotropies seen by the WMAP satellite [112] provide strong evidence that these primordial density perturbations do exist on scales much larger than the causal horizon at early times. The detailed distribution of primordial inhomogeneities is left as an unexplained initial condition in the standard hot big bang.

The primary success of inflation [113,114] is to give a model for the origin of the primordial density perturbations from vacuum fluctuations during a period of accelerated expansion at very early times. This relies on speculative and uncertain physics – in particular it requires some form of energy density with negative pressure in general relativity – but the unexpected discovery that the universe is accelerating today appears to show that cosmological inflation does happen. Zero-point vacuum fluctuations of any light, weakly coupled scalar field will be stretched up to super-Hubble scales during inflation and leave an approximately scale-invariant and Gaussian distribution of perturbations on large scales [115–118,85,119,73,74].

The simplest model for inflation is that it is driven by a vacuum (potential) energy density which is a function of one or more scalar fields. We can describe the homogeneous FRW solution using the inflaton field, r, which describes the evolution along the trajectory, $\hat{r}$ defined in Eq. (9.14), in a possibly multi-dimensional field space. For a sufficiently flat potential the evolution can be well described by the slow-roll approximation which assumes that the energy density is potential dominated

$$\mathcal{H}^2 \simeq \frac{8\pi G a^2}{3}U(r).\tag{10.1}$$

This is equivalent to requiring that the first slow-roll parameter is small:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{\mathcal{H}' - \mathcal{H}^2}{\mathcal{H}^2} \ll 1.\tag{10.2}$$

Note that the condition for accelerated expansion requires $\epsilon < 1$. In terms of the potential we have

$$\epsilon \simeq \frac{1}{16\pi G}\left(\frac{U_r}{U}\right)^2.\tag{10.3}$$

We also assume that the evolution is overdamped, such that

$$3\mathcal{H}r' \simeq -a^2 U_r.\tag{10.4}$$

This implies that we can neglect the decaying mode of the overdamped system and we have a unique trajectory in field space for a single field. This is a self-consistent approximation when the inflaton field is light compared with the Hubble scale, which requires that the second slow-roll parameter is small:

$$|\eta_{rr}| \ll 1 \quad \text{where } \eta_{rr} \equiv \frac{U_{rr}}{3H^2}.\tag{10.5}$$

In the two-field model described in Section 9.2, allowing for non-adiabatic perturbations, we can define three parameters [120], η_{rr}, η_{rs} and η_{ss}, describing the curvature of the potential, where in general we have

$$\eta_{IJ} \equiv \frac{1}{8\pi G} \frac{U_{IJ}}{U}. \tag{10.6}$$

The background slow-roll solution is described in terms of the slow-roll parameters by

$$\dot{r}^2 \simeq \frac{2}{3}\epsilon U, \qquad H^{-1}\dot{\theta} \simeq -\eta_{rs}, \tag{10.7}$$

while the perturbations on large scales (comoving wavenumber $k \ll \mathcal{H}$) obey

$$H^{-1}\dot{\delta r}_{\text{flat}} \simeq (2\epsilon - \eta_{rr})\,\delta r_{\text{flat}} - 2\eta_{rs}\delta s,$$
$$H^{-1}\dot{\delta s} \simeq -\eta_{ss}\delta s, \tag{10.8}$$

where we neglect spatial gradients. Although $U_s \simeq 0$ at lowest order in slow-roll, this does not mean that the inflaton and entropy perturbations decouple. $\dot{\theta}$ given by Eq. (10.7) is in general non-zero at first order in slow-roll and large-scale entropy perturbations do affect the evolution of the adiabatic perturbations when $\eta_{\sigma s} \neq 0$.

While the general solution to the two second-order perturbation equations (9.22) and (9.23) has four independent modes, the two first-order slow-roll equations (10.8) give the approximate form of the squeezed state on large scales. This has only two modes which we can describe in terms of the dimensionless comoving curvature and isocurvature perturbations:

$$\mathcal{R} \equiv \frac{\mathcal{H}}{r'}\delta r_{\text{flat}}, \qquad \mathcal{S} \equiv \frac{\mathcal{H}}{r'}\delta s. \tag{10.9}$$

The normalisation of $\mathcal{R}$ coincides with the standard definition of the comoving curvature perturbation, Eq. (7.46). The normalisation of the dimensionless entropy during inflation, $\mathcal{S}$, chosen here coincides with Ref. [120]. It can be related to the non-adiabatic pressure perturbation (9.5) on large scales

$$\delta P_{\text{nad}} \simeq -\epsilon \eta_{rs} \frac{H^2}{2\pi G} \mathcal{S}. \tag{10.10}$$

The slow-roll approximation can provide a useful approximation to the instantaneous evolution of the fields and their perturbations on large scales during slow-roll inflation, but is not expected to remain accurate when integrated over many Hubble times, where inaccuracies can accumulate. In single-field inflation the constancy of the comoving curvature perturbation after Hubble exit, which does not rely on the slow-roll approximation, is crucial in order to make accurate predictions of the primordial perturbations using the slow-roll approximation only around Hubble crossing. In a two-field model we must describe the evolution after Hubble exit in terms of a general transfer matrix:

$$\begin{pmatrix} \mathcal{R} \\ \mathcal{S} \end{pmatrix} = \begin{pmatrix} 1 & T_{\mathcal{R}\mathcal{S}} \\ 0 & T_{\mathcal{S}\mathcal{S}} \end{pmatrix} \begin{pmatrix} \mathcal{R} \\ \mathcal{S} \end{pmatrix}_*. \tag{10.11}$$

On large scales the comoving curvature perturbation still remains constant for the purely adiabatic mode, corresponding to $\mathcal{S} = 0$, and adiabatic perturbations remain adiabatic. These general results are enough to fix two of the coefficients in the transfer matrix, but $T_{\mathcal{R}\mathcal{S}}$ and $T_{\mathcal{S}\mathcal{S}}$ remain to be determined either within a given theoretical model, or from observations, or ideally by both. The scale dependence of the transfer functions depends upon the inflaton–entropy coupling at Hubble exit during inflation and can be given in terms of the slow-roll parameters as [120]

$$\frac{\partial}{\partial \ln k} T_{\mathcal{R}\mathcal{S}} = 2\eta_{rs} + (2\epsilon - \eta_{rr} + \eta_{ss})T_{\mathcal{R}\mathcal{S}},$$
$$\frac{\partial}{\partial \ln k} T_{\mathcal{S}\mathcal{S}} = (2\epsilon - \eta_{rr} + \eta_{ss})T_{\mathcal{S}\mathcal{S}}. \tag{10.12}$$

10.1. Initial power spectra

The expectation value of the fluctuations of a homogeneous field are given by

$$\langle \delta\varphi_I(\mathbf{k}_1)\delta\varphi_I(\mathbf{k}_2) \rangle = (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2) P_{\delta\varphi_I}(k_1), \tag{10.13}$$

where angle brackets denote the ensemble average (i.e., the average over infinitely many realisations of the field, which is equivalent to taking the spatial average in an infinite space). The dimensionless power spectrum of the field (equivalently the variance of the field per logarithmic range of k) is given by

$$\mathcal{P}_{\delta\varphi_I}(k) = \frac{4\pi k^3}{(2\pi)^3} P_{\delta\varphi_I}(k). \tag{10.14}$$

For weakly coupled, light fields we can neglect interactions on wavelengths below the Hubble scale, so that vacuum fluctuations correspond to [16]

$$P_{\delta\varphi_I}(k) = \frac{1}{2ka^2}. \tag{10.15}$$

This gives rise to the power spectrum of field fluctuations on the Hubble scale ($k = \mathcal{H} = aH$) during inflation given by

$$\mathcal{P}_{\delta\varphi_I} \simeq \left(\frac{H}{2\pi}\right)_*^2, \tag{10.16}$$

where we use a $*$ to denote quantities evaluated at Hubble exit. If a field has a mass comparable to the Hubble scale or larger then the vacuum fluctuations on wavelengths greater than the effective Compton wavelength are suppressed. In addition fluctuations in strongly interacting fields may develop correlations before Hubble exit. But during slow-roll inflation the correlation between vacuum fluctuations in weakly coupled, light fields at Hubble exit is suppressed by slow-roll parameters. This remains true under a local rotation in fields space to another orthogonal basis such as the instantaneous inflaton and entropy directions (9.18) and (9.19) in field space.

The curvature and isocurvature power spectra at Hubble exit are given by

$$\mathcal{P}_{\mathcal{R}}|_* \simeq \mathcal{P}_{\delta}|_* \simeq \left(\frac{H^2}{2\pi\dot\sigma}\right)_*^2 \simeq \frac{8}{3}\left(\frac{U}{\epsilon M_{\mathrm{Pl}}^4}\right)_*, \tag{10.17}$$

while the cross-correlation is first order in slow-roll [110,121],

$$\mathcal{C}_{\mathcal{R}\delta}|_* \simeq -2C\eta_{rs}\,\mathcal{P}_{\mathcal{R}}|_*, \tag{10.18}$$

where $C = 2 - \ln 2 - \gamma \approx 0.73$ and γ is the Euler number. The normalisation chosen for the dimensionless entropy perturbation in Eq. (10.9) ensures that the curvature and isocurvature fluctuations have the same power at horizon exit [120]. The spectral tilts at Hubble exit are also the same and are given by

$$\Delta n_{\mathcal{R}}|_* \simeq \Delta n_{\delta}|_* \simeq -6\epsilon + 2\eta_{rr} \tag{10.19}$$

where $\Delta n_X \equiv \mathrm{d} \ln \mathcal{P}_X/\mathrm{d} \ln k$.

The tensor perturbations (8.64) are decoupled from scalar metric perturbations at first order and hence the power spectrum has the same form as in single-field inflation. Thus the power spectrum of gravitational waves on super-Hubble scales during inflation is given by

$$\mathcal{P}_{\mathrm{T}}|_* \simeq \frac{16H^2}{\pi M_{\mathrm{Pl}}^2} \simeq \frac{128}{3}\frac{U_*}{M_{\mathrm{Pl}}^4}, \tag{10.20}$$

and the spectral tilt is

$$\Delta n_{\mathrm{T}}|_* \simeq -2\epsilon. \tag{10.21}$$

10.2. Primordial power spectra

The resulting primordial power spectra on large scales can be obtained simply by applying the general transfer matrix (10.11) to the initial scalar perturbations. The scalar power spectra probed by astronomical observations are thus given by [120]

$$\mathcal{P}_{\mathcal{R}} = (1 + T_{\mathcal{R}\delta}^2)\mathcal{P}_{\mathcal{R}}|_* \tag{10.22}$$

$$\mathcal{P}_{\delta} = T_{\delta\delta}^2\mathcal{P}_{\mathcal{R}}|_* \tag{10.23}$$

$$\mathcal{C}_{\mathcal{R}\delta} = T_{\mathcal{R}\delta}T_{\delta\delta}\mathcal{P}_{\mathcal{R}}|_*. \tag{10.24}$$

Note that the primordial curvature and isocurvature perturbations from inflation are in general correlated [122] (see also [80,123,124]). The cross-correlation can be given in terms of a dimensionless correlation angle:

$$\cos\Theta \equiv \frac{\mathcal{C}_{\mathcal{R}\delta}}{\sqrt{\mathcal{P}_{\mathcal{R}}\mathcal{P}_{\delta}}} = \frac{T_{\mathcal{R}\delta}}{\sqrt{1 + T_{\mathcal{R}\delta}^2}}. \tag{10.25}$$

We see that if we can determine the dimensionless correlation angle, Θ, from observations, then this determines the off-diagonal term in the transfer matrix

$$T_{\mathcal{R}\delta} = \cot\Theta, \tag{10.26}$$

and we can in effect measure the contribution of the entropy perturbation during two-field inflation to the resultant curvature primordial perturbation. In particular this allows us in principle to deduce from observations the power spectrum of the curvature perturbation at Hubble exit during two-field slow-roll inflation [120]:

$$\mathcal{P}_{\mathcal{R}}|_* = \mathcal{P}_{\mathcal{R}} \sin^2 \Theta. \tag{10.27}$$

The scale dependence of the resulting scalar power spectra depends both upon the scale dependence of the initial power spectra and of the transfer coefficients. The spectral tilts are given from Eqs. (10.22)–(10.24) by

$$\Delta n_{\mathcal{R}} = \Delta n_{\mathcal{R}}|_* + H_*^{-1}(\partial T_{\mathcal{R} \delta}/\partial t_*) \sin 2\Theta,$$
$$\Delta n_{\delta} = \Delta n_{\mathcal{R}}|_* + 2H_*^{-1}(\partial \ln T_{\delta \delta}/\partial t_*), \tag{10.28}$$
$$\Delta n_{\mathcal{C}} = \Delta n_{\mathcal{R}}|_* + H_*^{-1}[(\partial T_{\mathcal{R} \delta}/\partial t_*) \tan \Theta + (\partial \ln T_{\delta \delta}/\partial t_*)],$$

where we have used Eq. (10.26) to eliminate $T_{\mathcal{R} \delta}$ in favour of the observable correlation angle Θ. Substituting Eq. (10.19) for the tilt at Hubble exit, and Eq. (10.12) for the scale dependence of the transfer functions, we obtain [120]

$$\Delta n_{\mathcal{R}} \simeq -(6 - 4\cos^2 \Theta)\epsilon + 2\left(\eta_{rr} \sin^2 \Theta + 2\eta_{rs} \sin \Theta \cos \Theta + \eta_{ss} \cos^2 \Theta\right),$$
$$\Delta n_{\delta} \simeq -2\epsilon + 2\eta_{ss}, \tag{10.29}$$
$$\Delta n_{\mathcal{C}} \simeq -2\epsilon + 2\eta_{ss} + 2\eta_{rs} \tan \Theta.$$

Although the overall amplitude of the transfer functions are dependent upon the evolution after Hubble exit and through reheating into the radiation era, the spectral tilts can be expressed solely in terms of the slow-roll parameters at Hubble exit during inflation and the correlation angle, Θ, which can in principle be observed.

If the primordial curvature perturbation results solely from the adiabatic inflaton field fluctuations during inflation then we have $T_{\mathcal{R} \delta} = 0$ in Eq. (10.22) and hence $\cos \Theta = 0$ in Eq. (10.29), which yields the standard single-field result

$$\Delta n_{\mathcal{R}} \simeq -6\epsilon + 2\eta_{rr}. \tag{10.30}$$

Any residual isocurvature perturbations must be uncorrelated with the adiabatic curvature perturbation (at first order in slow-roll) with spectral index

$$\Delta n_{\delta} \simeq -2\epsilon + 2\eta_{ss}. \tag{10.31}$$

On the other hand, if the observed primordial curvature perturbation is produced due to some entropy field fluctuations during inflation, we have $T_{\mathcal{R} \delta} \gg 1$ and $\sin \Theta \simeq 0$. In a two-field inflation model any residual primordial isocurvature perturbations will then be completely correlated (or anti-correlated) with the primordial curvature perturbation and we have

$$\Delta n_{\mathcal{R}} \simeq \Delta n_{\mathcal{C}} \simeq \Delta n_{\delta} \simeq -2\epsilon + 2\eta_{ss}. \tag{10.32}$$

The gravitational wave power spectrum is frozen-in on large scales, independent of the scalar perturbations, and hence

$$\mathcal{P}_{\mathrm{T}} = \mathcal{P}_{\mathrm{T}}|_*. \tag{10.33}$$

Thus we can derive a modified consistency relation [16] between observables applicable in the case of two-field slow-roll inflation:

$$\frac{\mathcal{P}_{\mathrm{T}}}{\mathcal{P}_{\mathcal{R}}} \simeq -8\Delta n_{\mathrm{T}} \sin^2 \Theta. \tag{10.34}$$

This relation was first obtained in Ref. [125] at the end of two-field inflation, and verified in Ref. [126] for slow-roll models. But it was realised in Ref. [120] that this relation also applies to the primordial perturbation spectra in the radiation era long after two-field slow-roll inflation has ended and hence may be tested observationally.

More generally, if there is any additional source of the scalar curvature perturbation, such as additional scalar fields during inflation, then this could give an additional contribution to the primordial scalar curvature spectrum without affecting the gravitational waves, and hence the more general result is the inequality [127,103]:

$$\frac{\mathcal{P}_{\mathrm{T}}}{\mathcal{P}_{\mathcal{R}}} \leq -8\Delta n_{\mathrm{T}} \sin^2 \Theta. \tag{10.35}$$

This leads to a fundamental difference when interpreting the observational constraints on the amplitude of primordial tensor perturbations in multiple inflation models. In single-field inflation, observations directly constrain $r = [\mathcal{P}_{\mathrm{T}}/\mathcal{P}_{\mathcal{R}}]_*$ and hence, from Eqs. (10.17) and (10.20), the slow-roll parameter ϵ. However in multiple field inflation, non-adiabatic perturbations can enhance the power of scalar perturbations after Hubble exit and hence observational constraints on the amplitude of primordial tensor perturbations do not directly constrain the slow-roll parameter ϵ.

Current CMB data alone require $r < 0.55$ (assuming power-law primordial spectra) [112] which in single-field models is interpreted as requiring $\epsilon < 0.04$. But in multiple field models ϵ could be larger if the primordial density perturbation comes from non-adiabatic perturbations during inflation.

11. Non-linear evolution and non-Gaussianity

A powerful technique to calculate the primordial curvature perturbation resulting from many inflation models, including multi-field models, is to note that the curvature perturbation ζ defined in Eq. (7.61) can be interpreted as a perturbation in the local expansion [117,128,103,54]

$$\zeta = \delta N, \tag{11.1}$$

where δN is the perturbed expansion to uniform density hypersurfaces with respect to spatially flat hypersurfaces, which is given to first order by

$$\zeta = -H \frac{\delta \rho_{\text{flat}}}{\dot{\rho}}, \tag{11.2}$$

where $\delta \rho_{\text{flat}}$ must be evaluated on spatially flat ($\psi = 0$) hypersurfaces (see Sections 3.2 and 7.6).

An important simplification arises on large scales where anisotropy and spatial gradients can be neglected, and the local density, expansion, etc., obeys the same evolution equations as in a homogeneous FRW universe [129,103,130,91,90,131,54]. Thus we can use the homogeneous FRW solutions to describe the local evolution, which is known as the "separate universe" approach [129,103,130,91,131]. In particular we can evaluate the perturbed expansion in different parts of the universe resulting from different initial values for the fields during inflation using the homogeneous background solutions [103].

In the slow-roll approximation the integrated expansion on super-Hubble scales from some initial spatially flat hypersurface up to a subsequent fixed density hypersurface (say at the epoch of primordial nucleosynthesis) is some function of the local field values on the initial hypersurface, $N(\varphi_I|_\psi)$. More generally we expect this to hold whenever we can neglect the decaying mode for the field perturbations on super-Hubble scales. The resulting primordial curvature perturbation on the uniform density hypersurface is then

$$\zeta = \sum_I N_I \delta \varphi_{I\text{flat}}, \tag{11.3}$$

where $N_I \equiv \partial N / \partial \varphi_I$ and $\delta \varphi_{I\text{flat}}$ is the field perturbation on some initial spatially flat hypersurfaces during inflation (8.58). In particular the power spectrum for the primordial density perturbation in a multi-field inflation can be written (at leading order) in terms of the field perturbations after Hubble exit as

$$\mathcal{P}_\zeta = \sum_I N_I^2 \mathcal{P}_{\delta \varphi_{I\text{flat}}}. \tag{11.4}$$

This approach is readily extended to estimate the non-linear effect of field perturbations on the metric perturbations [130,90,132,54]. We can take Eq. (11.1) as our definition of the non-linear primordial curvature perturbation, ζ, so that in the radiation dominated era the non-linear extension of Eq. (11.2) is given by [54]

$$\zeta = \frac{1}{4} \ln \left(\frac{\rho_{\text{flat}}}{\rho_0} \right), \tag{11.5}$$

where $\rho_{\text{flat}}(t, \mathbf{x})$ is the perturbed (inhomogeneous) density evaluated on a spatially flat hypersurface and $\rho_0(t)$ is the background (homogeneous) density. See Refs. [131,133,134,51,135,136] for alternative approaches to define the non-linear extension of ζ.

This non-linear curvature perturbation as a function of the initial field fluctuations can simply be expanded as a Taylor expansion [132,137–139]

$$\zeta \simeq \sum_I N_I \delta \varphi_{I\text{flat}} + \frac{1}{2} \sum_{I,J} N_{IJ} \delta \varphi_{I\text{flat}} \delta \varphi_{J\text{flat}} + \frac{1}{6} \sum_{I,J,K} N_{IJK} \delta \varphi_{I\text{flat}} \delta \varphi_{J\text{flat}} \delta \varphi_{K\text{flat}} + \cdots \tag{11.6}$$

where we now identify (11.3) as the leading-order term.

We expect the field perturbations at Hubble exit to be close to Gaussian for weakly coupled scalar fields during inflation [140–142,137,143]. In this case the bispectrum of the primordial curvature perturbation at leading (fourth) order, can be written as [144,145]

$$B_\zeta(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}) = \frac{6}{5} f_{NL} \left[P_\zeta(k_1) P_\zeta(k_2) + P_\zeta(k_2) P_\zeta(k_3) + P_\zeta(k_3) P_\zeta(k_1) \right] \tag{11.7}$$

where $P_\zeta(k) = 2\pi^2 \mathcal{P}_\zeta(k) / k^3$, and the dimensionless non-linearity parameter is given, using the δN formalism, by[5] [132]

$$f_{NL} = \frac{5}{6} \frac{N_A N_B N^{AB}}{\left(N_C N^C \right)^2}. \tag{11.8}$$

---

[5] Note, that the factor "5/6" in Eq. (11.8) is a historical convention, due to the original definition [145] which was given in terms of the Newtonian potential, which on large scales in the matter era is given by $\Phi = -(3/5)\zeta$.

Similarly to the bispectrum, the connected part of the trispectrum in this case can be written as [139,138]

$$T_\zeta(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}, \mathbf{k_4}) = \tau_{NL} \left[P_\zeta(|\mathbf{k_1} + \mathbf{k_3}|) P_\zeta(k_3) P_\zeta(k_4) + (11 \text{ perms}) \right]$$
$$+ \frac{54}{25} g_{NL} \left[P_\zeta(k_2) P_\zeta(k_3) P_\zeta(k_4) + (3 \text{ perms}) \right] \tag{11.9}$$

where

$$\tau_{NL} = \frac{N_{AB} N^{AC} N^B N_C}{(N_D N^D)^3}, \tag{11.10}$$

$$g_{NL} = \frac{25}{54} \frac{N_{ABC} N^A N^B N^C}{(N_D N^D)^3}. \tag{11.11}$$

The expression for τ_{NL} was first given in [146]. Note that we have factored out products in the trispectrum with different k dependence in order to define the two k independent non-linearity parameters τ_{NL} and g_{NL}. This gives the possibility that observations may be able to distinguish between the two parameters [147].

In many cases there is single direction in field space, χ, which is responsible for perturbing the local expansion, $N(\chi)$, and hence generating the primordial curvature perturbation (11.6). For example this would be the inflaton field in single-field models of inflation, or it could be a late-decaying scalar field [95,97,98] as in the curvaton scenario [99,100]. In this case the curvature perturbation (11.6) is given by

$$\zeta \simeq N' \delta\chi_{\text{flat}} + \frac{1}{2} N'' \delta\chi_{\text{flat}}^2 + \frac{1}{6} N''' \delta\chi_{\text{flat}}^3 + \cdots, \tag{11.12}$$

and the non-Gaussianity of the primordial perturbation has the simplest "local" form

$$\zeta = \zeta_1 + \frac{3}{5} f_{NL} \zeta_1^2 + \frac{9}{25} g_{NL} \zeta_1^3 + \cdots \tag{11.13}$$

where $\zeta_1 = N' \delta\chi_{\text{flat}}$ is the leading-order Gaussian curvature perturbation and the non-linearity parameters f_{NL} and g_{NL}, are given by [132,148]

$$f_{NL} = \frac{5}{6} \frac{N''}{(N')^2}, \tag{11.14}$$

$$g_{NL} = \frac{25}{54} \frac{N'''}{(N')^3}. \tag{11.15}$$

The primordial bispectrum and trispectrum are then given by Eqs. (11.7) and (11.9), where the non-linearity parameters f_{NL} and g_{NL}, given in Eqs. (11.8) and (11.11), reduce to Eqs. (11.14) and (11.15) respectively, and τ_{NL} given in Eq. (11.10) reduces to

$$\tau_{NL} = \frac{(N'')^2}{(N')^4} = \frac{36}{25} f_{NL}^2. \tag{11.16}$$

Thus τ_{NL} is proportional to f_{NL}^2 (first shown in [147] using the Bardeen potential, and in [132] using this notation). However the trispectrum could be large even when the bispectrum is small because of the g_{NL} term [147,148].

In the case where the primordial curvature perturbation is generated solely by adiabatic fluctuations in the inflaton field, r, the curvature perturbation is non-linearly conserved on large scales [90,54,135] and we can calculate N', N'', N''', etc, at Hubble exit. In terms of the slow-roll parameters, we find

$$N' = \frac{H}{\dot\varphi} \simeq \frac{1}{\sqrt{2}} \frac{1}{m_{\text{Pl}}} \frac{1}{\sqrt{\epsilon}} \sim \mathcal{O}\left(\epsilon^{-1/2}\right), \tag{11.17}$$

$$N'' \simeq -\frac{1}{2} \frac{1}{m_{\text{Pl}}^2} \frac{1}{\epsilon} (\eta_{rr} - 2\epsilon) \sim \mathcal{O}(1), \tag{11.18}$$

$$N''' \simeq \frac{1}{\sqrt{2}} \frac{1}{m_{\text{Pl}}^3} \frac{1}{\epsilon\sqrt{\epsilon}} \left(\epsilon\eta_{rr} - \eta_{rr}^2 + \frac{1}{2}\xi_r^2 \right) \sim \mathcal{O}(\epsilon^{1/2}), \tag{11.19}$$

where we have used the reduced Planck mass $m_{\text{Pl}}^2 = (8\pi G)^{-1}$ and introduced the second-order slow-roll parameter $\xi_\sigma^2 = m_{\text{Pl}}^4 U_r U_{rrr}/U^2$. Hence the non-linearity parameters for single-field inflation, (11.14) and (11.15), are given by

$$f_{NL} = \frac{5}{6} (\eta_{rr} - 2\epsilon), \tag{11.20}$$

$$g_{NL} = \frac{25}{54} \left(2\epsilon\eta_{rr} - 2\eta_{rr}^2 + \xi_r^2 \right) \tag{11.21}$$

with τ_{NL} given by Eq. (11.16). Although there are additional contributions to the primordial bispectrum and trispectrum coming from the intrinsic non-Gaussianity of the field perturbations at Hubble exit, these are also suppressed by slow-roll

parameters in slow-roll inflation. Thus the primordial non-Gaussianity is likely to be too small to be ever observed in the conventional inflaton scenario of single-field slow-roll inflation [149,140]. Indeed any detection of primordial non-Gaussianity $f_{NL} > 1$ would appear to rule out this inflaton scenario.

However significant non-Gaussianity can be generated due to non-adiabatic field fluctuations. Thus far it has proved difficult to generate detectable non-Gaussianity in the curvature perturbation during conventional slow-roll inflation, even in multiple field models [150,151], though see Refs. [152,153]. But detectable non-Gaussianity might be produced in non-slow-roll [154] or non-canonical scalar field inflation [155,156], or when the curvature perturbation is generated from isocurvature field perturbations at the end of inflation [157,17], during inhomogeneous reheating [158–163], or some time after inflation has ended in the curvaton model [99,164,165,55,148].

12. Summary and outlook

Linear perturbations have become part of the standard toolbox of modern cosmology. Earlier confusion surrounding apparently different behaviour found in different coordinate bases has largely been resolved through the use of variables which have gauge-invariant definitions. In Section 7 we have emphasised the physical interpretation of these gauge-invariant variables.

The power spectrum of primordial perturbations revealed by the cosmic microwave background and large-scale galaxy surveys is a powerful probe of inflationary models of the very early universe, and a challenge for alternative theories. Linear theory enables us to relate the primordial spectra to quantum fluctuations in the metric and matter fields at much higher energies. In the simplest single-field inflation models, it is possible to equate the primordial density perturbation with the curvature perturbation, ζ defined in Eq. (7.61), during inflation which remains constant on large scales for adiabatic density perturbations. More generally, if one allows for non-adiabatic perturbations then it becomes necessary to allow for variation in the curvature perturbation, even on super-Hubble scales. Nonetheless it is still possible in many cases to identify the primordial density perturbation with a perturbed expansion in the δN approach described in Section 11 where the integrated expansion, N, is a function of the local field values on spatially flat slices during inflation. As one goes beyond the textbook examples, it becomes necessary to have a clear physical definition of the perturbation variables to consistently extend the background FRW equations to the inhomogeneous perturbations.

The new frontier in the study of cosmological perturbations is the study of non-linear primordial perturbations, at second order and beyond. Many of the familiar certainties of linear perturbation theory no longer apply. We have shown in Section 6 that quantities that were automatically gauge independent at first order (including the non-adiabatic pressure perturbation, anisotropic stress, and the tensor metric perturbation) become gauge dependent at second order. We have shown in Section 7 that it is possible to use the same methodology to construct gauge-invariant variables at second (and higher) order. A variable with an unambiguous physical meaning will have a gauge-invariant definition. The resulting gauge-invariant definitions inevitably become more complicated than those at first order and we have only presented explicit definitions at second order for a few cases. Likewise we have not attempted to present the second-order dynamical equations in a comprehensive manner as was done at first order in Section 8. Our aim has been to provide an introduction to some of the issues that arise at higher orders. Early works on non-linear and second-order perturbation theory include Refs. [166,8], more recently see also Refs. [167,168,11,169,13,37].

Non-linearities allow additional information to be gleaned from the primordial perturbations. Much effort is currently being devoted to the study of higher-order correlations. Non-Gaussianity in the distribution of primordial density perturbations would reveal interactions beyond the linear theory. Such interactions are minimal (suppressed by slow-roll parameters) in the simplest single-field inflation model, so any detection of primordial non-Gaussianity would cause a major upheaval in our thinking about the very early universe. In principle the δN approach can be easily extended to higher orders, simply by extending the Taylor series for the integrated expansion as a function of the field perturbations beyond linear order. This enables one to compute higher-order correlation functions for ζ as shown in Section 11. The challenge is then to develop transfer functions to relate the primordial ζ to observables beyond linear order [12,170–172], although large primordial non-Gaussianity (e.g., $f_{NL} \gg 1$) is expected to dominate over non-linearity in the transfer functions.

The Klein–Gordon equation in closed form at second order shows that at second-order scalar perturbations will also be sourced by terms quadratic in first-order field perturbations [173,174]. This and second-order perturbation theory in general, provide an alternative to using the δN formalism in calculating the primordial non-Gaussianity f_{NL} [175]. The main advantage of perturbation theory is that it can also be extended to sub-horizon scales, whereas the δN formalism is only valid on super-horizon scales, and in some cases it has been shown to be numerically more efficient [55].

Typically non-linear effects are going to be small, given that scalar metric perturbations are only of order 10^{-5}. Primordial tensor modes are smaller, and vector modes are effectively zero during scalar field driven inflation. But the additional information available from higher-order correlations and the use of optimised filters for specific forms of non-Gaussianity means that there are already impressive constraints on the degree of primordial Gaussianity.

Qualitatively new effects appear beyond linear order. The non-linearity of the field equations inevitably leads to mixing between scalar, vector and tensor modes and the existence of primordial density perturbations then inevitably generates vector and tensor modes [63,13,64,65,176]. As shown in Section 6, if we continue studies of scalar perturbations to higher order then the distinction between the different types of perturbations becomes gauge dependent and consistent computation will require careful (gauge-invariant) definition of the variables being used.

Acknowledgements

Appendix A. Definitions and notation

A.1. Notation

The sign convention is $(+++)$ in the classification of Misner, Thorne, and Wheeler [178].

Tensor indices:
Greek indices, such as α, β, ..., μ, ν, ..., run from 0 to 3. Latin indices, such as a, b, ..., i, j, ..., run from 1 to 3, that is only over spatial dimensions.
Spatial three-vectors are written in boldface, i.e. $\mathbf{v} \equiv v^i$, whenever convenient.
Throughout this work we use the units $c = \hbar = 1$.

A.2. Definitions

The connection coefficient is defined as

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma} \left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} \right). \tag{A.1}$$

The Riemann tensor is defined as

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\lambda\mu}\Gamma^{\lambda}_{\beta\nu} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\mu}. \tag{A.2}$$

The Ricci tensor is a contraction of the Riemann tensor and given by

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}, \tag{A.3}$$

and the Ricci scalar is given by contracting the Ricci tensor

$$R = R^{\mu}_{\mu}. \tag{A.4}$$

The Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{A.5}$$

The covariant derivatives are denoted by

$$_{;\mu} \equiv \nabla_{\mu} \quad \text{covariant differentiation with respect to } g_{\mu\nu}.$$

Partial derivatives are denoted by

$$X_{,i} \equiv \frac{\partial X}{\partial x^i}. \tag{A.6}$$

Symmetrisation and antisymmetrisation are, as usual, defined as

$$V_{(i,j)} \equiv \frac{1}{2} \left(V_{i,j} + V_{j,i} \right), \qquad V_{[i,j]} \equiv \frac{1}{2} \left(V_{i,j} - V_{j,i} \right). \tag{A.7}$$

A.3. Lie derivatives

The Lie derivatives with respect to a vector field ξ^{μ} of a scalar φ, a covariant vector v_{μ}, and a covariant tensor $t_{\mu\nu}$ are given by (see e.g. Ref. [39])

$$\pounds_{\xi}\varphi = \xi^{\lambda}\varphi_{,\lambda}, \tag{A.8}$$

$$\pounds_{\xi}v_{\mu} = v_{\mu,\alpha}\xi^{\alpha} + v_{\alpha}\xi^{\alpha}_{,\mu}, \tag{A.9}$$

$$\pounds_{\xi}t_{\mu\nu} = t_{\mu\nu,\lambda}\xi^{\lambda} + t_{\mu\lambda}\xi^{\lambda}_{,\nu} + t_{\lambda\nu}\xi^{\lambda}_{,\mu}. \tag{A.10}$$

A.4. Covariant derivatives

The covariant derivatives of a scalar φ, a covariant vector V_μ, and a covariant tensor $t_{\mu\nu}$ are given by (e.g. [39])

$$
\begin{aligned}
\varphi_{;\mu} &= \varphi_{,\mu}, \\
V_{\mu;\nu} &= V_{\mu,\nu} - \Gamma^\alpha_{\mu\nu} V_\alpha, \\
t_{\mu\nu;\lambda} &= t_{\mu\nu,\lambda} - \Gamma^\alpha_{\mu\lambda} t_{\alpha\nu} - \Gamma^\beta_{\nu\lambda} t_{\beta\mu}.
\end{aligned}
\tag{A.11}
$$

Appendix B. Components of connection coefficients and tensors

In the following no gauge is specified, and we leave quantities undecomposed whenever convenient (the decomposition rules are given in Eq. (2.4), (2.11) and (2.12)).

B.1. Connection coefficients

The connection coefficients up to and including second order are

$$
\Gamma^0_{00} = \frac{a'}{a} + \phi'_1 + \frac{1}{2}\phi'_2 - 2\phi_1\phi'_1 + \frac{a'}{a}B_{1k}B^{1k} + B^k_1 B'_{1k} + B^k_1 \phi_{1,k},
\tag{B.1}
$$

$$
\Gamma^0_{0i} = \phi_{1,i} + \frac{1}{2}\phi_{2,i} + \frac{a'}{a}\left(B_{1i} + \frac{1}{2}B_{2i}\right) - 2\phi_1\phi_{1,i} - 2\frac{a'}{a}\phi_1 B_{1i} + B^k_1 C'_{1ki} + \frac{1}{2}B^k_1\left(B_{1k,i} - B_{1i,k}\right),
\tag{B.2}
$$

$$
\Gamma^i_{00} = \frac{a'}{a}\left(B^i_1 + \frac{1}{2}B^i_2\right) + \left(B_1^{i'} + \frac{1}{2}B_2^{i'}\right) + \phi_{1,}^{\ i} + \frac{1}{2}\phi_{2,}^{\ i} - \phi'_1 B^i_1 - 2\frac{a'}{a}C_1^{ik}B_{1k} - 2C_1^{ik}B'_{1k} - 2C_1^{ik}\phi_{1,k},
\tag{B.3}
$$

$$
\begin{aligned}
\Gamma^i_{j0} =\ & \frac{a'}{a}\delta^i_j + C_{1j}^{\ i}{}' + \frac{1}{2}C_{2j}^{\ i}{}' + \frac{1}{2}\left(B_1^{\ i}{}_{,j} - B_{1j,}^{\ \ i}\right) + \frac{1}{4}\left(B_2^{\ i}{}_{,j} - B_{2j,}^{\ \ i}\right) \\
& - 2C_1^{ik}C'_{1jk} - B_1^{\ i}\left(\frac{a'}{a}B_{1j} + \phi_{1,j}\right) + C_1^i\left(B_{1j,k} - B_{1k,j}\right),
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
\Gamma^0_{ij} =\ & \left[\frac{a'}{a} - 2\frac{a'}{a}\left(\phi_1 + \frac{1}{2}\phi_2\right)\right]\delta_{ij} + C'_{1ij} + \frac{1}{2}C'_{2ij} + 2\frac{a'}{a}C_{ij} + \frac{a'}{a}C_{2ij} - \frac{1}{2}\left(B_{1j,i} + B_{1i,j}\right) - \frac{1}{4}\left(B_{2j,i} + B_{2i,j}\right) \\
& + B^k_1\left(C_{1jk,i} + C_{1ik,j} - C_{1ij,k}\right) + \phi_1\left[B_{1j,i} + B_{1i,j} - 4\frac{a'}{a}C_{1ij} - 2C'_{1ij}\right] + \delta_{ij}\frac{a'}{a}\left[4\phi_1^2 - B^k_1 B_{1k}\right],
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
\Gamma^i_{jk} =\ & C_{1k,j}^{\ \ i} + C_{1j,k}^{\ \ i} - C_{1jk,}^{\ \ \ i} + \frac{1}{2}\left(C_{2k,j}^{\ \ i} + C_{2j,k}^{\ \ i} - C_{2jk,}^{\ \ \ i}\right) - \frac{a'}{a}\delta_{kj}\left(B^i_1 + \frac{1}{2}B^i_2\right) + \frac{1}{2}B^i_1\left(B_{1k,j} + B_{1j,k}\right) \\
& + B_1^{\ i}\left[2\frac{a'}{a}\phi_1\delta_{kj} - C'_{1jk} - 2\frac{a'}{a}C_{1jk}\right] + 2\frac{a'}{a}\delta_{kj}B_{1l}C_1^{li} - 2C_1^{il}\left(C_{1kl,j} + C_{1jl,k} - C_{1jk,l}\right),
\end{aligned}
\tag{B.6}
$$

including scalar, vector, and tensor perturbations.

B.2. Energy–momentum tensor for N scalar fields

The energy–momentum tensor for N scalar fields with potential $U(\varphi_I)$ is then split into background, first, and second order perturbations, using Eq. (2.1), as

$$
T^\mu_\nu \equiv T^\mu_{(0)\nu} + \delta T^\mu_{(1)\nu} + \frac{1}{2}\delta T^\mu_{(2)\nu},
\tag{B.7}
$$

and we get for the components, from Eq. (5.10), at zeroth order

$$
T^0_{(0)0} = -\left(\sum_K \frac{1}{2a^2}\varphi_{0K}'^{\ 2} + U_0\right), \qquad T^i_{(0)j} = \left(\frac{1}{2a^2}\sum_K \varphi_{0K}'^{\ 2} - U_0\right)\delta^i_j,
\tag{B.8}
$$

at first order

$$
\begin{aligned}
\delta T^0_{(1)0} &= -\frac{1}{a^2}\sum_K\left(\varphi_{0K}'\delta\varphi_{1K}' - \varphi_{0K}'^{\ 2}\phi_1\right) - \delta U_1, \\
\delta T^0_{(1)i} &= -\frac{1}{a^2}\sum_K\left(\varphi_{0K}'\delta\varphi_{1K,i}\right), \\
\delta T^i_{(1)j} &= \frac{1}{a^2}\left[\sum_K\left(\varphi_{0K}'\delta\varphi_{1K}' - \varphi_{0K}'^{\ 2}\phi_1\right) - a^2\delta U_1\right]\delta^i_j,
\end{aligned}
\tag{B.9}
$$

and at second order in the perturbations

$$
\delta T^0_{(2)0} = -\frac{1}{a^2} \sum_K \left[\varphi'_{0K} \delta\varphi_{2K}{}' - 4\varphi'_{0K}\phi_1\delta\varphi_{1K}{}' - \varphi'_{0K}{}^2\phi_2 + 4\varphi'_{0K}{}^2\phi_1{}^2 + \delta\varphi_{1K}{}'^2 + a^2\delta U_2 \right.
$$
$$
\left. + \delta\varphi_{1K,l}\delta\varphi_{1K,}{}^l - \varphi'_{0K}{}^2 B_{1k}B_1{}^k \right],
$$

$$
\delta T^0_{(2)i} = -\frac{1}{a^2} \sum_K \left(\varphi'_{0K}\delta\varphi_{2K,i} - 4\phi_1\varphi'_{0K}\delta\varphi_{1K,i} + 2\delta\varphi_{1K}{}'\delta\varphi_{1K,i} \right), \tag{B.10}
$$

$$
\delta T^i_{(2)j} = \frac{1}{a^2} \sum_K \left[\varphi'_{0K}\delta\varphi_{2K}{}' - 4\varphi'_{0K}\phi_1\delta\varphi_{1K}{}' - \varphi'_{0K}{}^2\phi_2 + 4\varphi'_{0K}{}^2\phi_1{}^2 + \delta\varphi_{1K}{}'^2 - \delta\varphi_{1K,l}\delta\varphi_{1K,}{}^l \right.
$$
$$
\left. - \varphi'_{0K}{}^2 B_{1k}B_1{}^k - 2\varphi'_{0K}\delta\varphi_{1K,l}B_1{}^l - a^2\delta U_2 \right] \delta^i_j + \frac{2}{a^2} \left(\varphi'_{0K}B_1^i + \delta\varphi_{1K,}{}^i \right) \delta\varphi_{1K,j}.
$$

B.3. Energy–momentum tensor for fluid

We get for the components of the stress energy tensor, with indices lowered, in the background

$$
T_{00} = -a^2\rho_0, \qquad T_{0i} = 0, \qquad T_{ij} = a^2 P_0, \tag{B.11}
$$

at first order,

$$
{}^{(1)}\delta T_{00} = a^2 \left(\delta\rho_1 + 2\rho_0\phi_1 \right), \tag{B.12}
$$
$$
{}^{(1)}\delta T_{0i} = a^2 \left[-\left(\rho_0 + P_0\right) v_{1i} + \rho_0 B_{1i} \right], \tag{B.13}
$$
$$
{}^{(1)}\delta T_{ij} = a^2 \left[\delta P_1 \delta_{ij} + 2P_0 C_{1ij} + \pi_{1ij} \right], \tag{B.14}
$$

and at second order

$$
{}^{(2)}\delta T_{00} = a^2 \left[\delta\rho_2 + 2\rho_0\phi_2 + 4\phi_1\delta\rho_1 + 2\left(\rho_0 + P_0\right) v_{1k}v_1{}^k \right], \tag{B.15}
$$
$$
{}^{(2)}\delta T_{0i} = -a^2 \left[\left(\rho_0 + P_0\right)\left(v_{2i} + 2\phi_1 v_{1i} + 4C_{1ik}v_1{}^k \right) + \rho_0 B_{2i} + 2\delta\rho_1 B_{1i} + 2v_{1i}\left(\delta\rho_1 + \delta P_1\right) + 4\pi_{1ik}v_1{}^k \right], \tag{B.16}
$$
$$
{}^{(2)}\delta T_{ij} = a^2 \left[\delta P_2\, \delta_{ij} + 2P_0 C_{2ij} + 4\delta P_1 C_{1ij} + \left(\rho_0 + P_0\right)\left(v_{1i} + B_{1i} \right)\left(v_{1j} + B_{1j} \right) + \pi_{2ij} \right]. \tag{B.17}
$$

Appendix C. Geometry of spatial hypersurfaces

C.1. Components at first and second orders of shear, expansion, and acceleration

The calculation of the shear, defined above in Eq. (3.9), simplifies in the case of the unit normal vector field n^μ, that is for $n_i \equiv \mathbf{0}$,

$$
\sigma_{ij} = -n_0\Gamma^0_{ij} - \frac{1}{3}\theta\, g_{ij}, \tag{C.1}
$$

which gives (including vectors and tensors) at first order

$$
\delta^{(1)}\sigma_{00} = 0, \qquad \delta^{(1)}\sigma_{0i} = 0, \tag{C.2}
$$
$$
\delta^{(1)}\sigma_{ij} = a \left[C'_{1ij} - B_{1(i,j)} - \frac{1}{3}\delta_{ij}\left(C_{1k}{}^k - B_{1k,}{}^k \right) \right], \tag{C.3}
$$

and at second order

$$
\delta^{(2)}\sigma_{00} = 0, \tag{C.4}
$$
$$
\delta^{(2)}\sigma_{0i} = 2a \left[B_1^k \left(C'_{1ik} - B_{1(1,k)} \right) - \frac{1}{3}B_{1i}\left(C_{1k}{}'^k - B_{1k,}{}^k \right) \right], \tag{C.5}
$$

$$
\delta^{(2)}\sigma_{ij} = a \left[C'_{2ij} - B_{2(i,j)} + 2B_1^k \left(C_{1ki,j} + C_{1kj,i} - C_{1ij,k} \right) + 2\phi_1 \left(B_{1(i,j)} - C'_{1ij} \right) - \frac{4}{3}C_{1ij}\left(C_{1k}{}'^k - B_{1k,}{}^k \right) \right.
$$
$$
\left. + \frac{1}{3}\delta_{ij}\left\{ -C'_{2k}{}^k + B_{2k,}{}^k + 2\phi_1\left(C_{1k}{}'^k - B_{1k,}{}^k \right) + 4C_1^{kl}\left(C'_{1kl} - B_{1k,l} \right) - 2B_1^l\left(2C_{1lk,}{}^k - C_1{}^k{}_{k,l} \right) \right\} \right]. \tag{C.6}
$$

The expansion is given from Eq. (3.8) in the background as

$$\theta_0 = \frac{3a'}{a^2},$$ (C.7)

at first order

$$\delta\theta_1 = \frac{1}{a}\left[-3\frac{a'}{a}\phi_1 + C_{1k}^{\ k'} - B_{1k,}^{\ k}\right],$$ (C.8)

and at second order

$$\delta\theta_2 = \frac{1}{a}\left[-3\frac{a'}{a}\left(\phi_2 - 3\phi_1^2\right) + \left(C_{2k}^{\ k'} - B_{2k,}^{\ k}\right) + 2\phi_1\left(B_{1k,}^{\ k} - C_{1k}^{\ k'}\right)\right.$$
$$\left. - 3\frac{a'}{a}B_{1k}B_1^k - 4C_1^{kl}C_{1kl}' + 4C_1^{kl}B_{1l,k} + 4B_1^l C_{1lk,}^{\ k} - 2B_1^k C_{1\ l,k}^l\right].$$ (C.9)

The acceleration is given from Eq. (3.11) at first order as

$$a_{(1)0} = 0, \qquad a_{(1)i} = \phi_{1,i},$$ (C.10)

and at second order as

$$a_{(2)0} = 2B_1^k\phi_{1,k}, \qquad a_{(2)i} = \left[\phi_{2,i} + \left(B_{1k}B_1^k - 2\phi_1^2\right)_{,i}\right].$$ (C.11)

C.2. Curvature of spatial three-hypersurfaces

The intrinsic curvature of spatial three-hypersurfaces is given at first and second orders, respectively, by

$$\delta^{(3)}R_1 = \frac{4}{a^2}\nabla^2\psi_1,$$ (C.12)

$$\delta^{(3)}R_2 = \frac{1}{a^2}\left[4\nabla^2\psi_2 - 4C_{1km}^{\ \ m}C_{1\ ,n}^{kn} + 3C_{1mn,}^{\ \ k}C_{1\ ,k}^{mn} - C_{1\ k,n}^k C_{1\ m,}^{m\ n} + 4C_1^{mn}\left(C_{1mn,\ k}^{\ \ k} + C_{1\ k,mn}^k - C_{1mk,n}^{\ \ \ k} - C_{1kn,m}^{\ \ \ k}\right)\right.$$
$$\left. + 2\left(C_{1\ k,j}^k C_{1\ ,n}^{jn} + C_{1jk,}^{\ \ j}C_{1\ m,}^{m\ k} - C_{1\ n,m}^k C_{1\ ,k}^{mn}\right)\right],$$ (C.13)

where we used

$$2\left(C_{\ \ ,mn}^{mn} - C_{m,\ k}^{m\ k}\right) = 4\nabla^2\psi.$$ (C.14)

Appendix D. Governing equations

It is often convenient to have all relevant equations available "at a single glance". We therefore reproduce all governing equations given in previous sections in this appendix together. No gauge is specified, i.e. without choosing a particular hypersurface or gauge restrictions, and we leave quantities undecomposed whenever possible (the decomposition rules are given in Eqs. (2.4), (2.11) and (2.12)).

D.1. Background

Energy conservation for the α-fluid in the background is given from Eq. (4.21) as

$$\rho_{0\alpha}' = -3\mathcal{H}\left(\rho_{0\alpha} + P_{0\alpha}\right) + aQ_{0\alpha},$$ (D.1)

and the total energy conservation is then given by summing over the individual fluids and using Eq. (4.22) as

$$\rho_0' = -3\mathcal{H}\left(\rho_0 + P_0\right).$$ (D.2)

The Friedmann constraint is given from the 0–0 component equation (8.1) as

$$H^2 = \frac{8\pi G}{3}\rho_0 \qquad \mathcal{H}^2 = \frac{8\pi G}{3}a^2\rho_0.$$ (D.3)

The trace gives

$$\frac{a'^2}{a^2} - 2\frac{a''}{a} = 8\pi G a^2 P_0.$$ (D.4)

D.2. First order

In this subsection we give the governing equations on large scales in the general case without any gauge restrictions, i.e. without choosing a particular hypersurface.

D.2.1. Field equations
The 0–0 Einstein equation is given from Eq. (8.1) as

$$3\mathcal{H}\left(\mathcal{H}\phi_1 + \psi_1'\right) - \nabla^2\left(\psi_1 + \mathcal{H}\sigma_1\right) = -4\pi G a^2 \delta\rho_1. \tag{D.5}$$

The 0–i Einstein equation is

$$\mathcal{H}\phi_1 + \psi_1' = -4\pi G a^2 \left(\rho_0 + P_0\right)\left(v_1 + B_1\right), \tag{D.6}$$

the off-trace is

$$\sigma_1' + 2\mathcal{H}\sigma_1 + \psi_1 - \phi_1 = 8\pi G a^2 \Pi, \tag{D.7}$$

and the trace is

$$\psi_1'' + 2\mathcal{H}\psi_1' + \mathcal{H}\phi_1' + \left(2\frac{a''}{a} - \frac{a'^2}{a^2}\right)\phi_1 = 4\pi G a^2 \left(\delta P_1 + \frac{2}{3}\nabla^2\Pi\right). \tag{D.8}$$

D.2.2. Energy–momentum conservation
Energy and momentum conservation of the α-fluid is given from Eq. (4.21) at first order as

$$\delta\rho_\alpha' + 3\mathcal{H}(\delta\rho_\alpha + \delta P_\alpha) - 3\left(\rho_\alpha + P_\alpha\right)\psi' + a^{-1}(\rho_\alpha + P_\alpha)\nabla^2\left(V_\alpha + \sigma\right) = aQ_\alpha\phi + a\delta Q_\alpha. \tag{D.9}$$

The momentum conservation equation of the α-fluid is

$$V_\alpha' + \left[\frac{aQ_\alpha}{\rho_\alpha + P_\alpha}(1 + c_\alpha^2) - 3\mathcal{H}c_\alpha^2\right]V_\alpha + a\phi + \frac{a}{\rho_\alpha + P_\alpha}\left[\delta P_\alpha + \frac{2}{3}\nabla^2\Pi_\alpha - Q_\alpha V - f_\alpha\right] = 0. \tag{D.10}$$

Total energy and momentum conservation follows from the above, by summing over all individual fluids and using Eq. (4.22), and is given by

$$\delta\rho_1' + 3\mathcal{H}\left(\delta\rho_1 + \delta P_1\right) + \left(\rho_0 + P_0\right)\left[\nabla^2\left(\sigma_1 + v_1 + B_1\right) - 3\psi_1'\right] = 0, \tag{D.11}$$

$$\left[\left(\rho_0 + P_0\right)\left(v_1 + B_1\right)\right]' + \left(\rho_0 + P_0\right)\left[4\mathcal{H}\left(v_1 + B_1\right) + \phi_1\right] + \delta P_1 + \frac{2}{3}\nabla^2\Pi = 0. \tag{D.12}$$

D.3. Second order

D.3.1. Energy–momentum conservation
In the multi-fluid case, energy conservation of the α-fluid is given from Eq. (4.21) at second order as

$$\begin{aligned}
\delta\rho_{2\alpha}' &+ 3\mathcal{H}\left(\delta\rho_{2\alpha} + \delta P_{2\alpha}\right) + \left(\rho_{0\alpha} + P_{0\alpha}\right)\left(-3\psi_2' + \nabla^2 E_2' + \nabla^2 v_{2\alpha}\right) + 2\left(\delta\rho_{1\alpha} + \delta P_{1\alpha}\right)\left(-3\psi_1' + \nabla^2 E_1' + \nabla^2 v_{1\alpha}\right) \\
&+ 2\left(\delta\rho_{1\alpha} + \delta P_{1\alpha}\right)_{,k} v_{1\alpha}^k + 2\left(\rho_{0\alpha}' + P_{0\alpha}'\right)v_{1\alpha}^k\left(v_{1(\alpha)k} + B_{1k}\right) + 2\left(\rho_{0\alpha} + P_{0\alpha}\right)\left[\left(v_{1(\alpha)k}' + B_{1k}'\right)\left(2v_{1(\alpha)k} + B_{1k}\right)\right. \\
&+ \left.4\mathcal{H}v_{1(\alpha)}^k\left(v_{1(\alpha)k} + B_{1k}\right)\right] - 4\left(\rho_{0\alpha} + P_{0\alpha}\right)C_{1ij}'C_1^{ij} + 2\left(\rho_{0\alpha} + P_{0\alpha}\right)\left(2v_{1\alpha}^k\phi_{1,k} + \phi_1\nabla^2 v_{1\alpha}\right) \\
&+ 2\pi_{1(\alpha)ij}\left(C_1^{'ij} - 2\mathcal{H}C_1^{ij}\right) + 2\pi_{1(\alpha)}^{kl}v_{1(\alpha)k,l} + 2v_{1\alpha}^k\pi_{1(\alpha)kl,}^{\ l} + 2\left(\rho_{0\alpha} + P_{0\alpha}\right)\left(-3\psi_{1,l} + \nabla^2 E_{1,l}\right)v_{1\alpha}^l \\
&= a\left\{\delta Q_{2\alpha} + 2\phi_1\delta Q_{1\alpha} + Q_{0\alpha}\left(\phi_2 - \phi_1^2 + v_{1k}v_1^k\right) + \frac{4}{a}f_{1(\alpha)k}v_1^k\right\}. \tag{D.13}
\end{aligned}$$

For a single fluid we find by summing over the individual fluids and using Eq. (4.22)

$$\begin{aligned}
\delta\rho_2' &+ 3\mathcal{H}\left(\delta\rho_2 + \delta P_2\right) + \left(\rho_0 + P_0\right)\left(-3\psi_2' + \nabla^2 E_2' + \nabla^2 v_2\right) + 2\left(\delta\rho_1 + \delta P_1\right)\left(-3\psi_1' + \nabla^2 E_1' + \nabla^2 v_1\right) \\
&+ 2\left(\delta\rho_1 + \delta P_1\right)_{,k} v_1^k + 2\left(\rho_0' + P_0'\right)v_1^k\left(v_{1k} + B_{1k}\right) + 2\left(\rho_0 + P_0\right)\left[\left(v_{1k}' + B_{1k}'\right)\left(2v_{1k} + B_{1k}\right) + 4\mathcal{H}v_1^k\left(v_{1k} + B_{1k}\right)\right] \\
&- 4\left(\rho_0 + P_0\right)C_{1ij}'C_1^{ij} + 2\left(\rho_0 + P_0\right)\left(2v_1^k\phi_{1,k} + \phi_1\nabla^2 v_1\right) + 2\pi_{1ij}\left(C_1^{'ij} - 2\mathcal{H}C_1^{ij}\right) \\
&+ 2\pi_1^{kl}v_{1k,l} + 2v_1^k\pi_{1kl,}^{\ l} + 2\left(\rho_0 + P_0\right)\left(-3\psi_{1,l} + \nabla^2 E_{1,l}\right)v_1^l = 0. \tag{D.14}
\end{aligned}$$

List of symbols

a	Scale factor
a_μ	Acceleration
c_s	Adiabatic sound speed
c_α	Adiabatic sound speed of the α component
$f_{(J)}$	Momentum transfer perturbation of the Jth component
$g_{\mu\nu}$	Metric tensor
h_{ij}	Tensor metric perturbation
$\mathcal{P}_{\mu\nu}$	Projection tensor, $\mathcal{P}_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$
k^i	Comoving wave vector
k	Comoving wavenumber $k^2 \equiv k^i k_i$
$M_{\rm Pl}$	Planck mass $= G^{-1}$
$m_{\rm Pl}$	Reduced Planck mass $= (8\pi G)^{-1}$
n^μ	Unit time-like vector field
n_s	Spectral index of curvature perturbations
Δn_X	Scale dependence of perturbation spectrum of a quantity X
q^μ	Momentum current density (in rest frame)
q^i	Wave vector
ds	Infinitesimal line element
t	Coordinate time
u^μ	4-velocity
v	Scalar velocity perturbation
v_α	Scalar velocity perturbation of the α-component
$\bar{v}_J$	Rescaled perturbation in the Jth orthogonal field, $a\delta\bar{\sigma}_J$
v^i	Vector velocity perturbation
x^i	Spatial coordinate
B	Shift (scalar metric perturbation)
E	Scalar spatial metric perturbation
F_i	Vector spatial metric perturbation
G	Newton's constant
$G_{\mu\nu}$	Einstein tensor
H	Hubble parameter, $H \equiv \frac{\dot{a}}{a}$
$H_\nu^{(1)}$	Hankel function of the first kind of degree ν
$K_{\mu\nu}$	Extrinsic curvature
L	Lagrangian
$\mathcal{L}$	Lagrangian density
N	Number of e-folds (integrated expansion)
$\mathcal{P}_{\mu\nu}$	Projection tensor, $\mathcal{P}_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$
P	Pressure
P_α	Pressure of the α-component
$Q_{(J)}$	Energy transfer parameter of the Jth component
$Q_{(J)}^\mu$	Energy–momentum four-vector of the Jth component
$\mathcal{P}_X$	Power spectrum of a quantity X
R	Ricci scalar
$R_{\mu\nu}$	Ricci tensor
$^{(3)}R$	Intrinsic spatial curvature of three-hypersurface
$\mathcal{R}$	Curvature perturbation in comoving gauge
S	Action
S_i	Vector metric perturbation
S_{IJ}	Entropy perturbation
$T_{\mu\nu}$	Energy–momentum tensor
$U(\varphi)$	Potential of scalar field
α	Arbitrary scalar function (temporal gauge function)
β	Arbitrary scalar function (spatial gauge function)
δ^μ_ν	Kronecker delta
η	Conformal time, $a\mathrm{d}\eta \equiv \mathrm{d}t$
ζ	Curvature perturbation on uniform density hypersurface
γ^i	Arbitrary divergence-free vector function (spatial gauge function)
γ_{ik}	Metric tensor on spatial 3-hypersurface
κ	Curvature of background spacetime

λ	Comoving wavelength $k = \frac{2\pi}{k}$
λ_{phys}	Physical wavelength $\lambda_{\text{phys}} = a\lambda$
$\omega_{\mu\nu}$	Vorticity tensor
π^i	Anisotropic stress vector
π^{μ}_{ν}	Anisotropic stress tensor
$^{\text{tensor}}\pi^i_j$	Tensorial anisotropic stress tensor
φ	Scalar field
φ_I	One of the multiple scalar fields
ϕ	Lapse function (scalar metric perturbation)
ψ	Curvature perturbation (scalar metric perturbation)
ρ	Energy density
ρ_α	Energy density of α-component
σ	Shear scalar
$\sigma_{\mu\nu}$	Shear tensor
τ	Proper time, $d\tau^2 = ds^2$
θ	Expansion
$\bar{\xi}^i$	Arbitrary vector valued function
$\Gamma(x)$	Gamma function
Π	Scalar anisotropic stress tensor
Φ	Bardeen potential (lapse function in longitudinal gauge)
Ψ	Bardeen potential (curvature perturbation in longitudinal gauge)

Dimensions

It can be useful as a quick check of the validity of an equation or expression, particularly for the large expressions at second order, to check that all terms have the correct dimensions.

Quantity	Dimension
$[a]$	1
$[\varphi]$	T^{-1}
$[H]$	T^{-1}
$[U]$	T^{-4}
$[\dot{\varphi}]$	T^{-2}
$[G]$	T^2
$[\eta]$	T
$[x^i]$	T
$[\xi^\mu]$	T
$[\phi]$	1
$[\psi]$	1
$[B]$	T
$[E]$	T^2
$[C_{ij}]$	1
$[\mathcal{R}]$	1
$[\zeta]$	1
$[\rho]$	T^{-4}

Note, in geometric units $T \equiv L$.

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