

SMALLEST RELATIONAL MECHANICS MODEL OF QUANTUM COSMOLOGY

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Abstract

Relational particle mechanics are models in which there is, overall, no time, position, orientation (nor, sometimes, scale). They are useful for whole-universe modelling - the setting for quantum cosmology. This note concerns 3 particles in 1d in shape-scale split variables. The scale part parallels certain Friedmann equations, while in this note the shape part involves functions on the circle. The scale part is taken to be 'heavy' and 'slow' so the semiclassical approach applies and scale provides an approximate timestandard with respect to which the light physics runs. Relational particle mechanics moreover provide conceptual models of inhomogeneity, structure formation and nontrivial linear constraints (minisuperspace models do not and midisuperspace models only do at the cost of substantial complications).

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1 Introduction

Euclidean relational particle mechanics (RPM) [1, 2] has no absolute time, absolute position or absolute orientation (in the rotational sense). Similarity RPM is likewise [3, 4, 5, 6] but also with no absolute scale. RPM's are useful toy models of GR in a number of ways (listed in [5]) resembling it (particularly in the formulations [7]) to a comparable but different extent to the more habitually studied minisuperspace models. In addition to having a constraint quadratic in the momenta that leads to the frozen quantum equation aspect of the Problem of Time [8], RPM's (unlike minisuperspace) have nontrivial constraints that are linear in the momenta. In the case of Euclidean RPM, the latter is a zero total angular momentum constraint whereby one passes to the quotient of relative particle separations by rotations. These parallel the GR momentum constraint, its association with spatial diffeomorphisms and the quotient superspace of the space of Riemannian 3-metrics by these. Also RPM's (unlike minisuperspace) have notions of locality and clustering, making them more amenable to records-theoretic modelling [9] and some aspects of semiclassical quantum cosmological modelling [10, 8].

This note addresses the semiclassical quantum cosmology aspect. It concerns a regime in which there is a split into heavy, slow variables and light fast variables (in the Born-Oppenheimer and WKB senses respectively). In GR, these are, respectively, homogeneous and inhomogeneous modes. In Euclidean RPM, these are scale (moment of inertia of the model universe) and shape (clustering). The heavy slow variables provide an (approximate) timestandard with respect to which the light fast variables evolve.

RPM's are straightforward to study in 1 or 2 spatial dimensions: for N -particle similarity RPM, these have configuration spaces \mathbb{S}^{N-2} and $\mathbb{C}\mathbb{P}^{N-2}$ respectively [4]. For Euclidean RPM, one has the corresponding cones [2]. In 1-d with N particles, scale comes in via radius $\iota = \sqrt{I}$ (for I the moment of inertia of the system), so one has a Cartesian interpretation (the 2-d 3-particle sphere $\mathbb{C}\mathbb{P}^1 = \mathbb{S}^2$ is harder in these respects [2]). This note covers the 3-particle case.

2 The smallest RPM model of quantum cosmology

In terms of 2 mass weighted Jacobi coordinates ι_1, ι_2 and then passing to the corresponding polar coordinates ι, Φ , the relational kinetic term for 3 particles on a line becomes $\mathbb{T} = \{\iota^2 + \iota^2 \dot{\Phi}^2\}/2$, while this note's multi-HO potential maps to $\iota^2\{A + B\cos 2\Phi\}$ for $A = \{K_1 + K_2\}/4 > 0$ and $B = \{K_1 - K_2\}/4$ of whatever sign (for K_1, K_2 the Jacobi-Hooke coefficients divided by the Jacobi cluster masses). We only consider the case with a spring between particles 1 and 2 and an effective spring between particle 3 and the centre of mass of particles 1 and 2 (else there would be a third potential term).

The scale part of this obeys the analogue Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{2E}{a^2} - 2A \quad (1)$$

i.e. with a 'curvature' term and a 'cosmological' constant term, and so corresponding to the Milne in AdS model [11].

Then 1) the $B = 0$ case at the QM level with conformal ordering [12] is mathematically a 2-d isotropic HO, which has a standard solution, which in terms of our problem's original quantities, takes the following guise. The eigenvalues are

$$\frac{\sqrt{2E}}{\hbar\sqrt{K_1 + K_2}} = |d| + 2N + 1, \quad (2)$$

for d a relative dilation quantum number and N a quantum number that counts the nodes in the ‘radius’ ι , and the wavefunctions are

$$\Psi_{Nd}(\iota, \Phi) \propto \iota^{|\mathbf{d}|} \exp\left(-\frac{\sqrt{K_1 + K_2} \iota^2}{2\sqrt{2}\hbar}\right) L_N^{|\mathbf{d}|}\left(\frac{\sqrt{K_1 + K_2} \iota^2}{\sqrt{2}\hbar}\right) \exp(i\mathbf{d}\Phi). \quad (3)$$

2) Even $B \neq 0$ is exactly soluble by using Cartesian coordinates, in terms of which one obtains the usual Gaussian times Hermite polynomial form in each Cartesian coordinate. One can then dress this answer up in our problem’s significant shape-scale variables.

3) In parallel to the situation in cosmology, we declare $\iota = H$ and $\Phi = L$. Then the H-equation [which amounts to (1)] gives the approximate emergent WKB timestandard (= cosmic time here)

$$t^{\text{WKB}} = \sqrt{\frac{2}{K_1 + K_2}} \arcsin\left(\sqrt{\frac{\{K_1 + K_2\}}{4E}} \iota\right). \quad (4)$$

Next, the L-equation is, in terms of the rectified time

$$T = -\sqrt{\frac{2}{K_1 + K_2}} \cot\left(\sqrt{\frac{K_1 + K_2}{2}} t^{\text{WKB}}\right),$$

$$i\hbar \frac{\partial|\chi\rangle}{\partial T} = -\frac{\hbar^2}{2M} \frac{\partial^2|\chi\rangle}{\partial \Phi^2} + \frac{BM \cos 2\Phi}{\{1 + \{K_1 + K_2\}T^2/2\}^2} |\chi\rangle, \quad (5)$$

for ‘mass’ $M = 4E/\{K_1 + K_2\}$, which, for B small, i.e. $K_1 \approx K_2$, poses, about a very simple QM equation, a (fairly analytically tractable) T -dependent perturbation problem. This parallels Halliwell–Hawking’s work [10] while being simpler and so permitting rather more and rather more straightforward checks of various Problem of Time and quantum cosmological ideas.

3 Conclusion

This note’s analysis extends in many ways to the N -particle case and to tighter analogies with more commonly encountered cosmological models [2, 13]. The reason for studying this note’s ‘negative curvature balanced by negative cosmological constant’ type scenario is that it is ulteriorly exactly soluble [14], permitting in this case what are usually unavailable checks of the circumstances under which the various assumptions and approximations used in the semiclassical approach to the problem of time and quantum cosmology do and do not hold for the exactly solved model. This is work in progress [13].

To incorporate nontrivial constraints, 2-d models are considerably more useful [6, 15]; however, these models are harder, and so I am considering N particles on the line first. Records-theoretic study is best left to models with 4 or more particles.

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