IGC–11/9–1

Quantum gravity in the very early universe*

Martin Bojowald†

Institute for Gravitation and the Cosmos,
The Pennsylvania State University,
104 Davey Lab, University Park, PA 16802, USA

Abstract

General relativity describes the gravitational field geometrically and in a self-interacting way because it couples to all forms of energy, including its own. Both features make finding a quantum theory difficult, yet it is important in the high-energy regime of the very early universe. This review article introduces some of the results for the quantum nature of space-time which indicate that there is a discrete, atomic picture not just for matter but also for space and time. At high energy scales, such deviations from the continuum affect the propagation of matter, the expansion of the universe, and perhaps even the form of symmetries such as Lorentz or CP transformations. All these effects may leave traces detectable by sensitive measurements, as pointed out here by examples.

Processes in the very early universe require for their description general relativity (space is expanding) and quantum physics (the early universe is hot and dense). Sometimes, this even involves quantum physics not just of matter but of gravity. Gravity is described by the geometry of space-time, and so we need to quantize space and time. By experience from quantum mechanics, one possible consequence is that elementary constituents, or "atoms of space," arise for space-time.

Dimensional arguments can be used to arrive at a first estimate of direct effects. There is a unique length parameter, the Planck length $\ell_{\rm Pl}=\sqrt{G\hbar/c^3}\approx 10^{-35}{\rm m}$ and a unique mass parameter, the Planck mass $M_{\rm Pl}=\sqrt{\hbar c/G}\approx 10^{18}{\rm GeV}\approx 10^{-6}{\rm g}$, that can be formed solely by reference to the relevant fundamental constants, Newton's gravitational constant G, Planck's constant $\hbar$, and the speed of light c. At those scales, or, perhaps more intuitively, at the Planck density $\rho_{\rm Pl}=M_{\rm Pl}/\ell_{\rm Pl}^3$, quantum gravity becomes inevitable. Compared to the current density of the universe, at about an atom per cubic meter, the Planck density of roughly a trillion solar masses in the region of the size of a single proton, is huge. The relevance of quantum gravity for current physics may thus be questioned.

However, dimensional arguments can be misleading when large dimensionless parameters are involved. In the context of quantum gravity, perhaps suggesting some kind of

*Published as *Nuclear Physics* **A862–863** (2011) 98–103
†e-mail address: `bojowald@gravity.psu.edu`

elementary atoms of space, such a parameter can easily be seen to arise: the number of tiny, Planck-sized spatial atoms in a given macroscopic region under consideration. Similar questions, in which indirect evidence for phenomena on tiny scales can be found long before the resolution of observations becomes good enough for direct tests, have played important roles before in the history of physics. For instance, in 1905 Albert Einstein used an analysis of Brownian motion to find convincing evidence for atoms, and only fifty years later, in 1955, did Erwin Müller produce the first direct image of atoms using field ion microscopy. By that time, the overwhelming majority of physicists was already convinced of the reality of material atoms based on Einstein's arguments.

Returning to quantum gravity, the best microscope we currently have to magnify and probe the fundamental structures of space is the universe itself. By its own expansion, it enlarges spatial regions and eventually translates their properties into visible large-scale structures. This magnification process, of course, takes a long time and many other processes happen throughout; no direct image can be obtained in this way and a great deal of physics must be used to disentangle the form of original structures from what has emerged in the meantime. With a good understanding of all the physics involved one can begin to find indirect evidence for effects controlled by quantum gravity.

The physics of quantum gravity is not well-understood at present, and no complete theory is known. Nevertheless, several characteristic effects have been suggested which do not so much depend on theoretical details but are rather based on general expectations from fundamental properties of general relativity and quantum mechanics. One of the main such suggestions is the atomic nature of space, and it is of relevance for early-universe cosmology. An expanding discrete space grows not continuously but atom by atom. Implications are weak for a large universe, but may be noticeable by sensitive observations of events that happen sufficiently early.

Observations which have a chance of providing sufficient sensitivity with current technological means must first be found by analyzing available theories. As an example one may consider the abundances of light elements, which depend on the baryon-photon ratio during big-bang nucleosynthesis, an early-universe process of proton-neutron interconversion by the weak interaction. The baryon-photon ratio depends on the dilution behavior of radiation and (relativistic) fermions. If discrete expansion leads to modifications of the dilution behavior, small changes in the abundance of light elements would be expected. We will return to this example at a later stage.

Other examples for some chance of testing quantum gravity can be found in all the phases included in the standard model of cosmology:

The big bang: an extreme phase starting with Planckian density preceded, in the classical understanding of general relativity, by a singularity 13.8 billion years ago.

Inflation: an accelerated phase of expansion of currently unknown origin, happening at an energy scale about $10^{-10}\rho_{\rm Pl}$ at which particle production seeds all matter as seen in the cosmic microwave background (CMB) and the galaxy distribution.

Baryogenesis: the formation of baryons out of a primordial quark-gluon plasma, somehow expected to lead to the matter/antimatter asymmetry of the present universe.

Nucleosynthesis: the generation of nuclei as bound states of the light baryons, arising in relative quantities of about 75% hydrogen and deuterium, 25% helium, and just trace amounts of other light elements.

CMB release: once atoms neutralize, the universe becomes transparent 380,000 years after the big bang.

The succession of most of these phases is well supported by observations. However, the picture is incomplete, for the story begins with a singularity at which the equations of general relativity lose their meaning and unphysical conditions such as infinite densities and temperatures are reached. The singularity is a general consequence of the equations that govern a classical universe, in the simplest case described by the Friedmann and Raychaudhuri equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \quad , \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{1}$$

for the scale factor a providing the distance measure of the universe, and with the energy density ρ and pressure P of matter.

A simple singularity theorem can be obtained from this equation as follows: First, a simple rewrite implies $\dot{\mathcal{H}} = -4\pi G(\frac{1}{3}\rho + P) - \mathcal{H}^2$ for the Hubble parameter $\mathcal{H} = \dot{a}/a$. If we assume the so-called strong energy condition $\rho+3P \geq 0$ as a rather general requirement for the matter ingredients, we obtain the inequality $\mathrm{d}\mathcal{H}^{-1}/\mathrm{d}t \geq 1$ and thus $\mathcal{H}^{-1} \geq \mathcal{H}_0^{-1}+t-t_0$. If $\mathcal{H}_0^{-1}$ is negative, $\mathcal{H}^{-1}$ must be positive at $t_1 = t_0 - \mathcal{H}_0^{-1}$, and so $\mathcal{H}^{-1} = 0$ at some time when $\mathcal{H} \to \infty$ and $\rho \to \infty$ diverge: a future singularity resulting from collapse. Similarly, a past singularity is obtained if the universe is expanding at some time. More complicated singularity theorems can be demonstrated under more general conditions, dropping the symmetry assumption of an exactly isotropic universe and weakening the conditions posed for matter. Thus, singularities are generic in space-time dynamics.

These conclusions lead to the identification of several shortcomings of the standard model of cosmology despite its observational success: (i) Any singularity is unphysical and must be eliminated by improving the theory. (ii) Inflation assumes that matter starts out in an initial vacuum state. Is this assumption appropriate, especially when the density and temperature diverge at the "initial" singularity? (iii) With current theories, the matter/antimatter asymmetry that is supposed to form during baryogenesis is difficult to explain. If there was a prehistory of the universe before the big bang, as one possible scenario alternative to a singular one, more time existed for an asymmetry to build up. (iv) The matter equation of state is important for some aspects of big-bang and other phases, but is not well known for most of the scales between currently probed densities and the Planck density. This lack of knowledge does not so much affect the singular nature but plays a role for specific scenarios. To see what ingredients exactly we must bring under better control to discuss possible improvements of the standard model, we need more information about quantum gravity and the resulting space-time structure.

Gravity is "strongly interacting" at a fundamental, non-perturbative level. This statement may come as a surprise, given that gravity is much weaker than the other fundamental forces and can safely be ignored in particle interactions. However, particle physics already provides indications for the special nature of gravity: Its well-known non-renormalizability implies that gravity cannot be quantized as a weakly-interacting theory of gravitons on some background space-time. The weak form of gravity should rather arise as the long-range remnant of a more elementary theory. What exactly this elementary theory is is difficult to extract from the long-range physics of gravity that we know. Theoretical models are based on suitable principles for their mathematical formulation, for which different approaches exist, but no fully consistent version yet.

A quantization directly addressing the structure of space and time is loop quantum gravity [1], based crucially on the principle of background independence. Some part of the theory can be constructed by means analogous to those of lattice QCD, but with one crucial difference: General covariance implies that all states must be invariant under deformations of space (diffeomorphisms or coordinate changes). As a consequence, several new features (and complications) compared to QCD arise: (i) Regular lattices are too restrictive because they would be deformed when coordinates are changed. Instead lattices are "floating;" they are not assigned a fixed position in space. Only topological and combinatorial properties of their linking and knotting behavior can be relevant for gravity. (ii) No well-motivated restriction on the valence of lattice vertices exists (except the desired but possibly deluding simplicity of their mathematical description). (iii) Superpositions of different lattice states must be considered because the lattices correspond to states of a fundamental quantum theory, not to an approximation of such a theory. (iv) States of the continuum theory are described by lattices; they do not provide an approximation, and no continuum limit is to be taken. In this way, one obtains a fundamental lattice theory for quantum geometry. Geometrical excitations are, as we will see, generated by creation operators for lattice links. Near the continuum, physics can only be described by a highly excited many-particle state; in this sense the theory is "interacting". So far, the complicated resulting physics has mainly been analyzed in model systems, primarily obtained by assuming spatial symmetries.

To provide more technical details, we describe space-time geometry by an su(2)-valued "electric field" $\vec{E}_i$ and a "vector potential" $\underset{\rightarrow}{A}_i$ (using so-called Ashtekar–Barbero variables) with the following meaning.[3]

Electric field: Geometrically called a densitized triad, it determines spatial distances and angles by assigning three orthonormal vectors $\vec{E}_i$, $i = 1, 2, 3$, to each point in space.

Vector potential: $\underset{\rightarrow}{A}_i = \underset{\rightarrow}{\Gamma}_i + \gamma \underset{\rightarrow}{K}_i$ where $\underset{\rightarrow}{\Gamma}_i$ is related to the intrinsic curvature of space, and $\underset{\rightarrow}{K}_i$ to extrinsic curvature of space in space-time. These two contributions are

[3] In general relativity, we must distinguish between contravariant and covariant vector fields, denoted here by arrows above or below the letter, respectively. On a metric manifold one can uniquely transform between these two types of vector fields, but for gravity the metric follows from the fundamental fields. It is not available before those fields are known. Keeping track of the metric dependences is crucial for a background-independent formulation of quantum gravity.

added with a relative weighting γ, the real-valued Barbero–Immirzi parameter.

In addition to these geometrical properties and meanings of the fields, we have their canonically conjugate nature: $\vec{E}_i$ is the momentum of $\underset{\to}{A}_i$, $\{\underset{\to}{A}_i(x), \vec{E}_j(y)\} = 8\pi\gamma G\delta_{ij} \underset{\to}{\vec{\delta}}\, \delta(x,y)$ (using the identity matrix $\underset{\to}{\vec{\delta}}$). We can thus proceed by attempting a canonical quantization, with due observation of special properties resulting from symmetries of the theory, in particular general covariance.

As in lattice gauge theories, we define as basic variables holonomies $h_e = \mathcal{P}\exp(\frac{i}{2}\int_e \mathrm{d}\lambda \underset{\to}{A}_j \cdot \vec{t}_e\sigma^j)$ for the connection $\underset{\to}{A}^i$ along spatial curves e, with Pauli matrices σ^j. However, we use these objects in a way very different from lattice gauge theory; they will become creation operators of quantum geometry. To that end, we define a basic state ψ_0 by $\psi_0(\underset{\to}{A}_i) = 1$, that is it is independent of the connection.[4] Excited states are then obtained by the action of holonomies via multiplication in this connection representation. We present the formulas only in a simplified U(1)-example where $h_e(\underset{\to}{A}) = \exp(i\int_e \mathrm{d}\lambda \underset{\to}{A} \cdot \vec{t}_e)$ are just phase factors; SU(2) formulas as needed for gravity are analogous but more tedious. We thus write all excited states obtained in this way as $\psi_{e_1,k_1;\ldots;e_i,k_i} = \hat{h}_{e_1}^{k_1}\cdots\hat{h}_{e_i}^{k_i}\psi_0$. A general state is then labeled by a graph g, the collection of all curves used for holonomies to generate the state, and integers k_e as quantum numbers on the edges: $\psi_{g,k}(\underset{\to}{A}) = \prod_{e\in g} h_e(\underset{\to}{A})^{k_e} = \prod_{e\in g}\exp(ik_e\int_e \mathrm{d}\lambda \underset{\to}{A}\cdot\vec{t}_e)$.

The Ashtekar–Barbero connection has momenta $\vec{E}_i$ such that $\sum_i \vec{E}_i\otimes\vec{E}_i = (\det \vec{\vec{q}})^{-1}\cdot\vec{\vec{q}}$ gives the inverse spatial metric $\vec{\vec{q}}$. Quantizing $\vec{E}_i$, or rather the fluxes $\int_S \mathrm{d}^2y\underset{\to}{n}\cdot\vec{E}_i$ (with $\underset{\to}{n}$ the metric-independent co-normal to surfaces S), they naturally become derivative operators. At the level of states, flux operators measure the excitation levels k_e:

$$\int_S \mathrm{d}^2y\underset{\to}{n}\cdot\hat{\vec{E}}\psi_{g,k} = \frac{8\pi\gamma G\hbar}{i}\int_S \mathrm{d}^2y\underset{\to}{n}\cdot\frac{\delta\psi_{g,k}}{\delta\underset{\to}{A}(y)} = 8\pi\gamma\ell_{\mathrm{Pl}}^2\sum_{e\in g}k_e\mathrm{Int}(S,e)\psi_{g,k} \tag{2}$$

with the intersection number $\mathrm{Int}(S,e)$. From this equation one readily concludes that the $\psi_{g,k}$ are eigenstates of fluxes, with eigenvalues given by $8\pi\gamma\ell_{\mathrm{Pl}}^2$ times an integer. Spatial geometry is discrete: for gravity, fluxes representing the spatial metric have discrete spectra, and so do operators for area or volume constructed from them [3]. The Planck length $\ell_{\mathrm{Pl}} = \sqrt{G\hbar}$ together with the Barbero–Immirzi parameter determines the elementary discreteness scale. From computations of black-hole entropy one derives that γ is of the order one, but somewhat smaller than one [4].

So far, the quantum geometry we developed is only of space, not space-time. In order to see how the graph states evolve in time, possibly being reconnected and refined by the creation of new vertices, we need to quantize the Hamiltonian. Schematically, it has the form [5] $\hat{H}\psi_{g,k} = \sum_{v,IJK}\epsilon^{IJK}\mathrm{tr}(h_{v,e_I}h_{v+e_I,e_J}h^{-1}_{v+e_J,e_I}h^{-1}_{v,e_J}h_{v,e_K}[h^{-1}_{v,e_K},\hat{V}])\psi_{g,k}$ summing over vertices v of the graph g and triples (IJK) of edges. As is clear from this expression, there are creation operators (holonomies) as well as the volume operator $\hat{V}$. At this fundamental

[4] This state turns out to be normalizable by the inner product constructed via integration on spaces of connections [2].

level of elementary excitations of geometry, the theory is interacting, as promised: The Hamiltonian contains products of creation operators. Its action describes the dynamics of a discrete graph state, depending on the spatial geometry via the volume operator.

Gauge fields for fundamental forces other than gravity can be implemented in a very similar way, via independent types of holonomies for connections associated with the groups of the standard model. Fermions are represented as spinor degrees of freedom in the vertices of graphs, and the resulting matter Hamiltonian is added to $\hat{H}$ to form the total Hamiltonian. It turns out to be well-defined, without divergences [6]. However, the limit of a classical space-time remains poorly understood, and so it is difficult to say how exactly the theory breaks the bad spell of non-renormalizability.

Another main challenge remains, that of understanding space-time dynamics. The elementary interactions mediated by holonomies as creation operators must somehow conspire to result in the well-known long-range behavior of gravity. In a cosmological context, for instance, an expanding universe must result from the single tiny bricks of Planck cubes added on to space when holonomies act. The task is especially difficult owing to strong consistency requirements to ensure that discrete quantum geometry combines in the correct way with general covariance, normally thought of as a continuous group of space-time transformations that cannot leave discrete structures invariant.

Irrespective of the precise form of a consistent Hamiltonian, general properties of the dynamics, required to implement background independence, directly lead to several characteristic types of quantum corrections.

Inverse-volume corrections: Inverse metric components are corrected in all Hamiltonians because flux operators (2) have discrete spectra containing zero. Such operators do not have a densely defined inverse, which would be needed to quantize components of the inverse densitized triad. Operators with those inverses as their classical limit can be defined [6], but since they are not direct inverse operators they imply quantum corrections at small flux eigenvalues.

Holonomy corrections: Higher powers of curvature components arise from the substitution of connection components by holonomies. Covariant representations must also include higher time derivatives, which come from the last type of corrections.

Quantum back-reaction: As always in interacting quantum theories, the evolution of expectation values depends on the behavior of the whole state, for instance on the development of fluctuations or correlations. This quantum interaction can be captured in canonical effective equations [7].

While the first two types of corrections are characteristic of loop quantum gravity, the third one is generic for all kinds of interacting quantum systems. One can illustrate some features of those corrections, or more generally the dynamics of loop quantum gravity, by considering reduced models. If one requires isotropy of space, the key loop properties turn out to be preserved, but the dynamics obtained from the same constructions is much easier to analyze. In the resulting loop quantum cosmology [8], the Friedmann equation

(1) receives corrections by higher powers of the momentum p_a, the isotropic reduction of the connection, because it is, according to the basic premise of loop quantization, replaced by $\sin(\delta p_a)/\delta$ with some parameter δ akin to the edge length in holonomies.

Such a whole series of higher-order corrections seems difficult to control, but there is a special solvable model for matter given by a free, massless scalar, in which the series can be resummed to change (1) to $(\dot{a}/a)^2 = (8\pi G/3)\rho\,(1-\rho/\rho_0)$ [9] with ρ_0 of the order of $\rho_{\rm Pl}$. (Solvability is based on an underlying $\mathrm{sl}(2,\mathbb{R})$ symmetry obtained from the algebra $[\hat{V},\hat{J}] = i\delta\hbar\hat{H}$, $[\hat{V},\hat{H}] = -i\delta\hbar\hat{J}$, $[\hat{J},\hat{H}] = i\delta\hbar\hat{V}$ with the volume $\hat{V}$, $J = V\exp(i\delta\mathcal{H})$ for the Hubble parameter $\mathcal{H}$, and the Hamiltonian $\hat{H}$ [10].)

Exact solutions show some of the main implications, intuitively grasped as follows. In a discrete space, as it underlies the construction of this model, there is a finite capacity to store energy. When the limit is reached, gravity turns repulsive at high densities. A bounce at about the Planck density results which, if it is realized generally enough, can resolve the singularity problem. At present it remains unclear how general the mechanism to resolve singularities is; it applies at least in models in which the kinetic energy of matter dominates the potential term. It also remains to be determined at what density ρ_0 the bounce happens. The precise value depends on δ which is difficult to derive from the full Hamiltonian. However, there are internal consistency conditions by comparing the different corrections within the model, and those conditions suggest that the bounce density is less than Planckian [11].

An interesting consequence is that matter properties are relevant throughout cosmic evolution, including the bounce phase. Sometimes, one attempts to develop this bounce cosmology [12] as an alternative to inflation to explain the nearly scale-free spectrum of anisotropies. In some models, structure can be generated in the collapse phase and transmitted through the bounce. The transmission phase is difficult to control because it is very sensitive to quantum-gravity properties. Non-standard equations of state of exotic matter leave an imprint on the structures formed. Exotic matter may also play a role in the build-up of anisotropy.

We now return to the example of the sensitive phase of big-bang nucleosynthesis. In quantum gravity the Maxwell and Dirac Hamiltonians could be subject to different quantum corrections, and thus the relative dilution behavior may change. Only inverse-triad corrections have been implemented so far, which change the equations of state in the same way for photons and relativistic fermions [13]. Effects are thus not as strong as could have been expected, but they are nevertheless close to being interesting: A detailed analysis [14] provides an upper bound $\rho < 3/\ell_{\rm Pl}^3$ for the density of atoms of space, not off by orders of magnitude from the theoretical expectation of at most about one atom per Planck cube. This looks promising; however, the precision of big-bang nucleosynthesis observations at the present stage is difficult to improve. There is more potential in looking at details of the cosmic microwave background. Here, Hamiltonians are endowed with correction factors $\alpha \sim 1+\epsilon$ for inverse-triad components in loop quantum gravity. The parameter ϵ can be constrained by a CMB analysis, and so far is consistent with zero. However, there is a convergence of theoretical lower and observational upper bounds for the parameters [15] which should accelerate with new data.

Another test area of quantum gravity is black holes. In general relativity it is impossible, under very general assumptions on the equation of state, to stop the gravitational collapse of a heavy star. Gravity is always attractive, and thus becomes the dominant force when matter is sufficiently dense. In quantum gravity, the space-time dynamics changes, and as in the solvable cosmological model we have repulsive gravity at extremely high densities. Also for black holes, a non-singular collapse results, but one that still leads to a horizon trapping light [16]. However, the horizon disappears once the collapsing matter has traversed the high-density phase. Horizons, and thus by definition black holes, exist only for finite times. The horizon shrinks by Hawking evaporation, and eventually disappears, at which time one expects some kind of stellar explosion. Also here, specific models for collapse depend on the matter behavior, opening ways for tests.

The space-time structure in quantum gravity may even have implications for particle physics, especially for parity symmetry [13]. The vector potential is defined as $\underset{\rightarrow}{A}_i = \underset{\rightarrow}{\Gamma}_i + \gamma \underset{\rightarrow}{K}_i$ where $\underset{\rightarrow}{\Gamma}_i$ is parity-odd and $\underset{\rightarrow}{K}_i$ parity-even. Unless γ is a pseudoscalar, which for a fundamental constant would be rather unusual, there is a non-trivial and indefinite parity behavior of $\underset{\rightarrow}{A}_i$. Classically the equations of motion are parity invariant; they are, after all, equivalent to Einstein's equation. But there is so far no good reason to expect the invariance to be preserved after replacing $\underset{\rightarrow}{A}_i$ with $h_e(\underset{\rightarrow}{A}_i)$, implementing one form of quantum corrections. Parity is still to be checked by involved calculations that not only derive corrections in equations of motion but also ensure that they are consistent in the sense of covariance and anomaly freedom. If parity violation due to quantum gravity is found, it may be relevant for baryogenesis. In this context, it is also worth mentioning that some bounce models show a change of orientation at the densest moment (the universe "turns its inside out"). Parity breaking will then become relevant for the big-bang transition.

In order to discuss possible relationships between quantum gravity and the quark-gluon plasma, the main topic of these proceedings, the first thing to note is that there are still many orders of magnitude from quark-gluon plasma densities to the Planck scale. At best, indirect consequences can be expected as always in quantum gravity. The following suggestions can be made:

- The matter equation of state is important for collapse and bounce scenarios as exemplified in here, for instance for the build-up of anisotropy and the evolution of structure.

- The cosmological prehistory is relevant for baryogenesis: is it more reasonable to assume a matter/antimatter-symmetric initial state, or a more messy and non-symmetric one after the collapse of an entire universe?

- There are indications that symmetries such as parity or local Lorentz transformations are modified by quantum geometry, with implications for quantum field theory.

To summarize, we have considered a quantum theory of space-time as a gauge theory. A crucial new feature compared to other gauge theories is the important role of general covariance. In loop quantum gravity, this is seen to imply an (irregular) lattice structure even for the continuum theory. Direct effects are important only at extremely high densities, but indirect tests are conceivable in intermediate regimes; several examples have already been described in cosmology. For specific scenarios, the equation of state of matter is then required for details. There is certainly no observation yet or in the foreseeable future, but bounds on the theory are becoming interesting and have already ruled out some possibilities.

Acknowledgements

Part of this work was supported by NSF grant PHY0748336. The author is grateful to the organizers of the "6th International Conference on Physics and Astrophysics of Quark Gluon Plasma" (ICPAQGP 2010), where this work has been presented as a plenary talk.

References

[1] C. Rovelli, Quantum Gravity (Cambridge University Press, Cambridge, UK, 2004); T. Thiemann, Introduction to Modern Canonical Quantum General Relativity (Cambridge University Press, Cambridge, UK, 2007), gr-qc/0110034.

[2] A. Ashtekar and J. Lewandowski, J. Math. Phys. 36 (1995) 2170.

[3] C. Rovelli and L. Smolin, Nucl. Phys. B 442 (1995) 593, gr-qc/9411005.

[4] K.A. Meissner, Class. Quantum Grav. 21 (2004) 5245, gr-qc/0407052.

[5] T. Thiemann, Class. Quantum Grav. 15 (1998) 839, gr-qc/9606089.

[6] T. Thiemann, Class. Quantum Grav. 15 (1998) 1281, gr-qc/9705019.

[7] M. Bojowald and A. Skirzewski, Rev. Math. Phys. 18 (2006) 713, math-ph/0511043.

[8] M. Bojowald, Living Rev. Relativity 11 (2008) 4, gr-qc/0601085, `http://www.livingreviews.org/lrr-2008-4`;
Quantum Cosmology: A Fundamental Theory of the Universe (Springer, New York, 2011).

[9] P. Singh, Phys. Rev. D 73 (2006) 063508, gr-qc/0603043.

[10] M. Bojowald, Phys. Rev. D 75 (2007) 081301(R), gr-qc/0608100.

[11] M. Bojowald, Class. Quantum Grav. 26 (2009) 075020, arXiv:0811.4129.

[12] M. Novello and S.E.P. Bergliaffa, Phys. Rep. 463 (2008) 127.

[13] M. Bojowald and R. Das, Phys. Rev. D 78 (2008) 064009, arXiv:0710.5722.

[14] M. Bojowald, R. Das and R. Scherrer, Phys. Rev. D 77 (2008) 084003, arXiv:0710.5734.

[15] J. Mielczarek, T. Cailleteau, J. Grain and A. Barrau, Phys. Rev. D 81 (2010) 104049, arXiv:1003.4660; M. Bojowald, G. Calcagni and S. Tsujikawa, arXiv:1101.5391; arXiv:1107.1540.

[16] T. A. Roman and P. G. Bergmann, Phys. Rev. D 28 (1983) 1265–1277; A. Ashtekar and M. Bojowald, Class. Quantum Grav. 22 (2005) 3349, gr-qc/0504029; S. A. Hayward, Phys. Rev. Lett. 96 (2006) 031103, gr-qc/0506126.