

The Wave Function of the Universe and CMB Fluctuations

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Abstract

The Hartle-Hawking and Tunneling (Vilenkin) wave functions are treated in the Hamiltonian formalism. We find that the leading (i.e. quadratic) terms in the fluctuations around a maximally symmetric background, are indeed Gaussian (rather than inverse Gaussian), for both types of wave function, when properly interpreted. However the suppression of non-Gaussianities and hence the recovery of the Bunch-Davies state is not transparent.

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1 Introduction

Inflationary cosmology is not past eternal. Under the assumption that the average Hubble parameter $\langle H \rangle > 0$, i.e. that on average the universe has been expanding in the past, both null and time-like geodesics cannot be arbitrarily extended to the past [1]. In fact unlike the cosmological singularity theorems which relied on the use of the weak energy condition (violated by inflationary cosmology), this argument does not need such a condition.

On the other hand the standard calculation of the scalar and tensor fluctuation spectrum around the inflationary background assumes that in the far past (conformal time $\eta \rightarrow -\infty(1 - i\epsilon)$), the inflationary background remains valid, and correlation functions in any state of the system (defined in the interaction picture) at some conformal time τ is related to those in the “Bunch-Davies” (BD) vacuum state $|0\rangle$ by the “in-in” formula [2, 3],

$$\frac{\langle \Omega(\tau) | \hat{W}(\tau) | \Omega(\tau) \rangle}{\langle \Omega(\tau) | \Omega(\tau) \rangle} = \langle 0 | U^\dagger(\tau, -\infty) W(\tau) U(\tau, -\infty) | 0 \rangle, \quad (1)$$

$$U(\tau, -\infty) = T \exp \left(-\frac{i}{\hbar} \int_{-\infty(1-i\epsilon)}^{\tau} \hat{H}_{1I}(\tau') d\tau' \right). \quad (2)$$

In the above the Hamiltonian for fluctuations around the (time-dependent) inflationary background is taken to be of the form $H_0(t) + H_1(t)$, where the first term governs quadratic fluctuations and the second cubic and higher order fluctuations. Also \hat{H}_{1I} is H_1 in the interaction picture.

The important point to note here is that both H_0 and H_1 are time dependent, with this dependence given by the slow roll inflationary background metric. The projection onto the matrix element in the BD vacuum is a consequence of taking the limit of infinite negative conformal time. This however seems to be in apparent conflict with the theorem of [1] that inflation is not past eternal.

This motivates the search for some explanation as to the emergence of an inflationary background starting from “nothing”. This has been discussed since the early eighties and there are two main proposals. One is the “no boundary” wave function of Hartle and Hawking [4] (HH). The other is the tunneling (T) wave function of Vilenkin [5]¹. While both wave functions could in principle be valid solutions to the Wheeler DeWitt equation without truncation - in practice the explicit forms of the wave functions have only been obtained in the so-called mini-superspace model where only the temporal dependence of the fields is kept.

Recently this basis for justifying the BD state in the inflationary background has been questioned [9, 10, 11, 12]. Using the Picard-Lefschetz theory of saddle point approximations to the integral over the lapse N of the ADM formulation of general relativity, it was claimed in [9] that this method justifies the tunneling wave function but not the HH wave function. In the subsequent paper [10], the authors then argued that quadratic scalar/tensor fluctuations around both T and HH wave functions were unsuppressed, having an inverse Gaussian form. This seemed to cast doubt on the quantum cosmology justification for a smooth BD beginning to inflationary cosmology.

¹For a recent discussion and references to the early literature see [6]. This paper also addresses similar issues to those in this paper but only for the T wave function and from a different perspective. See also [7] and references therein for a related discussion, as well as [8] for a loop quantum gravity perspective.

In a rebuttal of these claims one of the original proponents of the HH wave function (Hartle) and collaborators [13], argued that there is a choice of contour that justifies the HH wave function. Furthermore that whereas the T wave function (following from the contour of [9]) was indeed unstable due to unsuppressed fluctuations (as argued a while ago also in the references quoted in [13]) the HH wave function is not. Subsequently in [11] it was argued that the contour chosen in [13] actually should pick up subdominant saddle points which restores the unsuppressed fluctuations in the HH wave function as well.

A new round of claims and counter claims were made this year (2018). Reference [14] considered a generalization of minisuperspace, replacing the round S_3 by axial Bianchi IX geometry. They argued, using a circular contour for the integration over the lapse N , that the HH wave function is well defined and hence that the original HH wave function was stable under deformations. This choice of contour was criticized in [12] on the grounds that it was not physically well motivated and leads to “mathematical and physical inconsistencies”. Finally in a very recent paper Vilenkin and Yamada [6] have argued that provided certain boundary/initial conditions on the scalar fluctuations are satisfied, the scalar field fluctuations around the tunneling wave function (T) are well behaved.

In this paper we will first discuss a simple problem which illustrates one of the main points which we wish to make, namely that there is no advantage to using the functional integral and the Picard - Lefschetz method to solve the WdW equation wherever it is possible to solve the equation directly in the semi-classical approximation, which is the case in all these examples. In particular we argue that one needs some physical input to decide what particular solution to pick, and that one cannot do this on mathematical grounds. We then review the mini-superspace solution for universe creation from “nothing” - both the HH and the T cases. Finally we discuss fluctuations around mini-superspace. We will find that, properly interpreted, both HH and T cases lead to suppressed Gaussian fluctuations for the wave function in the classical regime, contrary to the claims in [10, 11, 12]. On the other hand this wave function (in either HH or T cases) necessarily has non-Gaussian (i.e. cubic and higher powers of fluctuations) terms. In other words the emergence of the Bunch Davies vacuum wave function, which is just Gaussian, does not appear to have any explanation from these considerations.

2 Particle creation in an E-field.

In [15] Brown and Teitelboim (BT) use a Euclidean instanton to describe pair creation in an electric field (the Schwinger process) in 1+1 dimensions, and then extend it to brane nucleation in higher dimensions. We will just focus on the former in flat space and ignore the dynamics of the E-M field.

$$S = -m \int ds (-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{1/2} - e \int ds \dot{x}^\mu A_\mu, \quad \dot{x}^\mu \equiv \frac{dx^\mu}{ds}. \quad (3)$$

Introduce a proper time metric factor $N(s)$ to write the action in quadratic form and choose the gauge potential for a constant electric field E as $A_\mu = (Ex, 0)$ so

$$S = - \int ds \left\{ \frac{m}{2N(s)} (\dot{t}^2 - \dot{x}^2) + \frac{m}{2} N(s) \right\} - eE \int ds \dot{x}. \quad (4)$$

In the canonical formalism we have conjugate momenta

$$\pi_t = -\frac{m}{N}\dot{t} - eEx, \quad \pi_x = m\frac{\dot{x}}{N}, \quad \pi_N = 0,$$

and Hamiltonian

$$\mathcal{H} = N \left(-\frac{\pi_t^2}{2m} + \frac{\pi_x^2}{2m} - \frac{(eEx)^2}{2m} - \frac{e}{m}\pi_t Ex + \frac{m}{2} \right) \equiv NH. \quad (5)$$

The dynamics is given by Hamilton's equations for the phase space variables $\{\alpha\} = N, \pi_N, x, \pi_x, t, \pi_t$,

$$\dot{\alpha} = \{\alpha, \mathcal{H}\}, \quad (6)$$

with the following relations/constraints.

Poisson Brackets:

$$\{N, \pi_N\} = \{x, \pi_x\} = \{t, \pi_t\} = 1. \quad (7)$$

Primary constraints:

$$\pi_N \approx 0. \quad (8)$$

Secondary constraints:

$$\dot{\pi}_N = \{\pi_N, \mathcal{H}\} = -H \approx 0. \quad (9)$$

The equations of motion are

$$\begin{aligned} \dot{N} &= \{N, \mathcal{H}\} = 0, \quad \dot{t} = -\frac{N}{m}(\pi_t + eEx), \quad \dot{x} = N\frac{\pi_x}{m} \\ \dot{\pi}_t &= 0, \quad \dot{\pi}_x = N(eE)^2\frac{x}{m}. \end{aligned} \quad (10)$$

Since π_t is constant let us choose

$$\pi_t = \pi_{t0} = 0. \quad (11)$$

Also in the passage to QM $\pi_q \rightarrow -i\hbar\frac{\delta}{\delta q}$ etc. acting on the Schroedinger wave function Ψ . In particular $\pi_N \approx 0$ implies

$$\frac{\delta}{\delta N}\Psi = 0,$$

and $H \approx 0$ implies the ‘‘Wheeler-DeWitt’’ equation

$$H(q, -i\hbar\frac{\delta}{\delta q})\Psi(q) = 0.$$

(We are assuming there is no boundary in space and hence no boundary Hamiltonian.)

In the WKB approximation (ignoring the pre-factor)

$$\Psi \propto e^{\frac{i}{\hbar}S_{cl}} \quad (12)$$

where the classical action (evaluated on a solution) is

$$\begin{aligned} S_{cl} &= \int ds[\pi_t\dot{t} + \pi_x\dot{x} + \pi_N\dot{N} - \mathcal{H}] \\ &= \int \pi_x dx. \end{aligned} \quad (13)$$

In the last step we used (8)(11). Also from (9)(11) we have solving for π_x

$$\pi_x = \pm |eE| \sqrt{x^2 - \left(\frac{m}{eE}\right)^2}.$$

Evaluating the integral between the two turning points one gets (defining $\gamma = m/|eE|$)

$$\begin{aligned} S_{cl} &= \pm |eE| \int_{-\gamma}^{+\gamma} dx \sqrt{x^2 - \gamma^2} \\ &= \pm |eE| \left[\frac{1}{2} x \sqrt{x^2 - \gamma^2} - \frac{1}{2} \gamma^2 \ln(x + \sqrt{x^2 - \gamma^2}) \right]_{-\gamma}^{+\gamma} \\ &= \pm \frac{i\pi m^2}{2|eE|} (2n + 1), \quad n \in \mathcal{Z}. \end{aligned} \tag{14}$$

This gives a probability

$$P \propto |\Psi|^2 \propto e^{-\frac{\pi m^2}{|eE|}} \tag{15}$$

In agreement with Brown and Teitelboim's instanton calculation (and the leading term in Schwinger's calculation). Note that here we have rejected the possible positive sign in the exponent on physical grounds since otherwise we would have P rising with decreasing electric field E ! In other words there is no way that one can get the right sign from the formalism.

Let us redo the calculation in the Lagrangian formulation with the lapse gauge fixed to $N(s) = N$ (a constant as in [9]). The action is again given by (4) and the Lagrangian equations of motion are,

$$\ddot{t} = \alpha \dot{x}, \quad \ddot{x} = \alpha \dot{t}, \quad \alpha \equiv -\frac{eEN}{m},$$

from varying w.r.t. x^μ , and from varying w.r.t. N ,

$$-\frac{m}{2N^2}(\dot{t}^2 - \dot{x}^2) + \frac{m}{2} = 0.$$

Define $x^\pm = t \pm x$ so that the above equations become

$$\ddot{x}^\pm = \pm \alpha \dot{x}^\pm, \quad N^2 = \dot{x}^+ \dot{x}^-.$$

With appropriate choice of initial conditions we have the solutions

$$\dot{x}^\pm = \dot{x}_0^\pm e^{\pm \alpha s}, \quad x^\pm = \pm \frac{\dot{x}_0^\pm}{\alpha} e^{\pm \alpha s}, \quad N^2 = \dot{x}_0^+ \dot{x}_0^-. \tag{16}$$

The equation for the orbit is:

$$t^2 - x^2 = -\frac{\dot{x}_0^+ \dot{x}_0^-}{\alpha^2} = -\frac{m^2}{(eE)^2}, \tag{17}$$

where in the last step we used the last equation of (16). Note that at $t = 0$ we have

$$x(t = 0) = \pm \frac{m}{eE} \equiv \pm \gamma. \tag{18}$$

In the Brown-Teitelboim description of pair creation, the particle on the left propagates backwards in time (anti-particle) t while the particle on the right propagates forward. This is to be interpreted as pair creation at time $t = 0$ at the points given by (18). This scenario requires $x_0^+ x_0^- = -|x_0^+ x_0^-|$, which implies that the saddle points for the integration over N are pure imaginary,

$$N = \pm i \sqrt{|x_0^+ x_0^-|}. \quad (19)$$

Integrating the last term in the action over the tunneling trajectory gives an imaginary part to the action from the term,

$$eE \int_{x=-\alpha_c}^{x=\alpha_c} dstx = - \int_{-\gamma}^{\gamma} dxt(x) = \mp \int_{-\alpha_c^{-1}}^{\alpha_c^{-1}} dx \sqrt{x^2 - \gamma^2},$$

which is the same as before i.e. (14) and hence gives the same probability for pair creation (15).

The point of this exercise was to show that the picking one or other saddle point in (19) does not resolve the sign ambiguity that we had in the Hamiltonian discussion. As there this ambiguity comes from the two solutions for t , which in that case was from solving the Hamiltonian constraint while here it comes from solving the orbit equation (17).

3 Mini-superspace HH and T

3.1 Background wave function

Let us now discuss mini-superspace in the Hamiltonian formalism. The action is given by

$$S = \int_0^1 dt (-N^{-1} 3a\dot{a}^2 + (3ka - a^3\Lambda)N). \quad (20)$$

Note for future reference that the actual action (in 3+1 dimensions) has a factor of the unit three sphere volume and is $2\pi^2 S$. Also $\dot{x} \equiv \frac{dx}{dt}$. Change variable to $q = a^2$ $N \rightarrow N/a$:

$$S = \int dt \left(-\frac{3}{4N} \dot{q}^2 + N(3k - \Lambda q) \right),$$

$$\pi_N = 0, \quad \pi_q = -\frac{3}{2N} \dot{q}, \quad \dot{q} = -\frac{2N\pi_q}{3}, \quad (21)$$

$$\mathcal{H} = N \left(-\frac{\pi_q^2}{3} + (\Lambda q - 3k) \right) \equiv NH. \quad (22)$$

The primary and secondary constraints are,

$$\pi_N = 0, \quad \dot{\pi}_N = \{\pi_N, \mathcal{H}\} = H \approx 0.$$

So on a classical trajectory

$$\pi_q^2 = 3(\Lambda q - 3k), \quad \pi_q = \pm 3\sqrt{\frac{\Lambda}{3}q - k}.$$

The Wheeler-DeWitt equation for the system is obtained by putting $\pi_q \rightarrow -i\hbar\partial/\partial q$ in the Hamiltonian constraint and is

$$\left\{ +\frac{\hbar^2}{3} \frac{\partial^2}{\partial q^2} + (\Lambda q - 3) \right\} \Psi = 0. \quad (23)$$

Consider tunneling to a deSitter space with $\Lambda > 0$, $k = 1$. The classical action is

$$\begin{aligned} \int_{q_0}^{q_1} \pi_q dq &= \pm 3 \int_{q_0}^{q_1} dq \sqrt{\frac{\Lambda}{3} q - k} \\ &= \mp \frac{6i}{\Lambda} \left[\left(1 - \frac{\Lambda}{3} q_1\right)^{3/2} - \left(1 - \frac{\Lambda}{3} q_0\right)^{3/2} \right]. \end{aligned}$$

This is pure imaginary for “under the barrier” propagation $q_0, q_1 < \frac{3}{\Lambda}$. In the semi-classical approximation (and ignoring the fluctuations) this gives the transition amplitude (ignoring pre-factors!)

$$\begin{aligned} K(a_1; a_0) &\sim \exp \left[i2\pi^2 \int_{q_0}^{q_1} \pi_q dq \right] \\ &= \exp \left[\pm \frac{12\pi^2}{\Lambda} \left\{ \left(1 - \frac{\Lambda}{3} a_1^2\right)^{3/2} - \left(1 - \frac{\Lambda}{3} a_0^2\right)^{3/2} \right\} \right] \end{aligned} \quad (24)$$

Thus again the Hamiltonian analysis shows that both signs are allowed, i.e. both HH and T are valid solutions. The two signs come again from the fact that the H constraint is quadratic in π . As in the case of the particle in an E-field the sign has to be chosen on physical grounds. One might argue as in the latter case that the upper sign (T) is physically more plausible. Actually the probability to find a scale factor a_1 ,

$$P(a_1) \sim |\Psi(a_1)|^2 \quad (25)$$

depends on the initial wave function since

$$\Psi(a_1) = \int da_0 \mu(a_0) K(a_1; a_0) \Psi_0[a_0].$$

If we take the “initial” mini-superspace universe to be an eigenstate of the scale factor with zero scale factor (“nothing”), $\Psi_0 \propto \delta(a_0)$. So

$$P(a_1) \sim \exp \left[\mp \frac{24\pi^2}{\Lambda} \left\{ 1 - \left(1 - \frac{\Lambda}{3} a_1^2\right)^{3/2} \right\} \right]. \quad (26)$$

The upper sign gives a falling probability with increasing a_1 (tunneling (T) -Vilenkin) while the lower sign a rising probability (HH).

We should note in passing that this wave function is an asymptotic expression that is valid only in the region that is not only inside but also far from the turning point. In particular it cannot be used at the turning point $a = \sqrt{\frac{3}{\Lambda}}$ itself. We shall discuss this further in the next subsection.

The authors of [9] claim to fix the sign ambiguity by first integrating over q and then doing an integral over N and picking what they claim is the correct saddle points. Somehow this seems to imply that the above calculation must fail for the “wrong” sign! As in the particle in E field case it is clear from this analysis that this ambiguity cannot be fixed by pure mathematics.

3.2 Hartle-Hawking or Tunneling

We argued in the previous subsections that the choice of the overall sign of the exponent in (24)(26) is not determined by any mathematical consistency argument, but maybe fixed by physical considerations. To discuss this we need to match each under the barrier (real) wave function to the appropriate (linear combination of) oscillating wave functions. This is done by the standard WKB matching conditions.

In fact if one ignores the fluctuations, one simply has a linear potential and the exact solution is well-known (eg. [16]). Thus defining

$$z = \left(\frac{3}{\hbar^2 \Lambda^2} \right)^{1/3} (3 - \Lambda q), \quad (27)$$

the Wheeler-DeWitt equation (23) becomes the Airy function equation

$$\frac{d^2 f(z)}{dz^2} - z f(z) = 0.$$

The exact solution for the wave function is thus

$$\Psi(q) = A \text{Ai}(z(q)) + B \text{Bi}(z(q)). \quad (28)$$

The asymptotic behavior of the Airy functions in the classical region $z \ll -1$ i.e. $a \gg \sqrt{\frac{3}{\Lambda}}$ is

$$\text{Ai}(z) \sim \pi^{-1/2} |z|^{-1/4} \cos\left(\frac{2}{3}|z|^{3/2} - \frac{\pi}{4}\right), \quad (29)$$

$$\text{Bi}(z) \sim -\pi^{-1/2} |z|^{-1/4} \sin\left(\frac{2}{3}|z|^{3/2} - \frac{\pi}{4}\right). \quad (30)$$

On the other hand in the non-classical regime $z \gg 1$ (which of course can exist only for a very small cosmological constant $\Lambda \ll 1$), one has

$$\text{Ai}(z) \sim \frac{1}{2} \pi^{-1/2} |z|^{-1/4} \exp\left\{-\frac{2}{3} z^{3/2}\right\}, \quad (31)$$

$$\text{Bi}(z) \sim \pi^{-1/2} |z|^{-1/4} \exp\left\{\frac{2}{3} z^{3/2}\right\}. \quad (32)$$

Suppose that we interpret the observed condition of an expanding universe to mean that Ψ should be an eigenstate of momentum

$$-i\hbar \frac{d}{dq} \Psi = p \Psi,$$

where by identifying the eigenvalue with the classical momentum (21) we have

$$p = \pi_q^{\text{cl}} = -\frac{3}{2N} \dot{q} < 0,$$

for an expanding universe. Thus we need

$$\begin{aligned}\Psi_{\text{out}} &\sim e^{-i\frac{6}{\hbar\Lambda}\left(\frac{\Lambda q}{3}-1\right)^{3/2}+i\frac{\pi}{4}} \\ &\propto \text{Ai}(z(q)) + i\text{Bi}(z(q)),\end{aligned}\tag{33}$$

which gives $p = -3\left(\frac{\Lambda q}{3}-1\right)^{1/2} < 0$ and hence $\dot{a} > 0$. Thus the corresponding under the barrier (i.e. for $\frac{\Lambda q}{3} < 1$) wave function is

$$\begin{aligned}\Psi_{\text{under}}(q) &= A\pi^{-1/2} \left[\left(\frac{3}{\hbar^2\Lambda^2} \right)^{1/3} (3 - \Lambda q) \right]^{-1/4} \times \\ &\quad \left[\frac{1}{2}e^{-\frac{6}{\hbar\Lambda}\left(1-\frac{\Lambda q}{3}\right)^{3/2}} + ie^{\frac{6}{\hbar\Lambda}\left(1-\frac{\Lambda q}{3}\right)^{3/2}} \right].\end{aligned}\tag{34}$$

This is Vilenkin's tunneling wave function proposal. The condition that the observed universe is expanding (much as we observe electrons coming out of the nucleus in β -decay), is used to impose this outgoing boundary condition. Note that we have not normalized these functions. Indeed it is not clear to us whether wave functionals in a QFT (leave alone in quantum gravity) are even in principle normalizable. However the relative probability may still make sense. So the relevant probability for tunneling from "nothing" is

$$P_{\text{Tunnel}}(a; 0) \equiv \frac{|\Psi_{\text{out}}(q)|^2}{|\Psi_{\text{under}}(0)|^2} \sim \exp\left(-\frac{12}{\hbar\Lambda}\right)\tag{35}$$

On the other hand Hartle and Hawking argued that the under the barrier wave function is given by Euclidean quantum gravity. This amounts to demanding that the first term in (34) be the only allowed wave function (i.e. we must pick the solution Ai). Then in the large q regime

$$\Psi \sim \text{Ai}(z(q)) \rightarrow \cos\left[\frac{6}{\hbar\Lambda}\left(\frac{\Lambda q}{3}-1\right)^{3/2} + i\frac{\pi}{4}\right],\tag{36}$$

corresponding to a superposition of an expanding and contracting universe. In other words one does not have a classical expanding background and needs to appeal to some sort of decoherence argument to account for the observed expanding universe. In this case

$$P_{\text{HH}}(a; 0) \equiv \frac{|\Psi_{\text{out}}(q)|^2}{|\Psi_{\text{under}}(0)|^2} \sim \exp\left(\frac{12}{\hbar\Lambda}\right)\tag{37}$$

All this is well-known and is included here for completeness and to set the stage for identifying the fluctuation spectrum around these solutions.

3.3 Fluctuations

Now one might ask whether the ambiguity as to which wave function is physical might be fixed by the inclusion of tensor perturbations. According to [10, 11], the leading order calculation of tensor perturbations has the wrong (inverse Gaussian) sign for both wave functions, implying that

there is no smooth beginning to the universe in either case. This is contrary to the claim of [13] (and citations therein) that HH leads to Gaussian fluctuations, while the tunneling wave function is unstable to quadratic fluctuations.

The tensor modes are expanded in S_3 spherical harmonics labelled by integers l, m, n . Suppressing the last two indices the action for a mode ϕ_l in the deSitter background is to quadratic order [11]

$$S_l = \frac{1}{2} \int dt \left[q^2 \frac{\dot{\phi}_l^2}{N} - Nl(l+2)\phi_l^2 \right]. \quad (38)$$

So (dropping also the subscript l),

$$\pi_\phi = q^2 \frac{\dot{\phi}}{N}, \quad \dot{\phi} = \frac{N\pi_\phi}{q^2}, \quad (39)$$

giving the total Hamiltonian

$$\mathcal{H} = NH = N \left\{ -\frac{1}{3}\pi_q^2 + (\Lambda q - 3) + \frac{\pi_\phi^2}{2q^2} + \frac{1}{2}l(l+2)\phi^2 \right\}. \quad (40)$$

The WdW equation is: ($\pi_q \rightarrow -i\hbar\partial/\partial q$, $\pi_\phi \rightarrow -i\hbar\partial/\partial\phi$)

$$\left\{ +\frac{\hbar^2}{3} \frac{\partial^2}{\partial q^2} + (\Lambda q - 3) - \frac{\hbar^2}{2q^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{2}l(l+2)\phi^2 \right\} \Psi = 0. \quad (41)$$

Building on the solution in the absence of ϕ , we try

$$\Psi[q, \phi] \sim \exp \left\{ \frac{1}{\hbar} \left(\frac{6}{\Lambda} c_1 \left(1 - \frac{\Lambda}{3} q\right)^{3/2} + c_2 \frac{1}{2} q \phi^2 \sigma(q) \right) \right\}, \quad (42)$$

where $c_1 = \pm 1$, $c_2 = \pm 1$ and σ_l is a function that is to be determined by (41). Computing derivatives get;

$$\begin{aligned} \partial_q \Psi &= \frac{1}{\hbar} \left\{ -c_1 3 \left(1 - \frac{\Lambda}{3} q\right)^{1/2} + c_2 \frac{1}{2} \phi^2 (\sigma(q) + q\sigma'(q)) \right\} \Psi, \\ \frac{\hbar^2}{3} \partial_q^2 \Psi &= \left\{ (3 - \Lambda q) - c_1 c_2 \phi^2 (\sigma(q) + q\sigma'(q)) \left(1 - \frac{\Lambda}{3} q\right)^{1/2} + O(\phi^4) + O(\hbar) \right\} \Psi, \\ -\frac{\hbar^2}{2q^2} \partial_\phi^2 \Psi &= \left[-\frac{1}{2} \phi^2 \sigma^2(q) + O(\hbar) \right] \Psi. \end{aligned} \quad (43)$$

Thus ignoring the ϕ^4 and \hbar corrections the WdW equation (41) becomes

$$\left[-c_1 c_2 \phi^2 (\sigma(q) + q\sigma'(q)) \left(1 - \frac{\Lambda}{3} q\right)^{1/2} - \frac{1}{2} \phi^2 \sigma^2(q) + \frac{1}{2} l(l+2) \phi^2 \right] \Psi = 0. \quad (44)$$

Thus σ_l is required to be a solution of

$$-c_1 c_2 (\sigma(q) + q\sigma'(q)) \left(1 - \frac{\Lambda}{3} q\right)^{1/2} - \frac{1}{2} \sigma^2(q) + \frac{1}{2} l(l+2) = 0. \quad (45)$$

Putting

$$\sigma(q) = \frac{g_l}{(1 - \frac{\Lambda}{3}q)^{1/2} + f_l},$$

we see that (45) is satisfied with

$$f_l = \pm(l+1), \quad g_l = c_1 c_2 l(l+2). \quad (46)$$

Thus (since $c_2^2 = 1$) we have the four solutions for the wave function (up to terms $O(\phi^4)$, $O(\hbar)$ in the exponent)

$$\Psi \sim \exp \left\{ \frac{c_1}{\hbar} \left(\frac{6}{\Lambda} (1 - \frac{\Lambda}{3}q)^{3/2} + \frac{1}{2} q \phi^2 \frac{l(l+2)}{(1 - \frac{\Lambda}{3}q)^{1/2} \pm (l+1)} \right) \right\}. \quad (47)$$

$$= \exp \left\{ \frac{c_1}{\hbar} \left(\frac{6}{\Lambda} (1 - \frac{\Lambda}{3}q)^{3/2} + \frac{1}{2} q \phi^2 \frac{l(l+2) (-1 - \frac{\Lambda}{3}q)^{1/2} \pm (l+1)}{\frac{\Lambda}{3}q + l(l+2)} \right) \right\}. \quad (48)$$

Similarly we have for four possible solutions in the classically allowed region,

$$\begin{aligned} \Psi &\sim \exp \left\{ \frac{c_1}{\hbar} \left(\frac{6}{\Lambda} i \left(\frac{\Lambda}{3}q - 1 \right)^{3/2} - \frac{1}{2} q \phi^2 \frac{l(l+2)}{i \left(\frac{\Lambda}{3}q - 1 \right)^{1/2} \pm (l+1)} \right) \right\} \\ &= \exp \left\{ \frac{c_1}{\hbar} \left(\frac{6}{\Lambda} i \left(\frac{\Lambda}{3}q - 1 \right)^{3/2} + \frac{1}{2} q \phi^2 \frac{l(l+2) \left(i \left(\frac{\Lambda}{3}q - 1 \right)^{1/2} \mp (l+1) \right)}{\frac{\Lambda}{3}q + l(l+2)} \right) \right\} \end{aligned} \quad (49)$$

Note that these four solutions are in agreement with the solutions for the classical action including quadratic fluctuations given in eqn. (30) of [11]. It is clear that two of the four solutions (both under and outside the barrier) give Gaussian fluctuations while two give inverse Gaussian fluctuations. Obviously the general solution which would be a linear superposition of these four solutions will necessarily have components which would invalidate the smooth background which is the starting assumption of this analysis. Feldebrugge et al claim that their path integral argument (essentially involving an ordinary integral over the lapse N) necessarily includes a non-Gaussian component in both the T and HH solutions - in other words depending on the choice of contour (i.e. with a given sign for c_1 one needs to include both \pm signs (corresponding to the two signs for f_l in eqn. (46)) in the solution.

Nevertheless as we will discuss below once one uses the matching conditions to evaluate the wave function in the classical regime, it will be seen that we get different linear combinations of the under barrier wave functions. In fact one can choose to pick those linear combinations which admit only Gaussian fluctuations (with the appropriate choice of sign in f_l in equation (46)). This can be done for both the tunneling wave function as well as the Hartke-Hawking wave function. But in both cases the under barrier wave function necessarily involves inverse-Gaussian fluctuations. However the under the barrier wave functions are valid only for $q\Lambda, q\phi^2 \ll 1$, so it is not clear that this is a problem. Also in evaluating the tunneling probability we will only need the under the barrier wave function at $q = 0$.

Before we discuss this we would like to point out that the wave function necessarily contains non-Gaussian fluctuations. Even ignoring \hbar corrections there are $O(\phi^4)$ terms

$$\frac{1}{12} (\sigma(q) + q\sigma'(q))^2 \phi^4$$

in (43) which imply that the quadratic (in ϕ) term in the log of the wave function Ψ in (42) needs to include a quartic term in ϕ ! This would entail non-Gaussianities in the CMB spectrum that are most probably ruled out.

3.4 Wave function with fluctuations

Using the matching conditions for the zeroth order (in fluctuations) wave functions given by the asymptotic behavior of the Airy functions (29)(30)(31)(32), and the correlation that we found between the sign of the fluctuations and the particular solution to the zeroth order equation (47) (see also (61)(62)), we can now write down the wave function in the classically forbidden and allowed regions for the asymptotic regimes and for small ϕ fluctuations. To simplify the formulae let us first make the following definitions:

$$\lambda(q) \equiv \frac{6}{\Lambda} \left| \left(1 - \frac{\Lambda q}{3} \right) \right|^{3/2}, \quad \Delta \equiv \frac{\Lambda}{3} q + l(l+2) > 0$$

For $\frac{\Lambda q}{3} \ll 1$ we have

$$\begin{aligned} \Psi_{\text{in}}(q, \phi) = & A_+ e^{-\frac{1}{\hbar} \left\{ \lambda(q) + \frac{1}{2} q \phi^2 \frac{l(l+2) \left(-\left(1 - \frac{\Lambda}{3} q\right)^{1/2} + (l+1) \right)}{\Delta} \right\}} + A_- e^{-\frac{1}{\hbar} \left\{ \lambda(q) + \frac{1}{2} q \phi^2 \frac{l(l+2) \left(-\left(1 - \frac{\Lambda}{3} q\right)^{1/2} - (l+1) \right)}{\Delta} \right\}} \\ & B_+ e^{+\frac{1}{\hbar} \left\{ \lambda(q) + \frac{1}{2} q \phi^2 \frac{l(l+2) \left(-\left(1 - \frac{\Lambda}{3} q\right)^{1/2} + (l+1) \right)}{\Delta} \right\}} + B_- e^{+\frac{1}{\hbar} \left\{ \lambda(q) + \frac{1}{2} q \phi^2 \frac{l(l+2) \left(-\left(1 - \frac{\Lambda}{3} q\right)^{1/2} - (l+1) \right)}{\Delta} \right\}} \end{aligned} \quad (50)$$

Note that the A_+ and B_- terms have Gaussian fluctuations while the A_- and B_+ terms have inverse Gaussian fluctuations. Hence one might choose the solution with

$$A_- = B_+ = 0, \quad (51)$$

in order to avoid quadratic instabilities. Of course this is not a guarantee that the solutions are indeed stable since we have nothing to say about the sign of higher order fluctuations!

However these solutions go over in the asymptotic classical regime $\Lambda q/3 \gg 1$ to (using the matching conditions of the $\phi = 0$ theory)

$$\begin{aligned} \Psi_{\text{out}}(q, \phi) = & \left(A_+ - \frac{B_+}{2i} \right) e^{\frac{1}{\hbar} \left\{ i \lambda(q) - \frac{q}{2} \phi^2 \frac{l(l+2)}{D_+} \right\} - i \frac{\pi}{4}} + \left(A_+ + \frac{B_+}{2i} \right) e^{-\frac{1}{\hbar} \left\{ i \lambda(q) - \frac{q}{2} \phi^2 \frac{l(l+2)}{D_+} \right\} + i \frac{\pi}{4}} + \\ & \left(A_- - \frac{B_-}{2i} \right) e^{\frac{1}{\hbar} \left\{ i \lambda(q) - \frac{q}{2} \phi^2 \frac{l(l+2)}{D_-} \right\} - i \frac{\pi}{4}} + \left(A_- + \frac{B_-}{2i} \right) e^{-\frac{1}{\hbar} \left\{ i \lambda(q) - \frac{q}{2} \phi^2 \frac{l(l+2)}{D_-} \right\} + i \frac{\pi}{4}} \end{aligned} \quad (52)$$

with

$$\frac{1}{D_{\pm}} = \frac{-i(\Lambda q/3 - 1)^{1/2} \pm (l+1)}{\Delta}.$$

In the above eqn. (52) as well as in (50) the particular combination of the fluctuation term ($\propto \phi^2$) and the background term $\propto \lambda(q)$ is determined by the Wheeler DeWitt equation as indicated in (49). To avoid having solutions which are unstable to quadratic fluctuations in the classical region we need to set

$$A_+ + \frac{B_+}{2i} = 0, \quad A_- - \frac{B_-}{2i} = 0. \quad (53)$$

Clearly these conditions are compatible with the classically forbidden region conditions (51) only for a vanishing wave function!

On the other hand the expression (50) is only valid for small q and hence it is not clear that we need to take the inverse Gaussian component since the fluctuations (which are in fact suppressed by a power of q are as problematic. In any case the probability for tunneling is obtained by comparing Ψ_{out} for large q to Ψ_{in} at $q = 0$. So let us (for now) just impose the conditions (53) and use (50) only in the limit $q \rightarrow 0$.

Consider now Vilenkin's tunneling wave function case. The boundary condition here is that there is only an outgoing component in classical region corresponding to an expanding universe which means setting $A_+ - B_+/2i = A_- - B_-/2i = 0$. Imposing also the physical requirement that the quadratic fluctuations are suppressed (i.e. (53)) we get

$$\Psi_{\text{out}}^{(T)}(q, \phi) = 2A_- e^{-\frac{1}{\hbar} \left\{ i\lambda(q) - \frac{q}{2} \phi^2 \frac{l(l+2)}{D_-} \right\} + i\frac{\pi}{4}} \quad (54)$$

This should be compared with Ψ_{in} for the same values of the constants (i.e. $A_+ = B_+ = 0$, $B_-/2i = A_-$) giving,

$$\Psi_{\text{in}}^{(T)}(q \rightarrow 0) = A_- \left[e^{-\frac{6}{\hbar\Lambda}} + 2ie^{\frac{6}{\hbar\Lambda}} \right]$$

The probability of the universe emerging in a Bunch-Davies vacuum relative to remaining in a state of nothing (i.e with zero scale factor $a = \sqrt{q} = 0$) is (after restoring the $2\pi^2$ factor which we had dropped),

$$P^{(T)}(q, \phi) = \frac{|\Psi_{\text{out}}^{(T)}(q, \phi)|^2}{|\Psi_{\text{in}}^{(T)}(q \rightarrow 0)|^2} = e^{-24\pi^2/\hbar\Lambda} e^{-\frac{1}{\hbar} q \phi^2 \frac{l(l+1)(l+2)}{\Delta}}. \quad (55)$$

Now let us consider the Hartle-Hawking case. Here if we insist that the boundary conditions are given by Euclidean quantum gravity we would need the wave function to be real. In this case using (53) in (52) and imposing reality we get

$$\Psi_{\text{out}}^{(HH)} = 4A_- \cos \left[\frac{\lambda(q)}{\hbar} \left(1 + \frac{q\phi^2}{2} \frac{l(l+2)}{\Delta} + \frac{\pi}{4} \right) e^{-\frac{1}{\hbar} \phi^2 \frac{l(l+1)(l+2)}{2\Delta}} \right], \quad (56)$$

in agreement with eqn. 3.23 of [13]. Furthermore under the barrier solution in this case in the limit of zero scale factor is,

$$\Psi_{\text{in}}^{(HH)}(q \rightarrow 0) = 2A_- e^{-\frac{6}{\hbar\Lambda}}.$$

Thus we have (after restoring the factor of $2\pi^2$)

$$P^{(HH)}(q, \phi) = \frac{|\Psi_{\text{out}}^{(HH)}(q, \phi)|^2}{|\Psi_{\text{in}}^{(HH)}(q \rightarrow 0)|^2} = e^{24\pi^2/\hbar\Lambda} e^{-\frac{1}{\hbar} q \phi^2 \frac{l(l+1)(l+2)}{\Delta}} 4 \cos^2 \left[\frac{\lambda(q)}{\hbar} \left(1 + \frac{q\phi^2}{2} \frac{l(l+2)}{\Delta} + \frac{\pi}{4} \right) \right]. \quad (57)$$

The Hartle-Hawking wave function is time symmetric between an expanding and a contracting universe. The observed universe is of expanding so somehow the two branches must decohere - in which case the the only difference between the two probabilities is in the pre-factor - with the tunneling case (as is well-known) favoring a larger cosmological constant and the Hartle-Hawking case favoring a smaller CC. Of course given that these formula are actually valid only for small values of the CC this difference is probably not that significant.

3.5 Solving the Hamilton-Jacobi equation beyond quadratic order

In the previous subsection we just considered the quadratic fluctuations around the mini-superspace HH and tunneling solutions. However as we mentioned at the end of subsection (3.3) it is clear that the wave function necessarily contains non-Gaussianities.

To investigate the solutions systematically it is convenient to write

$$\Psi[q, \phi] = e^{\frac{i}{\hbar}S[q, \phi]} \quad (58)$$

Substituting in the WdW equation (41) we have the Hamilton-Jacobi (HJ) equation plus its quantum correction:

$$-\frac{1}{3}\left(\frac{\partial S}{\partial q}\right)^2 + (\Lambda q - 3) + \frac{1}{2q^2}\left(\frac{\partial S}{\partial \phi}\right)^2 + \frac{1}{2}l(l+2)\phi^2 + i\hbar\left(\frac{1}{3}\frac{\partial^2 S}{\partial q^2} - \frac{1}{2q^2}\frac{\partial^2 S}{\partial \phi^2}\right) = 0 \quad (59)$$

The limit $\hbar \rightarrow 0$ give the classical Hamilton-Jacobi equation. Ignoring the quantum correction we try a solution of the form (as before an overall factor of $2\pi^2$ is understood)

$$iS[q, \phi] = c_1 \left[\frac{6}{\Lambda} \left(1 - \frac{\Lambda}{3}q\right)^{3/2} + q \left(\sum_{n=1}^{\infty} \frac{1}{2n!} \sigma_n(q) \phi^{2n} \right) \right] \quad (60)$$

The HJ equation then gives

$$\begin{aligned} -2\left(1 - \frac{\Lambda}{3}q\right)^{1/2} \sum_{n=1}^{\infty} \frac{1}{2n!} (\sigma_n(q) + q\sigma'_n(q)) \phi^{2n} + \\ \left(\sum_{n=1}^{\infty} \frac{1}{2n!} (\sigma_n(q) + q\sigma'_n(q)) \phi^{2n} \right)^2 - \frac{1}{2} \left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)!} \sigma_n(q) \phi^{2n-1} \right)^2 \\ + \frac{1}{2}l(l+2)\phi^2 = 0 \end{aligned}$$

Equating powers of ϕ^2 gives a set of recursion relations which in principle can be solved iteratively to determine σ_n . For instance from the coefficient of ϕ^2 we get

$$-\left(1 - \frac{\Lambda}{3}q\right)^{1/2} (\sigma_1(q) + q\sigma'_1(q)) - \frac{1}{2}\sigma_1^2 + \frac{1}{2}l(l+2) = 0,$$

which is solved by

$$\sigma_1 = \frac{l(l+2)}{\left(1 - \frac{\Lambda}{3}q\right)^{1/2} + (l+1)}. \quad (61)$$

From the coefficient of ϕ^4 we get

$$\frac{2}{4!} \left(1 - \frac{\Lambda}{3}q\right)^{1/2} (\sigma_2 + q\sigma'_2) + \frac{1}{6}\sigma_1\sigma_2 = \frac{1}{4}(\sigma_1 + q\sigma'_1)^2.$$

Since $\sigma_1 + q\sigma'_1 \neq 0$ this shows that $\sigma_2 \neq 0$. Clearly the higher order terms will all be non-zero. Note also that in the classical regime $\left(\frac{\Lambda}{3}q > 1, a > a_{dS} \equiv \sqrt{\frac{3}{\Lambda}}\right)$ the sigma's are complex since as we remarked before:

$$\sigma_1 = \frac{l(l+2)[\pm(l+1) - i\sqrt{\frac{\Lambda}{3}q - 1}]}{\frac{\Lambda}{3}q - 1 + (l+1)^2}, \text{ etc.} \quad (62)$$

The above calculation seems to indicate that solution of the WdW equation in the mini-superspace approximation has non-Gaussian terms. It is however unclear whether all of them (like the quadratic fluctuation coming from σ) are suppressed for the expanding universe component of the wave function of the universe.

4 Conclusions

We have argued that there is no particular advantage to the saddle point (Picard-Lefschetz) method in solving the WdW equation in the minisuperspace truncation. We've also argued that whether or not one picks the Hartle-Hawking or the tunneling wave function is a matter of choosing one or the other boundary condition, and so must be determined by some physical input, and is not a matter of consistency of the saddle point method. Furthermore we've shown how to include fluctuations around the mini-superspace model and indeed the general solution does contain terms with inverse Gaussian fluctuations as shown by Feldbrugge et al [10, 11]. However we have argued that one can choose to set the arbitrary constants multiplying such terms in the classical region to zero, so properly interpreted, the quadratic fluctuations around both competing wave functions are indeed of Gaussian form and are suppressed. Of course in this case the under the barrier wave function does appear to have inverse Gaussian fluctuations but we have argued that these do not matter in the $q \rightarrow 0$ regime where we trust the calculation. On the other hand we have been unable to say anything definitive about non-Gaussianities - which of course need to be suppressed if the initial state of the universe is to lead to the observed CMB spectrum.

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