INTRODUCTORY LECTURES ON QUANTUM COSMOLOGY[†]

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March, 1990

CTP # 1845

ABSTRACT: We describe the modern approach to quantum cosmology, as initiated by Hartle and Hawking, Linde, Vilenkin and others. The primary aim is to explain how one determines the consequences for the late universe of a given quantum theory of cosmological initial or boundary conditions. An extensive list of references is included, together with a guide to the literature.

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1. INTRODUCTION

My intention in these lectures is to describe the practical business of actually doing quantum cosmology. That is, I will describe how, in the context of particular models, one determines the consequences for the late universe of a given theory of initial conditions.

What is the motivation for studying quantum cosmology? One possible motivation comes from quantum gravity. Cosmological models are simple examples to which quantum gravity ideas may be applied. Moreover, the very early universe is perhaps the only laboratory in which quantum gravity may be tested. A second motivation, and the main one for the purposes of these lectures, concerns initial conditions in cosmology. Although the hot big bang model explains some of the features of the observed universe, there are a number of features that it did not explain, such as its flatness, absence of horizons, and the origin of the density fluctuations required to produce galaxies. The inflationary universe scenario (Guth, 1981), which involves quantized matter fields on a classical gravitational background, provided a possible solution to the horizon and flatness problems. Moreover, by assuming that the matter fields start out in a particular quantum state, the desired density fluctuation spectrum may be obtained.[†] However, in the inflationary universe scenario, the question of initial conditions was largely ignored. Whilst it is certainly true that, as a result of inflation, the observed universe could have arisen from a much larger class of initial conditions than in the hot big bang model, it is certainly not true that it could have arisen from *any* initial state – one could choose an initial quantum state for the matter which did not lead to the correct density perturbation spectrum, and indeed, one could choose initial conditions for which inflation does not occur. In order to have a complete explanation of the presently observed state of the universe, therefore, it is necessary to face up to the question of initial conditions.

Now, as the evolution of the universe is followed backwards in time, the curvatures and densities approach the Planck scale, at which one would expect quantum gravitational effects to become important. Quantum cosmology, in which both the matter and gravitational fields are quantized, is therefore the natural framework in which to address the question of initial conditions.

[†] For a review, see, for example, Brandenberger (1987, 1989).

In a sentence, quantum cosmology is the application of quantum theory to the dynamical systems describing closed cosmologies. Historically, the earliest investigations into quantum cosmology were primarily those of by DeWitt (1967), Misner (1969a, 1969b, 1969c, 1970, 1972, 1973) and Wheeler (1963, 1968) in the 1960's. This body of work I shall refer to as the "old" quantum cosmology, and will not be discussed here. It is discussed in the articles by MacCallum (1975), Misner (1972) and Ryan (1972).

After the initial efforts by the above authors, quantum cosmology went through a bit of a lull in the 1970's. However, it was re-vitalized in the 1980's, primarily by Hartle and Hawking (Hartle and Hawking, 1983; Hawking, 1982, 1984a), by Vilenkin (1984, 1986, 1988) and by Linde (1984a, 1984b, 1984c). There were two things that these authors added to the old approach. Firstly, Hartle and Hawking introduced Euclidean functional integrals, and used a blend of canonical and path integral methods. Secondly, all of the above authors faced up squarely to the issue of boundary or initial conditions on the wave function of the universe. It is this modern approach to quantum cosmology that will be the subject of these lectures.

The central object of interest in quantum cosmology is the wave function of a closed universe,

$$\Psi[h_{ij}(\mathbf{x}), \Phi(\mathbf{x}), B] \tag{1.1}$$

This is the amplitude that the universe contains a three-surface B on which the three-metric is $h_{ij}(\mathbf{x})$ and the matter field configuration is $\Phi(\mathbf{x})$. From such an amplitude one would hope to extract various predictions concerning the outcome of large scale observations. To fix the amplitude (1.1), one first needs a theory of dynamics, such as general relativity. From this one can derive an equation analagous to the Schrodinger equation, called the Wheeler-DeWitt equation, which the wave function of the universe must satisfy. The Wheeler-DeWitt equation will have many solutions, so in order to have any predictive power, it is necessary to propose a law of initial or boundary conditions to single out just one solution. And fnally, one needs some kind of scheme to interpret the wave function. So these are the three elements that go into quantum cosmology: dynamics, initial conditions, interpretation.

One of the most basic observational facts about the universe we observe today is that it is described by classical laws to a very high degree of precision. Since in quantum cosmology the universe is taken to be fundamentally quantum mechanical in nature, one

of the most primitive predictions a quantum theory of initial conditions should make, is that the universe is approximately classical when it is large. Indeed, what we will typically find to be the case is that the wave function indicates the regions in which space-time is essentially classical, and those in which it is not. In the regions where spacetime is essentially classical, we will find that the wave function is peaked about a set of solutions to the classical Einstein equations and, as a consequence of the boundary conditions on the wave function, this set is a subset of the general solution. The boundary conditions, through the wave function, therefore set initial conditions on the classical solutions. We may then begin to ask whether or not the finer details of the universe we observe, such as the existence of an inflationary era, are consequences of the chosen theory of initial conditions. In addition, in the approximately classical region, we will recover from the Wheeler-DeWitt equation the familiar quantum field theory for the matter fields on a classical curved spacetime background. Moreover, we will find that the boundary conditions on the wave function of the universe single out a particular choice of vacuum state for the matter fields. We may then ask whether or not the chosen vacuum state is the appropriate one for the subsequent emergence of large scale structure.

These remarks will hopefully become clearer as we progress, but in brief, the theme of these lectures may be summarized as follows. The inflationary universe scenario – and indeed most other cosmological scenarios – will always depend to some extent on initial conditions. I would like to try and argue that, within the context of quantum cosmology, there exist natural quantum theories of initial or boundary conditions from which the appropriate initial conditions for inflation and the emergence of large scale structure follow.

Throughout the text I will give very few references. An extensive guide to the literature is contained in Section 13.

2. A SIMPLE EXAMPLE

Rather than begin with the general formalism of quantum cosmology, I am going to first consider a simple inflationary universe model. This will help clarify some of the rather vague remarks made above concerning the need for initial conditions. The model will be treated rather heuristically; the details will be attended to later.

Consider a universe described by a homogeneous isotropic Robertson-Walker metric

$$ds^2 = \sigma^2 \left[-N^2(t)dt^2 + e^{2\alpha(t)} d\Omega_3^2(k) \right] \tag{2.1}$$

where $\sigma^2 = 2/(3\pi m_p^2)$ and $d\Omega_3^2(k)$ is the metric on the spatial sections which have constant curvature $k = -1, 0, +1$. In quantum cosmology one is generally interested in closed ($k = +1$) universes, but for the moment we will retain all three values of k. The metric is described by a single scale factor, $e^{\alpha(t)}$. As matter source we will use a homogeneous minimally coupled scalar field $\sqrt{2}\pi\sigma\phi(t)$ with potential $2\pi^2\sigma^2 V(\phi)$. The Einstein-scalar action for this system is

$$S = \frac{1}{2} \int dt N e^{3\alpha} \left[-\frac{\dot{\alpha}^2}{N^2} + \frac{\dot{\phi}^2}{N^2} - V(\phi) + k e^{-2\alpha} \right] \tag{2.2}$$

(the full form of the Einstein-scalar action is given in the next section). By varying with respect to α, ϕ and N, one may derive the field equations and constraint, which, after some rearrangement, are conveniently written,

$$\ddot{\phi} = -3\dot{\alpha}\dot{\phi} - \frac{1}{2}V'(\phi) \tag{2.3}$$

$$\ddot{\alpha} = -2\dot{\phi}^2 - \dot{\alpha}^2 + V(\phi) \tag{2.4}$$

$$-\dot{\alpha}^2 + \dot{\phi}^2 + V(\phi) = k e^{-2\alpha} \tag{2.5}$$

in the gauge $N = 1$. We will not assume a precise form for $V(\phi)$, except that it is of the inflationary type; that is, that for some range of values of ϕ, $V(\phi)$ is large and $|V'(\phi)/V(\phi)| << 1$. This is satisfied, for example, for large ϕ in chaotic models, with $V(\phi) = m^2\phi^2$ or $\lambda\phi^4$, and for ϕ near the origin in models with a Coleman-Weinberg potential. It is important to note that the general solution to the system (2.3)-(2.5) will involve *three* arbitrary parameters.

For models in which the potential satisfies the above conditions, it is easily seen that there exist solutions for which $\dot{\phi} \approx 0$ and the potential then acts like a cosmological constant; thus the model undergoes inflation, $e^\alpha \approx e^{V^{\frac{1}{2}}t}$. However, whether or not such a solution arises is clearly a question of *initial conditions*: one needs to choose the initial value of $\dot{\phi}$ to be small, and one needs to choose the initial value of ϕ to be in the region for which $|V'(\phi)/V(\phi)| << 1$. It is therefore pertinent to ask, to what extent is inflation generic in a model of this type?

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To address this question, one needs a complete picture of the classical solutions. Clearly it would be very difficult to solve the field equations exactly, even for very simple choices of $V(\phi)$. However, one can often obtain useful information using the qualitative theory of dynamical systems. The sort of differential equations one encounters in cosmology can frequently be cast in the form

$$\dot{x} = f(x, y, z...), \quad \dot{y} = g(x, y, z...), \quad \dot{z} = ... \tag{2.6}$$

Eq.(2.6) gives the direction of the solutions at every point $(x, y, z...)$. By drawing arrows at a selection of points one may thus construct a complete picture of the entire family of trajectories which solve (2.6) without integrating explicitly.

This method may be applied to the field equations (2.3), (2.4) by writing $x = \dot{\phi}$, $y = \dot{\alpha}$, $z = \phi$ (the constraint (2.5) is not normally used so that the three cases $k = 0, -1, +1$ may be treated simultaneously). The resulting three-dimensional phase portrait is, however, rather difficult to construct.[†] Let us therefore make a simplification, which is to go straightaway to a region where the ϕ-dependence of $V(\phi)$ is negligible. This is like having a massless scalar field and a cosmological constant. One then has a two-dimensional system,

$$\dot{x} = -3xy, \quad \dot{y} = -2x^2 - y^2 + V \tag{2.7}$$

The constraint equation

$$x^2 - y^2 + V = ke^{-2\alpha} \tag{2.8}$$

simply indicates that the $k = 0$ solutions are the two curves $y = \pm\sqrt{x^2 + V}$, the $k = +1$ solutions lie between these curves and the $k = -1$ solutions lie outside these curves.

The phase portrait for this two-dimensional system is shown in Fig.1. The point of particular interest is the point $\dot{\alpha} = V^{\frac{1}{2}}$, $\dot{\phi} = 0$, on the $k = 0$ curve, because at this point the model undergoes inflation. This point is an attractor for all the expanding $k = 0$ and $k = -1$ solutions. The $k = +1$ solutions, however, with which one is primarily concerned in quantum cosmology, do not all end up on the attractor: if they start out away from the $k = 0$ curve with $|\dot{\phi}|$ large they recollapse before getting anywhere near the attractor. Inflation occurs, therefore, only for the subset of $k = +1$ solutions with reasonably small

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[†] In the case $k = 0$, one can eliminate $\dot{\alpha}$ using the constraint, and the phase portrait becomes two-dimensional. This has been constructed for various inflationary potentials by Belinsky et al.(1985) and Piran and Williams (1985).

initial $\dot{\phi}$. Furthermore, when $V(\phi)$ is allowed to vary with ϕ, there is also the issue of *sufficient* inflation. In the massive scalar field model, for example, even if $\dot{\phi} \approx 0$ initially, it is known that the universe inflates by the required factor e^{65} only for initial values of ϕ greater than about 4 (in Planck units) (Hawking, 1984a; Page, 1986a).

So this simple model allows one to see quite clearly how the occurence of inflation depends rather crucially on the initial values of ϕ and $\dot{\phi}$. Now let us consider the quantization of this model, still proceeding heuristically, to see how quantum cosmology may shed some light on this issue.

We wish to quantize the dynamical system described by the action (2.2), for the case $k = +1$. We begin by finding the Hamiltonian of the theory. The momenta conjugate to α and ϕ are defined in the usual way and are given by

$$\pi_\alpha = -e^{3\alpha}\frac{\dot{\alpha}}{N}, \quad \pi_\phi = e^{3\alpha}\frac{\dot{\phi}}{N} \tag{2.9}$$

The canonical Hamiltonian is defined in the usual way and is given by

$$H_c = \frac{1}{2}Ne^{-3\alpha}\left[-\pi_\alpha^2 + \pi_\phi^2 + e^{6\alpha}V(\phi) - e^{4\alpha}\right] \equiv NH \tag{2.10}$$

The Hamiltonian form of the action is given by

$$S = \int dt \left[\dot{\alpha}\pi_\alpha + \dot{\phi}\pi_\phi - NH\right] \tag{2.11}$$

This form of the action exposes the fact that the lapse function N is a Lagrange multiplier which enforces the constraint

$$H = 0 \tag{2.12}$$

This is just the phase-space form of the constraint (2.5). The constraint indicates the presence of a symmetry, in this case reparametrization invariance, about which we will have more to say later.

Proceeding naively, we quantize this system by introducing a wave function $\Psi(\alpha, \phi, t)$ and asking that it satisfy a time-dependent Schrödinger equation constructed from the canonical Hamiltonian (2.10):

$$i\frac{\partial\Psi}{\partial t} = H_c\Psi \tag{2.13}$$

To ensure that the symmetry corresponding to the constraint (2.12) be imposed at the quantum level, we will also ask that the wave function is annihilated by the operator version of (2.12):

$$H\Psi = \frac{1}{2}e^{-3\alpha}\left[\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} + e^{6\alpha}V(\phi) - e^{4\alpha}\right]\Psi = 0 \tag{2.14}$$

where the momenta in (2.12) have been replaced by operators using the usual substitutions. However, since $H_c = NH$, it follows from (2.13) and (2.14) that the wave function is independent of t; thus the entire dynamics of the wave function is in fact contained in (2.14) with $\Psi = \Psi(\alpha, \phi)$. The fact that the wave function does not depend on the time parameter t explicitly is actually characteristic of parametrized theories such as general relativity. (2.14) is called the Wheeler-DeWitt equation and is the central equation of interest in quantum cosmology.

Let us find some simple solutions to this equation. Let us go to a region for which $|V'(\phi)/V(\phi)| << 1$ and look for solutions which do not depend very much on ϕ, so we may ignore the ϕ derivative term in (2.14). The problem is then a standard one-dimensional WKB problem in α with a potential $U = e^{6\alpha}V(\phi) - e^{4\alpha}$. In the region $U << 0$, where the scale factor is small, there are WKB solutions of the form

$$\Psi(\alpha, \phi) \approx \exp\left(\pm\frac{1}{3V(\phi)}(1 - e^{2\alpha}V(\phi))^{3/2}\right) \tag{2.15}$$

This region, in which the wave function is exponential, is normally regarded as some kind of tunneling or classically forbidden region. In the region $U >> 0$, where the scale factor is large, there are WKB solutions of the form

$$\Psi(\alpha, \phi) \approx \exp\left(\pm\frac{i}{3V(\phi)}(e^{2\alpha}V(\phi) - 1)^{3/2}\right) \tag{2.16}$$

This region, in which the wave function is oscillatory, is usually thought of as a classically allowed region. One can impose boundary conditions in either region, and then match the solutions in the two regions using the usual WKB matching procedure.

Consider in a little more detail the oscillatory region, including the ϕ dependence. Let us look for solutions of the form $\Psi = e^{iS}$, where S is a rapidly varying function of α and ϕ. Inserting this in the Wheeler-DeWitt equation, one finds that, to leading order, S must obey the Hamilton-Jacobi equation

$$-\left(\frac{\partial S}{\partial\alpha}\right)^2 + \left(\frac{\partial S}{\partial\phi}\right)^2 + U(\alpha, \phi) = 0 \tag{2.17}$$

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We will assume that some set of boundary conditions are imposed on Ψ; thus a particular solution of the Hamilton-Jacobi equation (2.17) will be picked out. Compare (2.17) with the Hamiltonian constraint,

$$-\pi_\alpha^2 + \pi_\phi^2 + U(\alpha, \phi) = 0 \tag{2.18}$$

It invites the identification

$$\pi_\alpha = \frac{\partial S}{\partial \alpha}, \quad \pi_\phi = \frac{\partial S}{\partial \phi} \tag{2.19}$$

More precisely, one can in fact show that a wave function of the form e^{iS} predicts a strong correlation between coordinates and momenta of the form (2.19). Furthermore, using the relationship between velocities and momenta (2.9), and the fact that S obeys the Hamilton-Jacobi equation (2.17), one may show that (2.19) defines a set of trajectories in the $\alpha\phi$ plane which are solutions to the classical field equations and constraint, (2.3)-(2.5). That is, the wave function e^{iS} is strongly peaked about a set of solutions to the classical field equations.

For a given solution S of the Hamilton-Jacobi equation the first integral of the field equations (2.19) about which the wave function is peaked involves just *two* arbitrary parameters. Recall, however, that the general solution to the full field equations (2.3)-(2.5) involved *three* arbitrary parameters. For given S, therefore, *the wave function e^{iS} is strongly peaked about the two-parameter subset of the three-parameter general solution.* By imposing boundary conditions on the wave function a particular solution Ψ to the Wheeler-DeWitt equation is picked out, which in the WKB approximation picks out a particular solution S to the Hamilton-Jacobi equation; this in turn defines a two-parameter subset of the three-parameter general solutions. It is in this way that boundary conditions on the wave function of the universe effectively imply initial conditions on the classical solutions.

Let us see how this works for the particular solution (2.16). For $e^{2\alpha}V >> 1$, it is of the form e^{iS} with $S \approx -\frac{1}{3}e^{3\alpha}V^{\frac{1}{2}}$. According to the above analysis, this wave function is peaked about the trajectories defined by

$$\dot{\alpha} \approx V^{\frac{1}{2}}, \quad \dot{\phi} \approx 0 \tag{2.20}$$

(we could of course have taken the opposite sign for S – this leads to a set of contracting solutions). Eq.(2.20) integrates to yield

$$e^\alpha \approx e^{V^{\frac{1}{2}}(t-t_0)}, \quad \phi \approx \phi_0 = constant \tag{2.21}$$

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Here t_0 and ϕ_0 are the two arbitrary constants parametrizing this set of solutions. The constant t_0 is in fact irrelevant, because it is just the origin of unobservable parameter time. From (2.20) one may see that the wave function is peaked right on the inflationary attractor in Fig.1. So *this particular wave function* picks out the inflationary solutions.

One can actually get a little more out of the wave function in addition to (2.20). The wave function more generally is of the form $C(\alpha, \phi)e^{iS}$. The e^{iS} part, as we have discussed, shows that the wave function is peaked about a set of trajectories. These trajectories may be labeled by the value of the arbitrary constant ϕ_0. The prefactor effectively provides a measure on the set of possible values of ϕ_0, and may therefore be used to assess the relative likelihood of inflation. We will describe this in a lot more detail later.

From this simple model we have learned a few things that are in fact quite general. They are as follows:

1) Classical cosmology needs initial conditions. This is illustrated rather clearly using the phase-portrait of classical solutions, allowing one to see what sort of features are generic, and what sort of features are dependent on a specific choice of initial conditions.

2) In the quantized model, there is a region in which the wave function is exponential, indicating that this region is classically forbidden.[†]

3) There is a region in which the wave function is oscillatory, indicating that this region classically allowed. To be precise, the wave function in the oscillatory region is strongly peaked about a *set* of solutions to the classical field equations.

4) The set of solutions about which a given WKB wave function is peaked is a *subset* of the general solution to the field equations. That is, a particular solution to the Wheeler-DeWitt equation is peaked about a particular subset of the full set of solutions to the field equations. Moreover, the wave function provides a measure on the classical trajectories within this set. A general solution to the Wheeler-DeWitt equation would be peaked about a general solution to the field equations, so by simply quantizing the model one does not necessarily learn anything about initial conditions. One merely

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[†] In this particular model, and for the particular solution to the Wheeler-DeWitt equation we looked at, the classically forbidden region is at small values of the scale factor. This is in accord with the general belief that "quantum gravity effects become important when the universe is very small". This is, however, dependent on boundary conditions. There are other solutions to the Wheeler-DeWitt equation which are oscillatory for small scale factors. We shall return to this point in Section 6.

transfers the question of initial conditions on the classical solutions to the question of boundary conditions on the wave function of the universe. To have complete predictive power, therefore, one needs a quantum theory of boundary conditions.

In connection with the fourth point above, one might ask, have we really improved the situation with regard to initial conditions by going to the quantum theory? The answer is, I believe, yes, for at least two reasons. Firstly, as the simple model above indicates, a classical description of cosmology is not always valid. In attempting to impose classical initial conditions at small three-geometries, therefore, one might be imposing them in a region in which, from the point of view of the quantum theory, a classical description is not really appropriate. Secondly, a somewhat more aesthetic point. Classically, there is no obvious reason for choosing one set of initial conditions over another. No one choice stands out as being more natural or elegant than any other. In quantum cosmology, however, one can argue that certain quantum states for the universe have considerably more appeal than others on the grounds of simplicity or naturalness. I will leave this to the reader to judge for themselves when we come to discuss particular proposals for quantum theories of initial conditions.

This ends what has really been an introductory tour of quantum cosmology. In the following sections, we will go over essentially the same points but in greater generality and detail.

3. THE HAMILTONIAN FORMULATION
OF GENERAL RELATIVITY

We now procede to the general formalism of quantum cosmology. This begins with the Hamiltonian formulation of general relativity (Hanson et al., 1976; Misner et al.,1970; Teitelboim, 1990). One considers a three-surface on which the three-metric is h_{ij}, with some matter field configuration. We will take the three-surface to be compact, since we are considering only closed universes. The three-surface is embedded in a four-manifold on which the four-metric is $g_{\mu\nu}$. This embedding is described by the standard $(3+1)$ form of the four-metric,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(N^2 - N_i N^i\right)dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j \qquad (3.1)$$

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where N and N_i are the lapse and shift functions. (Our conventions are $\mu, \nu = 0, 1, 2, 3$ and $i, j = 1, 2, 3$). They describe the way in which the choice of coordinates on one three-surface is related to the choice on an adjacent three-surface, and are therefore arbitrary.

The action will be taken to be the standard Einstein-Hilbert action coupled to matter,

$$S = \frac{m_p^2}{16\pi} \left[\int_M d^4x (-g)^{\frac{1}{2}} (R - 2\Lambda) + 2 \int_{\partial M} d^3x h^{\frac{1}{2}} K \right] + S_{matter} \tag{3.2}$$

where K is the trace of the extrinsic curvature K_{ij} at the boundary ∂M of the four-manifold M, and is given by

$$K_{ij} = \frac{1}{2N} \left[-\frac{\partial h_{ij}}{\partial t} + 2D_{(i}N_{j)} \right] \tag{3.3}$$

Here, D_i is the covariant derivative in the three-surface. For a scalar field Φ, the matter action is

$$S_{matter} = -\frac{1}{2} \int d^4x (-g)^{\frac{1}{2}} \left[g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi) \right] \tag{3.4}$$

In terms of the (3+1) variables, the action takes the form

$$S = \frac{m_p^2}{16\pi} \int d^3x dt N h^{\frac{1}{2}} \left[K_{ij} K^{ij} - K^2 + {}^3R - 2\Lambda \right] + S_{matter} \tag{3.5}$$

In a perfectly standard way, one may derive the Hamiltonian form of the action,

$$S = \int d^3x dt \left[\dot{h}_{ij} \pi^{ij} + \dot{\Phi} \pi_\Phi - N\mathcal{H} - N^i \mathcal{H}_i \right] \tag{3.6}$$

where π^{ij} and π_Φ are the momenta conjugate to h_{ij} and Φ respectively. The Hamiltonian is a sum of constraints, with the lapse N and shift N^i playing the role of lagrange multipliers. There is the momentum constraint,

$$\mathcal{H}_i = -2D_j \pi_i^j + \mathcal{H}_i^{matter} = 0 \tag{3.7}$$

and the Hamiltonian constraint

$$\mathcal{H} = \frac{16\pi}{m_p^2} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{m_p^2}{16\pi} h^{\frac{1}{2}} ({}^3R - 2\Lambda) + \mathcal{H}^{matter} = 0 \tag{3.8}$$

where G_{ijkl} is the DeWitt metric and is given by

$$G_{ijkl} = \frac{1}{2} h^{-\frac{1}{2}} \left(h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl} \right) \tag{3.9}$$

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These constraints are equivalent, respectively, to the time-space and time-time components of the classical Einstein equations. The constraints play a central role in the canonical quantization procedure, as we shall see.

The arena in which the classical dynamics takes place is called superspace, the space of all three-metrics and matter field configurations $(h_{ij}(\mathbf{x}), \Phi(\mathbf{x}))$ on a three-surface[†]. Superspace is infinite dimensional, with a finite number of coordinates $(h_{ij}(\mathbf{x}), \Phi(\mathbf{x}))$ at every point $\mathbf{x}$ of the three-surface. The DeWitt metric (plus some suitable metric on the matter fields) provides a metric on superspace. It has the important property that its signature is hyperbolic at every point $\mathbf{x}$ in the three-surface. The signature of the DeWitt metric is independent of the signature of spacetime.

4. QUANTIZATION

In the canonical quantization procedure, the quantum state of the system is represented by a wave functional $\Psi[h_{ij}, \Phi]$, a functional on superspace. An important feature of this wave function is that is does not depend explicitly on the coordinate time label t. This is because the three-surfaces are compact, and thus their intrinsic geometry, specified by the three-metric, fixes more-or-less uniquely their relative location in the four-manifold. Another way of saying essentially the same thing, is to say that general relativity is an example of a *parametrized* theory, which means that "time" is already contained amongst the dynamical variables describing it, h_{ij}, Φ.

According to the Dirac quantization procedure, the wave function is annihilated by the operator versions of the classical constraints. That is, if one makes the usual substitutions for momenta

$$\pi^{ij} \to -i\frac{\delta}{\delta h_{ij}} \qquad \pi_\Phi \to -i\frac{\delta}{\delta \Phi} \tag{4.1}$$

one obtains the following equations for Ψ. There is the momentum constraint

$$\mathcal{H}_i \Psi = 2iD_j\frac{\delta\Psi}{\delta h_{ij}} + \mathcal{H}_i^{matter}\Psi = 0 \tag{4.2}$$

[†] This superspace has nothing to do with the superspace of supersymmetry. Also, earlier authors in quantum cosmology used a different definition of superspace: they defined it to be the space of all three-metrics, but factored out by the three-dimensional diffeomorphisms.

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and the Wheeler-DeWitt equation

$$\mathcal{H}\Psi = \left[-G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} - h^{\frac{1}{2}}(^3R - 2\Lambda) + \mathcal{H}^{matter} \right]\Psi = 0 \qquad (4.3)$$

where we have ignored operator ordering problems.

The momentum constraint implies that the wave function is the same for configurations $(h_{ij}(\mathbf{x}), \Phi(\mathbf{x}))$ that are related by coordinate transformations in the three-surface. To see this, let us restrict attention to the case of no matter, and consider the effect of shifting the argument of the wave function by a diffeomorphism in the three-surface, $x^i \rightarrow x^i - \xi^i$. One has

$$\Psi[h_{ij} + D_{(i}\xi_{j)}] = \Psi[h_{ij}] + \int d^3\mathbf{x} D_{(i}\xi_{j)}\frac{\delta\Psi}{\delta h_{ij}} \qquad (4.4)$$

Integrating by parts in the last term, and dropping the boundary term (since the three-manifold is compact), one finds that the change in Ψ is given by

$$\delta\Psi = -\int d^3\mathbf{x}\xi_j D_i\left(\frac{\delta\Psi}{\delta h_{ij}}\right) = \frac{1}{2i}\int d^3\mathbf{x}\xi_i \mathcal{H}^i\Psi \qquad (4.5)$$

showing that wave functions satisfying (4.2) are unchanged. The momentum constraint (4.2) is therefore the quantum mechanical expression of the invariance of the theory under three-dimensional diffeomorphisms.* Similarly, the Wheeler-DeWitt equation (4.3) is connected with the reparametrization invariance of the theory. This is a lot harder to show and we will not go into it here[†].

The Wheeler-DeWitt equation is a second order hyperbolic functional differential equation describing the dynamical evolution of the wave function in superspace. The part of the three-metric corresponding to the minus sign in the hyperbolic signature, and so to the "time" part, is the volume of the three-metric, $h^{\frac{1}{2}}$. The Wheeler-DeWitt equation will in general have a vast number of solutions, so in order to have any predictive power we need boundary conditions to pick out just one solution. This might involve, for example, giving the value of the wave function at the boundary of superspace.

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[*] This was first shown by Higgs (1958).

[†] The difficulty is essentially due to the fact that although wave functions $\Psi[h_{ij}]$ carry a representation of the three-dimensional diffeomorphism group, they do not carry a representation of the four-dimensional diffeomorphisms. A closely related fact is that the Poisson bracket algebra of the constraints is not that of the four-dimensional diffeomorphsims. For a discussion of these issues and their resolution, see Isham and Kuchař (1985a, 1985b), Kuchař (1986).

As an alternative to the canonical quantization procedure, one can construct the wave function using a path integral. In the path integral method, the wave function (or more precisely, some kind of propagator) is represented by a Euclidean functional integral over a certain class of four-metrics and matter fields, weighted by e^{-I}, where I is the Euclidean action of the gravity plus matter system. Formally, one writes

$$\Psi[\tilde{h}_{ij}, \tilde{\Phi}, B] = \sum_M \int \mathcal{D}g_{\mu\nu} \mathcal{D}\Phi e^{-I}. \qquad (4.6)$$

The sum is taken over some class of manifolds M for which B is part of their boundary, and over some class of four-metrics $g_{\mu\nu}$ and matter fields Φ which induce the three-metric $\tilde{h}_{ij}$ and matter field configuration $\tilde{\Phi}$ on the three-surface B (see Fig.2.). The sum over four-manifolds is actually very difficult to define in practice, so one normally considers each admissable four-manifold separately. The path integral permits one to construct far more complicated amplitudes than the wave function for a single three-surface (Hartle, 1990), but this is the simplest and most frequently used amplitude, and it is the only one that will be discussed here.

When the four-manifold has topology $\mathbb{R} \times B$, the path integral has the explicit form

$$\Psi[\tilde{h}_{ij}, \tilde{\Phi}, B] = \int \mathcal{D}N^\mu \int \mathcal{D}h_{ij} \mathcal{D}\Phi \delta[\dot{N}^\mu - \chi^\mu] \Delta_\chi \exp(-I[g_{\mu\nu}, \Phi]) \qquad (4.7)$$

Here, the delta-functional enforces the gauge-fixing condition $\dot{N}^\mu = \chi^\mu$ and Δ_χ is the associated Faddeev-Popov determinant. The lapse and shift N^μ are unrestricted at the endpoints. The three-metric and matter field are integrated over a class of paths $(h_{ij}(\mathbf{x}, \tau), \Phi(\mathbf{x}, \tau))$ with the restriction that they match the argument of the wave function on the three-surface B, which may be taken to be the surface $\tau = 1$. That is,

$$h_{ij}(\mathbf{x}, 1) = \tilde{h}_{ij}(\mathbf{x}), \quad \Phi(\mathbf{x}, 1) = \tilde{\Phi}(\mathbf{x}) \qquad (4.8)$$

To complete the specification of the class of paths one also needs to specify the conditions satisfied at the initial point, $\tau = 0$ say.

The expression, "Euclidean path integral" should be taken with a very large grain of salt for the case of gravitational systems. One needs to work rather hard to give the expression (4.6) a sensible meaning. In particular, in addition to the usual issues associated with defining a functional integral over fields, one has to deal with the fact that

the gravitational action is not bounded from below. This means that the path integral will not converge if one integrates over real Euclidean metrics. Convergence is achieved only by integrating along a complex contour in the space of complex four-metrics. The sum is therefore over complex metrics and is not even equivalent to a sum over Euclidean metrics in any sense. Furthermore, there is generally no unique contour and the outcome of evaluating the path integral could depend rather crucially on which complex contour one chooses. We will have more to say about this later on.

As we have already noted, the Wheeler-DeWitt equation and momentum constraints, (4.2), (4.3) are normally thought of as a quantum expression of invariance under four-dimensional diffeomorphisms. One ought to be able to see the analagous thing in the path integral, and in fact one can. The wave functions generated by the path integral (4.7) may formally be shown to satisfy the Wheeler-DeWitt equation and momentum constraints, providing that the path integral is constructed in an *invariant* manner. This means that the action, measure, and class of paths summed over should be invariant under diffeomorphisms (Halliwell and Hartle, 1990).

Which solution to the Wheeler-DeWitt equation is generated by the path integral will depend on how the initial conditions on the paths summed over are chosen, and how the contour of integration is chosen; thus the question of boundary conditions on the wave function in canonical quantization appears in the path integral as the question of choosing a contour and choosing a class of paths. No precise relationship is known, however.

Interpretation

To complete this discussion of the general formalism of quantum cosmology, a few words on interpretation are in order. Hartle has covered the basic ideas involved in interpreting the wave function. Here, I am just going to tell you how I am going to interpret the wave function without trying to justify it. The basic idea is that we are going to regard a strong peak in the wave function, or in a distribution constructed from the wave function, as a prediction. If no such peaks may be found, then we make no prediction. This will be sufficient for our purposes. References to the vast literature on this subject are given in Section 13.

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5. MINISUPERSPACE – GENERAL THEORY

Since superspace, the configuration space one deals with in quantum cosmology, is infinite dimensional, the full formalism of quantum cosmology is very difficult to deal with in practice. In classical cosmology, because the universe appears to be homogeneous and isotropic on very large scales, one's considerations are largely restricted to the region of superspace in the immediate vicinity of homogeneity and isotropy. That is, one begins by studying homogeneous isotropic (or sometimes anisotropic) metrics and then goes on to consider small inhomogeneous perturbations about them. In quantum cosmology one does the same. To be precise, one generally begins by considering a class of models in which all but a finite number of degrees of freedom of the metric and matter fields are "frozen" or "suspended". This is most commonly achieved by restricting the fields to be homogeneous. Such models are known as "minisuperspace" models and are characterized by the fact that their configuration space, minisuperspace, is finite dimensional. One is thus dealing with a problem of quantum mechanics, not of field theory. A very large proportion of the work done in quantum cosmology has concentrated on models of this type.

Clearly in the quantum theory there are considerable difficulties associated with the restriction to minisuperspace. Setting most of the field modes and their momenta to zero identically violates the uncertainty principle. Moreover, the restriction to minisuperspace is not known to be part of a systematic approximation to the full theory. At the humblest level, one can think of minisuperspace models not as some kind of approximation, but rather, as toy models which retain certain aspects of the full theory, whilst avoiding others, thereby allowing one to study certain features of the full theory in isolation from the rest. However, in these lectures we are interested in cosmological predictions. I am therefore going to take the stronger point of view that these models *do* have something to do with the full theory. In what follows I will therefore try to emphasize what aspects of minisuperspace models may be argued to transcend the restrictions to minisuperspace. We will return to the question of the validity of the minisuperspace "approximation" later on.

The simple model of the previous section was of course a minisuperspace model, in that we restricted the metric and matter field to be homogeneous and isotropic. More generally, minisuperspace usually involves the following: in the four-metric (3.1), the lapse is taken to be homogeneous, $N = N(t)$, and the shift is set to zero, $N^i = 0$, so that one

has

$$ds^2 = -N^2(t)dt^2 + h_{ij}(\mathbf{x}, t)dx^i dx^j \tag{5.1}$$

Most importantly, the three-metric h_{ij} is restricted to be homogeneous, so that it is described by a finite number of functions of t, $q^\alpha(t)$ say, where $\alpha = 0, 1, 2 \cdots (n-1)$. Some examples of possible ways in which the three-metric may be restricted are given below.

One could take a Roberston-Walker metric as we did in Section 2,

$$h_{ij}(\mathbf{x}, t)dx^i dx^j = a^2(t)d\Omega_3^2 \tag{5.2}$$

Here, $d\Omega_3^2$ is the metric on the three-sphere, and $q^\alpha = a$. One could take an anisotropic metric with spatial sections of topology $S^1 \times S^2$,

$$h_{ij}(\mathbf{x}, t)dx^i dx^j = a^2(t)dr^2 + b^2(t)d\Omega_2^2 \tag{5.3}$$

Here, $d\Omega_2^2$ is the metric on the two-sphere, r is periodically identified, and $q^\alpha = (a, b)$. More generally, one could consider Bianchi-type metrics,

$$h_{ij}(\mathbf{x}, t)dx^i dx^j = a^2(t)(e^\beta)_{ij}\sigma^i \sigma^j \tag{5.4}$$

Here, the σ^i are a basis of one-forms and the q^α consist of the scale factor a and the various components of the matrix β, which describe the degree of anisotropy. Many more models are cited in Section 13.

In terms of the variables describing the $(3+1)$ decomposition of the four-metric, (3.1), the Einstein action with cosmological constant (3.2) is

$$S[h_{ij}, N, N^i] = \frac{m_p^2}{16\pi} \int dt d^3x N h^{\frac{1}{2}} \left[K_{ij}K^{ij} - K^2 + {}^3R - 2\Lambda \right] \tag{5.5}$$

On inserting the restricted form of the metric described above one generally obtains a result of the form

$$S[q^\alpha(t), N(t)] = \int_0^1 dt N \left[\frac{1}{2N^2} f_{\alpha\beta}(q)\dot{q}^\alpha \dot{q}^\beta - U(q) \right] \equiv \int L dt \tag{5.6}$$

Here, $f_{\alpha\beta}(q)$ is the reduced version of the DeWitt metric, (3.6), and has indefinite signature, $(-, +, +, +...)$. The range of the t integration may be taken to be from 0 to 1 by shifting t and by scaling the lapse function. The inclusion of matter variables, restricted in some way, also leads to an action of this form, so that the q^α may include matter variables

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as well as three-metric components. The $(-)$ part of the signature in the metric always corresponds to a gravitational variable, however.

Restricting to a metric of the form (5.1) is not the only way of obtaining a minisuperspace model. Sometimes it will be convenient to scale the lapse by functions of the three-metric. Alternatively, one may wish to consider not homogeneous metrics, but inhomogeneous metrics of a restricted type, such as spherically symmetric metrics. Or, one may wish to use a higher-derivative action in place of (5.5). In that case, the action can always be reduced to first order form by the introduction of extra variables (*e.g.* $Q = \ddot{a}$, etc.). One way or another, one always obtains an action of the form (5.6). We will therefore take this action to be the defining feature of minisuperspace models. So from here onwards, our task is to consider the quantization of systems described by an action of the form (5.6).

The action (5.6) has the form of that for a relativistic point particle moving in a curved space-time of n dimensions with a potential. Varying with respect to q^α one obtains the field equations

$$\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{q}^\alpha}{N}\right) + \frac{1}{N^2}\Gamma^\alpha_{\beta\gamma}\dot{q}^\beta\dot{q}^\gamma + f^{\alpha\beta}\frac{\partial U}{\partial q^\beta} = 0 \qquad (5.7)$$

where $\Gamma^\alpha_{\beta\gamma}$ is the usual Christoffel connection constructed from the metric $f_{\alpha\beta}$. Varying with respect to N one obtains the constraint

$$\frac{1}{2N^2}f_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + U(q) = 0 \qquad (5.8)$$

These equations describe geodesic motion in minisuperspace with a forcing term.

It is important to note that the general solution to (5.7), (5.8) will involve $(2n - 1)$ arbitrary parameters.[†]

For consistency, (5.7) and (5.8) ought to be equivalent, respectively, to the 00 and ij components of the full Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi}{m_p^2}T_{\mu\nu} \qquad (5.9)$$

This is not, however, guaranteed. Inserting an ansatz for the metric into the action and then taking variations to derive the minisuperspace field equations does not necessarily

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[†] As in the model of Section 2, one of the parameters will be t_0, the origin of unobservable parameter time, so effectively one has $(2n - 2)$ physically relevant parameters.

yield the same field equations as are obtained by inserting the minisuperspace ansatz directly into (5.9). Our analysis is therefore restricted to minisuperspace models for which these two paths to the field equation give the same result. This excludes, for example, metrics in Bianchi Class B, but does include a suffcent number of interesting examples. When studying minisuperspace models, one should always check that the acts of taking variations and inserting an ansatz commute (and also that the $0j$ components of (5.9) are trivially satisfied).

The Hamiltonian is found in the usual way. One first defines canonical momenta

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} = f_{\alpha\beta} \frac{\dot{q}^\beta}{N} \tag{5.10}$$

and the canonical Hamiltonian is

$$H_c = p_\alpha \dot{q}^\alpha - L = N \left[\frac{1}{2} f^{\alpha\beta} p_\alpha p_\beta + U(q) \right] \equiv NH \tag{5.11}$$

where $f^{\alpha\beta}(q)$ is the inverse metric on minisuperspace. The Hamiltonian form of the action is

$$S = \int_0^1 dt \, [p_\alpha \dot{q}^\alpha - NH] \tag{5.12}$$

This indicates that the lapse function N is a Lagrange multiplier enforcing the Hamiltonian constraint

$$H(q^\alpha, p_\alpha) = \frac{1}{2} f^{\alpha\beta} p_\alpha p_\beta + U(q) = 0 \tag{5.13}$$

This is equivalent to the Hamiltonian constraint of the full theory (3.8), integrated over the spatial hypersurfaces. The momentum constraint, (3.7), is usually satisfied identically by the minisuperspace ansatz (modulo the above reservations).

Canonical Quantization

Canonical quantization involves the introduction of a time-independent wave function $\Psi(q^\alpha)$ and demanding that it is annihilated by the operator corresponding to the classical constraint (5.13). This yields the Wheeler-DeWitt equation,

$$\hat{H}(q^\alpha, -i\frac{\partial}{\partial q^\alpha})\Psi(q^\alpha) = 0 \tag{5.14}$$

Because the metric $f^{\alpha\beta}$ depends on q there is a non-trivial operator ordering issue in (5.14). This may be partially resolved by demanding that the quantization procedure is covariant in minisuperspace; *i.e.* that is is unaffected by field redefinitions of the three-metric and matter fields, $q^{\alpha} \to \tilde{q}^{\alpha}(q^{\alpha})$. This narrows down the possible operator orderings to

$$\hat{H} = -\frac{1}{2}\nabla^2 + \xi I\!R + U(q) \tag{5.15}$$

where ∇^2 and $I\!R$ are the Laplacian and curvature of the minisuperspace metric $f_{\alpha\beta}$ and ξ is an arbitrary constant.

The constant ξ may be fixed once one recognises that the minisuperspace metric (and indeed, the full superspace metric (3.9)) is not uniquely defined by the form of the action or the Hamiltonian, but is fixed only up to a conformal factor. Classically the constraint (5.13) may be multiplied by an arbitrary function of q, $\Omega^{-2}(q)$ say, and the constraint is identical in form but has metric $\tilde{f}_{\alpha\beta} = \Omega^2 f_{\alpha\beta}$ and potential $\tilde{U} = \Omega^{-2}U$. The same is true in the action (5.6) or (5.12) if, in addition to the above rescalings, one also rescales the laspe function, $N \to \tilde{N} = \Omega^{-2}N$. Clearly the quantum theory should also be insensitive to such rescalings. This is achieved if the metric dependent part of the operator (5.15) is *conformally covariant*; *i.e.* if the coefficient ξ is taken to be the conformal coupling

$$\xi = -\frac{(n-2)}{8(n-1)} \tag{5.16}$$

for $n \geq 2$ (Halliwell, 1988a; Moss, 1988; Misner, 1970). In what follows, we will be working almost exclusively in the lowest order semi-classical approximation, for which these issues of operator ordering are in fact irrelevant. However, I have mentioned this partially for completeness, but also because one often studies models in which considerable simplifications arise by suitable lapse function rescalings and field redefinitions, and one might wonder whether or not these changes of variables affect the final results.

Path Integral Quantization

The wave function may also be obtained using a path integral. To discuss the path integral, we first need to discuss the symmetry of the action. The Hamiltonian constraint, (5.13), indicates the presence of a symmetry, namely reparametrization invariance. This

is the left-over of the general covariance of the full theory, after the restriction to minisuperspace. More precisely, under the transformations

$$\delta_\epsilon q^\alpha = \epsilon(t)\{q^\alpha, H\}, \quad \delta_\epsilon p_\alpha = \epsilon(t)\{p_\alpha, H\}, \quad \delta_\epsilon N = \dot{\epsilon}(t) \tag{5.17}$$

it is not difficult to show that the action changes by an amount

$$\delta S = \left[\epsilon(t) \left(p_\alpha \frac{\partial H}{\partial p_\alpha} - H \right) \right]_0^1 \tag{5.18}$$

The action is therefore unchanged if the parameter $\epsilon(t)$ satisfies the boundary conditions $\epsilon(0) = 0 = \epsilon(1)$. This symmetry may be completey broken by imposing a gauge-fixing condition of the form

$$G \equiv \dot{N} - \chi(p_\alpha, q^\alpha, N) = 0 \tag{5.19}$$

where χ is an arbitrary function of p_α, q^α, N.

We may now write down the path integral. It has the form

$$\Psi(q^{\alpha\prime\prime}) = \int \mathcal{D}p_\alpha \mathcal{D}q^\alpha \mathcal{D}N \delta[G] \Delta_G e^{iS[p,q,N]} \tag{5.20}$$

where $S[p, q, N]$ is the Hamiltonian form of the action (5.12) and Δ_G is the Faddeev-Popov measure associated with the gauge-fixing condition (5.19), and guarantees that the path integral is independent of the choice of gauge-fixing function G. The integral is taken over a set of paths $(q^\alpha(t), p_\alpha(t), N(t))$ satisfying the boundary condition $q^\alpha(1) = q^{\alpha\prime\prime}$ at $t = 1$ with p_α and N free, and some yet to be specified conditions at $t = 0$.

The only really practical gauge to work in is the gauge $\dot{N} = 0$. Then it may be shown that $\Delta_G = constant.^\dagger$ The functional integral over N then reduces to a single ordinary integration over the constant N. One thus has

$$\Psi(q^{\alpha\prime\prime}) = \int dN \int \mathcal{D}p_\alpha \mathcal{D}q^\alpha e^{iS[p,q,N]} \tag{5.21}$$

Eq.(5.21) has a familiar form: it is the integral over all times N of an ordinary quantum mechanical propagator, or wave function,

$$\Psi(q^{\alpha\prime\prime}) = \int dN \psi(q^{\alpha\prime\prime}, N) \tag{5.22}$$

$\dagger$ This is easily seen: Δ_G is basically the determinant of the operator $\delta_\epsilon G/\delta\epsilon$. In the gauge $\dot{N} = 0$, this is the operator d^2/dt^2, which has constant determinant.

where $\psi(q^{\alpha\prime\prime}, N)$ satisfies the time-dependent Schrödinger equation with time coordinate N. From Eq.(5.22), it is readily shown that the wave function generated by the path integral satisfies the Wheeler-DeWitt equation. Suppose we operate on (5.22) with the Wheeler-DeWitt operator at $q^{\alpha\prime\prime}$. Then, using the fact that the integrand satisfies the Schrödinger equation, one has

$$\hat{H}''\Psi(q^{\alpha\prime\prime}) = \int dN i \frac{\partial \psi}{\partial N} = i \left[\psi(q^{\alpha\prime\prime}, N)\right]_{N_1}^{N_2} \tag{5.23}$$

where N_1, N_2 are the end-points of the N integral, about which we have so far said nothing. Clearly for the wave function to satisfy the Wheeler-DeWitt equation we have to *choose* the end-points so that the right-hand side of (5.23) vanishes. N is generally integrated along a contour in the complex plane. This contour is usually taken to be infinite, with $\psi(q^{\alpha\prime\prime}, N)$ going to zero at the ends, or closed *i.e.* $N_1 = N_2$. In both of these cases, the right-hand side of (5.23) vanishes and the wave function so generated satisfies the Wheeler-DeWitt equation. (In the closed contour case, attention to branch cuts may be needed.) Note that these ranges are invariant under reparametrizations of N. They would not be if the contour had finite end-points and the right-hand side would then not be zero. This is an illustration of the remarks in Section 4 concering the relationship between the Wheeler-DeWitt equation and the invariance properties of the path integral.

The representation (5.21) of the wave function is of considerable practical value in that it can actually be used to evaluate the wave function directly. But first, one normally rotates to Euclidean time, $\tau = it$. After integrating out the momenta, the resulting Euclidean functional integral has the form

$$\Psi(q^{\alpha\prime\prime}) = \int dN \int \mathcal{D}q^{\alpha} \exp\left(-I[q^{\alpha}(\tau), N]\right) \tag{5.24}$$

Here, I is the minisuperspace Euclidean action

$$I[q^{\alpha}(\tau), N] = \int_0^1 d\tau N \left[\frac{1}{2N^2} f_{\alpha\beta}(q)\dot{q}^{\alpha}\dot{q}^{\beta} + U(q)\right] \tag{5.25}$$

Although the part of this action which corresponds to the matter modes is always positive definite, the gravitational part is not. Recall that the minisuperspace metric has indefinite signature, the $(-)$ part corresponding to the conformal part of the three-metric, so the kinetic term is indefinite. Also, the potential, which is the integral of $2\Lambda - {}^3R$, is not positive definite. So complex integration contours are necessary to give meaning to (5.24).

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Here, however, we will work largely in the lowest order semi-classical approximation, which involves taking the wave function to be (a sum of terms) of the form $e^{-I_{cl}}$, where I_{cl} is the action of the classical solution $(q^\alpha(\tau), N)$ satisfying the prescribed boundary conditions. This solution may in fact be complex, and indeed will need to be if the wave function is to be oscillatory. Similarly, when working with the Wheeler-DeWitt equation, we will work largely in the WKB approximation, in which solutions of the above type are sought.

We are now in a position to comment on the validity of the minisuperspace "approximation". Providing we are sufficiently careful in making our minisuperspace ansatz, the classical solutions $(q^\alpha(\tau), N)$ will be solutions to the full field equations, and thus I_{cl} will be the action of a solution to the full Einstein equations. The lowest order semi-classical approximation to the minisuperspace wave function therefore coincides with the lowest order semi-classical approximation to the wave function of the full theory. This means that minisuperspace does give some indication as to what is going on in the full theory as long as we remain close to the lowest order semi-classical approximation.

The Probability Measure

Given a wave function $\Psi(q^\alpha)$ for a minisuperspace model we need to construct from it a probability measure with which to make predictions. The question is, which probability measure do we use? The Wheeler-DeWitt equation is a Klein-Gordon type equation. It therefore has associated with it a conserved current

$$J = \frac{i}{2}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) \tag{5.26}$$

It satisfies

$$\nabla \cdot J = 0 \tag{5.27}$$

by virtue of the Wheeler-DeWitt equation. Like the Klein-Gordon equation, however, the probability measure constructed from the conserved current can suffer from difficulties with negative probabilities. For this reason, some authors have suggested that the correct measure to use is

$$dP = |\Psi(q^\alpha)|^2 dV \tag{5.28}$$

where dV is a volume element of minisuperspace. However, this is also problematic, in that one of the coordinates q^{α} is, in some crude sense, "time", so that (5.28) is the analogue of interpreting $|\Psi(x,t)|^2$ in ordinary quantum mechanics as the probability of finding the particle in the space-*time* interval $dxdt$. One could conceivably make sense out of (5.28), but not before a careful discussion of the nature of time in ordinary quantum mechanics.[†]

For the moment we will not commit ourselves to either of these possibilities, but will keep each one in mind. We will just look for peaks in the wave function itself when asking for predictions. If the peak is sufficiently strong, one would expect any sensible measure constructed from the wave function to have the same peak.

6. CLASSICAL SPACETIME

We have described in the previous section two ways of calculating the wave function for minisuperspace models: the Wheeler-DeWitt equation and the path integral. Before going on to the evaluation of the wave function, it is appropriate to ask what sort of wave functions we are hoping to find. If the wave function is to correctly describe the late universe, then it must predict that spacetime is classical when the universe is large. The first question to ask, therefore, is "What, in the context of quantum cosmology, constitutes a prediction of classical spacetime?".

There are at least two requirements that must be satisfied before a quantum system may be regarded as classical:

1. The wave function must predict that the canonical variables are strongly correlated according to classical laws; *i.e.* the wave function (or some distribution constructed from it) must be strongly peaked about one or more classical configurations

2. The quantum mechanical interference between distinct such configurations should be negligible; *i.e.* they should *decohere.*

To exemplify both of these requirements, let us first consider a simple example from ordinary quantum mechanics. There, the most familiar wave functions for which the first

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[†] This line of thought has been pursued by numerous authors, including Caves (1986, 1987), Hartle (1988b) and Page(1989b).

requirement is satisfied are *coherent states*. These are single wave packets strongly peaked about a single classical trajectory, $\bar{x}(t)$, say. For example, for the simple harmonic oscillator, the coherent states are of the form

$$\psi(x,t) = e^{ipx} \exp\left(-\frac{(x - \bar{x}(t))^2}{\sigma^2}\right) \tag{6.1}$$

On being presented with a solution to the Schrodinger equation of this type, one might be tempted to say that it predicts classical behaviour, in that on measuring the position of the particle at a sequence of times, one would find it to be following the trajectory $\bar{x}(t)$. Suppose, however, one is presented with a solution to the Schrödinger equation which is a superposition of many such states:

$$\psi(x,t) = \sum_n c_n e^{ip_n x} \exp\left(-\frac{(x - \bar{x}_n(t))^2}{\sigma^2}\right) \tag{6.2}$$

where the $\bar{x}_n(t)$ are a set of distinct classical solutions. One might be tempted to say that this wave function corresponds to classical behaviour, and that one would find the particle to be following the classical trajectory $\bar{x}_n(t)$ with probability $|c_n|^2$. The problem, however, is that these wave packets may meet up at some stage in the future and interfere. One could not then say that the particle was following a definite classical trajectory. To ascribe a definite classical history to the particle, the interference between distinct states has to be destroyed. The way in which this may be achieved is a fascinating subject in itself, but we will say little about it here. We will concentrate mainly on the first requirement for classical behaviour.

Turn now to quantum cosmology. One might at first think that, in the search for the emergence of classical behaviour, the natural thing to do there is to try and construct the analogue of coherent states. This is rather hard to do, but has been achieved for certain simple models. Because the wave function does not depend on time explicitly, the analogue of coherent states are wave functions of the form

$$\Psi(q^\alpha) = e^{i\phi(q^\alpha)} \exp\left(-f^2(q^\alpha)\right) \tag{6.3}$$

where $f(q^\alpha) = 0$ is the equation of a single classical trajectory in minisuperspace. So in a two-dimensional model, for example, the wave function will consist of a sharply peaked ridge in minisuperspace along a single classical trajectory.

Wave functions of this type do not arise very naturally in quantum cosmology because they need very special boundary conditions. However, they do highlight a particular feature typical of wave functions in quantum cosmology that predict classical spacetime: they are peaked about an entire history. Moreover, although the original wave function does not carry a particular label playing the role of time, a notion of time may emerge for certain types of wave functions, such as (6.3): it is basically the affine parameter along the histories about which the wave function is peaked (*i.e.* the distance along the ridge in the case of (6.3)). So time, and indeed spacetime, are only derived concepts appropriate to certain regions of configuration spacetime and contingent upon initial conditions. The decoherence requirement may also be achieved in quantum cosmology, but this is rather complicated and will not be covered here.

As we have seen in the simple model of Section 2, the sort of wave functions most commonly arising in quantum cosmology are not of wavepacket form, but are of WKB form, and may be broadly classified as oscillatory, of the form e^{iS}, or exponential, of the form e^{-I}. It is the oscillatory wave functions that correspond to classical spacetime, whilst the exponential ones do not. Let us discuss why this is so.

Recall that the way we are interpreting the wave function is to regard a strong peak in the wave function, or in a distribution constructed from it, as a prediction. Classical spacetime, therefore, is predicted when the wave function, or some distribution constructed from it, becomes strongly peaked about one or more classical configurations. How do we identify such peaks? In the most general case, the wave function will be peaked not about some region of configuration space – e^{iS} is most certainly not – but about some correlation between coordinates and momenta. Perhaps the most transparent way of identifying such correlations is to introduce a quantum mechanical distribution function which depends on both coordinates and momenta, $F(p, q)$. The Wigner function is such a distribution function, and turns out to be very useful in quantum cosmology for identifying the correlations present in a given wave function. However, this would take rather a long time to explain. Here I will just report the result that the Wigner function shows that (i) a wave function of the form e^{-I} predicts no correlation between coordinates and momenta, and so cannot correspond to classical behaviour; and (ii) a wave function of the form e^{iS} predicts a strong correlation between p and q of the form

$$p_\alpha = \frac{\partial S}{\partial q^\alpha} \tag{6.4}$$

S is generally a solution to the Hamilton-Jacobi equation and, as we will demonstrate in detail below, (6.4) is then a first integral of the equations of motion. It thus defines a set of solutions to the field equations. A wave function of the form e^{iS}, therefore, is normally thought of a being peaked about not a single classical solution, but about a *set* of solutions to the field equations. It is in this sense that it corresponds to classical spacetime.

Given the peak about the correlation (6.4) for wave functions of the form e^{iS}, it may now be explicitly verified using a canonical transformation. For simplicity consider the one-dimensional case. A canonical transformation from (p, q) to $(\tilde{p}, \tilde{q})$ may be generated by a generating function $G_0(q, \tilde{p})$:

$$p = \frac{\partial G_0}{\partial q}, \quad \tilde{q} = \frac{\partial G_0}{\partial \tilde{p}} \tag{6.5}$$

In quantum mechanics, the transformation from the wave function $\Psi(q)$ to a new wave function $\tilde{\Psi}(\tilde{p})$ is given by

$$\tilde{\Psi}(\tilde{p}) = \int dq \, e^{-iG(q,\tilde{p})} \Psi(q) \tag{6.6}$$

Here, the generating function $G(q, \tilde{p})$ is not actually quite the same as $G_0(q, \tilde{p})$ above, but agrees with it to leading order in Planck's constant. Suppose $\Psi(q) = e^{iS(q)}$. Then a transformation to new variables

$$\tilde{p} = p - \frac{\partial S}{\partial q}, \quad \tilde{q} = q \tag{6.7}$$

may be achieved using the generating function $G_0(q, \tilde{p}) = q\tilde{p} + S(q)$. Inserting this in (6.6), it is easily seen that the wave function as a function of $\tilde{p}$ is of the form

$$\tilde{\Psi}(\tilde{p}) = \delta(\tilde{p}) \tag{6.8}$$

to leading order. As advertised, it is therefore strongly peaked about the configuration (6.4).

It is sometimes stated that wave functions of the form e^{-I} are not classical because they correpond to a Euclidean spacetime. It is certainly true that they are not classical, and it is certainly true that, if the wave function is a WKB solution, then I is the action of a classical Euclidean solution. However, this does not mean that they correspond to a Euclidean spacetime. In contrast to a wave function of the form e^{iS}, which is *peaked* about a set of classical Lorentzian solutions, a wave function e^{-I} is *not peaked* about a set

of Euclidean solutions. It is not classical quite simply because it fails to predict classical correlations between the Lorentzian momentum p and its conjugate q.

A much better way of discussing peaks in the wave function, or more generally, of discussing predictions arising from a given theory of initial conditions, is to use the path integral methods described by Hartle in his lectures (Hartle, 1990). Although conceptually much more satisfactory, they are somewhat cumbersome to use in practice. Moreover, they have not as yet been applied to any simple examples in quantum cosmology. For the moment it is therefore not inappropriate to employ the rather heuristic but quicker methods outlined above.

The General Behaviour of the Solutions

Having argued that classical spacetime is predicted, loosely speaking, when the wave function is oscillatory, our next task is to determine the regions of configuration space for which the wave function is oscillatory, and those for which the wave function is exponential. This will depend to some extent on boundary conditions, which we have not yet discussed, but one can get broad indications about the behaviour of the wave function by looking at the potential in the Wheeler-DeWitt equation So we are considering the Wheeler-DeWitt equation

$$\left[-\frac{1}{2}\nabla^2 + U(q)\right]\Psi(q) = 0 \tag{6.9}$$

Here, we have assumed that the curvature term has been absorbed into the potential. Compare (6.9) with the one-dimensional quantum mechanical problem

$$\left[\frac{d^2}{dx^2} + U(x)\right]\Psi(x) = 0 \tag{6.10}$$

In this case, one immediately sees that the wave function is exponential in the region $U << 0$ and oscillatory in the region $U >> 0$. The case of (6.9) is more complicated, however, in that there are n independent variables, and the metric has indefinite signature.

To investigate this in a little more detail, let us divide the minisuperspace coordinates q^α into a single "timelike" coordinate q^0 and $n-1$ "spacelike" coordinates $\mathbf{q}$. Then locally, the Wheeler-DeWitt equation will have the form

$$\left[\frac{\partial^2}{\partial q^{0^2}} - \frac{\partial^2}{\partial \mathbf{q}^2} + U(q^0, \mathbf{q})\right]\Psi(q) = 0. \tag{6.11}$$

The point now, is that the broad behaviour of the solution will depend not only on the sign of U, but also, loosely speaking, on whether it is the q^0-dependence of U or the $\mathbf{q}$-dependence of U that is most significant. More precisely, one has the following. Consider the surfaces of constant U in minisuperspace. They may be timelike or spacelike in a given region. First of all suppose that they are spacelike. Then in that local region, one can always perform a "Lorentz" rotation to new coordinates such that U depends only on the timelike coordinate in that region, $U \approx U(q^0)$. One can then solve approximately by separation of variables and, assuming one can go sufficiently far into the regions $U > 0$, $U < 0$ for the potential to dominate the separation constant, the solution will be oscillatory for $U >> 0$, exponential for $U << 0$. Similarly, in regions where the constant U surfaces are timelike, one may Lorentz-rotate to coordinates for which the potential depends only on the spacelike coordinates. The wave function is then oscillatory in the region $U << 0$ and exponential in the region $U >> 0$.

The above is only a rather crude way of getting an idea of the behaviour of the solutions. In particular, the assumptions about the separation constant need to be checked in particular cases, given the boundary conditions.

One may also determine the broad behaviour of the wave function by studying the path integral. In the Euclidean path integral representation of the wave function (5.24), one considers the propagation amplitude to a final configuration determined by the argument of the wave function, from an initial configuration determined by the boundary conditions. In the saddle-point approximation, the wave function is of the form $e^{-I_{cl}}$, where I_{cl} is the Euclidean action of the classical solution satisfying the above boundary conditions. Finding I_{cl} therefore involves the mathematical question of solving the Einstein equations as a boundary value problem. If the solution is real, it will have real action, and the wave function will be exponential. However, it appears to be most commonly the case for generic boundary data that no real Euclidean solution exists, and the only solutions are complex, with complex action. The wave function will then be oscillatory. The boundary value problem for the Einstein equations is actually a rather difficult mathematical problem about which very little appears to be known, in the general case.

In the minisuperspace case, qualitative information about the nature of the solution to the boundary value problem is readily obtained by inspecting the Euclidean version of the constraint equation (5.8). So for example, when looking for a solution between fixed values of q^α that are reasonably close together, one can see that the nature of the

solution depends not only on the sign of the potential, but also on whether the connecting trajectory is timelike or spacelike in minisuperspace.

The saddle-point appoximation to the path integral perhaps gives a more reliable indication than the Wheeler-DeWitt equation as to the broad behaviour of the wave function, in that the dependence on boundary conditions is more apparent.

At this stage it is appropriate to emphasize an important distinction between the above discussion and tunneling processes in ordinary quantum mechanics or field theory. In ordinary quantum mechanics or field theory, when considering tunneling at fixed energy, one has a constraint equation similar to (5.8), but with the important difference that its metric is *positive definite*. This has the consequence that at fixed energy, the configuration space is divided up into classically allowed and classically forbidden regions, and one can see immediately where they are by inspection of the potential in the constraint.

By contrast, for gravitational systems, the constraint (5.8) (or more generally, the Hamiltonian constraint (3.8)) has a metric of *indefinite* signature. This has the consequence that configuration space *is not* divided up into classically allowed and classically forbidden regions – the constraint alone does not rule out the existence of real Euclidean or real Lorentzian solutions in a given region of configuration space. One can only determine the nature of the solution (*i.e.* real Euclidean, real Lorentzian or complex) by solving the boundary value problem.

Further discussion of complex solutions and related issues may be found in Gibbons and Hartle (1989), Halliwell and Hartle (1989) and Halliwell and Louko (1989a, 1989b, 1990).

7. THE WKB APPROXIMATION

Having considered the general behaviour of the solutions to the Wheeler-DeWitt equation, we now go on to find the solutions more explicitly in the oscillatory region, using the WKB approximation. This will allow us to be more explicit in showing that, as we have already hinted a few times, the correlation (6.4) about which the wave function is peaked in the oscillatory region defines a set of solutions to the classical field equations.

We are interested in solving the Wheeler-DeWitt equation,

$$\left[-\frac{1}{2m_p^2} \nabla^2 + m_p^2 U(q) \right] \Psi(q) = 0 \tag{7.1}$$

For convenience, the Planck mass m_p has been reinstated, because we are going to use it as a large parameter in terms of which to do the WKB expansion. (If there is a cosmological constant in the problem one can sometimes use Λm_p^{-4} as a small parameter to control the WKB expansion, which has the advantage of being dimensionless.) Normally in the WKB approximation one looks for solutions that are strictly exponential or oscillatory, of the form e^{-I} or e^{iS}. However, in quantum cosmology one often uses the Wheeler-DeWitt equation hand-in-hand with the path integral. As noted above, in the saddle-point approximation to the path integral, one generally finds that the dominating saddle-points are four-metrics that are not real Euclidean, or real Lorentzian, but complex, with complex action. It is therefore most appropriate to look for WKB solutions to (7.1) of the form

$$\Psi(q) = C(q) e^{-m_p^2 I(q)} + O(m_p^{-2}) \tag{7.2}$$

where I and C are complex. Inserting (7.2) into (7.1) and equating powers of m_p, one obtains

$$-\frac{1}{2} (\nabla I)^2 + U(q) = 0 \tag{7.3}$$

$$2 \nabla I \cdot \nabla C + C \nabla^2 I = 0 \tag{7.4}$$

Here, ∇ denotes the covariant derivative with respect to q^α in the metric $f_{\alpha\beta}$, and the dot product is with respect to this metric. Let us split I into real and imaginary parts, $I(q) = I_R(q) - iS(q)$. Then the real and imaginary parts of (7.3) are

$$-\frac{1}{2} (\nabla I_R)^2 + \frac{1}{2} (\nabla S)^2 + U(q) = 0 \tag{7.5}$$

$$\nabla I_R \cdot \nabla S = 0 \tag{7.6}$$

Consider (7.5). We will return later to (7.4) and (7.6). We are interested in wave functions which correspond to classical spacetime. As we have discussed, to correspond to classical spacetime, the wave function should be of the form e^{iS} where S is a solution to the Lorentzian Hamilton-Jacobi equation,

$$\frac{1}{2} (\nabla S)^2 + U(q) = 0 \tag{7.7}$$

for then S defines an ensemble of classical trajectories, as will be shown in detail below. Evidently from (7.5) this is generally not the case for the S appearing in the wave function (7.2). However, if the imaginary part of I varies with q much more rapidly than the real part, *i.e.* if

$$|\nabla S| >> |\nabla I_R| \tag{7.8}$$

then it follows from (7.5) that S will be an *approximate* solution to the Lorentzian Hamilton-Jacobi equation, (7.7). Furthermore, the wave function (7.2) will then be predominantly of the form e^{iS} and, as we have already argued, it therefore indicates a strong correlation between coordinates and momenta of the form

$$p_\alpha = m_p^2 \frac{\partial S}{\partial q^\alpha} \tag{7.9}$$

Now we are in a position to show explicitly that (7.9) defines a first integral to the field equations. Clearly the momenta p_α defined by (7.9) satisfy the constraint (5.13), by virtue of the Hamilton-Jacobi equation, (7.7). To obtain the second order field equation, differentiate (7.7) with respect to q^γ. One obtains

$$\frac{1}{2} f^{\alpha\beta}_{,\gamma} \frac{\partial S}{\partial q^\alpha} \frac{\partial S}{\partial q^\beta} + f^{\alpha\beta} \frac{\partial S}{\partial q^\alpha} \frac{\partial^2 S}{\partial q^\beta \partial q^\gamma} + \frac{\partial U}{\partial q^\gamma} = 0 \tag{7.10}$$

The form of the second term in (7.10) invites the introduction of a vector

$$\frac{d}{ds} = f^{\alpha\beta} \frac{\partial S}{\partial q^\alpha} \frac{\partial}{\partial q^\beta} \tag{7.11}$$

When operated on q^γ it implies, via (7.9), the usual relationship between velocities and momenta, (5.10), provided that s is identified with the proper time, $ds = Ndt$. Using (7.11) and (7.9), (7.10) may now be written

$$\frac{dp_\gamma}{ds} + \frac{1}{2m_p^2} f^{\alpha\beta}_{,\gamma} p_\alpha p_\beta + m_p^2 \frac{\partial U}{\partial q^\gamma} = 0 \tag{7.12}$$

The field equation (5.7) is obtained after use of (5.10) and after raising the indices using the minisuperspace metric. We have therefore shown that the wave function (7.12), if it satisfies the condition (7.8), is strongly peaked about a *set* of solutions to the field equations, namely the set defined by the first integral (7.9).

Now we come to the most important point. For a given Hamilton-Jacobi function S, the solution to the first integral (7.9) will involve n arbitrary parameters. Recall, however,

that the general solution to the full field equations (5.7), (5.8) will involve $(2n-1)$ arbitrary parameters. The wave function is therefore strongly peaked about an n-parameter subset of the $(2n-1)$-parameter general solution. By imposing boundary conditions on the Wheeler-DeWitt equation a particular wave function is singled out. In the oscillatory region, this picks out a particular Hamilton-Jacobi function S. This in turn defines defines an n-parameter subset of the $(2n-1)$ parameter general solution. It is in this way that boundary conditions on the wave function of the universe effectively imply initial conditions on the classical solutions.

The Measure on the Set of Classical Trajectories

Suppose one now chooses an $(n-1)$-dimensional surface in minisuperspace as the beginning of classical evolution. Through (7.9), the wave function then effectively fixes the initial velocities on that surface. However, the wave function contains yet more information than just the initial velocities: it provides a probability measure on the set of classical trajectories about which the wave function is peaked. To see how this comes about, consider the remaining parts of the wave function, C and I_R. From the assumption, (7.8), (7.4) may be written

$$\nabla \cdot \left(|C|^2 \nabla S \right) = 0 \tag{7.13}$$

Moreover, we can combine this with (7.6) and write

$$\nabla \cdot \left(\exp(-2m_p^2 I_R)|C|^2 \nabla S \right) = 0 \tag{7.14}$$

This is a current conservation law,

$$\nabla \cdot J = 0 \tag{7.15}$$

where

$$J \equiv \exp(-2m_p^2 I_R)|C|^2 \nabla S \tag{7.16}$$

Loosely speaking, (7.15) implies that that the coefficient of ∇S in (7.16) provides a conserved measure on the set of classical trajectories about which the WKB wave function is peaked.

Eq.(7.16) is of course a special case of the Wheeler-DeWitt current

$$J = \frac{i}{2} \left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right) \tag{7.17}$$

which is conserved by virtue of the Wheeler-DeWitt equation (7.1), independently of any approximation. In suggesting that part of (7.16) provides a conserved probability measure on the set of classical trajectories, we are therefore aiming at using the conserved current as our probability measure. So now it is time to be precise about how the conserved current may successfully used, avoiding the difficulties with negative probabilities. The point is that it may be made to work only in the WKB approximation, although this is probably sufficient for all practical purposes. The following is based primarily on Vilenkin (1989) (see also Hawking and Page (1986) and Misner (1970, 1972)).

First we show how to construct a conserved measure. Consider a pencil B of the congruence of classical trajectories with tangent (co)vector ∇S, about which the wave function is peaked. Suppose it intersects an $(n-1)$-dimensional surface Σ_1 in $B \cap \Sigma_1$, and subsequently intersects a second surface Σ_2, in $B \cap \Sigma_2$. Now consider the volume V of minisuperspace swept out by the pencil of classical trajectories between the surfaces Σ_1 and Σ_2. Because $\nabla \cdot J = 0$, one may write

$$0 = \int_V dV \nabla \cdot J = \int_{\partial V} J \cdot dA \tag{7.18}$$

where dA is the element of area normal to the boundary of V. Since $J \cdot dA$ is non-zero only on the parts of the boundary of V consisting of the "ends", where the pencil B intersects Σ_1 and Σ_2, it follows that

$$\int_{B \cap \Sigma_1} J \cdot dA = \int_{B \cap \Sigma_2} J \cdot dA \tag{7.19}$$

This means that the flux of the pencil of trajectories across a hypersurface is in fact independent of the hypersurface. It suggests that we may use the quantity

$$dP = J \cdot d\Sigma \tag{7.20}$$

as a conserved probability measure on the set of classical trajectories with tangent vector ∇S, where Σ is some hypersurface that cuts across the flow.

Now we need to consider whether or not this definition of the probability measure gives positive probabilities. An intimately related problem is the choice of the surface Σ in (7.20).

In the case of the Klein-Gordon equation, one takes the surfaces Σ to be surfaces of constant physical time, $X^0 = constant$, and thus attempts to use J_0, the time-like

component of the current as a probability density. As is well-known, however, this may be negative. This is very significant in relativistic quantum mechanics, because it opens the way to the notion of antiparticles and second quantization.

The analagous thing to do in quantum cosmology would be to take the surfaces Σ to be surfaces of constant q^0, the timelike coordinate on minisuperspace. These would be surfaces for which the conformal part of the three-metric is constant. Once again one would find that the timelike component of J may be negative. However, this does not have the same significance as the Klein-Gordon case. It corresponds to the fact that in the classical theory one may have both expanding and collapsing universes. It is merely due to a bad choice of surfaces Σ, and does not oblige one to go to third quantization (the analogue of second quantization). For classical solutions which expand and recollapse, the flow will intersect a surface of constant conformal factor twice, in a different direction each time. What one really needs is a set of surfaces Σ which the flow intersects once and only once. For then the flux will pass through these surfaces always in the same direction and the probability (7.20) will be positive.

For a typical system in minisuperspace, the trajectories will generally go backwards and forwards in all coordinates q^α. However, merely by inspection of a typical congruence of classical paths, it is easy to see that one can always find a set of surfaces which the trajectories cross only once and in the same direction. The probability measure (7.20) will then remain of the same sign all along the congruence of classical trajectories (see Fig.3.).

One particularly simple choice that one might at first think of are the surfaces of constant S, which are clearly orthogonal to the congruence of classical trajectories. These do indeed provide a good set of surfaces for substantial regions of minisuperspace. However, this choice breaks down when the trajectories approach the surface $U(q) = 0$. Since both J and $d\Sigma$ are proportional to ∇S, dP is proportional to $(\nabla S)^2$, which vanishes at $U = 0$ by virtue of the Hamilton-Jacobi equation. But apart from this restriction there is still considerable freedom in the choice of surfaces Σ.

So we have seen that a sensible probability measure *can* be constructed from the conserved current, by suitable choice of the surfaces Σ. The important point to note, however, is that *it works only in the semi-classical regime*, when the wave function is of WKB form (7.2), subject to the condition (7.8).

Some words are in order on how the probability density (7.20) is to be used. It should

not be thought of as an *absolute* probability on the entire surface Σ. That is, it cannot be used to find the absolute probability that the universe will start out at some part of the surface Σ. Indeed, it is not clear that it is normalizable over a surface Σ stretching right out to the boundary of superspace; *i.e.* one would not expect to be able to write

$$\int_{\Sigma} J \cdot dA = 1 \tag{7.21}$$

unless very special boundary conditions were imposed on the wave function. Rather, (7.20) should be used to compute *conditional* probabilities. Such probabilities are used when answering questions of the type, "Given that the universe starts out in some finite subset s_1 of Σ, what is the probabilility that it will start out in the subset s_0 of s_1?". This conditional probability would be given by an expression of the form

$$P(s_0|s_1) = \frac{\int_{s_0} J \cdot dA}{\int_{s_1} J \cdot dA} \tag{7.22}$$

Each integral is finite because the domains of integration s_0, s_1 are finite, and the integrand will typically be bounded on these domains. The theory makes a prediction when conditional probabilities of this type are close to zero or one.

Finally, it should be noted that there is a certain element of circularity in our use of the conserved current as the probability measure. We have shown that the conserved current can provide a sensible probability measure in the semi-classical approximation. Beyond that it seems unlikely that it can be made to work. The problem, however, is that strictly speaking one really needs a probability measure in the first place to say what one means by "semi-classical", and to say that a given wave function is peaked about a given configuration. The resolution to this apparent dilemma is to use the measure $|\Psi|^2 \, dV$ from the very beginning, without any kind of approximations, and it is in terms of this that one dicusses the notion of semiclassical, and the peaking about classical trajectories. One may then apply this measure to non-zero volume regions consisting of slightly "thickened" $(n-1)$-dimensional hypersurfaces intersecting the classical flow. With care, it is then in fact possible to recover the probability measure $J \cdot d\Sigma$ discussed above, but only in the semi-classical approximation.

Let me now summarize this rather lengthy discussion of classical spacetime and the WKB approximation. In certain regions of minisuperspace, and for certain boundary conditions, the Wheeler-DeWitt equation will have solutions of the WKB form (7.2), for

which (7.8) holds. These solutions correspond to classical spacetime in that they are peaked about the set of solutions to the classical field equations satisfying the first integral (7.9). These classical solutions consist of a congruence of trajectories in minisuperspace with tangent vector ∇S. One may think of the wave function as imposing initial conditions on the velocities on some hypersurface Σ cutting across the flow of S. In addition, the quantity $J \cdot d\Sigma$ may be used as a probability measure on this surface; that is, it may be used to compute conditional probabilities that the universe will start out in some region of the surface Σ.

We will see how this works in detail in an example in the following sections.

8. BOUNDARY CONDITION PROPOSALS

Throughout the course of these lectures I have tried to emphasize the importance of boundary or initial conditions in quantum cosmology, although nothing we have done so far depends on a particular choice of boundary conditions. Now we come to discuss particular proposals and investigate their consequences.

A quantum theory of initial conditions involves selecting just one wave function of the universe from amongst the many that the dynamics allows; *i.e.* choosing a particular solution to the Wheeler-DeWitt equation. Numerous proposals have been made over the years. As long ago as 1967, DeWitt expressed a hope that mathematical consistency alone would lead to a unique solution to the Wheeler-DeWitt equation (DeWitt, 1967). Regretfully such a hope does not appear to have been realized. More recently, workers in the field have contented themselves with offering proposals motivated by considerations of simplicity, naturalness, analogies with simple quantum mechanical sytems etc. Here, we will concentrate on just two recent proposals that are the most comprehensive and the most studied. These are the "no-boundary" proposal of Hartle and Hawking (Hawking 1982, 1984a; Hartle and Hawking, 1983) and the "tunneling" boundary condition due primarily to Vilenkin and to Linde (Vilenkin, 1982, 1983, 1984, 1985a, 1985b, 1986, 1988; Linde, 1984a, 1984b, 1984c).

It should be stated at the outset that all known proposals for boundary conditions in quantum cosmology may be criticised on the grounds of lack of generality of lack of

precision, and these two are no exception. The issue of proposing a sensible theory of initial conditions which completely specifies a unique wave function of the universe for all conceivable situations, is to my mind still an open one.

The No-Boundary Proposal

The no-boundary proposal of Hartle and Hawking is expressed in terms of a Euclidean path integral. Before stating it, recall that a wave function $\Psi[\tilde{h}_{ij}, \tilde{\Phi}, B]$ satisfying the Wheeler-DeWitt equation and the momentum constraint may be generated by a path integral of the form

$$\Psi[\tilde{h}_{ij}, \tilde{\Phi}, B] = \sum_{M} \int \mathcal{D}g_{\mu\nu} \mathcal{D}\Phi \exp(-I[g_{\mu\nu}, \Phi]) \tag{8.1}$$

The sum is over manifolds M which have B as part of their boundary, and over metrics and matter fields $(g_{\mu\nu}, \Phi)$ on M matching the arguments of the wave function on the three-surface B. When M has topology $I\!R \times B$, this path integral has the form

$$\Psi[\tilde{h}_{ij}, \tilde{\Phi}, S] = \int \mathcal{D}N^{\mu} \int \mathcal{D}h_{ij} \mathcal{D}\Phi \delta[\dot{N}^{\mu} - \chi^{\mu}] \Delta_{\chi} \exp(-I[g_{\mu\nu}, \Phi]) \tag{8.2}$$

The lapse and shift N^{μ} are unrestricted at the end-points. The three-metric and matter field are integrated over a class of paths $(h_{ij}(\mathbf{x}, \tau), \Phi(\mathbf{x}, \tau))$ with the restriction that they match the argument of the wave function on the three-surface B, which may be taken to be the surface $\tau = 1$. That is,

$$h_{ij}(\mathbf{x}, 1) = \tilde{h}_{ij}(\mathbf{x}), \quad \Phi(\mathbf{x}, 1) = \tilde{\Phi}(\mathbf{x}) \tag{8.3}$$

To complete the specification of the class of paths one also needs to specify the conditions satisfied at the initial point, $\tau = 0$ say.

The no-boundary proposal of Hartle and Hawking is an essentially toplogical statement about the class of histories summed over. To calculate the no-boundary wave function, $\Psi_{NB}[\tilde{h}_{ij}, \tilde{\Phi}, B]$, we are instructed to regard the three-surface B as the *only* boundary of a compact four-manifold M, on which the four-metric is $g_{\mu\nu}$ and induces $\tilde{h}_{ij}$ on S, and the matter field configuration is Φ and matches the value $\tilde{\Phi}$ on S. We are then instructed to peform a path integral of the form (8.1) over all such $g_{\mu\nu}$ and Φ and over all such M (see Fig.4.).

For manifolds of the form $I\!R \times B$, the no-boundary proposal in principle tells us what conditions to impose on the histories $(h_{ij}(\mathbf{x}, \tau), \Phi(\mathbf{x}, \tau))$ at the initial point $\tau = 0$ in the path integral (8.2). Loosely speaking, one is to choose initial condition ensuring the closure of the four-geometry. However, although the four-dimensional geometric picture of what is going on here is intuitively very clear, the initial conditions one needs to impose on the histories in the (3+1) picture are rather subtle. They basically involve setting the initial three-surface volume, $h^{\frac{1}{2}}$, to zero, but also involve conditions on the derivatives of the remaining components of the three-metric and the matter fields, which have only been given in certain special cases.[†]

There is a further issue concerning the contour of integration. As discussed earlier a complex contour of integration is necessary if the path integral is to converge. Although convergent contours are readily found, convergence alone does not lead one automatically to a unique contour, and the value of the wave function may depend, possibly quite crucially, on which contour one chooses. The no-boundary proposal does not obviously offer any guidelines as to which contour one should take.

Because of these difficulties of precision in defining the no-boundary wave function, I am going to allow myself considerable license in my interpretation of what this proposal actually implies for practical calculations.

As far as the closure conditions goes, the following is, I think, a reasonable approach to take for practical purposes. The point to note is that one rarely goes beyond the lowest order semi-classical approximation in quantum cosmology. That is, for all practical purposes, one works with a wave function of the form $\Psi = e^{-I_{cl}}$, where I_{cl} is the action of a (possibly complex) solution to the Euclidean field equations. The reason one does this is partly because of the difficulty of computing higher order corrections; but primarily, it is because our present understanding of quantum gravity is rather poor and if these models have any range of validity at all, they are unlikely to be valid beyond the lowest order semi-classical approximation. What this means is that in attempting to apply the no-boundary proposal, one need only concern oneself with the question of finding initial conditions

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[†] Some earlier statements of the Hartle-Hawking proposal also used the word "regular", *i.e.* demanded that the sum be over regular geometries and matter fields. This is surely inappropriate because in a functional integral over fields, most of the configurations included in the sum are not even continuous, let alone differentiable. They may, however, be regular at the saddle-points, and we will exploit this fact below.

that correspond to the no-boundary proposal at the *classical* level. In particular, we are allowed to impose regularity conditions on the metric and matter fields. To be precise, we will impose initial conditions on the histories which ensure that (i) the four-geometry closes, and (ii) the *saddle-points* of the functional integral correspond to metrics and matter fields which are *regular* solutions to the classical field equations matching the prescribed data on the bounding three-surface B. There is a lot more one could say about this, but these conditions will be sufficient for our purposes. For a more detailed discussion of these issues see Halliwell and Louko (1990) and Louko (1988b).

Consider next the contour of integration. Because we will only be working in the semiclassical approximation, we do not have to worry about finding convergent contours. Nevertheless, the contour becomes an issue for us if the solution to the Einstein equations satisfying the above boundary conditions is not unique. For then the path integral will have a number of saddle-points, each of which may contribute to the integral an amount of order $e^{-I_{cl}^k}$, where I_{cl}^k is the action of the solution corresponding to saddle-point k. Without choosing a contour and performing a detailed contour analysis it is unfortunately not possible to say which saddle-points will generally provide the dominant contributions. We therefore have no general guidelines to offer here.

We will see how exactly these issues arise in the simple example discussed below.

We now calculate the no-boundary wave function explicitly for the scalar field model described in Section 2. In the gauge $\dot{N} = 0$, the minisuperspace path integral for the no-boundary wave function is

$$\Psi_{NB}(\tilde{a}, \tilde{\phi}) = \int dN \int \mathcal{D}a \mathcal{D}\phi \exp\left(-I[a(\tau), \phi(\tau), N]\right)$$

where I is the Euclidean action for the scalar field model,

$$I = \frac{1}{2} \int_0^1 d\tau N \left[-\frac{a}{N^2} \left(\frac{da}{d\tau}\right)^2 + \frac{a^3}{N^2} \left(\frac{d\phi}{d\tau}\right)^2 - a + a^3 V(\phi) \right] \tag{8.4}$$

The Euclidean field equations may be written,[†]

$$\frac{1}{N^2 a} \frac{d^2 a}{d\tau^2} = -\frac{2}{N^2} \left(\frac{d\phi}{d\tau}\right)^2 - V(\phi) \tag{8.5}$$

[†] Because the path integral representation of the wave function involves an ordinary integral over N, not a functional integral, the constraint (8.7) does not immediately follow from extremizing the action (8.4) with respect to the variables integrated over. Rather, the saddle-point condition is $\partial I/\partial N = 0$, and one actually obtains the *integral over time* of

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$$\frac{1}{N^2}\frac{d^2\phi}{d\tau^2} + \frac{3}{Na}\frac{da}{d\tau}\frac{d\phi}{d\tau} - \frac{1}{2}V'(\phi) = 0 \tag{8.6}$$

$$\frac{1}{N^2}\left(\frac{da}{d\tau}\right)^2 - \frac{a^2}{N^2}\left(\frac{d\phi}{d\tau}\right)^2 - 1 + a^2V(\phi) = 0 \tag{8.7}$$

The integral (8.4) is taken over a class of paths $(a(\tau), \phi(\tau), N)$ satisfying the final condition

$$a(1) = \tilde{a}, \quad \phi(1) = \tilde{\phi} \tag{8.8}$$

and a set of initial conditions determined by the no-boundary proposal, discussed below. The constant N is integrated along a closed or infinite contour in the complex plane and is not restricted by the boundary conditions. We are interested only in the semi-classical approximation to the above path integral, in which the wave function is taken to be of the form

$$\Psi(\tilde{a}, \tilde{\phi}) = \exp(-I_{cl}(\tilde{a}, \tilde{\phi})) \tag{8.9}$$

(or possibly a sum of wave functions of this form). Here $I_{cl}(\tilde{a}, \tilde{\phi})$ is the action of the solution to the Euclidean field equations $(a(\tau), \phi(\tau), N)$, which satisfies the final condition (8.5) and, in accordance with the above interpretation of the no-boundary proposal, is regular and respects the closure condition.

Consider, then, the important issue of determining the initial conditions on the paths that correspond to the closure condition and ensure that the solution is regular. Consider first $a(\tau)$. The Euclidean four-metric is

$$ds^2 = N^2 d\tau^2 + a^2(\tau)d\Omega_3^2 \tag{8.10}$$

We want the four-geometry to close off in a regular way. Imagine making the three-sphere boundary smaller and smaller. Then eventually we will be able to smoothly close it off with flat space. Compare, therefore, (8.10) with the metric on flat space in spherical coordinates

$$ds^2 = dr^2 + r^2 d\Omega_3^2 \tag{8.11}$$

(8.7). The form of (8.7) as written is obtained once one realizes that the integrand is in fact constant, by virtue of the other two field equations, hence the integral sign may be dropped. However, writing the constraint *with* the integral over time highlights the fact that the field equations and constraint contain two functions and one constants worth of information. This is precisely the right amount of information to determine the two functions $(a(\tau), \phi(\tau))$ and the constant N in terms of the boundary data.

From this, one may see that for (8.10) to close off in a regular way as $a \to 0$, we must have

$$a(\tau) \sim N\tau, \quad as \quad \tau \to 0 \tag{8.12}$$

This suggests that the conditions that must be satisfied at $\tau = 0$ are

$$a(0) = 0, \quad \frac{1}{N}\frac{da}{d\tau}(0) = 1 \tag{8.13}$$

(8.13) are the conditions that are often stated in the literature. However, this is in general too many conditions. In general, we would not expect to be able to find a classical solution satisfying the boundary data of fixed a on the final surface, fixed a on the initial surface *and* and fixed $da/d\tau$ on the initial surface. We might of course be able to do this at the classical level, for certain special choices of boundary data, but such conditions could not be elevated to quantum boundary conditions on the full path integral. One of these condition must be dropped. Since the main requirement is that the geometry closes, let us drop the condition on the derivative and keep the condition that $a(0) = 0$. On the face of it, this seems to allow the possibility that the four-geometry may not close off in a regular fashion. Consider, however, the constraint equation (8.7). It implies that if the solution is to be regular, then $da/d\tau \to \pm 1$ as $a \to 0$. The regularity condition is therefore recovered when the constraint equation holds. This guarantees that the saddle-points will indeed be regular four-geometries, if we only impose $a(0) = 0$.

Now consider the scalar field $\phi(\tau)$. Consider the equation it satisfies, (8.6). It is not difficult to see that if the solution is to be regular as $a \to 0$, then $\phi(\tau)$ must satisfy the initial condition

$$\frac{d\phi}{d\tau}(0) = 0 \tag{8.14}$$

So the sole content of the no-boundary proposal, for this model, is the initial condition (8.14) and the condition $a(0) = 0$.

Our task is now to solve the field equations (8.5)-(8.7) for the solution $(a(\tau), \phi(\tau), N)$, subject to the boundary conditions (8.9), (8.14) and $a(0) = 0$, and then calculate the action[†] of the solution.

[†] The action (8.4) is the appropriate one when a and ϕ are fixed on both boundaries. If one wants to fix instead derivatives of the fields on the boundary, as (8.14) requires, then (8.4) must have the appropriate boundary terms added. The correct boundary term does in fact vanish in the case under consideration here, although this is a point that generally needs to be treated quite carefully.

For definiteness, let us assume that the potential $V(\phi)$ is of the chaotic type (*i.e.* U-shaped) and let us go to the large ϕ region at which $|V'/V| << 1$. It is not difficult to see that the approximate solution to the scalar field equation (8.6), subject to the boundary conditions (8.8), (8.14), is

$$\phi(\tau) \approx \tilde{\phi} \tag{8.15}$$

Similarly, the approximate solution to the second order equation for $a(\tau)$, (8.5), satisfying the boundary conditions $a(0) = 0$, $a(1) = \tilde{a}$, is

$$a(\tau) \approx \frac{\tilde{a}\sin(V^{\frac{1}{2}}N\tau)}{\sin(V^{\frac{1}{2}}N)} \tag{8.16}$$

Finally, we insert (8.15), (8.16) into the constraint (8.7) to obtain a purely *algebraic* equation for the lapse, N. It is

$$\sin^2(V^{\frac{1}{2}}N) = \tilde{a}^2 V \tag{8.17}$$

There are an infinite number of solutions to this equation. If $\tilde{a}^2 V < 1$, they are real, and are conveniently written

$$N = N_n^\pm \equiv \frac{1}{V^{\frac{1}{2}}}\left[(n + \frac{1}{2})\pi \pm \cos^{-1}(\tilde{a}V^{\frac{1}{2}})\right] \tag{8.18}$$

where $n = 0, \pm 1, \pm 2, ...$ and $\cos^{-1}(\tilde{a}V^{\frac{1}{2}})$ lies in its principal range, $(0, \pi/2)$. For the moment, we set $n = 0$. We will return later to the significance of the other values of n.

With $n = 0$, the solution for the lapse inserted into the solution for $a(\tau)$, (8.16), now reads

$$a(\tau) \approx \frac{1}{V^{\frac{1}{2}}}\sin\left[\left(\frac{\pi}{2} \pm \cos^{-1}(\tilde{a}V^{\frac{1}{2}})\right)\tau\right] \tag{8.19}$$

We now have the complete solution to the field equations subject to the above boundary conditions. It is (8.15), (8.19), together with the solution for the lapse (8.18). The action of the solution is readily calculated. It is

$$I_\pm = -\frac{1}{3V(\tilde{\phi})}\left[1 \pm \left(1 - \tilde{a}^2 V(\tilde{\phi})\right)^{3/2}\right] \tag{8.20}$$

It is not difficult to see that these two solutions represent the three-sphere boundary being closed off with sections of four-sphere. As expected, the action is negative. The $(-)/(+)$ sign corresponds to the three-sphere being closed of by less than/more than half of a four-sphere. The classical solution is therefore not unique.

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Because the classical solution is not unique, we are faced with the problem of which solution to take in the semi-classical approximation to the wave function. Naively, one might note that the (+) saddle-point has most negative action, and will therefore provide the dominant contribution. However, as briefly mentioned earlier, this depends on the contour of integration. One can only say that the (+) saddle-point provides the dominant contribution if the chosen integration contour in the path integral may be distorted into a steepest-descent contour along which the (+) saddle-point is the global maximum. In their original paper, Hartle and Hawking (1983) gave heuristic arguments, based on the conformal rotation, which suggest that the contour was such that it could not be distorted to pass through the (+) saddle-point and was in fact dominated by the (-) saddle-point. For the moment let us accept these arguments. They thus obtained the following semi-classical expression for the no-boundary wave function:

$$\Psi_{NB}(a, \phi) \approx \exp\left(\frac{1}{3V(\phi)}\left[1 - \left(1 - a^2 V(\phi)\right)^{3/2}\right]\right) \tag{8.21}$$

(where we have dropped the tildes, to avoid the notation becoming too cumbersome). (8.21) is indeed an approximate solution to the Wheeler-DeWitt equation for the model, (2.14), in the region $a^2 V(\phi) < 1$. Using the WKB matching procedure, it is readily shown that the corresponding solution in the region $a^2 V(\phi) > 1$ is

$$\Psi_{NB}(a, \phi) \approx \exp\left(\frac{1}{3V(\phi)}\right)\cos\left[\frac{1}{3V(\phi)}\left(a^2 V(\phi) - 1\right)^{3/2} - \frac{\pi}{4}\right] \tag{8.22}$$

This completes the calculation of the no-boundary wave function.[†]

Some further remarks are in order. First, the contour of integration. The path integral for the no-boundary wave function as discussed above has two saddle-points, and Hartle and Hawking argued that it is the saddle-point corresponding to less than half a four-sphere that provides the dominant contribution. However, their heuristic argument is not, in my opinion, totally convincing.

A more detailed analysis of this situation by myself and Jorma Louko exposed the assumptions that Hartle and Hawking implicitly made to arrive at the above answer (Halliwell and Louko, 1989a). By a suitable choice of variables, and by working with a cosmological

[†] The reader familiar with the literature will note that this is not the derivation given by Hartle and Hawking (1983). However, I have presented it in this way to emphasize certain points which will be discussed in what follows.

constant instead of a scalar field, we were able to evaluate the minisuperspace path integral for this model exactly. In particular, we were able to determine convergent contours explicitly for the model, and thus see whether or not certain saddle-points did or did not yield the dominant contribution to the path integral. What we found is that there are a number of inequivalent contours along which the path integral converges, each dominated by different saddle-points, and thus leading to different forms for the wave function. No one contour was obviously preferred. In particular, the no-boundary proposal did not indicate which contour one was supposed to take. A contour yielding the above form for the wave function could be found, but it was not obvious why one should take that particular one. So the essential conclusion here is that the no-boundary proposal as it stands does not fix the wave function uniquely. There are, so to speak, many no-boundary wave functions, each corresponding to a different choice of contour. The wave function is therefore only fixed uniquely after one has put in some *extra* information fixing the contour.

As an example, in the simple model above one could *define* the no-boundary wave function to be as defined by Hartle and Hawking, with the *additional* piece of information that one is to take the contour dominated by the less-than-half saddle-point. A more general statement is however not currently available. A possible approach to this problem is that of Halliwell and Hartle (1989), which involved restricting the possible contours on the grounds of mathematical consistency and physical predictions.

The second issue that deserves further comment is the equation for the lapse, (8.17), and there are a number of points to be made here. Firstly, we considered only $\tilde{a}^2 V(\tilde{\phi}) < 1$, so that the solution was real. One may allow $\tilde{a}^2 V(\tilde{\phi}) > 1$, in which case N, the scale factor (8.16) and the action become complex – the action is essentially (8.20) with $\tilde{a}^2 V(\tilde{\phi})$ continued into the range $\tilde{a}^2 V(\tilde{\phi}) > 1$. Complex saddle-points are generally expected in this sort of problem. Indeed, they are essential if the wave function is to be oscillatory, and thus predict classical spacetime. Secondly, we restricted to the solutions with $n = 0$. What is the significance of the other solutions? Consider first the case of n positive. It is not difficult to see that for values of $n > 0$, the solution (8.16) undergoes many oscillations. More precisely, a^2, which appears in the metric, expands to a maximum size and then "bounces" each time it reaches zero. The geometric picture of these saddle-points is therefore of linear chains of contiguous spheres (Halliwell and Myers, 1989; Klebanov et al., 1989).

What about the saddle-points with $n < 0$? These saddle-points have negative lapse.

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Because the action changes sign under $N \to -N$, the action of these saddle-points has the "wrong" sign. However, these saddle-points are otherwise identical to the ones with positive lapse – their four-metrics are the same. Moreover, they have a perfectly legitimate place as saddle-points of the path integral. They are not artefacts of this model. They arise because the action, by virtue of the presence of the $\sqrt{g}$ factor, is double-valued in the space of complex four-metrics. Carrying the metric once around the branch point returns one to a physically identical solution to the Einstein equations, but with action of the opposite sign. So to every physically significant solution there corresponds *two* saddle-points. Because one has to integrate over complex metrics for convergence, both saddle-points are candidate contributants to the path integral.

So finally it seems sensible to ask, why did we not include any of these extra saddle-points, *i.e.* $n = \pm 1, \pm 2, ...$, in the calculation of the no-boundary wave function? The answer is that one can, by a suitable choice of contour. However, the saddle-points with N negative (or more generally, with $Re(\sqrt{g})$ negative), lead to difficulties with the recovery of quantum field theory in curved spacetime if they dominate the path integral, because a normally positive matter action will become negative definite on the gravitational background corresponding to such a saddle-point. For this reason, the contour should not be chosen in such a way that it is dominated by a negative N saddle-point (Halliwell and Hartle, 1989). This leaves the saddle-points with $n = 0, 1, 2, 3...$ The saddle-points corresponding to the linear chains of spheres, $n = 1, 2, ..$ may contribute with a suitable choice of contour, but one could exclude them by taking the definition of the no-boundary wave function offered above (*i.e* demand that the contour be dominated by the saddle-point corresponding to less than half of a four-sphere).

These issue involving the contour are still very much up in the air, and I would regard the question of choosing a sensible contour for the no-boundary wave function as at this moment an open one.

The Tunneling Boundary Condition

The other proposal we will consider here is the so-called "tunneling" boundary condition advocated primarily by Vilenkin (1982, 1983, 1984, 1985a, 1985b, 1986, 1988) and Linde (1984a, 1984b, 1984c). I will concentrate on Vilenkin's formulation, which is the

most comprehensive. The tunneling boundary condition attempts to draw most strongly the analogy between the quantum creation of the universe and tunneling in ordinary quantum mechanics. Vilenkin has offered various formulations of this boundary condition, not all of which are obviously equivalent. The most detailed is the "outgoing modes" formulation, which we now discuss (Vilenkin, 1988).

The outgoing modes statement of the tunneling boundary condition proposal is a statement about the behaviour of the solutions to the Wheeler-DeWitt equation at the boundary of superspace. In brief, the idea is as follows. In a manner analagous to that in which solutions to the Klein-Gordon equation are classified as positive or negative frequency, Vilenkin attempts to classify the solutions to the Wheeler-DeWitt equation as "ingoing" or "outgoing" at the boundary. The proposal is then that the wave function should consist solely of outgoing modes at the parts of the boundary of superspace which correspond to singular four-geometries. A regularity condition, that Ψ be everywhere bounded, is also imposed.

This is perhaps a little vague, so let us discuss it more carefully. First, consider the nature of the boundary of superspace. The boundary of superspace will generally consist of configurations which are in some sense singular, *e.g.* $h^{\frac{1}{2}}$ will be zero or infinite, or quantities such as Φ, or $(\partial_i \Phi)^2$ may be infinite. However, this does not necessarily mean that a four-geometry which has that three-geometry as a slice is singular. For example, $h^{\frac{1}{2}}$ vanishes at the north and south pole of a four-sphere, but the four-geometry is perfectly regular. Let us therefore divide the boundary of superspace into two regions. The first region consists of three-geometries having singularities attributable to the slicing of a regular Euclidean[†] four-geometry. That is, there exists a regular Euclidean[†] four-geometry of which the singular three-geometry is a slice. Call these parts of the boundary non-singular. The second part of the boundary is what remains, and will be referred to as the singular boundary. This part of the boundary *does* correspond to singularities of the four-geometry. A more detailed mathematical discussion of this point can be given, using Morse theory, but the above is sufficient for our purposes.

Now let us discuss the notion of ingoing and outgoing modes. Solutions to the Klein-Gordon equation of relativistic quantum mechanics may be expanded in terms of mode

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[†] The description "Euclidean" was not given explicitly in Vilenkin (1988), but it appears to have been tacitly assumed there and elsewhere

functions $e^{ip \cdot x}$, and these modes may be classified as positive or negative frequency, with respect to the timelike Killing vector $-i\partial/\partial t$. More precisely, the mode solutions are eigenfunctions of this Killing vector and the classification depends on the sign of the eigenvalue. The positive and negative frequency modes may also be characterized by the sign of J_0, the timelike component of the conserved current

$$J = \frac{i}{2}\left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*\right), \quad \nabla \cdot J = 0 \tag{8.23}$$

One might hope to do an analagous thing for the Wheeler-DeWitt equation. In the general case, one meets with an immediate difficulty. This is that to say what one means by positive and negative frequency on the whole of superspace, one needs a timelike Killing vector. However, it is a mathematical property of superspace that it has has no Killing vectors at all, so positive and negative frequency modes cannot in general be defined (Kuchař, 1981).

Despite this obstruction, one can still make considerable progress by restricting attention to certain approximate forms for the wave function, or by restricting attention to certain regions of superspace, such as close to the boundary. One is, for example, primarily interested in the solution in the oscillatory region. There, one expects solutions to the Wheeler-DeWitt equation of the form

$$\Psi = \sum_n C_n e^{iS_n} \tag{8.24}$$

where the S_n are solutions to the Hamilton-Jacobi equation. The current for the mode $C_n e^{iS_n}$ is

$$J_n = -|C_n|^2 \nabla S_n \tag{8.26}$$

This mode is thus defined to be outgoing at the boundary if $-\nabla S_n$ points outward there. If the wave function is not oscillatory in the neighbourhood of the boundary, then the definition of outgoing modes is more problematic, if not impossible.

Now let us give a more precise statement of Vilenkin's outgoing modes proposal for the tunneling wave function, Ψ_T:

Ψ_T is the solution to the Wheeler-DeWitt equation that is everywhere bounded and consists solely of outgoing modes at singular boundaries of superspace.

Despite the apparent vagueness in its definition, and the obstruction of principle to making it more general, the Vilenkin outgoing modes form of the tunneling boundary condition appears in simple minisuperspace models to be intuitively reasonably clear, and it has been quite successful in defining a unique solution to the Wheeler-DeWitt equation.

Now let us calculate the tunneling wave function, using the above proposal, for the scalar field model. The Wheeler-DeWitt equation may be written

$$\left[\frac{\partial^2}{\partial a^2} - \frac{1}{a^2} \frac{\partial^2}{\partial \phi^2} + a^4 V(\phi) - a^2 \right] \Psi(a, \phi) = 0 \tag{8.27}$$

The minisuperspace for this model is the two-dimensional space with coordinates (a, ϕ), with $0 < a < \infty$, $-\infty < \phi < \infty$. The only non-singular part of the boundary is $a = 0$ with ϕ finite. The rest is singular, and consists of configurations with one or both of a and ϕ infinite. Writing $a = e^\alpha$, minisuperspace, which is now just flat two-dimensional Minkowski space, is conveniently represented on the usual conformal diagram. The non-singular boundary is mapped to the single point i^-, past timelike infinity. The remaining singular part of the boundary is mapped to $\mathcal{I}^\pm$, future and past null infinity, and i^0, i^+, spacelike and future timelike infinity (see Fig.5.). The basic idea of the outgoing modes prescription is that probability flux is injected into superspace at i^- with finite ϕ and $a = 0$, and flows out of superspace across the singular boundaries.

We will again work in the region for which the scalar field potential depends only very slowly on ϕ. So provisionally we impose the restriction $|V'/V| << 1$, although this condition will be revised below. Next, note that as a goes to zero, the coefficient of the second derivative with respect to ϕ in (8.27) blows up. If, as the boundary conditions demand, we are to get a regular solution, it seems reasonable to insist that $\Psi(a, \phi)$ becomes independent of ϕ for small a. We will therefore neglect the second derivative with respect to ϕ in (8.27).

Consider first the solution in the oscillatory region, $a^2 V(\phi) > 1$. The WKB solutions are proportional to e^{iS}, or e^{-iS}, where $S = (a^2 V(\phi) - 1)^{3/2}/3V(\phi)$. The first has probability flux $J \sim -\nabla S$, pointing back towards i^-, the second has $J \sim \nabla S$, pointing outwards away from i^-. The latter, if evolved in the forward direction would eventually reach the singular boundary at which it would be outgoing. The former, however, corresponds to the time reverse of this so is ingoing at the boundary. This means that the outgoing modes prescription implies that only an outgoing wave should be present in the classically

allowed region, so the wave function should be proportional to e^{-iS}. This implies that the tunneling wave function in the oscillatory region is of the form

$$\Psi_T(a, \phi) \approx A(\phi) \exp\left(-\frac{i}{3V(\phi)} \left(a^2 V(\phi) - 1\right)^{3/2}\right) \tag{8.30}$$

Here we have included, as we may, the slowly varying ϕ-dependent factor $A(\phi)$. It turns out that this needs to be included to ensure that the solution is regular.

By the WKB matching procedure, one may determine the solution corresponding to (8.30) in the exponential region, $a^2 V(\phi) < 1$. It is

$$\Psi_T(a, \phi) \approx A(\phi) \exp\left(\frac{1}{3V(\phi)} \left(1 - a^2 V(\phi)\right)^{3/2}\right)$$

$$- iA(\phi) \exp\left(-\frac{1}{3V(\phi)} \left(1 - a^2 V(\phi)\right)^{3/2}\right) \tag{8.31}$$

The second term is exponentially smaller than the first, so may be neglected. Now consider what happens to the solution as a goes to zero. For regularity, we need $\partial\Psi/\partial\phi \to 0$ as $a \to 0$. This can only be achieved by choosing the function $A(\phi)$ to be

$$A(\phi) = \exp\left(-\frac{1}{3V(\phi)}\right) \tag{8.32}$$

With this choice, $\Psi_T \sim e^{-\frac{1}{2}a^2}$ for small a, which is regular for all values of ϕ.

We should now check that all this is consistent with the approximation of neglecting the second derivative with respect to ϕ in the Wheeler-DeWitt equation. Inserting the approximate solution with $A(\phi)$ given by (8.32) into (8.27), it may be shown that the solution is valid in the region for which $|V'(\phi)| << a^{-2}$. If $a^2 V(\phi) < 1$, this is actually an improvement on the original condition, $|V'/V| << 1$. In particular, it means that the solution is valid for arbitrarily rapid dependence of the potential on ϕ as a goes to zero. This would not have been true had we not multiplied the wave function by (8.32). So the revised restriction under which our approximations are valid is

$$|V'(\phi)| << \max\left[|V(\phi)|, a^{-2}\right] \tag{8.33}$$

The final expression for the tunneling wave function is given by

$$\Psi_T(a, \phi) \approx \exp\left(-\frac{1}{3V(\phi)} \left[1 - \left(1 - a^2 V(\phi)\right)^{3/2}\right]\right) \quad for \quad a^2 V(\phi) < 1 \tag{8.34}$$

$$\Psi_T(a, \phi) \approx \exp\left(-\frac{1}{3V(\phi)}\right) \exp\left(-\frac{i}{3V(\phi)}\left(a^2 V(\phi) - 1\right)^{3/2}\right) \quad for \quad a^2 V(\phi) > 1 \quad (8.35)$$

This completes the calculation of the tunneling wave function.

Mention should also be made of an alternative, not so well-known version of the tunneling boundary condition, also due to Vilenkin. This is that the wave function is given by a *Lorentzian* path integral over geometries which close off in the past,

$$\Psi_T = \int \mathcal{D}g_{\mu\nu} e^{iS} \quad (8.36)$$

where S is the Lorentzian action. The phrase "close off in the past" is taken to mean that the histories summed over have vanishing initial three-volume, and also that the lapse function in the path integral (4.7) (or (5.21)) is integrated not over an infinite range, but over a half-infinite range, from 0 to ∞. The wave function thus calculated is then not quite a solution to the Wheeler-DeWitt equation, but is a Green function of the Wheeler-DeWitt operator; *i.e.* one obtains a delta-function on the right-hand side of Eq.(5.23), although this delta-function is pushed to the boundary of superspace where $h^{\frac{1}{2}} = 0$. This is in keeping with the idea that the tunneling wave function involves probability flux being injected into superspace at the non-singular boundary. It seems reasonable to interpret this proposal as being essentially the same as the no-boundary proposal, in which a particular choice for the contour is made. Namely, that the contour is chosen to be the complex contour which may be distorted to lie along the real Lorentzian axis. It is not obviously equivalent to the outgoing modes version of the tunneling proposal, however, and actually fails to coincide precisely in some models (Halliwell and Louko, 1990).

Linde's version of the tunneling proposal (Linde, 1984a, 1984b, 1984c) also appears to involve a Lorentzian path integral as a starting point. Because the usual Wick rotation to Euclidean time leads to a minus sign in front of the kinetic term for the scale factor in the action, Linde proposed that the Wick rotation should be performed in the "wrong" direction. It may be argued that this involves choosing the lapse contour to be the distortion into the region $Re(N) < 0$ of the contour running up the positive imaginary axis (Halliwell and Louko, 1989a, 1990). This proposal is therefore identical to Vilenkin's path integral version of the tunneling proposal.

Finally, an interesting point. Vilenkin observed that the full Wheeler-DeWitt equation is invariant under the transformation

$$h_{ij} \to e^{i\pi} h_{ij}, \quad V(\Phi) \to e^{-i\pi} V(\Phi) \quad (8.37)$$

That is, given a solution $\Psi[h_{ij}, \Phi]$, a second solution may be generated from it using the above transformation. In particular, Vilenkin noticed that the no-boundary and tunneling wave functions for the scalar field minisuperspace model are related by this transformation:

$$\Psi_{NB} = \Psi_T(V \to e^{-i\pi}V, a \to e^{i\pi/2}a) \tag{8.38}$$

(to see this explicitly one has to use the Airy functions of which (8.34) and (8.35) are asymptotic forms). The possible significance of this observation is the following: as I have tried to emphasize, there are considerable difficulties of precision and generality in the definitions of the no-boundary and tunneling wave functions. If, however, one succeeded in defining *one* of these wave functions in a much more precise, more general way, then the other could be *defined* by the transformation (8.37).

9. NO-BOUNDARY VS. TUNNELING

Let us now compare the no-boundary and tunneling wave functions. For convenience we record their explicit forms in the oscillatory region, in a range of ϕ for which $V(\phi)$ is slowly varying. To be definite, let us take the potential $V(\phi)$ to be of the chaotic inflationary type. Let us introduce

$$S = \frac{1}{3V(\phi)} \left(a^2 V(\phi) - 1 \right)^{3/2} - \frac{\pi}{4} \tag{9.1}$$

The tunneling wave function is

$$\Psi_T \approx \exp\left(-\frac{1}{3V(\phi)} \right) e^{-iS} \tag{9.2}$$

The no-boundary wave function is

$$\Psi_{NB} \approx \exp\left(+\frac{1}{3V(\phi)} \right) \left[e^{-iS} + e^{iS} \right] \tag{9.3}$$

There are two differences. The first is that the no-boundary wave function is real, being a sum of a WKB component and its complex conjugate, whilst the Vilenkin wave function consists of just one WKB component.[†] If one component corresponds to expanding

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[†] The fact that the no-boundary wave function is real corresponds to the fact that it is in a sense CPT invariant, and has implications for the arrow of time in cosmology (Hawking, 1985; Page, 1985).

solutions, then the other corresponds to collapsing solutions, although it is not possible to say which one is which. It may be argued that these components have negligible interference, so each component may be considered separately (Halliwell, 1989b). One may thus compare a single component of the no-boundary wave function with the Vilenkin wave function.

The second, and more important difference, is the sign difference in the prefactor. Both wave functions are peaked about the same set of solutions to the field equations, namely those satisfying the first integral $p = \nabla S$, with S given by (9.1). As we have shown, these solutions are initially inflationary, with $a(t) \approx e^{V^{\frac{1}{2}}t}$, $\phi(t) \approx \phi_0 = constant$. These solutions may be labeled by their initial values of ϕ, ϕ_0. Although all the solutions undergo *some* inflation, the *amount* by which they inflate depends on ϕ_0. For example, if the potential is $V(\phi) = m^2\phi^2$, then sufficient inflation is obtained only for values of ϕ_0 in excess of about 4 (in Planck units) (Hawking, 1984a; Page, 1986a).

To see which initial values of ϕ are most favoured by each wave function, we need to study the measure on the set of paths, $J \cdot d\Sigma$. Because the trajectories have $\dot{\phi} \approx 0$, J points largely in the a direction. A suitable choice of surfaces Σ is therefore surfaces of constant a, at least locally. The probability measure is thus given by

$$dP = J \cdot d\Sigma \approx \exp\left(\pm\frac{2}{3V(\phi)}\right)d\phi \tag{9.4}$$

(with $(+)$ for the no-boundary wave function, $(-)$ for the tunneling wave function). With this measure, we now have to ask the right questions. As discussed previously, we cannot take this to be an absolute measure on the initial values of ϕ. Rather, it should be thought of as a conditional probability measure. So we must first decide what conditions to impose; that is, in what range of values of ϕ are we to ask for predictions?

First of all consider what happens if ϕ is very small initially, close to zero (for convenience, we restrict attention to positive ϕ in what follows). Universes starting out with a very small initial value of ϕ will very rapidly reach a small maximum size and then recollapse in a short period of time. One would not expect large scale structure and indeed, observers, to exist in such universes. It therefore seems reasonable to impose the condition that the universe expands out to a "reasonable" size. This is somewhat vague, but what it means is that we restrict attention to initial values of ϕ greater than some exceedingly small value ϕ_{min}, say. This restriction has the consequence that the no-boundary $(+)$

measure (9.4) is now bounded (it was previously unbounded at $\phi = 0$) and it is peaked about ϕ_{min}.

Now consider very large values of ϕ. For a chaotic potential at least, as ϕ becomes very large the scalar field energy density $V(\phi)$ will approach the Planck energy density, $V(\phi) \sim 1$. If minisuperspace models are to have any validity at all, it seems unlikely that they can be trusted in the range of ϕ for which $V(\phi) > 1$. So our second condition is to ask for predictions only in the region $\phi < \phi_p$, where $V(\phi_p) = 1$. For the potential $V(\phi) = m^2\phi^2$, m is normally taken to be about 10^{-4}, so $\phi_p \sim 10^4$.

Our task is now to ask for predictions with the condition that the initial value of ϕ lies in the range $\phi_{min} < \phi < \phi_p$. For a chaotic potential, there will be a value of ϕ in this range, larger than ϕ_{min}, call it ϕ_{suf}, for which sufficient inflation is achieved if $\phi_0 > \phi_{suf}$, and it is not achieved if $\phi_0 < \phi_{suf}$. For the massive scalar field, $\phi_{suf} \sim 4$. A pertinent question to ask, therefore is this: "What is the probability that $\phi_0 > \phi_{suf}$, given that $\phi_{min} < \phi_0 < \phi_p$?" It is given, using (9.4), by the following expression.

$$P(\phi_0 > \phi_{suf} | \phi_{min} < \phi_0 < \phi_p) = \frac{\int_{\phi_{suf}}^{\phi_p} d\phi \exp\left(\pm\frac{2}{3V(\phi)}\right)}{\int_{\phi_{min}}^{\phi_p} d\phi \exp\left(\pm\frac{2}{3V(\phi)}\right)} \tag{9.5}$$

This is effectively the probability of sufficient inflation.

It is reasonably easy to see the result of evaluating (9.5) by merely looking at the plot of the two probability distributions, $\exp\left(\pm\frac{2}{3V(\phi)}\right)$ (see Fig.6.). Consider first the tunneling wave function $(-)$. The integrand becomes very small as ϕ approaches ϕ_{min} and it is clear that by far the largest contribution to the integral in the denominator comes from the region $\phi > \phi_{suf}$. One therefore has $P \approx 1$, and sufficient inflation *is* a prediction of the tunneling wave function.

Now consider the no-boundary wave function $(+)$. The integrand diverges as ϕ approaches zero, but is cut off by ϕ_{min}. If, as we are assuming, ϕ_{min} is very small, the main contribution to the integral in the denominator will come from the region very close to ϕ_{min}. One therefore has $P << 1$ for the no-boundary wave function, and sufficient inflation *is not* a prediction.

The above conclusion about the no-boundary wave function is not the one reached by Hawking and Page in their analysis (Hawking and Page, 1986). They concluded that

sufficient inflation has probability unity. The difference with the analysis given here (which is based on that of Vilenkin (1989)), is that Hawking and Page did not restrict to $\phi < \phi_p$. For $\phi > \phi_p$, the integrands in (9.4) level off to 1; thus although in the range $\phi_{min} < \phi < \infty$ the integrands in the denominator are strongly peaked at ϕ_{min}, the contribution to the integral from this region is overwhelmingly outweighed by that from very large values of ϕ. (9.4) would therefore yield the value 1 for both wave functions.

A new aspect to the no-boundary/tunneling debate was recently exposed by Grishchuk and Rozhansky (1989) (see also Grishchuk and Rozhansky (1988)). They asked whether the above calculation of the no-boundary wave function, which involves the approximate solutions to the classical Euclidean field equations for the scalar field model, is really valid down to ϕ close to zero. The conclusion they came to is that they are not, and that the above expression for the no-boundary wave function makes sense only in the range $\phi > \phi_*$, for some critical value of ϕ, ϕ_* which they estimated. Although less than ϕ_{suf}, ϕ_* is much greater than ϕ_{min}, the lower bound we imposed to ensure that the universe expanded to a "reasonable" size. The value of ϕ_* is model-dependent. For the massive scalar field model $\phi_* \approx 1$.

Their conclusion was reached by giving a more careful treatment of the motion of the scalar field, which was taken to be approximately constant in the above analysis. For large ϕ, $|V'/V| << 1$, and the Euclidean solution for $a(\tau)$ is given approximately by (8.19). The trajectories start at $a = 0$ with some value of ϕ, expand, and then turn around and recollapse. In particular, along the curve $a^2 V(\phi) = 1$, neighbouring Euclidean trajectories intersect – they form a *caustic*. Because the real Euclidean trajectories dominating the path cannot reach immediately beyond the caustic, *i.e.* into the region $a^2 V(\phi) > 1$, a complex solution is necessary in order to get there. This means that the wave function becomes oscillatory in this region. Suppose however, one follows the caustic to smaller values of ϕ. It departs from the curve $a^2 V(\phi) = 1$, and in fact has a singularity at $\phi = \phi_*$, breaking into two branches there. This seems to invalidate the form of the wave function used above, and in fact, Grishchuk and Rozhansky claimed that it implies that the wave function fails to predict the emergence of any real Lorentzian trajectories for $\phi < \phi_*$. Moreover, their analysis also applies, they claim, to the tunneling wave function.

What this all means is that the conditions used above in the calculation of the probability of sufficient inflation should be replaced by the conditions $\phi_* < \phi < \phi_p$. Most importantly the region very close to $\phi = \phi_{min}$, in which the no-boundary and tunneling

wave function differ most severely, is excised. This has the consequence that the predictions of these two wave functions are not as different as previously believed. Although the predictions of the tunneling wavefunction are little affected by this result, for the no-boundary wave function it is now not so obvious that $P << 1$. In particular, what one would hope to find is that $\phi_* > \phi_{suf}$. This would have the consequence that *all* the classical Lorentzian solutions the wave function corresponds to have sufficient inflation; thus sufficient inflation would be predicted with probability 1, irrespective of whether an upper cut-off is imposed. The value of ϕ_* is, however, model dependent, and a model for which $\phi_* > \phi_{suf}$ is yet to be found.[†]

This is an interesting development which deserves further study.

10. BEYOND MINISUPERSPACE

For most of these lectures, we have largely concentrated on the application of the formalism of quantum cosmology to minisuperspace models. These models, with the appropriate boundary conditions, have been reasonably successful – in predicting inflation, for example. However, the universe we see today is not exactly described by the homogeneous metrics of the type considered in minisuperspace models. There are local deviations from homogeneity because matter is clumped into galaxies and other large scale structures. In conventional galaxy formation scenarios, this large scale structure can arise as a result of small density perturbations $\delta\rho/\rho \sim 10^{-4}$ in an otherwise homogeneous universe at very early times. The hot big bang model offered no explanation as to the origin of these perturbations, but had to assume them as initial conditions. The inflationary universe scenario shed considerable light on the situation by showing that they could have arisen from pre-inflationary quantum fluctuations in the scalar field hugely amplified by inflation. To be more precise, the density fluctuations in inflationary universe models are calculated from a quantity of the form $\langle 0|\Phi^2|0\rangle$, using standard methods of quantum field theory in a curved (usually de Sitter-like) spacetime. However, a point that was not emphasized in the early studies of this problem is that the form and magnitude of the density

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[†] More detailed calculations with the massive scalar field model, for which $\phi_* \sim 1$ and $\phi_{suf} \sim 4$ indicate that the previous conclusions (*i.e.* pre-Grishchuk-Rozhansky) concering the probability of inflation are in fact largely unaffected (J.Fort, private communication).

fluctuations calculated in this way depend rather crucially on the particular vacuum state $|0\rangle$ one uses, and in most curved spacetimes, there is no unique natural choice. Since this is clearly a question of initial conditions, one would expect to gain new insight into this issue from the perspective of quantum cosmology. It is therefore of considerable interest to go beyond minisuperspace to the full, infinite dimensional superspace. It would of course be very difficult to do this in full generality, but for the purposes of describing density fluctuations and gravitational waves, it is sufficient consider linearized fluctuations about a homogeneous isotropic minisuperspace background. This is the subject of this section.

We will find that there are two things that come out of this. Firstly, we will see that in the semi-classical limit, quantum cosmology reduces to the familiar formalism of quantum field theory for the fluctuations on a classical minisuperspace background. Secondly, the boundary conditions on the wave function of the universe imply a particular choice of vacuum state for the quantum fields.

Quantum Field Theory in Curved Spacetime

Before going on to study perturbations about minisuperspace in quantum cosmology, let us begin by reviewing some basic aspects of quantum field theory in curved spacetime (see, for example, Birrell and Davies (1982)). For definiteness, let us consider scalar field theory described by the action

$$S_m = -\frac{1}{2} \int d^4x \sqrt{-g} \left[(\partial \Phi)^2 + m^2 \Phi^2) \right] \tag{10.1}$$

This theory is normally quantized in the Heisenberg picture in a given background by introducing a set of mode functions $u_k(\mathbf{x}, t)$ satisfying a wave equation of the form

$$\left(\Box - m^2 \right) u_k(\mathbf{x}, t) = 0 \tag{10.2}$$

The field operator $\hat{\Phi}$ is then expanded in terms of these mode functions

$$\hat{\Phi}(\mathbf{x}, t) = \sum_k \left(\hat{a}_k u_k(\mathbf{x}, t) + \hat{a}_k^\dagger u_k^*(\mathbf{x}, t) \right) \tag{10.3}$$

where $\hat{a}_k^\dagger$ and $\hat{a}_k$ are the usual creation and annihilation operators. The vacuum state is then defined to be the state $|0\rangle$ for which

$$\hat{a}_k|0\rangle = 0 \tag{10.4}$$

The vacuum state is determined by the choice of mode functions u_k.

In Minkowski space, there is a unique vacuum state which is invariant under the Poincare group, and so is the agreed vacuum state for all inertial observers. However, in an arbitrary curved spacetime, there is no unique vacuum state. Any expectation value will generally depend rather crucially on the particular choice of state.

There is another perhaps less familiar way of doing quantum field theory in curved spacetime which is closer to quantum cosmology than the Heisenberg picture outlined above. This is the functional Schrodinger picture (Brandenberger, 1984; Burges, 1984;, Floreanini et al., 1987; Freese et al., 1985; Guth and Pi, 1985; Ratra, 1985). This picture is based very much on the $(3 + 1)$ decomposition we also used for quantum cosmology earlier. The $(3 + 1)$ form of the scalar field action (10.1) (in the gauge $N^i = 0$) is

$$S_m = \frac{1}{2} \int d^3x dt N h^{\frac{1}{2}} \left[\frac{\dot{\Phi}^2}{N^2} - h^{ij} \partial_i \Phi \partial_j \Phi - m^2 \Phi^2 \right] \tag{10.5}$$

Defining canonical momenta π_Φ in the usual way, one readily derives the Hamiltonian

$$H_m = \frac{1}{2} \int d^3x N h^{\frac{1}{2}} \left[h^{-1} \pi_\Phi^2 + h^{ij} \partial_i \Phi \partial_j \Phi + m^2 \Phi^2 \right] \tag{10.6}$$

In the functional Scrödinger quantization, the quantum state of the scalar field is represented by a wave functional $\Psi_m[\Phi(\mathbf{x}), t]$, a functional of the field configuration $\Phi(\mathbf{x})$ on the surface $t = constant$. The evolution of the quantum state is governed by the functional Schrödinger equation

$$i \frac{\partial \Psi_m}{\partial t} = H_m \Psi_m \tag{10.7}$$

where the operator appearing on the right-hand side is the Hamiltonian (10.6) with the momenta replaced by operators in the usual way,

$$\pi_\Phi(\mathbf{x}) \to -i \frac{\delta}{\delta \Phi(\mathbf{x})} \tag{10.8}$$

There are two differences between the representation of states in the two picture outlined above. Firstly, Heisenberg picture states are time-independent, whereas Schrödinger picture states are not (at least, in the flat space case – in curved backgrounds Heisenberg states may acquire time-dependence through the gravitational field). They are related by

$$|\Psi_S(t)\rangle = \exp\left(-i \int^t dt' H_m(t') \right) |\Psi_H\rangle \tag{10.9}$$

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Secondly, the Schrödinger picture states are represented at each moment of time by wave functionals $\Psi[\Phi(\mathbf{x})]$ rather than abstract Hilbert space elements $|\Psi\rangle$. The relationship between these two is found by introducing a complete set of *field states* $|\Phi(\mathbf{x})\rangle$, defined to be the eigenstates of the field operator $\hat{\Phi}$ at a moment of time

$$\hat{\Phi}|\Phi(\mathbf{x})\rangle = \Phi(\mathbf{x})|\Phi(\mathbf{x})\rangle \tag{10.10}$$

The wave functionals $\Psi[\Phi(\mathbf{x})]$ are then defined to be the coefficients in the expansion of the abstract Hilbert space elements in terms of the complete set of field states:

$$|\Psi_S\rangle = \int \mathcal{D}\Phi(\mathbf{x})|\Phi(\mathbf{x})\rangle\langle\Phi(\mathbf{x})|\Psi_S\rangle$$

$$\equiv \int \mathcal{D}\Phi(\mathbf{x})|\Phi(\mathbf{x})\rangle\Psi_S[\Phi(\mathbf{x})] \tag{10.11}$$

The question of choosing a vacuum state $|0\rangle$ in the Heisenberg picture becomes the question of choosing a solution to the functional Schrödinger equation (10.7) in the functional Schrödinger picture.

With these preliminaries in mind, let us now turn to perturbations about minisuperspace.

Inhomogeneous Perturbations about Minisuperspace

Now we will study inhomogeneous perturbations about minisuperspace. We primarily follow Halliwell (1987b), Halliwell and Hawking (1985), and Hartle (1986), but many more references are given in Section 13. To see how this works, it is simplest to consider a particular example. Namely, we will consider perturbations about the scalar field model considered earlier. There, the minisuperspace ansatz involved writing

$$h_{ij} = e^{2\alpha}\Omega_{ij}, \quad \Phi(\mathbf{x}, t) = \phi(t)$$

$$N(\mathbf{x}, t) = N_0(t), \quad N^i(\mathbf{x}, t) = 0 \tag{10.12}$$

where Ω_{ij} is the metric on the unit three-sphere. To go beyond this perturbatively we write

$$h_{ij} = e^{2\alpha}\left(\Omega_{ij} + \epsilon_{ij}\right), \Phi(\mathbf{x}, t) = \phi(t) + \delta\phi(\mathbf{x}, t)$$

$$N(\mathbf{x}, t) = N_0(t) + \delta N(\mathbf{x}, t) \tag{10.13}$$

and in addition, we allow non-zero $N^i(\mathbf{x}, t)$, which is regarded as a small perturbation. The easiest way to deal with the inhomogeneous perturbations is to expand in harmonics on the three-sphere. So, for example, one writes the scalar field perturbation as

$$\delta\phi(\mathbf{x}, t) = \sum_{nlm} f_{nlm}(t) Q^n_{lm}(\mathbf{x}) \tag{10.14}$$

where Q^n_{lm} are three-sphere harmonics. They satisfy

$$^{(3)}\Delta Q^n_{lm} = -(n^2 - 1) Q^n_{lm} \tag{10.15}$$

where $^{(3)}\Delta$ is the Laplacian on the three-sphere. The sum in (10.14) excludes the homogeneous mode, $n = 1$. The details of this expansion are not important in what follows, and may be found in Halliwell and Hawking (1985).

Inserting the above ansatz into the Einstein-scalar action, and expanding to quadratic order in the perturbations, one obtains a result of the form

$$S[g_{\mu\nu}, \Phi] = S_0[q^\alpha, N_0] + S_2[q^\alpha, N_0, \epsilon_{ij}, \delta\phi, \delta N, N^i] \tag{10.16}$$

where as before, we use q^α to denote the minisuperspace coordinates. S_0 is the original minisuperspace action and S_2 is the action of the perturbations, and is quadratic in them. The total Hamiltonian following from (10.16) is then found to be of the form

$$H_T = N_0 \left(H_0 + \int d^3x \mathcal{H}_{(2)} \right)$$

$$+ \int d^3x \delta N(\mathbf{x}) \mathcal{H}_{(1)}(\mathbf{x}) + \int d^3x N^i(\mathbf{x}) \mathcal{H}_i(\mathbf{x}) \tag{10.17}$$

From this one may see that first of all, there is a non-trivial momentum constraint at every point $\mathbf{x}$ in the three-surface

$$\mathcal{H}_i(\mathbf{x}) = 0 \tag{10.18}$$

It is linear in the perturbations. Secondly, the Hamiltonian constraint has split into two parts. There is a part linear in the perturbations, at each point $\mathbf{x}$ of the three-surface,

$$\mathcal{H}_{(1)}(\mathbf{x}) = 0 \tag{10.19}$$

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and there is a part consisting of the original minisuperspace Hamiltonian plus a term quadratic in the perturbations integrated over the three-surface:

$$H_0 + \int d^3x \mathcal{H}_{(2)} \equiv H_0 + H_2 = 0 \tag{10.20}$$

Quantization procedes by introducing a wave function $\Psi(q^\alpha, \epsilon_{ij}, \delta\Phi)$ and insisting that it be annihilated by the operator versions of the constraints, (10.18)-(10.20). The procedure is complicated by the fact that, in addition to the invariance under reparametrizations present in the minisuperspace case, the perturbations involve gauge degree of freedom. There are numerous ways of dealing with this. For example, the gauge degrees of freedom for the perturbations generated by (10.18) and (10.19) may be fixed classically, and the constraints solved for the physical degrees of freedom of the perturbations. This then leaves only the constraint (10.20), which now depends only on the minisuperspace variable and the unconstrained physical degrees of freedom of the metric and matter perturbations. One way or another, (10.20) ends up being the most important equation, and it is this that we now concentrate on.

A useful example to bear in mind is the case of purely scalar field perturbations about a purely gravitational background consisting of a Robertson-Walker metric with scale factor e^α. Then the Hamiltonian H_2 is given by

$$H_2 = \sum_{nlm} \frac{1}{2} e^{-3\alpha} \left[-\frac{\partial^2}{\partial f_{nlm}^2} + (m^2 e^{6\alpha} + (n^2 - 1)e^{4\alpha}) f_{nlm}^2 \right] \tag{10.21}$$

after expansion in harmonics.

The Wheeler-DeWitt equation resulting from (10.20) is of the form

$$\left[-\frac{1}{2m_p^2} \nabla^2 + m_p^2 U(q) + H_2 \right] \Psi(q, \Phi) = 0 \tag{10.22}$$

For convenience, we will consider only scalar field perturbations Φ, but what follows is equally applicable to the case of gravitational wave perturbations. The operator ∇ operates only on q^α, not on the perturbations. We are interested in the solution to the Wheeler-DeWitt equation in the region of superspace where the minisuperspace variables q^α are approximately classical, but the perturbations may be quantum mechanical. We therefore look for solutions of the form

$$\Psi(q, \Phi) = \exp\left(im_p^2 S_0(q) + iS_1(q) \right) \psi(q, \Phi) + O(m_p^{-2}) \tag{10.23}$$

where $S_0(q)$ is real,[†] but S_1 and ψ may be complex. Inserting (10.23) into (10.22), and equating powers of the Planck mass, one obtains the following. At lowest order, once again one gets the Hamilton-Jacobi equation for S_0,

$$\frac{1}{2}(\nabla S_0)^2 + U(q) = 0. \tag{10.24}$$

This shows that, to lowest order in m_p^2, the wave function (10.23) is, as in the minisuperspace case, peaked about the ensemble of solutions to the classical field equations with Hamilton-Jacobi function S_0. It is convenient to introduce the tangent vector to these classical solutions,

$$\frac{\partial}{\partial t} = \nabla S_0 \cdot \nabla \tag{10.25}$$

At the next order, one obtains the equation:

$$\psi \left[\nabla S_0 \cdot \nabla S_1 - \frac{i}{2} \nabla^2 S_0 \right] = i \nabla S_0 \cdot \nabla \psi - H_2 \psi \tag{10.26}$$

This is one equation for the two unknowns S_1 and ψ, so there is the freedom to impose some restrictions on them. We are anticipating that the ψ will be matter wave functionals for the scalar field Φ. Let us therefore introduce an inner product between matter wave functionals, at each point of minisuperspace, q^α:

$$(\psi_1, \psi_2) = \int \mathcal{D}\Phi(\mathbf{x}) \psi_1^*(q, \Phi(\mathbf{x})) \psi_2(q, \Phi(\mathbf{x})) \tag{10.27}$$

Note that this involves an integral only over $\Phi(\mathbf{x})$ and *not* over the minisuperspace variables q^α. This is acceptable because we expect the appropriate matter wave functionals to be normalizable in the matter modes. We do not, however, expect any part of the wave function to be normalizable in the large, minisuperspace modes, so we do not attempt to introduce an inner product involving an integral over q^α. Using the freedom available in ψ, let us demand that

$$\frac{d}{dt}(\psi, \psi) = 0 \tag{10.28}$$

That is, the norm of ψ is preserved along the classical minisuperspace trajectories. This seems like a reasonable restriction if we are to recover quantum field theory for matter. We may therefore take $(\psi, \psi) = 1$. Differentiating out (10.28), it is readily seen that

$$\left(i \frac{\partial \psi}{\partial t}, \psi \right) = \left(\psi, i \frac{\partial \psi}{\partial t} \right) \tag{10.29}$$

[†] As in Section 7, we could also allow a slowly varying exponential prefactor, but this may be absorbed into the definition of S_1

and hence that this quantity is real.

Armed with this information, let us now return to (10.26). Taking the inner product of (10.26) with ψ, and making the reasonable assumption that the perturbation Hamiltonian H_2 is hermitian in the above inner product, we quickly discover that, apart from the ψ, the left-hand side must be real. Its imaginary part must therefore vanish,

$$\nabla S_0 \cdot \nabla (Im S_1) - \frac{1}{2} \nabla^2 S_0 = 0 \tag{10.30}$$

If we write $C = \exp(-Im S_1)$, then C is the usual real minisuperspace WKB prefactor, obeying (6.26), and is unaffected by the perturbations.

Subtracting (10.30) from (10.26), and using the definition (10.25), one obtains the following equation for ψ:

$$i\frac{\partial \psi}{\partial t} = \left[H_2 + \frac{\partial}{\partial t}(Re S_1) \right] \psi \tag{10.31}$$

Finally, by writing $\tilde{\psi} = e^{iRe S_1}\psi$, we discover that $\tilde{\psi}$ obeys the functional Schrödinger equation along the classical trajectories in minisuperspace about which the wave function is peaked:

$$i\frac{\partial \tilde{\psi}}{\partial t} = H_2 \tilde{\psi} \tag{10.32}$$

This derivation may be concisely summarized as follows: the WKB solution to the Wheeler-DeWitt equation (10.22) is of the form

$$\Psi(q, \Phi) = C(q) e^{im_p^2 S_0(q)} \tilde{\psi}(q, \Phi) \tag{10.33}$$

where $S_0(q)$ is a solution to the Hamilton-Jacobi equation, indicating that the wave function to leading order is peaked about a set of classical trajectories, $C(q)$ is the usual unperturbed WKB prefactor, and $\tilde{\psi}$ satisfies the functional Schödinger equation (10.32) along the classical trajectories about which the wave function is peaked.

What we have shown, therefore, is that the Wheeler-DeWitt equation reduces, in the semi-classical limit, to the familiar formalism of quantum field theory for the fluctuations Φ in a fixed classical background. This shows that quantum cosmology is consistent with the standard approach, which involves quantum field theory on a fixed background. See Section 13 for references to the large number of papers on this issue.

11. VACUUM STATES FROM QUANTUM COSMOLOGY

We have shown that quantum cosmology reduces, in the semi-classical limit, to the formalism of quantum field theory for the matter modes in a fixed curved spacetime background. So far we have therefore done little new, except to demonstrate consistency with that we already know. However, there is a bonus. Boundary conditions on the wave function define a particular solution to the Wheeler-DeWitt equation of the form (10.33), where $\tilde{\psi}$ is a solution to the functional Schrödinger equation for the perturbations. This means that boundary conditions on the wave function of the universe will pick out a particular solution to the functional Schrödinger equation; that is, *they define a particular vaccum state for matter*, with which to do quantum field theory.

The natural question to ask now, is what is the nature of the vacuum state picked out by the no-boundary and tunneling boundary conditions in a given background? The background of particular interest as far as inflation is concerned is de Sitter space, or spacetimes that are very nearly de Sitter. For that background it may be shown that the vacuum state defined by both of these proposals is a vacuum state known as the "Euclidean" or "Bunch-Davies" vacuum. This is the vacuum state that is often assumed when calculating density fluctuations, and leads to a reasonable spectrum for the emergence of large scale structure.

Before seeing exactly how the above proposals define this vacuum state, let us first explain how it is defined.[†]

De Sitter-Invariant Vacua

Minkowski space has as its isometry group the 10 parameter Poincare group. There is a vacuum which is invariant under this group, and thus is the agreed vacuum for all inertial observers. It is unique, up to trivial Bogoliubov transformations. The isometry group of de Sitter space, which also has 10 parameters, is the de Sitter group, $SO(4,1)$. In choosing vacuum states with which to do quantum field theory in de Sitter space, it is therefore natural to seek vacua invariant under the de Sitter group.

[†] For a useful discussion of de Sitter-invariant vacua, see Allen (1985), and references therein.

A convenient way of characterizing vacua is through the symmetric two-point function in a state $|\lambda\rangle$:

$$G_\lambda(x, y) = \langle\lambda|(\Phi(x)\Phi(y) + \Phi(y)\Phi(x))|\lambda \tag{11.1}$$

The state $|\lambda\rangle$ is then said to be de Sitter invariant if the two-point function depends on x and y only through $\mu(x, y)$, the geodesic distance between x and y:

$$G_\lambda(x, y) = f_\lambda(\mu) \tag{11.2}$$

Using the fact that Φ obeys the Klein-Gordon equation, a second order ordinary differential equation for $f_\lambda(\mu)$ is readily derived. From it, it may be shown that there is not just one de Sitter-invariant vacuum, but there is a one-parameter family of inequivalent de Sitter-invariant vacua.

For this one-parameter family, the function $f_\lambda(\mu)$ generally has two poles: one when y is on the light-cone of x, the other when y is on the light cone of $\bar{x}$, the point in de Sitter space antipodal to x. However, amongst the one-parameter family, there is one member for which $f_\lambda(\mu)$ has just one pole, when y is on the light-cone of x. This member is called the "Euclidean" or "Bunch-Davies" vacuum, and has the nicest analytic properties. As mentioned above, it is this one that is always used in calculations of density fluctuations in inflationary universe models.

There is another equivalent way of characterizing the Euclidean vacuum that will be most convenient for our purposes. This is a definition in terms of a particular choice of mode functions. Suppose we expand the scalar field operator in terms of a set of modes functions $\{u_{nlm}(\mathbf{x}, t)\}$, say,

$$\hat{\Phi}(\mathbf{x}, t) = \sum_{nlm} \left(u_{nlm}(\mathbf{x}, t)\hat{a}_{nlm} + u^*_{nlm}(\mathbf{x}, t)\hat{a}^\dagger_{nlm} \right) \tag{11.3}$$

The vacuum state $|0\rangle$ corresponding to this particular choice of mode functions is defined by

$$\hat{a}_{nlm}|0\rangle = 0 \tag{11.4}$$

To define the Euclidean vacuum, one first chooses the mode functions

$$u_{nlm}(\mathbf{x}, t) = y_n(t)Q^n_{lm}(\mathbf{x}) \tag{11.5}$$

where the $Q^n_{lm}(\mathbf{x})$ are three-sphere harmonics, and the $y_n(t)$ satisfy the equation

$$\ddot{y}_n + 3\frac{\dot{a}}{a}\dot{y}_n + \left(\frac{n^2 - 1}{a^2} + m^2\right) y_n = 0 \tag{11.6}$$

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Here, $a(t) = H^{-1}\cosh(Ht)$ is the scale factor for de Sitter space. The normalization of the $y_n(t)$ is fixed through the Wronskian condition

$$y_n\dot{y}_n^* - y_n^*\dot{y}_n = \frac{i}{a^3} \tag{11.7}$$

The Euclidean section of de Sitter space is the four-sphere, and may be obtained by writing $t = -i(\tau - \frac{\pi}{2H})$, which turns $a(t) = H^{-1}\cosh(Ht)$ into $a(\tau) = H^{-1}\sin(H\tau)$. The Euclidean vacuum is then defined by the requirement that the $y_n(t)$ are *regular* on the Euclidean section. The $y_n(t)$ actually become real on the Euclidean section, so one may equivalently demand that the $y_n^*(t)$ are regular there.

There is a third possible way of dicussing de Sitter-invariant vacua, which is conceptually the most transparent way. This is to explicitly construct the de Sitter generators and demand that the state be annihilated by them, but we will not consider this here (Burges, 1984; Floreanini et al., 1986).

The No-Boundary Vacuum State

Now let us explicitly calculate the matter state wave functional for a massive minimally coupled scalar field in a de Sitter background, using the no-boundary proposal. We follow Laflamme (1987a). We regard all the modes of the scalar field, including the homogeneous mode, as perturbations on a homogeneous isotropic background with scale factor $a(t)$, driven by a cosmological constant. The no-boundary wave function is given by a path integral of the form

$$\Psi_{NB}(\tilde{a}, \tilde{\Phi}) = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\Phi \exp\left(-I_g[g_{\mu\nu}] - I_m[g_{\mu\nu}, \Phi]\right) \tag{11.8}$$

In the saddle-point approximation to the integral over metrics, this leads to an expression of the form

$$\Psi_{NB}(\tilde{a}, \tilde{\Phi}) \approx \exp\left(-I_g[\bar{g}_{\mu\nu}]\right) \int \mathcal{D}\Phi \exp\left(-I_m[\bar{g}_{\mu\nu}, \Phi]\right) \tag{11.9}$$

where $\bar{g}_{\mu\nu}$ is the saddle-point metric. When $aH < 1$, $\bar{g}_{\mu\nu}$ is real and is the metric on the section of four-sphere closing off a three-sphere of radius a. When $aH > 1$, $\bar{g}_{\mu\nu}$ is complex, and corresponds to a section of de Sitter space with minimum radius a matched onto half a four-sphere at its maximum radius.

Comparing (11.9) with (10.23), one may see that the matter wave functionals are given by the path integral

$$\psi[\tilde{a}, \tilde{\Phi}] = \int \mathcal{D}\Phi \exp\left(-I_m[\bar{g}_{\mu\nu}, \Phi]\right) \tag{11.10}$$

The no-boundary proposal implies that the integral over matter modes is over fields $\Phi(\mathbf{x}, \tau)$ that match $\tilde{\Phi}(\mathbf{x})$ on the three-sphere boundary. As in Section 8, we shall demand that the saddle-point of the functional integral over Φ in (11.10) corresponds to a *regular* solution to the scalar field equation on the given background geometry.

The scalar field is most easily handled by expanding in three-sphere harmonics

$$\Phi(\mathbf{x}, \tau) = \sum_{nlm} f_{nlm}(\tau) Q^n_{lm}(\mathbf{x}) \tag{11.11}$$

In terms of the coefficients $f_{nlm}(\tau)$, the Euclidean action is

$$I_m[a(\tau), \Phi] = \frac{1}{2} \sum_{nlm} \int_0^1 d\tau N a^3 \left[\frac{1}{N^2} \left(\frac{df_{nlm}}{d\tau} \right)^2 + \left(\frac{n^2 - 1}{a^2} + m^2 \right) f^2_{nlm} \right]$$

$$\equiv \sum_{nlm} I_{nlm}[a(\tau), f_{nlm}] \tag{11.12}$$

The Euclidean field equations are

$$\frac{d^2 f_{nlm}}{d\tau^2} + \frac{3}{a}\frac{da}{d\tau}\frac{df_{nlm}}{d\tau} - N^2 \left(\frac{n^2 - 1}{a^2} + m^2 \right) f_{nlm} = 0 \tag{11.13}$$

Here, $a(\tau), N$ is the solution to the field equation and constraint for the background satisfying $a(0) = 0$, $a(1) = \tilde{a}$. Explicitly,

$$a(\tau) = \frac{1}{H} \sin(NH\tau), \quad N = \frac{1}{H} \left(\frac{\pi}{2} - \cos^{-1}(\tilde{a}H) \right) \tag{11.14}$$

The solutions to (11.13) may be written down explicitly in terms of hypergeometric functions, although this is not necessary for our purposes. They are regular everywhere, with the possible exception of the region near $\tau = 0$. In this region, $a(\tau) \sim N\tau$, and it is easily shown that the solutions to (11.13) behave like τ^{-n-1}, or τ^{n-1}. Clearly only one of these is regular. It may be picked out by imposing the initial condition

$$f_{nlm}(0) = 0, \quad \text{for} \quad n = 2, 3, ..., \quad \text{and} \quad \frac{df_{nlm}}{d\tau}(0) = 0, \quad \text{for} \quad n = 1. \tag{11.15}$$

These are the initial conditions on the histories implied by the no-boundary proposal. The histories also satisfy the final condition

$$f_{nlm}(1) = \tilde{f}_{nlm} \tag{11.16}$$

Because the modes decouple, we may write

$$\psi[\tilde{a}, \tilde{\Phi}(\mathbf{x})] = \prod_{nlm} \psi_{nlm}(\tilde{a}, \tilde{f}_{nlm}) \tag{11.17}$$

From (11.10) it then follows that

$$\psi(\tilde{a}, \tilde{f}_{nlm}) = \int \mathcal{D}f_{nlm} e^{-I_{nlm}} \tag{11.18}$$

Because I_{nlm} is quadratic in the scalar field modes, the path integral (11.18) may be evaluated exactly to yield an expression of the form

$$\psi(\tilde{a}, \tilde{f}_{nlm}) = A_{nlm}(\tilde{a}) \exp\left(-\bar{I}_{nlm}(\tilde{a}, \tilde{f}_{nlm})\right) \tag{11.19}$$

Here, $\bar{I}_{nlm}(\tilde{a}, \tilde{f}_{nlm})$ is the action of the solution to the Euclidean field equations satisfying the boundary conditions (11.15), (11.16). Let us denote this solution by $g_n(\tau)$. It is independent of l, m, because the field and equations and boundary conditions are. Then it is readily shown that

$$I_{nlm}(\tilde{a}, \tilde{f}_{nlm}) = \frac{1}{2}\left[a^3(\tau)g_n(\tau)\frac{dg_n(\tau)}{d\tau}\right]_0^1 = \frac{1}{2}\tilde{a}^3\tilde{f}_{nlm}^2\left[\frac{1}{g_n}\frac{dg_n}{d\tau}\right]_{\tau=1} \tag{11.20}$$

The matter wave functional defined by the no-boundary proposal is therefore given by (11.18), with

$$\psi_{nlm}(\tilde{a}, \tilde{f}_{nlm}) = A_{nlm}(\tilde{a}) \exp\left(-\frac{1}{2}\tilde{a}^3\tilde{f}_{nlm}^2\left[\frac{1}{g_n}\frac{dg_n}{d\tau}\right]_{\tau=1}\right) \tag{11.21}$$

The key point to note is that it involves the expression $\dot{g}_n/g_n$, evaluated at the upper end-point, where the $g_n(\tau)$ are solutions to the field equations which are *regular* on the Euclidean section.

We now need to show that this matter wave functional corresponds to the Euclidean vacuum state defined above. This basically involves determining what the vacuum state $|0\rangle$ defined by (11.4) looks like in the functional Schrödinger picture. To this end, first

compare the expansions (11.3) and (11.11) of the scalar field. Turning (11.11) into an operator, one may therefore write

$$\hat{f}_{nlm}(t) = y_n(t)\hat{a}_{nlm} + y_n^*(t)\hat{a}_{nlm}^\dagger \tag{11.22}$$

The momentum operator conjugate to this is

$$\hat{\pi}_{nlm}(t) = a^3\dot{\hat{f}}_{nlm} = a^3\dot{y}_n(t)\hat{a}_{nlm} + a^3\dot{y}_n^*(t)\hat{a}_{nlm}^\dagger \tag{11.23}$$

(11.22) and (11.33) are readily inverted to yield

$$\hat{a}_{nlm} = -iy_n^*\left(a^3\frac{\dot{y}_n^*}{y_n^*}\hat{f}_{nlm} - \hat{\pi}_{nlm}\right) \tag{11.24}$$

By inserting a complete set of field states $\{|f_{nlm}\rangle\}$ in (11.4), we thus obtain the following equation for the vacuum state $\psi_{nlm}(f_{nlm}) \equiv \langle f_{nlm}|0\rangle$:

$$\left(a^3\frac{\dot{y}_n^*}{y_n^*}f_{nlm} + i\frac{\partial}{\partial f_{nlm}}\right)\psi_{nlm}(f_{nlm}) = 0 \tag{11.25}$$

It is readily solved to yield

$$\psi_{nlm} = \exp\left(\frac{i}{2}a^3\frac{\dot{y}_n^*}{y_n^*}f_{nlm}^2\right) \tag{11.26}$$

This, therefore, is the Euclidean vacuum in the functional Schrödinger picture. Going to the Euclidean section, one thus obtains

$$\psi_{nlm} = \exp\left(-\frac{1}{2}a^3\frac{1}{y_n^*}\frac{dy_n^*}{d\tau}f_{nlm}^2\right) \tag{11.27}$$

The equivalence of (11.27) and (11.21) immediately follows from the definition of the Euclidean vacuum, which is that the y_n, and hence the y_n^*, are solutions to the field equations which are regular on the Euclidean section. This completes the demonstration that the vacuum state defined by the no-boundary proposal is the de Sitter-invariant Euclidean vacuum.

A more heuristic argument for the de Sitter invariance of the no-boundary matter wave functionals may also be given. This argument shows that the de Sitter invariance is an inevitable consequence of the very geometrical nature of the no-boundary proposal, and is therefore true of most types of matter fields (D'Eath and Halliwell, 1987).

Suppose one asks for the quantum state of the matter field on a three-sphere of radius $a < H^{-1}$. The no-boundary state is defined by a path integral of the form (11.10). One

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sums over all matter fields regular on the section of four-sphere interior to the three-sphere which match the prescribed data on the three-sphere boundary. The resulting state will depend on the geometry only through the radius of the three-sphere, and not on its intrinsic location or orientation on the four-sphere. One thus has the freedom to move the three-sphere around on the four-sphere without changing the quantum state – at each location one is summing over exactly the same field configurations to define it. These different locations are related to each other by the isometry group of the four-sphere, $SO(5)$. It follows that the state is $SO(5)$-invariant on the Euclidean section. On continuation back to the Lorentzian section, one thus finds that the state is invariant under $SO(4,1)$, the de Sitter group; that is, the state is de Sitter invariant. This argument may be made mathematically precise, although we will not go into that here.

It may be shown that the tunneling wave function picks out the same vacuum state. This follows essentially from the imposition of a regularity requirement on the matter wave functionals (Vachaspati, 1989; Vachaspati and Vilenkin, 1988; Vilenkin, 1988).

12. SUMMARY

The purpose of these lectures has been to describe the route from a quantum theory of cosmological boundary conditions to a classical universe with the potential for evolving into one similar to that in which we live.

We began in Section 2 with a brief introductory tour of quantum cosmology by way of a simple example. This simple model illustrated the need for a quantum theory of initial conditions. The general formalism of quantum cosmology was briefly outlined in Sections 3 and 4. The full theory is very difficult to handle in practice, so in Section 5, we restricted to the case of minisuperspace models. The canonical and path integral formalism for minisuperspace models was described. In Section 6, we discussed the most important prediction a quantum theory of cosmology should make – the emergence of classical spacetime. The emergence of classical spacetime is very much contingent on boundary conditions on the wave function, and occurs only in particular regions of configuration space. These ideas were further developed in Section 7, in which the WKB approximation was described. Wave functions of oscillatory WKB form correspond to classical spacetime in that they

are peaked about a set of classical solutions to the Einstein equations. Moreover, this set of solutions is a subset of the general solution; thus boundary conditions on the wave function of the universe effectively imply initial conditions on the set of classical solutions. We discussed the way in which the wave function may be used to construct a measure on this set of classical solutions.

In Section 8, certain boundary condition proposals were described – the no-boundary proposal of Hartle and Hawking, and the tunneling boundary condition of Linde and of Vilenkin. Each of these proposals suffers from imprecision or lack of generality, although with a certain amount of license, each may be successfully used to calculate wave functions in simple models. We calculated the no-boundary and tunneling wave functions for the scalar field model introduced in Section 2. These wave functions were compared in Section 9. The two wave functions are peaked about the same set of classical solutions, but they give rather different measures on this set of solutions. In particular, they may give very different values for the likelihood of sufficient inflation. The comparison of these two wave functions was inconclusive, but this merely reflects the fact that no consensus of opinion has yet emerged.

In Sections 10 and 11 we described how one goes beyond minisuperspace by considering inhomogeneous perturbations. There are two things that come out of this. First, one finds that in the limit in which gravity becomes classical, one recovers quantum field theory for the perturbations in a fixed classical gravitational background. Secondly, boundary conditions on the wave function of the universe are found to imply a particular choice of vacuum state for the perturbations. In particular, in the case of a de Sitter background, the no-boundary and tunneling proposals pick out the de Sitter-invariant Euclidean vacuum. The density perturbations arising from this particular choice are of the correct form for the subsequent emergence of large scale structure.

Finally, I would like to emphasize the rather open-ended nature of many of the issues in quantum cosmology covered in these lectures. One might get the impression from reading the literature on the subject that certain aspects of the field are complete and neatly tied up beyond criticism. In my opinion this is most certainly not the case, and I have tried to indicate areas of difficulty at the appropriate points throughout the text. There is, I believe, considerable scope for development and improvement in many parts of the field. For example, the methods used in quantum cosmology to extract predictions from the wave function, as described in Section 6, are rather crude, and it would be much more

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satisfying to apply methods such as those described by Hartle in his lectures (Hartle, 1990). Another example concerns the use of the path integral in quantum cosmology. Although the role it plays is supposedly very central, especially in the formulation of the no-boundary proposal, it is I think reasonable to say that, with but a few exceptions, its use in quantum cosmology has been for the most part rather heuristic. A more careful approach using the path integral in a serious way would very desirable. Furture investigation of these and other issues is likely to be very profitable.

ACKNOWLEDGEMENTS

13. A GUIDE TO THE LITERATURE

General

Some of the earlier works in the field of quantum cosmology include those of DeWitt (1967), Misner (1969a, 1969b, 1969c, 1970, 1972, 1973) and Wheeler (1963, 1968). Early reviews are those of MacCallum (1975), Misner (1972) and Ryan (1972). More recent introductory or review accounts are those of Fang and Ruffini (1987), Fang and Wu (1986), Halliwell (1988b), Hartle (1985d, 1986), Hawking (1984b), Linde (1989a, 1989b), Narlikar and Padmanabhan (1986) and Page (1986a).

Minisuperspace Models

The literature contains a vast number of papers on minisuperspace. Models with **scalar fields** have been considerd by Blyth and Isham (1975), del Campo and Vilenkin (1989b), Carow and Watamura (1985), Christodoulakis and Zanelli (1984b), Esposito and Platania (1988), Fakir (1989), Gibbons and Grishchuk (1988), Gonzalez-Diaz (1985), Hartle and Hawking (1983), Hawking (1984a), Hawking and Wu (1985), Moss and Wright (1984), Page (1989a), Poletti (1989), Pollock (1988a), Yokoyama et al. (1988) and Zhuk (1988). The scalar field model of Section 2 is described in, for example, Hawking (1984a) and Page (1986a).

Anisotropic minisuperspace models are considered in the papers by Amsterdamski (1985), Ashtekar and Pullin (1990), Berger (1975, 1982, 1984, 1985, 1988, 1989), Berger and Vogeli (1985), Bergamini and Giampieri (1989), del Campo and Vilenkin (1989a), Duncan and Jensen (1988), Fang and Mo (1987), Furusawa (1986), Halliwell and Louko (1990), Hawking and Luttrell (1984), Hussain (1987, 1988), Kodama (1988b), Laflamme (1987b), Laflamme and Shellard (1987), Louko (1987a, 1987b, 1988a), Louko and Ruback (1989), Louko and Vachaspati (1988), Matsuki and Berger (1989), Misner (1969c, 1973), Moss and Wright (1985) and Schleich (1988).

The extension to **Kaluza-Klein** theories has been considered by Beciu (1985), Bleyer at al. (1989), Carow-Watamura et al. (1987), Halliwell (1986, 1987a), Hu and Wu (1984, 1985, 1986), Ivashchuk et al.(1989), Lonsdale (1986), Matzner and Mezzacappa (1986), Okada and Yoshimura (1986), Pollock (1986), Shen (1989a), Wu (1984, 1985a, 1985b, 1985c) and Wudka (1987a).

In these lectures we concentrated on Einstein gravity. Minisuperspace models involving **higher derivative** actions have been studied by Coule and Mijić (1988), Hawking (1987a), Hawking and Luttrell (1984b), Horowitz (1985), Hosoya (1989), Mijić et al. (1989), Pollock (1986, 1988b, 1989b) and Vilenkin (1985a).

Other minisuperspace models not obviously falling into any of the above categories include those of Brown (1989), Li and Feng (1987), Liu and Huang (1988), Mo and Fang (1988) and Wudka (1987b).

The question of the **validity** of minisuperspace, when considered as an approximation to the full theory, has been addressed by Kuchař and Ryan (1986, 1989).

Inhomogeneous Peturbations about Minisuperspace

Perturbative models of the type described in Section 10 have been studied by Anini (1989a, 1989b), Banks et al.(1985), D'Eath and Halliwell (1987), Fischler et al. (1985), Halliwell and Hawking (1985), Morris (1988), Ratra (1989), Rubakov (1984), Shirai and Wada (1988), Vachaspati and Vilenkin (1988), Vilenkin (1988) and Wada (1986, 1986c, 1987).

An important feature of this type of model is the **derivation of the Schrödinger equation** from the Wheeler-DeWitt equation and the emergence of **quantum field theory in curved spacetime** This sort of issue has been considered by Banks (1985), Brout (1987), Brout et al. (1987), Brout and Venturi (1989), DeWitt (1967), Halliwell (1987c), Halliwell and Hawking (1985), Laflamme (1987a), Lapchinsky and Rubakov (1979), Vachaspati (1989) and Wada (1987).

In Section 10 we only derived the dynamics of the perturbation modes on a minisuperspace background. However, one can go one step further than that and ask how the perturbation modes react back on the minisuperspace background. In principle, one may thus attempt to derive the **semi-classical Einstein equations**. This area seems to be somewhat confused, and no completely clear derivation has yet been given. The relevant papers are those of Brout (1987), Brout et al. (1987), Brout and Venturi (1989), Castagnino et al. (1988), Halliwell (1987b), Hartle (1986), Padmanabhan (1989a), Padmanabhan (1989c), Padmanabhan and Singh (1988) and Singh and Padmanabhan (1989).

Black Holes and Spherically Symmetric Systems

One is normally interested in cosmological models, but spherically symmetric systems, including black holes have been studied by Allen (1987), Fang and Li (1986), Laflamme (1987b), Nagai (1989), Nambu and Sasaki (1988) and Rodrigues et al. (1989). The connection between the path integral for the no-boundary wave function and that for the partition function for a black hole in a box is discussed by Halliwell and Louko (1990).

Quantum Cosmology and String Theory

String-inspired models have been studied by Enqvist et al. (1987, 1989), Gonzalez-Diaz (1988), Lonsdale and Moss (1987) and Pollock (1989a, 1989b). The formal resemblances between quantum cosmology and string theory have been explored by Birmingham and Torre (1987), Luckock et al. (1988) and Matsuki and Berger (1989).

Fermionic Matter and Supersymmety

Most papers involve bosonic matter sources, but the inclusion of fermions and supersymmetric aspects have been studied by Christodoulakis and Papadopoulos (1988), Christodoulakis and Zanelli (1984b), D'Eath and Halliwell (1987), D'Eath and Hughes (1988), Elitzur et al. (1986), Furlong and Pagels (1987), Isham and Nelson (1974), Macias et al. (1987), Shen (1989b) and Shen and Tan (1989).

Interpretation

The rather basic interpretation mentioned in Section 4 (that we regard a strong peak in the wave function as a prediction) comes from Hartle (1986), Geroch (1984) and Wada (1988a). Other relevant papers include those of Barbour and Smolin (1989), Barrow and Tipler (1986), DeWitt and Graham (1973), Drees (1987), Ellis et al. (1989), Everett (1957), Gell-Mann and Hartle (1989), Halliwell (1987b, 1989b), Hartle (1988a, 1988b, 1988c, 1990), Kazama and Nakayama (1985), Markov and Mukhanov (1988), Tipler (1986, 1987), Wald and Unruh (1988), Vilenkin (1989) and Wada (1986a, 1988b).

The **decoherence** requirement discussed in Section 6, for quantum cosmology, has been considered by Calzetta (1989), Fukuyama and Morikawa (1989), Gell-Mann and Hartle (1989), Halliwell (1989b), Joos (1986), Kiefer (1987, 1988, 1989a, 1989c), Mellor (1989), Padmanabhan (1989b), Morikawa (1989) and Zeh (1986, 1988, 1989a, 1989b). Further discussions of this and related issues are those of Hu (1989) (which also includes extensive references on statistical effects) and Kandrup (1988).

Decoherence as considered in the above references involves the notion of diagonalization of a reduced density matrix. **Density matrices** in quantum cosmology have been

considered in a somewhat different context by Hawking (1987b), Page (1986b).

For more general discussions of decoherence in quantum mechanics, see Gell-Mann and Hartle (1990), Joos and Zeh (1985), Unruh and Zurek (1989) and Zurek (1981, 1982).

In an attempt to see how classical behaviour emerges, some authors have constructed **wavepacket** solutions to the Wheeler-DeWitt equation, including Kiefer (1988, 1989d), Kazama and Nakayama (1985) and Wada (1985).

The first requirement for classical behaviour discussed in Section 6 (peaking about classical configurations) was discussed using the **Wigner** function by Halliwell (1987b), Kodama (1988a) and Singh and Padmanabhan (1989). Use of the Wigner function in this way has been criticised by Anderson (1990). A somewhat different approach using the Wigner function is that of Calzetta and Hu (1989).

The Issue of Time

Various authors have addressed the issue of time in quantum cosmology and quantum gravity more generally. The sorts of question one is interested in are along the following lines: Does the theory possess an intrinsic time? If it does not, can one quantize it? Does time emerge from a theory that has no time in it to start with ? Many of these questions are discussed by Banks (1985), Brout (1987), Brout et al. (1987), Brout and Venturi (1989), Brown and York (1989), Castagnino (1989), Englert (1989), Fukuyama and Kamimura (1988), Fukuyama and Morikawa (1989), Greensite (1989a, 1989b), Halliwell (1989a), Hartle (1988a, 1988b, 1988c, 1990), Jacobson (1989), Kuchař (1989), Sorkin (1987, 1989) and Unruh and Wald (1988).

A related issue is the connection of the cosmological **arrow of time** with the thermodynamic arrow in quantum cosmology. This has been studied by Fukuyama and Morikawa (1989), Hawking (1985), Page (1984, 1985), Qadir (1987), Wada (1989) and Zeh (1986, 1988, 1989a, 1989b).

Path Integrals and the Wheeler-DeWitt Equation

The explicit construction of the path integral for the wave function of the universe and the derivation of the associated Wheeler-DeWitt equation have been considered by

Barvinsky (1986), Barvinsky and Ponomariov (1986), Barvinsky (1987), Halliwell (1988), Halliwell and Hartle (1990), Teitelboim (1980, 1982, 1983a, 1983b, 1983c) and Woodard (1989). The detailed construction of the path integral described in Section 4 (Eq. (4.7)) is described by Teitelboim (1982, 1983a). The discussion of the minisuperspace path integral in Section 5 is based on Halliwell (1988).

The issue of finding **complex contours** to make the Euclidean path integral converge has been studied by Gibbons, Hawking and Perry (1978), Halliwell and Hartle (1989), Halliwell and Louko (1989a, 1989b, 1990), Halliwell and Myers (1989), Hartle (1984, 1989), Hartle and Schleich (1987), Mazur and Mottola (1989) and Schleich (1985, 1987, 1989).

Other papers involving path integrals are those of Arisue et al. (1987), Berger (1985), Berger and Vogeli (1985), Duncan and Jensen (1988), Farhi (1989), Giddings (1990), Hajicek (1986a, 1986b), Hartle (1984, 1988a, 1988b, 1988c), Louko (1988a, 1988b, 1988c, 1988d), Narlikar and Padmanabhan (1983) and Suen and Young (1989).

Quantization Methods and Superspace

One most commonly uses the Dirac quantization procedure in quantum cosmology, in which one takes the wave function to be annihilated by the operator versions of the constraints. However, one could in principle use the ADM (or reduction) method, in which one solves the constraints classical before quantizing. The connections between these methods for systems like gravity has been considered by Ashtekar and Horowitz (1982), Gotay (1986), Gotay and Demaret (1983), Gotay and Isenberg (1980), Hajicek (1989), Isenberg and Gotay (1981) and Kaup and Vitello (1974).

The properties of **superspace** and quantization methods in it have been discussed by DeWitt (1970), Fisher (1970), Giulini (1989), Isham (1976), and Kuchař (1981). The article by Kuchař also contains a useful guide to the literature on canonical quantization.

Topological Aspects

Goncharov and Bytsenko (1985, 1987), Gurzadyan and Kocharyan (1989), Li Miao (1986), Mkrtchyan (1986), and Starobinsky and Zel'dovich (1984), considered the possibilities of non-trivial topologies in quantum creation of the universe. Other interesting

toplogical aspects of the no-boundary proposal have been considered by Hartle and Witt (1988) (see also Louko and Ruback (1989)).

Singularities

Numerous authors have been interested in singularities in quantum cosmology and their possible avoidance, including Laflamme and Shellard (1987), Lemos (1987), Louko (1987a), Narlikar (1983, 1984) and Smith and Bergman (1988).

Boundary Condition Proposals

We concentrated exclusively on the boundary condition proposals of Hartle and Hawking (Hartle and Hawking, 1983; Hawking 1982, 1984a), Linde (1984a, 1984b, 1984c) and Vilenkin (1982, 1983, 1984, 1985b, 1986, 1988), but there are others (see for example, Suen and Young (1989)).

Quantum Creation of the Universe

Some of the older papers on quantum creation of the universe are those of Atkatz and Pagels (1982), Brout, Englert and Gunzig (1978, 1979), Brout, Englert and Spindel (1979), Casher and Englert (1981), Gott (1982) and Tryon (1973). Various aspects of the quantum creation of the universe as a tunneling event have been explored by Goncharov et al. (1987), Grishchuk (1987), Grishchuk and Sidorov (1988, 1989), Grishchuk and Zel'dovich (1982), Lavrelashvili, Rubakov, Serebryakov and Tinyakov (1989), Lavrelashvili, Rubakov, and Tinyakov (1985), Rubakov (1984) and Rubakov and Tinyakov (1988).

Measures

The measure coming from quantum cosmology on sets of inflationary solutions, and also classical measures, have been studied Gibbons et al. (1987) and Hawking and Page (1986, 1988). Gibbons and Grishchuk (1988) introduced a measure on the set of solutions to the Wheeler-DeWitt equation.

Operator Ordering

The issue of operator ordering in the Wheeler-DeWitt equation has been studied in minisuperspace by Halliwell (1988), Misner (1972) and Moss (1988). More generally, see Christodoulakis and Zanelli (1986a, 1986b, 1987), Friedman and Jack (1988), Hawking and Page (1986) and Tsamis and Woodard (1987).

Creating a Universe in the Laboratory

The possibility of quantum creation of an inflationary universe in the laboratory has bee studied by Farhi et al. (1989) and Fischler et al. (1989). See also Hiscock (1987) and Sato et al. (1982).

Miscellaneous

Regge calculus minisuperspace models have been studied by Hartle (1985a, 1985b, 1985c, 1989). In (2+1) dimensions, gravity becomes essentially quantum mechanical. This has been studied from a quantum cosmology viewpoint by Hosoya and Nakao (1989) and Martinec (1984). Considerable simplifications appear to occur in general relativity using the **Ashtekar variables** (Ashtekar, 1987). Their application to cosmologies has been considerd by Ashtekar and Pullin (1990), Hussain and Smolin (1989) and Kodama (1988b). The relationship between the wave function of the universe and the **stochastic approach** to inflation have been studied by Goncharov et al. (1987), Goncharov and Linde (1986) and Mijić (1988a, 1988b, 1989). Many classical cosmologies exhibit **chaos**. Quantization of such cosmologies has been studied by Berger (1989) and Furusawa (1986). Finally, mention should be made of the extensive contributions of Narlikar, Padmanabhan and collaborators, much of which concentrates on quantization of the conformal part of the metric, including Narlikar (1981, 1983, 1984), Padmanabhan (1981, 1982a, 1982b, 1983a, 1983b, 1983c, 1983d, 1983e, 1983f, 1984a, 1984b, 1985a, 1985b, 1986, 1987, 1988), Padmanabhan and Narlikar (1981, 1982), Padmanabhan et al. (1989), Singh and Padmanabhan (1987).

REFERENCES

The following is a list of almost 400 papers in the field of quantum cosmology, plus a small number of other references relevant to these lectures (the latter are generally distinguised by the fact that their title is not included). A substantially similar list of references may be found in "A Bibliography of Papers on Quantum Cosmology" (*Int.J.Mod.Phys.A*, 1990, to appear).

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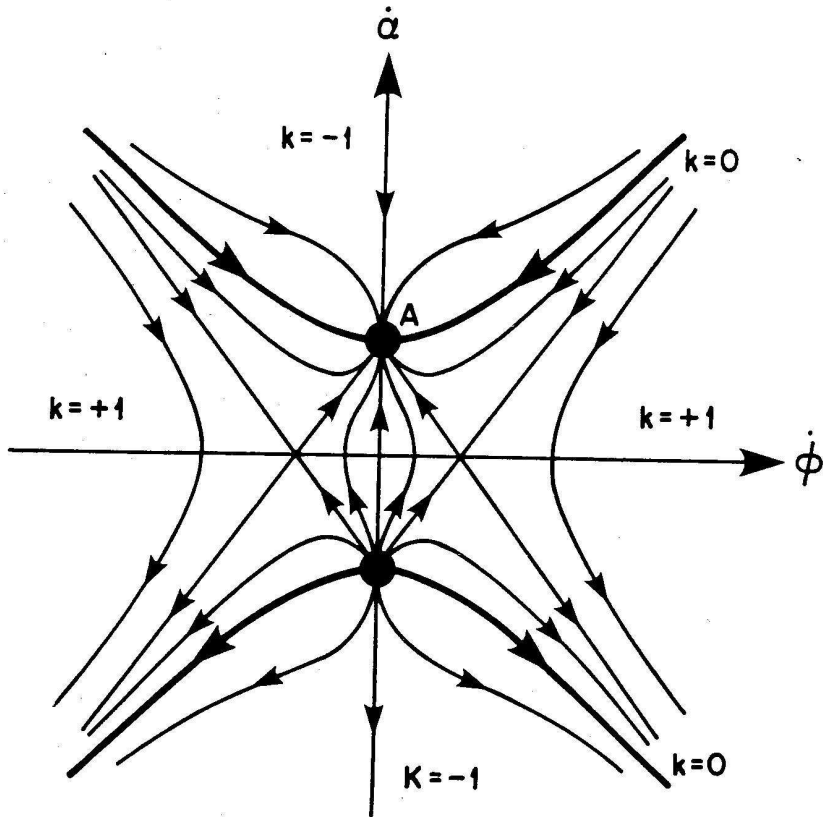
Fig.1: The phase portrait of the system (2.7)
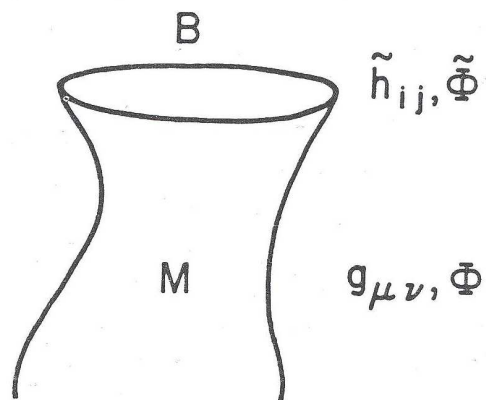
Fig.2: A pictorial representation of the histories summed over in the calculation of the wave function $\Psi[\tilde{h}_{ij}, \tilde{\Phi}]$.
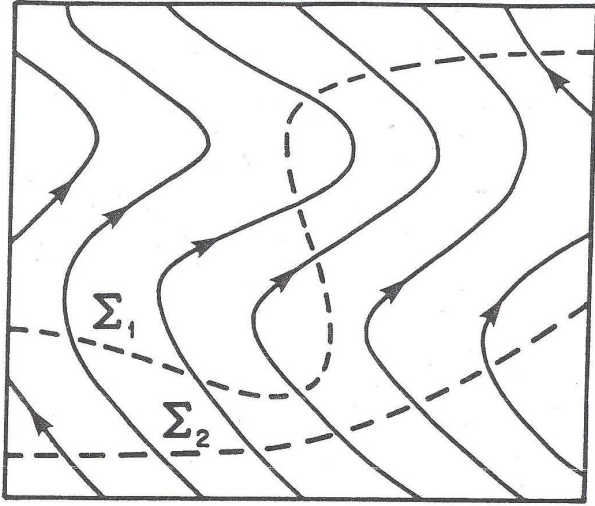
Fig.3: The integral curves of the current J (the bold lines) and some possible choices for hypersurfaces Σ (the dashed lines). Σ_1 is a bad choice because the flow of J intersects Σ_1 more than once. Σ_2 is a good choice because the flow intersects it once and only once.
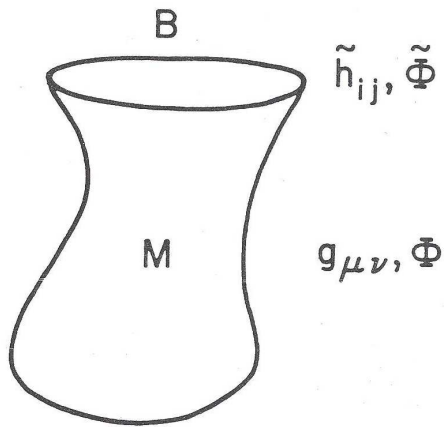

Fig.4: A pictorial representation of the class of histories summed over in the calculation of the no-boundary wave function.
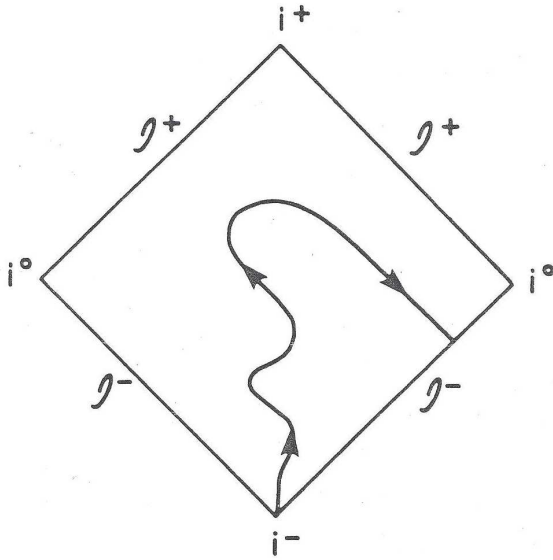
Fig.5: *The conformal diagram of minisuperspace for the scalar field model. The current of a solution satisfying the outgoing modes condition is shown. It enters at the non-singular boundary, i^-, and flows out across $\mathcal{I}^-$, part of the singular boundary.*
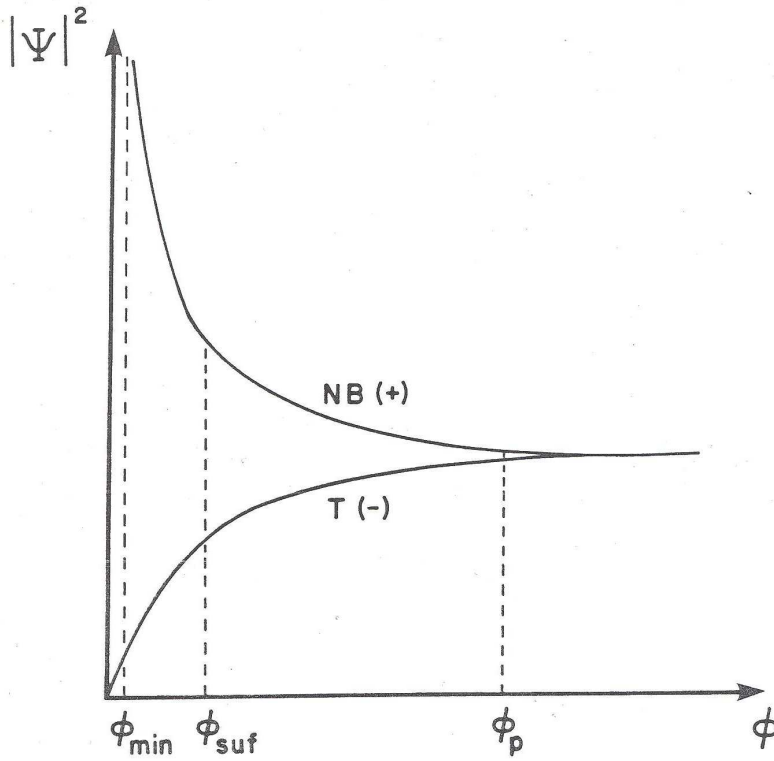

Fig.6: *A plot of $|\Psi|^2$ against ϕ on a hypersurface of constant scale factor, for the tunneling wave function and (one component of) the no-boundary wave function.*