

## Quantum Cosmology for the XXI<sup>st</sup> Century: A Debate

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Quantum cosmology from the late sixties into the early XXI<sup>st</sup> century is reviewed and appraised in the form of a debate, set up by two presentations on mainly the Wheeler–DeWitt quantization and on loop quantum cosmology, respectively. (Open) questions, encouragement and guiding lines shared with the audience are provided here.

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### 1. *Raison d'être* (Paulo Vargas Moniz)

Quantum cosmology is the application of quantum theory to the Universe as a whole. As such, it plays a special role in physics. First, it is based on a quantum theory of gravity about which no general agreement has been reached. Second, as part of cosmology, it has not yet been susceptible to observational tests.

Providing a corresponding summary was a major aim of the debate reported here<sup>a</sup>. More precisely, we wanted to present a viewpoint on past progress, the current situation, and new approaches to proceed.<sup>2</sup> Hence the presentation by Claus Kiefer as summarized in Sec. 2. On the other hand, we intended this to be an interchange of ideas, dynamically including and challenging the audience.

A subsequent talk was delivered by Martin Bojowald (see Sec. 3), in which the more recent progress of a new school<sup>3</sup> — bringing novel ideas into the current century, together with methods as well as challenges to quantum cosmology — was widely acknowledged: by importing techniques and results from loop quantum gravity, quantum cosmology did gain new momentum, and new results for quantum cosmology have been appearing in the literature ever since.

The subject of quantum cosmology therefore enjoys renewed interest, but does perhaps not involve as large a set of groups and researchers as it deserves. For this reason the debate, whose two lecturers are not in opposing camps, but rather act

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<sup>a</sup> I decided on a more pro-active and all-inclusive dynamical discussion of quantum cosmology. Quite like the “Presidential debates” that were seen in the run for office at the White House,<sup>1</sup> in which an audience of citizens could directly put questions.

as colleagues striving for progress within overlapping schools, has left ample time for open questions and fair criticism as well as some prudent scepticism from the audience. The overarching question of the debate was: can quantum cosmology take a main stance and central stage in XXI<sup>st</sup>-century research? A (tentative!) summary of the questions and constructive replies is collected in Sec. 4.

In a slowly progressing, only thinly populated research area, earlier results are often forgotten, overlooked or ignored, then rediscovered, rederived or reproduced in disguise. An appraisal of preceding contributions may, at a pragmatic level, save some work and, more idealistically, provide unity. This is one of our aims. We invite all unhappy readers of these notes to send us comments about serious omissions or misrepresentations, which may find a place in a future review.

Was this debate worth it? We think so. We add some thoughts in Sec. 5, proposing where new directions could be sought, even if other results from other sometimes and somewhat unforeseen directions may turn up in the meantime, then further promote the subject and continue to lead its progress. But anyway, that is how research most of the time makes significant progress: **Unexpectedly**.

## 2. A legacy from the XX<sup>th</sup> century (Claus Kiefer)

### 2.1. *Why quantum cosmology?*

Why should or even must one apply quantum theory to the Universe as a whole? Is it not sufficient to consider the standard picture of cosmology describing the Universe as expanding from a dense hot phase in the past to its present state with galaxies and clusters of galaxies? After all, there exist at present no observations for which a quantum cosmological explanation seems to be mandatory.

There is, in fact, support for quantum cosmology for at least two reasons.<sup>b</sup> First, general relativity is incomplete in that it predicts the occurrence of singularities in a wide range of situations. This concerns the origin of the Universe (“Big-Bang”) but also its final fate; some models using dark energy as an explanation for the current acceleration of the Universe predict singularities in the future. Therefore, a more general theory is needed in order to encompass these situations. The general belief is that this theory is a quantum theory of gravity, for it was, after all, quantum mechanics which rescued the atom from the singularities of classical electrodynamics.

The second reason derives from a general feature of quantum theory. Except in microscopic cases, most quantum systems cannot be considered as isolated. They interact with their natural environment, as a result of which a globally entangled state results that includes the variables of the system and the environment. For macroscopic systems, this entanglement leads to the emergence of classical properties for the system – a process called *decoherence*.<sup>8</sup> Since the environment

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<sup>b</sup>In the following, I shall heavily rely on my earlier contributions to this topic, in particular on [2] as well as on [4–6] and the references therein. A classic introduction to quantum cosmology is [7].

of a system is again coupled to its environment, the only truly closed quantum system is the Universe as a whole. One arrives in this way straightforwardly at the notion of a “wave function of the Universe”. This notion is conceptually independent of any particular interaction and a direct consequence of the central feature of any quantum theory – entanglement. As such it was already employed, at least in spirit, by Hugh Everett in his presentation of the “relative-state interpretation”<sup>9</sup> and can be found also at other places.<sup>c</sup>

Since gravity is the dominating interaction at cosmic scales, quantum cosmology must, however, be based on a theory of quantum gravity.<sup>2</sup> But which theory?

## 2.2. Which framework should one use?

At present we do not have a final framework for a quantum theory of gravity. Among the existing approaches, one can mainly distinguish between the direct quantization of Einstein’s theory of general relativity and string theory (or M-theory). The latter is more ambitious in the sense that it aims at a unification of all interactions within a single quantum framework. Quantum general relativity, on the other hand, attempts to construct a consistent, non-perturbative, quantum theory of the gravitational field on its own.

The fundamental length scales that are connected with these theories are the Planck length,  $l_P = \sqrt{G\hbar}$ , or the string length,  $l_s$ .<sup>d</sup> It is generally assumed that the string length is somewhat larger than the Planck length. Although not fully established in quantitative detail, quantum general relativity should follow from superstring theory for scales  $l \gg l_s > l_P$ . Can one, in spite of this uncertainty about the fundamental theory, say something reliable about quantum gravity? In [11] I have made the point that this is indeed possible. The situation can be compared to the role of the quantum mechanical Schrödinger equation. Although this equation is not fundamental (it is non-relativistic, it is not field-theoretic), important insights can be drawn from it. For example, in the case of the hydrogen atom, one has to impose boundary conditions for the wave function at the origin  $r \rightarrow 0$ , that is, at the centre of the atom. This is certainly not a region where one would expect non-relativistic quantum mechanics to be exactly valid, but its consequences, in particular the resulting spectrum, are empirically correct to an excellent approximation.

Erwin Schrödinger has found his equation by “guessing” a wave equation from which the Hamilton–Jacobi equation of classical mechanics can be recovered in the limit of small wavelengths, analogously to the limit of geometric optics from wave optics. The same approach can be applied to general relativity. One can start from the Hamilton–Jacobi version of Einstein’s equations and “guess” a wave equation from which they can be recovered in the classical limit. The only assumption that

<sup>c</sup>To quote from the monograph<sup>10</sup> of Feynman and Hibbs, p. 58: “All of history’s effect upon the future of the Universe could be obtained from a single gigantic wave function.”

<sup>d</sup>We set  $c = 1$  throughout.

is required is the universal validity of quantum theory, that is, its linear structure. It is not yet needed for this step to impose a Hilbert-space structure (a linear space with a scalar product). Such a structure is employed in quantum mechanics because of the probability interpretation for which one needs a scalar product and its conservation in time (unitarity). The status of this interpretation in quantum gravity remains open. We should, however, keep in mind that it is exactly the normalization of quantum states which is crucial in obtaining the correct spectra for atoms and other systems.

The result of this approach is quantum geometrodynamics. Its central equation is the Wheeler–DeWitt equation, first discussed by Bryce DeWitt and John Wheeler in the 1960s. In a shorthand notation, it is of the form

$$\mathcal{H}\Psi = 0 , \quad (1)$$

where  $\mathcal{H}$  denotes the full Hamiltonian for both the gravitational field (here described by the three-metric) as well as all non-gravitational fields.<sup>e</sup> For the detailed structure of this equation I can refer, for example, to the classic paper by DeWitt<sup>12</sup> and Wheeler<sup>13</sup> or my review in [2]. Two properties are especially important for our purpose here. First, this equation does not contain any classical time parameter  $t$ . The reason is that space-time as such has disappeared in the same way as particle trajectories have disappeared in quantum mechanics; here, only space (the three-geometry) remains. Second, inspection of  $\mathcal{H}$  exhibits the local hyperbolic structure of the Hamiltonian, that is, the Wheeler–DeWitt equation possesses locally the structure of a Klein–Gordon equation (that is, a wave equation). In the vicinity of Friedmann Universes, this hyperbolic structure is not only locally present, but also globally. One can thus define a new time variable which exists only intrinsically and which can be constructed from the three-metric (and non-gravitational fields) itself. It is this absence of external time that could render the probability interpretation and the ensuing Hilbert-space structure obsolete in quantum gravity, for no conservation of probability may be needed.<sup>f</sup> Of course, a Hilbert-space structure *is* needed in the semiclassical limit discussed below.

In the following I shall briefly review the key points in the application of the Wheeler–DeWitt equation to quantum cosmology.

### 2.3. Wheeler–DeWitt equation and boundary conditions

Cosmology can only be dealt with if one makes simplifying assumptions. Since the Universe looks approximately homogeneous and isotropic on large scales, one can impose this assumption on the metric of space-time. As a result, one

<sup>e</sup>Strictly speaking, one has the quantized Hamiltonian constraint as well as quantized diffeomorphism constraints.

<sup>f</sup>The situation is different for an isolated quantum gravitational system such as a black hole; there, the semiclassical time of the rest of the Universe enters the description.<sup>14</sup>

obtains the Friedmann–Lemaître models usually employed. Such models are called “minisuperspace models”.

In the case of a Friedmann Universe with a homogeneous scalar field  $\phi$ , the Wheeler–DeWitt equation reads (see e.g. [2,7] or the Appendix of [4] for a derivation)

$$\mathcal{H}\Psi = \left\{ \frac{2\pi G\hbar^2}{3} \frac{\partial^2}{\partial \alpha^2} - \frac{\hbar^2}{2} \frac{\partial^2}{\partial \phi^2} + e^{6\alpha} \left( V(\phi) + \frac{\Lambda}{8\pi G} \right) - 3e^{4\alpha} \frac{k}{8\pi G} \right\} \Psi(\alpha, \phi) = 0, \quad (2)$$

with cosmological constant  $\Lambda$  and curvature index  $k = \pm 1, 0$ . The variable  $\alpha = \ln a$ , where  $a$  stands for the scale factor, is introduced to obtain a convenient form of the equation.

The general structure of the Wheeler–DeWitt equation concerning the concept of time produces a peculiar notion of determinism at the level of quantum cosmology. Despite the absence of an *external* time parameter, the equation is of hyperbolic form thus suggesting to use the 3-volume  $v$  or  $\alpha = \frac{1}{3} \ln v$  as an *intrinsic* time parameter. Exchanging the classical differential equations in time for a differential equation hyperbolic in  $\alpha$  alters the determinism of the theory: the wave function is evolved from small  $\alpha$  to large  $\alpha$ , but not along a classical trajectory parametrized by  $t$ . This has important consequences for a classically recollapsing Universe, because in the quantum theory both Big-Bang and Big-Crunch correspond to  $\alpha \rightarrow -\infty$  and are thus conceptually indistinguishable.

Implementing boundary conditions in quantum cosmology differs from the situation in both general relativity and ordinary quantum mechanics. In the following I shall briefly review two of the most widely discussed boundary conditions: the “no-boundary proposal” and the “tunnelling proposal”.<sup>2</sup>

Also called the “Hartle–Hawking proposal”,<sup>15</sup> the no-boundary proposal is essentially of a topological nature. It is originally based on a Euclidean path integral representation for the wave function, in which it is assumed that the integration is over compact manifolds with only one boundary – the boundary on which the wave functional is defined. The term “no-boundary proposal” arises from the fact that there is no other boundary; there is, in particular, no boundary corresponding to  $a \rightarrow 0$ . In order to guarantee convergence, it is in general necessary to integrate over complex metrics and to associate distinguished solutions with particular contours in the complex plane.<sup>16</sup>

Except for the simplest cases, the path integral cannot be evaluated exactly. One therefore has to resort to semiclassical (“saddle point”) approximations. For the above model (2), taking  $\Lambda = 0$ , one gets in this approximation the following expression for the wave function (here,  $\hbar = 1 = G$ ),

$$\Psi_{\text{NB}} \propto (a^2 V(\phi) - 1)^{-1/4} \exp\left(\frac{1}{3V(\phi)}\right) \cos\left(\frac{(a^2 V(\phi) - 1)^{3/2}}{3V(\phi)} - \frac{\pi}{4}\right). \quad (3)$$

The tunnelling proposal emerged from the work by Alexander Vilenkin and others, cf. [17,18] and references therein. It is most easily formulated in

minisuperspace. In analogy with, for example, the process of  $\alpha$ -decay in quantum mechanics, it is proposed that the wave function consists solely of *outgoing* modes. “Outgoing” means that the sign of the phase in the wave function is distinguished from the outset. In contrast to the no-boundary proposal, the tunnelling proposal thus leads to complex wave functions.

In the above model one get the following wave function:

$$\Psi_T \propto (a^2 V(\phi) - 1)^{-1/4} \exp\left(-\frac{1}{3V(\phi)}\right) \exp\left(-\frac{i}{3V(\phi)}(a^2 V(\phi) - 1)^{3/2}\right). \quad (4)$$

Consequences of this difference to (3) arise, for example, if one asks for the probability of an inflationary phase to occur in the early Universe: whereas the tunnelling proposal seems to favour the occurrence of such a phase, the no-boundary proposal seems to disfavour it. No final word on this issue has, however, been spoken. It is interesting that the tunnelling proposal allows the possibility that the Standard-Model Higgs field can play the role of the inflaton if a nonminimal coupling of the Higgs field to gravity is invoked.<sup>18</sup>

The application of these boundary conditions (as well as others discussed in the literature) is thus mainly restricted to the semiclassical realm. What one would like to do is to formulate a proper boundary condition for the Wheeler–DeWitt equation (2), that is, formulate a boundary condition from which a unique solution follows. This is by no means a simple task.<sup>2</sup> For the class of hyperbolic partial differential equations, to which (2) belongs, a proper boundary value problem is the Cauchy problem: specify  $\psi$  and  $\partial\psi/\partial\alpha$  at constant scale factor, and a unique solution results. This is, however, not a proper boundary problem in the case of classically recollapsing Universes. There, the wave function must tend to zero for large scale factor; otherwise, the correspondence with the classical theory is lost. The specification of  $\psi$  at  $a = \text{constant}$  and demanding  $\psi \rightarrow 0$  for  $a \rightarrow \infty$  will, however, not lead to a unique solution. It will in general not lead to any solution at all unless specific (“quantized”) values for the parameters are chosen, see, for example, the simple model discussed in [19]. A mathematical discussion of these issues can be found in [20].

#### 2.4. Inclusion of inhomogeneities and the semiclassical picture

Realistic models require the inclusion of further degrees of freedom; after all, our Universe is not homogeneous. This is usually done by adding a large number of multipoles describing density perturbations and small gravitational waves.<sup>2,21</sup> One can then derive an approximate Schrödinger equation for these multipoles, in which the time parameter  $t$  is defined through the minisuperspace variables (here,  $a$  and  $\phi$ ). The derivation is performed by a Born–Oppenheimer type of approximation scheme. The result is that the total state (a solution of the Wheeler–DeWitt equation) is of the form

$$\Psi \approx C(a, \phi) \exp(iS_0(a, \phi)/\hbar) \prod_n \psi_n(a, \phi, x_n), \quad (5)$$

where  $\{x_n\}$  stands for the inhomogeneities (“multipoles”). In short, one has that

- $S_0$  obeys the Hamilton–Jacobi equation for  $a$  and  $\phi$  and thereby defines a classical space-time which is a solution to Einstein’s equations (this order is formally similar to the recovery of geometrical optics from wave optics via the eikonal equation);  $C(a, \phi)$  denotes a slowly varying amplitude.
- The multipole wave functions  $\psi_n$  obey approximate Schrödinger equations,

$$i\hbar \frac{\partial \psi_n}{\partial t} := i\hbar \nabla S_0 \cdot \nabla \psi_n \approx \mathcal{H}_n \psi_n, \quad (6)$$

where the  $\mathcal{H}_n$  denote the Hamiltonians for the multipole degrees of freedom. The  $\nabla$ -operator on the left-hand side of (6) is a shorthand notation for derivatives with respect to the minisuperspace variables (here:  $a$  and  $\phi$ ). Semiclassical time  $t$  is thus defined in this limit from dynamical variables and is *not* prescribed from the outside;  $t$  controls the dynamics in this approximation.

- The next order of the Born–Oppenheimer scheme yields quantum gravitational correction terms proportional to  $G$ .<sup>2,22</sup> The presence of such terms may in principle lead to observable effects, for example, in the anisotropy spectrum of the cosmic microwave background radiation. These terms result from quantum dynamical features; quantum geometry may lead to additional corrections as seen in Section 3 on loop quantum cosmology.

The Born–Oppenheimer expansion scheme distinguishes a state of the form (5) from its complex conjugate. In fact, in a generic situation where the total state is real, being for example a superposition of (5) with its complex conjugate, both states will decohere from each other, that is, they will become dynamically independent.<sup>8</sup> This is a type of symmetry breaking, in analogy to the occurrence of parity violating states in chiral molecules. It is through this mechanism that the  $i$  in the Schrödinger equation emerges. Quite generally one can show how a classical geometry emerges from quantum gravity in the sense of decoherence:<sup>8</sup> irrelevant degrees of freedom (such as density perturbations or small gravitational waves) interact with the relevant ones (such as the scale factor or the relevant part of the density perturbations), which leads to quantum entanglement. Integrating out the irrelevant variables (which are contained in the above multipoles  $\{x_n\}$ ) produces a density matrix for the relevant variables, in which non-diagonal (interference) terms become small.<sup>23</sup> One can show that the Universe assumes classical properties at the onset of inflation.<sup>24</sup> The quantum fluctuations out of which eventually the galaxies form acquire classical properties through a similar mechanism.<sup>25</sup>

Due to the linear structure of quantum gravity, the total quantum state is a superposition of many macroscopic branches even in the semiclassical situation, each branch containing a corresponding version of the observer (the various versions of the observer usually do not know of each other due to decoherence). This is often referred to as the “many-worlds (or Everett) interpretation of quantum theory”,<sup>9</sup> although only one *quantum* world (described by the full  $\Psi$ ) exists.<sup>8</sup>

### 2.5. Arrow of time and structure formation

Although most fundamental laws are invariant under time reversal, there are several classes of phenomena in Nature that exhibit an arrow of time.<sup>26</sup> It is generally expected that there is an underlying master arrow of time behind these phenomena, and that this master arrow can be found in cosmology. If there existed a special initial condition of low entropy and if time proceeded in terms of the scale factor  $a$ , statistical arguments could be invoked to demonstrate that the entropy of the Universe will increase with increasing size.

There are several subtle issues connected with this problem. First, one does not yet know a general expression for the entropy of the gravitational field; the only exception is the black-hole entropy, which is given by the expression

$$S_{\text{BH}} = \frac{k_{\text{B}}A}{4G\hbar} = k_{\text{B}} \frac{A}{4l_{\text{P}}^2}, \quad (7)$$

where  $A$  is the surface area of the event horizon,  $l_{\text{P}}$  is again the Planck length and  $k_{\text{B}}$  denotes Boltzmann's constant. According to this formula, the most likely state for our Universe would result if all matter would assemble into a gigantic black hole; this would maximize (7), cf. [5]. More generally, Roger Penrose has suggested<sup>27</sup> to use the Weyl tensor as a measure of gravitational entropy, which expresses the very special nature of the Big-Bang (small Weyl tensor) and the generic nature of a Big-Crunch (large Weyl tensor). Entropy would thus increase from Big-Bang to Big-Crunch. (See [26] for a detailed exposition and references.)

Second, since these boundary conditions apply in the very early (or very late) Universe, the problem has to be treated within quantum gravity. But as we have seen, there is no external time in quantum gravity – so what does the notion “arrow of time” mean?

We shall address this issue in quantum geometrodynamics, but given that only the type of equations will be referred to the situation should not be very different in loop quantum cosmology or string cosmology. An important observation is that the Wheeler–DeWitt equation exhibits a fundamental asymmetry with respect to the “intrinsic time” defined by the sign of the kinetic term. Very schematically, one can write this equation as

$$\mathcal{H}\Psi = \left( \frac{\partial^2}{\partial \alpha^2} + \sum_i \left[ -\frac{\partial^2}{\partial x_i^2} + \underbrace{V_i(\alpha, x_i)}_{\rightarrow 0 \text{ for } \alpha \rightarrow -\infty} \right] \right) \Psi = 0, \quad (8)$$

where again  $\alpha = \ln a$ , and the  $\{x_i\}$  again denote inhomogeneous degrees of freedom describing perturbations of the Friedmann Universe (see above);  $V_i(\alpha, x_i)$  are the potentials of the inhomogeneities. The important property of the equation is that the potential becomes small for  $\alpha \rightarrow -\infty$  (where the classical singularities would occur), but complicated for increasing  $\alpha$ . In the general case (not restricting to small inhomogeneities), this may be further motivated by the BKL-conjecture according



to which spatial gradients become small near a spacelike singularity.<sup>28</sup> The Wheeler–DeWitt equation thus possesses an asymmetry with respect to “intrinsic time”  $\alpha$ . One can in particular impose the simple boundary condition

$$\Psi \stackrel{\alpha \rightarrow -\infty}{\rightarrow} \psi_0(\alpha) \prod_i \psi_i(x_i), \quad (9)$$

which would mean that the degrees of freedom are initially *not* entangled. Defining an entropy as the entanglement entropy between relevant degrees of freedom (such as  $\alpha$ ) and irrelevant degrees of freedom (such as most of the  $\{x_i\}$ ), this entropy vanishes initially but increases with increasing  $\alpha$  because entanglement increases due to the presence of the potential. In the semiclassical limit where  $t$  is constructed from  $\alpha$  (and other degrees of freedom), cf. (6), entropy increases with increasing  $t$ . This, then, would *define* the direction of time and would be the origin of the observed irreversibility in the world. The expansion of the Universe would then be a tautology. Due to the increasing entanglement, the Universe rapidly assumes classical properties for the relevant degrees of freedom due to decoherence. Decoherence is here calculated by integrating out the  $\{x_i\}$  in order to arrive at a reduced density matrix for  $\alpha$ .<sup>23,24</sup>

This process has interesting consequences for a classically recollapsing Universe.<sup>26,29</sup> Since Big-Bang and Big-Crunch correspond to the same region in configuration space ( $\alpha \rightarrow -\infty$ ), an initial condition for  $\alpha \rightarrow -\infty$  would encompass both regions. This would mean that the above initial condition would always correlate increasing size of the Universe with increasing entropy: the arrow of time would formally reverse at the classical turning point; Big-Bang and Big-Crunch would be identical regions in configuration space. As it turns out, however, a reversal cannot be observed because the Universe would enter a quantum phase.<sup>29</sup> Further consequences concern black holes in such a Universe because no horizon and no singularity would ever form.

## 2.6. Transition to the XXI<sup>st</sup> century

The main application to quantum cosmology in the last ten years is motivated by loop quantum gravity and is described in the next section. But work on the quantum geometrodynamical Wheeler–DeWitt equation is also going on. This is at least in part due to the reasons given at the beginning of my section above. But it is also related to the fact that loop quantum cosmology, too, addresses a constraint equation of the form  $\mathcal{H}\Psi = 0$ , although it is now a difference equation; for large-enough scale factors one expects that this difference equation is approximately given by the Wheeler–DeWitt equation.

Work on the standard Wheeler–DeWitt equation includes supersymmetric quantum cosmology (see the remarks towards the end of our contribution), path-integral methods (see, for example, [30]), or investigations of singularity avoidance[31–33]. An overview of other recent developments can be found in [34].

### 3. From the last decade(s) (Martin Bojowald)

Several ones of the general issues in quantum cosmology are to be faced by any approach, irrespective of its details and technicalities. Among those are, starting at a rather fundamental level and proceeding to more practical problems, (i) the interpretation of the wave function of the Universe and of the observable ingredients it contains, (ii) quantum dynamics of the wave function formulated by a Wheeler–DeWitt-type equation, (iii) the role of “quantum geometry” underlying the dynamical concepts and the associated understanding of quantum space–time, and finally (iv) the detailed prescription of feasible test procedures and potential observational consequences of the whole framework.

Many of these questions, in accordance with their general nature, have already been addressed in depth in Sec. 2. But during the last one or two decades, some of these issues have been approached specifically within the realm of loop quantum gravity,<sup>35–37</sup> providing several new insights. Especially the background independent notion of quantum geometry, realized within this framework not just in reduced models of quantum cosmology but in a general setting, has emphasized the importance of point (iii) above. But all these questions are interlinked, and thus new ingredients have resulted from loop quantum gravity also for the other issues.

By a change of perspective, loop quantum gravity has made significant progress regarding one of the major issues that has so far stubbornly remained out of reach for Wheeler–DeWitt quantizations: the rigorous formulation of a *kinematical* quantum representation for general, unrestricted geometries, going beyond exactly symmetric models or perturbative multipole expansions around them. Seen from the general viewpoint of quantum field theory, the loop representation implements background independence by introducing basic operators without reference to a background space–time metric. As always, basic canonical fields should be smeared (that is, spatially integrated) for well-defined quantum representations; otherwise delta-functions with their infinities appear in the classical Poisson brackets.

For a background-independent quantization of gravity, the smearing must be done in such a way that no metric other than the physical one is used for integration measures. The only possibility known (so far) has been provided by loop quantum gravity:<sup>38</sup> use holonomies  $h_e(A) = \mathcal{P} \exp(\int_e A_a^i \dot{e}^a \tau_i dt)$  of the Ashtekar connection  $A_a^i$  along spatial curves  $e$  and fluxes  $F_S^{(f)}(E) = \int_S E_i^a n_a f^i d^2y$  of the densitized triad  $E_i^a$  along spatial surfaces  $S$  (with  $\mathfrak{su}(2)$ -generators  $\tau_i$  proportional to the Pauli matrices and surface-supported smearing functions  $f^i$ ). The fields  $A_a^i$  and  $E_i^a$  are canonically conjugate to each other, giving rise to the holonomy-flux algebra under taking Poisson brackets. A unique<sup>39</sup> diffeomorphism covariant quantum representation results, in which fluxes, representing spatial geometry via the densitized triad, turn out to have discrete spectra,<sup>40</sup> and holonomy operators are not continuous in the edge length. Unlike in the Wheeler–DeWitt representation, connection (or curvature) components cannot be represented directly.

These are the main lessons from loop quantum gravity that loop quantum

cosmology<sup>3</sup> attempts to incorporate in cosmological models.

### 3.1. Difference equation

The Wheeler–DeWitt equation (2) is obtained by quantizing  $a$  (or  $\alpha = \ln a$ ) and its momentum in the well-known quantum mechanical way. This procedure is not compatible with what we have seen from full loop quantum gravity, where connections, and thus the momenta of metric components, cannot be represented directly but must rather refer to holonomies.

In an isotropic context, the connection reduces to  $A_a^i = c\delta_a^i$ , conjugate to an isotropic densitized triad  $E_i^a = p\delta_i^a$ . For spatially flat models, the isotropic connection component is related to the scale factor by  $c \propto \dot{a}$ , and it is canonically conjugate to  $p$ , determining the metric via  $a = \sqrt{|p|}$ . Loop quantum cosmology, following full loop quantum gravity, then provides a quantization only of isotropic holonomies as functions of the isotropic connection component  $c \propto \dot{a}$ , *not of  $c$  directly*. Only functions of the form  $\exp(i\delta(a)\dot{a})$ , as matrix elements of holonomies along straight curves of length  $\delta(a)$ , can be turned into operators.<sup>5</sup>

One can view non-linear holonomies as contributing to higher-curvature terms, the leading order reproducing the Friedmann dynamics. Adding suitable powers of  $\delta(a)\dot{a}$ , which are small when the Hubble distance  $a/\dot{a}$  is large compared to the comoving edge length  $a\delta(a)$ , makes the Friedmann equation loop quantizable. Once quantized, the exponentials of holonomies act as shift operators on the spectrum  $\{\mu\}$  of  $\hat{p}$ . As the resulting dynamical equation for wave functions  $\psi_\mu(\phi)$  depending on matter fields  $\phi$  as well as triad eigenvalues  $\mu$  one thus obtains a discrete evolution equation, a difference equation of the form<sup>42</sup>

$$C_+(\mu)\psi_{\mu+\delta(\mu)}(\phi) + C_0(\mu)\psi_\mu(\phi) + C_-(\mu)\psi_{\mu-\delta(\mu)}(\phi) = \mathcal{H}_\phi(\mu)\psi_\mu(\phi). \quad (10)$$

All coefficients, including the matter Hamiltonian  $\mathcal{H}_\phi(\mu)$ , can be computed explicitly for a specific choice of the regularization and for the lattice-refining curve parameters  $\delta(a)$ . Owing to these choices, the coefficients are subject to quantization ambiguities. Several qualitative aspects of the difference equation and its solutions are nevertheless robust. When the discreteness is not relevant, e.g. in low-curvature regimes in which the wave function does not oscillate strongly, one can Taylor-expand and reproduce the Wheeler–DeWitt equation as its continuum limit.<sup>43</sup>

At this stage, the usual wave function issues of quantum cosmology arise. But there are also new ones of mathematical nature, related to an analysis of the resulting difference equations. Especially in anisotropic models<sup>44</sup> quantizing the Bianchi cosmologies, these equations can be rather more complicated than the

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<sup>5</sup>One might expect the length parameter  $\delta$  to be a constant, such as a regulator chosen once and for all. However, in a reduction of quantum gravity states,  $\delta$  arises from the lengths of links in lattice-like structures, which are being refined as the dynamics of an expanding Universe unfolds. In minisuperspace models, this can be faithfully mimicked only by working with a phase-space dependent edge length, such as  $\delta(a)$ .<sup>41</sup>

isotropic one shown here. Numerical techniques, especially for the non-equidistant types of difference equations that arise for complicated forms of  $\delta(\mu)$  in (10), are being developed.<sup>45</sup> Another recent development is the formulation of a path-integral picture of loop quantum cosmology, in this case making contact with spin foam models as the covariant version of loop quantum gravity.<sup>46</sup>

### 3.2. Singularity resolution

With its new mathematical structure, loop quantum cosmology also makes possible applications regarding physical space-time effects. A basic new phenomenon brought about by the difference-equation nature of the dynamics of loop quantum cosmology concerns the singularity problem. Classically, the initial-value problem underlying isotropic cosmological models becomes ill-posed when certain quantities diverge at the Big-Bang singularity. No unique extension beyond the singularity can be found for a classical geometry, and so the singularity appears as an insurmountable border to space-time and to our knowledge of earliest stages of the Universe.

In quantum cosmology, a similar problem is to be addressed for the wave function, which replaces space-time geometry as the information carrier. Sometimes, arguments for singularity resolution can be put forward in the Wheeler–DeWitt context. But generic statements are difficult to obtain, and there are models in which this quantization does not resolve the singularity problem: all states follow exactly the classical trajectories into the singularity.<sup>h</sup> Stronger ingredients are needed.

In loop quantum cosmology, new qualitative features arise. First, just kinematically we have an extension of minisuperspace across the singularity: the space of isotropic geometries is given not by a half-line  $a > 0$  with the singular  $a = 0$  at a boundary (or pushed to infinity by  $\alpha = \ln a$ ), but by the whole real axis of densitized-triad components  $p \in \mathbb{R}$ . (Both signs are allowed for  $p$  since it is the component of a densitized triad, which changes sign under a reversion of orientation.) Physics might still break down at  $p = 0$  (or  $\mu = 0$  at the quantum level), but there is now a clear way of finding out what happens by analyzing the difference equation. It turns out that the classical singularity is resolved:<sup>47</sup> any evolving wave function continues through  $\mu = 0$ . Starting from initial values for the wave function somewhere in a well-understood regime, dynamics extends it uniquely across the classical singularity. Moreover, initial conditions for solutions follow automatically; they are derived dynamically, not imposed by hand.<sup>48</sup>

From the wave function, one may attempt to reconstruct the behavior of space-time beyond the singularity. In general the geometry may not be of the classical form we know. Here, one should use observables rather than the wave function, but that analysis is not required for a general statement of singularity resolution: Any wave function is uniquely extended, and even if we lack complete observables or an

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<sup>h</sup>Specifically, a free massless scalar field in a spatially flat isotropic space-time presents a solvable system free of quantum back-reaction. Quantum states follow exactly the classical trajectories and cannot avoid the singularity.

explicit physical inner product, we know that the wave function, and thus quantum geometry, exists beyond the classical singularity in a unique way.

In more specific models, fixing even the matter content, further properties of the mechanisms of singularity resolution can be elucidated. There are two different kinds of statements: (i) A geometrical notion of *discrete internal time*  $\mu$ , as embodied by the difference equation, implies an upper bound for energy density; wave functions must have a minimum wave length to find support on the discrete time lattice. (ii) The gravitational force then becomes repulsive at high densities, sometimes resulting in a “bounce”. Then, one obtains mean values for, say, the volume in a dynamical state that do not collapse into the classical singularity but are turned into re-expansion. Showing that a bounce<sup>49</sup> in the strict sense is realized, based on details of quantum geometries, is more difficult and obtained in fewer models,<sup>50–52</sup> than showing the boundedness of energy densities. Several main issues are to be addressed for geometrical pictures of singularity resolution: linking dynamical wave functions to observables, proper normalizations of states by a physical inner product, properties of sufficiently general states as they may be realized in generic quantum regimes near a classical singularity, and the sensitivity to perturbations.

The first bounce solutions at the physical Hilbert-space level of loop quantum cosmology were obtained numerically in models with a free, massless scalar as matter source.<sup>50</sup> Resulting bounce pictures looked surprisingly tame, with hardly any spreading or deformations of semiclassical wave packets throughout the whole, supposedly violent Big-Bang phase. Initially, this seemed exciting, for it suggested a comparatively simple analysis of everything concerning the Big-Bang. However, it soon became clear that the models used, as well as the states considered, were rather special: they are very close to exactly solvable, harmonic systems, ones with only weak quantum back-reaction.

The underlying exactly solvable bounce model<sup>53</sup> is obtained for a spatially flat isotropic Universe. Such a model can be formulated in a rather general way based on a power-law assumption  $\delta(a) = \delta_0 a^{2x}$  for the lattice-refinement behavior in (10), parameterized by a real number  $x$  (negative for refinement rather than coarsening to take place). If we define the variables  $V := a^{2(1-x)}$  and  $J := a^3 \exp(i\delta_0 a^{2x} \dot{a})$ , which obey a linear Poisson algebra, the difference equation of loop quantum cosmology with an energy density  $p_\phi^2/2a^3$  can be seen to equate the scalar momentum to  $p_\phi \propto |\text{Im}J|$ . Now taking  $\phi$  as internal time, instead of the scale factor as often used in the Wheeler–DeWitt context,  $p_\phi$  is realized as a linear Hamiltonian. Linear models are harmonic: like the harmonic oscillator they are free of quantum back-reaction. Only a small number of parameters is required to understand the evolution of expectation values in any state. With the corrections from loop quantum cosmology, all solutions for  $\langle \hat{V} \rangle(\phi)$ , computed in a state required only to be semiclassical once, bounce.

As an exactly solvable one, this model is very special, as special as the harmonic oscillator in quantum mechanics. Rather than exhibiting general properties, such models are important as the basis of a systematic perturbation theory,<sup>55</sup> the canonical generalization of low-energy effective actions. In quantum cosmology, this

gives rise to an effective Friedmann equation<sup>54</sup>

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \left( \rho \left( 1 - \frac{\rho_Q}{\rho_{\text{crit}}} \right) \pm \frac{1}{2} \sqrt{1 - \frac{\rho_Q}{\rho_{\text{crit}}}} \eta (\rho - P) + \frac{(\rho - P)^2}{\rho + P} \eta^2 \right) \quad (11)$$

to describe the evolution of the scale factor  $a$ , now seen as the expectation value obtained from a state in quantum cosmology. The classical terms of the Friedmann equation are clearly recognizable, showing that the classical limit is achievable in certain regimes:  $\eta$ , parameterizing quantum correlations, must be small, and  $\rho_Q$ , the energy density with quantum corrections from quantum fluctuations, must be small compared to the critical density  $\rho_{\text{crit}} = 3/8\pi G(a\delta(a))^2$ . In  $\rho_{\text{crit}}$ , which arises from holonomy corrections, the refinement function  $\delta(a)$  enters.

We note that the derivation of Eq. (11) relies on the implementation of a physical inner product, or equivalently on using appropriate reality conditions. In this way, one ensures that the square root appearing in (11) is real. An upper bound  $\rho_Q \leq \rho_{\text{crit}}$  then follows irrespective of the state realized at those densities.<sup>54</sup> Secondly, if the state is such that  $\eta$  is very small when  $\rho_Q \sim \rho_{\text{crit}}$ , then there is a bounce ( $\dot{a} = 0$ ) when energy densities reach the critical value. In the very specific situation of a stiff fluid with  $\rho = P$ , the case first analyzed numerically,<sup>50</sup> a bounce at the critical density is realized irrespective of the value of  $\eta$ . It remains unclear how well the bounce result can be generalized to situations in which the potential energy of matter dominates over the kinetic contribution.

### 3.3. Observational contact?

For reliable predictions, one must understand all effects and quantum corrections, have some control on the quantum-to-classical transition, and manage a systematic form of perturbation theory including effective equations with quantum back-reaction and inhomogeneities. In particular, it is not sufficient to base general expectations on what has been seen to dominate only in a specific class of models. A general and systematic analysis of the whole theory is required which, needless to say, is only in its beginning stages for loop quantum gravity.

Effective equations must be sufficiently general in order to deal with proper dynamical states. There is no general reason to assume near-Gaussian (or uncorrelated) states as they occur in expansions around ordinary free field theory. A large class of underlying states must be encompassed; and as seen in Wheeler-DeWitt models, results can sensitively depend on the form of a state.<sup>56</sup>

In order to implement inhomogeneities consistently, one must face the anomaly issue, presenting strong consistency conditions from the usual overdetermined set as given classically by Einstein's equations. Simply modifying general relativistic equations is easily possible in homogeneous models, but as soon as inhomogeneities are included, perturbation equations must be delicately matched to the modified background equations. If the corrected set remains consistent, the system is anomaly-free. For reliable conclusions, every quantum correction considered in

homogeneous models, for instance the modification due to holonomies, must be shown to have a consistent formulation with inhomogeneities. Since homogeneous equations can be modified at will, as far as the anomaly problem is concerned, producing specific bounce models or bounded densities is not that difficult. What is non-trivial, and still lacking for several quantum-geometry corrections, is a consistent embedding in a set of anomaly-free equations for inhomogeneities.

The anomaly problem is related to the gauge problem: Fixing the space-time gauge before quantization or before the derivation of quantum corrections evades consistency issues and can be used to reduce the over-determinedness of equations. But crucial effects can easily be overlooked, especially since quantum gravity leads to corrections of constraints which generate gauge transformations. Referring to the gauge before one even knows the quantum corrected transformations can easily be misleading. Several examples exist by now in which perturbative inhomogeneities of certain forms can be implemented consistently.<sup>57-59</sup> The resulting equations imply properties, such as non-conservation of power on large scales or effective anisotropic stresses, that are important for observational consequences but would not appear in this form for gauge-fixed treatments. The anomaly problem makes any analysis rather complicated, but it also provides us with a chance to analyze quantum space-time on its smallest scales. Results then have important implications for full quantum gravity.

#### 4. ..., *citoyens!* (All authors)

The general set of queries and contributions (or concerns) from the audience could be presented in a somewhat fictionalized debate as follows:

*What exactly does it mean to quantize cosmology?* Most investigations of quantum cosmology happen in a minisuperspace framework, which starts with a classical truncation to finitely many degrees of freedom, then quantizing them by techniques borrowed from quantum mechanics (cf. Section 2). In some cases, quantization has been completed by arriving at a Hilbert-space of states with a representation of a complete set of observables. But those models are based on global internal times from matter, such as a free massless scalar or dust, ingredients which can hardly be considered general. It remains unclear what mathematical structures are needed for a general form of quantum cosmology.

*How can you at all trust results from those severely truncated quantum cosmological models?* Indeed, quantum cosmology is a truncation rather than an approximation, drastically cutting off unwanted degrees of freedom instead of providing a harmonious embedding of a simplified model within a fuller framework. There is currently no well-defined approximation scheme that would show under which conditions terms ignored in full expressions could be considered small.

The main difficulty is the lack of empirical tests by which to assess the validity of approximations, following the example of theoretical condensed matter physics which, too, employs severe truncations in many cases. Instead, theoretical



investigations have been performed,<sup>60</sup> more recently also using path integrals.<sup>30</sup> Some support for minisuperspace models comes from the BKL conjecture,<sup>28</sup> according to which the dynamics of space–time near a spacelike singularity is dominated by its ultralocal behavior: time derivatives seem to dominate over spatial ones. While fields can still have large spatial variations, the evolution of the geometry at a point seems to depend only on the geometry at that point. If this were true and held even in quantum gravity, minisuperspace models would indeed capture the main dynamical features at least in the approach to a spacelike singularity.

*What does it actually mean to “resolve” singularities? Shouldn’t one consider curvature invariants and make sure that they all remain bounded?* Curvature divergence is a feature of most of the solutions of general relativity, but it is not a property shown by singularity theorems; they use geodesic incompleteness as the key criterion. Focusing on curvature divergence for singularity resolution may thus be too restrictive or even misleading. Alternatives are specific forms of boundary conditions, or the quantum hyperbolicity condition of loop quantum cosmology.

*How does one make sure that a state achieves semiclassicality at large volume?* Semiclassical states play an important role in developing quantum gravity, to check the correct classical limit, and to find suitable regimes for low-energy physics. In the Wheeler–DeWitt approach, the semiclassical limit is well understood at a formal level, cf. Section 2.4. Most methods to construct semiclassical states in loop quantum gravity<sup>62</sup> remain at the kinematical level (and thus do *not* satisfy all constraints). Quantum cosmological models are often simple enough so as to allow the construction of dynamical coherent states and to see how semiclassicality takes place in a dynamical context: how fast can an initial state spread or change shape?

There are, of course, all the conceptual issues discussed in Section 2. When and how does decoherence act, that is, when can superpositions of different semiclassical states be treated as dynamically independent components? What is the relevance for the arrow of time? It seems obvious that quantum cosmology is in severe conflict with a Copenhagen-type of interpretation, which assumes the presence of a classical world from the outside. Are there alternatives to the Everett interpretation?

As also emphasized in Sections 2 and 3, quantum effects are not a priori restricted to the Planck scale. They may even occur for large Universes. Sizable quantum effects are responsible for the singularity avoidance of scenarios containing a Big-Rip, a Big- Brake, or other classically singular situations. At small volume, the dynamics of states may be much more violent than in semiclassical regimes. A general discussion of intuitive singularity-avoidance mechanisms, such as bounces, suffers considerably from the lack of knowledge about dynamical semiclassical states.

*Can quantum cosmology make realistic (that is, falsifiable) predictions?* There are arguments that quantum cosmology can predict inflation, combined with a reasonable spectrum of primordial fluctuations; see [18] and the references therein. Also bounce pictures in general terms lead to potential signatures.<sup>61</sup> Such statements are often related to proposals for initial states, which can simultaneously be used to address the singularity problem. One may also envisage quantum



gravitational corrections to standard cosmology scenarios, which could exhibit themselves in the CMB anisotropy spectrum.<sup>2</sup> In this sense, quantum cosmology has been progressing in a (self-)consistent manner facing the reality of astronomical observations. However, a clear observation or a (currently feasibly detectable) signature from a quantum Universe has not yet been possible. Only the availability of clear-cut empirical tests would confirm quantum cosmology as a viable approach to understand the foundations of cosmology.

### 5. *Nouvelle Vague?* (All authors)

Some progress has already been made in understanding the central conceptual issues such as the problem of time, the interpretation of the wave function, or the connection of quantum cosmology with full quantum gravity. As long as the full theory is not known, however, these insights have to remain preliminary. If it turned out, for example, that quantum gravity would be nonlinear,<sup>27</sup> most (if not all) of these results (as well as those of string theory) would become obsolete.

It is therefore at this stage quite difficult, if not impossible or at least uncertain, to make either predictions or indicating where quantum cosmology can be further developed in a significant manner in the years (or decades) to come. In the following we provide some possible lines, bearing in mind the risk of being proven wrong by other advances or even observational evidence.

The “(string) landscape” issue has driven efforts from quantum cosmology; see, for example, [63] and the references therein. Assuming that it is a robust property of string theory, one expects that the landscape picture requires elements of statistical theory and quantum selection rules for transitions. Could it be that quantum cosmology would merely provide an “average” perspective and that a more “field theory”-like structure (e.g. a third quantization or inputs from statistical physics) would benefit quantum cosmology and the manner in which the (wave function of the) Universe is discussed? These are wide-open issues.

We point out the interesting contribution of supersymmetric quantum cosmology.<sup>64–70</sup> It brings additional structure to the framework and still has potential for fresh ideas, or even new problems with tentative solutions, especially on how we probe the very structure of space-time.<sup>64</sup> If supersymmetry is discovered at particle accelerators, investigating the early Universe within any quantum cosmology school may require the implementation of supersymmetry.

In loop quantum cosmology, the link to the full theory has turned out to be essential, even though so far it is rather weakly built. Details of the equations of loop quantum cosmology can often be restricted by knowing how ingredients may arise from the equations of full loop quantum gravity. Effects are then important not just at ultra-high densities but even in tame regimes, for instance, in the context of the correct classical limit. Issues such as lattice refinement or anomaly-free extensions to include inhomogeneities make crucial use of properties of the fully theory — and then provide important feedback on the feasibility of full constructions.

Let us finally quote some questions for which we expect to obtain (or think we already know) an answer from quantum cosmology:

- How does one have to impose boundary conditions?
- Is the classical singularity really being avoided, and how so?
- Will there be a genuine quantum phase in the future?
- How does the appearance of our current classical Universe follow?
- Can the arrow of time be understood from quantum cosmology?
- How does the formation of structure proceed?
- Can inflation itself be understood from quantum cosmology?
- Can quantum cosmology be justified from full quantum gravity?
- Which consequences can be drawn for the interpretation of quantum theory in general and for quantum information in particular?
- Can quantum cosmology be experimentally tested?

We want to emphasize again that quantum cosmology will become an established part of physics only if it can and will be experimentally tested. We are optimistic but we do not know when this will happen. But let us finish by a quote from Erwin Schrödinger<sup>71</sup> as a motivation for continuing to ask questions about quantum cosmology: “. . . or else, one might seriously worry that just where we forbid further questions there could still be quite a bit worth knowing to ask about.”<sup>i</sup>

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<sup>i</sup> “. . . sonst wäre ernstlich zu befürchten, daß es dort, wo wir das Weiterfragen verbieten, wohl doch noch einiges Wissenswerte zu fragen gibt.”

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