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Classical Group Field Theory

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Abstract

The ordinary formalism for classical field theory is applied to dynamical group field theories. Focusing first on a local group field theory over one copy of $SU(2)$ and, then, on more involved nonlocal theories (colored and non colored) defined over a tensor product of the same group, we address the issue of translation and dilatation symmetries and the corresponding Noether theorem. The energy momentum tensor and dilatation current are derived and their properties identified for each case.

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1 Introduction

Group field theories (GFTs) are usually defined as tensor field theories over a group manifold. Introduced in the beginning of the 90's [1], they rapidly become pertinent candidates for quantum gravity [2, 3, 4]. In a nutshell, GFTs provide a framework for addressing the problem of the emergence of the topology and the metric properties of spacetime [4].

Since the inception of GFT, several studies have been led using the path integral approach [5]. Interesting facts pertaining to the renormalization program such as power counting theorems [6]-[13], an emergent locality principle [12, 14], Ward-Takahashi identities for unitary symmetries [15] have been highlighted. Furthermore, a large $1/N$ topological/combinatorial expansion [16][17] for colored GFT models [18, 19, 20] portends new and fertile contacts with models in statistical mechanics [21]-[25].

Classical aspects of GFT have been also examined. For instance, the equations of motion for GFT models possessing a trivial kinetic term[1] have been solved, thus providing an explicit class of instantons for these theories [26]. These solutions were used to compute a class of effective actions. We can also emphasize that, using a group Fourier transform (for seminal works on this topic and more applications, [27] affords a recent review), GFTs can be seen as noncommutative field models with a set of diffeomorphisms which turns out to be related to a deformed Poincaré group [28, 29]. Interestingly, one notices that the group manifold initially associated with the background space on which the fields are defined, becomes finally a curved momentum space via this group Fourier transform [30]. The direct space generated after inverse transform is flat whereas the algebra of fields is traded for a noncommutative algebra endowed with a $\star$-product. Nevertheless, other interesting properties at the classical level have been yet investigated. Another contribution emphasizes to what extent the dynamics matters in the renormalization program for GFT [14]. Implementing a dynamical part for GFTs will certainly affect the Noether analysis for symmetries for these models whereas Noether currents in the absence of dynamics are trivial, in general. Thus the present contribution addresses, in a more traditional field theory spirit and by reconsidering the group as a base manifold, the issue of the classical formalism for GFTs.

We should emphasize that some anterior investigations have been carried out on field theories on a Lie group regarding both classical symmetries and also some of their quantum implications. For example, a ϕ^4 quantum field theory on the affine group has been studied in [31]. Furthermore, diffeomorphisms and Weyl transformations in a curved ϕ^4 theory and their implications have a long history (see for instance [32][33]). We will not use, in the present contribution, the same route but will focus either on the gauge invariance of fields or on the nonlocal feature of GFT. Indeed, these two features of GFTs, and among these the nonlocal character of GFTs which should be regarded as the most peculiar, will have drastic consequences on the regular properties of classical symmetries that one is accustomed in ordinary field theory. It can be anticipated that the core notion of local conservation of currents will be drastically affected as it is the case in other well known nonlocal field theories. This is indeed the case of the so called field theories on a noncommutative spacetime [34]. For instance, in any Moyal type noncommutative field theories, the action possesses Poincare currents with an explicit breaking for the local conservation property [35]-[39].

[1] The kinetic term of the most basic GFT is trivial: it is simply composed by a mass term. A typical nontrivial dynamics is governed, for instance, by a Laplacian on the group manifold.

In this paper, we study the classical dynamical GFT over $D = 1$ and then $D = 3$ copies (that we shortly call dimensions or rank) of $SU(2)$. The case $D = 2$ turns out to be equivalent to the situation $D = 1$ due to the gauge invariance condition on fields. Our results can be extended without ambiguity in any dimension. As it will stand the action simply describes a gauge invariant tensor scalar field theory over D copies of the sphere S^3 with a local (for $D = 1$) or nonlocal (for $D = 3$) interaction. The main issue in the present study is to show that the procedure for solving equations of motion or for studying the symmetries of the action finds an extension in a curved, tensored and nonlocal theory. The translation symmetry and the corresponding energy momentum tensor (EMT) have been worked out. We find that the EMT appears to be symmetric in a certain sense but not locally conserved for $D > 1$ in ordinary GFT. However, we surprisingly find that the colored GFT possesses a covariantly conserved quantity obtained by integrating some EMT components. We then address another interesting question which is the implementation of dilatation symmetry or scale transformation at the GFT level. Requiring the invariance under dilatations, in the way we perform it here, yields a radically different action from the translation invariant action. We compute and characterize the current tensor associated with this transformation.

The content of this paper is the following: Section 2 is devoted to the model presentation and the first steps of the classical study: we solve the equation of motion for free fields for the dynamical Boulatov model in $D = 3$. Section 3 thoroughly undertakes the Noether theorem for translation and dilatation symmetries for a $1D$ GFT as a guiding example for more complicated higher rank GFTs. Section 4 discusses the same symmetries for general GFTs. Section 5 deals with colored models and their particular characteristics. Section 6 summarizes our results and also provides outlooks of this work. Finally, a detailed appendix collects the proofs of our claims and useful identities invoked along the text.

2 The dynamical Boulatov-Ooguri model

The prominent properties of the dynamical three dimensional GFT over $G = SU(2)$ are quickly introduced (for more details on the general formulation see [2, 3, 4]). This will be followed by the resolution of the field equation of motion without coupling constant. The Noether analysis of symmetries will be differed to the next sections. This section admits a straightforward extension in any GFT dimension $D > 1$.

The model - The fields belong to the Hilbert space of square integrable and gauge invariant functions on G^3 which satisfy

$$\phi(g_1 h, g_2 h, g_3 h) = \phi(g_1, g_2, g_3) \ , \quad \forall h \in G \ . \tag{1}$$

The shorthand notation $\phi(g_1, g_2, g_3) = \phi_{1,2,3}$ will be used henceforth.

The action S_{3D} is formed by a kinetic term and interacting part. The kinetic term has the form

$$S_{\text{kin},\,3D}[\phi] := \int [\prod_{\ell=1}^{3} dg_\ell] \ \left[\frac{1}{2} \sum_{s=1}^{3} \mathbf{g}_s^{ij} \nabla_{(s)\,i} \ \phi_{1,2,3} \nabla_{(s)\,j} \ \phi_{1,2,3} + \frac{1}{2} m^2 \phi_{1,2,3} \ \phi_{1,2,3} \right] , \tag{2}$$

where dg_i denotes the Haar measure on $G = SU(2)$, the operator $\nabla^i_{(s)}$ represents the covariant derivative (acting here merely as a partial derivative on above fields) defined with the Levi-

Civita connection on $S^3 \simeq SU(2)$. The index (s) will always refer to the tensor structure and so to the particular group element g_s with respect to which one derivates.[2] The labels i,j refer to the local coordinates and, therefore, are lowered or raised by the S^3 metric $\mathbf{g}_{ij}$. Note that the Haar measure of $SU(2)$, dg can be written in a more standard fashion with respect to a theory on a curved background as $dg = (2\pi^2)^{-1}\sqrt{|\det \mathbf{g}|}d\theta d\varphi_1 d\varphi_2$ with $\mathbf{g} = d\theta^2 + \sin^2\theta(d\varphi_1^2 + \sin^2\varphi_1 d\varphi_2^2)$.

The interaction in D dimensional GFTs is nonlocal and dually associated with a D-simplex. For $D=3$ dimensions, the interaction is

$$S_{\text{int},\,3D}[\phi] := \frac{\lambda}{4}\int [\prod_{\ell=1}^{6} dg_\ell]\ \phi_{1,2,3}\,\phi_{3,4,5}\,\phi_{5,2,6}\ \phi_{6,4,1}\ , \tag{3}$$

with a particular pairing of the six variables according to the pattern of the edges of a tetrahedron.

By reducing the kinetic part to a pure massive term and considering the interaction term (3), one gets a model belonging to the class Boulatov-Ooguri models [1]. We will refer to models including Laplacian dynamics as dynamical GFT models.

Formally, a Lagrangian density can be defined as

$$\mathcal{L}_{3D} = \frac{1}{2}\sum_s \nabla^i_{(s)}\ \phi_{1,2,3}\nabla_{(s)\ i}\ \phi_{1,2,3} + \frac{1}{2}m^2\phi_{1,2,3}\ \phi_{1,2,3} + \frac{\lambda}{4}\ \phi_{1,2,3}\phi_{3,4,5}\,\phi_{5,2,6}\,\phi_{6,4,1}\ . \tag{4}$$

The density (4) should be integrated here over six copies of the group (one copy for each g_i, $i = 1,2,\dots,6$), this is the base manifold of the present GFT. Remark that, the kinetic part does not involve g_j, $j = 4,5,6$, but the integration of these three variables is without any effect since the Haar measure is normalized.

Defining the quantity (4) as the Lagrangian density will not affect the remaining developments. Indeed, one should keep in mind that, in the Noether procedure, the field variations are taken with respect to the action and provide equations of motion for fields. The latter are, together with the data of infinitesimal field variations under a given transformation and boundary conditions, the main ingredients in order to apply the Noether theorem. Current calculations have to be performed, as it should be in ordinary field theory, by varying all quantities invoked in the action, up to a surface term. It turns out that, for the field symmetries treated hereafter, what we have called formal Lagrangian density appears as a natural object which is a part of that surface term. This is in exact agreement with the appearance of any ordinary Lagrangian in the computation of a Noether current, for instance the EMT.

The equation of motion for the field results from the action variation:

$$0 = \frac{\delta S_{\text{kin},\,3D}}{\delta\phi_{1,2,3}} + \frac{S_{\text{int},\,3D}}{\delta\phi_{1,2,3}} = -\sum_s \Delta_{(s)}\phi_{1,2,3} + m^2\phi_{1,2,3} + \lambda\int[\prod_{\ell=4}^{6} dg_i]\ \phi_{3,4,5}\,\phi_{5,2,6}\,\phi_{6,4,1}\ , \tag{5}$$

with $\Delta_{(s)}$ being the Laplace operator on the group. Remark that in order to get (5), we implicitly used an integration by parts and the fact that the sphere does not have a boundary.

[2] It will be also called strand index in the following, $s = 1,2,\dots,D$.

Furthermore, one should also rename cyclically the group arguments in the interaction in order to vary properly this nonlocal term.

Colored GFT - Colored GFT models [18][19] have mainly the same definition as above with the crucial attribute that fields possess an extra "color" index ϕ^a. We will choose them to be complex valued functions. The number of colors being the number of fields in the interaction. For the Boulatov colored model, we have $a = 1, 2, 3, 4$. More generally for a D dimensional GFT, the color indices are $a = 1, 2, , \dots, D+1$. All field properties remain the same as previously and a Lagrange density for the $3D$ theory can be given as

$$\begin{aligned}\mathcal{L}^{\rm color} &= \sum_{a=1}^{4}\left[\sum_s \nabla^i_{(s)}\;\bar\phi^a_{1,2,3}\nabla_{(s)\;i}\;\phi^a_{1,2,3} + m^2\bar\phi^a_{1,2,3}\;\phi^a_{1,2,3}\right] \\ &\quad +\lambda\;\phi^1_{1,2,3}\phi^2_{3,4,5}\;\phi^3_{5,2,6}\;\phi^4_{6,4,1} + \bar\lambda\;\bar\phi^1_{1,2,3}\bar\phi^2_{3,4,5}\;\bar\phi^3_{5,2,6}\;\bar\phi^4_{6,4,1}\;. \end{aligned} \tag{6}$$

Important quantum topological aspects lie in the "coloring" of GFT [19][40]. For the present work, we will indeed see that even at the level of classical analysis, implementing this extra color index to field might lead to an improvement of the features of the Noether currents for a given symmetry.

Solving the tensor Klein-Gordon equation - In [26], treating the Boulatov model, a class of solutions for the equation of motion has been found. In the present situation, another issue due to the dynamics arises. However the equivalent of Klein-Gordon equation can be again worked out. We have to solve

$$-\sum_s \Delta_{(s)}\phi_{1,2,3} + m^2\phi_{1,2,3} = 0\;, \tag{7}$$

for gauge invariant fields. Using Peter-Weyl decomposition (see Appendix A for a summary of following notations), the above equation is equivalent to

$$\sum_{j_a,m_a,n_a}\phi^{j_a}_{m_a,n_a}\left(-\sum_{a=1}^{3}C(j_a) + m^2\right)\prod_{a=1}^{3}\sqrt{d_{j_a}}D^{j_a}_{m_a,n_a}(g_a) = 0\;, \tag{8}$$

with $C(j_a) = j_a(j_a+1)$ denoting the Casimir or eigenvalue of $\Delta_{(a)}$. A solution of the Klein-Gordon equation for $D = 3$ GFT is therefore

$$\phi_{1,2,3} = \sum_{j_a,m_a,n_a}\phi^{j_a}_{m_a,n_a}\delta_{\sum_{a=1}^{3}C(j_a)-m^2,0}\prod_{a=1}^{3}\sqrt{d_{j_a}}D^{j_a}_{m_a,n_a}(g_a)\;, \tag{9}$$

where $\phi^{j_a}_{m_a,n_a}$ is assumed to satisfy also (A.3). For large spin j_a, the solutions (9) are such that only modes ϕ^{j_a} with $j_1^2 + j_2^2 + j_3^2 = m^2$ remain in the field expansion.

3 Translations and dilatations: 1D GFT

In this section, as a preliminary and essential study to the full picture for any GFT dimension, we first start by Noether theorem for translations and dilatations for GFT in one dimension.

The latter theory is local and the analysis in this local framework will be compared to the analogous for a GFT in any dimension $D \geq 3$ which is nonlocal.

In $1D$ GFT, the gauge invariant condition (1) for fields should be abandoned as it is equivalent to the requirement of constant fields. The bottom line is the data of an action over one copy of G of the form

$$S_{1D}[\phi] = \int dg\, \mathcal{L}_{1D}(\phi, \nabla\phi)\;, \qquad \mathcal{L}_{1D} = \frac{1}{2}\mathbf{g}^{ij}\nabla_i\phi(g)\nabla_j\phi(g) + \frac{1}{2}m^2\phi^2(g) + \frac{\lambda}{4}\phi^4(g)\;. \tag{10}$$

3.1 Translations and EMT

A right translation[3] symmetry by an element h is simply the right group multiplication $g \mapsto gh$. Under this symmetry, a field transforms as

$$\phi(g) \mapsto \phi(gh)\;. \tag{11}$$

At the infinitesimal level, given a local coordinate system, the variation of any field is given by

$$\delta_X\,\phi = X \cdot \partial\,\phi = \sum_{i=1}^{3} X^i \partial_i\,\phi\;. \tag{12}$$

The operator[4]

$$W(X)(\cdot) = \int d\theta d\varphi^1 d\varphi^2\, \left(\delta_X \mathbf{g}^{ij}\,\frac{\delta}{\delta \mathbf{g}^{ij}}(\cdot) + \delta_X\phi\,\frac{\delta}{\delta\phi}(\cdot)\right) \tag{13}$$

acting on the action S_{1D} (10) allows one to define the Noether theorem for a given symmetry with parameter X for which an infinitesimal field variation $\delta_X\phi$ is given. Operators of the kind (13) prove to be useful tool either in the situation of a gauge symmetry (and are indeed related to Ward identity operators when acting on a partition function), or when one deals with nonlocal interaction as appear in noncommutative geometry or matrix models [35]-[39]. In the following and according to the context, this operator will take different forms and will enable us to treat the nonlocal interaction properly.

Considering (12), one obtains after some algebra (Appendix B.1 provides details of the derivations)

$$\frac{\partial}{\partial X^i} W(X) S_{1D} = -\sum_k \int d\theta d\varphi^1 d\varphi^2\;\; \partial_k\big(\sqrt{|\det \mathbf{g}|}\;\mathbf{g}^{kj} T_{ij}\big)\;, \tag{14}$$

where T_{ij} is the EMT given in a covariant form as

$$T_{ij} = \nabla_i\phi\;\nabla_j\phi - \mathbf{g}_{ij}\mathcal{L}_{1D}\;. \tag{15}$$

[3] Left translations can be carried out in a similar manner.

[4] In the following, the normalization $1/(2\pi^2)$ of the Haar measure will be dropped.

The properties of the EMT are quite straightforward: T_{ij} is symmetric and covariantly conserved. Using the equation of motion, it can be proved (see Appendix B.1) that

$$\nabla^i T_{ij} = 0 \; . \tag{16}$$

Nevertheless, the sense of conserved charges remains unclear at this level. Indeed, there is *a priori* no preferred coordinate embodying the ordinary role of time and no obvious partial integration on the remaining variables for which a correct conserved quantity could be generated from (16).

For a massless theory, the trace of the EMT (15) is not vanishing. Note that also the usual EMT in a massless ϕ_4^4 theory is not traceless. A traceless EMT can be only built by adding a correction to the original EMT. Here, the naive improvement procedure by adding an extra term to the EMT such that

$$\hat{T}_{ij} = T_{ij;\, m=0} + \frac{1}{\beta}(\mathbf{g}_{ij}\nabla^k\nabla_k - \nabla_i\nabla_j)\phi^2 \tag{17}$$

yields still a symmetric tensor but it is neither traceless nor covariantly conserved (the obstruction of that local conservation can be expressed in terms of the Ricci tensor associated with the connection). Insisting on the traceless improvement procedure for the EMT (15), one can perform the following modification:

$$\hat{T}'_{ij} = T_{ij;\, m=0} + \frac{1}{\beta}\mathbf{g}_{ij}\phi\nabla^k\nabla_k\phi + \frac{1}{\beta'}\nabla_i\phi\nabla_j\phi \; , \tag{18}$$

such that Tr $\hat{T}' = 0$ is recovered for for $\beta' = 2$ and $\beta = 4$. Note that, $\hat{T}'$, even though symmetric, is not covariantly conserved.

3.2 Dilatations and current vector

Group dilatations - Unlike in the flat and noncompact space case, the notion of dilatation symmetry on a compact manifold like the sphere is not an obvious concept. We use here an idea familiar to wavelet analysis on the two-sphere [41] for discussing the concept of dilatations on the sphere S^3. We will show that these dilatations can be implemented for particular GFT models.

Let a be a real strictly positive number. Given a group element $g = g(\theta, \vec{n}) \in G \simeq S^3$, characterized by the class angle θ and the unit vector $\vec{n} \in S^2$, one defines the map $d_a : G \to G$ such that $g \mapsto g_a$ with

$$g_a = g_a(\theta_a, \vec{n}) \; , \qquad \tan\frac{\theta_a}{2} = a \tan\frac{\theta}{2} \; . \tag{19}$$

More intuitively, the group element g_a can be viewed as follows: given $g \in S^3$, project g on the tangent $3D$ hyperplane at the North pole by a stereographic projection from the South pole; apply an usual Euclidean dilatation by a to the projected element in the flat space and then project back the result onto the sphere S^3 by the inverse stereographic projection. Remark that the stereographic projection is not well defined at the South pole and this will also have consequences in the formulation with some undefined ratios.

Under this mapping, the θ dependence of the Haar measure undergoes (Appendix C.1 provides justifications of the following results)

$$d\theta(\sin\theta)^2 \mapsto d\theta_a(\sin\theta_a)^2 = (\mu(a,\theta))^3 d\theta(\sin\theta)^2\ , \qquad \mu(a,\theta) = \frac{2a}{(1-a^2)\cos\theta + 1 + a^2}\ . \tag{20}$$

In fact, restricted to the two-sphere, dilatations of this kind together with translations belong to a subgroup of the Lorentz group $SO_0(3,1)$, the component of $SO(3,1)$ connected to the identity, which acts conformally on S^2 [41]. For our situation of the three-sphere, we foresee that dilatations and translations will reasonably belong to a subgroup of conformal group acting of S^3 [42].

Discussing infinitesimal variations, the angle θ transforms as

$$\delta_\epsilon\theta = 2\arctan[(1+\epsilon)\tan\frac{\theta}{2}] - \theta = 2\epsilon\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \epsilon\sin\theta\ . \tag{21}$$

Dilatations and current vector - Scale invariance for fields for $1D$ GFT corresponds to the requirement

$$\phi(g) \mapsto \widetilde{\phi}(g_a) = \mu(a,\theta)^{-1}\phi(g)\ . \tag{22}$$

Infinitesimally, the above transformation finds the variation (see (C.42) in Appendix C.2)

$$\delta_\epsilon\phi(g) = -\epsilon[\cos\theta + \sin\theta\partial_\theta]\phi(g)\ . \tag{23}$$

One notices that the group field dilatations (22) might be different from the so-called canonical Weyl transformations considered in [33].

Considering the infinitesimal generator associated with this transformation (acting on fields)

$$\mathcal{D} = \cos\theta + \sin\theta\partial_\theta \tag{24}$$

together with translation generators ∂_j, we have

$$[\partial_j, \mathcal{D}] = \delta_{j\theta}\mathcal{D}'\ , \qquad [\mathcal{D}', \partial_j] = \delta_{j\theta}\mathcal{D}\ , \qquad [\mathcal{D}, \mathcal{D}'] = -\partial_\theta\ , \qquad \mathcal{D}' := (-\sin\theta + \cos\theta\partial_\theta)\ . \tag{25}$$

The generator $\mathcal{D}'$ can be seen as a rotation of $\mathcal{D}$ by an angle of $\pi/2$ and so defines a generator of a distinct dilatation seen from another pole (West, up to a sign). Hence on the algebra of fields, the translation generator ∂_θ, $\mathcal{D}$ and $\mathcal{D}'$ associated each with a different dilatation, form a closed $\mathfrak{so}(2,1)$ Lie algebra of vector fields. This can be seen by first multiplying each generator by the complex i and then rename $K_0 = i\partial_\theta$, $K_1 = i\mathcal{D}$ and $K_2 = i\mathcal{D}'$. Note also that other translations generators $\partial_{j'}$, $j' \neq \theta$, just span a central extension to be added to this Lie algebra.

Since the Haar measure transforms according to (20), a scale invariant action is of the form

$$S^{\text{scale}}_{1D}[\phi] \ = \ \int dg\Big[(\sin\theta)^{-1}\frac{\mathbf{g}^{kl}}{2}(\partial_k\ (\sin\theta\phi))(\partial_l\ (\sin\theta\phi)) + \frac{\lambda}{4}\sin\theta\phi^4\Big]$$

$$= \int dg\Big[\ (\sin\theta)^{-1}\frac{\mathbf{g}^{kl}}{2}\left\{\delta_{k,\theta}\delta_{l,\theta}[\cos\theta\phi]^2 + 2\delta_{l,\theta}\cos\theta\sin\theta\phi\partial_k\phi + (\sin\theta)^2\partial_k\phi\partial_l\phi\right\} + \frac{\lambda}{4}\sin\theta\phi^4\Big]. \tag{26}$$

It is worth emphasizing that, due to the explicit appearance of the coordinate θ in the Lagrangian, we expect a breaking of the ordinary notion of local conservation of current in this theory. Note also that a mass term could be also included but, for simplicity, we will not consider it.

The field equation of motion reads

$$\frac{\delta S^{\rm scale}_{1D}}{\delta\phi} = 0 = (\bullet)\frac{(\cos\theta)^2}{\sin\theta}\phi + (\bullet)\cos\theta\partial_\theta\phi - \partial_\theta[(\bullet)\cos\theta\phi] - \widetilde{\Delta}\phi + (\bullet)\lambda\sin\theta\phi^3\ , \tag{27}$$
$$(\bullet) := \sqrt{|\det\mathbf{g}|}\ ,\quad \widetilde{\Delta}\phi := \partial_k\left\{(\bullet)\sin\theta\mathbf{g}^{kl}\partial_l\phi\right\},$$

where $\widetilde{\Delta}$ is a modified Laplacian. The metric variation will be not considered this time and rather consider the functional operator for solely field dilatations given by

$$W(\epsilon)(\cdot) = \int d\theta d\varphi^1 d\varphi^2\ \ \delta_\epsilon\phi\,\frac{\delta}{\delta\phi}(\cdot)\ . \tag{28}$$

We have to evaluate the variation of the action under (28)

$$\frac{\partial}{\partial\epsilon}W(\epsilon)S^{\rm scale}_{1D} = \frac{\partial}{\partial\epsilon_i}\int d\theta d\varphi^1 d\varphi^2\Big\{\ (-\epsilon\mathcal{D}\phi)\times \Big[(\bullet)\frac{(\cos\theta)^2}{\sin\theta}\phi + (\bullet)\cos\theta\partial_\theta\phi - \partial_\theta[(\bullet)\cos\theta\phi] - \widetilde{\Delta}\phi + (\bullet)\lambda\sin\theta\phi^3\Big]\Big\} \tag{29}$$

and will prove that this can be computed as a surface term. A direct, though lengthy, calculation (see Appendix C.3) yields the current

$$D_j = \sin\theta\ [\cos\theta + \sin\theta\,\nabla_\theta]\,\phi\ \nabla_j\phi + \mathbf{g}_{j\theta}\cos\theta\phi\,[\cos\theta + \sin\theta\nabla_\theta]\,\phi - \mathbf{g}_{j\theta}\sin\theta\mathcal{L}^{\rm scale}_{1D}\ , \tag{30}$$

that we put in another form

$$D_j = \nabla_\theta(\sin\theta\phi)\nabla_j(\sin\theta\phi) - \mathbf{g}_{j\theta}\sin\theta\mathcal{L}^{\rm scale}_{1D}\ . \tag{31}$$

Concerning the local conservation property, as expected, we find that the current is not covariantly conserved (a proof of this can be found in Appendix C.3). The breaking term for the covariant conservation to hold can be written as

$$\begin{aligned}\nabla^j D_j &= \cos\theta\sin\theta\,\Big[-(\cot\theta)^2\phi^2 + \nabla_\theta\phi\ \nabla_\theta\phi + \frac{\lambda}{2}\phi^4\Big]\\ &= 2\cos\theta\,\Big[-\frac{1}{2}\frac{(\cos\theta)^2}{\sin\theta}\phi^2 + \frac{1}{2}\sin\theta\nabla_\theta\phi\ \nabla_\theta\phi + \frac{\lambda}{4}\sin\theta\phi^4\Big].\end{aligned} \tag{32}$$

The breaking expression comes mainly from partial derivative in θ of factors containing an explicit θ dependence in the initial Lagrangian $\mathcal{L}^{\rm scale}_{1D}$. A non-trivial task, going beyond the scope of this paper, is to understand the breaking (32) in terms of coordinate dependent regular Lagrangian systems [43].

As a remark, from the expression (31), one could be tempted to argue that the current D_j should be related to an EMT $\widetilde{T}_{ij}$ by just a field redefinition $\phi \to \widetilde{\phi} = \sin\theta\phi$. Then one ought to check that $\sin\theta\mathcal{L}^{\rm scale}_{1D}$ is the correct Lagrangian of the form $\widetilde{\mathcal{L}_{1D}} = (1/2)\mathbf{g}^{ij}\nabla_i\widetilde{\phi}\nabla_j\widetilde{\phi} + (\lambda/4)\widetilde{\phi}^4$ so that the EMT in this theory can be related to the current by $\widetilde{T}_{\theta j} = D_j$ and thereby $\nabla^j\widetilde{T}_{\theta j} = \nabla^j D_j = 0$ should reasonably hold. We then compute

$$\sin\theta\mathcal{L}^{\rm scale}_{1D} = \frac{\mathbf{g}^{kl}}{2}\partial_k\widetilde{\phi}\partial_l\widetilde{\phi} + \frac{\lambda}{4}(\sin\theta)^{-2}\widetilde{\phi}^4 \tag{33}$$

and clearly find that $\sin\theta\mathcal{L}^{\rm scale}_{1D} \neq \widetilde{\mathcal{L}_{1D}}$. The new Lagrangian $\sin\theta\mathcal{L}^{\rm scale}_{1D}$ as a function of $\widetilde{\phi}$ and θ is not translation invariant even up to a surface term. This means that dilatation invariance cannot be reduced, at least in that naive way, to translation invariance.

Finally, in the ordinary ϕ^4 theory, the dilatation current d_j can be related to the improved local and traceless energy momentum tensor by $d_\mu = x^\nu\hat{T}_{\nu\mu}$, where $\hat{T}_{\nu\mu} = T_{\nu\mu} + (1/6)(\eta_{\mu\nu}\partial^\kappa\partial_k - \partial_\nu\partial_\mu)\phi^2$, $T_{\nu\mu}$ being the ordinary EMT and η the Minkowski metric. Here trying to obtain an analogous relation, one may write at best

$$D_j = \sin\theta\, T^{\rm scale}_{\theta j} + \sin\theta\cos\theta\phi\ \nabla_j\phi + \mathbf{g}_{j\theta}\cos\theta\phi\,(\cos\theta + \sin\theta\nabla_\theta)\,\phi\,, \tag{34}$$

and, remarkably, $\sin\theta$ plays the old role of the coordinate position in flat spacetime.

4 Translations and dilatations: 3D case

4.1 Translations and EMT

Right translations conserve the sense of Section 3.1. In a higher rank tensor GFT, each group argument can be translated by a fixed quantity, but fields will transform according to a particular rule. It has been highlighted recently that diffeomorphisms can be implemented for GFT models through quantum deformed symmetries. The deformed Poincare group acts on the base manifold defined by the Lie algebra dual to the group [28]. Translations in Lie algebra or "metric" variables have been scrutinized by Baratin *et al.* still in the form of a quantum symmetry realization [29]. For colored GFT models in $3D$ and $4D$, quantum translation invariance corresponds to the invariance under translations of the vertices of the Euclidean tetrahedron representing the GFT interaction. The same symmetry reflects, in the group formulation, the fact that the boundary connection associated with field variables is flat. Even though the fields used in this section are non colored and so the geometric interpretation of translations will certainly differ, the way of implementing group translations here (by acting one some particular field arguments) is somehow similar to the anterior formalisms.

First, we need to introduce complex valued fields and consider the new Lagrangian density $\mathcal{L}_{3D} = \mathcal{L}_{{\rm kin},\,3D} + \mathcal{L}_{{\rm int},\,3D}$ with

$$\mathcal{L}_{{\rm kin},\,3D} = \sum_{s=1}^{3}\mathbf{g}_s^{ij}\nabla_{(s)\ i}\ \bar{\phi}_{1,2,3}\nabla_{(s)\ j}\ \phi_{1,2,3} + m^2\bar{\phi}_{1,2,3}\ \phi_{1,2,3}\ , \tag{35}$$

$$\mathcal{L}_{{\rm int},\,3D} = \frac{\lambda}{2}\phi_{1,2,3}\bar{\phi}_{5,4,3}\phi_{5,2,6}\bar{\phi}_{1,4,6}\ . \tag{36}$$

The action defined by

$$S_{\text{kin},\,3D}[\phi] := \int [\prod_{\ell=1}^{3} dg_\ell] \mathcal{L}_{\text{kin},\,3D} \;, \qquad S_{\text{int},\,3D}[\phi] = \int [\prod_{\ell=1}^{6} dg_\ell]\; \mathcal{L}_{\text{int},\,3D} \;, \tag{37}$$

is real and may be written $S_{3D} = S_{\text{kin},\,3D}[\phi] + S_{\text{int},\,3D}[\phi] := \int [\prod_{\ell=1}^{6} dg_\ell]\; \mathcal{L}_{3D}$. The equation of motion for fields $\phi_{1,2,3}$ can be inferred from (5) after renaming properly some variables in the interaction.

Under right translations by h_ℓ, $\ell = 1,2,3$, the fields transform according to

$$\begin{array}{l} \phi(g_1,g_2,g_3) \mapsto \phi(g_1h_1,g_2h_2,g_3h_3) \;, \\ \bar\phi(g_5,g_4,g_3) \mapsto \bar\phi(g_5h_1,g_4h_2,g_3h_3) \;, \\ \phi(g_5,g_2,g_6) \mapsto \phi(g_5h_1,g_2h_2,g_6h_3) \;, \\ \bar\phi(g_1,g_4,g_6) \mapsto \bar\phi(g_1h_1,g_4h_2,g_6h_3) \;. \end{array} \tag{38}$$

Note that the group arguments with labels $(1,5)$, $(2,4)$ and $(3,6)$ are shifted by the same amount. This defines a correct field symmetry. Actually, there is a simpler field transformation which can be extracted from the above mappings by just shifting either the first, the second or the last field argument. Thus an ordinary D dimensional GFT will have D such basic translations. The results and conclusions obtained with the "3-translation" (38) are in some sense more general and will again hold for any of these simpler symmetries.

Infinitesimal variations of a tensor field can be inferred from the variations of a field defined over a single copy of the group. Three translations defined by the group elements h_ℓ, $\ell = 1,2,3$, yield the infinitesimal variations

$$\delta_{X_{(1)},X_{(2)},X_{(3)}}\, \phi_{1,2,3} = \sum_s \; X_{(s)} \cdot \partial_{(s)}\; \phi_{1,2,3} = \sum_{s,i} X^i_{(s)} \partial_{(s)\;i}\, \phi_{1,2,3} \;. \tag{39}$$

Henceforth, $\delta_{X_{(1)},X_{(2)},X_{(3)}}$ will be simply denoted δ_X. The following operator

$$W(X)(\cdot) = \int [\prod_{\ell=1}^{6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \;\left[\sum_{s=1}^{6} \delta_X \mathbf{g}^{ij}_s \,\frac{\delta}{\delta \mathbf{g}^{ij}_s}(\cdot) + \delta_X \bar\phi_{1,2,3}\, \frac{\delta}{\delta\bar\phi_{1,2,3}}(\cdot) + \delta_X \phi_{1,2,3}\, \frac{\delta}{\delta\phi_{1,2,3}}(\cdot) \right] \tag{40}$$

generalizing (13) will act again on the action S_{3D} in order to find the Noether current for translation symmetry with parameters $X_{(s)}$.

Considering (39), some calculations yield (see Appendix B.2)

$$\frac{\partial}{\partial X^i_{(s)}} W(X) S_{3D} = -\sum_{s',k} \int [\prod_{\ell=1}^{6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \;\; \partial_{(s')\;k} ([\prod_{\ell=1}^{6} \sqrt{|\det \mathbf{g}_\ell|}]\; \mathbf{g}^{kj}_{s'} T_{(s,s');\,(i,j)}) \;, \tag{41}$$

where $T_{(s,s');\,(i,j)}$ is the "stranded" EMT given by

$$T_{(s,s');(i,j)} = \partial_{(s)\;i}\phi_{1,2,3}\, \partial_{(s')\;j}\bar\phi_{1,2,3} + \partial_{(s)\;i}\bar\phi_{1,2,3}\, \partial_{(s')\;j}\phi_{1,2,3} - \delta_{s,s'} \mathbf{g}_{s'\;ij}\mathcal{L}_{3D} - \delta_{s+[\alpha_s],s'}\mathbf{g}_{s'\;ij}\mathcal{L}_{\text{int},\,3D} \;, \tag{42}$$

with indices such that $s = 1,2,3$, $s' = 1,2,\dots,6$ and $([\alpha_1],[\alpha_2],[\alpha_3]) = (4,2,3)$. In any dimension D, the EMT will hold this form, the strand indices will become $s = 1,2,\dots,D$

and $s' = 1, 2, \ldots, D(D+1)/2$ whereas the indices $[\alpha_s]$ have to be combinatorially well chosen. For a basic field transformation acting, for example, only on the strands $(1,5)$, the corresponding EMT is nothing but the component $T_{(1,s');(i,j)}$.

First, the EMT (42) is symmetric under the permutation $(s,i) \leftrightarrow (s',j)$, if $s' = 1,2,3$. For $s' = 4,5,6$, the EMT breaks down to the interaction Lagrangian and therefore this component remains also symmetric. However, the EMT (42) turns out to be not covariantly conserved (see Appendix B.2). This is mainly due to the fact that the nonlocal interaction clashes with the specific way that the field symmetry is imposed in (38) (we will come back to this fact in depth in the next paragraph). Such an oddity prevents to form proper equations of motion for fields thanks to which, usually, the local conservation could be guaranteed. It could be asked if removing the dynamical part in the definition of the Lagrange density will not help to recover the conservation of the EMT. It can be shown, in that particular instance, that the EMT reduces to the Lagrangian itself, and in the non colored case, this quantity is still not locally conserved. The local conservation breaking of the EMT and, in fact, of all other currents as we will find in the subsequent analysis, has a deeper reason to hold.

Let us digress a little from our present purpose and understand what is the main reason why nonlocal field theories under a group of transformations will generally fail to obey the Noether theorem. The local conservation of currents by Noether theorem assumes different features of the initial theory. For the sake of clarity, let us consider a theory with local fields described by an action $S[\phi, \partial\phi]$ which computes up to surface term $\partial_\mu T^\mu$ under a group of field symmetry Q

$$Q \triangleright S[\phi, \partial\phi] = \int_\Omega \partial_\mu T^\mu = 0 \; . \tag{43}$$

The last equation is valid on-shell and the integration domain Ω should be any subspace of the domain M of fields $\phi : \Omega \subset M \to \mathbb{C}$. From the arbitrariness of Ω, one concludes to the local continuity equation:

$$\partial_\mu T^\mu = 0 \; . \tag{44}$$

However, there may exist nonlocal theories having an action $S^{\text{nonlocal}}[\phi, \partial\phi]$ invariant under a group of transformations, the surface term can be still computed but now the equation (43) may be only valid on the entire domain of the fields, namely

$$Q \triangleright S[\phi, \partial\phi] = \int_M \partial_\mu T^\mu = 0 \; , \tag{45}$$

then nothing else can be said about the local conservation properties of the current T^μ. In general, the corresponding current is not locally conserved. Nevertheless, depending on the structure of the field theory, the current could appear by itself a locally conserved quantity or one may use different tools in order to regularize or to improve the properties of that current. This last case includes, again, the so-called noncommutative field theories defined over the algebra of fields equipped with a Moyal star products [34]: the noncommutative tensor $i\theta^{\mu\nu} = [x^\mu, x^\nu]_\star$ can be chosen in a specific way to extract from the initial non locally conserved EMT of the scalar $\phi^4_\star$ theory, a conserved four-momentum. Obviously, the success of getting the relevant local properties of currents in these kinds of theories is strongly dependent on the type of framework one is dealing with.

In the present setting of nonlocal GFTs, we are in a similar situation. Indeed, let us scrutinize the translations given in (38) by reducing them to the minimal translation, for instance $(1,5)$. This transformation acts on the same field on two distinct points of the manifold: it can be called a "nonlocal translation". The associated surface term is computed using the structure of, at least, two Haar integrals over, at least, two group copies in order to vary properly the nonlocal interaction (involving two different group variables g_1 and g_5; in fact, the computations involve all the six copies of the group). As a simple consequence, an ordinary local conservation which occurs on a local group copy of the EMT will be explicitly broken. An improvement procedure, if exists, would be necessary in order to recover that local property. Note that, as shown in Appendix (B.2), integrating the local conservation breaking term on the full manifold vanishes again. By this, one is ensured of the validity of a vanishing divergence on the entire manifold as the calculation of the surface term claims this fact. Nevertheless, we will see that for colored GFTs, the EMT appear locally conserved because, in comparison to the above nonlocal translation, in colored GFT one can implement "local translation" (the meaning of these will become clear in the following). This confirms the point of view that the local conservation of the currents in nonlocal field theories becomes actually model dependent.

Let us investigate alternatively the meaning of group translations for fields invariant under permutation of their arguments and understand if this extra feature could not improve the currents. Hence, for any permutation $\sigma \in \Sigma_3$, we assume $\phi_{1,2,3} = \phi_{\sigma(1),\sigma(2),\sigma(3)}$. We can think about the consequences of having such an invariance on translations in a combinatoric way. We focus on a translation with respect to one field argument, say $g_1 \to g_1 h$. g_1 appears twice in the interaction, in $\phi_{1,2,3}$ and in $\bar\phi_{1,4,6}$, and will be translated in both of these fields. The same translation remains to be defined on two fields: $\bar\phi_{5,4,3}$ and $\phi_{5,2,6}$. The transformation could only affect $g_{5,4}$ in the former and $g_{5,6}$ in the latter field (otherwise, g_2 or g_3 will be assumed to be transformed and then the situation become more nonlocal with the, by now, expected features). Then, two transformations could be consistently implemented: one of which is the minimal translation involving $(1,5)$ as performed earlier and a second one defined by

$$\begin{aligned} &\phi(g_1,g_2,g_3) \to \phi(g_1 h,g_2,g_3)\;, \qquad &&\bar\phi(g_1,g_4,g_6) \to \bar\phi(g_1 h,g_4,g_6)\;, \\ &\bar\phi(g_5,g_4,g_3) \to \bar\phi(g_5,g_4 h,g_3)\;, \qquad &&\phi(g_5,g_2,g_6) \to \phi(g_5,g_2,g_6 h)\;. \end{aligned} \tag{46}$$

As peculiar as it appears, considering (46) as a valid field translation, one agrees that it is even more nonlocal than the minimal $(1,5)$ translation. The transformation acts on the same field on three distinct points. It becomes obvious that all previous broken currents will remain broken under this transformation. Otherwise, if one disputes the fact that g_4 and g_6 have been translated and then should be translated in the remaining terms, a rapid checking shows that the said translation is equivalent to a translation of all field arguments by a unique element. This transformation is nothing but the identity from the gauge invariance of the fields. Thus, considering fields invariant under permutation of their arguments leads to the same conclusions as given in the above analysis or to triviality. Nevertheless, we can comment that, through any new meaning that one could give to "nonlocal translations," that we did not investigated in depth in this paper focusing on the most nontrivial situations, it is not excluded that properties of currents can be improved using new types of transformations.

Next, for a massless theory, let us evaluate the trace of the EMT (42) in the following sense:

$$\mathrm{Tr}\; T_{m=0} = \sum_{s=1}^{3} \Big[\mathbf{g}_s^{ij} T_{(s,s);(i,j);\, m=0} + \mathbf{g}_{s+[\alpha_s]}^{ij} T_{(s,s+[\alpha_s]);(i,j);\, m=0} \Big] \,, \tag{47}$$

where $T_{(s,s');(i,j);\, m=0}$ is defined from $\mathcal{L}_{3D,m=0}$. A trace of this form is justified by the fact that a contribution for each strand represented in the Lagrangian is needed. The calculations of this trace yield (in covariant notations)

$$\mathrm{Tr}\; T_{m=0} = \sum_{s=1}^{3} [\nabla^i_{(s)} \bar{\phi}_{1,2,3}\; \nabla_{(s)\; i} \phi_{1,2,3}] - 9(\mathcal{L}_{3D,m=0} + \mathcal{L}_{\mathrm{int},\,3D}) \tag{48}$$

which is not a vanishing quantity. A traceless EMT can be built by considering instead

$$\hat{T}_{(s,s');(i,j)} = T_{(s,s');(i,j);\, m=0} + \frac{1}{\beta} \delta_{s,s'} \mathbf{g}_{s'\; ij} \sum_{s''=1}^{3} \phi_{1,2,3} \Delta_{(s'')} \bar{\phi}_{1,2,3} + \frac{1}{\beta'} \nabla_{(s)\; i} \bar{\phi}_{1,2,3}\; \nabla_{(s')\; j} \phi_{1,2,3} \,. \tag{49}$$

The trace of this tensor is

$$\mathrm{Tr}\; \hat{T} = \frac{-8\beta' + 1}{\beta'} \sum_{s=1}^{3} \nabla^i_{(s)} \bar{\phi}_{1,2,3} \nabla_{(s)\; i} \phi_{1,2,3} - 9\lambda \phi_{123} \bar{\phi}_{543} \phi_{526} \bar{\phi}_{146} + \frac{9}{\beta} \sum_{s=1}^{3} \phi_{1,2,3} \Delta_{(s)} \bar{\phi}_{1,2,3}. \tag{50}$$

The improved EMT $\hat{T}$ is traceless if $\beta' = 1/8$ and $\beta = 1$ after integration of the variables coined by $4, 5, 6$. We should emphasize that $\hat{T}$ is not covariantly conserved.

In general, a traceless property of a locally conserved EMT hints at a scale invariant theory. Scale or dilatation symmetry is an important feature of any field theory because it preludes, in certain cases, to a larger conformal symmetry. The latter has a cortege of physical implications among which rests, for example, the existence of universality classes having for fixed point a particular conformal theory [44]. Hence, before demanding conformal invariance for GFT, one could investigate if simpler symmetries can be implemented at this level. A way to test if a theory is scale invariant, is to prove that its conserved EMT is traceless. Showing that the EMT is not locally conserved and not traceless[5], we anticipate the fact that GFT as given by the action (37) will be not scale invariant and hence the ordinary dilatation (and so the larger conformal) symmetry of ϕ^4_4 theory will be explicitly broken in GFTs. This is again due to the nonlocality of these theories.

4.2 Dilatations and current tensor

Boulatov-Ooguri model - We first study GFTs without dynamics in order to make the following developments more comprehensible. Consider a $3D$ GFT model equipped with a quadratic part $\mathcal{L}^0_{\mathrm{kin}}[\phi] := \bar{\phi}_{1,2,3} \phi_{1,2,3}$ and an interaction part as given by (36). The total Lagrange density is $\mathcal{L}^0$ and we put the mass to $m = 1$.

[5]Of course, one of these features could have been a sufficient condition for claiming that the dilatation symmetry of the ordinary ϕ^4_4 is explicitly broken. However, we have shown that the lack of traceless property of the EMT can be improved hence only the first condition should be considered as the actual reason for the breaking of scale symmetry.

Demanding scale invariance of the action implies that the fields transform as

$$\phi(g_1,g_2,g_3)\mapsto\widetilde{\phi}(g_{a\,1},g_{a\,2},g_{a\,3})=[\prod_{s=1}^{3}\mu(a_s,\theta_s)^{c_s}]\phi(g_1,g_2,g_3)\;. \tag{51}$$

Given (20), the scale invariance of the interaction (quartic in fields) and trivial kinetic term is achieved by $3+2c_s=0$. Thus the scaling dimension for fields is characterized by $c=-3/2$. Furthermore, as it was the case for translation symmetry, we use complex fields and require that group elements defined by the couples $(1,5)$, $(2,4)$ and $(3,6)$ are all submitted to the same dilatations.

Setting $a_s=1+\epsilon_s$, where ϵ_s is an infinitesimal parameter, angle and field infinitesimal variations (Appendix C.2 gives useful details pertaining to the following identities) are of the form:

$$\begin{aligned}&\delta_{\epsilon_s}\theta_s=\epsilon_s\sin\theta_s\;,\\&\delta_\epsilon\phi_{1,2,3}:=\delta_{\epsilon_1,\epsilon_2,\epsilon_3}\phi=\sum_{s=1}^{3}\delta_{\epsilon_s}\phi_{1,2,3}=-\sum_{s=1}^{3}\epsilon_s\big(-c\cos\theta_s+\sin\theta_s\partial_{(s)\;\theta}\big)\phi_{1,2,3}\;.\end{aligned} \tag{52}$$

We introduce the functional differential operator

$$W(\epsilon)(\cdot)=\int[\prod_{\ell=1}^{3}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\quad\Big[\delta_\epsilon\phi_{1,2,3}\,\frac{\delta}{\delta\phi_{1,2,3}}(\cdot)+\delta_\epsilon\bar{\phi}_{1,2,3}\,\frac{\delta}{\delta\bar{\phi}_{1,2,3}}(\cdot)\Big] \tag{53}$$

and evaluate for the action $S^0=\int[\prod_\ell dg_\ell]\mathcal{L}^0$

$$\begin{aligned}&\frac{\partial}{\partial\epsilon_i}W(\epsilon)S^0=\frac{\partial}{\partial\epsilon_i}\int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\prod_{s=1}^{6}\sqrt{|\det\mathbf{g}_s|}\Big\{\\&\Bigg[-\sum_{s=1}^{3}\epsilon_s\big[-c\cos\theta_s+\sin\theta_s\partial_{(s)\;\theta}\big]\phi_{1,2,3}\Bigg]\big[\bar{\phi}_{1,2,3}+\lambda\bar{\phi}_{5,4,3}\phi_{5,2,6}\bar{\phi}_{1,4,6}\big]+(\phi\leftrightarrow\bar{\phi})\Big\}\;.\end{aligned} \tag{54}$$

In the last above line, $(\phi\leftrightarrow\bar{\phi})$ is the symmetric of the previous expression under complex conjugation. The expansion of the last line yields

$$\begin{aligned}&\frac{\partial}{\partial\epsilon_i}W(\epsilon)S^0=-\int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\\&\Bigg\{\partial_{(i)\;\theta}\Bigg[[\prod_{s=1}^{6}\sqrt{|\det\mathbf{g}_s|}]\sin\theta_i\mathcal{L}^0\Bigg]+\partial_{(i+[\alpha_i])\;\theta}\Bigg[[\prod_{s=1}^{6}\sqrt{|\det\mathbf{g}_s|}]\sin\theta_{i+[\alpha_i]}\mathcal{L}_{\text{int},\,3D}\Bigg]\Bigg\}\;.\end{aligned} \tag{55}$$

The dilatation current vector becomes a "reduced" and "stranded" quantity with two components

$$D_s=\sin\theta_s\mathcal{L}^0\;,\qquad\tilde{D}_s=\sin\theta_{s+[\alpha_s]}\mathcal{L}_{\text{int},\,3D}\;. \tag{56}$$

This fact merely comes from the absence of a true dynamics in the model. In this "trivial" situation, the EMT reduces to the Lagrangian itself plus the interaction again, namely

$T^0 = (T_1^0, T_2^0) = (-\mathcal{L}^0, -\mathcal{L}_{\text{int},3D})$. From this point, one infers a slightly generalized formula for dilatation current in Boulatov-Ooguri tensor theory in terms of

$$D_s = -\sin\theta_s T_1^0, \qquad \tilde{D}_s = -\sin\theta_{s+[\alpha_s]} T_2^0 \tag{57}$$

where $\sin\theta_s$ should be seen as the coordinate position. Properties of the EMT T^0 and the current D_s are direct: they are not locally conserved.

Dynamical GFT - Incorporating nontrivial dynamical part, we have to define a correct scaling of derivative on fields. Consider a kinetic part of the form

$$\begin{aligned} S^{\text{scale}}_{\text{kin},3D}[\phi] &= \int [\prod_{\ell=1}^{3} dg_\ell]\ \mathcal{L}^{\text{scale}}_{\text{kin},3D} \\ \mathcal{L}^{\text{scale}}_{\text{kin},3D} &:= \sum_{s=1}^{3} (\sin\theta_s)^{\gamma_s} \mathbf{g}_s^{ij} (\nabla_{(s)\,i}\ (\sin\theta_s)^{\beta_s} \bar{\phi}_{1,2,3})(\nabla_{(s)\,j}\ (\sin\theta_s)^{\beta_s} \phi_{1,2,3})\ , \end{aligned} \tag{58}$$

where the degrees γ_s, β_s have to be chosen in order to satisfy the scale invariance. It can easily inferred that $\gamma_s = \gamma = -1$ and $\beta_s = \beta = 3/2$. The interaction part remains as $S_{\text{int},3D}$ and $\mathcal{L}_{\text{int},3D}$ so that $S^{\text{scale}}_{3D} = S^{\text{scale}}_{\text{kin},3D} + S_{\text{int},3D}$ and $\mathcal{L}^{\text{scale}}_{3D} = \mathcal{L}^{\text{scale}}_{\text{kin},3D} + \mathcal{L}_{\text{int},3D}$.

A direct evaluation (see Appendix C.4) shows that the dilatation current is a tensor defined by

$$\begin{aligned} &D_{(s,s');j} = \\ &\sin\theta_s \Big\{ \sin\theta_{s'}^{\frac{1}{2}} \Big[\partial_{(s)\,\theta}\phi_{1,2,3}\, \partial_{(s')\,j} (\sin\theta_{s'}^{\beta} \bar{\phi}_{1,2,3}) + \partial_{(s)\,\theta}\bar{\phi}_{1,2,3}\, \partial_{(s')\,j} (\sin\theta_{s'}^{\beta} \phi_{1,2,3}) \Big] \\ &-\delta_{s,s'} \mathbf{g}_{s'\,j\theta} \mathcal{L}^{\text{scale}}_{3D} \Big\} - \delta_{s+[\alpha_s],s'} \mathbf{g}_{s'\,j\theta}\ \sin\theta_{s+[\alpha_s]} \mathcal{L}_{\text{int},3D} + \beta\cos\theta_s \partial_{(s')\,j} \left((\sin\theta_{s'})^2 \bar{\phi}_{1,2,3}\phi_{1,2,3} \right) . \end{aligned} \tag{59}$$

This tensor is not covariantly conserved and its breaking involves both the nonlocal interaction and the fact that the Lagrangian contains explicit coordinate dependence.

Note that there exist other types of GFT interactions which are scale invariant under (19). For instance, the following interaction

$$\tilde{S}_{\text{int}}[\phi] := \frac{\lambda}{4} \int [\prod_{\ell=1}^{4} dg_\ell]\ \phi_{1,2,3}\, \phi_{3,2,4}\, \phi_{3,4,1}\, \phi_{4,2,1}\ , \tag{60}$$

assigns each group variables g_i (appearing three times in the interaction) to a vertex in the tetrahedron. This vertex is indeed shared by three triangles, each triangle being represented by a field. Hence, this model should be equivalent to a colored GFT model. A straightforward inspection using (60) proves that the scaling dimension of the fields is such that $c = -\beta_s = -1$ so that a kinetic term of the form (58) with $\gamma_s = -1$ would be scale invariant. Remark that the problem of non locally conserved quantities will be not necessarily solved by considering these interactions.

The pattern followed by the field arguments in the interaction (60) and the pattern of the vertex of Carrozza and Oriti [45] are exactly the same. In the latter, the field colors are explicitly given. In fact, it can be shown that the assignment of a group variable to each vertex of a tetrahedron and a particular pattern should be enough to ensure the equivalence between the said vertex and a colored vertex. The proof of this statement can be given as follows: let us consider the following interaction (we change once again the vertex in order to recover a typical colored theory)

$$\tilde{S}^{\rm col}_{\rm int}[\phi] := \frac{\lambda}{4}\int [\prod_{\ell=1}^{4} dg_\ell]\; \phi_{2,3,4}\,\phi_{1,4,3}\,\phi_{2,1,4}\,\phi_{3,2,1}\;. \tag{61}$$

To each field ϕ_{abc}, $a,b,c \in \{1,2,3,4\}$, we assign another redundant index $d \in \{1,2,3,4\} \setminus \{a,b,c\}$. After the procedure, the same vertex can be recast as

$$\tilde{S}^{\rm col}_{\rm int}[\phi] := \frac{\lambda}{4}\int [\prod_{\ell=1}^{4} dg_\ell]\; \phi^1_{2,3,4}\,\phi^2_{1,4,3}\,\phi^3_{2,1,4}\,\phi^4_{3,2,1}\;. \tag{62}$$

Assign, once again, to each field variable the redundant index of the same field. This procedure yields a colored vertex in the sense of Gurau [18]. At the quantum level, the gluing rules between these new colored fields can be imposed as one may require (for instance, only fields with the same redundant index have to be glued). Conversely, given a colored model, in order to obtain, the model described by (61), affect each color to a vertex of a tetrahedron, then remove the index from the vertex (the said index is indeed redundant). The model obtained by this process coincides with (61).

5 Translations and dilatations: Colored GFT

5.1 Translations and EMT

We consider now a colored GFT with Lagrangian of the form (6). Due the freedom of having colored fields, a right translation for only the fields ϕ^1 and ϕ^4 can be defined such that

$$\begin{array}{l}\phi^1(g_1,g_2,g_3) \mapsto \phi^1(g_1h,g_2,g_3)\;,\\ \phi^4(g_6,g_4,g_1) \mapsto \phi^4(g_6,g_4,g_1h)\;,\end{array} \tag{63}$$

whereas the fields of color $2,3$ remain non modified. At the infinitesimal level, (63) gives

$$\delta_X \phi^{a=1,4} = (X^i \cdot \partial_{(1)\,i})\,\phi^{a=1,4}\;, \tag{64}$$

where $\partial_{(1)}$ refers to only to derivative with respect to the strand 1 involving the group element g_1.

The coloring of GFT defines the "minimal" symmetry for the GFT action in the sense that we can transform only one group argument. This feature both simplifies somehow the derivations and, in fact, radically differs from the above translations for non colored GFTs as we will see. In order to recover the full symmetry of the action, one simply has to identify pairs of field arguments which can be transformed independently. Thus, besides of (63), other possible field transformations are

$$\begin{array}{lcl}\phi^1(g_1,g_2,g_3) \mapsto \phi^1(g_1,g_2h,g_3) & \text{and} & \phi^3(g_5,g_2,g_6) \mapsto \phi^3(g_5,g_2h,g_6)\;,\\ \phi^1(g_1,g_2,g_3) \mapsto \phi^1(g_1,g_2,g_3h) & \text{and} & \phi^2(g_3,g_4,g_5) \mapsto \phi^2(g_3h,g_4,g_5)\;,\end{array}$$

$$
\begin{array}{lll}
\phi^2(g_3,g_4,g_5) \mapsto \phi^2(g_3,g_4h,g_5) & \text{and} & \phi^4(g_6,g_4,g_1) \mapsto \phi^4(g_6,g_4h,g_1)\ , \\
\phi^2(g_3,g_4,g_5) \mapsto \phi^2(g_3,g_4,g_5h) & \text{and} & \phi^3(g_5,g_2,g_6) \mapsto \phi^3(g_5h,g_2,g_6)\ , \\
\phi^3(g_5,g_2,g_6) \mapsto \phi^3(g_5,g_2,g_6h) & \text{and} & \phi^4(g_6,g_4,g_1) \mapsto \phi^4(g_6h,g_4,g_1)\ .
\end{array} \qquad (65)
$$

For such a D dimensional GFT, there will be $D(D+1)/2$ of such basic transformations, one for each pair of group arguments in the interaction.

We write the equations of motion of the colors 1 and 4:

$$
\begin{aligned}
0 = \frac{\delta S^{\rm color}}{\delta \phi^1_{1,2,3}} &= -\sum_{s=1}^{3} \Delta_{(s)} \bar\phi_{1,2,3} + m^2 \bar\phi^1_{1,2,3} + \lambda \int [\prod_{\ell=4}^{6} dg_\ell]\ \phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1}\ , \\
0 = \frac{\delta S^{\rm color}}{\delta \phi^4_{6,4,1}} &= -\sum_{s=1,4,6} \Delta_{(s)} \bar\phi^4_{6,4,1} + m^2 \bar\phi^4_{6,4,1} + \lambda \int [\prod_{\ell\neq 1,4,6} dg_\ell]\ \phi^1_{1,2,3}\phi^2_{3,4,5}\phi^3_{5,2,6}\ .
\end{aligned} \qquad (66)
$$

For the present purpose, the following functional operator will be used to compute the EMT:

$$
\begin{aligned}
W(X)(\cdot) &= \int [\prod_{\ell=1}^{3} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \Big\{ \delta \mathbf{g}_1^{ij} \frac{\delta(\cdot)}{\delta \mathbf{g}_1^{ij}} + \delta\phi^1_{1,2,3} \frac{\delta(\cdot)}{\delta\phi^1_{1,2,3}} + \delta\bar\phi^1_{1,2,3} \frac{\delta(\cdot)}{\delta\bar\phi^1_{1,2,3}} \Big\} \\
&+ \int [\prod_{\ell=1,4,6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\ \delta\phi^4_{6,4,1} \frac{\delta(\cdot)}{\delta\phi^4_{6,4,1}} + \delta\bar\phi^4_{6,4,1} \frac{\delta(\cdot)}{\delta\bar\phi^4_{6,4,1}} \Big\}.
\end{aligned} \qquad (67)
$$

Varying the action up to a surface term, the following two-component EMT has been identified (see Appendix B.3)

$$
\begin{aligned}
T^{(1)}_{(1,s);(i,j)} &= \partial_{(1)\ i}\phi^1_{1,2,3}\ \partial_{(s)\ j}\bar\phi^1_{1,2,3} + \partial_{(1)\ i}\bar\phi^1_{1,2,3}\ \partial_{(s)\ j}\phi^1_{1,2,3} - \delta_{1,s}\mathbf{g}_{s\ ij}\mathcal{L}^{\rm color}\ , \\
T^{(4)}_{(1,s);(i,j)} &= \partial_{(1)\ i}\phi^4_{6,4,1}\ \partial_{(s)\ j}\bar\phi^4_{6,4,1} + \partial_{(1)\ i}\bar\phi^4_{6,4,1}\ \partial_{(s)\ j}\phi^4_{6,4,1}\ .
\end{aligned} \qquad (68)
$$

More generally, for any pair of colors (a,b) sharing a common group argument labelled by g_s, the EMT for a translation in g_s will be of the form:

$$
\begin{aligned}
T^{(a)}_{(s,s');(i,j)} &= \partial_{(s)\ i}\phi^a_{1,2,3}\ \partial_{(s')\ j}\bar\phi^a_{1,2,3} + \partial_{(s)\ i}\bar\phi^a_{1,2,3}\ \partial_{(s')\ j}\phi^a_{1,2,3} - \delta_{s,s'}\mathbf{g}_{s'\ ij}\mathcal{L}^{\rm color}\ , \\
T^{(b)}_{(s,s');(i,j)} &= \partial_{(s)\ i}\phi^b_{1',2',3'}\ \partial_{(s')\ j}\bar\phi^b_{1',2',3'} + \partial_{(s)\ i}\bar\phi^b_{1',2',3'}\ \partial_{(s')\ j}\phi^b_{1',2',3'}\ .
\end{aligned} \qquad (69)
$$

Moreover, the components $T^{(1)}$ and $T^{(4)}$ satisfy the relation (see Appendix B.3)

$$
\nabla^j_{(1)} \int [\prod_{\ell=2}^{6} dg_\ell] \Big[T^{(1)}_{(1,1);(i,j)} + T^{(4)}_{(1,1);(i,j)} \Big] = 0 \qquad (70)
$$

and this means that

$$
\int [\prod_{\ell=2}^{6} dg_\ell] \Big[T^{(1)}_{(1,1);(i,j)} + T^{(4)}_{(1,1);(i,j)} \Big], \qquad (71)
$$

being still function of g_1, is a conserved current.

The existence of such a conserved quantity for GFT models can be easily understood. This mainly comes from the definition of translations: they involve an unique field argument whereas all remaining field variables becomes integrated. We can call these local translations. On the symmetry point of view, the colored theory with its minimal[6] symmetry acts as a kind of local theory in $\phi^1(g_1,-)$ and $\phi^4(-,g_1)$. Hence, the fact the EMT is covariantly conserved here can be translated into a local integral in the sense of (43). Another kind analysis supporting this idea will be definitely interesting and useful. For instance, one should check that, in the noncommutative dual space, colored GFTs endowed with Laplacian dynamics which are invariant under translations will have a conserved EMT.

5.2 Dilatations and current tensor

Let us assume once again that only ϕ^1 and ϕ^4 are subjected to the dilatation $g_1 \to g_{a\,1}$:

$$\begin{aligned} \phi^1(g_1,g_2,g_3) &\mapsto \mu(a,\theta_1)^c \phi^1(g_1,g_2,g_3)\;, \\ \phi^4(g_6,g_4,g_1) &\mapsto \mu(a,\theta_1)^c \phi^4(g_6,g_4,g_1)\;. \end{aligned} \tag{72}$$

The action invariant under these dilatations is defined by the Lagrangian density

$$\begin{aligned} &\mathcal{L}^{\text{color,scale}} = \\ &(\sin\theta_1)^{-1}\mathbf{g}_1^{ij}\partial_{(1)\,i}[(\sin\theta_1)^{-c}\bar\phi^1_{1,2,3}]\partial_{(1)\,j}[(\sin\theta_1)^{-c}\phi^1_{1,2,3}] + \sum_{s=2}^{3}\mathbf{g}_s^{ij}\partial_{(s)\,i}\bar\phi^1_{1,2,3}\partial_{(s)\,j}\phi^1_{1,2,3} \\ +\;& (\sin\theta_1)^{-1}\mathbf{g}_1^{ij}\partial_{(1)\,i}[(\sin\theta_1)^{-c}\bar\phi^4_{6,4,1}]\partial_{(1)\,j}[(\sin\theta_1)^{-c}\phi^4_{6,4,1}] + \sum_{s=4,6}\mathbf{g}_s^{ij}\partial_{(s)\,i}\bar\phi^4_{6,4,1}\partial_{(s)\,j}\phi^4_{6,4,1} \\ +\;& \sum_{s=3,4,5}\mathbf{g}_s^{ij}\partial_{(s)\,i}\bar\phi^2_{3,4,5}\partial_{(s)\,j}\phi^2_{3,4,5} + \sum_{s=5,2,6}\mathbf{g}_s^{ij}\partial_{(s)\,i}\bar\phi^3_{5,2,6}\partial_{(s)\,j}\phi^3_{5,2,6} \\ +\;& \lambda\,\phi^1_{1,2,3}\phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1} + \bar\lambda\,\bar\phi^1_{1,2,3}\bar\phi^2_{3,4,5}\bar\phi^3_{5,2,6}\bar\phi^4_{6,4,1}\;, \end{aligned} \tag{73}$$

where we omit to write mass terms even though they can be also included. Indeed, they possess the same scaling behaviour as the interaction itself.

Requiring an invariant action implies that $c=-3/2$. The associated current can be derived using

$$\begin{aligned} W(\epsilon)(\cdot) &= \int [\prod_{\ell=1}^{3} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\;\Big[\delta_\epsilon\phi^1_{1,2,3}\,\frac{\delta}{\delta\phi^1_{1,2,3}}(\cdot) + \delta_\epsilon\bar\phi^1_{1,2,3}\,\frac{\delta}{\delta\bar\phi^1_{1,2,3}}(\cdot)\Big] \\ &+ \int [\prod_{\ell=1,4,6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\;\Big[\delta_\epsilon\phi^4_{1,2,3}\,\frac{\delta}{\delta\phi^2_{1,2,3}}(\cdot) + \delta_\epsilon\bar\phi^4_{1,2,3}\,\frac{\delta}{\delta\bar\phi^2_{1,2,3}}(\cdot)\Big]. \end{aligned} \tag{74}$$

The current tensor for this symmetry possesses distinct components (derivations are given in Appendix C.5)

$$D^{(1)}_{(1);\,j} \;=\; \Big[(\sin\theta_1)^2[\,\beta\cos\theta_1 + \sin\theta_1\,\partial_{(1)\,\theta}\,]\phi^1_{1,2,3}\;\partial_{(1)\,j}\bar\phi^1_{1,2,3}$$

[6]Minimal in a sense that we previously gave, namely, a single translation symmetry on one group argument and keeping all the remaining variables fixed and afterwards integrated. But, at the end, collecting all these minimal symmetries, the colored theory will certainly fall into a category of theories with the maximal number of symmetries. For instance, the non colored case has D independent field translations/dilatations whereas the colored theory has $D(D+1)/2$ of such transformations.

$$
\begin{aligned}
&+\beta \mathbf{g}_{1\ j\theta}\cos\theta_1\sin\theta_1\big[\,\beta\cos\theta_1+\sin\theta_1\,\partial_{(1)\ \theta}]\phi^1_{1,2,3}\ \bar\phi^1_{1,2,3}+(\phi\leftrightarrow\bar\phi)\big] \\
&-\mathbf{g}_{1\ j\theta}\ \sin\theta_1\mathcal{L}^{\text{color,scale}\ (1,4)}\ , \\
D^{(1)}_{(s);\ j} \ =\ &[\,\beta\cos\theta_1+\sin\theta_1\,\partial_{(1)\ \theta}\,]\phi^1_{1,2,3}\ \partial_{(s)\ j}\bar\phi^1_{1,2,3}\ +\ (\phi\leftrightarrow\bar\phi)\ ,\quad s=2,3, \qquad (75)
\end{aligned}
$$

while the components $D^{(4)}_{(s);\ j}$ can be obtained from $D^{(1)}_{(1);\ j}$ and $D^{(1)}_{(s=2,3);\ j}$ by taking the symmetry $(\phi^1_{1,2,3}\leftrightarrow\phi^4_{6,4,1})$ without the Lagrangian term $\mathcal{L}^{\text{color,scale}\ (1,4)}$. The latter is defined from $\mathcal{L}^{\text{color,scale}}$ by considering only terms containing the fields of colors 1 and 4. The dilatation current component $D^{(1)}_{(1);\ j}$ can be written also as

$$
\begin{aligned}
D^{(1)}_{(1);\ j} \ =\ &\partial_{(1)\ \theta}[(\sin\theta_1)^\beta\bar\phi^1_{1,2,3}]\partial_{(1)\ j}[(\sin\theta_1)^\beta\phi^1_{1,2,3}]+\partial_{(1)\ \theta}[(\sin\theta_1)^\beta\bar\phi^1_{1,2,3}]\partial_{(1)\ j}[(\sin\theta_1)^\beta\phi^1_{1,2,3}] \\
-\ &\sin\theta_1\mathcal{L}^{\text{color,scale}\ (1,4)}\ . \qquad (76)
\end{aligned}
$$

Finally, we can note that the dilatation current is not covariantly conserved due to the explicit coordinate dependence but not because of the nonlocal interaction:

$$
\int[\prod_{\ell=2}^{6}dg_\ell]\left[\sum_{s=1,2,3}\nabla^j_{(s)}D^{(1)}_{(s)\ j}+\sum_{s=1,2,3}\nabla^j_{(s)}D^{(4)}_{(s)\ j}\right]\neq 0\ . \qquad (77)
$$

The explicit expression of the breaking is given in Appendix C.5.

6 Summary and outlooks

The classical formalism, i.e. the extension of Klein-Gordon field equation and the group symmetry study, for dynamical GFT over tensor copies of $SU(2)$ has been investigated in this paper. We find that the GFTs exhibit peculiarities that one could expect when dealing with nonlocal models. For translation symmetry, the EMT for the general GFT (without colors) proves to be symmetric but not locally conserved for any dimension, save $D=1$. As a matter of fact, for $D=1$ locality is recovered and, from that, the EMT is covariantly conserved. In contrast and astonishingly, the genuine colored GFTs which are also nonlocal possess a covariantly conserved quantity obtained by integrating and summing some EMT components. This is another feature advocating in favor of these colored theories. In all situations (with and without color), the EMT possesses a nonvanishing trace, even though the latter property could be improved but the meaning of the resulting tensor remains unclear. We also discuss a specific way that group dilatations could be implemented at the GFT level. Indeed, dilatation symmetry can been consistently settled in GFT with cost to put an explicit dependence of the class angle θ coordinate of the $SU(2)$ group variable $g(\theta,\vec{n})$ in the Lagrangian. In any dimension $D\geq 1$ colored or not, this dissipative term explicitly breaks the conservation of the dilatation current. Remark that there should be an alternative way to implement dilatation on the sphere according to the work by Okuyama [42] or by using Weyl transformations that we did not consider here. Nevertheless, the issue of nonlocality and the current local conservation breaking will be in all cases difficult to circumvent.

It can be desirable to find some improvement procedures adapted for the non-colored GFTs in order to render the EMTs and other currents covariantly conserved for $D\geq 1$. In the case of noncommutative field theory defined with a Moyal $\star$-product with its induced nonlocality, improvement procedures have been highlighted to treat the breaking term of the EMT local conservation [37]-[39]. Nevertheless, these methods were successful due to the specificity of the Moyal field algebra. Here, clearly the issue is different and deserves a better understanding. Another important field symmetry, that we did not discuss and could be viewed as rotation in the context, would be the one associated with group adjoint transformation $g \to hgh^{-1}$. There are more infinitesimal vector field generators associated with such transformations and so more involved becomes the computation of the "angular momentum" tensor. The fact that the EMTs found here are symmetric is encouraging for the local conservation of the angular momentum tensor but only in the color case. Finally, new vertices leading to orientable graphs for GFTs have been highlighted recently [46]. It could be interesting to see if the techniques used in this paper could be also implemented on these models and could lead to other or more regular properties.

The present analysis has revealed that one of the simplest notions of symmetry, such as translations, for dynamical non colored GFTs are more difficult to implement and so remain to be understood. In contrast, dynamical colored GFTs have quite regular properties with a conserved quantity for translation symmetry. A key point revealed by this analysis is that dynamics for colored GFTs is not incompatible with the notion of symmetries. A natural question is the implications of such symmetries at the quantum level. We should comment that this study could be useful for the larger program aiming at renormalizing GFTs [5]. In particular, among the most interesting candidates for that program are colored tensor theories. Here we have proved that colored theories endowed with dynamics have a well defined notion of translation symmetry. As a result, they will have more constrained Ward-Takahashi identities in relation with that particular symmetry. These will be useful for both perturbative renormalization and also nonperturbative features. Indeed, Ward-Takahashi identities (which can be called quantum versions of conservation laws) provide useful relations between correlation functions which should hold even after renormalization. These identities have been successfully used in quantum electrodynamics renormalization (giving a link between the renormalized vertex three-point function and the renormalized wave function). More recently, they prove to be one of the main ingredients for the proof of asymptotic safety at all order of perturbation theory for particular matrix models [47][48]. Note that Ward-Takahashi identities have been already discussed for non dynamical colored GFTs under unitary symmetries [15]. In the present work, dynamical colored tensored theories have been investigated and should provide other kinds of Ward-Takahashi identities with respect to translations.

Last, GFT symmetries can be inquired in another way by establishing a bridge between our formulation and the metric representation as described by Baratin and co-workers [29]. For simplicity, we will restrict to the $1D$ GFT formalism. Consider a function $\phi : G \to \mathbb{C}$ of one group variable and use the group Fourier transform [30] in order to obtain its Lie algebra representation:

$$\widehat{\phi}(x) = \int_G \phi(g) e_g(x) dg \ , \quad e_g : \mathfrak{su}(2) \sim \mathbb{R}^3 \to U(1) \ , \quad e_g(x) = e^{i\mathrm{Tr}(xg)} \ , \quad x = \vec{x} \cdot \vec{\tau} \ , \tag{78}$$

where $\vec{x} \in \mathbb{R}^3$, and $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ is the vector of anti-Hermitian $\mathfrak{su}(2)$ generators. Let $g \to gh$ be a group translation such that under this transformation fields get modified according to $\phi(g) \to \phi'(g' = gh)$, then the transformed dual field becomes

$$\widehat{(\phi')}(x) = \left(\widehat{\phi} \star e_h\right)(x) \;, \tag{79}$$

where the explicit nature of the $\star$-product can be found in [30]. Thus, the group translation invariance for fields can be translated as a $U_\star(1)$ noncommutative gauge invariance in the Lie algebra formulation. Note that the latter result is in contrast with translations $x \to x + a$ directly stated in the Lie algebra formulation which yield a field transformation corresponding to a plane wave multiplication: $[\widehat{\phi}(x) \to \widehat{\phi}'(x + a)] \;\leftrightarrow\; [\phi(g) \to \phi'(g) = e_g(a)\phi(g)]$. Group dilatations will certainly have a noncommutative analogue but this deserves to be investigated. Remarkably, there is an analogue to our Noether theorem in this noncommutative setting. An infinitesimal translation corresponding to $x \to x + a$, where $a \in \mathbb{R}^3$, is totally similar to an ordinary infinitesimal translation is the common spacetime: $\delta_\epsilon \, \phi = \sum_{i=1}^{3} \epsilon^i \partial_i \, \phi$. A main difference between the group and the Lie algebra formalisms lies in the type of field algebra under consideration. While in the former theory, the field algebra is commutative, in the latter, the same algebra becomes noncommutative. However, it has been defined a Noether procedure for noncommutative field algebras equipped with $\star$-product ([35]-[39] and for more general discussion on Noether's theorem in noncommutative geometry see [49]). In the present context, one needs to introduce the operator

$$W_\star(\epsilon)(\cdot) = \int d\theta d\varphi^1 d\varphi^2 \; \left(\delta_\epsilon \phi \star \frac{\delta}{\delta \phi}(\cdot)\right)(x) \;. \tag{80}$$

Applying the operator $\frac{\delta}{\delta\phi}(\cdot)$ on the action should produce the noncommutative version of the equations of motion for fields. Note that, due to the noncommutativity theory, it is still possible to introduce other types of operators $W_\star(\epsilon)$ by considering for instance $[\frac{\delta}{\delta\phi} \star \delta_\epsilon \phi](\cdot)](x)$ or by symmetrizing both the latter and (80). Obviously dealing with colored fields, the operator (80) should be extended for that purpose. An interesting exercise would be to compute and the characterize all noncommutative Noether quantities issued from the operator (80) and the similar for other field transformations. We expect that currents will share the similar (broken or not) features as given in this work.

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Appendix

A Gauge invariant fields

Consider a gauge invariant field, namely a field satisfying

$$\phi(g_1, g_2, g_3) = \int dh\ \phi(g_1 h, g_2 h, g_3 h)\ . \tag{A.1}$$

This condition translates in Fourier modes via Peter-Weyl theorem as

$$\sum_{j_a,m_a,n_a} \phi^{j_a}_{m_a,n_a} \prod_a \sqrt{d_{j_a}} D^{j_a}_{m_a,n_a}(g_a) = \sum_{j_a,m_a,k_a,n_a} \phi^{j_a}_{m_a,k_a} [\prod_a \sqrt{d_{j_a}} D^{j_a}_{m_a,n_a}(g_a)] \int dh \prod_a D^{j_a}_{n_a,k_a}(h) \tag{A.2}$$

where $a = 1,2,3$, $\phi^{j_a}_{m_a,n_a}$ is a notation for $\phi^{j_1,j_2,j_3}_{m_1,n_1;m_2,n_2;m_3,n_3}$, j_a indexes half spin representation, $j_a \in \frac{1}{2}\mathbb{N}$, n_a, m_a, k_a are ordinary associated magnetic momenta each constrained to be inside $[-j_a, +j_a]$, $d_{j_a} = 2j_a + 1$ is the dimension of the representation space. $D^j_{mn}(g)$ denotes the Wigner matrix element of g in the representation j. The factor $\sqrt{d_{j_a}}$ is chosen as a normalization convention.

Computing the last integral, one gets by simple identification:

$$\phi^{j_a}_{m_a,n_a} = \sum_{k_a} \phi^{j_a}_{m_a,k_a} \begin{pmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}\ , \tag{A.3}$$

with the arrays denoting the rotation invariant Wigner $3j$-symbols. A field with coefficients as (A.3) indeed exists and satisfies (A.1) due the orthogonality relation of $3j$ symbols. A simple $\phi^{j_a}_{m_a,n_a}$ is given for instance by the product of two $3j$-symbols with coefficients following the pattern (A.3). For the general D dimensional gauge invariant fields, the integral $\int dh \prod_{a=1}^D D^{j_a}_{n_a,k_a}(h)$ gives an invariant intertwiner which can be recoupled in more involved Wigner $3nj$-symbols.

B Group translations

In this appendix, we give the main identities leading to the formulas of EMTs in $1D$, $3D$ and colored GFTs and to the (non)local conservation property of these tensors. The first subsection explains also the method of computation of these quantities.

B.1 Ward operator action for translations for 1D GFT

In this paragraph, an explicit calculation of the EMT is given for a $1D$ GFT by applying the Ward operator method in a curved background like S^3 parametrized by $(\theta, \varphi_1, \varphi_2)$. The ensuing tensor study could be deduced from this point.

For an infinitesimal translation of parameter X, we can set in a given local coordinate system $\delta_X \phi = X^i \partial_i \phi$. Then one introduces the operator

$$W(X)(\cdot) = \int d\theta d\varphi_1 d\varphi_2\ \left(\delta_X \mathbf{g}^{ij} \frac{\delta(\cdot)}{\delta \mathbf{g}^{ij}} + \delta_X \phi \frac{\delta(\cdot)}{\delta \phi} \right) \tag{B.4}$$

that acts on the action S_{1D} (10) such that

$$\begin{aligned}\frac{\partial}{\partial X^\rho}W(X)S_{1D} &= \frac{\partial}{\partial X^\rho}\int d\theta d\varphi_1 d\varphi_2 \Big\{\delta_X \mathbf{g}^{ij}\left[\frac{\partial\sqrt{|\det\mathbf{g}|}}{\partial \mathbf{g}^{ij}}\mathcal{L}_{1D} + \sqrt{|\det\mathbf{g}|}\frac{\partial\mathcal{L}_{1D}}{\partial\mathbf{g}^{ij}}\right] \\ &\quad + \sqrt{|\det\mathbf{g}|}\left(X^\kappa\partial_\kappa\phi\right)\left(-\Delta\phi + m^2\phi + \lambda\phi^3\right)\Big\} . \end{aligned} \tag{B.5}$$

Keeping in mind that the Laplacian contains an inverse factor of the metric determinant, one gets

$$\begin{aligned}&\frac{\partial}{\partial X^\rho}W(X)S_{1D} = \delta^\kappa_\rho\int d\theta d\varphi_1 d\varphi_2\ \Big\{\partial_\kappa\mathbf{g}^{ij}\left[-\frac{1}{2}\sqrt{|\det\mathbf{g}|}\mathbf{g}_{ij}\mathcal{L}_{1D} + \sqrt{|\det\mathbf{g}|}\frac{1}{2}\partial_i\phi\partial_j\phi\right] \\ &-\partial_i(\partial_\kappa\phi\mathbf{g}^{ij}\sqrt{|\det\mathbf{g}|}\partial_j\phi) + (\partial_i\partial_\kappa\phi)\mathbf{g}^{ij}\sqrt{|\det\mathbf{g}|}\partial_j\phi + \sqrt{|\det\mathbf{g}|}\partial_\kappa(m^2\frac{1}{2}\phi + \frac{\lambda}{4}\phi^4)\Big\} . \end{aligned} \tag{B.6}$$

It is customary in a field theory to identify the EMT from the variations of the metric $\mathbf{g}^{ij}$. The EMT already appears in the above expression up to some factor. However, in this paper, we will not use this route preferring instead to get a final surface term. To this end, further computations invoking both metric and field variations are in order:

$$\begin{aligned}&\frac{\partial}{\partial X^\rho}W(X)S_{1D} = \delta^\kappa_\rho\int d\theta d\varphi_1 d\varphi_2\ \Big\{\partial_\kappa\mathbf{g}^{ij}\left[-\frac{1}{2}\sqrt{|\det\mathbf{g}|}\mathbf{g}_{ij}\mathcal{L}_{1D} + \sqrt{|\det\mathbf{g}|}\frac{1}{2}\partial_i\phi\partial_j\phi\right] \\ &-\partial_i(\partial_\kappa\phi\mathbf{g}^{ij}\sqrt{|\det\mathbf{g}|}\partial_j\phi) - \sqrt{|\det\mathbf{g}|}(\partial_\kappa\mathbf{g}^{ij})(\frac{1}{2}\partial_i\phi\partial_j\phi) + \sqrt{|\det\mathbf{g}|}\partial_\kappa\mathcal{L}_{1D}\Big\} \\ &= -\int d\theta d\varphi_1 d\varphi_2\ \partial_i\{\sqrt{|\det\mathbf{g}|}\mathbf{g}^{ij}\left(\partial_\rho\phi\partial_j\phi - \mathbf{g}_{\rho j}\mathcal{L}_{1D}\right)\} . \end{aligned} \tag{B.7}$$

Finally, this expression is of the form of a surface term and we identify the EMT

$$T_{\rho j} = \partial_\rho\phi\partial_j\phi - \mathbf{g}_{\rho j}\mathcal{L}_{1D} . \tag{B.8}$$

We verify, in a covariant script, that

$$\begin{aligned}&\nabla^i T_{ij} = \nabla^i(\nabla_i\phi\nabla_j\phi) - \nabla_j\left(\frac{1}{2}\mathbf{g}^{kl}\nabla_k\phi\nabla_l\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4\right) \\ &= (\nabla^i\nabla_i\phi)\nabla_j\phi + \nabla_i\phi\nabla^i\nabla_j\phi - \frac{1}{2}\mathbf{g}^{kl}\nabla_j(\nabla_k\phi\nabla_l\phi) - \nabla_j\phi\left(m^2\phi + \lambda\phi^3\right) = 0 , \end{aligned} \tag{B.9}$$

where, we use the property of the Levi-Civita connection, and, in last resort, the field equation of motion $-\Delta\phi + m^2\phi + \lambda\phi^3 = 0$.

This method of deriving the EMT will be extended in the subsequent situations dealing with tensor models.

B.2 EMT for GFT in 3D

EMT calculation - We provide here the main stages of calculations leading to (42).

First, we recall that the infinitesimal variation for a tensor field ϕ under a "3-translation" is given by $\delta_X \phi_{1,2,3} := \delta_{X_{(1)},X_{(2)},X_{(3)}} \phi_{1,2,3} = \sum_{s=1}^{3} X^i_{(s)} \partial_{(s)\ i} \phi_{1,2,3}$. Then, we require an operator symmetrization for the interaction part:

$$
\begin{aligned}
&\lambda \int [\prod_{\ell=1}^{6} dg_\ell] \ (\delta_X \phi_{1,2,3})\, \bar\phi_{5,4,3} \phi_{5,2,6} \bar\phi_{1,4,6} + \big(\delta_X \bar\phi_{1,2,3}\big)\, \phi_{5,4,3} \bar\phi_{5,2,6} \phi_{1,4,6} = \\
&\frac{\lambda}{2} \int [\prod_{\ell=1}^{6} dg_\ell] \Big\{ (\delta_X \phi_{1,2,3})\, \bar\phi_{5,4,3} \phi_{5,2,6} \bar\phi_{1,4,6} + (\delta_X \phi_{5,2,6})\, \phi_{1,2,3} \bar\phi_{5,4,3} \bar\phi_{1,4,6} \\
&+ \big(\delta_X \bar\phi_{1,2,3}\big)\, \phi_{5,4,3} \bar\phi_{5,2,6} \phi_{1,4,6} + \big(\delta_X \bar\phi_{5,2,6}\big)\, \phi_{5,4,3} \phi_{1,4,6} \bar\phi_{1,2,3} \Big\} \\
&= \frac{\lambda}{2} \int [\prod_{\ell=1}^{6} dg_\ell] \Big\{ \sum_{s=1}^{3} \big(X^i_{(s)} \partial_{(s)\ i} \phi_{1,2,3} \big)\, \bar\phi_{5,4,3} \phi_{5,2,6} \bar\phi_{1,4,6} \\
&+ \sum_{s=1}^{3} \big(X^i_{(s)} \partial_{(s+\alpha_s)\ i} \bar\phi_{5,4,3} \big)\, \phi_{1,2,3} \phi_{5,2,6} \bar\phi_{1,4,6} + \sum_{s=1}^{3} \big(X^i_{(s)} \partial_{(s+\alpha_s)\ i} \phi_{5,2,6} \big)\, \bar\phi_{5,4,3} \phi_{1,2,3} \bar\phi_{1,4,6} \\
&+ \sum_{s=1}^{3} \big(X^i_{(s)} \partial_{(s+\alpha_s)\ i} \bar\phi_{1,4,6} \big)\, \bar\phi_{5,4,3} \phi_{5,2,6} \phi_{1,2,3} \Big\} \\
&= \frac{\lambda}{2} \int [\prod_{\ell=1}^{6} dg_\ell] \sum_{s=1}^{3} X^i_{(s)} \Big(\partial_{(s)\ i} + \partial_{(s+[\alpha_s])\ i} \Big) \phi_{1,2,3} \bar\phi_{5,4,3} \phi_{5,2,6} \bar\phi_{1,4,6}\ .
\end{aligned}
\tag{B.10}
$$

where the index $\alpha_s = 0, 2, 3, 4$ has to be chosen appropriately. In the last equality, the notation $[\alpha_s]$ means that we fix $([\alpha_1], [\alpha_2], [\alpha_3]) = (4, 2, 3)$. It is remarkable that, under integral over all six variables, we can exchange $\bar\phi_{1,2,3}$ for $\bar\phi_{5,4,3}$ and $\bar\phi_{5,2,6}$ for $\bar\phi_{1,4,6}$ by just renaming the variables (this is a set of discrete symmetries which will be also used in the sequel).

We introduce the notations $(\bullet) = \prod_{s=1}^{6} \sqrt{|\det \mathbf{g}_s|}$ and $(\bullet)_{\check s} = \prod_{k\neq s} \sqrt{|\det \mathbf{g}_k|}$, so that

$$
(\bullet) = (\bullet)_{\check s} \sqrt{|\det \mathbf{g}_s|}\ , \qquad \prod_{\ell=1}^{6} [dg_\ell] = \frac{1}{2\pi^2} \prod_{\ell=1}^{6} [d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] (\bullet)\ . \tag{B.11}
$$

The factor $1/(2\pi^2)$ will be omitted in the following. The EMT can be computed as follows

$$
\begin{aligned}
W(X) S_{3D} = &\int [\prod_{\ell=1}^{6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \ \Bigg(\sum_{s=1}^{6} \delta_X \mathbf{g}^{ij}_s \frac{\delta}{\delta \mathbf{g}^{ij}_s} S_{3D} \\
&+ \sum_{s'=1}^{3} X^k_{(s')} \partial_{(s')\ k} \phi_{1,2,3} \Big[- \sum_{s=1}^{3} \partial_{(s)\ j} ((\bullet)_{\check s} \sqrt{|\det \mathbf{g}_s|} \mathbf{g}^{jl}_s \partial_{(s)\ l} \bar\phi_{1,2,3}) \Big] \\
&+ (\bullet) \sum_{s'=1}^{3} X^k_{(s')} \partial_{(s')\ k} \phi_{1,2,3} \big[m^2 \bar\phi_{1,2,3} + \lambda\, \bar\phi_{5,4,3} \phi_{5,2,6} \bar\phi_{1,4,6}) \big] + (\phi \leftrightarrow \bar\phi) \Bigg)\ ,
\end{aligned}
\tag{B.12}
$$

where the part including the variation $\delta_X \bar\phi$ is not explicitly displayed but appears symbolically as $(\phi \leftrightarrow \bar\phi)$. The corresponding terms can be carried out in a symmetric manner.

Adding all contributions, combining the mass terms and the interaction using (B.10), it can be seen that

$$
\begin{aligned}
W(X)S_{3D} = \int [\prod_{\ell=1}^{6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \;\Big\{ \sum_s \delta_X \mathbf{g}^{ij}_s \frac{\delta}{\delta \mathbf{g}^{ij}_s} S_{3D} \\
+\Big\{ -\sum_{s',s=1}^{3} X^k_{(s')} \partial_{(s)\;j} \left[\partial_{(s')\;k}\phi_{1,2,3}\;(\bullet)\mathbf{g}^{jl}_s \partial_{(s)\;l}\bar\phi_{1,2,3}\right] \\
+ \sum_{s',s=1}^{3} X^k_{(s')} \partial_{(s)\;j} \left[\partial_{(s')\;k}\phi_{1,2,3}\right]\;(\bullet)\mathbf{g}^{jl}_s \partial_{(s)\;l}\bar\phi_{1,2,3} \;+ (\phi \leftrightarrow \bar\phi)\Big\} \qquad \text{(B.13)} \\
+(\bullet)\sum_{s'=1}^{3} X^k_{(s')} \left[m^2 \partial_{(s')\;k}(\bar\phi_{1,2,3}\phi_{1,2,3}) + \frac{\lambda}{2}(\partial_{(s')\;k} + \partial_{(s'+[\alpha_{s'}])\;k})\phi_{1,2,3}\bar\phi_{5,4,3}\phi_{5,2,6}\bar\phi_{1,4,6}) \right] \Big\} .
\end{aligned}
$$

Focusing now on the dynamical part and metric variations, one has:

$$
\begin{aligned}
K = \int [\prod_{\ell=1}^{6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \;\Big\{ \int [\prod_{\ell'=1}^{6} d\theta_{\ell'} d\varphi^1_{\ell'} d\varphi^2_{\ell'}] \sum_{s=1}^{6} \delta_X \mathbf{g}^{ij}_s \frac{\delta}{\delta \mathbf{g}^{ij}_s} ((\bullet)_{\check s} \sqrt{|\det \mathbf{g}_s|}) \mathcal{L}_{3D} \\
+ \int [\prod_{\ell'=1}^{6} d\theta_{\ell'} d\varphi^1_{\ell'} d\varphi^2_{\ell'}] \sum_{s=1}^{6} (\bullet) \delta_X \mathbf{g}^{ij}_s \frac{\delta \mathcal{L}_{\mathrm{kin},\,3D}}{\delta \mathbf{g}^{ij}_s} \\
+\Big\{ -\sum_{s',s=1}^{3} X^k_{(s')} \partial_{(s)\;j} \left[\partial_{(s')\;k}\phi_{1,2,3}\;(\bullet)\mathbf{g}^{jl}_s \partial_{(s)\;l}\bar\phi_{1,2,3}\right] + (\phi \leftrightarrow \bar\phi)\Big\} \\
+(\bullet)\;\sum_{s',s=1}^{3} X^k_{(s')} \partial_{(s')\;k} \left[\mathbf{g}^{jl}_s \partial_{(s)\;j}\phi_{1,2,3}\;\partial_{(s)\;l}\bar\phi_{1,2,3}\right] \\
-(\bullet)\;\sum_{s',s=1}^{3} X^k_{(s')} \partial_{(s')\;k} \left[\mathbf{g}^{jl}_s\right] \partial_{(s)\;j}\phi_{1,2,3}\;\partial_{(s)\;l}\bar\phi_{1,2,3} \Big\} \;. \qquad \text{(B.14)}
\end{aligned}
$$

Canceling the variation $\delta\mathcal{L}_{\mathrm{kin},\,3D}/\delta\mathbf{g}^{ij}_s$ with its partner coming from the field variations and recomposing the Lagrangian, by injecting expression K (B.14) into (B.12), we get

$$
\begin{aligned}
W(X)S_{3D} = \\
\int [\prod_{\ell=1}^{6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \Big[\Big\{ -\sum_{s',s=1}^{3} X^k_{(s')} \partial_{(s)\;j} \left[(\bullet)\partial_{(s')\;k}\phi_{1,2,3}\mathbf{g}^{jl}_s \partial_{(s)\;l}\bar\phi_{1,2,3}\right] + (\phi \leftrightarrow \bar\phi)\Big\} \\
+\sum_{s'=1}^{3} X^k_{(s')} \Big\{ \partial_{(s')\;k}[(\bullet)\mathcal{L}_{3D}] + \partial_{(s'+[\alpha_{s'}])\;k}[(\bullet)\mathcal{L}_{\mathrm{int},\,3D}] \Big\} \Big] \;. \qquad \text{(B.15)}
\end{aligned}
$$

We are now in position to provide the EMT for group translations of the GFT by just deriving the above expression by some infinitesimal parameter: $X^k_{(s)}$, $s, k = 1, 2, 3$,

$$
\frac{\partial}{\partial X^k_{(s)}} W(X)S_{3D} = -\int [\prod_{\ell=1}^{6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \sum_{s'=1}^{6} \partial_{(s')\;j}(\bullet)\mathbf{g}^{jl}_{s'} \Big\{
$$

$$\partial_{(s)\ k}\phi_{1,2,3}\ \partial_{(s')\ l}\bar{\phi}_{1,2,3}+\partial_{(s)\ k}\bar{\phi}_{1,2,3}\ \partial_{(s')\ l}\phi_{1,2,3}-\delta_{s,s'}\mathbf{g}_{s'\ lk}\mathcal{L}_{3D}\ -\delta_{s+\alpha_s,s'}\mathbf{g}_{s'\ lk}\mathcal{L}_{\mathrm{int},\,3D}\Big\}. \tag{B.16}$$

Hence the EMT is given by

$$T_{(s,s');(i,j)}=\partial_{(s)\ i}\phi_{1,2,3}\,\partial_{(s')\ j}\bar{\phi}_{1,2,3}+\partial_{(s)\ i}\bar{\phi}_{1,2,3}\,\partial_{(s')\ j}\phi_{1,2,3}-\delta_{s,s'}\mathbf{g}_{s'\ ij}\mathcal{L}_{3D}-\delta_{s+[\alpha_s],s'}\mathbf{g}_{s'\ ij}\mathcal{L}_{\mathrm{int},\,3D}\ , \tag{B.17}$$

for $s=1,2,3$, $s'=1,2,\dots,6$, and $i,j=1,2,3$.

Covariant conservation - The fact that the EMT is not covariantly conserved is proved here. We have in covariant notations:

$$\begin{aligned}
&\sum_{s'=1}^{6}\nabla^{j}_{(s')}T_{(s,s');(ij)}=\\
&\Big\{\sum_{s'=1}^{3}\Big(\nabla^{j}_{(s')}\nabla_{(s)\ i}\phi_{1,2,3}\ \nabla_{(s')\ j}\bar{\phi}_{1,2,3}+\nabla_{(s)\ i}\phi_{1,2,3}\ \nabla^{j}_{(s')}\nabla_{(s')\ j}\bar{\phi}_{1,2,3}\Big)+(\phi\leftrightarrow\bar{\phi})\Big\}\\
&-\nabla_{(s)\ i}\left(\sum_{s'=1}^{3}\mathbf{g}^{kl}_{s'}\nabla_{(s')\ k}\phi_{1,2,3}\nabla_{(s')\ l}\bar{\phi}_{1,2,3}+m^2\bar{\phi}_{1,2,3}\phi_{1,2,3}+\frac{\lambda}{2}\phi_{1,2,3}\bar{\phi}_{5,4,3}\phi_{5,2,6}\bar{\phi}_{1,4,6}\right)\\
&-\nabla_{(s+[\alpha_s])\ i}(\frac{\lambda}{2}\phi_{1,2,3}\bar{\phi}_{5,4,3}\phi_{5,2,6}\bar{\phi}_{1,4,6})\\
&=\sum_{s'=1}^{3}\Big(\nabla_{(s)\ i}\phi_{1,2,3}\ \nabla^{j}_{(s')}\nabla_{(s')\ j}\bar{\phi}_{1,2,3}\Big)-m^2\bar{\phi}_{1,2,3}(\nabla_{(s)\ i}\phi_{1,2,3})\\
&+\sum_{s'=1}^{3}\Big(\nabla_{(s)\ i}\bar{\phi}_{1,2,3}\ \nabla^{j}_{(s')}\nabla_{(s')\ j}\phi_{1,2,3}\Big)-m^2\phi_{1,2,3}(\nabla_{(s)\ i}\bar{\phi}_{1,2,3})\\
&-\frac{\lambda}{2}(\nabla_{(s)\ i}\phi_{1,2,3})\bar{\phi}_{5,4,3}\phi_{5,2,6}\bar{\phi}_{1,4,6}-\frac{\lambda}{2}\phi_{1,2,3}(\nabla_{(s)\ i}\bar{\phi}_{5,4,3})\phi_{5,2,6}\bar{\phi}_{1,4,6}\\
&-\frac{\lambda}{2}\phi_{1,2,3}\bar{\phi}_{5,4,3}(\nabla_{(s)\ i}\phi_{5,2,6})\bar{\phi}_{1,4,6}-\frac{\lambda}{2}\phi_{1,2,3}\bar{\phi}_{5,4,3}\phi_{5,2,6}(\nabla_{(s)\ i}\bar{\phi}_{1,4,6})\\
&-\frac{\lambda}{2}\phi_{1,2,3}(\nabla_{(s+[\alpha_s])\ i}\bar{\phi}_{5,4,3})\phi_{5,2,6}\bar{\phi}_{1,4,6}-\frac{\lambda}{2}\phi_{1,2,3}\bar{\phi}_{5,4,3}(\nabla_{(s+[\alpha_s])\ i}\phi_{5,2,6})\bar{\phi}_{1,4,6}\\
&-\frac{\lambda}{2}\phi_{1,2,3}\bar{\phi}_{5,4,3}\phi_{5,2,6}(\nabla_{(s+[\alpha_s])\ i}\bar{\phi}_{1,4,6})\ .
\end{aligned} \tag{B.18}$$

Using the equation of motion of $\phi_{1,2,3}$ and $\bar{\phi}_{1,2,3}$ after integrating by g_4, g_5 and g_6, (B.18) becomes

$$\begin{aligned}
&\int[\prod_{\ell=4}^{6}dg_\ell]\Big\{+\frac{\lambda}{2}(\nabla_{(s)\ i}\phi_{1,2,3})\bar{\phi}_{5,4,3}\phi_{5,2,6}\bar{\phi}_{1,4,6}-\frac{\lambda}{2}\phi_{1,2,3}(\nabla_{(s)\ i}\bar{\phi}_{5,4,3})\phi_{5,2,6}\bar{\phi}_{1,4,6}\\
&-\frac{\lambda}{2}\phi_{1,2,3}\bar{\phi}_{5,4,3}(\nabla_{(s)\ i}\phi_{5,2,6})\bar{\phi}_{1,4,6}-\frac{\lambda}{2}\phi_{1,2,3}\bar{\phi}_{5,4,3}\phi_{5,2,6}(\nabla_{(s)\ i}\bar{\phi}_{1,4,6})\\
&-\frac{\lambda}{2}\phi_{1,2,3}(\nabla_{(s+[\alpha_s])\ i}\bar{\phi}_{5,4,3})\phi_{5,2,6}\bar{\phi}_{1,4,6}-\frac{\lambda}{2}\phi_{1,2,3}\bar{\phi}_{5,4,3}(\nabla_{(s+[\alpha_s])\ i}\phi_{5,2,6})\bar{\phi}_{1,4,6}\\
&-\frac{\lambda}{2}\phi_{1,2,3}\bar{\phi}_{5,4,3}\phi_{5,2,6}(\nabla_{(s+[\alpha_s])\ i}\bar{\phi}_{1,4,6})+\lambda(\nabla_{(s)\ i}\bar{\phi}_{1,2,3})\phi_{5,4,3}\bar{\phi}_{5,2,6}\phi_{1,4,6}\Big\}
\end{aligned} \tag{B.19}$$

which is not a vanishing quantity. Hence the EMT is not covariantly conserved and the breaking term is clearly a factor of the coupling constant λ. As a proof that actually the

EMT is globally (covariantly) conserved, we have to check that, for $s = 1, 2, 3$, the above remainder can only vanish under integration over the full six group copies. Due the particular symmetric form of the strands, it is sufficient to check that claim for $s = 1$. We have:

$$\int [\prod_{\ell=4}^{6} dg_\ell] \Big\{ \frac{\lambda}{2} (\nabla_{(1)\ i} \phi_{1,2,3}) \bar\phi_{5,4,3} \phi_{5,2,6} \bar\phi_{1,4,6} - \frac{\lambda}{2} \phi_{1,2,3} \bar\phi_{5,4,3} \phi_{5,2,6} (\nabla_{(1)\ i} \bar\phi_{1,4,6}) \\ - \frac{\lambda}{2} \phi_{1,2,3} (\nabla_{(5)\ i} \bar\phi_{5,4,3}) \phi_{5,2,6} \bar\phi_{1,4,6} - \frac{\lambda}{2} \phi_{1,2,3} \bar\phi_{5,4,3} (\nabla_{(5)\ i} \phi_{5,2,6}) \bar\phi_{1,4,6} \\ + \lambda (\nabla_{(1)\ i} \bar\phi_{1,2,3}) \phi_{5,4,3} \bar\phi_{5,2,6} \phi_{1,4,6} \Big\} . \tag{B.20}$$

We use two further integrations in g_1 and g_3, in order to put the last expression, in the form

$$\int [\prod_{\ell=1,3,4,5,6} dg_\ell] \Big\{ - \lambda \phi_{1,2,3} \bar\phi_{5,4,3} \phi_{5,2,6} (\nabla_{(1)\ i} \bar\phi_{1,4,6}) + \lambda (\nabla_{(1)\ i} \bar\phi_{1,2,3}) \phi_{5,4,3} \bar\phi_{5,2,6} \phi_{1,4,6} \Big\} , \tag{B.21}$$

and, finally, in order to cancel this term, we need the last integration in g_2.

B.3 EMT for the colored model

EMT calculation - Let us start by considering the functional operator (67) where the variations of the fields are $\delta_X \phi^a_{1,2,3} = X^i \partial_i \phi^a_{1,2,3}$ and equations of motion (66). We introduce further notations $(\bullet)_{a,b,c} = \prod_{s=a,b,c} \sqrt{|\det \mathbf{g}_s|}$. Using the field variations and equations of motion, the interaction has to be reconstructed as follows

$$\lambda \int [\prod_{\ell=1}^{6} dg_\ell] \left[\delta_X \phi^1_{1,2,3} \phi^2_{3,4,5} \phi^3_{5,2,6} \phi^4_{6,4,1} + \delta_X \phi^4_{6,4,1} \phi^1_{1,2,3} \phi^2_{3,4,5} \phi^3_{5,2,6} + (\phi \leftrightarrow \bar\phi) \right] \\ = \lambda \int [\prod_{\ell=1}^{6} dg_\ell] \left[X^i \partial_{(1)\ i} (\phi^1_{1,2,3} \phi^2_{3,4,5} \phi^3_{5,2,6} \phi^4_{6,4,1}) + (\phi \leftrightarrow \bar\phi) \right] . \tag{B.22}$$

The EMT can be computed from a similar routine as previously performed. We sum up the main steps:

$$W(X) S^{\rm color} = \int [\prod_{\ell=1}^{3} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \Big\{ \delta_X \mathbf{g}_1^{ij} \frac{\delta}{\delta \mathbf{g}_1^{ij}} S^{\rm color} \\ + X^k \partial_{(1)\ k} \phi^1_{1,2,3} \left[- \sum_{s=1}^{3} \partial_{(s)\ j} ((\bullet)_{\check s} \sqrt{|\det \mathbf{g}_s|} \mathbf{g}_s^{jl} \partial_{(s)\ l} \bar\phi^1_{1,2,3}) \right] \\ + (\bullet)_{1,2,3} X^k \partial_{(1)\ k} \phi^1_{1,2,3} \left[m^2 \bar\phi^1_{1,2,3} + \lambda \int [\prod_{\ell=4}^{6} dg_\ell] \ \phi^2_{3,4,5} \phi^3_{5,2,6} \phi^4_{6,4,1} \right] + (\phi \leftrightarrow \bar\phi) \Big\} \\ + \int [\prod_{\ell=1,4,6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \Big\{ X^k \partial_{(1)\ k} \phi^4_{6,4,1} \left[- \sum_{s=1}^{3} \partial_{(s)\ j} ((\bullet)_{\check s} \sqrt{|\det \mathbf{g}_s|} \mathbf{g}_s^{jl} \partial_{(s)\ l} \bar\phi^4_{6,4,1}) \right] \\ + (\bullet)_{1,4,6} X^k \partial_{(1)\ k} \phi^4_{6,4,1} \left[m^2 \bar\phi^4_{6,4,1} + \lambda \int [\prod_{\ell \neq 1,4,6} dg_\ell] \ \phi^1_{1,2,3} \phi^2_{3,4,5} \phi^3_{5,2,6} \right] + (\phi \leftrightarrow \bar\phi) \Big\} \tag{B.23}$$

Adding the contributions to the mass term and using (B.22), we get

$$
\begin{aligned}
&W(X)S^{\rm color} = \int [\prod_\ell d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \Big\{ \delta_X \mathbf{g}_1^{ij} \Big[(\bullet)_{\check{1}} \frac{\delta\sqrt{|\det \mathbf{g}|_1}}{\delta \mathbf{g}_1^{ij}} \mathcal{L}^{\rm color} + (\bullet) \frac{\delta \mathcal{L}^{\rm color}_{\rm kin}}{\delta \mathbf{g}_1^{ij}} \Big] \\
&+\Big\{ - X^k \sum_{s=1}^{3} \partial_{(s)\ j} \big[\partial_{(1)\ k} \phi^1_{1,2,3}\ (\bullet) \mathbf{g}_s^{jl} \partial_{(s)\ l} \bar\phi^1_{1,2,3} \big] \\
&-X^k \sum_{s=1,4,6} \partial_{(s)\ j} \big[\partial_{(1)\ k} \phi^4_{6,4,1}\ (\bullet) \mathbf{g}_s^{jl} \partial_{(s)\ l} \bar\phi^4_{6,4,1} \big] \\
&+X^k \sum_{s=1}^{3} (\bullet) \mathbf{g}_s^{jl} \partial_{(1)\ k} \big[\partial_{(s)\ j} \phi^1_{1,2,3}\ \partial_{(s)\ l} \bar\phi^1_{1,2,3} \big] \qquad \text{(B.24)} \\
&+X^k \sum_{s=6,4,1} (\bullet) \mathbf{g}_s^{jl} \partial_{(1)\ k} \big[\partial_{(s)\ j} \phi^4_{6,4,1}\ \partial_{(s)\ l} \bar\phi^4_{6,4,1} \big] + (\phi \leftrightarrow \bar\phi) \Big\} \qquad \text{(B.25)} \\
&+(\bullet) X^k \Big[m^2 \partial_{(1)\ k} (\bar\phi^1_{1,2,3} \phi^1_{1,2,3} + \bar\phi^4_{6,4,1} \phi^4_{6,4,1}) \\
&+\Big(\lambda \partial_{(1)\ k} (\phi^1_{1,2,3} \phi^2_{3,4,5} \phi^3_{5,2,6} \phi^4_{6,4,1}) + (\phi \leftrightarrow \bar\phi) \Big) \Big] \Big\} \ ,
\end{aligned}
$$

where we used the symmetric part in $\bar\phi$ for completing the partial derivative. Expanding the metric variations, one has

$$
\begin{aligned}
&\delta_X \mathbf{g}_1^{ij} \Big[(\bullet)_{\check{1}} \frac{\delta\sqrt{|\det \mathbf{g}|_1}}{\delta \mathbf{g}_1^{ij}} \mathcal{L}^{\rm color} + (\bullet) \frac{\delta \mathcal{L}^{\rm color}}{\delta \mathbf{g}_1^{ij}} \Big] = \\
&X^k \partial_{(1)\ k} \mathbf{g}_1^{ij} \Big[-\frac{1}{2} (\bullet)_{\check{1}} \sqrt{|\det \mathbf{g}|_1} \mathbf{g}_{1\ ij} \mathcal{L}^{\rm color} \big] \qquad \text{(B.26)} \\
&+(\bullet) [\partial_{(1)\ i} \bar\phi^1_{1,2,3} \partial_{(1)\ l} \phi^1_{1,2,3} + \partial_{(1)\ i} \bar\phi^4_{6,4,1} \partial_{(1)\ l} \phi^4_{6,4,1} \Big] . \qquad \text{(B.27)}
\end{aligned}
$$

The term (B.26) completes the derivative of $(\bullet)\mathcal{L}^{\rm color}$ while the second (B.27) cancels exactly the term with $-\partial_{(1)\ k}\mathbf{g}_s^{ij}$ obtained after integrating by parts (B.24) and (B.25). We obtain

$$
\begin{aligned}
&\frac{\partial}{\partial X^\rho} W(X) S = -\delta^{\rho k} \int [\prod_\ell d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \Big\{ \\
&\Big[\sum_{s=1}^{3} \partial_{(s)\ j} \big[\partial_{(1)\ k} \phi^1_{1,2,3}\ (\bullet) \mathbf{g}_s^{jl} \partial_{(s)\ l} \bar\phi^1_{1,2,3} \big] \\
&+ \sum_{s=1,4,6} \partial_{(s)\ j} \big[\partial_{(1)\ k} \phi^4_{6,4,1}\ (\bullet) \mathbf{g}_s^{jl} \partial_{(s)\ l} \bar\phi^4_{6,4,1} \big] + (\phi \leftrightarrow \bar\phi) \Big] \\
&-\partial_{(1)\ k} \ [(\bullet) \mathcal{L}^{(1,4)}] - [\partial_{(1)\ k} \mathbf{g}_1^{ij}] [-\frac{1}{2} (\bullet)_{\check{1}} \sqrt{|\det \mathbf{g}|_1} \mathbf{g}_{1\ ij} \mathcal{L}^{(\check{1},\check{4})}] \Big\}, \qquad \text{(B.28)}
\end{aligned}
$$

where, by definition,

$$
\begin{aligned}
\mathcal{L}^{(1,4)} :=& \sum_{s=1}^{3} \mathbf{g}_s^{ij} \partial_{(s)\ i} \bar\phi^1_{1,2,3} \partial_{(s)\ j} \phi^1_{1,2,3} + \sum_{s=1,4,6} \mathbf{g}_s^{ij} \partial_{(s)\ i} \bar\phi^4_{6,4,1} \partial_{(s)\ j} \phi^4_{6,4,1} \\
&+m^2 \big[\bar\phi^1_{1,2,3} \phi^1_{1,2,3} + \bar\phi^4_{6,4,1} \phi^4_{6,4,1} \big] + \lambda\ \phi^1_{1,2,3} \phi^2_{3,4,5} \phi^3_{5,2,6} \phi^4_{6,4,1} + \bar\lambda\ \bar\phi^1_{1,2,3} \bar\phi^2_{3,4,5} \bar\phi^3_{5,2,6} \bar\phi^4_{6,4,1} \ ,
\end{aligned}
$$

$$\mathcal{L}^{(\check{1},\check{4})} := \sum_{s=3,4,5} \mathbf{g}_s^{ij} \partial_{(s)\ i}\bar{\phi}^2_{3,4,5} \partial_{(s)\ j}\phi^2_{3,4,5} + \sum_{s=5,2,6} \mathbf{g}_s^{ij} \partial_{(s)\ i}\bar{\phi}^3_{5,2,6} \partial_{(s)\ j}\phi^3_{5,2,6}$$
$$+m^2 \left[\bar{\phi}^2_{3,4,5}\bar{\phi}^2_{3,4,5} + \bar{\phi}^3_{5,2,6}\phi^3_{5,2,6}\right] . \qquad \text{(B.29)}$$

Since $\mathcal{L}^{(\check{1},\check{4})}$ does not contain the variable g_1, the last term in (B.28) computes to a surface term $\partial_{(1)\ k}[(\bullet)\mathcal{L}^{(\check{1},\check{4})}]$. Thence, the variations (B.28) can be written

$$\frac{\partial}{\partial X^\rho} W(X) S =$$
$$-\int [\prod_{\ell=1}^{6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \Big\{ \sum_{s=1}^{3} \partial_{(s)\ j}(\bullet) \mathbf{g}_s^{jl} \Big[\Big(\partial_{(1)\ \rho}\phi^1_{1,2,3}\ \partial_{(s)\ l}\bar{\phi}^1_{1,2,3} + (\phi \leftrightarrow \bar{\phi}) \Big) - \delta_{s,1} \mathbf{g}_{s\ l\rho} \mathcal{L}^{\text{color}} \Big]$$
$$+ \sum_{s=1,4,6} \partial_{(s)\ j}(\bullet) \mathbf{g}_s^{jl} \left[\partial_{(1)\ \rho}\phi^4_{6,4,1}\ \partial_{(s)\ l}\bar{\phi}^4_{6,4,1} + (\phi \leftrightarrow \bar{\phi}) \right] \Big\}. \qquad \text{(B.30)}$$

From these last lines, the EMT can be readily identified as a two-component tensor

$$\begin{aligned} T^{(1)}_{(1,s);(i,j)} &= \partial_{(1)\ i}\phi^1_{1,2,3}\ \partial_{(s)\ j}\bar{\phi}^1_{1,2,3} + \partial_{(1)\ i}\bar{\phi}^1_{1,2,3}\ \partial_{(s)\ j}\phi^1_{1,2,3} - \delta_{1,s} \mathbf{g}_{s\ ij} \mathcal{L}^{\text{color}} , \\ T^{(4)}_{(1,s);(i,j)} &= \partial_{(1)\ i}\phi^4_{6,4,1}\ \partial_{(s)\ j}\bar{\phi}^4_{6,4,1} + \partial_{(1)\ i}\bar{\phi}^4_{6,4,1}\ \partial_{(s)\ j}\phi^4_{6,4,1} . \end{aligned} \qquad \text{(B.31)}$$

Of course it is a matter of choice to put the Lagrangian term in one or the other component.
Covariant conservation - The conservation property of the EMT should be checked. We first evaluate:

$$\sum_{s=1}^{3} \nabla^j_{(s)} T^{(1)}_{(1,s);(i,j)} + \sum_{s=1,4,6} \nabla^j_{(s)} T^{(4)}_{(1,s);(i,j)} =$$
$$\sum_{s=1}^{3} \Big[\nabla_{(1)\ i}\phi^1_{1,2,3}\ \nabla^j_{(s)} \nabla_{(s)\ j}\bar{\phi}^1_{1,2,3} + \nabla_{(1)\ i}\bar{\phi}^1_{1,2,3}\ \nabla^j_{(s)} \nabla_{(s)\ j}\phi^1_{1,2,3} \Big]$$
$$+ \sum_{s=1,4,6}^{3} \Big[\nabla_{(1)\ i}\phi^4_{6,4,1} \nabla^j_{(s)} \nabla_{(s)\ j}\bar{\phi}^4_{6,4,1} + \nabla_{(1)\ i}\bar{\phi}^4_{6,4,1} \nabla^j_{(s)} \nabla_{(s)\ j}\phi^4_{6,4,1} \Big]$$
$$-\Big[m^2 \left[\nabla_{(1)\ i}\bar{\phi}^1_{1,2,3}\phi^1_{1,2,3} + \nabla_{(1)\ i}\bar{\phi}^4_{6,4,1}\phi^4_{6,4,1} \right]$$
$$+\lambda \left[\nabla_{(1)\ i}\phi^1_{1,2,3} \right] \phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1} + \lambda\ \phi^1_{1,2,3}\phi^2_{3,4,5}\phi^3_{5,2,6} [\nabla_{(1)\ i}\phi^4_{6,4,1}] + (\phi \leftrightarrow \bar{\phi}) \Big]. \qquad \text{(B.32)}$$

Integrating first by g_4, g_5 and g_6, and using equations of motion of ϕ^1 and $\bar{\phi}_1$, we obtain

$$\int [\prod_{\ell=4}^{6} dg_\ell] \Big[\sum_{s=1}^{3} \nabla^j_{(s)} T^{(1)}_{(1,s);(i,j)} + \sum_{s=1,4,6} \nabla^j_{(s)} T^{(4)}_{(1,s);(i,j)} \Big] = \qquad \text{(B.33)}$$
$$\int [\prod_{\ell=4}^{6} dg_\ell] \Big\{ \nabla_{(1)\ i}\phi^4_{6,4,1} \Big[\sum_{s=1,4,6} \nabla^j_{(s)} \nabla_{(s)\ j}\bar{\phi}^4_{6,4,1} - m^2 \bar{\phi}^4_{6,4,1} - \lambda \int dg_5\ \phi^1_{1,2,3}\phi^2_{3,4,5}\phi^3_{5,2,6} \Big]$$
$$+(\phi \leftrightarrow \bar{\phi}) \Big\} .$$

Performing a second integration with respect to g_2 and g_3, using this time equations of motion of ϕ^4 and $\bar{\phi}^4$, one gets

$$\int [\prod_{\ell=2}^{6} dg_\ell] \Big[\sum_{s=1}^{3} \nabla^j_{(s)} T^{(1)}_{(1,s);(i,j)} + \sum_{s=1,4,6} \nabla^j_{(s)} T^{(4)}_{(1,s);(i,j)} \Big] = 0 \; . \tag{B.34}$$

The following quantity

$$\int [\prod_{\ell=2}^{6} dg_\ell][T^{(1)}_{(1,1);(i,j)} + T^{(4)}_{(1,1);(i,j)}] \tag{B.35}$$

is therefore covariantly conserved. Indeed, starting from (B.34), a calculation yields

$$\begin{aligned}
&0 = \nabla^j_{(1)} \int [\prod_{\ell=2}^{6} dg_\ell] \Big[T^{(1)}_{(1,1);(i,j)} + T^{(4)}_{(1,1);(i,j)} \Big] + \\
&\int [\prod_{\ell=2}^{6} dg_\ell] \Big[\sum_{s=2,3} \nabla^j_{(s)} T^{(1)}_{(1,s);(i,j)} + \sum_{s=4,6} \nabla^j_{(s)} T^{(4)}_{(1,s);(i,j)} \Big] \\
&0 = \nabla^j_{(1)} \int [\prod_{\ell=2}^{6} dg_\ell] \Big[T^{(1)}_{(1,1);(i,j)} + T^{(4)}_{(1,1);(i,j)} \Big] \\
&+ \sum_{s=2,3} \int \prod_{\ell=2,3} [d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \partial_{(s)\ k} [(\bullet) \mathbf{g}^{kj} T^{(1)}_{(1,s);(i,j)}] \\
&+ \sum_{s=4,6} \int \prod_{\ell=4,6} [d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \partial_{(s)\ k} [(\bullet) \mathbf{g}^{kj} T^{(4)}_{(1,s);(i,j)}] \; ,
\end{aligned} \tag{B.36}$$

where we used the fact that $\nabla_{(s)\ j}$ and $\nabla_{(s')\ i}$ commute for $s \neq s'$, and some integrations by parts for trading covariant derivatives for partial derivatives. Thus (B.35) is a conserved current.

C Group dilatations

C.1 Dilatations on the sphere S^D

Consider the sphere S^D with spherical local coordinates $(\theta, \phi_1, \phi_2, \dots, \phi_{D-1})$, and the transformation $d_a : \theta \mapsto \theta_a$ such that

$$\tan \frac{\theta_a}{2} = a \tan \frac{\theta}{2} \; . \tag{C.37}$$

Note that this transformation is invertible $(d_a)^{-1} = d_{\frac{1}{a}}$. We define the mapping on S^D

$$(\theta, \phi_1, \phi_2, \dots, \phi_{D-1}) \mapsto (y^0 = \theta_a, y^1 = \phi_1, y^2 = \phi_2, \dots, y^{D-1} = \phi_{D-1}) \tag{C.38}$$

$$\theta_a = 2 \arctan\{a \tan \frac{\theta}{2}\} \tag{C.39}$$

The mapping defines a conformal transformation of the sphere with metric tensor $\mathbf{g}$ if the metric induced by (C.38) satisfies, $\forall p \in S^D$,

$$\mathbf{g}_{\mu\nu}|_{(\theta_a,\phi_i)} \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta} = \mu^2(\theta,\phi_i)\ \mathbf{g}_{\alpha\beta}|_{(\theta,\phi_i)}\ . \tag{C.40}$$

The first component of the induced metric can be computed as, for $t = \theta/2$,

$$\mathbf{g}_{\theta\theta}|_{(\theta_a,\phi_i)} \frac{\partial \theta_a}{\partial \theta}\frac{\partial \theta_a}{\partial \theta} = 1\cdot\left(\frac{\partial \theta_a}{\partial \theta}\right)^2 = \mu^2(a,\theta) \tag{C.41}$$
$$\frac{1}{2}(1+\tan^2 t_a)d\theta_a = \frac{a}{2}(1+\tan^2 t)d\theta\ , \quad \frac{d\theta_a}{d\theta} = \frac{2a}{(1-a^2)\cos\theta + 1 + a^2} = \mu(a,\theta)\ .$$

The other metric components are of the form $\mathbf{g}_{\phi_i,\phi_i}|_{(\theta_a,\phi_i)}\cdot 1$, $i = 1,\dots,D-1$, such that one can prove that the metric tensor is conformally invariant. Indeed, the central points for that are: (1) $\sin^2\theta_a$ is a factor shared by all these components and (2) $\sin\theta_a/\sin\theta$ scales as $d\theta_a/d\theta$. Hence the relation (C.40) is verified.

C.2 Infinitesimal dilatations

Under an infinitesimal dilatation with parameter such that $a = 1+\epsilon$, $\delta_\epsilon\theta = \epsilon\sin\theta$, a field with scaling factor c defined on a single copy of $G \simeq S^3$ transforms as

$$\delta_\epsilon\phi(g) = \tilde{\phi}(d_a d_{\frac{1}{a}}(g)) - \phi(g) = \left[\tfrac{2(1+\epsilon)}{(1-(1+2\epsilon))\cos[\theta-\epsilon\sin\theta]+1+(1+2\epsilon)}\right]^c \phi(\theta-\epsilon\sin\theta) - \phi(\theta)$$
$$= -\epsilon\big(-c\cos\theta + \sin\theta\partial_\theta\big)\phi(g)\ . \tag{C.42}$$

Thus $\mathcal{D}(\cdot) := [-c\cos\theta + \sin\theta\partial_\theta](\cdot)$ is the generator of this dilatation. For tensor fields, it can be shown similarly that the corresponding operator becomes

$$\delta_\epsilon\phi_{1,2,3} = \sum_{s=1}^{3}\delta_{\epsilon_s}\phi_{1,2,3} = \sum_{s=1}^{3} -\epsilon_s\big(-c\cos\theta_s + \sin\theta_s\,\partial_{(s)\ \theta}\big)\phi_{1,2,3}\ , \tag{C.43}$$
$$\mathcal{D}_{(s)} := -c\cos\theta_s + \sin\theta_s\,\partial_{(s)\ \theta}\ . \tag{C.44}$$

C.3 Dilatation current for 1D GFT

Current calculation - Consider the operator (28), we evaluate the variation of the action under this operator using the equation of motion (27):

$$\frac{\partial}{\partial\epsilon}W(\epsilon)S^{\rm scale}_{1D} = \frac{\partial}{\partial\epsilon}\int d\theta d\varphi^1 d\varphi^2\Big\{\ (-\epsilon\mathcal{D}\phi)\times$$
$$\Big[(\bullet)\frac{(\cos\theta)^2}{\sin\theta}\phi + (\bullet)\cos\theta\partial_\theta\phi - \partial_\theta[(\bullet)\cos\theta\phi] - \widetilde{\Delta}\phi + (\bullet)\lambda\sin\theta\phi^3\Big]\Big\}. \tag{C.45}$$

First, we recombine the interaction in a surface term

$$A_0 = \int d\theta d\varphi^1 d\varphi^2\Big\{\ (-\epsilon\mathcal{D}\phi)\ \Big[(\bullet)\lambda\sin\theta\phi^3\Big]\Big\} = -\epsilon\int d\theta d\varphi^1 d\varphi^2\ \partial_\theta\left[(\bullet)\frac{\lambda}{4}(\sin\theta)^2\phi^4\right], \tag{C.46}$$

and then reduce the following terms

$$
\begin{aligned}
&B_0 = \int d\theta d\varphi^1 d\varphi^2 \Big\{ (-\epsilon\mathcal{D}\phi) \Big[(\bullet)\frac{(\cos\theta)^2}{\sin\theta}\phi + (\bullet)\cos\theta\partial_\theta\phi - \partial_\theta[(\bullet)\cos\theta\phi]\Big]\Big\} \\
&= -\epsilon\int d\theta d\varphi^1 d\varphi^2 \Big\{ -\partial_\theta\Big[(\bullet)\left(\cos\theta + \sin\theta\partial_\theta\right)\phi\cos\theta\phi\Big] \\
&+(\bullet)\frac{(\cos\theta)^3}{\sin\theta}\phi^2 + (\bullet)(\cos\theta)^2\partial_\theta\frac{1}{2}\phi^2 \\
&+(\bullet)\Big[\{-\cos\theta\sin\theta\phi^2 + 3(\cos\theta)^2\phi\partial_\theta\phi + \cos\theta\sin\theta\partial_\theta[\phi\partial_\theta\phi]\Big]\Big\} \\
&= \epsilon\int d\theta d\varphi^1 d\varphi^2 \Big\{\partial_\theta(\bullet)\Big[\cos\theta\{(\cos\theta + \sin\theta\partial_\theta)\phi\}\phi - \cos\theta(\cos\theta+\sin\theta\partial_\theta)\frac{1}{2}\phi^2\Big] \\
&\quad -(\bullet)(\sin\theta)^2\phi\partial_\theta\phi\Big\}\ .
\end{aligned}
\tag{C.47}
$$

Evaluating the Laplacian term, one finds:

$$
\begin{aligned}
&C_0 = \epsilon\int d\theta d\varphi^1 d\varphi^2\ \mathcal{D}\phi\ \widetilde{\Delta}\phi \\
&= \epsilon\int d\theta d\varphi^1 d\varphi^2\ \Big\{\partial_k\left\{\left[\cos\theta + \sin\theta\,\partial_\theta\,\right]\phi\ (\bullet)\sin\theta\,\mathbf{g}^{kl}\partial_l\phi\right\} \\
&-\partial_k\left\{\left[\cos\theta + \sin\theta\,\partial_\theta\,\right]\phi\right\}\ (\bullet)\sin\theta\,\mathbf{g}^{kl}\partial_l\phi\ \Big\} \\
&= \epsilon\int d\theta d\varphi^1 d\varphi^2\ \Big\{\partial_k\left\{\left[\cos\theta + \sin\theta\,\partial_\theta\right]\phi\ (\bullet)\sin\theta\,\mathbf{g}^{kl}\partial_l\phi\right\} \\
&+(\bullet)\sin\theta\phi\,\partial_\theta\phi - \partial_\theta\Big[(\bullet)\ \frac{1}{2}(\sin\theta)^2\ \mathbf{g}^{kl}\partial_k\phi\partial_l\phi\Big]\Big\}\ ,
\end{aligned}
\tag{C.48}
$$

where we use some integrations by parts and the fact that $\partial_\theta[\sin^2\theta\mathbf{g}^{kl}] = 2\delta_{k,\theta}\delta_{l,\theta}\cos\theta\sin\theta$.

One notices that in the last expression, the term which is not of the form of a surface term cancels exactly the similar expression in (C.47). Adding the three contributions A_0 (C.46), B_0 (C.47) and C_0 (C.48), we get

$$
\begin{aligned}
&\frac{\partial}{\partial\epsilon}W(\epsilon)S_{1D}^{\rm scale} = \int d\theta d\varphi^1 d\varphi^2\Big\{\partial_k\left\{(\bullet)\sin\theta\,\mathbf{g}^{kl}\left[\cos\theta + \sin\theta\,\partial_\theta\right]\phi\ \partial_l\phi\right\} \\
&-\partial_\theta\ (\bullet)\Big[\sin\theta\Big(\ \frac{1}{2}\sin\theta\ \mathbf{g}^{kl}\partial_k\phi\partial_l\phi + \frac{(\cos\theta)^2}{\sin\theta}\frac{1}{2}\phi^2 + \cos\theta\phi\partial_\theta\phi + \frac{\lambda}{4}\sin\theta\phi^4\Big) \\
&-\cos\theta\{(\cos\theta+\sin\theta\partial_\theta)\phi\}\phi\Big]\ \Big\} \\
&= \int d\theta d\varphi^1 d\varphi^2\Big\{\partial_k\left\{(\bullet)\sin\theta\,\mathbf{g}^{kl}\left[\cos\theta + \sin\theta\,\partial_\theta\right]\phi\ \partial_l\phi\right\} \\
&-\partial_\theta\ (\bullet)\Big[\ \sin\theta\mathcal{L}_{1D}^{\rm scale} - \cos\theta\{(\cos\theta+\sin\theta\partial_\theta)\phi\}\phi\Big]\ \Big\} \\
&= \int d\theta d\varphi^1 d\varphi^2\ \ \partial_k\Big\{\ (\bullet)\mathbf{g}^{kl}\Big[\sin\theta\left[\cos\theta + \sin\theta\,\partial_\theta\right]\phi\ \partial_l\phi \\
&-\mathbf{g}_{l\theta}\sin\theta\mathcal{L}_{1D}^{\rm scale} + \mathbf{g}_{l\theta}\cos\theta\phi\left(\cos\theta+\sin\theta\partial_\theta\right)\phi\Big]\Big\}\ .
\end{aligned}
\tag{C.49}
$$

The dilatation current can be written

$$
D_j \ = \ \sin\theta\left[\cos\theta + \sin\theta\,\partial_\theta\right]\phi\ \partial_j\phi + \mathbf{g}_{j\theta}\cos\theta\phi\left(\cos\theta+\sin\theta\partial_\theta\right)\phi - \mathbf{g}_{j\theta}\sin\theta\mathcal{L}_{1D}^{\rm scale}\ ,
$$

$$= \quad \partial_\theta(\sin\theta\phi)\partial_j(\sin\theta\phi) - \mathbf{g}_{j\theta}\sin\theta\mathcal{L}^{\rm scale}_{1D}\ . \tag{C.50}$$

Covariant conservation - The equation of motion for this model can be calculated further as:

$$0 = -(\bullet)\frac{\cos^2\theta - \sin^2\theta}{\sin\theta}\phi - \widetilde{\Delta}\phi + (\bullet)\lambda\sin\theta\phi^3\ . \tag{C.51}$$

We compute still using the Levi-Civita connection:

$$\begin{aligned}
&\nabla^j D_j = \big[\cos\theta + \sin\theta\,\nabla_\theta\big]\phi\,\frac{1}{(\bullet)}\widetilde{\Delta}\phi \\
&+ \big[-\sin\theta + \cos\theta\,\nabla_\theta\big]\phi\,(\sin\theta\nabla_\theta\phi) + \big(\big[\cos\theta + \sin\theta\,\nabla_\theta\big]\nabla^j\phi\big)\,(\sin\theta\nabla_j\phi) \\
&- \sin\theta\phi\,[\cos\theta + \sin\theta\nabla_\theta]\,\phi + \cos\theta(\nabla_\theta\phi)\,[\cos\theta + \sin\theta\nabla_\theta]\,\phi \\
&+ \cos\theta\phi\,[-\sin\theta + \cos\theta\nabla_\theta]\,\phi + \cos\theta\phi\,[\cos\theta + \sin\theta\nabla_\theta]\,\nabla_\theta\phi \\
&- \cos\theta\Big[\frac{1}{2}\frac{(\cos\theta)^2}{\sin\theta}\phi^2 + \cos\theta\phi\partial_\theta\phi + \frac{1}{2}\sin\theta\mathbf{g}^{kl}\nabla_k\phi\nabla_l\phi + \frac{\lambda}{4}\sin\theta\phi^4\Big] \\
&- \sin\theta\Big[\ \frac{1}{2}[(-2) - \cot^2\theta]\cos\theta\phi^2 + \frac{(\cos\theta)^2}{\sin\theta}\phi\nabla_\theta\phi \\
&+ (\nabla_\theta\phi)[\cos\theta\nabla_\theta\phi] + \phi[-\sin\theta\nabla_\theta\phi + \cos\theta\nabla_\theta\nabla_\theta\phi] \\
&+ \frac{1}{2}\nabla_\theta\{\sin\theta\mathbf{g}^{kl}\nabla_k\phi\nabla_l\phi\} + \frac{\lambda}{4}[\cos\theta\phi^4 + 4(\sin\theta\phi^3)\nabla_\theta\phi]\Big]\ .
\end{aligned} \tag{C.52}$$

Canceling the equation of motion, substituting the remaining term in $\widetilde{\Delta}$ making use of the equation of motion and performing some direct simplifications yields

$$\nabla^j D_j = \cos\theta\sin\theta\,\Big[-(\cot\theta)^2\phi^2 + \nabla_\theta\phi\ \nabla_\theta\phi + \frac{\lambda}{2}\phi^4\Big]\ , \tag{C.53}$$

which is not vanishing without further assumptions. Hence the dilatation current is not conserved as expected from a system with an explicit coordinate dependence in the Lagrangian unless the field satisfies both the equation of motion and (C.53) equated to zero. The latter statement is far to be an obvious issue or to even have nontrivial solutions for fields on $SU(2)$. In order to have a taste of that problem and for simplicity, by considering only class angle fields $\phi = \phi(\theta)$, the system to be solved is of the form

$$\left\{\begin{array}{c}
-(\cos\theta)^2\phi^2 + (\sin\theta)^2(\phi')^2 + \frac{\lambda}{2}(\sin\theta)^2\phi^4 = 0 \\[2ex]
(\cos^2\theta - \sin^2\theta)\phi + 3\cos\theta\sin\theta\phi' + \sin^2\theta\phi'' - \lambda\sin^2\theta\phi^3 = 0
\end{array}\right. \tag{C.54}$$

C.4 Dilatation current for GFT in 3D

In this section, we compute the dilatation current for a dynamical GFT in $3D$.

We need again to symmetrize the variation operator on the complex interaction, recalling that $\delta_\epsilon\phi_{1,2,3}$ assumes the form (C.43):

$$\lambda\int[\prod_{\ell=1}^{6} dg_\ell]\ \big(\delta_\epsilon\phi_{1,2,3}\bar\phi_{5,4,3}\phi_{5,2,6}\bar\phi_{1,4,6} + \delta_\epsilon\bar\phi_{1,2,3}\phi_{5,4,3}\bar\phi_{5,2,6}\phi_{1,4,6}\big) =$$

$$\frac{\lambda}{2}\int[\prod_{\ell=1}^{6}dg_\ell]\,\Big\{\,(\delta_\epsilon\phi_{1,2,3})\,\bar\phi_{5,4,3}\phi_{5,2,6}\bar\phi_{1,4,6}+(\delta_\epsilon\phi_{5,2,6})\,\phi_{1,2,3}\bar\phi_{5,4,3}\bar\phi_{1,4,6}$$

$$+\left(\delta_\epsilon\bar\phi_{5,4,3}\right)\phi_{1,2,3}\phi_{5,2,6}\bar\phi_{1,4,6}+\left(\delta_\epsilon\bar\phi_{1,4,6}\right)\phi_{1,2,3}\bar\phi_{5,4,3}\phi_{5,2,6}\Big\}$$

$$=-\frac{\lambda}{2}\int[\prod_{\ell=1}^{6}dg_\ell]\,\Big\{\sum_{s=1}^{3}\left(\epsilon_{(s)}\mathcal{D}_{(s)}\phi_{1,2,3}\right)\bar\phi_{5,4,3}\phi_{5,2,6}\bar\phi_{1,4,6}$$

$$+\sum_{s=1}^{3}\left(\epsilon_{(s)}\mathcal{D}_{(s+\alpha_s)}\bar\phi_{5,4,3}\right)\phi_{1,2,3}\phi_{5,2,6}\bar\phi_{1,4,6}+\sum_{s=1}^{3}\left(\epsilon_{(s)}\mathcal{D}_{(s+\alpha_s)}\phi_{5,2,6}\right)\bar\phi_{5,4,3}\phi_{1,2,3}\bar\phi_{1,4,6}$$

$$+\sum_{s=1}^{3}\left(\epsilon_{(s)}\mathcal{D}_{(s+\alpha_s)}\bar\phi_{1,4,6}\right)\bar\phi_{5,4,3}\phi_{5,2,6}\phi_{1,2,3}\Big\}$$

$$=-\frac{\lambda}{2}\int[\prod_{\ell=1}^{6}dg_\ell]\sum_{s=1}^{3}\epsilon_{(s)}\Big(\mathcal{D}^{(2)}_{(s)}+\mathcal{D}^{(2)}_{(s+[\alpha_s])}\Big)\phi_{1,2,3}\bar\phi_{5,4,3}\phi_{5,2,6}\bar\phi_{1,4,6}\;, \tag{C.55}$$

$$\mathcal{D}^{(2)}_{(s)}:=-2c\cos\theta_s+\sin\theta_s\,\partial_{(s)\;\theta}\;, \tag{C.56}$$

where the definition of indices α_s and $[\alpha_s]$ remains the same as in the section dealing with translations.

Let us consider the action $S^{\rm scale}_{{\rm kin},\,3D}[\phi]$ (58) without mass,[7] with $\gamma=-1$, $\beta=3/2$. The equation of motion for the field $\phi_{1,2,3}$ can be inferred as:

$$\frac{\delta S^{\rm scale}_{3D}}{\delta\phi_{1,2,3}}=\sum_{s=1}^{3}\Big\{(\bullet)_{1,2,3}\beta^2(\cos\theta_s)^2\bar\phi_{1,2,3}+(\bullet)_{1,2,3}\beta\cos\theta_s\sin\theta_s\partial_{(s)\;\theta}\bar\phi_{1,2,3} \tag{C.57}$$

$$-\beta\partial_{(s)\;\theta}\left[(\bullet)_{1,2,3}\cos\theta_s\sin\theta_s\bar\phi_{1,2,3}\right]-\widetilde\Delta_{(s)}\bar\phi_{1,2,3}\Big\}+(\bullet)_{1,2,3}\int[\prod_{\ell=4}^{6}dg_\ell]\lambda\bar\phi_{5,4,3}\phi_{5,2,6}\bar\phi_{1,4,6}\;,$$

$$(\bullet)_{1,2,3}:=\prod_{s=1}^{3}\sqrt{|\det\mathbf{g}_s|}\;,\quad\widetilde\Delta_{(s)}\phi_{1,2,3}:=\partial_{(s)\;k}\left\{(\bullet)_{1,2,3}(\sin\theta_s)^2\mathbf{g}_s^{kl}(\partial_{(s)\;l}\phi_{1,2,3})\right\},$$

where $\widetilde\Delta_{(s)}$ is again a modified Laplacian due the presence of the sine function. The functional operator for dilatations given by (53) allows us to compute the variations of the action up to a surface term:

$$\frac{\partial}{\partial\epsilon_i}W(\epsilon)S^{\rm scale}_{3D}=\frac{\partial}{\partial\epsilon_i}\int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{\left(-\sum_{s=1}^{3}\epsilon_s\mathcal{D}_s\phi_{1,2,3}\right)\times$$

$$\Big[\sum_{s=1}^{3}\Big\{(\bullet)\beta^2(\cos\theta_s)^2\bar\phi_{1,2,3}+(\bullet)\beta\cos\theta_s\sin\theta_s\partial_{(s)\;\theta}\bar\phi_{1,2,3}$$

$$-\beta\partial_{(s)\;\theta}\left[(\bullet)\cos\theta_s\sin\theta_s\bar\phi_{1,2,3}\right]-\widetilde\Delta_{(s)}\bar\phi_{1,2,3}\Big\}+(\bullet)\lambda\bar\phi_{5,4,3}\phi_{5,2,6}\bar\phi_{1,4,6}\Big]\;+(\phi\leftrightarrow\bar\phi)\Big\}.$$

By first recombining the variations of the interaction, we get an expression like (C.55):

$$A=\int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{-\sum_{s=1}^{3}\epsilon_s[\mathcal{D}_s\phi_{1,2,3}]\Big[(\bullet)\lambda\bar\phi_{5,4,3}\phi_{5,2,6}\bar\phi_{1,4,6}\Big]+(\phi\leftrightarrow\bar\phi)\Big\}$$

[7]In fact, a mass term can be included but for simplicity purpose, we do not consider a massive field.

$$= -\int [\prod_{\ell=1}^{6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \sum_{s=1}^{3} \epsilon_s (\bullet) \Big\{ \mathcal{D}^{(2)}_s + \mathcal{D}^{(2)}_{s+[\alpha_s]} \Big\} \frac{\lambda}{2} \phi_{1,2,3} \bar\phi_{5,4,3} \phi_{5,2,6} \bar\phi_{1,4,6}$$

$$= -\sum_{s=1}^{3} \epsilon_s \int [\prod_{\ell=1}^{6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \Big\{ \partial_{(s)\ \theta} + \partial_{(s+[\alpha_s])\ \theta} \Big\} [(\bullet) \frac{\lambda}{2} \phi_{1,2,3} \bar\phi_{5,4,3} \phi_{5,2,6} \bar\phi_{1,4,6}] \ . \quad \text{(C.58)}$$

Second, we treat the terms with no or an unique derivative in the kinetic part:

$$B = -\sum_{s'=1}^{3} \epsilon_{s'} \int [\prod_{\ell=1}^{6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \sum_{s=1}^{3} \Big\{$$
$$\mathcal{D}_{s'} \phi_{1,2,3} \Big[(\bullet) \beta^2 (\cos\theta_s)^2 \bar\phi_{1,2,3} - \beta \partial_{(s)\ \theta} [(\bullet) \cos\theta_s \sin\theta_s] \bar\phi_{1,2,3} \Big] + (\phi \leftrightarrow \bar\phi) \Big\}$$
$$= -\sum_{s'=1}^{3} \epsilon_{s'} \int [\prod_{\ell=1}^{6} d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell] \sum_{s=1}^{3} \Big\{$$
$$+ \Big[-\beta \partial_{(s)\ \theta} [(\bullet) \big(\beta \cos\theta_{s'} + \sin\theta_{s'}\ \partial_{(s')\ \theta} \big) \phi_{1,2,3} \cos\theta_s \sin\theta_s \bar\phi_{1,2,3}] + (\phi \leftrightarrow \bar\phi) \Big]$$
$$+ (\bullet) 3\beta^2 (\cos\theta_s)^2 \cos\theta_{s'} \bar\phi_{1,2,3} \phi_{1,2,3} + (\bullet) \beta^2 (\cos\theta_s)^2 \sin\theta_{s'} \partial_{(s')\ \theta} [\bar\phi_{1,2,3} \phi_{1,2,3}]$$
$$+ (\bullet) \delta_{s,s'} \beta \cos\theta_s \sin\theta_s \big(-2\beta \sin\theta_{s'} \bar\phi_{1,2,3} \phi_{1,2,3} \big) \quad \text{(C.59)}$$

$$+ (\bullet) \delta_{s,s'} \beta \cos\theta_s \sin\theta_s \cos\theta_{s'} (\bar\phi_{1,2,3} \partial_{(s')\ \theta} \phi_{1,2,3} + \phi_{1,2,3} \partial_{(s')\ \theta} \bar\phi_{1,2,3})$$
$$+ (\bullet) 2\beta^2 \cos\theta_s \sin\theta_s \cos\theta_{s'} (\bar\phi_{1,2,3} \partial_{(s)\ \theta} \phi_{1,2,3} + \phi_{1,2,3} \partial_{(s)\ \theta} \bar\phi_{1,2,3})$$
$$+ (\bullet) \beta \sin\theta_{s'} \cos\theta_s \sin\theta_s [\partial_{(s')\ \theta} (\bar\phi_{1,2,3} \partial_{(s)\ \theta} \phi_{1,2,3}) + \partial_{(s')\ \theta} (\phi_{1,2,3} \partial_{(s)\ \theta} \bar\phi_{1,2,3})] \Big\}. \quad \text{(C.60)}$$

Since $\partial_{(s')\ \theta}[(\bullet)\sin\theta_{s'}\cos^2\theta_s] = (\bullet)\big(3\cos\theta_{s'}\cos^2\theta_s - 2\delta_{s,s'}\sin^2\theta_{s'}\cos\theta_{s'}\big)$, the intermediate line (C.59) reduces to a surface term

$$(\bullet) 3\cos\theta_{s'} \sum_{s=1}^{3} \beta^2 (\cos\theta_s)^2 \bar\phi_{1,2,3}\phi_{1,2,3} + (\bullet)\sin\theta_{s'} \sum_{s=1}^{3} \beta^2 (\cos\theta_s)^2 \partial_{(s')\ \theta}[\bar\phi_{1,2,3}\phi_{1,2,3}] \quad \text{(C.61)}$$
$$-2(\bullet)\sin\theta_{s'} \sum_{s=1}^{3} \delta_{s,s'} \beta^2 \cos\theta_s \sin\theta_s \bar\phi_{1,2,3}\phi_{1,2,3} = \partial_{(s')\ \theta} \left[\sin\theta_{s'} (\bullet) \sum_{s=1}^{3} \beta^2 (\cos\theta_s)^2 \bar\phi_{1,2,3}\phi_{1,2,3} \right]$$

Furthermore, using

$$\partial_{(s')\ \theta}[(\bullet)\sin\theta_{s'}\cos\theta_s\sin\theta_s] = (\bullet)\left[3\cos\theta_{s'}\cos\theta_s\sin\theta_s + \delta_{s,s'}\sin\theta_{s'}(-\sin^2\theta_{s'} + \cos^2\theta_{s'})\right], \quad \text{(C.62)}$$

we have

$$3\cos\theta_{s'}(\bullet) \sum_{s=1}^{3} \beta\cos\theta_s\sin\theta_s (\bar\phi_{1,2,3}\partial_{(s)\ \theta}\phi_{1,2,3} + \phi_{1,2,3}\partial_{(s)\ \theta}\bar\phi_{1,2,3})$$
$$+ \sum_{s=1}^{3} (\bullet)\delta_{s,s'}\beta\cos\theta_s\sin\theta_s\cos\theta_{s'} (\bar\phi_{1,2,3}\partial_{(s)\ \theta}\phi_{1,2,3} + \phi_{1,2,3}\partial_{(s)\ \theta}\bar\phi_{1,2,3})$$
$$+ \sin\theta_{s'}(\bullet) \sum_{s=1}^{3} \beta\cos\theta_s\sin\theta_s \partial_{(s')\ \theta} (\bar\phi_{1,2,3}\partial_{(s)\ \theta}\phi_{1,2,3} + \phi_{1,2,3}\partial_{(s)\ \theta}\bar\phi_{1,2,3})$$

$$= \partial_{(s')\,\theta}\Big[\sin\theta_{s'}\sum_{s=1}^{3}(\bullet)\beta\cos\theta_s\sin\theta_s(\bar\phi_{1,2,3}\partial_{(s)\,\theta}\phi_{1,2,3}+\phi_{1,2,3}\partial_{(s)\,\theta}\bar\phi_{1,2,3})\Big]$$
$$+\beta\sin^3\theta_{s'}(\bar\phi_{1,2,3}\partial_{(s')\,\theta}\phi_{1,2,3}+\phi_{1,2,3}\partial_{(s')\,\theta}\bar\phi_{1,2,3})\,. \tag{C.63}$$

Hence, the quantity B can be rewritten as

$$B = -\sum_{s'=1}^{3}\epsilon_{s'}\int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{ \tag{C.64}$$
$$\Big[-\sum_{s=1}^{3}\beta\partial_{(s)\,\theta}[(\bullet)\big(\beta\cos\theta_{s'}+\sin\theta_{s'}\,\partial_{(s')\,\theta}\big)\phi_{1,2,3}\cos\theta_s\sin\theta_s\bar\phi_{1,2,3}]+(\phi\leftrightarrow\bar\phi)\Big]$$
$$+\partial_{(s')\,\theta}\Big[\sin\theta_{s'}(\bullet)\sum_{s=1}^{3}\beta^2(\cos\theta_s)^2\bar\phi_{1,2,3}\phi_{1,2,3}\Big]$$
$$+\partial_{(s')\,\theta}\Big[\sin\theta_{s'}(\bullet)\sum_{s=1}^{3}\beta\cos\theta_s\sin\theta_s\partial_{(s)\,\theta}(\bar\phi_{1,2,3}\phi_{1,2,3})\Big]+\beta\sin^3\theta_{s'}\partial_{(s')\,\theta}(\bar\phi_{1,2,3}\phi_{1,2,3})\Big\}.$$

Last, the Laplacian terms have to be calculated as follows:

$$C = \sum_{s'=1}^{3}\epsilon_{s'}\int[\prod_{\ell=1}^{3}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\sum_{s=1}^{3}\Big[\mathcal{D}_{s'}\phi_{1,2,3}\;\widetilde\Delta_{(s)}\bar\phi_{1,2,3}\;+(\phi\leftrightarrow\bar\phi)\Big]$$
$$= \sum_{s'=1}^{3}\epsilon_{s'}\int[\prod_{\ell=1}^{3}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\sum_{s=1}^{3}\Big\{$$
$$\Big[\partial_{(s)\,k}\big\{\big[\beta\cos\theta_{s'}+\sin\theta_{s'}\,\partial_{(s')\,\theta}\big]\phi_{1,2,3}\;(\bullet)(\sin\theta_s)^2\,\mathbf{g}^{kl}_s\partial_{(s)\,l}\bar\phi_{1,2,3}\big\}$$
$$+(\bullet)\delta_{s,s'}\delta_{k,\theta}\beta(\sin\theta_{s'})^3\phi_{1,2,3}\,\mathbf{g}^{kl}_s\partial_{(s)\,l}\bar\phi_{1,2,3}$$
$$-(\bullet)\beta\cos\theta_{s'}(\sin\theta_s)^2\partial_{(s)\,k}\phi_{1,2,3}\,\mathbf{g}^{kl}_s\partial_{(s)\,l}\bar\phi_{1,2,3}$$
$$-(\bullet)\delta_{s,s'}\delta_{k,\theta}\cos\theta_{s'}(\sin\theta_s)^2\partial_{(s)\,\theta}\phi_{1,2,3}\,\mathbf{g}^{kl}_s\partial_{(s)\,l}\bar\phi_{1,2,3}+(\phi\leftrightarrow\bar\phi)\Big]$$
$$-(\bullet)\sin\theta_{s'}\partial_{(s')\,\theta}[\sin^2\theta_s\,\mathbf{g}^{kl}_s\partial_{(s)\,k}\phi_{1,2,3}\partial_{(s)\,l}\bar\phi_{1,2,3}]$$
$$+(\bullet)\sin\theta_{s'}\partial_{(s')\,\theta}[\sin^2\theta_s\,\mathbf{g}^{kl}_s]\partial_{(s)\,k}\phi_{1,2,3}\partial_{(s)\,l}\bar\phi_{1,2,3}\;\Big\}. \tag{C.65}$$

The identity $\partial_{(s')\,\theta}(\sin^2\theta_s\mathbf{g}^{kl}_s)=\delta_{s,s'}\delta_{k,\theta}\delta_{l,\theta}2\cos\theta_{s'}\sin\theta_{s'}$, allows one to rewrite (C.65) as

$$C = \sum_{s'=1}^{3}\epsilon_{s'}\int[\prod_{\ell=1}^{3}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{ \tag{C.66}$$
$$\Big[\sum_{s=1}^{3}\partial_{(s)\,k}\big\{\big[\beta\cos\theta_{s'}+\sin\theta_{s'}\,\partial_{(s')\,\theta}\big]\phi_{1,2,3}\;(\bullet)(\sin\theta_s)^2\mathbf{g}^{kl}_s\partial_{(s)\,l}\bar\phi_{1,2,3}\big\}+(\phi\leftrightarrow\bar\phi)\Big]$$
$$-\partial_{(s')\,\theta}\left((\bullet)\sin\theta_{s'}\sum_{s=1}^{3}\sin^2\theta_s\mathbf{g}^{kl}_s\partial_{(s)\,k}\phi_{1,2,3}\partial_{(s)\,l}\bar\phi_{1,2,3}\right)+(\bullet)\beta(\sin\theta_{s'})^3\partial_{(s')\,\theta}(\bar\phi_{1,2,3}\phi_{1,2,3})\,.$$

The non-like surface term appearing in (C.66) cancels the extra term appearing in (C.63). Summing all contributions, A (C.58), B (C.64) and C (C.66) affords

$$\frac{\partial}{\partial\epsilon_q}W(\epsilon)S^{\rm scale}_{3D} = \int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{$$

$$
\begin{aligned}
&\Big[\sum_{s=1}^{3}\partial_{(s)\ k}\left\{(\bullet)(\sin\theta_s)^2[\,\beta\cos\theta_q+\sin\theta_q\,\partial_{(q)\ \theta}\,]\phi_{1,2,3}\ \mathbf{g}_s^{kl}\partial_{(s)\ l}\bar\phi_{1,2,3}\right\}\\
&+\sum_{s=1}^{3}\beta\partial_{(s)\ \theta}[(\bullet)\cos\theta_s\sin\theta_s\big(\,\beta\cos\theta_q+\sin\theta_q\,\partial_{(q)\ \theta}\big)\phi_{1,2,3}\ \bar\phi_{1,2,3}]+(\phi\leftrightarrow\bar\phi)\Big]\\
&-\partial_{q\ \theta}[(\bullet)\sin\theta_q\mathcal{L}^{\rm scale}_{3D}]-\partial_{q+[\alpha_q]\ \theta}[(\bullet)\sin\theta_{q+[\alpha_q]}\mathcal{L}_{\rm int,\,3D}]\Big\}\\
&=\int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{\\
&\Big[\sum_{s=1}^{3}\partial_{(s)\ k}(\bullet)\mathbf{g}_s^{kl}\Big\{(\sin\theta_s)^2[\,\beta\cos\theta_q+\sin\theta_q\,\partial_{(q)\ \theta}\,]\phi_{1,2,3}\ \partial_{(s)\ l}\bar\phi_{1,2,3}\\
&+\beta\mathbf{g}_{s\ l\theta}\cos\theta_s\sin\theta_s[\,\beta\cos\theta_q+\sin\theta_q\,\partial_{(q)\ \theta}]\phi_{1,2,3}\ \bar\phi_{1,2,3}+(\phi\leftrightarrow\bar\phi)\Big]\\
&-\delta_{q,s}\mathbf{g}_{s\ l\theta}\ \sin\theta_q\mathcal{L}^{\rm scale}_{3D}-\delta_{q+[\alpha_q],s}\mathbf{g}_{s\ l\theta}\ \sin\theta_{q+[\alpha_q]}\mathcal{L}_{\rm int,\,3D}\Big\}\ . \qquad\qquad (C.67)
\end{aligned}
$$

The current for this symmetry becomes a stranded tensor expressed by

$$
\begin{aligned}
&D_{(s,s');\ j}=\\
&\sin\theta_s\Big\{\sin\theta_{s'}^{\frac12}\Big[\partial_{(s)\ \theta}\phi_{1,2,3}\partial_{(s')\ j}(\sin\theta^\beta_{s'}\bar\phi_{1,2,3})+\partial_{(s)\ \theta}\bar\phi_{1,2,3}\partial_{(s')\ j}(\sin\theta^\beta_{s'}\phi_{1,2,3})\Big] \qquad (C.68)\\
&-\delta_{s,s'}\mathbf{g}_{s'\ j\theta}\mathcal{L}^{\rm scale}_{3D}\Big\}-\delta_{s+[\alpha_s],s'}\mathbf{g}_{s'\ j\theta}\ \sin\theta_{s+[\alpha_s]}\mathcal{L}_{\rm int,\,3D}+\beta\cos\theta_s\partial_{(s')\ j}\left((\sin\theta_{s'})^2\bar\phi_{1,2,3}\phi_{1,2,3}\right).
\end{aligned}
$$

Again due to both the presence of the nonlocal interaction and the explicit coordinate appearance in the Lagrangian, the dilatation current is not covariantly conserved.

C.5 Dilatation current for the colored model

Current calculation - We start by giving the equations of motion for the fields ϕ^1 and ϕ^4, using $\mathcal{L}^{\rm color,scale}=\mathcal{L}^{{\rm color,scale}(1,4)}+\mathcal{L}^{{\rm color,scale}(\check1,\check4)}$ in the form (73), with $-c=\beta=3/2$, with

$$
\begin{aligned}
&\mathcal{L}^{{\rm color,scale}(1,4)}=\\
&(\sin\theta_1)^{-1}\mathbf{g}_1^{ij}\partial_{(1)\ i}[(\sin\theta_1)^\beta\bar\phi^1_{1,2,3}]\partial_{(1)\ j}[(\sin\theta_1)^\beta\phi^1_{1,2,3}]+\sum_{s=2}^{3}\mathbf{g}_s^{ij}\partial_{(s)\ i}\bar\phi^1_{1,2,3}\partial_{(s)\ j}\phi^1_{1,2,3}\\
&+(\sin\theta_1)^{-1}\mathbf{g}_1^{ij}\partial_{(1)\ i}[(\sin\theta_1)^\beta\bar\phi^4_{6,4,1}]\partial_{(1)\ j}[(\sin\theta_1)^\beta\phi^4_{6,4,1}]+\sum_{s=4,6}\mathbf{g}_s^{ij}\partial_{(s)\ i}\bar\phi^4_{6,4,1}\partial_{(s)\ j}\phi^4_{6,4,1}\\
&+\lambda\ \phi^1_{1,2,3}\phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1}+\bar\lambda\ \bar\phi^1_{1,2,3}\bar\phi^2_{3,4,5}\bar\phi^3_{5,2,6}\bar\phi^4_{6,4,1}\ ,\\
&\mathcal{L}^{{\rm color,scale}(\check1,\check4)}=\sum_{s=3,4,5}\mathbf{g}_s^{ij}\partial_{(s)\ i}\bar\phi^2_{3,4,5}\partial_{(s)\ j}\phi^2_{3,4,5}+\sum_{s=5,2,6}\mathbf{g}_s^{ij}\partial_{(s)\ i}\bar\phi^3_{5,2,6}\partial_{(s)\ j}\phi^3_{5,2,6}\ . \qquad (C.69)
\end{aligned}
$$

The equation of motion obtained for $\phi^1_{1,2,3}$ is

$$
\frac{\delta S^{\rm color,scale}}{\delta\phi^1_{1,2,3}}=(\bullet)_{1,2,3}\beta^2(\cos\theta_1)^2\bar\phi^1_{1,2,3}+(\bullet)_{1,2,3}\beta\cos\theta_1\sin\theta_1\partial_{(1)\ \theta}\bar\phi^1_{1,2,3}
$$

$$-\beta\partial_{(1)\ \theta}\left[(\bullet)_{1,2,3}\cos\theta_1\sin\theta_1\bar\phi^1_{1,2,3}\right]-\widetilde\Delta_{(1)}\bar\phi^1_{1,2,3}-(\bullet)_{1,2,3}\sum_{s=2,3}\Delta_{(s)}\bar\phi^1_{1,2,3}$$

$$+\lambda(\bullet)_{1,2,3}\int[\prod_{\ell=4}^{6}dg_\ell]\phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1}\ , \qquad (C.70)$$

$$(\bullet)_{a,b,c}:=\prod_{s=a,b,c}\sqrt{|\det\mathbf{g}_s|}\ ,\quad \widetilde\Delta_{(1)}\phi_{1,2,3}:=\partial_{(1)\ k}\left\{(\bullet)_{1,2,3}(\sin\theta_1)^2\mathbf{g}_1^{kl}(\partial_{(1)\ l}\phi_{1,2,3})\right\}.$$

The equation of motion of ϕ^4 and complex conjugate fields are therefore obvious from (C.70). The functional operator for dilatations is given by (74) where the infinitesimal field variations possess an unique parameter ϵ. Let us evaluate the variations of the action up to the point we obtain a surface term:

$$\frac{\partial}{\partial\epsilon}W(\epsilon)S^{\text{color,scale}}=\frac{\partial}{\partial\epsilon}(-\epsilon)\int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{$$
$$\mathcal{D}_{(1)}\phi^1_{1,2,3}\Big[(\bullet)\beta^2(\cos\theta_1)^2\bar\phi^1_{1,2,3}+(\bullet)\beta\cos\theta_1\sin\theta_1\partial_{(1)\ \theta}\bar\phi^1_{1,2,3}$$
$$-\beta\partial_{(1)\ \theta}\left[(\bullet)\cos\theta_1\sin\theta_1\bar\phi^1_{1,2,3}\right]-\widetilde\Delta_{(1)}\bar\phi^1_{1,2,3}-(\bullet)\sum_{s=2,3}\Delta_{(s)}\bar\phi^1_{1,2,3}+(\bullet)\lambda\phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1}\Big]$$
$$+\mathcal{D}_{(1)}\phi^4_{6,4,1}\Big[(\bullet)\beta^2(\cos\theta_1)^2\bar\phi^4_{6,4,1}+(\bullet)\beta\cos\theta_1\sin\theta_1\partial_{(1)\ \theta}\bar\phi^4_{6,4,1}$$
$$-\beta\partial_{(1)\ \theta}\left[(\bullet)\cos\theta_1\sin\theta_1\bar\phi^4_{6,4,1}\right]-\widetilde\Delta_{(1)}\bar\phi^4_{6,4,1}-(\bullet)\sum_{s=4,6}\Delta_{(s)}\bar\phi^4_{6,4,1}+(\bullet)\lambda\phi^1_{1,2,3}\phi^2_{3,4,5}\phi^3_{5,2,6}\Big]$$
$$+(\phi\leftrightarrow\bar\phi)\Big\}. \qquad (C.71)$$

Following the same steps as in Appendix C.4, the variations of the interaction can be recombined as

$$A\ =\ -\epsilon\int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell](\bullet)\Big[\mathcal{D}^{(2)}_{(1)}[\lambda\phi^1_{1,2,3}\phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1}]+(\phi\leftrightarrow\bar\phi)\Big] \qquad (C.72)$$
$$=\ -\epsilon\int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{\partial_{(1)\ \theta}\{(\bullet)\sin\theta_1\ [\lambda\phi^1_{1,2,3}\phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1}]\}+(\phi\leftrightarrow\bar\phi)\Big\}.$$

Second, we treat the terms with no or a single derivative in the kinetic part:

$$B=-\epsilon\int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{$$
$$\Big[\Big(\mathcal{D}_{(1)}\phi_{1,2,3}\Big[(\bullet)\beta^2(\cos\theta_1)^2\bar\phi^1_{1,2,3}-\beta\partial_{(1)\ \theta}[(\bullet)\cos\theta_1\sin\theta_1]\bar\phi^1_{1,2,3}\Big]$$
$$+(\phi^1_{1,2,3}\leftrightarrow\phi^4_{6,4,1})\Big)\ +\ (\phi\leftrightarrow\bar\phi)\Big]\Big\}$$
$$=-\epsilon\int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{$$
$$\Big[-\beta\partial_{(1)\ \theta}[(\bullet)\big(\beta\cos\theta_1+\sin\theta_1\,\partial_{(1)\ \theta}\big)\phi^1_{1,2,3}\cos\theta_1\sin\theta_1\bar\phi^1_{1,2,3}]+(\phi\leftrightarrow\bar\phi)\Big]$$

$$
\begin{aligned}
&+\partial_{(1)\ \theta}\Big[\beta^2(\bullet)\sin\theta_1(\cos\theta_1)^2\bar\phi^1_{1,2,3}\phi^1_{1,2,3}\Big]\\
&+\partial_{(1)\ \theta}\Big[\beta(\bullet)\cos\theta_1(\sin\theta_1)^2\partial_{(1)\ \theta}(\bar\phi_{1,2,3}\phi^1_{1,2,3})\Big]+\beta(\bullet)\sin^3\theta_1\partial_{(1)\ \theta}(\bar\phi^1_{1,2,3}\phi^1_{1,2,3})\\
&+(\phi^1_{1,2,3}\leftrightarrow\phi^4_{6,4,1})\Big\}. \qquad \text{(C.73)}
\end{aligned}
$$

Last, the Laplacian terms have to be calculated following the same steps as done for the case without color. We find:

$$
\begin{aligned}
&C=\epsilon\int[\prod_{\ell=1}^{3}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{\\
&\Big[\mathcal{D}_1\phi^1_{1,2,3}\ \widetilde{\Delta}_{(1)}\bar\phi^1_{1,2,3}+(\bullet)_{1,2,3}\sum_{s=2,3}\mathcal{D}_1\phi^1_{1,2,3}\ \Delta_{(s)}\bar\phi^1_{1,2,3}+(\phi\leftrightarrow\bar\phi)\Big]\ +(\phi^1_{1,2,3}\leftrightarrow\phi^4_{6,4,1})\Big\}\\
&=\epsilon\int[\prod_{\ell=1}^{3}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{\\
&\Big[\partial_{(1)\ k}\left\{[\,\beta\cos\theta_1+\sin\theta_1\,\partial_{(1)\ \theta}\,]\phi^1_{1,2,3}\ (\bullet)(\sin\theta_1)^2\ \mathbf{g}_1^{kl}\partial_{(1)\ l}\bar\phi^1_{1,2,3}\right\}\\
&+\sum_{s=2,3}\partial_{(s)\ k}[(\beta\cos\theta_1+\sin\theta_1\partial_{(1)\ \theta})\phi^1_{1,2,3}\ (\bullet)\mathbf{g}_s^{kl}\partial_{(s)\ l}\bar\phi^1_{1,2,3}]+(\phi\leftrightarrow\bar\phi)\Big]\\
&+(\bullet)\beta(\sin\theta_1)^3\partial_{(1)\ \theta}(\phi^1_{1,2,3}\ \bar\phi^1_{1,2,3})\\
&-\partial_{(1)\ \theta}\Big[\sum_{s=2,3}(\bullet)\sin\theta_1\mathbf{g}_s^{kl}\partial_{(s)\ k}\phi^1_{1,2,3}\ \partial_{(s)\ l}\bar\phi^1_{1,2,3}\Big]\\
&-\partial_{(1)\ \theta}\Big[(\bullet)\sin\theta_1[(\sin\theta_1)^2\ \mathbf{g}_1^{kl}\partial_{(1)\ k}\phi^1_{1,2,3}\partial_{(1)\ l}\bar\phi^1_{1,2,3}]\Big]+(\phi^1_{1,2,3}\leftrightarrow\phi^4_{6,4,1})\ \Big\}\ . \qquad \text{(C.74)}
\end{aligned}
$$

and again the non-like surface term in (C.74) cancels the extra term in (C.73). By adding all contributions, A (C.72), B (C.73) and C (C.74), one writes

$$
\begin{aligned}
&\frac{\partial}{\partial\epsilon}W(\epsilon)S^{\text{color,scale}}=\int[\prod_{\ell=1}^{6}d\theta_\ell d\varphi^1_\ell d\varphi^2_\ell]\Big\{\\
&\partial_{(1)\ k}(\bullet)\mathbf{g}_1^{kl}\Big\{\Big[\Big((\sin\theta_1)^2[\,\beta\cos\theta_1+\sin\theta_1\,\partial_{(1)\ \theta}\,]\phi^1_{1,2,3}\ \partial_{(1)\ l}\bar\phi^1_{1,2,3}\\
&+\beta\mathbf{g}_{1\ l\theta}\cos\theta_1\sin\theta_1[\,\beta\cos\theta_1+\sin\theta_1\,\partial_{(1)\ \theta}]\phi^1_{1,2,3}\bar\phi^1_{1,2,3}+(\phi\leftrightarrow\bar\phi)\Big)\\
&+(\phi^1_{1,2,3}\leftrightarrow\phi^4_{6,4,1})\Big]-\mathbf{g}_{1\ l\theta}\sin\theta_1\mathcal{L}^{\text{color,scale}(1,4)}\Big\}\\
&+\sum_{s=2,3}\partial_{(s)\ k}(\bullet)\mathbf{g}_s^{kl}\Big[\Big([\beta\cos\theta_1+\sin\theta_1\partial_{(1)\ \theta}]\phi^1_{1,2,3}\ \partial_{(s)\ l}\bar\phi^1_{1,2,3}]+(\phi\leftrightarrow\bar\phi)\Big)\\
&+(\phi^1_{1,2,3}\leftrightarrow\phi^4_{6,4,1})\Big]\Big\}. \qquad \text{(C.75)}
\end{aligned}
$$

The current tensor for this symmetry possesses the distinct components:

$$
\begin{aligned}
D^{(1)}_{(1);\ j}\ =\ &\Big[(\sin\theta_1)^2[\,\beta\cos\theta_1+\sin\theta_1\,\partial_{(1)\ \theta}\,]\phi^1_{1,2,3}\ \partial_{(1)\ j}\bar\phi^1_{1,2,3}\\
&+\beta\mathbf{g}_{1\ j\theta}\cos\theta_1\sin\theta_1[\,\beta\cos\theta_1+\sin\theta_1\,\partial_{(1)\ \theta}]\phi^1_{1,2,3}\ \bar\phi^1_{1,2,3}+(\phi\leftrightarrow\bar\phi)\Big]\\
&-\mathbf{g}_{1\ j\theta}\ \sin\theta_1\mathcal{L}^{\text{color,scale}(1,4)}\ ,
\end{aligned}
$$

$$D^{(1)}_{(s);\,j} \;=\; [\,\beta\cos\theta_1 + \sin\theta_1\,\partial_{(1)\,\theta}\,]\phi^1_{1,2,3}\;\partial_{(s)\,j}\bar\phi^1_{1,2,3} \;+\; (\phi\leftrightarrow\bar\phi)\;, \quad s=2,3, \qquad (C.76)$$

and the other components $D^{(4)}_{(s);\,j}$ can be obtained from $D^{(1)}_{(1);\,j}$ and $D^{(1)}_{(s=2,3);\,j}$ by taking the symmetry $(\phi^1_{1,2,3}\leftrightarrow\phi^4_{6,4,1})$ and omitting the Lagrangian part. We can rewrite the dilatation tensor component $D^{(1)}_{(1);\,j}$ in the more compact form:

$$\begin{aligned} D^{(1)}_{(1);\,j} \;=\;& \partial_{(1)\,\theta}[(\sin\theta_1)^{\beta}\bar\phi^1_{1,2,3}]\,\partial_{(1)\,j}[(\sin\theta_1)^{\beta}\phi^1_{1,2,3}] + \partial_{(1)\,\theta}[(\sin\theta_1)^{\beta}\phi^1_{1,2,3}]\,\partial_{(1)\,j}[(\sin\theta_1)^{\beta}\bar\phi^1_{1,2,3}] \\ -\;& \mathbf{g}_{1\,j\theta}\sin\theta_1\mathcal{L}^{\text{color,scale}(1,4)}\;. \qquad (C.77)\end{aligned}$$

Covariant conservation - We write the equation of motion for the color 1 field as

$$\begin{aligned} 0 = \beta\left[-\beta\cos\theta_1^2+\sin^2\theta\right]\bar\phi^1_{1,2,3} - \frac{1}{(\bullet)_{1,2,3}}\widetilde\Delta_{(1)}\bar\phi^1_{1,2,3} - \sum_{s=2,3}\Delta_{(s)}\bar\phi^1_{1,2,3} \\ +\lambda\int[\prod_{\ell=4}^{6}dg_\ell]\phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1}\;.\end{aligned}$$

In a covariant form, we evaluate

$$\begin{aligned}
&\sum_{s=1,2,3}\nabla^j_{(s)}D^{(1)}_{(s)\,j} + \sum_{s=1,4,6}\nabla^j_{(s)}D^{(4)}_{(s)\,j} = \\
&\nabla^j_{(1)}\Big[[\beta\cos\theta_1+\sin\theta_1\nabla_{(1)\,\theta}]\bar\phi^1_{1,2,3}\;[(\sin\theta_1)^2\nabla_{(1)\,j}\phi^1_{1,2,3}] \\
&+\beta\delta_{\theta j}\sin\theta_1\cos\theta_1\phi^1_{1,2,3}[\beta\cos\theta_1+\sin\theta_1\nabla_{(1)\,\theta}]\bar\phi^1_{1,2,3} + (\phi\leftrightarrow\bar\phi)\Big] \\
&+\sum_{s=2,3}\Big[[\,\beta\cos\theta_1+\sin\theta_1\,\nabla_{(1)\,\theta}\,]\nabla^j_{(s)}\phi^1_{1,2,3}\;\nabla_{(s)\,j}\bar\phi^1_{1,2,3} \\
&+[\,\beta\cos\theta_1+\sin\theta_1\,\nabla_{(1)\,\theta}\,]\phi^1_{1,2,3}\;\nabla^j_{(s)}\nabla_{(s)\,j}\bar\phi^1_{1,2,3} \;+\; (\phi\leftrightarrow\bar\phi)\Big] \\
&+\nabla^j_{(1)}\Big[[\beta\cos\theta_1+\sin\theta_1\nabla_{(1)\,\theta}]\bar\phi^4_{6,4,1}\;[(\sin\theta_1)^2\nabla_{(1)\,j}\phi^4_{6,4,1}] \\
&+\beta\delta_{\theta j}\sin\theta_1\cos\theta_1\phi^4_{6,4,1}[\beta\cos\theta_1+\sin\theta_1\nabla_{(1)\,\theta}]\bar\phi^4_{6,4,1} + (\phi\leftrightarrow\bar\phi)\Big] \\
&+\sum_{s=4,6}\Big[[\,\beta\cos\theta_1+\sin\theta_1\,\nabla_{(1)\,\theta}\,]\nabla^j_{(s)}\phi^4_{6,4,1}\;\nabla_{(s)\,j}\bar\phi^4_{6,4,1} \\
&[\,\beta\cos\theta_1+\sin\theta_1\,\nabla_{(1)\,\theta}\,]\phi^4_{6,4,1}\;\nabla_{(s)\,j}\nabla^j_{(s)}\bar\phi^4_{6,4,1} \;+\; (\phi\leftrightarrow\bar\phi)\Big] \\
&-\cos\theta_1\mathcal{L}^{(1,4)} \\
&-\sin\theta_1\Big[\\
&\Big(2\cos\theta_1\sin\theta_1\nabla^j_{(1)}\bar\phi^1_{1,2,3}\nabla_{(1)\,j}\phi^1_{1,2,3} \\
&+(\sin\theta_1)^2\mathbf{g}_1^{jk}[\nabla_{(1)\,\theta}\nabla_{(1)\,j}\bar\phi^1_{1,2,3}]\nabla_{(1)\,k}\phi^1_{1,2,3} + (\sin\theta_1)^2\mathbf{g}_1^{jk}\nabla_{(1)\,j}\bar\phi^1_{1,2,3}[\nabla_{(1)\,\theta}\nabla_{(1)\,k}\phi^1_{1,2,3}] \\
&+\beta[-(\sin\theta_1)^2+(\cos\theta_1)^2]\bar\phi^1_{1,2,3}\nabla_{(1)\,\theta}\phi^1_{1,2,3} \\
&+\beta\cos\theta_1\sin\theta_1[\nabla_{(1)\,\theta}\bar\phi^1_{1,2,3}\nabla_{(1)\,\theta}\phi^1_{1,2,3} + \bar\phi^1_{1,2,3}\nabla_{(1)\,\theta}\nabla_{(1)\,\theta}\phi^1_{1,2,3}] \\
&+\beta[-(\sin\theta_1)^2+(\cos\theta_1)^2]\phi^1_{1,2,3}\nabla_{(1)\,\theta}\bar\phi^1_{1,2,3}
\end{aligned}$$

$$
\begin{aligned}
&+\beta\cos\theta_1\sin\theta_1[\nabla_{(1)\;\theta}\phi^1_{1,2,3}\nabla_{(1)\;\theta}\bar{\phi}^1_{1,2,3}+\phi^1_{1,2,3}\nabla_{(1)\;\theta}\nabla_{(1)\;\theta}\bar{\phi}^1_{1,2,3}] \\
&+\beta^2[-2\sin\theta_1\cos\theta_1]\bar{\phi}^1_{1,2,3}\phi^1_{1,2,3}+\beta^2(\cos\theta)^2[\nabla_{(1)\;\theta}\bar{\phi}^1_{1,2,3}\phi^1_{1,2,3}+\bar{\phi}^1_{1,2,3}\nabla_{(1)\;\theta}\phi^1_{1,2,3}] \\
&+\sum_{s=2,3}[\nabla_{(1)\;\theta}\nabla^j_{(s)}\bar{\phi}^1_{1,2,3}\nabla_{(s)\;j}\phi^1_{1,2,3}+\nabla^j_{(s)}\bar{\phi}^1_{1,2,3}\nabla_{(1)\;\theta}\nabla_{(s)\;j}\phi^1_{1,2,3}] \\
&+(\phi^1\leftrightarrow\phi^4)\Big) \\
&\lambda\;[\nabla_{(1)\;\theta}\phi^1_{1,2,3}]\phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1}+\phi^1_{1,2,3}\phi^2_{3,4,5}\phi^3_{5,2,6}[\nabla_{(1)\;\theta}\phi^4_{6,4,1}] \\
&+\bar{\lambda}\;[\nabla_{(1)\;\theta}\bar{\phi}^1_{1,2,3}]\bar{\phi}^2_{3,4,5}\bar{\phi}^3_{5,2,6}\bar{\phi}^4_{6,4,1}+\bar{\phi}^1_{1,2,3}\bar{\phi}^2_{3,4,5}\bar{\phi}^3_{5,2,6}[\nabla_{(1)\;\theta}\bar{\phi}^4_{6,4,1}]\Big]
\end{aligned}
\tag{C.78}
$$

which yields after canceling equations of motion of ϕ^1, $\bar{\phi}^1$, ϕ^4 and $\bar{\phi}^4$, by integrating all variables save g_1 and trading the remaining modified Laplacian using once again the equations of motion:

$$
\begin{aligned}
&\int[\prod_{\ell=2}^{6}dg_\ell]\Big[\sum_{s=1,2,3}\nabla^j_{(s)}D^{(1)}_{(s)\;j}+\sum_{s=1,4,6}\nabla^j_{(s)}D^{(4)}_{(s)\;j}\Big]= \\
&=\int[\prod_{\ell=2}^{6}dg_\ell]\Big\{\Big[\Big(\beta\cos\theta_1\bar{\phi}^1_{1,2,3}\Big[\frac{1}{(\bullet)_{123}}\widetilde{\Delta}_{(1)}\phi^1_{1,2,3}+\sum_{s=2,3}\nabla^j_{(s)}\nabla_{(s)\;j}\phi^1_{1,2,3}\Big]+(\phi\leftrightarrow\bar{\phi})\Big) \\
&+2\cos\theta_1(\sin\theta_1)^2\nabla_{(1)\;\theta}\bar{\phi}^1_{1,2,3}\;\nabla_{(1)\;\theta}\phi^1_{1,2,3}+\beta^2\cos\theta[(\cos\theta_1)^2-2(\sin\theta_1)^2]\phi^1_{1,2,3}\bar{\phi}^1_{1,2,3} \\
&+2\cos\theta_1\sum_{s=2,3}\nabla^j_{(s)}\bar{\phi}^1_{1,2,3}\;\nabla_{(s)\;j}\phi^1_{1,2,3}\;+\;(\phi^1\leftrightarrow\phi^4)\Big]\Big\} \\
&-\Big[\lambda\cos\theta_1\phi^1_{1,2,3}\phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1}+\bar{\lambda}\cos\theta_1\bar{\phi}^1_{1,2,3}\bar{\phi}^2_{3,4,5}\bar{\phi}^3_{5,2,6}\bar{\phi}^4_{6,4,1}\Big]\Big\} \\
&=\int[\prod_{\ell=2}^{6}dg_\ell]\Big\{\Big[2\cos\theta_1(\sin\theta_1)^2\nabla_{(1)\;\theta}\bar{\phi}^1_{1,2,3}\;\nabla_{(1)\;\theta}\phi^1_{1,2,3} \\
&+2\cos\theta_1\sum_{s=2,3}\nabla^j_{(s)}\bar{\phi}^1_{1,2,3}\;\nabla_{(s)\;j}\phi^1_{1,2,3}-\frac{9}{2}(\cos\theta_1)^3\bar{\phi}^1_{1,2,3}\phi^1_{1,2,3}\;+\;(\phi^1\leftrightarrow\phi^4)\Big] \\
&+\frac{1}{2}\Big[\lambda\cos\theta_1\phi^1_{1,2,3}\phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1}+\bar{\lambda}\cos\theta_1\bar{\phi}^1_{1,2,3}\bar{\phi}^2_{3,4,5}\bar{\phi}^3_{5,2,6}\bar{\phi}^4_{6,4,1}\Big]\Big\}.
\end{aligned}
\tag{C.79}
$$

One can compare the latter expression with the breaking (C.53) for the $1D$ case and discover than they have in fact the same structure.

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