
Gauge Theories and Fiber Bundles

Applications to Particle Dynamics

by

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1 INTRODUCTION

A theory defined by an action which is invariant under a time-dependent group of transformations can be called a gauge theory. Well known examples of such theories are those defined by the Maxwell and Yang-Mills Lagrangians. It is widely believed nowadays that the fundamental laws of physics have to be formulated in terms of gauge theories.

The underlying mathematical structures of gauge theories are known to be geometrical in nature and the local and global features of this geometry have been studied for a long time in mathematics under the name of fibre bundles. It is now understood that the global properties of gauge theories can have a profound influence on physics. For example, instantons and monopoles are both consequences of properties of geometry in the large, and the former can lead to, e.g., CP violation, while the latter can lead to such remarkable results as the creation of fermions out of bosons. Some familiarity with global differential geometry and fibre bundles seems therefore very desirable to a physicist who works with gauge theories. One of the purposes of the present work is to introduce the physicist to these disciplines using simple examples.

There exists a certain amount of literature written by general relativists and particle physicists which attempts to explain the language and techniques of fibre bundles. Generally, however, in these admirable reviews, the concepts are illustrated by field theoretic examples like the gravitational and the Yang-Mills systems. This practice tends to create the impression that the subtleties of gauge invariance can be understood only through the medium of complicated field theories. Such an impression, however, is false and simple systems with gauge invariance occur in plentiful quantities in the mechanics of point particles and extended objects. Further, it is often the case that the large scale properties of geometry play an essential role in determining the physics of these systems. They are thus ideal to commence studies of gauge theories from a geometrical point of view. Besides, such systems have an intrinsic physical interest as they deal with particles with spin, interacting charges and monopoles, particles in Yang-Mills fields, etc.. We shall present an exposition of these systems and use them to introduce the reader to the mathematical concepts which underlie gauge theories. Many of these examples are known to exponents of geometric quantization, but we suspect that, due in part to mathematical difficulties, the wide community of physicists is not very familiar with their publications. We admit that our own acquaintance with these publications is slight. If we are amiss in giving proper credit, the reason is ignorance and not deliberate intent.

The matter is organized as follows. After a brief introduction to the concept of gauge

invariance and its relationship to determinism in Section 2, we introduce in Chapters 3 and 4 the notion of fibre bundles in the context of a discussion on spinning point particles and Dirac monopoles. The fibre bundle language provides for a singularity-free global description of the interaction between a magnetic monopole and an electrically charged test particle. Chapter 3 deals with a non-relativistic treatment of the spinning particle. The non-trivial extension to relativistic spinning particles is dealt with in Chapter 5. The free particle system as well as interactions with external electromagnetic and gravitational fields are discussed in detail. In Chapter 5 we also elaborate on a remarkable relationship between the charge-monopole system and the system of a massless particle with spin. The classical description of Yang-Mills particles with internal degrees of freedom, such as isospinor colour, is given in Chapter 6. We apply the above in a discussion of the classical scattering of particles off a 't Hooft-Polyakov monopole. In Chapter 7 we elaborate on a Kaluza-Klein description of particles with internal degrees of freedom. The canonical formalism and the quantization of most of the preceding systems are discussed in Chapter 8. The dynamical systems given in Chapters 3-7 are formulated on group manifolds. The procedure for obtaining the extension to super-group manifolds is briefly discussed in Chapter 9. In Chapter 10, we show that if a system admits only local Lagrangians for a configuration space Q , then under certain conditions, it admits a global Lagrangian when Q is enlarged to a suitable $U(1)$ bundle over Q . Conditions under which a symplectic form is derivable from a Lagrangian are also found.

The list of references cited in the text is, of course, not complete, but it is instead intended to be a guide to the extensive literature in the field.

2 THE MEANING OF GAUGE INVARIANCE

Below we will deal with systems which exhibit a gauge symmetry. It is thus useful to clarify the distinction between ordinary symmetries and gauge symmetries at the beginning.

2.1 The Action

The action S is a functional of fields with values in a suitable range space. The domain of the fields is a suitable parameter space. Thus for a non-relativistic particle, the range space may be \mathbf{R}^3 , a point of which denotes the coordinate of the particle. The parameter space is \mathbf{R} , a point of which denotes an instant of time. The fields are functions from \mathbf{R} to \mathbf{R}^3 :

$$\mathcal{F}(\mathbf{R}, \mathbf{R}^3) = q, \quad q = (q_1, q_2, q_3), \quad q(t) \in \mathbf{R}^3. \quad (2.1)$$

Thus each field q assigns a point $q(t)$ in \mathbf{R}^3 to each instant of time t .

For a real scalar field theory in Minkowski space M^4 , the parameter space is M^4 , the range space is \mathbf{R} and the set of fields $\mathcal{F}(\mathbf{R}^4, \mathbf{R})$ is the set of functions from \mathbf{R}^4 to \mathbf{R} .

Let us denote the parameter space by D , the range space by R and the set of fields by $\mathcal{F}(D, R)$. Then the action S is a functional on $\mathcal{F}(D, R)$. It assigns to each $f \in \mathcal{F}(D, R)$ a number $S[f]$. For instance, in the non-relativistic example cited above,

$$S[f] = \frac{m}{2} \int dt \frac{dq_i(t)}{dt} \frac{dq_i(t)}{dt}. \quad (2.2)$$

The action also depends on the limits of the time integration. Since these limits are not important for us, they have here been ignored. If necessary, they can be introduced by restricting D suitably. In this case, for example, instead of \mathbf{R} , we can choose for D the interval $t_1 \leq t \leq t_2$.

The concept of a global symmetry group $G = \{g\}$ may be defined as follows: Suppose G is a group with a specified action $r \rightarrow gr$ on $R \equiv \{r\}$. Then, G has a natural action $f \rightarrow gf$ on $\mathcal{F}(D, R)$, where $(gf)(t) = gf(t)$. This group of transformations on $\mathcal{F}(D, R)$ is the global group associated with G . We denote it by the same symbol G . We say further that G is a global symmetry group if

$$S[f] = S[gf], \quad (2.3)$$

up to surface terms. For reasons of simplicity we shall assume hereafter that G is a Lie group.

As an example, consider the non-relativistic free particle with $D = \{t \mid -\infty < t < \infty\}$, $R = \mathbf{R}^3$ and $G = SO(3)$. The rotation group has a standard action on \mathbf{R}^3 . It can be "lifted" to the action $q \rightarrow gq$ on $\mathcal{F}(D, R)$, where

$$[gq](t) = gq(t) \quad [\equiv (g_{ij}q_j(t))] . \quad (2.4)$$

Thus in the usual language, g is a global rotation. Further, $SO(3)$ is a global symmetry group since for the action (2.4)

$$S[q] = S[gq] . \quad (2.5)$$

In contrast, the gauge group \mathcal{G} associated with G is defined to be the set of all functions from D to G , i.e. $\mathcal{G} = \mathcal{F}(D, G) = \{h\}$, where for $d \in D$, $d \xrightarrow{h} h(d) \in G$. The group multiplication in \mathcal{G} is defined by $(hh')(d) = h(d)h'(d)$. This group, as well, has a natural action on $\mathcal{F}(D, R)$, i.e., $(hf)(d) = h(d)f(d)$. If S is invariant under \mathcal{G} (up to surface terms), i.e., $S[f] = S[hf] + \text{possible surface terms}$, then the gauge group is a gauge symmetry group.

It is possible that the sort of boundary conditions we impose on the set of functions in the gauge group can have serious consequence for the theory (see, e.g., Ref.[2]). If we do not impose any particular boundary conditions so that the boundary conditions are "free", \mathcal{G} will contain constant functions and the associated global group G may be thought of as the subgroup of \mathcal{G} of these constant functions.

Let \mathcal{G} be a gauge symmetry group and Γ be a global symmetry group not associated with \mathcal{G} . Now recall that the parameter space contains a parameter which we identify as time t . The profound difference between the gauge symmetry group \mathcal{G} and Γ is due to the fact that \mathcal{G} contains time-dependent symmetries unlike Γ . It affects the deterministic aspects of the theory and also has its impact on Noether's derivation of conservation laws. These twin aspects are manifested as constraints in the Hamiltonian frame work Ref.[3]. We can illustrate these remarks as follows:

a) Determinism

A trajectory, in our language, is a function $\bar{f} \in \mathcal{F}(D, R)$ such that

$$\delta S[\bar{f}] = 0 . \quad (2.6)$$

Suppose \bar{f} is a possible trajectory for a specified set of initial conditions $d^k \bar{f}/dt^k|_{t=0}$, $k = 1, 2, \dots, n$. Since \mathcal{G} is a gauge symmetry group, $h\bar{f}$ is also a trajectory. Further since

the time dependence of h is at our disposal, we can choose h such that

$$\left. \frac{d^k(h\bar{f})(t)}{dt^k} \right|_{t=0} = \left. \frac{d^k\bar{f}(t)}{dt^k} \right|_{t=0} \quad k = 0, 1, \dots, n . \quad (2.7)$$

This does not constrain h to be trivial for all time. Here we assume, of course, that \mathcal{G} acts non-trivially on fields. The conclusion is that there are several possible trajectories for specified initial conditions. In this sense, the theory does not determine the future from the present if the state of the system is given by the values of \bar{f} and its derivatives at a given time.

In the customary formulation, determinism is restored by considering only those functions which are invariant under \mathcal{G} . These gauge invariant functions and their derivatives at a given time are then defined to constitute the observables of the theory. Such a definition of observables seems to have little direct bearing on whether they are accessible to experimental observation. It is a definition which is internal to the theory and dictated by requirements of determinism.

In a Hamiltonian formulation with no constraints, the specification of Cauchy data, i.e., a point of phase space, allows us to uniquely specify the future state of the system, at least for sufficiently small times. The existence of a gauge symmetry group for the action S thus means that S should lead to a constrained Hamiltonian dynamics. An orderly way to treat such a dynamics is due to Dirac [3]. We will have occasion to use it later.

b) Conservation Laws

The infinitesimal variation of S under a gauge transformation is characterized by arbitrary functions ϵ_α . If \mathcal{G} is a symmetry group, Noether's argument shows that there is a charge formally written in the form

$$Q = \int_{\bar{D}} dt \epsilon_\alpha Q_\alpha , \quad (2.8)$$

which is a constant of motion

$$\frac{dQ}{dt} = 0 . \quad (2.9)$$

Here \bar{D} is a fixed time slice of D . Since the ϵ_α 's are arbitrary functions, we can conclude that [4]

$$Q_\alpha = 0 . \quad (2.10)$$

Thus the generators Q_α of the gauge group vanish identically.

In electro-magnetism, the analogues of Eq.(2.10) are the Gauss law

$$\nabla \cdot \mathbf{E} + j_0 = 0 . \quad (2.11)$$

and the vanishing of the canonical momentum Π^0 conjugate to A^0 . The non-Abelian generalizations of these equations are well known [5].

In the Hamiltonian framework, the equations $Q_\alpha = 0$ become first class constraints. Quantization of the system then often becomes highly non-trivial in their presence.

2.2 The Lagrangian

The configuration space is usually identified with $\mathcal{F}(\bar{D}, R)$, where \bar{D} now is a fixed time-slice of D . It is clear however that for precision we should write \bar{D}_t for the time slice at time t . The customary hypothesis is that \bar{D}_t for different t are diffeomorphic and that there is a natural identification of points of \bar{D}_t for different times. Under these circumstances, (which we now assume, we are justified in writing \bar{D}).

As an example, consider a field theory in Minkowski space M^4 . Slices at different times t give different three dimensional subspaces $\mathbf{R}_t^3 \subseteq \mathbf{R}^3$. Without further considerations, there is no natural identification of points of these spaces, that is, there is as yet no obvious meaning to the identity of spatial points for observations at different times. What is done in practice is as follows: On M^4 , there is an action of the time translation group $\{U_\tau \mid -\infty < \tau < \infty\}$. The latter maps \mathbf{R}_t^3 to $\mathbf{R}_{t+\tau}^3$ in a smooth, invertible way. We then identify all points in \mathbf{R}_t^3 and $\mathbf{R}_{t+\tau}^3$ which are carried into each other by time translation U_τ . In terms of the conventional coordinates (\mathbf{x}, t) ,

$$U_\tau(\mathbf{x}, t) = (\mathbf{x}, t + \tau) , \quad (2.12)$$

and we think of \mathbf{x} as referring to the same three dimensional point for all times.

A field $f \in \mathcal{F}(\bar{D}, R)$ restricted to a given time t is a function on \bar{D}_t . Since we have an identification of points of \bar{D}_t for different t , the field f can be regarded as a one-dimensional family of functions $f_t \in \mathcal{F}(\bar{D}, R)$ parameterized by time. We have thus established a correspondence

$$\mathcal{F}(D, R) \rightarrow \mathcal{F}(\mathbf{R}, \mathcal{F}(\bar{D}, R)) , \quad (2.13)$$

between functions appropriate to the action principle and curves in the configuration space $\mathcal{F}(\bar{D}, R)$.

The Lagrangian is a function, or more precisely a functional, of "coordinates and velocities". That is, it is a function of a point $\alpha \in \mathcal{F}(D, R)$ on the configuration space and of the tangent, denoted by $\dot{\alpha}$, to this space at this point. This new space, a point of which is a point and a tangent at the point of the configuration space, is the tangent bundle $TF(\bar{D}, R)$ on the configuration space.

When the action is reconstructed from the Lagrangian by the formula

$$S = \int dt L[\alpha(t), \dot{\alpha}(t)] , \quad (2.14)$$

we are integrating L along a curve in the tangent bundle. This curve is not arbitrary since we require that $\dot{\alpha}(t) = d\alpha(t)/dt$. Such a curve in the tangent bundle is the "lift of a curve" from the configuration space. It is defined by a "second order" vector field in the tangent bundle. With this restriction on curves, a curve on the tangent bundle is uniquely determined by a curve $\alpha_t \in \mathcal{F}(\bar{D}, R)$. Since such a curve in turn defines a function in $\mathcal{F}(D, R)$, we recover the original interpretation of the action as a functional on $\mathcal{F}(\bar{D}, R)$.

We need to investigate the action of the gauge group on the tangent bundle. It turns out that in its action on the tangent bundle, the gauge group, in its simplest version, is associated to the global group

$$G \ltimes \underline{G} = \{ (h, l) | h \in G, l \in \underline{G} \} . \quad (2.15)$$

where G is the associated global group, and \underline{G} is its Lie algebra and the group multiplication is

$$(h', l')(h, l) = (h'h, l' + adh'l) . \quad (2.16)$$

Here ad is the adjoint representation of G on \underline{G} . In the notation common in physics literature

$$adh'l = h'lh'^{-1} . \quad (2.17)$$

Thus $G \ltimes \underline{G}$ is the semi-direct product of G with \underline{G} . This result has been discussed before by Sudarshan and Mukunda [3].

We denote the gauge group associated to G at a given time by $\hat{\mathcal{G}}$. The elements of $\hat{\mathcal{G}}$ are functions $\mathcal{F}(\bar{D}, G) = \{h\}$ with group multiplication defined by

$$(hh')(\bar{d}) = h(\bar{d})h'(\bar{d}) , \bar{d} \in \bar{D} . \quad (2.18)$$

The Lie algebra \underline{G} is a group under addition and the associated gauge group at a given time is denoted by $\hat{\underline{G}}$. Finally the gauge group associated to $G \ltimes \underline{G}$ at a given time is denoted by $\hat{\mathcal{G}} \ltimes \hat{\underline{G}}$.

These remarks can be established by examining the way the action of the gauge group "projects down" to an action on coordinates and velocities. A function $f \in \mathcal{F}(D, R)$ is transformed to hf . Thus the curve at $\alpha_t \in \mathcal{F}(\bar{D}, R)$ is transformed into $(h\alpha)_t$, where h is time-dependent. Thus a point of the tangent bundle is transformed according to

$$(\alpha, \dot{\alpha}) \rightarrow (h\alpha, \frac{d(h\alpha)}{dt}) = (h\alpha, h\dot{\alpha} + l(h\alpha)) , \quad (2.19)$$

where $h \in \hat{\mathcal{G}}$, $l = \dot{h}h^{-1} \in \underline{\hat{\mathcal{G}}}$. In Eq.(2.19), the time-dependence of h and l have disappeared since we are examining the action at a point of $T\mathcal{F}(\bar{D}, R)$. In writing Eq.(2.19), we have also assumed that the action of the gauge group is local in time, that is

$$(h\alpha)_t = h(t)\alpha(t) . \quad (2.20)$$

If $(h\alpha)_t$ depends on $h(t)$ as well as (say) its derivatives $d^k h(t)/dt^k$, Eq.(2.19) will have to be modified. For Yang-Mills theories, this actually happens (see below). We prefer to illustrate the idea without this complication. With this assumption we can write

$$(h, l) \in \hat{\mathcal{G}} \times \underline{\hat{\mathcal{G}}}, (h, l)(\alpha, \dot{\alpha}) = (h\alpha, h\dot{\alpha} + l(h\alpha)) . \quad (2.21)$$

The group multiplication Eq.(2.16) follows from

$$\begin{aligned} (h', l')(h\alpha, h\dot{\alpha} + l(h\alpha)) &= (hh'\alpha, hh'\dot{\alpha} + (h'lh'^{-1})(h'h\alpha) + l'(h'h\alpha)) \\ &= (hh'\alpha, h'h\dot{\alpha} + (l' + adh'l)(h'h\alpha)) = (h'h, l' + adh'l)(\alpha, \dot{\alpha}) . \end{aligned} \quad (2.22)$$

The preceding considerations are easily illustrated by YangMills theory where the vector potential A_μ has values in the Lie algebra $\underline{\mathcal{G}}$ of the gauge group G and transforms as follows:

$$A_\mu \rightarrow hA_\mu h^{-1} + h\partial_\mu h^{-1} . \quad (2.23)$$

Thus at a fixed time

$$(h, l)A_i = hA_i h^{-1} , \quad (2.24)$$

and

$$(h, l)A_0 = hA_0 h^{-1} - l , \quad (2.25)$$

where

$$l = \dot{h}h^{-1} . \quad (2.26)$$

The group multiplication law Eq.(2.16) follows by considering the application of (h', l') to the left-hand side of Eqs. (2.24) and (2.25).

The transformation Eq.(2.25) on the configuration space variable A_0 is not local in time since (2.26) involves dh/dt . Nonetheless, the group multiplication Eq.(2.16) is unaffected.

The space on which the group is supposed to act, however, is not the space of A_μ , but of (A_μ, \dot{A}_μ) . If we consider the subspace (A_i, \dot{A}_i) , since (2.24) does not involve \dot{h} , we find the group $\hat{G} \times \hat{G}$. However, the argument has to be modified if \dot{A}_0 is considered since its transformation involves \dot{l} . An element of the gauge group is now a triple (h, l, \dot{l}) with the action

$$(h, l, \dot{l})(A_0, \dot{A}_0) = (hA_0h^{-1} - l, h\dot{A}_0h^{-1} + [l, hA_0h^{-1}] - \dot{l}) , \quad (2.27)$$

and the multiplication law

$$(h_1, l_1, \dot{l}_1)(h_2, l_2, \dot{l}_2) = (h_1h_2, l_1 + h_1l_2h_1^{-1}, \dot{l}_1 + [l_1, h_1h_2h_1^{-1}] + h_1\dot{l}_2h_1^{-1}) . \quad (2.28)$$

The action of (h, l, \dot{l}) on (A_i, \dot{A}_i) is obtained from taking the derivative of Eq.(2.24). In this action, \dot{l} is passive. The general gauge group G_L at the Lagrangian level can thus in general involve $\dot{l}, \ddot{l}, \ddot{\ddot{l}}, \dots$.

The subgroup of constant functions from \bar{D} to G is what is called the global symmetry group. Since it is isomorphic to G , we can denote it by the same symbol G . It is a subgroup of \hat{G} if there are no boundary conditions on functions in \hat{G} , that is if all constant functions are allowed in \hat{G} . Thus, with free boundary conditions, we can conclude the following: Since observables are invariant under \hat{G} , they are invariant under the global group G . That is, all observables are globally neutral.

2.3 The Hamiltonian

The Hamiltonian framework provides an algebraic formulation of the classical theory in terms of Poisson brackets (PB's). It is an essential step in the quantization of the classical theory according to Ref.[6].

In this section, we qualitatively describe the preliminaries to Dirac's procedure for setting up the canonical formalism in the presence of constraints. Concrete examples will be worked out in the subsequent chapters. In the canonical formalism, we start with defining a "cotangent bundle" $T^*\mathcal{F}(\bar{D}, R)$ on the configuration space $\mathcal{F}(\bar{D}, R)$ and PB's between functions on this bundle. This construction is carried out whether or not there are constraints present in the theory. A point in this bundle is labeled by (α, p) where $\alpha \in \mathcal{F}(\bar{D}, R)$ is a point of the configuration space and p is the "conjugate

momentum variable". It is also a function on \bar{D} . The PB's involving α and p are conventional.

If we are given a Lagrangian L , then it defines a map $T\mathcal{F}(\bar{D}, R) \rightarrow T^*\mathcal{F}(\bar{D}, R)$ by the formula

$$(\alpha, \dot{\alpha}) \rightarrow \left(\alpha, \frac{\partial L}{\partial \dot{\alpha}}\right). \quad (2.29)$$

The Lagrangian is non-singular if this map is one-to-one onto $T^*\mathcal{F}(\bar{D}, R)$. In that case, when α and $\dot{\alpha}$ range over the allowed values, all of $T^*\mathcal{F}(\bar{D}, R)$ is recovered and every point of $T^*\mathcal{F}(\bar{D}, R)$ specifies a state of the system.

It is then an elementary result that the time-evolution on $T^*\mathcal{F}(\bar{D}, R)$ can be generated by the formula

$$\dot{x} = \{x, H\}, \quad x \in T^*\mathcal{F}(\bar{D}, R). \quad (2.30)$$

where H is the Hamiltonian for the system under consideration and is constructed by the Legendre transform from L and $\{\cdot, \cdot\}$ is the Poisson bracket.

As we remarked earlier, in gauge theories, the image of the map $(\alpha, \dot{\alpha}) \rightarrow (\alpha, \partial L/\partial \dot{\alpha})$ is not all of $T^*\mathcal{F}(\bar{D}, R)$, but only a sub-manifold \mathcal{M} of this space. That is, there are functions $\phi_n, n = 1, 2, \dots$, on $T^*\mathcal{F}(\bar{D}, R)$ such that ϕ_n is zero on \mathcal{M} :

$$\phi_n\left(\alpha, \frac{\partial L}{\partial \dot{\alpha}}\right) \equiv 0. \quad (2.31)$$

We note that not all functions on $T^*\mathcal{F}(\bar{D}, R)$ need to have zero PB's with ϕ_n on \mathcal{M} , i.e., $\{f, \phi_n\}$ need not vanish on \mathcal{M} for all functions f . For instance, in electro-dynamics the electric field \mathbf{E} is conjugate to the potential \mathbf{A} and Gauss' law

$$\nabla \cdot \mathbf{E} + j_0 = \phi_1. \quad (2.32)$$

is a (secondary) constraint. But obviously,

$$\{A_i(\mathbf{x}), \phi_1(\mathbf{y})\} \neq 0. \quad (2.33)$$

on \mathcal{M} . Note that we must first evaluate the PB's and then substitute $\phi_n = 0$. Due to the existence of such f , we cannot set $\phi_n = 0$ before evaluating PB's. Thus we cannot eliminate redundant degrees of freedom using Eq.(2.31) without trouble from the Poisson bracket algebra.

A systematic method to treat the constraints is due to Dirac. References [3] contain a detailed exposition of the method. In Chapter 8 we will have occasion to illustrate the method in specific examples.

3 NON-RELATIVISTIC PARTICLES WITH SPIN

A classical non-relativistic particle with spin is an example of an elementary system where the utility of the fibre bundle formalism can be illustrated. The Hamiltonian description of such systems is well known (see, e.g., Ref.[3]) and is recalled below. The construction of a Lagrangian description however is not quite straightforward. One such construction involves the use of non-trivial fibre bundles. Below we will only discuss particles with zero electric charge. In Chapter 5 we return to a relativistic description of charged particles.

3.1 The Hamiltonian Description

Let $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{p} = (p_1, p_2, p_3)$, and $\mathbf{S} = (S_1, S_2, S_3)$ denote the coordinate, the momentum and the spin of the particle. Here we therefore impose the constraint

$$S^2 \equiv S_i S_i = \lambda^2 , \quad (3.1)$$

where λ is a constant. The Poisson brackets are

$$\{x_i, x_j\} = \{p_i, p_j\} = 0 , \quad (3.2)$$

$$\{x_i, p_j\} = \delta_{ij} , \quad (3.3)$$

$$\{S_i, S_j\} = \epsilon_{ijk} S_k . \quad (3.4)$$

If the particle is free, the Hamiltonian of the system is

$$H_0 = \frac{\mathbf{p}^2}{2m} , \quad (3.5)$$

where m is the mass of the particle. If there is an external, not necessarily homogeneous, magnetic field $\mathbf{B} = (B_1, B_2, B_3)$ present, and the particle has a magnetic moment μ , the Hamiltonian has the following form:

$$H = H_0 + \mu \mathbf{S} \cdot \mathbf{B} . \quad (3.6)$$

The equations of motion for the free particle and the interacting particle are, respec-

tively

$$\dot{x}_i = \frac{p_i}{m}, \quad (3.7)$$

$$\dot{p}_i = 0, \quad (3.8)$$

$$\dot{S}_i = 0, \quad (3.9)$$

and

$$\dot{x}_i = \frac{p_i}{m}, \quad (3.10)$$

$$\dot{p}_i = -\mu S_j \partial_i B_j, \quad (3.11)$$

$$\dot{S}_i = \mu \epsilon_{ijk} B_j S_k. \quad (3.12)$$

Here we make use of the notation $\partial_i \equiv \partial/\partial x_i$.

3.2 The Lagrangian Description

If we know the Hamiltonian description, it is often possible to find the Lagrangian of the system by a Legendre transformation. We can perform the Legendre transformation provided we can find coordinates for the phase space which are canonical. By this we mean the following. Let Q denote the configuration space of the system under consideration. The phase space T^*Q , in our case, is eight-dimensional. A canonical system of coordinates for this space is by definition of the form

$$T^*Q = \left\{ (Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3, P_4) \right\}, \quad (3.13)$$

where

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0, \quad (3.14)$$

and

$$\{Q_i, P_j\} = \delta_{ij}. \quad (3.15)$$

For our system we can, of course, set

$$Q_i = x_i, \quad P_i = p_i, \quad i = 1, 2, 3. \quad (3.16)$$

It remains to find Q_4 and P_4 . They will depend on \mathbf{S} and perhaps \mathbf{x} and \mathbf{p} . One can show, however, that there exists no such coordinates Q_4 and P_4 which are smooth

functions of \mathbf{S} . From the constraint Eq.(3.1), \mathbf{S} spans a 2-dimensional sphere. It is well-known that a 2-dimensional sphere cannot be globally coordinatized by a set of coordinates (Q_4, P_4) (see, e.g., Ref.[7]). Any choice of Q_4 and P_4 will therefore show a singularity for at least one \mathbf{S} . This singularity is the analogue of the Dirac string [8, 9] in the theory of magnetic monopoles. We refer to Section 3.3 for a proof of this result.

Thus we cannot find a global Lagrangian by a Legendre transformation when we have a constraint like Eq.(3.1). For a local Lagrangian description we refer to Ref.[10]. Although it is not possible to find a global Lagrangian by a Legendre transformation, the above system does admit a global Lagrangian description by enlarging the configuration space. We shall now construct it and point out some of its novel features. The canonical formalism for this Lagrangian is the one discussed above. We will discuss the derivation of this formalism in Chapter 8.

Let $\Gamma = \{s\}$ denote the usual spin 1/2 representation of the rotation group (see, e.g., Ref.[11]). Thus we have

$$s^\dagger s = \mathbf{1} \ , \quad \det s = 1 \ . \quad (3.17)$$

The configuration space Q for the Lagrangian will be the product space $\mathbf{R}^3 \times \Gamma$. The points of \mathbf{R}^3 as usual correspond to the position coordinates of the particle while a point $s \in \Gamma$ is related to the spin degrees of freedom S_i through

$$S_i \sigma_i = \lambda s \sigma_3 s^{-1} \ . \quad (3.18)$$

Here σ_i , $i = 1, 2, 3$, are the three Pauli matrices. As a consequence of this definition, the constraint Eq.(3.1) is fulfilled as an identity.

The Lagrangian of the free spinning particle then is

$$L_0 = \frac{1}{2} m \dot{\mathbf{x}}^2 + i\lambda \text{Tr}(\sigma_3 s^{-1} \dot{s}) \ , \quad (3.19)$$

where the dot indicates differentiation with respect to time.

We now verify that L_0 gives the correct equations of motion. Variation of the coordinate \mathbf{x} leads in a known way to

$$m \ddot{x}_i = 0 \ , \quad i = 1, 2, 3 \ . \quad (3.20)$$

Variation of s can be performed as follows. The most general variation of s can be written in the form

$$\delta s = i \varepsilon \cdot \sigma s \ , \quad (3.21)$$

where

$$\varepsilon \cdot \sigma = \varepsilon_i \sigma_i \quad . \quad (3.22)$$

This is so because $i\varepsilon \cdot \sigma$ is a generic element of the Lie algebra of Γ and the general variation of s is induced by such an element. Equations (3.17) and (3.21) imply

$$\delta s^{-1} = -i s^{-1} \varepsilon \cdot \sigma \quad . \quad (3.23)$$

Hence, for the variation Eq.(3.21),

$$\delta L_0 = -\lambda \operatorname{Tr} (\sigma_3 s^{-1} \dot{\varepsilon} \cdot \sigma s) = -2S_i \dot{\varepsilon}_i \quad . \quad (3.24)$$

After a trivial partial integration, this yields the required equation of motion

$$\dot{S}_i = 0 \quad . \quad (3.25)$$

If the particle has a magnetic moment μ , the Lagrangian in the presence of an external magnetic field \mathbf{B} is

$$L = L_0 - \frac{\mu}{2} \operatorname{Tr}(\mathcal{S}\mathcal{B}) \equiv L_0 - \mu S_i B_i \quad , \quad (3.26)$$

where

$$\mathcal{S} \equiv S_i \sigma_i = \lambda s \sigma_3 s^{-1} \quad (3.27)$$

and

$$\mathcal{B} = B_i \sigma_i \quad . \quad (3.28)$$

In Eq.(3.26), during variations, we should regard S_i as a function of s . Now the variation of \mathbf{x} gives

$$\delta L = m \dot{x}_i \delta \dot{x}_i - \mu S_j \partial_i B_j \delta x_i \quad (3.29)$$

where

$$\partial_i B_j \equiv \frac{\partial B_j}{\partial x_i} \quad . \quad (3.30)$$

Hence

$$m \ddot{x}_i = -\mu S_j \partial_i B_j \quad . \quad (3.31)$$

The variation Eq.(3.21) of s gives in this case

$$\delta L_B = - \operatorname{Tr}(\mathcal{S} \sigma \cdot \dot{\varepsilon}) - \frac{i\mu}{2} \operatorname{Tr}([\mathcal{S}, \mathcal{B}] \varepsilon \cdot \sigma) \quad , \quad (3.32)$$

where we have used the cyclic property of the trace operation, i.e.,

$$\text{Tr}(A[B, C]) = \text{Tr}(B[C, A]) \quad . \quad (3.33)$$

Thus

$$\dot{S}_i = \mu \varepsilon_{ijk} B_j S_k \quad . \quad (3.34)$$

Equations (3.31) and (3.34) are the same as those given by the Hamiltonian discussed above.

3.3 Gauge Properties of L_0 and L

The Lagrangian L_A , $A = 0, 1$, where $L_1 \equiv L$, exhibits gauge invariance under a gauge group \mathcal{G} which we now discuss in some detail.

Let $U(1) = \{\exp(i\sigma_3\theta/2)\}$ and consider the transformation

$$s \rightarrow s \exp(i\sigma_3\theta/2) \quad , \quad (3.35)$$

where θ in general is time-dependent. Under this transformation, L_A changes only by the time derivative of a function, that is,

$$L_A \rightarrow L_A + \lambda \dot{\theta} \quad . \quad (3.36)$$

We distinguish this invariance property of a Lagrangian function from the conventional one where the last term in Eq.(3.36) is absent by saying that L_A is weakly invariant under the gauge transformation Eq.(3.35). This weak invariance of L_A clearly suggests that the equations of motion involve only variables invariant under the gauge transformation Eq.(3.35). For dynamical variables, ‘‘invariance’’ under the transformation Eq.(3.35), of course, has the conventional meaning. We may note here that the equations of motion Eqs.(3.31) and (3.34) in fact only contain the gauge invariant dynamical variables x_i and S_i .

Since L_A changes under the gauge transformation Eq.(3.35), it is not possible to write it as it stands in terms of gauge invariant quantities only. We can instead attempt to eliminate the additional gauge degree of freedom in L_A by fixing the gauge. This means the following: We can now show [12] that any gauge invariant is a function of S_i and, of course, of x_i . Gauge fixing means that for each $\mathbf{S} = (S_1, S_2, S_3)$, we try to find an element $s(\mathbf{S}) \in SU(2)$ such that

$$S_i \sigma_i = \lambda s(\mathbf{S}) \sigma_3 s(\mathbf{S})^{-1} \quad . \quad (3.37)$$

If such an $s(\mathbf{S})$ existed, we could substitute $s(\mathbf{S})$ for s in the Lagrangian L_A and thereby eliminate the gauge degree of freedom. We can show, however, that there exists no such choice of $s(\mathbf{S})$ which is continuous for all \mathbf{S} . The reason for this is as follows. The vectors \mathbf{S} which satisfy the normalization conditions $S_i S_i = \lambda^2$ span the two sphere S^2 . The existence of a smooth $s(\mathbf{S})$ with the property Eq.(3.37) means that

$$SU(2) = S^2 \times U(1) \quad , \quad (3.38)$$

since any point in Γ can then be written in the form $s(\mathbf{S}) \exp(i\sigma_3\theta/2)$. But $SU(2)$ is simply connected while $U(1)$ on the right hand side of Eq.(3.38) is infinitely connected, and so the right hand side of Eq.(3.38) is infinitely connected. Here we recall that $U(1)$ is topologically identical to the circle S^1 . Hence (3.38) and a smooth $s(\mathbf{S})$ do not exist. Thus we have the remarkable situation that a Lagrangian for a non-relativistic spinning system exists only if the space of coordinates and spin variables is non-trivially enlarged to include a $U(1)$ gauge degree of freedom (at least in our approach).

It is often stated in the literature that $U(1)$ gauge invariance implies electromagnetism. But the $U(1)$ gauge invariance of the Lagrangian L_A seems to have little to do with electro-magnetism. In the sections which follow, we will encounter other weakly gauge invariant Lagrangians in contexts which seem equally remote from Abelian or non-Abelian gauge fields. Thus the assertions in the literature seem to require qualifications.

3.4 Principal Fibre Bundles

The Lagrangians L_A are associated with what in differential geometry is called a principal fibre bundle structure. We now discuss this bundle structure .

As we have seen above, the configuration space appropriate to the Lagrangian L_A is the group space $SU(2) = \{s\}$. More accurately, it is $\mathbf{R}^3 \times SU(2)$. But \mathbf{R}^3 being, in this case, the set of positions of the particle under consideration, is not relevant in the present context and will be simply ignored. On the space $SU(2)$, there is the action of the group $U(1)$, i.e. there is an action

$$s \rightarrow s \exp(i\sigma_3\theta/2) \quad (3.39)$$

of $U(1)$. Under this action, L_A is weakly invariant for time dependent θ 's. If we now define the projection map π by

$$\pi : SU(2) \rightarrow S^2 \quad , \quad (3.40)$$

where

$$\pi : s \rightarrow \lambda s \sigma_3 s^{-1} \equiv S_i \sigma_i \quad , \quad (3.41)$$

weak invariance of L_A implies that the equations of motion depend only on the base manifold $S^2 = \{\mathbf{S}\}$.

Thus we have the following mathematical structure:

- 1.) A manifold $SU(2)$ which topologically is the same as the three sphere S^3 ,
- 2.) the action of a structure group $U(1)$ on the manifold $SU(2)$,
- 3.) the projection map π from $SU(2)$ to the base manifold S^2 . Further,
- 4.) the $U(1)$ action is free, that is, $sg = s$ for $g \in U(1)$ implies that g equals the identity element e of the structure group $U(1)$.

Note that the projection π maps all the right cosets

$$sU(1) \equiv s \cdot \{\exp(i\sigma_3\theta/2)\} \quad (3.42)$$

to a single point on the base space S^2 . This right coset is just the orbit of s under the action of the $U(1)$ group. It is also easy to check that distinct orbits have distinct images on S^2 and that the mapping is onto S^2 . That is, the space $SU(2)/U(1)$ of the right cosets can be identified with the base space S^2 . Thus, if we define an equivalence relation \sim by the statement

$$s_1 \sim s_2 \text{ if } s_1 g = s_2 \text{ for some } g \in U(1) \quad , \quad (3.43)$$

then π is just the map from $SU(2)$ to the space of equivalence classes generated by the relation \sim , that is, to the space $SU(2)/U(1)$.

The preceding features define a principal fibre bundle, as denoted by $U(1) \rightarrow S^3 \rightarrow S^2$, with the bundle space $S^3 \equiv SU(2)$ as a manifold, structure group $U(1)$ and base space S^2 . It is a well-known structure in mathematics - the Hopf fibration of the two sphere S^2 (see, e.g., Ref.[13]).

We now give the general definition of a principle fibre bundle $G \rightarrow E \rightarrow B$. For details, see for, e.g., Daniel and Viallet, Ref.[2] and Ref.[14]. It consists of a bundle space E , a structure group G , a base space B and a projection map Π from E onto B . The group $G = \{g\}$ has an action on the bundle space E :

$$E \ni p \rightarrow pg \in E \quad . \quad (3.44)$$

This action is required to be free, that is,

$$pg = p, \text{ for any } p, \text{ implies that } g \text{ equals the identity } e \text{ of } G. \quad (3.45)$$

The projection Π is just the identification of points related by the G -action. Thus

$$\Pi(p) = \Pi(pg), \quad (3.46)$$

while

$$\Pi(p) = \Pi(q) \quad (3.47)$$

implies that

$$q = pg \quad (3.48)$$

for some $g \in G$. We can think of B as the space of G -orbits in E .

A global section is a map

$$\varphi : B \rightarrow E \quad (3.49)$$

such that

$$\pi \circ \varphi = \text{identity map on } B. \quad (3.50)$$

Thus for $b \in B$, $\varphi(b)$ is in E and

$$\Pi(\varphi(b)) = b \text{ for all } b \text{ in } B. \quad (3.51)$$

A local section is defined analogously by restricting the domain of definition of the map $B \rightarrow E$ to an open set in B . For suitable open sets in B , a local section always exists. In fact, there is always a covering $\{V_\alpha\}$ of B by open sets V_α where $\bigcup_\alpha V_\alpha = B$, such that each V_α admits a local section.

The principal fibre bundle E is said to be trivial if $E = B \times G$. A principal fibre bundle is trivial *if and only if* it admits a global section. Note that a point p in a trivial bundle is of the form $p = (b, g)$, where $b \in B$ and $g \in G$, while the group acts on E as follows:

$$(b, g) \rightarrow (b, gg'), \quad g' \in G. \quad (3.52)$$

Thus the projection map is just

$$\Pi(b, g) = b. \quad (3.53)$$

3.5 Gauge Fixing

In the conventional treatment of gauge theories (see, e.g., Ref.[5]) there is a procedure called gauge fixing. It may be explained in the following way. Suppose the configuration space of the Lagrangian is $\{\xi\}$. Here ξ can be a multi-component, as well as a space-time dependent field. In the latter case, the considerations which follow are only formal. Suppose the gauge group \mathcal{G} is described by the set $\{g\}$, a time-dependent, and also possibly space-dependent, group, and has the action

$$\xi \rightarrow \xi g \quad (3.54)$$

on $\{\xi\}$. Fixing the gauge consists of picking exactly one point from each orbit $\{\xi g\}$. This is accomplished by imposing a condition of the form

$$\chi(\xi) = 0 \quad (3.55)$$

on ξ . Here χ , of course, can be multi-component, $\chi = (\chi_1, \chi_2, \dots, \chi_n)$. Thus Eq.(3.55) can actually be many conditions. The equation (3.55) defines a surface M . From the previous remarks, it is clear that the surface M must be such that each orbit cuts M once and exactly once.

If the action (3.54) is free, the previous discussion shows that M is a global section in a principal fibre bundle. In this case, M exists if and only if $\{\xi\}$ is a trivial bundle. Global gauge fixing is possible only in such a case.

In general, the action of the gauge group \mathcal{G} on $\{\xi\}$ can be quite involved. Thus:

a) The action of \mathcal{G} may not be free. Then the orbit $\xi\mathcal{G}$ is not diffeomorphic to \mathcal{G} since some elements of \mathcal{G} leave ξ fixed, that is, some degrees of freedom of \mathcal{G} disappear in the map

$$g \rightarrow \xi g \quad . \quad (3.56)$$

b) The little group, also called the stability group or the isotropy group, \mathcal{G}_ξ of ξ is the set

$$\mathcal{G}_\xi = \{g \in \mathcal{G} | g = \xi g\} \quad . \quad (3.57)$$

It may happen that two distinct points ξ and ξ' have little groups \mathcal{G}_ξ and $\mathcal{G}_{\xi'}$ which are not conjugate in \mathcal{G} . That is, there exist no element $\bar{g} \in \mathcal{G}$, such that

$$\bar{g}\mathcal{G}_\xi\bar{g}^{-1} \equiv \{\bar{g}g\bar{g}^{-1} | g \in \mathcal{G}_\xi\} = \mathcal{G}_{\xi'} \quad . \quad (3.58)$$

In fact, \mathcal{G}_ξ and $\mathcal{G}_{\xi'}$ may not even be isomorphic. An example is the action of the connected Lorentz group L_+^\uparrow on the Minkowski space M^4 . In this case, if for instance

$x \in M^4$ is time-like, the little group is $SO(3)$, while if x is space-like, the little group is the connected 2+1 Lorentz group. In case b), the different orbits are not diffeomorphic.

If the orbits $\xi\mathcal{G}$ for different ξ are diffeomorphic, we have a fibration of the space $\{\xi\}$ by the group \mathcal{G} . If there are topologically distinct orbits, we have a singular fibration of the space $\{\xi\}$ by the group \mathcal{G} .

In Yang-Mills field theory, there are some results which show the non-existence of a global gauge condition [15, 16], that is, of a global surface M with the properties discussed above. These results are usually proved either when the Euclidean space-time is compactified to the four sphere S^4 or its time-slices are compactified to three spheres S^3 . The physical meaning of such a compactification is obscure to us [2].

It may be noted that in principle, it is unnecessary to fix a gauge. The orbits of \mathcal{G} in $\{\xi\}$ are well defined. We can work on the space of these orbits. That is, \mathcal{G} defines an equivalence relation \sim on $\{\xi\}$, ξ and ξ' being equivalent if they are connected by the \mathcal{G} -action that is,

$$\xi \sim \xi' \iff \xi' = \xi g \text{ for some } g \in \mathcal{G} . \quad (3.59)$$

The space of orbits is just the space $\{\xi\}$ with \mathcal{G} -equivalent points identified, that is, $\{\xi\}/\sim$. Thus for the spinning particle system discussed above, it is unnecessary to fix a gauge. In fact, a global gauge does not exist for this system since the bundle is non-trivial. For each fixed time, the space $\{\xi\}$ in this case is the three sphere S^3 , the group G which is gauged is $U(1)$ and the space of orbits S^3/\sim is S^2 . This example also shows that even if a global gauge does not exist, the space of orbits, or the space of gauge invariant variables, can still be well defined.

However, the sort of systems (like the spinning particle) we discuss in the present work are rather exceptional. Here we can readily identify the space of gauge invariant variables in a concrete way. In field theoretical problems, this usually turns out to be difficult to do. The practice in these problems is to fix the gauge by some convenient procedure. We have seen that a global gauge fixing is not always possible. Such a circumstance can cause difficulties in such problems during quantization.

Recently a perturbation theory for gauge fields without gauge fixing has been developed [17] based on the Langevin equation of non-equilibrium statistical mechanics. We will not, however, enter into its discussion here.

4 MAGNETIC MONOPOLES

In this chapter, we discuss the classical formalism for the description of a non-relativistic charged particle in the field of a point-like Dirac magnetic monopole Ref.[8, 9, 18]. This system as well illustrates the utility of the fibre bundle formalism in an elementary context. See in this context also Refs.[20, 21, 22].

4.1 Equations of Motion

Let $\mathbf{x} = (x_1, x_2, x_3)$ denote the relative coordinates and m the reduced mass of the system. We assume that the magnetic field is Coulomb-like, i.e.,

$$B_i = g \frac{x_i}{4\pi r^3} \quad . \quad (4.1)$$

Then the conventional Lorentz force equation, for a particle with an electric charge $q = -e$, reads

$$m\ddot{x}_i = n \frac{1}{r^3} \varepsilon_{ijk} x_j \dot{x}_k \quad . \quad (4.2)$$

Here $r \equiv |\mathbf{x}|$ is the radial coordinate, ε_{ijk} is the Levi-Civita symbol, $4\pi n$ is the product eg of the electric and magnetic charges e and g , and dots denote time differentiation.

The equation (4.2) reveals a remarkable structure when written in terms of radial and angular variables. Let

$$x_i = r \hat{x}_i \quad , \quad \sum_i \hat{x}_i^2 = 1 \quad . \quad (4.3)$$

Then Eq.(4.2) is equivalent to

$$\ddot{r} = r \sum_i \dot{\hat{x}}_i^2 \quad , \quad (4.4)$$

$$\frac{d}{dt} [m \varepsilon_{ijk} x_j \dot{x}_k + n \hat{x}_i] = 0 \quad . \quad (4.5)$$

The radial equation (4.4) has the same form as for a non-relativistic free particle. But from Eq.(4.5), the conserved angular momentum

$$J_i = m \varepsilon_{ijk} x_j \dot{x}_k + n \hat{x}_i \quad (4.6)$$

has an additional piece $n \hat{x}_i$ as compared to that of the free particle. It can be interpreted as contributing a helicity

$$\hat{x}_i J_i = n \quad (4.7)$$

along the line joining the particle and the monopole.

4.2 The Hamiltonian Formalism

It is much easier to write down a Hamiltonian description of this system than it is to write a Lagrangian description. We describe the former in this section.

Let $B = \{\mathbf{x} \in \mathbf{R}^3 \mid r \equiv |\mathbf{x}| \neq 0\}$ denote the configuration space. Note that we have excluded the origin $r = 0$ from B . Thus the electric charge and the magnetic monopole are forbidden to occupy the same space-time point. The phase space T^*B can be chosen to have coordinates (\mathbf{x}, \mathbf{v}) , where $\mathbf{v} = (v_1, v_2, v_3)$ denotes the relative velocity of the electric charge and the magnetic monopole.

The equation of motion Eq.(4.2) is readily verified to be produced by the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}m \sum_i v_i^2 \\ &\equiv \frac{1}{2}mv^2 \quad , \end{aligned} \tag{4.8}$$

provided the Poisson brackets (PB's) are chosen as follows:

$$\{x_i, x_j\} = 0 \quad , \tag{4.9}$$

$$\{x_i, v_j\} = \delta_{ij}/m \quad , \tag{4.10}$$

$$\{v_i, v_j\} = -\frac{n}{m^2} \varepsilon_{ijk} \frac{x_k}{r^3} \quad . \tag{4.11}$$

Note that since the right-hand side of Eq.(4.2) is proportional to the magnetic field, the PB Eq.(4.11) is conventional for velocities in the presence of a magnetic field.

As was the case for the spinning non-relativistic particle, a global Lagrangian can be found if a canonical system of coordinates (Q, P) for T^*B can be found. It may again be shown, however, that no such global system of coordinates exists [20]. Thus, it is not possible to construct a global Lagrangian by application of a simple Legendre transformation.

4.3 The Lagrangian Formalism

The global Lagrangian can be constructed by enlarging the configuration space B appropriate to the Hamiltonian to a $U(1)$ bundle E over B . This Lagrangian exhibits weak gauge invariance under time dependent $U(1)$ transformations. As a consequence,

the equations of motion are defined entirely on B . The structure of the Lagrangian formalism bears a strong resemblance to the one for the non-relativistic spinning system, although there are important points of difference as well.

Let $\{s\}$ denote the set of all two-by-two unitary unimodular matrices, i.e., elements of the $SU(2)$ group in the defining representation. The space E is

$$E = R^1 \times SU(2) \equiv \{(r, s)\} \quad . \quad (4.12)$$

Here r is the radial variable with the restriction $r > 0$. So the electric charge and the magnetic monopole are again forbidden to occupy the same spacetime point. The relation of s to the relative coordinates x_i is given by

$$\hat{X} = \sigma_i \hat{x}_i = s \sigma_3 s^{-1} \quad . \quad (4.13)$$

In the Lagrangian below, the basic dynamical variables are r and s . So, wherever x_i occurs, it is to be regarded as written in terms of r and s .

The Lagrangian is

$$L = \frac{1}{2}m \sum_i \dot{x}_i^2 + in \text{Tr} \sigma_3 s^{-1} \dot{s} \quad (4.14)$$

$$= \frac{1}{2}mr^2 + \frac{1}{4}mr^2 \text{Tr} \dot{\hat{X}}^2 + in \text{Tr} \sigma_3 s^{-1} \dot{s} \quad , \quad (4.15)$$

In writing Eq.(4.15) the identity $\text{Tr} \hat{X} \dot{\hat{X}} = 0$ has been used. Variation of r in Eq.(4.15) leads directly to Eq.(4.4). The most general variation of s is

$$\delta s = i\varepsilon_i \sigma_i s \quad , \quad \varepsilon_i \text{ real.} \quad (4.16)$$

Hence

$$\delta \hat{X} = i[\varepsilon \cdot \sigma, \hat{X}] \quad , \quad \varepsilon \cdot \sigma = \varepsilon_i \sigma_i \quad , \quad (4.17)$$

and

$$\delta \text{Tr} \sigma_3 s^{-1} \dot{s} = i \text{Tr} \dot{\varepsilon} \cdot \sigma \hat{X} \quad . \quad (4.18)$$

The variation of s in Eq.(4.15) thus leads to

$$\text{Tr} \varepsilon \cdot \sigma \frac{d}{dt} \left\{ -\frac{1}{2} [\hat{X}, mr^2 \dot{\hat{X}}] + n \hat{X} \right\} = 0 \quad , \quad (4.19)$$

where we have used the identity Eq.(3.33). The bracketed expression in Eq.(4.19) is a linear combination of Pauli matrices and ε_i is arbitrary. Therefore,

$$\frac{d}{dt} \left\{ -\frac{i}{2} [\hat{X}, mr \dot{\hat{X}}] + n \hat{X} \right\} = 0 \quad , \quad (4.20)$$

which is equivalent to

$$\frac{dJ_i}{dt} = 0 \quad , \quad (4.21)$$

that is, to Eq.(4.5).

Thus L leads to both the equations of motion Eq.(4.4) and Eq.(4.5).

4.4 Gauge Properties of L

The Lagrangian L shows a weak gauge invariance under gauge transformations associated with the $U(1)$ group

$$U(1) = \{g = e^{i\sigma_3\theta/2}\} \quad . \quad (4.22)$$

This is similar to the weak gauge invariance of the Lagrangian for the spinning systems. Under the gauge transformation

$$s \rightarrow se^{i\sigma_3\theta/2} \quad , \quad \theta = \theta(t) \quad , \quad (4.23)$$

we have the weak gauge invariance

$$L \rightarrow L - n\dot{\theta} \quad . \quad (4.24)$$

As for the spinning system, associated with L , there is the fibre bundle structure

$$U(1) \rightarrow S^3 \rightarrow S^2 \quad . \quad (4.25)$$

Again, it is impossible to fix a gauge globally so as to eliminate the $U(1)$ gauge degree of freedom. This is because L is only weakly gauge invariant, and $S^3 \neq S^2 \times U(1)$. Thus there does not exist an $s(\hat{X}) \in SU(2)$ which is continuous for all \hat{X} such that

$$\hat{X} = s(\hat{X})\sigma_3s(\hat{X})^{-1} \quad . \quad (4.26)$$

It is of course possible to find an $s(\hat{X})$ which fails to be continuous only at one point, say the south pole $[\hat{x} = (0, 0, -1)]$. Such an $s(\hat{X})$ is

$$s(\hat{X}) = \frac{1}{2}\{\alpha\mathbf{1} - \frac{1}{\alpha}[\sigma_3, \hat{X}]\} \quad ,$$

$$\alpha = [2(1 + \hat{x}_3)]^{1/2} \quad . \quad (4.27)$$

It is easy to verify that $s(\hat{X})$ appearing in Eq.(4.27) is a unimodular unitary matrix and fulfills Eq.(4.26). Note that $s(\hat{X})$ in Eq.(4.27) is, however, not differentiable at the

south pole. Substitution of Eq.(4.27) into the interaction term appearing in Eq.(4.15) yields

$$in\text{Tr } \sigma_3 s(\hat{X})^{-1} \dot{s}(\hat{X}) = n \varepsilon_{3ij} \dot{\hat{x}}_i \dot{\hat{x}}_j / (1 + \hat{x}_3) \quad (4.28)$$

which is a conventional form of the interaction Lagrangian with a string singularity along the x_3 axis.

Alternatively, we can cover the two sphere $S^2 = \{\hat{X}\}$ by two coordinate patches U_1 and U_2 and find group elements $s_i(\hat{X})$ which are defined and continuous in U_1 and U_2 and which fulfill Eq.(4.26). Substitution of $s_i(\hat{X})$ for s in Eq.(4.14) leads to Lagrangians L_i defined on U_i . In the intersection region $U_1 \cap U_2$, in view of Eq.(4.26),

$$[s_1(\hat{X})^{-1} s_2(\hat{X})] \sigma_3 [s_1(\hat{X})^{-1} s_2(\hat{X})]^{-1} = \sigma_3 \quad . \quad (4.29)$$

This means that s_i differ from each other in the overlapping region by a gauge transformation,

$$s_1(\hat{X}) = s_2(\hat{X}) e^{i \frac{\sigma_3}{2} \theta} \quad (4.30)$$

for some $\theta = \theta(t)$. Hence L_1 and L_2 differ by the total time derivative of a function in $U_1 \cap U_2$:

$$L_1 = L_2 - n \dot{\theta} \quad . \quad (4.31)$$

Such a (singularity free) formulation which works with two local Lagrangians is the non-relativistic analogue of the work of Wu and Yang [22].

5 RELATIVISTIC SPINNING PARTICLES

In this chapter, we give the Lagrangian description for relativistic spinning particles, which is formulated on the Poincaré group manifold [24]. It describes a particle which, after quantization, is associated with any particular irreducible representation of the connected Poincaré group \mathcal{P}_+^\uparrow . The Lagrangian formalism can be generalized [24, 25] to include couplings with both electro-magnetism and gravity. We recover the usual equations of motion for the two systems, i.e., the Bargmann-Michel-Telegdi [26] equations for electro-magnetism and the Mathisson - Papapetrou [27] equations for gravitation. The latter equations have been generalized to include coupling to torsion [28]. Such systems are also recovered from our formalism.

5.1 The Configuration Space

The Lagrangian is associated with a non-trivial principal fibre bundle structure $U(1) \rightarrow L_+^\uparrow \rightarrow L_+^\uparrow/U(1)$, which is obtained in the following way. The bundle space is the connected component of the Lorentz group L_+^\uparrow . The structure group is, as usual, $U(1)$. It acts on $L_+^\uparrow = \{\Lambda\}$ by right multiplication, i.e.,

$$\Lambda \rightarrow \Lambda g, \quad g \in U(1). \quad (5.1)$$

Thus the base space is the space of left cosets $L_+^\uparrow/U(1)$. As in previous sections, we can infer from connectivity arguments that $L_+^\uparrow \neq (L_+^\uparrow/U(1)) \times U(1)$. Thus the bundle is non-trivial.

The configuration space Q for the Lagrangian is the connected Poincaré group, i.e.,

$$\mathcal{P}_+^\uparrow = \{z, \Lambda\} \mid z = (z^0, z^1, z^2, z^3) \in R^4, \quad \Lambda = [\Lambda^a_b] \in L_+^\uparrow\}. \quad (5.2)$$

Here z^a is interpreted as the components of the space-time coordinate of the particle. The interpretation of Λ is as follows. If p_a and S_{ab} are the momentum and spin components of the particle, we write

$$p_a = m\Lambda_{a0}, \quad m > 0, \quad (5.3)$$

and

$$\frac{1}{2}S_{ab}\sigma^{ab} = \lambda\Lambda\sigma_{12}\Lambda^{-1} \equiv -iS, \quad (5.4)$$

where the matrix elements of σ^{ab} are given by

$$(\sigma^{ab})_{cd} = -i(\delta_c^a\delta_d^b - \delta_d^a\delta_c^b), \quad (5.5)$$

and λ is a constant. These equations are valid for a time-like four vector p_a . The cases where the four vector p_a is not time-like will be treated later. Note that by the definitions above,

$$S_{ab} = \lambda(\Lambda_{a1}\Lambda_{b2} - \Lambda_{a2}\Lambda_{b1}) , \quad (5.6)$$

and

$$p_0 = m\Lambda_{00} > 0 , \quad p_a p^a = -m^2 . \quad (5.7)$$

Therefore we obtain

$$\frac{1}{2}S_{ab}S^{ab} = \lambda^2 , \quad (5.8)$$

and

$$p_a S^{ab} = 0 . \quad (5.9)$$

Here the Latin indices are raised and lowered by the Lorentzian metric

$$\eta = (-1, 1, 1, 1) . \quad (5.10)$$

5.2 The Lagrangian for a Free Spinning Particle

The Lagrangian for a massive spinning particle is then given by

$$L_p = p_a \dot{z}^a + i\frac{\lambda}{2}\text{Tr} \left[\sigma_{12}\Lambda^{-1}\dot{\Lambda} \right] , \quad (5.11)$$

where p_a is defined in the equation (5.3). The dynamical variables z^a , p_a and Λ in Eq.(5.11) are all functions of the parameter τ which parametrize the space-time trajectory of the particle. The dot indicates differentiation with respect to τ . Note that the action $\int L_p d\tau$ by construction is invariant under reparametrizations $\tau \rightarrow f(\tau)$.

Let us first derive the equations of motion. The variation of z^a is standard and leads to

$$\dot{p}_a = 0 . \quad (5.12)$$

The most general variation of Λ is, as usual,

$$\delta\Lambda = i\varepsilon \cdot \sigma\Lambda , \quad (5.13)$$

where

$$\varepsilon \cdot \sigma = \varepsilon^{ab}\sigma_{ab} . \quad (5.14)$$

This implies

$$\delta\Lambda^{-1} = -i\Lambda^{-1}\varepsilon \cdot \sigma . \quad (5.15)$$

Hence

$$\delta L_p = -i \text{Tr} [k \varepsilon \cdot \sigma] + \frac{i}{2} \text{Tr} \left[S \frac{d}{d\tau} (\varepsilon \cdot \sigma) \right] , \quad (5.16)$$

where the matrix k is defined by $k_{ab} = \dot{z}_a p_b$. The traces have a conventional meaning, i.e.,

$$\text{Tr} [k \varepsilon \cdot \sigma] = \sum_{ab} k^{ab} (\varepsilon \cdot \sigma)_{ba} . \quad (5.17)$$

After a partial integration in the Eq.(5.16) and use of Eq.(5.12), we obtain the equation for the conservation of total angular momentum:

$$\frac{d}{d\tau} M^{ab} = 0 , \quad (5.18)$$

where

$$M^{ab} = z^a p^b - z^b p^a + S^{ab} . \quad (5.19)$$

The proof that L_p actually describes a particle which is associated with an irreducible representation of the connected Poincaré group \mathcal{P}_+^\uparrow follows by showing that mass and spin have definite values. The mass has a definite value due to Eq.(5.7). Note also that the sign of p_0 is fixed by Eq.(5.3) since $\Lambda \in L_+^\uparrow$. Thus L_p does not describe a particle which can have both positive and negative energies. Both signs can be obtained by abandoning the condition that $\Lambda_{00} > 0$.

We can show that the spin has a definite value from equations (5.8) and (5.9). The latter shows that in the rest frame of the particle, the spin tensor S^{ab} has only space components. The former shows that the magnitude of this spin tensor has a definite numerical value. Thus the spin 3-vector $S_i \equiv \varepsilon_{ijk} S_{jk}/2$ has a definite value in the particle rest frame. In general, by computing the square of the Pauli-Lubanski vector W_a , i.e.,

$$W_a = \frac{1}{2} \varepsilon_{abcd} M^{bc} p^d , \quad (5.20)$$

where ε_{abcd} is the usual anti-symmetric tensor with $\varepsilon_{0123} = 1$, we find

$$W_a W^a = \frac{1}{2} m^2 \lambda^2 . \quad (5.21)$$

It is important to realize that the preceding equations imply

$$p_a = \frac{m \dot{z}_a}{\sqrt{-\dot{z} \cdot \dot{z}}} , \quad (5.22)$$

and

$$\dot{S}_{ab} = 0 . \quad (5.23)$$

Thus the conventional relation between momentum and velocity is recovered, and Eq.(5.12) becomes the usual equation of motion when written in terms of z_a . The derivation of these results relies on Eq.(5.18) which can be rewritten as

$$\dot{z}^a p^b - \dot{z}^b p^a + \dot{S}^{ab} = 0 . \quad (5.24)$$

in view of Eq.(5.12). It also relies on the time derivative of Eq.(5.9), i.e.,

$$p_a \dot{S}^{ab} = 0 . \quad (5.25)$$

Multiplication of Eq.(5.24) by p_a shows that p_a and \dot{z}_a are, in fact, parallel. The constant of proportionality can be determined by using the normalization condition $p_a p^a = -m^2$ and the condition that $p_0 > 0$. This gives Eq.(5.22). Now Eq.(5.22) applied to Eq.(5.24) yields Eq.(5.23).

The canonical quantization of the Lagrangian Eq.(5.11) will be carried out in Section 8.3.

5.3 The Spinning Particle in an Electro-Magnetic Field

We now discuss the coupling of electro-magnetism to spinning particles [25]. In order that our system reduces to the standard formulation in the limit of zero spin and electric charge $q = e$, the minimal coupling term $eA_a(z)\dot{z}^a$ must be present in the interaction Lagrangian. Here $A_a(z)$ is the electro-magnetic potential. When the spin is non-zero, an additional coupling to the electro-magnetic field of the form $cF_{ab}(z)S^{ab}$ may be present, where $F_{ab} = \partial_a A_b - \partial_b A_a$ and c is a constant. As we will see below, the constant c is associated with the gyro-magnetic ratio of the particle. One possible choice for the electro-magnetic interaction Lagrangian is therefore

$$L_{EI} = eA_a(z)\dot{z}^a + c\sqrt{-\dot{z} \cdot \dot{z}}F_{ab}(z)S^{ab} . \quad (5.26)$$

The second term in Eq.(5.26) is the generalization of the interaction term in the Hamiltonian Eq.(3.6) for a non-relativistic, spinning particle. The factor $\sqrt{-\dot{z} \cdot \dot{z}}$ in the second term of Eq.(5.26) was inserted in order to retain the invariance under reparametrization transformations $\tau \rightarrow f(\tau)$. As will be shown later, alternatives to Eq.(5.26) are possible.

The equations of motion are obtained by varying Λ and z in the total action

$$S = \int d\tau L_P + \int d\tau L_{EI} . \quad (5.27)$$

Variations in Λ now lead to

$$\dot{S}^{ab} + \dot{z}^a p^b - \dot{z}^b p^a = c\sqrt{-\dot{z} \cdot \dot{z}} (F^{ac}(z)S_c^b - F^{bc}(z)S_c^a) , \quad (5.28)$$

where M^{ab} is defined in Eq.(5.19). Variations of z^a

$$\dot{p}_a = eF_{ab}(z)\dot{z}^b + c\sqrt{-\dot{z} \cdot \dot{z}}S_{cd}\partial_a F^{cd}(z) + c\frac{d}{d\tau} \left(\frac{\dot{z}_a}{\sqrt{-\dot{z} \cdot \dot{z}}} S \cdot F(z) \right) , \quad (5.29)$$

where we have introduced the notation $S \cdot F(z) \equiv S_{ab}F^{ab}(z)$. Note that we no longer have the usual relationship between momentum and velocity Eq.(5.22). In general, the velocity and momentum variables are, in fact, not even parallel. This follows after substitution of Eq.(5.28) and Eq.(5.29) into the condition

$$\dot{S}^{ab}p_b + S^{ab}\dot{p}_b = 0 . \quad (5.30)$$

We find,

$$\begin{aligned} p_a = & -\frac{1}{p \cdot \dot{z}} \left(m^2 \dot{z}_a + c\sqrt{-\dot{z} \cdot \dot{z}} (p_b F^{bc}(z)S_{ca} + S_a^b S_{cd}\partial_b F^{cd}(z)) \right. \\ & \left. + cS_{ab}\frac{d}{d\tau} \left(\frac{\dot{z}^b}{\sqrt{-\dot{z} \cdot \dot{z}}} S \cdot F(z) \right) + eF_{bc}(z)\dot{z}^b S_{ab} \right) . \end{aligned} \quad (5.31)$$

In order to compare this system with that of Bargmann et al. in Ref.[26], let us examine the weak and homogeneous field limit. Upon substituting Eqs. (5.28) and (5.31) into (5.29), we then find the Lorentz equation of motion, i.e.,

$$m\frac{d}{d\tau} \left(\frac{\dot{z}^b}{\sqrt{-\dot{z} \cdot \dot{z}}} \right) = eF^{ab}(z)\dot{z}_b . \quad (5.32)$$

The equation for the spin precession can be expressed in terms of the Pauli-Lubanski vector as defined in Eq.(5.20). Substituting Eqs.(5.28), (5.31), and (5.31) into Eq.(5.29) into the equation

$$\dot{W}_a = \frac{1}{2}\varepsilon_{abcd} \left(\dot{S}^{bc}p^d + S^{bc}\dot{p}^d \right) , \quad (5.33)$$

and again keeping terms which are at most linear in the homogeneous field we find

$$\dot{W}_a = -2c\sqrt{-\dot{z} \cdot \dot{z}}F_{ab}W^b - \frac{2c + e/m}{\sqrt{-\dot{z} \cdot \dot{z}}} (\dot{z}^c F_{cb}W^b) \dot{z}_a . \quad (5.34)$$

Equations (5.32) and (5.34) and are the Bargmann-Michel-Telegdi equations for a spinning particle [26] with the identification

$$c = -\frac{eg_e}{4m} , \quad (5.35)$$

where g_e being the gyro-magnetic ratio.

The field equations for this system are obtained by adding the usual free field action, i.e.,

$$S_E = -\frac{1}{4} \int d^4x F_{ab}(x) F^{ab}(x) , \quad (5.36)$$

to Eq.(5.27). By varying the electro-magnetic potentials A_a and integrating by parts, we find

$$\partial_a F^{ab}(x) = -q \int d\tau \delta^4(x - z(\tau)) \dot{z}^b + 2c \int d\tau \sqrt{-\dot{z} \cdot \dot{z}} \partial_a \delta^4(x - z(\tau)) S^{ab} , \quad (5.37)$$

where, as above, we use the notation $\partial_a \equiv \partial/\partial x^a$. The second term on the right hand side of Eq.(5.37) represents the dipole contribution to the field in the sense of Papapetrou [27, 28] (see also in this context the work by Bailyn and Ragusa [29] and references therein).

As was stated above the interaction Lagrangian Eq.(5.26) is not uniquely determined (for a related discussion see Ref.[30]). For instance, we can replace the second term in Eq.(5.26) by [31]

$$- \frac{c}{m} p_a \dot{z}^a S_{bc} F^{bc}(z) . \quad (5.38)$$

This term preserve all the symmetries of the previous system, yet it gives a different set of equations of motion. In the limit of a weak homogeneous field the two systems can, however, be shown to be equivalent. Note that the term in Eq.(5.38) can be absorbed in the first term in L_P in Eq.(5.11), through a "renormalization" of the mass m :

$$m \rightarrow M(\alpha) = m + \frac{e g_e}{4m} ; \quad \alpha = F_{ab} S^{ab} . \quad (5.39)$$

In fact, if we no longer restrict ourselves to Lagrangians which are linear in $F_{ab} S^{ab}$, we can consider the case where the mass $M(\alpha)$ is an arbitrary function of α (which may be relevant when one is considering particles with an anomalous magnetic moment [32]). In this case the total particle Lagrangian would be

$$L_P + L_{EI} = L_p = M(\alpha) \Lambda_{a0} \dot{z}^a + i \frac{\lambda}{2} \text{Tr} \left[\sigma_{12} \Lambda^{-1} \dot{\Lambda} \right] + e A_a(z) \dot{z}^a , . \quad (5.40)$$

The resulting equations of motion are

$$\dot{S}^{ab} + \dot{z}^a p^b - \dot{z}^b p^a = \frac{d \ln M(\alpha)}{d\alpha} p_c \dot{z}^c (F^{ac}(z) S_c^b - F^{bc}(z) S_c^a) , \quad (5.41)$$

where we have corrected for a printing error in Eq.(31) of Ref.[25], and

$$\dot{p}_a = e F_{ab}(z) \dot{z}^b + \frac{d}{d\alpha} (\ln M(\alpha)) p_c \dot{z}^c \partial_a F_{bd}(z) S^{bd} . \quad (5.42)$$

These equations have also been considered by Dixon [33]. Even though the equations Eq.(5.41) and Eq.(5.42) correspond to a large class of systems, depending on the choice of $M(\alpha)$, they all lead to the Bergmann-Michel-Telegdi equations in the weak and homogeneous field limit. Here the identification of the particle's mass and the gyro-magnetic ratio g_e are given by

$$m = M(0) , \quad (5.43)$$

and

$$g_e = \frac{4}{e} \frac{dM(\alpha)}{d\alpha} \Big|_{\alpha=0} , \quad (5.44)$$

which are, of course, consistent with the specific choice Eq.(5.35).

5.4 The Spinning Particle in a Gravitational Field

It is rather straightforward to generalize L_P to include gravitational effects. It is then convenient to regard the gravitational field as a gauge field [34], i.e., the Poincaré group is regarded as a local symmetry group. Let $h = h_\mu^a$ be the vierbein fields and A_μ^{ab} the corresponding Yang-Mills potentials for the Lorentz group. Our notation is as follows. A Latin index like a is a tangent space index and a Greek index like μ is a curved space index. The metric tensor is $g_{\mu\nu} = \eta_{ab} h_\mu^a h_\nu^b$ and $\delta_\nu^\mu = h_a^\mu h_\nu^a$. The action now is

$$S = \int d\tau L_P + \int d^4x L_F , \quad (5.45)$$

where

$$L_P = m \Lambda_{ao} h_\mu^a \dot{z}^\mu + i \frac{\lambda}{2} \text{Tr}[\sigma_{12} \Lambda^{-1} D_\tau \Lambda] , \quad (5.46)$$

and

$$L_F = -\frac{1}{16\pi G} F_{\mu\nu}^{ab} h_\mu^a h_\nu^b \det(h) . \quad (5.47)$$

Here the Yang-Mills components of the field strength $F_{\mu\nu}^{ab}$ are given by

$$F_{\mu\nu} \equiv \frac{i}{2} F_{\mu\nu}^{ab} \sigma_{ab} = [D_\mu, D_\nu] . \quad (5.48)$$

where D_μ is the covariant derivative

$$D_\mu = \partial_\mu + \frac{i}{2} A_{\mu\nu}^{ab} \sigma_{ab} , \quad (5.49)$$

and

$$D_\tau \Lambda = \dot{\Lambda} + \dot{z}^\mu A_\mu \Lambda , \quad (5.50)$$

with $A_\mu \equiv A_\mu^{ab} \sigma_{ab}$. Furthermore, in Eq.(5.47), G is Newton's constant.

The equations of motion are found as follows. If we vary Λ as in Eq.(5.13), we see that Eq.(5.16) is replaced by

$$\delta L_P = -i \text{Tr}[J \varepsilon \cdot \sigma] + \frac{i}{2} \text{Tr}\left[S \frac{d}{d\tau}(\varepsilon \cdot \sigma) - [\dot{z}^\mu A_\mu, S] \varepsilon \cdot \sigma\right], \quad (5.51)$$

where

$$J^{ab} = h^{a\mu} \dot{z}_\mu m \Lambda^{b0}. \quad (5.52)$$

We thus find the equation for spin precession [35]

$$J^{ab} - J^{ba} + (D_\tau S)^{ab} = 0, \quad (5.53)$$

where

$$(D_\tau S) = \frac{dS}{d\tau} + [\dot{z}_\mu A^\mu, S]. \quad (5.54)$$

Variation of the coordinate z_μ leads to the Mathisson-Papapetrou equation in the presence of torsion [27, 28]. We find

$$\delta L_P = -\delta z_\mu (\dot{p}_\mu - \dot{z}_\lambda \partial_\mu h^{a\lambda} p_a) + \delta \dot{z}_\mu \frac{1}{2} \text{Tr}[S A^\mu] + \delta z_\mu \frac{1}{2} \text{Tr}[S \partial_\mu A^\lambda] \dot{z}_\lambda, \quad (5.55)$$

where $p_\mu = h_\mu^a p_a$. Partial integration in the second term and substitution from Eq.(5.53) leads to

$$\dot{p}_\mu - \dot{z}_\lambda \partial_\mu h^{a\lambda} p_a + \dot{z}_a A_\mu^{ab} p_b - \frac{1}{2} \text{Tr}[S F_{\mu\nu}] \dot{z}^\nu = 0. \quad (5.56)$$

This is actually the same equation as the Mathisson-Papapetrou equation in the presence of torsion, i.e.,

$$(D_\tau p)_a - h_a^\mu \dot{z}^\nu ((D_\mu h_\nu)^b - (D_\nu h_\mu)^b) p_b - \frac{1}{2} h_a^\mu \text{Tr}[S F_{\mu\nu}] \dot{z}^\nu = 0. \quad (5.57)$$

In the Equation (5.57) we make use of the notation

$$(D_\tau p)_a \equiv \frac{dp_a}{d\tau} + \eta_{ab} \dot{z}^\lambda A_\lambda^{bc} p_c, \quad (5.58)$$

and

$$(D_\mu h_\nu)^b \equiv \partial_\mu h_\nu^b + A_\mu^{bc} h_{c\nu}. \quad (5.59)$$

For a discussion of the field equations, we refer to Ref.[24].

5.5 General Irreducible Representations of \mathcal{P}_+^\uparrow

To find Lagrangian descriptions for other unitary irreducible representations of the Poincare group \mathcal{P}_+^\uparrow it is sufficient to alter the definitions of p_a and S_{ab} . For example, in order to describe a massless particle, like the photon or a massless neutrino, we may replace $m\Lambda_{a0}$ in Eq.(5.7) by

$$(\Lambda_{a0} + \Lambda_{a3})\omega , \quad (5.60)$$

where ω corresponds to the angular "frequency" of the massless particle. Equation (5.60) ensures that

$$p_a p^a = 0 . \quad (5.61)$$

For a massless particle, the Pauli-Lubanski vector W_a , as given by the Equation (5.20), obeys the condition that $W_a W^a = 0$, and since $p_a W^a = 0$, it is easy to show the following identity

$$W_a = \lambda p_a . \quad (5.62)$$

It follows that for $\lambda = 1/2$ and $\lambda = 1$, we get a "neutrino" and a "photon" of definite helicity. The sign of the helicity can, of course, be reversed by reversing the sign of λ .

The tachyonic representations are obtained by choosing

$$p_a = \rho \Lambda_{a3} . \quad (5.63)$$

Different values of ρ and λ give different irreducible representations as may be seen from the values of the invariants $p_a p^a$ and $W_a W^a$:

$$p_a p^a = \rho^2 , \quad (5.64)$$

and

$$W_a W^a = -\rho^2 \lambda^2 . \quad (5.65)$$

For the irreducible representations with zero four-momentum, we set

$$p_a = 0 , \quad (5.66)$$

and choose the Lagrangian to be

$$L_P = \frac{i}{2} \text{Tr} [K \Lambda^{-1} \dot{\Lambda}] , \quad (5.67)$$

where $K = K_{ab} \sigma^{ab}$ is a fixed element of the Lie algebra. From the variation Eq.(5.13), we find the equation of motion

$$\frac{d}{d\tau} S_{ab} = 0 , \quad (5.68)$$

where

$$\frac{1}{2}S_{ab}\sigma^{ab} = \Lambda K \Lambda^{-1} . \quad (5.69)$$

Thus S is the spin angular momentum. The irreducible representations are characterized by the two invariants [36], i.e., $S_{ab}S^{ab}/2$ and $S_{ab}^*S^{ab}/2$. Since

$$\frac{1}{2}S_{ab}S^{ab} = 2K_{ab}K^{ab} , \quad (5.70)$$

and

$$\frac{1}{2}S_{ab}^*S^{ab} = 2K_{ab}^*K^{ab} , \quad (5.71)$$

we can, classically, get any real values for these invariants by choosing values for K_{ab} appropriately. To quantize the system (see Chapter 8), we are, as usual, obliged to give them values which are appropriate for unitary irreducible representations [36].

5.6 Relation Between the Charge-Monopole System and the Massless Spinning Particle System

In this section we point out some striking analogies between the charge-monopole system and the system of a massless particle of fixed helicity. The similarities of the two systems become evident when the roles of coordinates and velocities are interchanged.

The analogies are as follows:

1) The angular momentum of a charged particle in the field of monopole contains a helicity n (see Chapter 4 and Eq.(4.7)) along the direction joining the monopole and the charge. The angular momentum of a massless particle of spin λ contains helicity λ along the direction of the momentum of the particle.

2) The components of the position vector of the charge-monopole system commute, but the components of the velocity vector do not (at least not for finite charge-monopole separation). Thus the system cannot be localized in velocity space. Furthermore, there is no globally defined momentum vector, consequently a globally defined momentum space wave function cannot be defined. For a massless particle, on the other hand, the components of momenta commute, i.e.,

$$[p_i, p_j] = 0 . \quad (5.72)$$

But the components of position do not

$$[x_i, x_j] = -i\lambda\varepsilon_{ijk}\frac{p_k}{p^3} . \quad (5.73)$$

Using Eqs. (5.72) and (5.73), along with the canonical commutation relation Eq.(5.76) as given below, we can verify that $J_i = \varepsilon_{ijk}x_jp_k + \lambda p_i/p$ generates rotations for this system. Equation (5.73) is analogous to the commutation relation for the components of velocity for the charge-monopole system Eq.(4.11). It is consistent with the fact that the photon cannot be localized [37]. With the Hamiltonian $H = |\mathbf{p}|$, we are lead to the equations of motion

$$[x_i, x_j] = -i\lambda\varepsilon_{ijk}\frac{p_k}{p^3} . \quad (5.74)$$

and

$$\frac{d}{dt}(\varepsilon_{ijk}x_jp_k + \lambda p_i/p) , \quad (5.75)$$

if we supplement the commutation relations Eqs.(5.72) and (5.73) with the canonical commutation relation

$$[x_i, p_j] = i\delta_{ij} . \quad (5.76)$$

3) The non-trivial topology of the charge-monopole system depends on the fact that their relative spatial separation cannot become zero. As a consequence, the configuration space has the topology $R \times S^1$. The unusual topological features of the charge-monopole system can be characterized in terms of this bundle. If the relative coordinate is allowed to vanish as well, the configuration space becomes R^3 , which does not admit non-trivial $U(1)$ bundles.

In contrast, since for a massless particle its three momentum cannot be transformed to zero by Lorentz transformations, the origin in momentum space should be excluded. The topology of \mathbf{p} is thus $R^1 \times S^1$. For a non-zero helicity, its Lagrangian description is facilitated by making use of the $U(1)$ bundle $R^1 \times S^3$ over $R^1 \times S^2$. In the photon Lagrangian, the entire Lorentz group appears to play the role of the bundle space. Consider, however, the translation group T_2 as generated by

$$\Pi_1 = M_{10} + M_{13} , \quad (5.77)$$

and

$$\Pi_2 = M_{20} + M_{23} . \quad (5.78)$$

The photon Lagrangian is invariant under the transformations

$$\Lambda \rightarrow \Lambda \exp(i\alpha^a(x)\Pi_a) . \quad (5.79)$$

Thus it can be globally written on $L_+/T_2 = R^1 \times S^3$ by factoring these gauge degrees of freedom. The Euclidean group generated by σ_3 , Π_1 , and Π_2 is the familiar stability group of the four momentum $(1, 0, 0, 1)$.

4) From the expression for the conserved angular momentum, we see that under a parity transformation, $\lambda \rightarrow -\lambda$ for both systems under consideration. Thus a charge-monopole system with a fixed value of e and g ($e(g)$ being the electric (magnetic) charge), or a massless particle of fixed helicity, is incompatible with parity invariance.

5) There are no bound states in the charge-monopole system. For large times, the motion approaches that of two free particles, i.e., as $t \rightarrow \infty$,

$$\mathbf{x}(t) \rightarrow \mathbf{v}t + \mathbf{x}_0 + \mathcal{O}(t^{-2}) . \quad (5.80)$$

where $\mathbf{x}(t)$ is the trajectory in the relative coordinate, and \mathbf{x}_0 and \mathbf{v} are constant vectors. It follows that as $t \rightarrow \infty$ the conserved angular momentum $J_i = \varepsilon_{ijk}x_k p_k + \lambda \hat{\mathbf{x}}_i$, where $\mathbf{p} = m\mathbf{v}$, approaches the value

$$J_i = \varepsilon_{ijk}(\mathbf{x}_0)_j x_k p_k + \lambda \hat{\mathbf{p}}_i , \quad (5.81)$$

which has the same form as that for a massless particle. In Ref.[38], the preceding limit for the charge-monopole system was discussed in detail. It was shown that the commutation relations of \mathbf{x}_0 and \mathbf{p} are the same as those in the Eqs.(5.72), (5.73), and (5.76).

A canonical formalism was, furthermore, developed in Ref.[38] for a free, non-relativistic particle with no internal degrees of freedom. This formalism was unusual in that upon quantization, the angular momentum contained a helicity λ in the direction of the three momentum \mathbf{p} . If λ is chosen as half integral, the system thus becomes "fermionic". Such a system resembles a massless particle or the large time limit of the charge-monopole system. In Ref.[38] no Lagrangian formulation of the system was given. We may notice here that it is just the non-relativistic analogue of the photon Lagrangian i.e.,

$$L = p\hat{p}_k \dot{x}_k - i\text{Tr}[\sigma_3 s^{-1} \dot{s}] . \quad (5.82)$$

Here $s \in SU(2)$, the momentum is $p_k = p\hat{p}_k$ and $\sigma_k p_k k = s\sigma_3 s^{-1}$. Thus p is not an independent variable, but is defined in terms of the dynamical group element s .

6 YANG-MILLS PARTICLES

The classical description of a charged particle in an electro-magnetic field is well-known. The motion of the particle is described by the Lorentz force equation, while the dynamics of the field is described by the Maxwell equations. The non-Abelian generalization of these equations is due to Wong [39]. Instead of an electric charge, the corresponding Yang-Mills particle carries a spin-like variable \mathbf{I} which transforms under the adjoint representation of the internal symmetry group. The Wong equations provide a coarser level of description than a non-Abelian gauge field theory since they treat the sources only as particles. Hence they may be more tractable than a gauge field theory and may also reveal important features of the latter. For such reasons, there is currently a growing interest in the Wong equations. Below, we first recall the Wong equations. Then the Hamiltonian and Lagrangian descriptions of these equations are discussed. The Lagrangian description in our approach [12, 40] requires the use of non-trivial fiber bundles.

6.1 The Wong Equations

The Wong equations, with a gauge coupling e , are

$$m \frac{d}{d\tau} \left[\frac{\dot{z}_a}{\sqrt{-\dot{z}^2}} \right] = -e F_{ab}^\alpha(z) I_\alpha \dot{z}^b, \quad (6.1)$$

and

$$(D^a)_{\alpha\beta} F_{ab}^\beta(x) = e \int d\tau \delta^4(x - z(\tau)) \dot{z}_b(\tau) I_\alpha(\tau). \quad (6.2)$$

Here, $z^a = z^a(\tau)$ denotes the particle trajectory in Minkowski space, while $F_{ab}^\beta \equiv \partial_a A_b^\beta - \partial_b A_a^\beta + e c_{\beta\alpha\gamma} A_a^\alpha A_b^\gamma$ and $(D_a)_{\alpha\beta} \equiv \delta_{\alpha\beta} \partial_a + ie [T(\gamma)]_{\alpha\beta} A_a^\gamma$ with the adjoint representation $[T(\gamma)]_{\alpha\beta} = -ic_{\alpha\beta\gamma}$, are the usual Yang-Mills tensor and covariant derivative, respectively. The range of the indices α, β and γ is equal to the dimension n of the internal symmetry group G . The vector $\mathbf{I} = \mathbf{I}(\tau)$ transforms under the adjoint representation of G . From Eq.(6.2) and the identity

$$[D^a, D^b]_{\alpha\beta} F_{ab}^\beta = 0, \quad (6.3)$$

one finds the following consistency condition on \mathbf{I} :

$$\frac{d}{d\tau} I_\alpha(\tau) - e \dot{z}^a(\tau) A_a^\rho(z(\tau)) c_{\rho\alpha\beta} I_\beta(\tau) = 0, \quad (6.4)$$

Here $c_{\rho\alpha\beta}$ are the structure constants of G .

It is known [12] that the spectrum of the Casimir invariants constructed out of \mathbf{I} determines the irreducible representations (IRRs) of G which occur in the quantum-mechanical Hilbert space. We want to describe a particle which belongs to a definite IRR of G . Thus we impose also the constraint: Casimir invariants of \mathbf{I} have definite numerical values. It is easy to show that this constraint is consistent with the time evolution of \mathbf{I} according to Eq.(6.4).

It is instructive to verify that the preceding equations reduce to the Lorentz and Maxwell equations when $G = U(1)$. In this case \mathbf{I} has only one component, say I_1 . Since, in this case, $c_{\rho\alpha\beta} = 0$, the component I_1 is a constant of motion by Eq.(6.4) and it can be assigned a definite numerical value, say λ . Identifying $e\lambda$ with the electric charge, Eqs.(6.1) and (6.2) are seen to reduce to the Lorentz and Maxwell equations of motion.

6.2 The Hamilton Formalism

The Hamiltonian is a simple generalization of the electrodynamic Hamiltonian. It is

$$H = H_F + H_P , \quad (6.5)$$

where

$$H_P = [(p_i - eA_i^\alpha(z)I_\alpha)^2 + m^2c^4]^{1/2} + eA_0^\alpha(z)I_\alpha , \quad (6.6)$$

and H_F is the Hamiltonian for the Yang-Mills field [5]. The latter is well-known. In writing Eq.(6.6), we have identified z_0 with time $\tau \equiv t$. The Poisson brackets (PBs) involving z_i 's and p_i 's, $i = 1, 2, 3$, are conventional. They have also zero PBs with I_α . The PBs involving I_α alone are

$$\{I_\alpha, I_\beta\} = c_{\alpha\beta\gamma}I_\gamma . \quad (6.7)$$

It is then straightforward to verify that the Hamiltonian Eq.(6.5) and the PBs in Eq.(6.7) lead to the required equations of motion.

6.3 The Lagrangian Formalism

The presence of a spin-like variable \mathbf{I} whose Casimir invariants are fixed suggests in analogy to previous sections that a Lagrangian can be found on a configuration space E which contains additional gauge degrees of freedom. This is indeed the case. The

space E turns out to be $\mathbf{R}^3 \otimes G$ where \mathbf{R}^3 is the usual space of spatial coordinates and G is the internal symmetry group.

We assume as usual that G is a compact and connected Lie group with a simple Lie algebra \underline{G} . Let $\Gamma = \{s\}$ be a faithful unitary representation of G . The associated Lie algebra $\underline{\Gamma}$ has a basis $T(\rho)$ ($\rho = 1, 2, \dots, n$) with $T(\rho)^\dagger = T(\rho)$. More precisely, this is a basis for $i\underline{G}$. We choose $T(\rho)$ so that the normalization condition

$$\text{Tr}[T(\rho)T(\sigma)] = \delta_{\rho\sigma} , \quad (6.8)$$

is fulfilled.

The commutation relations of $T(\rho)$ are

$$[T(\rho), T(\sigma)] = ic_{\rho\sigma\lambda}T(\lambda) . \quad (6.9)$$

The Lagrangian for the particle dynamics is

$$L = -m[-\dot{z}^2]^{1/2} - i\text{Tr}[Ks^{-1}(\tau)D_\tau s(\tau)] . \quad (6.10)$$

Here $s \equiv s(\tau) \in \Gamma$ represents the novel degrees of freedom in L . The covariant derivative D_τ is defined by

$$D_\tau = \frac{d}{d\tau} - ie\dot{z}^a A_a , \quad A_a \equiv A_a^\alpha(z(\tau))T(\alpha) , \quad (6.11)$$

where A_a^α are the Yang-Mills potentials. The matrix K is defined by

$$K = K_\rho T(\rho) , \quad (6.12)$$

where K_ρ are real valued constants. Their specific values determine the IRR of G to which the particle belongs.

The Yang-Mills Lagrangian

$$- \frac{1}{4} \int d^3x F_{ab}^\alpha F^{\alpha ab} , \quad (6.13)$$

can be added to L . We omit it here since the treatment of the Yang-Mills is standard (see, e.g., Ref.[5]).

The definition of the internal vector $I \equiv \mathbf{I}$ in terms of s and K is

$$I = I_\alpha T(\alpha) = sKs^{-1} . \quad (6.14)$$

The resemblance of equations (6.10) and (6.14) to the corresponding equations in the previous sections should be noted.

Let us now derive the equations of motion. The general variation of s is, as usual,

$$\delta s = i\varepsilon \cdot T s, \quad \varepsilon \cdot T = \varepsilon_\alpha T(\alpha). \quad (6.15)$$

For this variation,

$$\delta L = \text{Tr}[K s^{-1}(\dot{\varepsilon} \cdot T + ie[\varepsilon \cdot T, \dot{z} A_a])s]. \quad (6.16)$$

This becomes after a partial integration

$$\delta L = -\text{Tr}[\varepsilon \cdot T(D_\tau I)], \quad (6.17)$$

where

$$D_\tau I \equiv \frac{dI}{d\tau} - ie[\dot{z}^a A_a, I]. \quad (6.18)$$

Since $D_\tau I \in \underline{\Gamma}$ and ε_α are arbitrary, the variation of s leads to Eq.(6.4), i.e.,

$$D_\tau I = 0. \quad (6.19)$$

The Euler-Lagrange equation for the variation of z^a can be obtained from

$$\begin{aligned} \frac{d}{d\tau} \frac{\partial L}{\partial \dot{z}^a} &= m \frac{d}{d\tau} \left[\frac{\dot{z}^a}{(-\dot{z}^2)^{1/2}} \right] - \frac{d}{d\tau} \text{Tr}[iIA^a] \\ &= \frac{\partial L}{\partial z^a} = -e \text{Tr}[I \partial^a A^b] \dot{z}_b. \end{aligned} \quad (6.20)$$

In view of Eq.(6.19), we thus find Eq.(6.1), i.e

$$m \frac{d}{d\tau} \left[\frac{\dot{z}^a}{(-\dot{z}^2)^{1/2}} \right] = -e \text{Tr}[IF^{ab}] \dot{z}_b, \quad (6.21)$$

where $F_{ab} \equiv F_{ab}^\alpha T(\alpha)$.

The variation of A_a gives Eq.(6.2) in a standard way. Note that for this variation, the relevant term in the interaction has the conventional form

$$- e \dot{z}^a I_\alpha A_a^\alpha(z). \quad (6.22)$$

It is again helpful to understand the form of L when the gauge group is $U(1)$, i.e., $s = \exp(i\psi)$ where ψ is a real-valued function of τ . Then we can treat K as a constant number and L differs from the usual electro-magnetic Lagrangian by a term proportional to $d\psi/d\tau$. Since the latter is a time-derivative of a function, we thus see that for the $U(1)$ gauge group, L is equivalent to the usual Lagrangian.

6.4 Gauge Properties of L

The Lagrangian L is invariant under the usual Yang-Mills gauge transformation. Thus if $h(x) \in \Gamma$, it is invariant under

$$\begin{aligned} A_a(x) &\rightarrow h(x)A_a(x)h(x)^{-1} + \frac{i}{e}h(x)\partial_a h(x)^{-1} \\ s(\tau) &\rightarrow h[z(\tau)]s(\tau) . \end{aligned} \quad (6.23)$$

It is also weakly invariant under a novel gauge group. The latter acts only on s and not on A_a . It depends in general on the nature of K . We thus explain it under two headings: (A) the generic case and (B) the non-generic case. In the discussion which follows we assume that $K \neq 0$.

A. The Generic Case

Let $H = \{g\}$ denote all elements in Γ with the property

$$gKg^{-1} = K . \quad (6.24)$$

Thus H is the stability group of K under the adjoint action.

In the generic case, the Lie algebra \underline{H} corresponding to H is just the Cartan subalgebra containing K . If \underline{C} is an a priori chosen Cartan subalgebra, then in this case, there is a $t \in \Gamma$ such that

$$tHt^{-1} = \underline{C} . \quad (6.25)$$

For example, if $G = SU(2)$, Γ is its two-dimensional irreducible representation and $K = \sigma_3$, then $H = U(1) = \{\exp[i\sigma_3\theta/2]\}$. On the other hand, if Γ is the adjoint representation of $SU(3)$, so that $\Gamma = SU(3)/Z_3$, and $K = I_3$, then \underline{H} is spanned by I_3 and Y (with a standard $SU(3)$ notation).

It can be shown that "most" K are of this sort. The closure of the set of such K is all of the Lie algebra $\underline{\Gamma}$ [41]. Note that for the generic case the group H and the Lie algebra \underline{H} are Abelian.

Under the gauge transformation

$$s \rightarrow sg , \quad g \in H , \quad (6.26)$$

with s and g τ -dependent, we find

$$L \rightarrow L - i\text{Tr}[Kg^{-1}g] . \quad (6.27)$$

The extra term is the time derivative of a function since H is Abelian. For instance, we can choose a basis K, L_α , where $\alpha = 1, 2, \dots, k$ for \underline{H} such that

$$\text{Tr}[KL_\alpha] = 0 . \quad (6.28)$$

Then we can write

$$g = e^{i\Theta K} e^{i\Theta_\alpha L_\alpha} . \quad (6.29)$$

For this form of g ,

$$-i\text{Tr}[Kg^{-1}\dot{g}] = \text{Tr}[K^2\dot{\Theta}] , \quad (6.30)$$

in view of Eq.(6.28). Thus L is weakly invariant under H . The principal fiber bundle structure relevant to L is

$$H \rightarrow \Gamma \rightarrow \Gamma/H , \quad (6.31)$$

where $\Gamma/H = \{sH\}$ is the space of left cosets. Thus Γ and Γ/H are the bundle and base spaces and H is the structure group.

These principal fiber bundles are never trivial. For instance, if Γ is the defining representation of $SU(2)$ and $H = U(1) = \exp(i\Theta\sigma_3/2)$ we get the Hopf fibration of the two sphere. The non-triviality of the bundle can also be seen in general. Since H , being Abelian is the product of $U(1)$ s (modulo perhaps a discrete group), it is infinitely connected. But Γ , being the representation of a simple compact Lie group, is finitely connected. Thus $\Gamma \neq \Gamma/H \times H$.

It follows that it is impossible to fix the gauge globally in this problem. However L is invariant under gauge transformations of the form $\{\exp(i\Theta_\alpha L_\alpha)\}$ [Cf. equations (6.29) and (6.30)]. Thus the corresponding gauge degrees of freedom can be eliminated and L can be written in terms of a configuration space $\Gamma/\{\exp(i\Theta_\alpha L_\alpha)\}$. Since L is only weakly invariant if $\Theta \neq 0$, the gauge degree of freedom for the $U(1)$ gauge group $\{\exp(iK\Theta)\}$ cannot be so eliminated.

We note here the possibility of a topological problem which can prevent the elimination of the gauge degrees of freedom associated with $\Gamma/\{\exp(i\Theta_\alpha L_\alpha)\}$. It can occur that the ratios of the eigenvalues of K are not all rational. Then $\{\exp(i\Theta K)\}$ is isomorphic to the non-compact group of translations on R^1 . The topology of the latter is incompatible with the topology of the compact H . Thus, in this case, the decomposition $g = \exp(i\Theta K) \exp(i\Theta_\alpha L_\alpha)$ is incompatible with the topology of H and we cannot eliminate these gauge degrees of freedom in a smooth way.

Note that since Γ is a faithful representation of G we can replace Γ by G in much of the preceding discussion.

B. The Non-Generic Case

The non-zero elements in the complement of the generic K 's in $\underline{\Gamma}$ constitute the non-generic K 's [12]. The stability group

$$H\{g \in \Gamma | gKg^{-1} = K\} , \quad (6.32)$$

for a non-generic K is larger than that generated by the Cartan subalgebra containing K . For example, if $G = SU(3)$ and $K = Y$, then $H = U(2)$. A basis for H is I_1, I_2, I_3, Y . There are no non-generic elements for $SU(2)$.

Let K be non-generic with stability group H . We can still choose a basis K, L_α , where $\alpha = 1, 2, \dots, k$ for \underline{H} with the property (6.28). Now

$$\text{Tr}[K[L_\alpha, L_\beta]] = \text{Tr}[L_\alpha[L_\beta, K]] = 0 . \quad (6.33)$$

If we write

$$L_\alpha, L_\beta] = d_{\alpha\beta\gamma}L_\gamma + \xi K . \quad (6.34)$$

it follows that

$$\text{Tr}[K^2] = 0 . \quad (6.35)$$

But $\text{Tr}[K^2] = \text{Tr}[KK^\dagger] > 0$. Thus $\xi = 0$. The conclusion is that \underline{H} is the direct sum of two Lie algebras:

$$\underline{H} = \underline{H}_0 + \underline{H}_1 . \quad (6.36)$$

The algebra \underline{H}_0 is one dimensional and is spanned by K . The algebra \underline{H}_1 has a basis L_α , where $\alpha = 1, 2, \dots, k$. For the $SU(3)$ example above, $\underline{H}_0 = \underline{U}(1)$ with basis Y and $\underline{H}_1 = \underline{SU}(2)$ with basis I_1, I_2, I_3 .

A general gauge transformation is of the form

$$g = e^{i\Theta K} e^{i\Theta_\alpha L_\alpha} . \quad (6.37)$$

Under $s \rightarrow sg$

$$\begin{aligned} L &\rightarrow L - i\text{Tr}[Kg^{-1}\dot{g}] \\ &= L + \text{Tr}[K^2\dot{\Theta}] + \text{Tr}[e^{-i\Theta_\alpha L_\alpha} \frac{d}{d\tau} e^{i\Theta_\alpha L_\alpha}] . \end{aligned} \quad (6.38)$$

Since \underline{H}_1 is a Lie algebra, the term within the parentheses in Eq.(6.38) is in \underline{H}_1 . Hence the last term is zero by Eq.(6.28). It follows that L is weakly invariant under gauge transformations due to H . The gauge group in this case is in general non-Abelian.

The principal fiber bundle structure is

$$H \rightarrow \Gamma \rightarrow \Gamma/H , \quad (6.39)$$

as in the generic case. It is non-trivial because H is infinitely connected. The latter statement is proved as follows: Let H_0 and H_1 be the groups associated with \underline{H}_0 and \underline{H}_1 . Then the group for $\underline{H}_0 + \underline{H}_1$ is $H = H_0 \otimes H_1$, possibly modulo some discrete finite group. Thus H is infinitely connected. For example, if $\Gamma = SU(3)$ and $K = Y$, then H is $U(2)$ which is infinitely connected. Notice that H is then not $SU(2) \otimes U(1)$, but $SU(2) \otimes U(1)/Z_2$.

As in the generic case, L is invariant under \underline{H}_1 . Thus the \underline{H}_1 gauge freedom can be eliminated and L can be written as a function on Γ/\underline{H}_1 . After this partial elimination of gauge freedom, there still remains the \underline{H}_0 gauge freedom and the principal fiber bundle structure

$$H_0 \rightarrow \Gamma/\underline{H}_1 \rightarrow \Gamma/H . \quad (6.40)$$

This remaining gauge freedom cannot be eliminated.

The gauge group can thus be reduced to $U(1)$ in L both the generic and non-generic case with the possible exception as noted in the section on the elimination of some gauge degrees of freedom in the generic case. In fact, in almost all our examples from particle mechanics, the gauge group is either $U(l)$ or can be reduced to $U(l)$ by a process similar to the one above. In Chapter 10 we prove a general theorem which shows that under certain assumptions nothing more involved than $U(l)$ bundles need appear in mechanics. That is, we show that a global Lagrangian can always be found by enlarging the space of degrees of freedom appropriate to the equations of motion to at most a $U(1)$ bundle on this space.

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6.5 An Application: Scattering off 't Hooft-Polyakov Monopole

As an aside, we now illustrate how one can apply the preceding formalism to probe a specific Yang-Mills field configuration. The field configuration of interest is that of the 't Hooft-Polyakov monopole solution [42]. When placed at large distances from the monopoles center, the Yang-Mills particle is known to behave similarly to that of an electric charge in a Dirac monopole field [43]. This follows quite simply through the use of the Lagrangian Eq.(6.10), as is shown below.

In this example $G = SU(2)$ and we may set $T(\alpha) = \sigma_\alpha/\sqrt{2}$. At large distances from the center of the 't Hooft-Polyakov monopole the gauge potentials A_a take the form

$$\begin{aligned} A_i(x) &= \frac{1}{2e|x|^2} \varepsilon_{\alpha ij} x_j \sigma_\alpha , \\ A_0(x) &= 0 , \quad |x|^2 = x_i x_i , \quad i = 1, 2, 3 . \end{aligned} \quad (6.41)$$

Here we shall restrict the discussion to non-relativistic particles. Upon substituting into L we obtain

$$\begin{aligned} L &= \frac{1}{2} m \dot{z}_i^2 - i \text{Tr}[K s^{-1} \dot{s}] - \frac{i}{4} \text{Tr}[s K s^{-1} [\hat{z}, \frac{d}{d\tau} \hat{z}]] , \\ \hat{z} &= \frac{z_i \sigma_i}{r} , \quad r = |z| . \end{aligned} \quad (6.42)$$

In analogy with Chapter 4, let us write

$$\hat{z} = t \sigma_3 t^{-1} , \quad (6.43)$$

where where $t \in \Gamma$ will be regarded as a dynamical variable defining \hat{z} . Notice that the dynamics of this system will not be altered if we make the replacement

$$s = t u , \quad u \in \Gamma , \quad (6.44)$$

in Eq.(6.42). Variations of s can be implemented through variations of u . They can also be implemented through variations of t , which will simultaneously rotate the particle and its isospin \mathbf{I} . Clearly, the above two variations are equivalent to varying s and t independently.

Thus an equivalent Lagrangian for this system is

$$L = \frac{1}{2} m \dot{z}_i^2 - i \text{Tr}[K(tu)^{-1} \frac{d}{d\tau}(tu)] - \frac{i}{4} \text{Tr}[tu K(tu)^{-1} [\hat{z}, \frac{d}{d\tau} \hat{z}]] \quad (6.45)$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{4} m r^2 \text{Tr}[(\frac{d}{d\tau} \hat{z})^2] - \frac{1}{2} \text{Tr}[I_t \sigma_3] \text{Tr}[\sigma_3 t^{-1} \dot{t}] - i \text{Tr}[K u^{-1} \dot{u}] , \quad (6.46)$$

using the fact that $\text{Tr}[\hat{z} d\hat{z}/d\tau] = 0$, and where we have defined $I_t \equiv t^{-1} \dot{t}$. Let us now take up the equations of motion. Variations of the coordinate z_i will be performed through variations of r and t [cf. Chapter 4]. Variations of u yield

$$\dot{I}_t - \frac{1}{2} \text{Tr}[\sigma_3 t^{-1} \dot{t}] [I_t, \sigma_3] = 0 . \quad (6.47)$$

The isospin vector I_t thus precesses around the third direction in internal space. The precessional frequency depends on the position of the particle through the variable t . By taking the trace of Eq.(6.47) with σ_3 , we find

$$\text{Tr}[\sigma_3 I_t] = -2n = \text{constant}. \quad (6.48)$$

The remaining equations of motion are obtained from variations in the first three terms in Eq.(6.46). Notice that the first three terms are identical to the Lagrangian Eq.(4.15) describing a charged particle in a Dirac monopole field with the assignment Eq.(6.48). Thus the Yang-Mills particle behaves as a charged particle in a Dirac monopole field with additional internal dynamics given by Eq.(6.47). Here the correspondence is

$$\frac{eg}{2\pi} \leftrightarrow -\text{Tr}[\sigma_3 I_t]. \quad (6.49)$$

Unlike the charge-monopole system of Chapter 4, n is not a fixed number in the Lagrangian, but rather a dynamical quantity which obeys the inequality:

$$n^2 \leq \frac{1}{2}\text{Tr}[I_t^2] = \frac{1}{2}\text{Tr}[I^2]. \quad (6.50)$$

We thus expect that in the quantum mechanical system for the particle, a spectrum in n will appear, consistent with the inequality (6.49)[43].

7 KALUZA-KLEIN THEORY

The unified field theory of Kaluza and Klein [44] has been experiencing a revival of interest since the development of gauge field theories in elementary particle physics. Here the dynamical fields, denoted collectively by ψ , depend both on a space-time coordinate x and a group element s , i.e., $\psi \equiv \psi(x, s)$. If $M^4 = \{x\}$ is the Minkowski space and $G = \{s\}$ denotes the internal symmetry group, the fields are thus defined on the principal fibre bundle $M^4 \times G$ [45, 46].

In this section, we will discuss such theories in the context of particle mechanics. For extended objects [47], the Kaluza-Klein formalism can be generalized in a straightforward manner [48].

7.1 Kaluza-Klein Description of Point Particles

The conventional description of the Kaluza-Klein formalism is as follows. Let x^a denote the space-time coordinate of the particle. Let $G = s$ be a semi-simple, compact Lie group represented by unitary matrices. Here we wish to use G to describe the internal degrees of freedom of the particle. The natural metric to be used on $M^4 \times G$ is a combination of the invariant line element on M^4 and the left invariant metric on G [45]. The Lagrangian is chosen to be

$$L = -m \left(-\dot{x}^2 - \lambda \text{Tr}[s^{-1} \dot{s} s^{-1} \dot{s}] \right)^{1/2} . \quad (7.1)$$

Here m and λ are constants, and $x^a \equiv x^a(\tau)$, $s \equiv s(\tau)$. Geometrically, this Lagrangian has the following meaning. Let us enlarge the Minkowski M^4 to $M^4 \times G$ and regard the latter as the configuration space. Recall that the Lagrangian for a free particle possessing no internal symmetries is proportional to the invariant length in M^4 . Similarly, the Lagrangian (7.1) is proportional to the invariant length on $M^4 \times G$.

The system given by Eq.(7.1) has the following properties:

i) The states in the quantum system described by L belong to a reducible representation of G . This differs from the quantum system for the Yang-Mills particle described in Chapter 6 (also Cf. Sec.8.4).

ii) The square of the momentum, p^2 , depends on the quadratic Casimir operator. This leads to a mass spectrum for the particle.

Regarding *i)* we note that quantum mechanical Hilbert space carries the regular representation (see Sec.8.5). Thus the multiplicity of an irreducible representation is equal to its dimension by the theorem of Peter and Weyl [11].

We can show *ii*) by computing the operator p_a which generates translations, and the internal generators I_α from L , and showing that they are algebraically related. Thus we find

$$p_a = \frac{\partial L}{\partial \dot{x}^a} = \frac{m^2 \dot{x}_a}{L} . \quad (7.2)$$

The generators I_α can be found by examine the variation

$$\delta s = i\varepsilon_\alpha T(\alpha)s , \quad (7.3)$$

where $T(\alpha)$ are the hermitian generators of G , which fulfill

$$\text{Tr}[T(\alpha)T(\beta)] = \delta_{\alpha\beta} . \quad (7.4)$$

It follows that

$$\delta L = -i\varepsilon_\alpha \frac{m^2 \lambda}{L} \text{Tr}[T(\alpha) \dot{s}s^{-1}] . \quad (7.5)$$

Thus the quantities

$$I_\alpha = i \frac{m^2 \lambda}{L} \text{Tr}[T(\alpha) \dot{s}s^{-1}] , \quad (7.6)$$

are conserved. In Section 8.5 they will be shown to generate internal symmetry transformations. Now

$$p_a p^a = \frac{m^4}{L^2} \dot{x}^2 , \quad (7.7)$$

and furthermore,

$$I_\alpha I_\alpha = -\frac{m^4 \lambda^4}{L^2} \text{Tr}[s^{-1} \dot{s}s^{-1} \dot{s}] , \quad (7.8)$$

where we have used the completeness of the generators, i.e .

$$T(\alpha) \text{Tr}[T(\alpha)s^{-1} \dot{s}] = s^{-1} \dot{s} . \quad (7.9)$$

Hence we obtain

$$p_a p^a - \frac{1}{\lambda} I_\alpha I_\alpha = -m^2 . \quad (7.10)$$

By defining the mass M as $p^2 = M^2$, we can rewrite Eq.(7.10) as follows

$$M^2 = m^2 - \frac{1}{\lambda} I_\alpha I_\alpha . \quad (7.11)$$

If λ is less than zero, the M^2 -spectrum increases with the quadratic Casimir operator. If λ is larger than zero, the mass M becomes imaginary for some value of $I_\alpha I_\alpha$, L becomes complex and the system is inconsistent.

7.2 Reformulation of the Kaluza-Klein Theory

We shall now formulate the Kaluza-Klein Lagrangian in a different way. Although the classical equations of motion for this new system are identical to those discussed in the previous section, the corresponding quantum theories differ. Unlike in the previous section, the quantum mechanical Hilbert space derived from the following Lagrangian carries an irreducible representation of the group G .

The idea here is a simple generalization of the Lagrangian formalism used to describe the relativistic point particle as discussed in detail in Chapter 5. For the latter, if the mass is m and the spin is zero, the Lagrangian has the form

$$L = p_a \dot{x}^a, \quad (7.12)$$

where $p_a p^a = -m^2$. We can generalize Eq.(7.12) to

$$L = p_a \dot{x}^a + i \text{Tr} [K s^{-1} \dot{s}], \quad (7.13)$$

where p_a is now defined by

$$p_a = \left(m^2 + \frac{1}{\lambda} \text{Tr} [K^2] \right)^{1/2} \Lambda_{a0}. \quad (7.14)$$

If $K = K_\alpha T(\alpha)$ is treated as a dynamical variable we recover precisely the system discussed in the previous section, where the quantum mechanical Hilbert space carries the left regular representation of G . Here, however, K will be treated as a constant.

The equivalence of Eqs.(7.14) and (7.1) at the classical level is now shown by proving that for the Lagrangian Eq.(7.13), $\text{Tr} [K^2]$ is the quadratic Casimir operator of the generators of G . Now consider the variation Eq.(7.3) of s for which

$$\delta (i \text{Tr} [K s^{-1} \dot{s}]) = -\text{Tr} [s K s^{-1} T(\alpha) \dot{\epsilon}_\alpha]. \quad (7.15)$$

Consequently, the following charges J_α are conserved

$$J_\alpha \equiv \text{Tr} [T(\alpha) s K s^{-1}]. \quad (7.16)$$

J_α actually form the generators of G on the quantum mechanical Hilbert space. The desired result

$$p_a p^a = -m^2 - \frac{1}{\lambda} J_\alpha J_\alpha, \quad (7.17)$$

then follows from Eq.(7.14).

Note that the system described here is equivalent to the description of a free Wong particle (Cf. Chap.6), except for the constraint (7.17), which resulted from the redefinition of momentum. The constraint (7.17) is rather arbitrary. In fact, we can easily arrange for any mass-internal symmetry relation by changing the function appearing in front of Λ_{a0} in Eq.(7.14)). In order to see this we notice that if $C_n(J)$, $n = 1, 2, \dots, \text{rank}(G)$, denotes the Casimir invariants of G , then by Eq.(7.16), $C_n(J) = C_n(K)$. It follows that by setting

$$p_a = f(C_1(K), C_2(K), \dots)\Lambda_{a0}, \quad (7.18)$$

for a suitable function f , we can get any mass spectrum. For a conventional formulation of theories of this kind we refer the reader to the work by N. Mukunda et al. [49]. Note that the procedure of redefining momenta (or actually the mass) of a particle was also found to be useful in introducing an anomalous magnetic moment for a spinning particle (Cf. Sec.5.3).

7.3 Interaction with External Fields

Above we have considered a non-interacting particle with internal degrees of freedom. The incorporation of external fields is straightforward and as a result we can obtain the Wong equations [39]. In order to achieve this result we replace the time derivatives of the group element $s(t)$ in Eq.(7.1) by the corresponding covariant derivative Eq.(6.11), i.e., we consider the Lagrangian [48]

$$L = -m \left(-\dot{x}^2 - \lambda \text{Tr} [s^{-1} D_\tau s s^{-1} D_\tau s] \right)^{1/2}. \quad (7.19)$$

The equation of motion for the non-Abelian charges

$$I_\alpha = i \frac{m^2 \lambda}{L} \text{Tr} [\text{T}(\alpha) (D_\tau s) s^{-1}], \quad (7.20)$$

is, as before, obtained by considering the variation Eq.(7.3), i.e., $\delta s = i \varepsilon_\alpha \text{T}(\alpha) s$. The analogue of the Eq.(7.5) is then

$$\begin{aligned} \delta L = & -i \dot{\varepsilon}_\alpha \frac{m^2 \lambda}{L} \text{Tr} [\text{T}(\alpha) (D_\tau s) s^{-1}] \\ & + \varepsilon_\alpha \frac{m^2 \lambda}{L} \text{Tr} [[\dot{x}^a A_a, \text{T}(\alpha) (D_\tau s) s^{-1}]] . \end{aligned} \quad (7.21)$$

We obtain the equation of motion

$$\frac{dI_\alpha(\tau)}{d\tau} = i [\dot{x}^a(\tau) A_a(x(\tau)), I_\alpha(\tau)], \quad (7.22)$$

i.e., the Eq.(6.19).

The Euler-Lagrange equation for $x^a(\tau)$ is

$$\begin{aligned} \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{x}_a} \right) &= m^2 \frac{d}{d\tau} \left(\frac{\dot{x}^a}{L} \right) - \frac{d}{d\tau} (IA^a) \\ &= -e \text{Tr} [I \partial^a A_b] \dot{x}_b, \end{aligned} \quad (7.23)$$

as in the derivation of Eq.(6.20). By choosing the parameter τ in such a way that $L = m$, we obtain the Lorentz-Maxwell-Wong equation (6.1) in the proper time gauge.

Finally, we notice that in the presence of an external field the mass-internal symmetry relation Eq.(7.10) is changed to

$$(p_a + e I_\alpha A_a^\alpha) (p^a + e I_\alpha A^{a\alpha}) - \frac{1}{\lambda} I_\alpha I_\alpha = -m^2. \quad (7.24)$$

8 THE CANONICAL FORMALISM AND QUANTIZATION

In this section we carry out the canonical quantization for the various systems discussed in the previous chapters. Since all the Lagrangians presented here are singular, i.e., there exist constraints amongst the corresponding phase-space variables, we will rely on Dirac's quantization procedure. For extensive reviews on this procedure, see Ref.[3].

A common feature of all the systems presented here is that elements of a group G appear as dynamical variables. A method of treating group elements for setting up the canonical formalism was given in Refs.[3] and [12]. We recall it below.

Let $s \in G$ be parametrized by a set of variables (local coordinates) $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ so that $s = s(\xi)$, n being the dimension of G . The functional form of $s(\xi)$ will not be important for us. We can then regard the Lagrangian as a function of ξ and $\dot{\xi}$ as well as of any other configuration space variables present in the system and of their velocities.

We first note a preliminary identity. Let us define a set of functions $f(\varepsilon) = (f_1(\varepsilon), f_2(\varepsilon), \dots, f_n(\varepsilon))$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, by

$$e^{iT(\alpha)\varepsilon_\alpha} s(\xi) = s[f(\varepsilon)], \quad f(0) = \xi \quad , \quad (8.1)$$

where $T(\alpha)$'s form a basis for the Lie algebra of the group with

$$[T(\alpha), T(\beta)] = i c_{\alpha\beta\gamma} T(\gamma) \quad . \quad (8.2)$$

Differentiating Eq.(8.1) with respect to ε_α and setting $\varepsilon = 0$, we find

$$iT(\alpha)s(\xi) = \frac{\partial s(\xi)}{\partial \xi_\beta} N_{\beta\alpha}(\xi) \quad , \quad (8.3)$$

where

$$N_{\beta\alpha}(\xi) = \left. \frac{\partial f_\beta(\varepsilon)}{\partial \varepsilon_\alpha} \right|_{\varepsilon=0} \quad . \quad (8.4)$$

Here $\det N \neq 0$, for if not, there exist χ_α , not all zero, such that $N_{\rho\sigma}\chi_\sigma = 0$. By Eq.(8.3), $\chi_\sigma T(\sigma)s(\xi) = 0$, and hence $\chi_\sigma T(\sigma) = 0$. But this contradicts the linear independence of the $T(\alpha)$'s.

Now the coordinates ξ_α and their conjugate momenta π_α fulfill the Poisson bracket (PB) relations

$$\begin{aligned}\{\xi_\alpha, \xi_\beta\} &= \{\pi_\alpha, \pi_\beta\} = 0 \quad , \\ \{\xi_\alpha, \pi_\beta\} &= \delta_{\alpha\beta} \quad .\end{aligned}\tag{8.5}$$

Since N is nonsingular, we can replace the phase space variables π_α by t_α where

$$t_\alpha = -\pi_\beta N_{\beta\alpha} \quad .\tag{8.6}$$

From Eqs.(8.3) and (8.5) it follows that

$$\{t_\alpha, s\} = i T(\alpha) s \quad ,\tag{8.7}$$

$$\{t_\alpha, s^{-1}\} = -i s^{-1} T(\alpha) \quad ,\tag{8.8}$$

$$\{t_\alpha, t_\beta\} = c_{\alpha\beta\gamma} t_\gamma \quad .\tag{8.9}$$

To prove Eq.(8.9) note that from the Jacobi identity and Eq.(8.5) it follows that

$$\begin{aligned}\{\{t_\alpha, t_\beta\}, s\} &= -\{\{t_\beta, s\}, t_\alpha\} - \{\{s, t_\alpha\}, t_\beta\} \\ &= i c_{\alpha\beta\gamma} T(\gamma) s \quad .\end{aligned}\tag{8.10}$$

Thus

$$\{t_\alpha, t_\beta\} = c_{\alpha\beta\gamma} t_\gamma + F \quad ,\tag{8.11}$$

where $\{F, s(\xi)\} = 0$. Consequently F is independent of the π 's. Substituting $\pi_\alpha = 0$ in Eq.(8.11), we find $F = 0$. This proves Eq.(8.9). It also follows from a direct calculation using Eq.(8.3) and Eq.(8.6).

The PB's Eqs.(8.7), (8.8), and (8.9) involving t_α and s are simple and do not require a particular parameterization for $s(\xi)$. We therefore find it convenient to use these variables in canonically quantizing the systems below.

8.1 Non-Relativistic Spinning Particles

Here we show how the Hamiltonian description for a spinning particle (Eqs.(3.1)-(3.6)) is obtained from the Lagrangian Eq.(3.19) [Eqs.(3.26)]. Now G is $SU(2) = \{s\}$ and $T(i) = \sigma_i/2$. The phase-space coordinates are x_i, p_i, s and t_i , where p_i is canonically conjugate to x_i . From Eq.(3.19) [(3.26)] we obtain the following primary constraint:

$$\phi_i = t_i - S_i \approx 0 \quad ,\tag{8.12}$$

where S_i is defined in Eq.(3.18). From Eqs.(8.7), (8.8), and (8.9),

$$\{\phi_i, \phi_j\} = \varepsilon_{ijk}(\phi_k - S_k) \quad . \quad (8.13)$$

Applying Dirac's procedure, the following Hamiltonian is obtained from the Lagrangian Eq.(3.19):

$$H = \frac{p^2}{2m} + \phi_i \eta_i \quad , \quad (8.14)$$

where η_i are Lagrange multiples. From the requirement that

$$\{\phi_i, H\} = 0 \quad , \quad (8.15)$$

on the reduced phase space, we find that there exist no secondary constraints. Instead, we obtain conditions on η_i , i.e.,

$$\varepsilon_{ibc} \eta_b t_c = 0 \quad . \quad (8.16)$$

Since those $\eta_i = \eta_i^{(t)}$ in a direction parallel to t_i are left arbitrary, only those variables which have a weakly zero PB's have a well defined time-evolution. Only such variables are of physical interest. We will call them observables. Of course, x_i and p_i are observables. In addition, so are t_i and S_i . This follows from

$$\{t_i, \phi_j\} = \varepsilon_{ijk} \phi_k \quad , \quad (8.17)$$

and

$$\{s, \eta_i^{(t)} \phi_i\} = -\frac{i}{2} \eta_i^{(t)} \sigma_i s = \frac{i}{2} s \sigma_3 \quad . \quad (8.18)$$

Eq.(8.18), which is weakly valid, corresponds to an infinitesimal version of the $U(1)$ gauge transformation discussed in Chap.3. Hence only those functions of s which are invariant under gauge transformations (3.36) are also observables. But these are precisely S_i or functions thereof. However, we can eliminate S_i by applying the constraints. Thus a complete set of observables on the reduced phase space are

$$x_i, p_i \quad \text{and} \quad t_i \quad , \quad (8.19)$$

since S_i can be eliminated via the constraints. In so doing note that

$$t_i t_i = \lambda^2 \quad . \quad (8.20)$$

It remains to compute the Dirac Bracket (DB's) for the variables (8.20). But these are identical to the corresponding PB's since all variables (8.20) have weakly zero PB

with the constraints (Cf. Eq.(8.17)). Consequently, we have recovered the Hamiltonian description for a non-relativistic spinning particle Eqs.(3.1)-(3.5). Note that instead of eliminating S_i via the constraints, we could have eliminated t_i . In this case the DB's involving S_i do differ from the corresponding PB's. It can be shown that DB's for two S_i 's are given by Eq.(3.4) Consequently, both procedures are equivalent. It is straightforward to repeat the above analysis in the case where a spinning particle with magnetic moment μ is placed in an external magnetic field.

In passing to the quantum mechanical system, as usual, we replace the Poisson bracket by $-i$ times the commutator bracket. Now the particular representation which occurs in the quantum theory is determined by λ (Cf. Eq.(8.20)). This implies that i) only one irreducible representation (IRR) appears in the theory, and ii) quantization is possible only if λ^2 is restricted to having the values

$$t^2 = l(l+1) \quad , \quad l = 0, \frac{1}{2}, 1, \dots \quad (8.21)$$

Note ii) is similar to the Dirac charge quantization condition which occurs in magnetic monopole theory.

8.2 Magnetic Monopoles

The canonical quantization for the magnetic monopole theory proceeds in a similar fashion to the preceding section. The essential difference is due to equation Eq.(4.14), which constrains the configuration space variables for the monopole. Consequently, the independent phase space coordinates now consist of r , p_r , s and t_i , where p_r is canonically conjugate to r . From Eq.(4.15) we find only one primary constraint,

$$\phi \equiv \hat{x}_i t_i - n \approx 0, \quad (8.22)$$

where \hat{x}_i is defined in Eq.(4.13). Computing the Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{1}{2mr^2} (t_i t_i - n^2) + \eta \phi. \quad (8.23)$$

Here η is a Lagrange multiplier. The constraint Eq.(8.22) is rotationally invariant, i.e., $\{\phi, t_i\} = 0$. Hence the requirement that $\{H, \phi\} = 0$ on the reduced phase space leads to no secondary constraints.

As before, observables are those variables which have zero PB's with ϕ . Among them are

$$x_i, p_i, t_i \quad \text{and} \quad \hat{x}_i. \quad (8.24)$$

The latter follows from

$$\{\phi, s\} = \frac{i}{2}s\sigma_3, \quad (8.25)$$

which is analogous to (8.18). As before, this corresponds to a $U(1)$ gauge transformation, and only those functions of s which are invariant under such transformations are observables. But these are precisely \hat{x}_i or functions thereof, so Eq.(8.24) corresponds to a complete set of observables subject to the constraint (8.22).

A representation for the quantum theory can be constructed as follows. Let us regard the wave functions as functions of r and s :

$$\psi \equiv \psi(r, s). \quad (8.26)$$

The position coordinates are diagonal in this representation in view of Eq.(4.13). The momentum p_r acts as the usual differential operator on ψ . The operators t_i are the differential operators which represent the elements $\sigma_i/2$ in the left regular representation of $SU(2)$, i.e.,

$$[\exp(i\theta_k t_k) \psi](r, s) = \psi(r, \exp(-\frac{i}{2}\theta_k t_k)s). \quad (8.27)$$

The constraint (8.22) is taken into account by imposing the condition

$$\hat{x}_i t_i \psi = n\psi, \quad (8.28)$$

on the wave functions. In view of Eqs.(8.27) and (4.14), this means

$$\psi(r, \exp(-\frac{i}{2}\theta_k t_k)s) = \psi(r, s) \exp(i\theta n), \quad (8.29)$$

The scalar product of wave functions is

$$(\psi, \chi) \equiv \int_0^\infty dr r^2 \int_{SU(2)} d\mu(s) \psi^*(r, s) \chi(r, s), \quad (8.30)$$

where $d\mu$ is the invariant Haar measure on $SU(2)$.

Let $\{D^j(s)\}$ be the representation of $SU(2)$ with angular momentum j . Wave functions ψ with finite norm have the expansion [11]

$$\psi(r, s) = \sum_j \sum_{\rho, \sigma} \alpha_{\rho\sigma}^j D_{\rho\sigma}^j(s). \quad (8.31)$$

Here $D_{\rho\sigma}^j(s)$ are the matrix elements of $\{D^j(s)\}$ in the conventional basis with the third component of angular momentum diagonal.

The constraint (8.29) means that in Eq.(8.31), only those $\alpha_{\rho\sigma}^j$ with $\sigma = -n$ are non-zero. Thus

$$\{D_{\rho,-n}^j\} \quad , \quad \text{fixed } n \quad , \quad (8.32)$$

is a basis for expansions of the form (8.31). Since σ is necessarily integral or half-integral, we have the Dirac quantization condition

$$2n = \text{integer} \quad . \quad (8.33)$$

In (8.32), j and ρ are half-integral if $2n$ is odd and integral if $2n$ is even.

The quantum mechanics outlined here is essentially equivalent to conventional treatments.

8.3 Relativistic Spinning Particles

In this section we shall only be concerned with free relativistic spinning particles. The group G is now the connected component of the Lorentz group $L_+^\uparrow = \{\Lambda_b^a\}$ with generators σ_{ab} with matrix elements $(\sigma_{ab})_{cd}$ obtained from Eq.(5.5). Here Eq.(8.2) reads

$$[\sigma_{ab}, \sigma_{cd}] = i(-\eta_{bc}\sigma_{ad} + \eta_{bd}\sigma_{ac} + \eta_{ac}\sigma_{bd} - \eta_{da}\sigma_{bc}) \quad . \quad (8.34)$$

In addition Eqs.(8.9) and (8.7) are replaced by

$$[t_{ab}, t_{cd}] = (-\eta_{bc}t_{ad} + \eta_{bd}t_{ac} + \eta_{ac}t_{bd} - \eta_{da}t_{bc}) \quad , \quad (8.35)$$

and

$$\{t_{ab}, \Lambda\} = i\sigma_{ab}\Lambda \quad . \quad (8.36)$$

A. Spinless Particles

For simplicity, we begin with the case where the spin is absent, i.e., $\lambda = 0$ in Eq.(5.11). The phase space coordinates are given by z_a , π_a , Λ , and t_{abd} , where π_a is canonically conjugate to z_a . The primary constraints are

$$\phi_{ab} = t_{ab} \approx 0 \quad , \quad (8.37)$$

and

$$\theta_a = p_a - \pi_a \approx 0 \quad . \quad (8.38)$$

where p_a is defined in Eq.(5.3). The equation (5.37) follows because there are no time-derivatives of Λ appearing in the Lagrangian. The constraints obey the PB algebra:

$$\{\theta_a, \theta_b\} = 0, \quad (8.39)$$

and

$$\{\phi_{ab}, \theta_c\} = i(\sigma_{ab})_{cd} p^d, \quad (8.40)$$

along with Eq.(8.35).

Because of the reparametrization symmetry of the Lagrangian, the Hamiltonian consists solely of the constraints (for a discussion of this issue, see, for example, Ref.[3]), i.e.,

$$H = \rho^{ab} \phi_{ab} + \kappa^a \theta_a, \quad (8.41)$$

where ρ^{ab} and κ^a are Lagrange multipliers. Once again, there are no secondary constraints. Instead ρ^{ab} and κ^a are restricted by

$$\rho^{ab} p_b = 0, \quad (8.42)$$

and

$$\kappa_a p_b - \kappa_b p_a = 0, \quad (8.43)$$

on the reduced phase space. In deriving Eqs.(8.42) and (8.43) we have used the representation for σ_{ab} given by Eq.(5.5). Eqs.(8.42) and (8.43) imply that

$$\rho_{ia} = \varepsilon_{ijk} r_j \Lambda_{ka}^{-1}, \quad i, j, k = 1, 2, 3, \quad (8.44)$$

and

$$\kappa_a = k p_a, \quad (8.45)$$

where r_i , $i = 1, 2, 3$ and k are undetermined constants. This, in turn, implies that four linearly independent combinations of Eqs.(8.37) and (8.38) form first class constraints, namely

$$\phi_i \equiv \varepsilon_{ijk} \phi_j \Lambda_{ka}^{-1}, \quad (8.46)$$

and

$$\phi_0 \equiv \theta_a \Lambda_0^a. \quad (8.47)$$

Observables, by definition, have zero PB's with ϕ_a . Among them are

$$\pi_a \text{ and } J_{ab} = z_a \pi_b - z_b \pi_a, \quad (8.48)$$

where we have applied the constraints. Additional observables can be formed from p_a and t_{ab} , however, these degrees of freedom can be eliminated via the constraints Eqs.(8.37) and (8.38). There exist six independent observables amongst the remaining ten degrees of freedom for the system. The former are exactly given by Eqs.(8.48), since four constraints now exist on the variables π_a and J_{ab} ,

$$\pi_a \pi^a = -m^2, \quad (8.49)$$

and

$$W^a = 0, \quad W^a \equiv \frac{1}{2} \varepsilon^{abcd} \pi_b J_{cd}. \quad (8.50)$$

Note that Eq.(8.50) yields three relations since $\pi_a W^a$ is identically zero. Equations (8.49) and ((8.50) indicate that the above system describes a particle of mass m and spin zero.

It remains to compute the DB's for the variables (8.48). We first define J_{ab}^* :

$$J_{ab}^* = J_{ab} + \phi_{abd}, \quad (8.51)$$

which, along with π_a , form a complete set of first class variables. Consequently, all DB's involving J_{ab}^* and π_a are identical to the corresponding PB's. Equivalently, we can define a DB with J_{ab} according to

$$\{J_{ab}, \cdot\}^* \equiv \{J_{ab}^*, \cdot\}. \quad (8.52)$$

Using Eq.(8.52), we obtain the usual Poincare algebra for π_a and J_{ab} ,

$$\begin{aligned} \{\pi_a, \pi_a\}^* &= 0, \\ \{J_{ab}, \pi_c\}^* &= \eta_{ac} \pi_b - \eta_{bc} \pi_a, \\ \{J_{ab}, J_{cd}\}^* &= \eta_{ac} J_{bd} + \eta_{bd} J_{ac} + \eta_{ad} J_{cb} + \eta_{bc} J_{da}. \end{aligned} \quad (8.53)$$

Note that the equations (8.49) and (8.50) lie in the center of the algebra generated by π_a and J_{ab} . So if desired, one can eliminate redundant variables from π_a and J_{ab} , by hand, without conflict with their DB's (8.53).

Since the Hamiltonian is simply a linear combination of the constraints (8.46) and (8.47), it generates no time-evolution for π_a and J_{ab} . So if desired, we can declare that π_0 generates time-translations. Also we can identify

$$x_i = \frac{1}{\pi_0} J_{i0}, \quad i = 1, 2, 3, \quad (8.54)$$

as the space coordinate of the particle. It fulfills

$$\{x_i, \pi_j\}^* = \delta_{ij}, \quad (8.55)$$

and

$$\{x_i, x_j\}^* = 0. \quad (8.56)$$

In proving Eq.(8.56), a direct computation yields

$$\{x_i, x_j\}^* = \frac{1}{\pi_0^2} \left(-J_{ij} + \frac{1}{\pi_0} (J_{i0}\pi_j - J_{j0}\pi_i) \right). \quad (8.57)$$

The result is then obtained after applying the definition for J_{ab} (Cf. Eq.(8.48)).

B. Spinning Particles

In this case, we consider non-zero values for λ in Eq.(5.11). Here, Eq.(8.37) is replaced by

$$\phi_{ab} = t_{ab} - S_{ab} \approx 0, \quad (8.58)$$

where S_{ab} is given by Eq.(5.4). Equation (8.58), along with (8.38), form the primary constraints for this system. Their PB' are given by (8.39), (8.40) and

$$\begin{aligned} \{\phi_{ab}, \phi_{cd}\}^* &= \eta_{bc}(\phi_{da} - S_{da}) - \eta_{ad}(\phi_{bc} - S_{bc}) \\ &+ \eta_{ac}(\phi_{bd} - S_{bd}) - \eta_{dc}(\phi_{ca} - S_{ca}). \end{aligned} \quad (8.59)$$

The Hamiltonian, once again, consists solely of the constraints, i.e., Eq.(8.41). Again there are no secondary constraints, and, instead, the Lagrange multipliers are restricted by (8.42) and

$$\kappa_a p_b - \kappa_b p_a = 2(S_{ac}\rho_b^c - S_{bc}\rho_a^c), \quad (8.60)$$

on the reduced phase space. After applying the definitions for S_{ab} and p_a (Cf. Eqs.(5.3) and (5.4)), we find

$$\frac{m}{2\lambda} \left(\tilde{\kappa}_a \eta_{b0} + \eta_{a1} \tilde{\rho}_{b2} - \eta_{a2} \tilde{\rho}_{b1} - \tilde{\kappa}_b \eta_{a0} - \eta_{b1} \tilde{\rho}_{a2} + \eta_{b2} \tilde{\rho}_{a1} \right) = 0, \quad (8.61)$$

where $\tilde{\kappa} \equiv \Lambda^{-1}\rho$ and $\tilde{\rho} \equiv \Lambda^{-1}\rho\Lambda$. Eqs.(8.42) and (8.61) along with

$$\tilde{\rho}_{ab} = -\tilde{\rho}_{ba}, \quad (8.62)$$

imply that all components of $\tilde{\kappa}$ and $\tilde{\rho}$ vanish except for $\tilde{\kappa}_0$ and $\tilde{\rho}_{12}$. Consequently, there are two first class constraints: Eq.(8.47) and

$$\phi_{ab}\Lambda^{a1}\Lambda^{b2} \approx 0 . \quad (8.63)$$

Once again p_a and J_{ab} (Cf. Eq.(8.48)) are observables for the system. However, they no longer form a complete set of observables. Since now only two first class constraints can be found, there exist a total of eight observables for the system. But there are only six independent degrees of freedom in p_a and J_{ab} . Additional observables for this system are S_{ab} . Note that there are four constraining equations on S_{ab} :

$$S_{ab}\pi^b = 0 , \quad (8.64)$$

$$\frac{1}{2}S_{ab}S^{ab} = \lambda^2 . \quad (8.65)$$

Eq.(8.64), which holds on the reduced phase space, contains a total of three constraints since $S_{ab}\pi^a\pi^b$ vanishes identically. Thus two independent degrees of freedom remain in S_{ab} , i.e., Π_a , J_{ab} and S_{ab} form a complete set of observables.

Alternatively, the five independent degrees of freedom in J_{ab} and S_{ab} can be expressed more compactly by

$$M_{ab} \equiv J_{ab} + S_{ab} . \quad (8.66)$$

Now M_{ab} contains all five degrees of freedom since

$$W_a W^a = m^2 \lambda^2 , \quad (8.67)$$

where

$$W^a = \frac{1}{2}\varepsilon^{abcd}\pi_b M_{cd} , \quad (8.68)$$

is the only constraining equation on M_{ab} .

Equation (8.67) indicates that particle has a fixed spin λ . It, along with (8.49), can be used to eliminate, by hand, the redundant degrees of freedom from π_a and M_{ab} . This follows because (8.49) and (8.67) lie in the center of the algebra generated π_a and M_{ab} . remains to be shown that this algebra is, once again, the Poincaré algebra.

With this in mind, we define

$$M_{ab}^* \equiv M_{ab} + \phi_{ab} = J_{ab} + t_{ab} , \quad (8.69)$$

which, along with π_a , form a complete set of first class a variables. DB's involving M_{ab} can the be defined by

$$\{M_{ab} , \cdot\}^* \equiv \{M_{ab}^* , \cdot\} , \quad (8.70)$$

while DB's involving π_a are equivalent to the corresponding PB's. Using Eq.(8.70) we then verify that π_a and M_{ab} generate the Poincaré algebra. the Poincare algebra .

Again, if desired, we can declare that time-translations are generated by π_0 . The standard canonical DB's (8.55) and (8.56) are obtained after defining the space coordinate x_i according to (see Sudarshan and Mukunda, Ref.[3], p.439-454):

$$x_i = \frac{1}{\pi_0} \left(M_{i0} - \frac{\varepsilon_{ijk}\pi_j W_k}{m(m - \pi_0)} \right) . \quad (8.71)$$

Note that Eq.(8.71) does reduce to Eq.(8.54) in the limit of zero spin. In addition, spin 3-vectors \tilde{S}_i with the usual brackets

$$\{\tilde{S}_i, \tilde{S}_j\}^* = \varepsilon_{ijk}\tilde{S}_k , \quad (8.72)$$

can be defined in terms of the Poincaré generators [3]:

$$\tilde{S}_i = -\frac{1}{m} \left(W_i - \frac{W_j \pi_j \pi_i}{\pi_0(m - \pi_0)} \right) . \quad (8.73)$$

The variables \tilde{S}_i differs from $S_i = \varepsilon_{ijk}S_{jk}/2$ which can be reconstructed in terms of the Poincaré generators. The latter variables do not satisfy Eq. (8.72).

In conclusion, the eight degrees of freedom in π_a and M_{ab} can be expressed in terms of x_i , π_i , and \tilde{S}_i , which have standard bracket relations. The Hamiltonian for π_0 is

$$H = \sqrt{\pi_i^2 + m^2} , \quad (8.74)$$

where we have chosen the positive root in eliminating the constraint (8.49). In terms of the variables x_i , π_i , and \tilde{S}_i the constraint (8.67) translates to

$$\tilde{S}_i \tilde{S}_i = \lambda^2 . \quad (8.75)$$

This system represents the obvious generalizations of the non-relativistic spinning particle system described in Sections 3.1 and 8.1. As before, we find that only one IRR appears in the quantum theory, and quantization is possible only if λ^2 is restricted to having the values given in Eq.(8.21).

8.4 Yang-Mills Particles

For simplicity we shall specialize to the case of non-relativistic particles. Consequently, we replace the first term in Eq.(6.10) by $m\dot{x}_i^2/2$. Now the phase space coordinates are

x_i , p_i , s , and t_α . Here $s \in \Gamma$, where Γ being a faithful unitary representation of an arbitrary compact connected Lie group G .

The Hamiltonian for this system is

$$H = \frac{1}{2m} (p_i - eA_i^\alpha t_\alpha)^2 + eA_0^\alpha t_\alpha + \eta_\alpha \phi_\alpha , \quad (8.76)$$

where the primary constraints are given by

$$\phi_\alpha = I_\alpha - t_\alpha \approx 0 . \quad (8.77)$$

In deriving Eq.(8.76) we have used the constraints to rearrange terms. As usual, there are no secondary constraints and the η 's are restricted by

$$c_{\sigma\rho\lambda}\eta_\rho t_\alpha = 0 , \quad (8.78)$$

on the constrained surface. Let there be k independent vectors $\{\eta_\rho = \eta_\rho^{(A)}, A = 1, 2, \dots, k\}$ satisfying Eq.(8.78). The first class constraints of the theory are

$$\phi^A = \eta_\rho^{(A)} \phi_\rho . \quad (8.79)$$

Observables have zero PB's with ϕ^A . They consist of

$$x_i , p_i , t_\alpha . \quad (8.80)$$

Note that the I_α 's are also observables. This follows from

$$s, \phi^A = -\eta(A)s , \quad (8.81)$$

where $\eta(A) = \eta_\alpha^A T(\alpha)$ generate the stability group of $t = t_\alpha T(\alpha)$ under the adjoint action. This group is isomorphic to the group H (Cf. Sec.6.4). From Eq.(8.81), only those functions of s which are invariant under the action of the little group of t are of interest. These must be functions of I . However, the I 's can be eliminated via the constraint. Thus we are left with variables (8.80). Since they all have weakly zero PB with ϕ_α , all DB's involving these variables are identical to the corresponding PB's.

As in Section 8.1, not all the t_α 's are independent. From Eqs.(6.14) and (8.77), t is constrained to lie on a certain orbit in $\underline{\Gamma}$. These orbits are labeled by the constants K_α . Using Eq.(6.14) any function of the t_α which is a constant on the orbits can be written as a function of the K_α 's. In particular, the Casimir invariants can be expressed in terms of K . For the case of $G = SU(2)$, we are left with one constraint, which is analogous to (8.20).

The particular representation which occurs in the quantum theory is determined by the Casimir invariants, which in turn are determined by the K_α 's. Once again, only one IRR appears in the quantum theory and quantization is possible only if the Casimir invariants formed out of the K_α 's are restricted to a certain discrete set.

8.5 Kaluza-Klein Formulation

As was true in several other cases the Lagrangian here (Cf. Eq.(7.1)) contains a reparametrization symmetry. Here we shall remove it by fixing $x^0 = \tau$. The Hamiltonian for this system is

$$H = \sqrt{m^2 + p_i^2 - \frac{1}{\lambda} t_\alpha^2} , \quad (8.82)$$

where x_i, p_i , and t_α are the usual phase space variables. Unlike the previously discussed systems, there are no constraints on the phase space variables. This is due to the fact that t_α can be expressed in terms of ss^{-1} (Cf. Eq.(7.6). Here $I_\alpha = t_\alpha$).

In the previous section $t_\alpha T(\alpha)$ was constrained to lie on certain orbits in the Lie algebra. These orbits determined which IRR was to appear in the quantum theory. Now there are no constraints on the variables t_α and, consequently, all IRR's appear in the quantum theory.

In setting up the quantum theory, we can write down wave-functions which are functions of s as well as x_i :

$$\psi = \psi(s, x) . \quad (8.83)$$

This follows since all components $s_{\alpha\beta}$ can be simultaneously diagonalized. Then the t_α 's are differential operators which represent the generators $T(\alpha)$ in the left regular representation of the group. In particular,

$$(\exp(i\theta_\alpha L_\alpha) \psi)(s, x) = \psi(\exp(-i\theta_\alpha T(\alpha)) s, x) . \quad (8.84)$$

The scalar product with respect to which the t_α 's are Hermitian is given by

$$(\phi, \psi) = \int d\mu(s) d^3x \phi^*(s, x) \psi(s, x) , \quad (8.85)$$

where $d\mu(s)$ is the invariant Haar measure of the group. The left regular representation is highly reducible. Every irreducible representation occurs with a multiplicity equal to its own dimension.

If an irreducible representation of the Kaluza-Klein system is desired, we must deal with the formulation given in Section 7.2. As was noted earlier, the system there is identical to that of the Yang-Mill particle with the mass

$$\sqrt{m^2 + \frac{1}{\lambda} \text{Tr}[K^2]} . \quad (8.86)$$

Excluding this additional requirement, the quantization of such a system is identical to that discussed in Section 8.4.

9 PSEUDO-CLASSICAL DESCRIPTION

In the previous sections, we have seen how to describe the spin and isospin degrees of freedom of a particle in terms of dynamical group elements $g(\xi) \in G$. It has been pointed out however, that Grassmann variables, i.e., anti-commuting c -numbers, can be utilized for the same purpose. Such a formulation is usually, referred to as pseudo-classical mechanics. Upon quantization the anti-commuting c -numbers leads to certain irreducible representations of some symmetry group G . In the case of spin degrees of freedom it was discussed by Volkov, Peletminskii [50] and Martin [51] that the classical Grassmann variables, are replaced by Pauli matrices after quantization. These considerations for point particles (and extended objects) have recently been discussed in much detail in the literature [52, 53]. They have also been applied to internal degrees of freedom [40, 54, 55].

The algebra of the anti-commuting Grassmann variables can be used to extend the notion of Lie algebras to graded Lie-algebras [56]. The notation of graded Lie algebras, usually referred to as super-symmetry, has been extended to field theory leading to global and local (i.e., super-gravity) super-symmetric field theories. For a review of this very dynamic field of research and for further references see, e.g., Ref.[57]. In this Chapter we apply some of these concepts to the description of the systems discussed in the previous chapters.

9.1 Non-Relativistic Spinning Particles

The Lagrangian for a free, non-relativistic spinning particle involving dynamical anti-commuting Grassmannian variables, $f_a(\tau)$, is [53, 54]

$$L_0 = \frac{1}{2}\dot{x}_i^2 + \frac{1}{2}f_a\dot{f}_a . \quad (9.1)$$

The equations of motion derived from Eq.(9.1) are

$$m\ddot{x}_a = 0 , \quad \dot{f}_a = 0 . \quad (9.2)$$

The orbital angular momentum $L_a = \varepsilon_{abc}x_ap_b$ and spin S_a , as defined by

$$S_a = -\frac{i}{2}\varepsilon_{abc}f_b\dot{f}_c . \quad (9.3)$$

Note that the Lagrangian (9.1) is weakly invariant under the transformations

$$x_a \rightarrow x_a - i\varepsilon\frac{f_a}{\sqrt{m}} , \quad f_a \rightarrow f_a + \varepsilon\sqrt{m}\dot{x}_a , \quad (9.4)$$

where ϵ is a c-number Grassmann parameter. We define (9.4) to be a "super-symmetry" transformation. Under (9.4),

$$L_0 \rightarrow L_0 + i \frac{\sqrt{m}}{2} \frac{d}{dt} (f_a \dot{x}_a \epsilon) , \quad (9.5)$$

Thus the action $\int d\tau L_0$ is invariant.

Interactions can be added to the Lagrangian (9.1) in a straightforward manner, although in general, they will not be invariant under (9.4). For example, consider the interaction of a particle having magnetic moment μ (and no charge) with an external electro-magnetic field \mathbf{B} . The form of the Lagrangian is then the same as in Eq.(3.26), i.e.,

$$L = L_0 - \mu S_a B_a . \quad (9.6)$$

A variation of the coordinate x_a leads to the equation of motion Eq.(3.31). A variation of f_a leads to

$$\dot{f}_a + \mu \varepsilon_{abc} f_b B_c = 0 . \quad (9.7)$$

Using the definition (9.3) of the spin angular momentum, we obtain the spin precession equation Eq.(3.34), i.e.,

$$\dot{S}_a = \mu \varepsilon_{abc} B_b S_c . \quad (9.8)$$

The interaction given in Eq.(9.3) is not invariant under the super-symmetry transformation (9.4), since it transforms according to

$$-\mu \dot{S}_a B_a \rightarrow -\mu \dot{S}_a B_a + i \mu \sqrt{m} \epsilon \varepsilon_{abc} B_a \dot{x}_b f_c + i \frac{\mu \epsilon}{\sqrt{m}} S_a f_b \partial_b B_a . \quad (9.9)$$

On the other hand, if we add the term $q A_a \dot{x}_a$ for a particle with charge $q = -e$, which transforms according to

$$-e A_a \dot{x}_a \rightarrow -e A_a \dot{x}_a + i \frac{e}{\sqrt{m}} \partial_b A_a \epsilon f_b \dot{x}_a + i \frac{e}{\sqrt{m}} \dot{f}_a A_a , \quad (9.10)$$

to the interaction Lagrangian (9.6), and set

$$\mu = \frac{e}{m} , \quad (9.11)$$

the weak invariance under Eq.(9.4) is restored. Here

$$F_{ab} = \varepsilon_{abc} B_c = \partial_a A_b - \partial_b A_c . \quad (9.12)$$

Next we give a super-field formulation [58, 59] of the above systems. It provides us with a systematic method for constructing Lagrangians which are invariant under the super-symmetry transformations as in Eq.(9.4). We define $X_a(t, \Theta)$ to be a "super-coordinate", i.e., it depends on super-space parameters t and Θ , the latter being a Grassmann parameter. We then identify the coefficients of the Taylor expansion of $X_a(t, \Theta)$ in Θ with x_a and f_a/\sqrt{m} , i.e.,

$$X_a(t, \Theta) = x_a(t) + i \frac{\Theta}{\sqrt{m}} f_a(t) . \quad (9.13)$$

Now consider the following "super-charge" operator

$$Q \equiv \left(i \frac{\partial}{\partial \Theta} - \Theta \frac{\partial}{\partial t} \right) . \quad (9.14)$$

It follows that

$$[Q, Q]_+ = -2i \frac{\partial}{\partial t} , \quad (9.15)$$

i.e., the anti-commutator of two super-charges yields the "energy operator". Equation (9.15) thereby expresses a general property of super-symmetry algebras [57]. Furthermore, it can be easily verified that the super-charge Q induce translations in super-space (t, Θ) according to

$$\delta X_a(t, \Theta) \equiv i\epsilon Q X_a(t, \Theta) = X_a(t + i\epsilon\Theta, \Theta - \epsilon) - X_a(t, \Theta) . \quad (9.16)$$

The transformation defined by the Equation (9.16) applied to the super-coordinate $X_a(t, \Theta)$ is identical to the super-symmetry transformations (9.16) applied to x_a and f_a .

For the purpose of constructing weakly invariant Lagrangians, we now note the following:

i) Let $Y = Y(t, \Theta) + \Theta\eta(t)$ be a super-coordinate, which undergoes the transformation

$$\delta Y_a(t, \Theta) = i\epsilon Q Y_a(t, \Theta) . \quad (9.17)$$

Then $\eta(t)$ is invariant under this transformation up to a total time derivative. This is analogous to the transformation properties of the D -term in 3 + 1 dimensional super-symmetry [57].

ii) Let $P(t, \Theta)$ be a (fermionic-) bosonic operator which (anti-) commutes with Q . Then if the super-field Y transforms according to Eq.(9.17), so does $P(t, \Theta)Y$. Examples of first-order differential operators P fulfilling this property are

$$\frac{\partial}{\partial t}, \quad d_{\Theta} \equiv \frac{\partial}{\partial \Theta} - i\Theta \frac{\partial}{\partial t}, \quad (9.18)$$

the former being bosonic while the latter is fermionic.

iii) If Y and Z transform according to Eq.(9.17), then so does the product YZ .

Let $L_{0*} = L_{0*}(t, \Theta)$ transform under super-symmetry according to Eq.(9.17). Then the coefficient of L_{0*} is invariant up to a time derivative. Since we desire an invariant quantity which is bosonic, L_{0*} should be fermionic. A choice for L_{0*} which is quadratic in first order derivations of X_a is

$$L_{0*} = i \frac{m}{2} \frac{\partial X_a}{\partial t} d_{\Theta} X_a. \quad (9.19)$$

The Θ coefficient of $L_{0*}(t, \Theta)$ can be extracted by integrating over Θ and utilizing the usual rule [60]

$$\int d\Theta \Theta = 1, \quad \int d\Theta = 0, \quad (9.20)$$

Applying this to Eq.(9.19), we then find

$$\int d\Theta L_{0*}(t, \Theta) = L_0(t), \quad (9.21)$$

where $L_0(t)$ is the free particle Lagrangian (9.1).

Next we consider adding an interaction term to ((9.19)). We first take up the case of a particle interacting with a scalar (bosonic) potential $V = V(X)$. The latter transforms under super-symmetry according to Eq.(9.17). Since $V(X)$ is bosonic it must appear in $L_* = L_{0*} + L_{I*}$ times a fermionic operator. The latter must anti-commute with Q . Thus interactions like $L_{I*} = \Theta V(X)$ are excluded since they explicitly break the super-symmetry invariance. On the other hand, interactions like $L_{I*} = d_{\Theta} V(X)$ preserve the super-symmetry invariance. However, integrating with respect to Θ leaves only a total time derivative so no interaction results. Consequently, it appears difficult to construct super-symmetric invariant version of a particle interacting with a scalar potential. The latter is possible, however, for other treatments of the super-symmetric point particle (Cf . Ref.[61]).

In our formalism we can quite easily write down the interaction of the particle with a vector potential $A_i = A_i(X)$. Here we simply make the replacement

$$\begin{aligned} \frac{\partial}{\partial t} X_a &\rightarrow \frac{\partial}{\partial t} X_a - 2\frac{e}{m} A_a(X) , \\ d_{\Theta} X_a &\rightarrow d_{\Theta} X_a , \end{aligned} \tag{9.22}$$

in Eq.(9.19). Expanding the total Lagrangian in Θ and performing the Θ -integral according to the rule Eq.(9.20), one obtains

$$L(t) = L_0(t) - e\dot{x}_a A_a(x) - \frac{e}{m} S_a B_a . \tag{9.23}$$

This is identical to the Lagrangian (9.23) added with (9.10), with the restriction that the electric charge $q = -e$ and the magnetic moment μ are related according to Eq.(9.11).

Here we notice that the Lagrangian in Eq.(9.23), or Eq.(9.11), leads to the conclusion that the gyro-magnetic ratio of the particle is 2. This is actually a general feature of super-symmetric point particles (see, e.g., Refs.[62, 25, 40]). In super-symmetric field theories, where the super-symmetry is unbroken, this situation corresponds to the anomalous magnetic moment being zero (see, e.g., Ref. [63]).

In the above we have used a hermitian Grassmannian variable Θ to describe the spin degrees of freedom. One can also develop a non-relativistic super-symmetry by making use of a complex Grassmann variable. As was shown by Witten [61] and discussed by other authors [64, 65] super-symmetric quantum theories can be useful for studying the non-perturbative breaking of super-symmetry.

9.2 Super-Symmetric Point Particles in the Field of a Magnetic Monopole

In this Section we will super-symmetrize [61] the global Lagrangian of Chapter 4. This can be achieved by applying the rules given in Section 9.1 for constructing, weakly invariant super-symmetric Lagrangians. In Chapter 4 a dynamical group element $s(t)$ entered in the construction of the global Lagrangian (4.14). We can write $s(t)$ in the form

$$s(t) = \exp(iT(a)\varepsilon_a(t)) , \tag{9.24}$$

where $T(a) = \sigma_a/2$. Similarly, we can define a group element s_* on the super-space (t, Θ) , according to

$$s_*(t) = \exp(iT(a)\eta_a(t, \Theta)) , \quad (9.25)$$

where $\eta_a(t, \Theta)$ is now a "super-field". If we write $\eta_a(t, \Theta) = \varepsilon_a(t) - 2\Theta\xi_a(t)$, then Eq.(9.25) can be expressed by

$$s_*(t) = (1 + \Theta\xi)s(t) , \quad (9.26)$$

where $\xi = \xi_a\sigma_a$. Note that no conditions have to be placed on ξ_a other than it being an odd Grassmann variable in order that Eq.(9.26) be consistent with $s_*^\dagger s_* = 1$ and $\det(s_*) = 1$.

A natural extension of the Lagrangian (4.14) is

$$L_*(t) = L_{0*} - n\text{Tr} [\sigma_3 s_*^\dagger d_\theta s_*] , \quad (9.27)$$

where L_{0*} is given by the Eq.(9.19). We must, furthermore, generalize (4.13), i.e., the relation between the relative coordinate x_a and the dynamical group element s , to the (t, Θ) -space. This extension can now be easily achieved after constructing the following polar decomposition of the super-coordinate $\mathbf{X}(t, \theta)$:

$$\mathbf{X}(t, \theta) = \mathbf{x}(t, \theta) + i\Theta \frac{\mathbf{f}}{\sqrt{m}} = R_*(t, \Theta) \hat{\mathbf{X}}_*(t, \theta) , \quad (9.28)$$

where

$$R_*(t, \Theta) = r(t) + i \frac{\Theta}{\sqrt{m}} \hat{\mathbf{x}}(t) \cdot \mathbf{f}(t) , \quad (9.29)$$

and

$$\hat{\mathbf{X}}_*(t, \theta) = \hat{\mathbf{x}}(t) + i \frac{\Theta}{r\sqrt{m}} (\mathbf{f}(t) - \hat{\mathbf{x}}(t)(\hat{\mathbf{x}}(t) \cdot \mathbf{f}(t))) . \quad (9.30)$$

In Eqs.(9.29) and (9.30) $\mathbf{x} = r\hat{\mathbf{x}}$. The super-symmetric generalization of the Eq.(4.13) then is

$$\hat{X}_* = \hat{X}_{*a}\sigma_a = s_*\sigma_3 s_*^\dagger . \quad (9.31)$$

The Eqs.(9.19), (9.26), (9.27), and (9.31) lead to the following Lagrangian

$$L = \int d\Theta L_*(t, \Theta) = \frac{1}{2}m\dot{x}_a^2 + \frac{1}{2}f_a\dot{f}_a + in\text{Tr} [\sigma_3 s^\dagger \dot{s}] + 2ni\varepsilon_{abc}\hat{x}_a\dot{\xi}_b\xi_c . \quad (9.32)$$

By making use of the constraint Eq.(9.31) and the explicit form of $s_*(t, \Theta)$, as given by Eq.(9.26), we obtain

$$\hat{X}_*(t, \Theta) = \hat{x}(t) + \Theta[\xi(t), \hat{x}(t)] , \quad (9.33)$$

where $\hat{x} = s\sigma_3 s^\dagger$. Eqs.(9.30) and (9.33) then lead to the following relationship

$$\varepsilon_{abc}\hat{x}_a f_b f_c = 4mr^2 \varepsilon_{abc}\hat{x}_a \xi_b \xi_c , \quad (9.34)$$

i.e., the Lagrangian (9.32) can now be written in the following form

$$L = \frac{1}{2}m \left(\dot{r}^2 + r^2 \dot{\hat{x}}_a^2 \right) + \frac{1}{2}f_a f_a + in \text{Tr} [\sigma_3 s^\dagger \dot{s}] - \frac{e}{m} S_a B_a . \quad (9.35)$$

Here \mathbf{B} is the magnetic field of the monopole, i.e.,

$$B_a = \frac{g}{4\pi} \frac{x_a}{r^2} , \quad (9.36)$$

and $4\pi n = eg$ (Cf. with Section 4.1). In the expression (9.35) \hat{x}_a is to be regarded as a function of s (Cf. Eq.(4.13)). For $f_a = 0$, (9.35) becomes the Lagrangian (4.15).

In order to obtain the equations of motion we consider variations of the dynamical variables r , f_a , and s . The variation of r in Eq.(9.35) gives

$$m\ddot{r} = r\dot{\hat{x}}_a^2 + \frac{2n}{mr^3} \mathbf{S} \cdot \hat{\mathbf{x}} . \quad (9.37)$$

A variation of the Grassmann variables f_a leads to a spin precession equation (Cf. Eq.(9.8))

$$\dot{S}_a = \frac{e}{m} \varepsilon_{abc} B_b S_c . \quad (9.38)$$

Again for variations in s , we take (Cf. Eqs.(3.21) and (3.21))

$$\delta S = i\epsilon_k \sigma_k s . \quad (9.39)$$

By the Eq.(4.13), (9.39) will induce an infinitesimal rotation of the unit vector $\hat{\mathbf{x}}$ as given by Eq.(4.17), i.e.,

$$\delta \hat{x}_a = -2\varepsilon_{abc} \epsilon_b \hat{x}_c . \quad (9.40)$$

We therefore obtain the following result due to the variation of (9.39):

$$\delta L = 2\epsilon_a \left(\frac{d}{dt} (L_a + n\hat{x}_a) + \frac{e}{m} \varepsilon_{abc} S_b B_c \right) . \quad (9.41)$$

Eqs.(9.38) and (9.41) can now be combined to give

$$\frac{d}{dt}(L_a + n\hat{x}_a + S_a) = 0 , \quad (9.42)$$

i.e., angular momentum conservation. After a long but straightforward calculation, the Eqs.(9.37), (9.38), and (9.42) can be combined to yield the following equation (Cf. Eq.(3.11)):

$$m\ddot{\mathbf{x}} = -e\dot{\mathbf{x}} \times \mathbf{B} - \frac{e}{m}\nabla(\mathbf{S} \cdot \mathbf{B}) , \quad (9.43)$$

which is the equation of motion for a spinning particle in a non-homogeneous magnetic field (Cf. Section 3.2 and Ref.[66]) The equation (9.43) can also be obtained directly from the Lagrangian (9.35) by considering simultaneous variations of s and r . Here one makes use of the relation

$$2n\dot{\hat{x}}_a\epsilon_a = -e(\dot{\mathbf{x}} \times \mathbf{B}) \cdot \delta\mathbf{x} , \quad (9.44)$$

where $\delta\mathbf{x} = -2\epsilon \times \mathbf{x} + \delta r\hat{\mathbf{x}}$. Equation (9.35) follows from (9.40) and the explicit form of the magnetic monopole field 9.36. As expected (Cf. Section 9.1) the gyro-magnetic ratio of the particle is 2 according to the Eqn.(9.38). Here we also notice that although $\mathbf{L} + n\hat{\mathbf{x}}$ is not conserved, its projection along the \mathbf{x} -direction is. In fact, the latter is just n . This fact will turn out to be important when we quantize the system (Cf. Section 9.5) .

9.3 The Super-Symmetric Hopf Fibration

In Chapter 4 we have seen that the non-trivial $U(1)$ bundle on the two-sphere S^2 could be used to find a global Lagrangian description of magnetic monopoles. Let us recall how these bundles, here denoted by L_M , were constructed [13, 62, 67]. For some related work see also Refs.[68, 69, 70] . In Section 4.4 we regarded $SU(2)$ as a $U(1)$ bundle over S^2 , where the action of the $U(1)$ group corresponded to the gauge transformation Eq.(4.23), i.e.,

$$s(t) \rightarrow s(t) \exp(i\sigma_3\alpha(t)/2) \quad . \quad (9.45)$$

The projection map from the $SU(2)$ s bundle to the two-sphere S^2 is given by Eq.(4.13), i.e.,

$$s(t) \rightarrow s(t)\sigma_3s^\dagger(t) = \hat{X}(t) \quad . \quad (9.46)$$

Now consider the following cyclic subgroup of $SU(2)$

$$Z_M = \{z_k = \exp\left(i\sigma_3 \frac{2\pi k}{M}\right) \mid k = 0, 1, \dots, M_1\} \quad , \quad (9.47)$$

where M is a positive integer. Z_M has an action on $s \in SU(2)$ which commutes with the projection (9.46), i.e., if

$$s \rightarrow sz_k \quad , \quad (9.48)$$

then

$$s \rightarrow s\sigma_3 s^\dagger \rightarrow sz_k \sigma_3 (sz_k)^\dagger = \hat{X} \quad . \quad (9.49)$$

The $U(1)$ -bundles over the two-sphere S^2 are then generated by the quotient of $SU(2)$ with the group-action (9.48) [71]. A function f on L_M can now be regarded as a function on $SU(2)$ which is Z_M invariant, i.e.,

$$f(sz) = f(s) \quad , \quad (9.50)$$

for all $z_k \in Z_M$. In view of the fact that the wave functions the charge-monopole system have the property of being $Z_{|2n|}$ invariant they can be regarded as on $R^1 \times Z_{|2n|}$. In this sense, there is a topological interpretation of the Dirac quantization condition Eq.(8.33) (Cf. Ref.[70]).

In the present chapter we have constructed the super-symmetric generalization of the fibrations of S^3 as discussed above. Let us here briefly examine the corresponding mathematical structure. The super-symmetric version $SU(2)_*$ of $SU(2)$ is defined by letting the group parameters become super-fields as indicated by the Eq.(9.25). Let

$$U(1)_* = \{\exp(i\sigma_3 \gamma)\} \quad , \quad (9.51)$$

where γ is an even Grassmann variable. $U(1)_*$ has a right-handed action on $SU(2)_*$, i.e.,

$$s_* \rightarrow s_* \exp(i\sigma_3 \alpha) \quad . \quad (9.52)$$

The projection map (9.25) can therefore be generalized to

$$s_* \rightarrow s_* \sigma_3 s_*^\dagger = \sigma_a \hat{x}_{*a} \quad , \quad (9.53)$$

where the image of the map (Cf. Eq.(9.30)) is the super-symmetric version S_*^2 of the two-sphere S^2 . The bundle which describes the spinning charge-monopole system is then

$$L_{M*} = SU(2)_*/Z_M \quad . \quad (9.54)$$

As will be shown in Section 9.5, the Dirac quantization condition Eq.(8.33) is full-filled also in this case, i.e., we choose $M = |2n|$.

The $U(1)_*$ gauge transformation will induce a transformation on the ξ -variables defined in the Eq.(9.26). In fact, if we write

$$\exp(i\sigma_3\gamma(t, \Theta)) = (1 + \Theta\sigma_3\beta(t)) \exp(i\sigma_3\alpha(t)) \quad , \quad (9.55)$$

then by the projection map (9.31) and (9.26) ξ will transform according to

$$\xi(t) \rightarrow \xi(t) + \beta(t)\hat{x}(t) \quad , \quad (9.56)$$

i.e., ξ undergoes a translation parallel to \hat{x} . Since X_* is gauge invariant, it determines ξ only up to a transformation (9.56). Eq.(9.33) is consistent with this observation.

9.4 Super-Symmetric Yang-Mills-Particles

In the present Section we will combine the description of Yang-Mills particles (Cf. Chapter 6) with the super-symmetry discussed above. For simplicity, we will restrict ourselves to non-relativistic particles, but the discussion can easily be generalized to the relativistic case [72].

The free part of the Lagrangian will again be given by Eq.(9.19). We now extend the minimal coupling prescription Eq.(9.2) to the non-Abelian case, where the Yang-Mills vector potential is a matrix (Cf. Eq.(6.11)). The super-symmetric generalization of the Lagrangian Eq.(6.10) is therefore

$$L_* = L_{0*} + L_{I*} \quad , \quad (9.57)$$

where L_{0*} is given by Eq.(9.19) and the minimal coupling term L_{I*} is

$$L_{I*} = \text{Tr} [K s_*^\dagger(t, \Theta) D_{(t, \Theta)} s_*(t, \Theta)] \quad . \quad (9.58)$$

Here we have generalized the covariant derivative appearing in Eq.(6.11) to

$$D_{(t, \Theta)} = d_\Theta - ie(d_\Theta X(t, \Theta))A_a(X(t, \Theta)) \quad . \quad (9.59)$$

Next we expand the dynamical group element $s_*(t, \Theta)$ (Cf. Eq.(9.26)):

$$s_*(t, \Theta) = (1 + \Theta\xi(t))s(t) \quad , \quad \xi(t) = \xi_a(t)T(a) \quad , \quad (9.60)$$

and the super-coordinate $X(t, \Theta)$ according to Eq.(9.13). We then integrate Eq.(9.57) with respect to Θ , i.e.,

$$L(t) = \int d\Theta L_*(t, \Theta) = L_0(t) + L_I(t) \quad . \quad (9.61)$$

The result is that $L_0(t)$ is given by Eq.(9.1) and that

$$L_I = -\text{Tr} [K s^\dagger D_t s] - i \frac{e}{m} \text{Tr} [I f_a f_b \partial_b A_a] - \text{Tr} \left[I \xi \xi + \frac{e}{\sqrt{m}} [I, \xi] A_a f_a \right] \quad ,$$

where I is given by Eq.(6.14) and D_t is the same as in Eq.(6.11). Since the Lagrangian Eq.(9.61) does not contain time derivatives of the dynamical variable ξ , it plays the role of an auxiliary field (see, e.g., Ref.[57]). The ξ -variable in the Lagrangian is necessary in order that successive super-symmetric transformations, induced by the translations

$$t \rightarrow t + i\epsilon\Theta \quad , \quad \Theta \rightarrow \theta - \epsilon \quad , \quad (9.62)$$

close without the use of the equations of motion (see, e.g., Ref.[58]). We are allowed to substitute the equation of motion for ξ , i.e.,

$$[\xi, I] = -\frac{e}{\sqrt{m}} f_a [A_a, I] \quad , \quad (9.63)$$

back into the Lagrangian Eq.(9.62). Equation (9.63) leads to

$$\text{Tr}[I \xi \xi] = \frac{e^2}{2m} f_a f_b \text{Tr}[I [A_a, A_b]] \quad . \quad (9.64)$$

After substituting (9.63) and (9.64) into Eq.(9.62), we then find

$$L_I = -\text{Tr}[K s^\dagger D_t s] = -\frac{e}{m} \mathbf{S} \cdot \text{Tr}[I \mathbf{B}] \quad , \quad (9.65)$$

where \mathbf{B} is the non-Abelian magnetic field strength.

Concerning the equations of motion as derived from the Lagrangian (9.65), or (9.62), we notice that the spin precession Eq.(9.38) will be modified according to

$$\dot{S}_a = \frac{e}{m} \varepsilon_{abc} B_b^\alpha I_\alpha S_c \quad . \quad (9.66)$$

Thus the gyro-magnetic ratio is 2 as expected.

9.5 Canonical Formulation and Quantization of Pseudo-Classical Systems

In deriving the canonical formalism for the preceding systems, we follow the methods used in Chapter 8. For treating the fermionic variables, we shall apply the methods of Ref.[54], which are as follows.

Let χ_a denote the momenta conjugate to f_a . If C and D are any anti-commuting variables, then PB is defined according to

$$\{C, D\} \equiv - \left(\frac{\partial C}{\partial f_a} \frac{\partial D}{\partial \chi_a} + \frac{\partial C}{\partial \chi_a} \frac{\partial D}{\partial f_a} \right) \quad . \quad (9.67)$$

Hence,

$$\{f_a, f_b\} = \{\chi_a, \chi_b\} = 0 \quad , \quad \{f_a, \chi_b\} = -\delta_{ab} \quad . \quad (9.68)$$

The remaining PB's are defined in the usual way.

For the non-relativistic particle interacting with a magnetic field (Cf. Eqs.(9.1) and (9.23)),

$$\chi_a = \frac{\partial L}{\partial \dot{f}_a} = -\frac{i}{2} f_a \quad . \quad (9.69)$$

Thus we obtain the primary constraints

$$\zeta_a = \chi_a + \frac{i}{2} f_a \approx 0 \quad . \quad (9.70)$$

The Hamiltonian is

$$H = \frac{1}{2m} (p_a - eA_a)^2 + \frac{e}{m} \mathbf{S} \cdot \mathbf{B} + \lambda_a \zeta_a \quad , \quad (9.71)$$

where λ_a are Lagrange multipliers. The requirement that $\{\zeta_a, H\} \approx 0$ determines the λ_a 's, i.e.,

$$\lambda_a = \frac{e}{m} \varepsilon_{abc} B_b f_c \quad , \quad (9.72)$$

and thus leads to no secondary constraints.

The constraints ζ_a are second class, since

$$\{\zeta_a, \zeta_b\} = -i\delta_{ab} \quad . \quad (9.73)$$

They may be eliminated by introducing the DB's [54]:

$$\{f_a, f_b\}^* = -i\delta_{ab} \quad , \quad \{f_a, \chi_b\}^* = -\frac{1}{2}\delta_{ab} \quad , \quad (9.74)$$

as well as

$$\{\chi_a, \chi_b\}^* = \frac{i}{4}\delta_{ab} \quad . \quad (9.75)$$

The DB's which involve x_a or p_a are all equal to the corresponding PB's. Thus we can replace PB's by DB's and then eliminate χ_a via Eq.(9.69).

The generator of the super-symmetry transformation on the phase space variables is

$$Q = \frac{1}{\sqrt{m}}f_a(p_a - eA_a) \quad , \quad (9.76)$$

since

$$\{f_a, Q\}^* = -\frac{i}{\sqrt{m}}(p_a - eA_a) \quad , \quad \{\chi_a, Q\}^* = \frac{1}{\sqrt{m}}f_a \quad . \quad (9.77)$$

(Cf. with Eq.(9.4).) Furthermore, the Hamiltonian Eq.(9.71) can be expressed by

$$H = \frac{1}{2i}\{Q, Q\}^* \quad . \quad (9.78)$$

In passing to the quantum theory we replace the DB's in Eq.(9.74) by $(-i)$ times the anti-commutator brackets (and the remaining DB's by $(-i)$ times the commutator brackets). In particular

$$[f_a, f_b]_+ = \delta_{ab} \quad . \quad (9.79)$$

It is known, as a consequence [54], that an IRR of the f_a 's is obtained in the quantum theory by the identification

$$f_a = -\frac{1}{\sqrt{2}}\sigma_a \quad , \quad (9.80)$$

with σ_a 's being the Pauli matrices. Consequently, the spin of the particle is 1/2. Furthermore, Eq.(9.78) becomes

$$H = Q^2 \quad . \quad (9.81)$$

For the monopole system described in Section 9.2, we replace the above variables x_a and p_a by r , p_r , s and t_a (p_r and t_a are canonically conjugate to r and x_a , respectively (Cf. Section 8.2). The variables t_α and s again satisfy the Poisson bracket relations Eqs.(8.7), (8.8), and (8.9). For this system, in addition to the constraint Eq.(9.70) we have Eq.(8.22), i.e.,

$$\phi \equiv \hat{x}_i t_i - n \approx 0 \quad . \quad (9.82)$$

The Hamiltonian is now

$$H = \frac{p_r^2}{2m} + \frac{1}{2mr^2}(t_a t_a - n^2) + \frac{e}{m}\mathbf{S} \cdot \mathbf{B} + \lambda_a \zeta_a + \eta \phi \quad , \quad (9.83)$$

λ_a and η being Lagrange multipliers. As before there are no secondary constraints. The constraints ζ_a are once again second class while ϕ is first class. The former are eliminated via the DB's Eq.(9.74, while the gauge symmetry generated by the latter is eliminated by working on the reduced phase space, which is coordinatized by r , p_r , t_a and \hat{x}_a (Cf. Section 8.2).

For the monopole system we can express the super-symmetry generator globally by

$$Q = \frac{1}{\sqrt{m}} \left(f_a \hat{x}_a p_r - \frac{1}{r} \varepsilon_{abc} t_a f_b \hat{x}_c \right) , \quad (9.84)$$

as compared with Eq.(9.76). After applying the constraint (9.82), we can once again show that Hamiltonian is given by Eq.(9.78).

In passing to the quantum theory we again make the identification Eq.(9.79), yielding the spin-half particle. The quantization of the remaining variables is the same as in Section 8.2. In particular, quantization is possible only if $2n = \text{integer}$.

Next, we take up the canonical quantization of the super-symmetric Yang-Mills particle described in Section 9.4. We pick up the discussion with the interaction Lagrangian Eq.(9.65), where the auxiliary variables ξ have already been eliminated. The corresponding phase space for this system is spanned by x_a , s , f_a and the canonically conjugate variables p_a , t_α , and χ_α . The variables t_α and s again satisfy the Poisson bracket relations Eqs.(8.7), (8.8), and (8.9). The bosonic variables are constrained by equation (8.77), i.e.,

$$\phi_\alpha = I_\alpha - t_\alpha \approx 0 , \quad (9.85)$$

while the fermionic variables are constrained by Eq.(9.70). The Hamiltonian for this system is

$$H = \frac{1}{2m} (p_a - e A_a^\alpha t_\alpha)^2 + \frac{e}{m} S_a t_\alpha B_a^\alpha + \lambda_a \zeta_a + \eta_\alpha \phi_\alpha . \quad (9.86)$$

The treatment of the bosonic constraints and the fermionic constraints have both been previously discussed (the former in Section 8.4). Here the super-symmetry generator is

$$Q = \frac{1}{\sqrt{m}} (p_a - e A_a^\alpha(x) t_\alpha) f_a . \quad (9.87)$$

It can have a non-trivial action on the isospin variables when an external field is present

$$\{t_\alpha, Q\}^* = -\frac{e}{\sqrt{m}} f_a c_{\alpha\beta\gamma} A_a^\beta(x) t_\gamma . \quad (9.88)$$

The quantum theory for the above system describes a particle of spin-half and isospin which is determined by the value of the constants K_a .

10 LOCAL AND GLOBAL LAGRANGIANS

In the previous chapters, we considered systems which admit a global Hamiltonian description. That is, these systems have a globally defined Hamiltonian or energy function, and the corresponding symplectic form (or equivalently, the Poisson bracket) is globally defined. However, these systems do not admit global canonical coordinates. Thus a global Lagrangian cannot be found in terms of the variables which occur in the Hamiltonian description. Now by a theorem of Darboux [73], local canonical coordinates always exist. Thus, locally, the Legendre transform can be made and a Lagrangian can be found. These local Lagrangians are defined on coordinate neighbourhoods and are, in general, not defined globally. In previous Chapters, in effect, we have constructed global Lagrangians from these local ones by introducing additional gauge degrees of freedom, that is, a principal fibre bundle structure.

In this Chapter we now give a systematic method for finding the global Lagrangian when the system admits local Lagrangians and a global Hamiltonian description. The analysis presented here is similar to an analysis used in context of geometric quantization.

Three striking results emerge from the analysis:

i) The construction in terms of $U(1)$ fibre bundles works only if, classically, a certain "quantization" is fulfilled. For the system of several charges and monopoles, this result has been proved by Friedman and Sorkin [20]. For that system, the condition is

$$\frac{e_i g_j}{e_k g_l} = \text{a rational number} , \quad (10.1)$$

where e_i and g_i are electric and magnetic charges. Note that this implies that electric and magnetic charges (and hence their product) are separately quantized. (Take $g_j = g_l$ to get the first result. Take $g_e = e_k$ to get the second result.) Note also that Eq.(10.1) is weaker than Dirac's result [8, 9]

$$e_i g_j = 2\pi k , \quad k \text{ integer} , \quad (10.2)$$

the proof of which requires quantum mechanics.

ii) Once the quantization condition is fulfilled, a global Lagrangian can be found by introducing $U(1)$ gauge degrees of freedom, that is a $U(1)$ fibre bundle. It is interesting that in such a case nothing more involved than a $U(1)$ fibre bundle is required or the construction of the global Lagrangian.

iii) Global Lagrangians can be constructed even if the quantization condition is not fulfilled and hence the fibre bundle approach fails. The fibre bundle construction is a special case of this more general construction.

In the proof of these results, we use the language of differential geometry because of its convenience. We have done so sparingly however, so that a reader with a small familiarity with differential geometry can follow the argument.

10.1 The Fibre Bundle Construction

Before discussing the main result, we first recall the proof of a theorem due to Weil [74]. For our purposes, Weil's result can be stated as follows: Let Ω be a closed two-form on Q , i.e.,

$$\Omega = \Omega_{ij} dx^i \wedge dx^j , \quad (10.3)$$

and

$$d\Omega = 0 \text{ or } \partial_i \Omega_{jk} + \partial_j \Omega_{ki} + \partial_k \Omega_{ij} = 0 . \quad (10.4)$$

Further, for every closed two-surface M in Q , let

$$\int_M \Omega = 2\pi\nu\lambda , \quad \nu = 0, \pm 1, \pm 2, \dots . \quad (10.5)$$

Here λ is the same for all M and ν is characteristic of M . Then there exists a $U(1)$ bundle E on Q , and a form $\tilde{\Omega}$ on E with the following properties:

1) $\tilde{\Omega}$ is exact, i.e., $\tilde{\Omega} = d\Lambda$.

Here Λ is a globally defined one-form on E .

2) $\tilde{\Omega}$ is "gauge invariant" and hence projects down to a form on Q .

3) The latter is precisely Ω .

Here by gauge invariance we mean the following: Let ϕ and ϕ' and be two sections (Cf. Chapter 3) from a coordinate neighbourhood in Q to E . Then the pull backs $\phi^*\tilde{\Omega}$ and $\phi'^*\tilde{\Omega}$ are equal. Stated in another way, let $\pi : E \rightarrow Q$ be the projection map from the bundle E to the base Q , then $\tilde{\Omega} = \pi^*\Omega$.

In conventional classical mechanics, where global canonical coordinates exist, the symplectic form

$$dp_i \wedge dq^i , \quad (10.6)$$

is necessarily exact:

$$dp_i \wedge dq^i = d(p_i dq) . \quad (10.7)$$

Weil's result gives us conditions under which a non-exact symplectic form can be turned into an exact one. This is accomplished by introducing gauge degrees of freedom. Note in this context the "quantization" of the integrals in Eq.(10.5). The origin of the classical quantization condition is this equation.

If $\lambda = 0$, then Ω is exact. We shall also assume hereafter that $\lambda \neq 0$. We shall also assume that Q is paracompact. Under this technical assumption, Q has a contractible covering $\{U_\alpha\}$ by coordinate neighbourhoods U_α . In such a covering, each of the sets $U_\alpha, U_\alpha \cap U_\beta, U_\alpha \cap U_\beta \cap U_\gamma, \dots$, is either empty or can be smoothly contracted to a point. The proof of the converse to the Poincaré lemma [73] is therefore valid on each of these sets. It follows from Eq.(10.3) that

$$\Omega|_{U_\alpha} = d\Theta_\alpha, \quad (10.8)$$

where $\Omega|_{U_\alpha}$ is the restriction of Ω to U_α . Also, since $d(\Theta_\alpha - \Theta_\beta) = 0$ on $U_\alpha \cap U_\beta$, we have

$$\Theta_\alpha - \Theta_\beta = df_{\alpha\beta} \text{ on } U_\alpha \cap U_\beta, \quad (10.9)$$

where

$$d(f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}) = 0 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma. \quad (10.10)$$

Equation (10.10) states that $f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}$ a constant on $U_\alpha \cap U_\beta \cap U_\gamma$. Suppose further that

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 2\pi n_{\alpha\beta\gamma} \lambda, \quad (10.11)$$

where $n_{\alpha\beta\gamma}$ takes integer values. Then the map $F : Q \rightarrow U(1)$ as defined by

$$F(f_{\alpha\beta}) \equiv g_{\alpha\beta} = \exp\left(\frac{if_{\alpha\beta}}{\lambda}\right), \quad (10.12)$$

fulfills the cocycle property

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma. \quad (10.13)$$

The functions $g_{\alpha\beta}$ are defined on $U_\alpha \cap U_\beta$ and have values in $U(1)$. Hence they define a $U(1)$ bundle on Q .

It may be shown [74] that Eq.(10.11) is equivalent to Eq.(10.5). Thus with Eq.(10.13), we have a $U(1)$ bundle on Q . It is defined as follows. Let x and x' be the co-ordinates of the same point p in $U_\alpha \cap U_\beta$ for the coordinate systems appropriate to U_α and U_β . Then $(x, h^{(\alpha)})$ and $(x', h^{(\alpha)}g_{\alpha\beta})$ define the same point in the fibre over p in the bundle

space E . Here $h^{(\alpha)} \in U(1)$. Such a definition of principal fibre bundles is equivalent to the definition we gave in Chapter 3.

Let

$$m_\alpha = -i\lambda(h^{(\alpha)})^{-1}dh^{(\alpha)} . \quad (10.14)$$

The form m_α is defined on the fibres over U_α in the coordinate system appropriate to U_α . We have,

$$m_\alpha - m_\beta = i\lambda g_{\alpha\beta}^{-1}dg_{\alpha\beta} = -df_{\alpha\beta} , \quad (10.15)$$

on $U_\alpha \cap U_\beta$. Comparison of Eq.(10.9) and Eq.(10.15) shows that

$$\Theta_\alpha + m_\alpha = \Theta_\beta + m_\beta . \quad (10.16)$$

Thus the one-form

$$\Theta = \Theta_\alpha + m_\alpha , \quad (10.17)$$

is globally defined on E . Further, since

$$dm_\alpha = 0 , \quad (10.18)$$

we can write $\Omega = d\Theta$ if we regard Ω as a form on E . (More correctly, it is the form $\tilde{\Omega}$ in the statement of the theorem). The theorem is thus proved.

In the statement of our result, we regard the Hamiltonian or energy and the symplectic form as defined in terms of coordinates and velocities (and not in terms of coordinates and momenta). We define Q to be the configuration space for a dynamical system. Let $\{U_\alpha\}$ be a contractible covering of Q (again assumed to be paracompact) by coordinate neighbourhoods U_α and TU_α be the tangent bundle (the space of coordinates and velocities) over U_α . Suppose now that the following is true:

- i)* The dynamical system admits local Lagrangians $L^{(\alpha)}$ defined on TU_α .
- ii)* The energy function H is defined globally on $TQ = \cup_\alpha TU_\alpha$. In local coordinates, this means

$$\frac{\partial L^{(\alpha)}}{\partial \dot{x}_i} \dot{x}_i - L^{(\alpha)} = \frac{\partial L^{(\beta)}}{\partial \dot{x}_i} \dot{x}_i - L^{(\beta)} , \quad (10.19)$$

on $TU_\alpha \cap TU_\beta$ (assumed not to be empty).

- iii)* The symplectic ω exists globally, that is

$$d \left[\frac{\partial L^{(\alpha)}}{\partial \dot{x}_i} dx_i \right] = d \left[\frac{\partial L^{(\beta)}}{\partial \dot{x}_i} dx_i \right] , \quad (10.20)$$

on $TU_\alpha \cap TU_\beta$.

iv) The integral of ω over any closed two-dimensional surface M in Q fulfills an analogue of Eq.(10.5):

$$\int_M \omega = 2\pi\nu\lambda \quad , \quad \nu = 0, \pm 1, \pm 2, \dots \quad . \quad (10.21)$$

Here λ is the same for all TQ and ν is characteristic of TQ. Then there exists a $U(1)$ bundle E on Q and a global Lagrangian on TE for this system.

Both assumptions *ii)* and *iii)* are necessary conditions for the existence of a Hamiltonian description. A system of charges and monopoles fulfills these conditions. Condition *iv)* is surprising in a classical context since it "quantizes" certain integrals of ω . We shall show that for a system of charges and monopoles, it coincides with the Friedman-Sorkin condition mentioned previously.

To prove our result we can proceed as follows. If $\psi_\alpha = (\partial L^{(\alpha)}/\partial \dot{x}_i)dx_i$ then by Eq.(10.20),

$$d(\psi_\alpha - \psi_\beta) = d \left[\frac{\partial}{\partial \dot{x}_i} (L^{(\alpha)} - L^{(\beta)}) dx_i \right] = 0 \quad . \quad (10.22)$$

Hence, $\psi_\alpha - \psi_\beta$ can be regarded as a closed one-form on $U_\alpha \cap U_\beta$. Since $U_\alpha \cap U_\beta$ is contractible,

$$\psi_\alpha - \psi_\beta = df_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta \quad , \quad (10.23)$$

where $f_{\alpha\beta}$ fulfills Eq.(10.11) by Eq.(10.21). As before, we can construct a $U(1)$ bundle E on Q and forms m_α with the property (10.15). Hence the form χ defined by

$$\chi = \kappa_\alpha + m_\alpha \quad , \quad (10.24)$$

exists globally on E and

$$d\chi = \omega \quad , \quad (10.25)$$

or more precisely $d\chi = \pi^*\omega$ where π is the projection $\pi : E \rightarrow Q$.

Now by the energy condition Eq.(10.19),

$$L^{(\alpha)} - L^{(\beta)} = \frac{\partial f_{\alpha\beta}}{\partial x_i} \dot{x}_i \quad . \quad (10.26)$$

Thus the Lagrangian

$$\tilde{L} = L^{(\alpha)} - i\lambda(h^{(\alpha)})^{-1} \frac{dh^{(\alpha)}}{dt} \quad , \quad (10.27)$$

is globally defined on TE . Since the last term is (locally) the time-derivative of a function, $L^{(\alpha)}$ and $L^{(\beta)}$ also give the same equations of motion. The result is thus proved.

Let us understand the result in terms of the charge-monopole system. If

$$\{\xi_i, \xi_j\} = \omega^{ij}(\xi) , \quad (10.28)$$

are the PB's for a system of coordinates $\xi = (\xi^1, \xi^2, \dots, \xi^{2n})$ for the phase space, the symplectic form is

$$\omega = \frac{1}{2} \omega_{ij}(\xi) d\xi^i \wedge d\xi^j , \quad (10.29)$$

where

$$\omega_{ij} \omega^{jk} = \delta_i^k . \quad (10.30)$$

For a system of one charge and one monopole, the PB's are given in Chapter 4 by Eqs.(4.8) - (4.10). With coordinates $(x_1, x_2, x_3, v_1, v_2, v_3)$, they imply that

$$\omega = 2mdv_i \wedge dx_i + \frac{1}{2} F_{ij} dx_i \wedge dx_j , \quad (10.31)$$

where

$$F_{ij} = n \frac{\varepsilon_{ijk} x_k}{r^3} , \quad (10.32)$$

If M is a closed surface in Q not enclosing the monopole, it follows by Stokes theorem that $\int_M \omega = 0$. If M is a 2-sphere S^2 (with outward orientation) which encloses the monopole we get $\int_M \omega = -4\pi n$. Multiple integrations over S^2 with different orientations effectively correspond to different M . Thus, in general,

$$\int_M \omega = 4\pi n \nu_n , \quad (10.33)$$

where ν_n is an integer.

In comparing with Eq.(10.21) we may set $\nu = \nu_n$ and $\lambda = 2n$. Consequently, the requirement Eq.(10.21) imposes no restrictions on the system and only defines λ . This, however, is not the case when more than one monopole is present.

If an additional monopole is introduced to the above system we must add

$$- n' \varepsilon_{ijk} \frac{x'_k}{r'^3} dx'_i \wedge dx'_j , \quad (10.34)$$

to Eq.(10.32). Here x' corresponds to the distance between the electric charge and the additional monopole. Now

$$\int_M \omega = 4\pi n \nu_n + 4\pi n' \nu_{n'} , \quad (10.35)$$

where ν_n and $\nu_{n'}$ are integers. Consequently, Eq.(10.21) implies that

$$\lambda\nu = 4\pi n\nu_n + 4\pi n'\nu_{n'} . \quad (10.36)$$

Since Eq.(10.36) holds for any M , we can choose it such that $\nu_{n'} = 0$. It then follows that λ equals n times a rational number. Similarly, by choosing MM such that $\nu_n = 0$, we can conclude that λ equals n' times a rational number. But then

$$\frac{n}{n'} = \text{a rational number} , \quad (10.37)$$

which is consistent with Eq.(10.1). Only here $e_i = e_j e$.

The following brief remarks about the consequences of discarding the global energy condition Eq.(10.19) may be of interest. If this condition is abandoned, the global nature of symplectic form Eq.(10.29) implies only that

$$L^{(\alpha)} - L^{(\beta)} = \frac{\partial f_{\alpha\beta}}{\partial x_i} \dot{x}_i + \rho_{\alpha\beta} , \quad (10.38)$$

where $\rho_{\alpha\beta}$ does not depend on \dot{x}_i . Further,

$$i) \quad \rho_{\alpha\beta} = -\rho_{\beta\alpha} \text{ and } \rho_{\alpha\beta} + \rho_{\beta\gamma} + \rho_{\gamma\alpha} = 0.$$

$ii)$ Since $L^{(\alpha)}$ and $L^{(\beta)}$ give the same equations of motion on $TU_\alpha \cap TU_\beta$, the $\rho_{\alpha\beta}$'s are actually constants and hence are globally defined.

Now let ϕ_α be a partition of unity subordinated to the covering $\{U_\alpha\}$, i.e.,

$$\text{Supp}[\phi_\alpha] = U_\alpha , \quad \phi_\alpha \geq 0 , \quad \sum_\alpha \phi_\alpha = 1 . \quad (10.39)$$

Then the globally defined functions

$$\kappa_\alpha = \sum_\lambda \rho_{\alpha\lambda} \phi_\lambda , \quad (10.40)$$

$$\kappa_\alpha - \kappa_\beta = \rho_{\alpha\beta} , \quad (10.41)$$

in view of $ii)$ above. Thus

$$\hat{L} = \hat{L}^{(\alpha)} - \kappa_\alpha , \quad (10.42)$$

fulfill an equation of the form Eq.(10.26) and

$$\hat{L} = \hat{L} - i\lambda(\hbar^{(\alpha)})^{-1} \frac{d\hbar^{(\alpha)}}{dt} , \quad (10.43)$$

is globally defined on TE . Also ω has the usual relation to \hat{L} . However, since κ_α can depend on x , \hat{L} may not give the original equations of motion.

10.2 Global Lagrangians without the Quantization Condition

The discussion which follows is taken from Ref.[75].

The variational-principles which follow often involve the phase space as the space $Q = \{\xi\}$. They are thus often related to Hamilton's variational principle.

We shall discuss Hamiltonian systems. Thus a globally defined Hamiltonian H and a globally defined symplectic form ω [Cf. Eq.(10.29)] are assumed to exist. Further ω is closed and non-degenerate, i.e.,

$$d\omega = 0 \quad , \quad (10.44)$$

and

$$\det \omega_{ij} \neq 0 \quad . \quad (10.45)$$

The Hamilton equations of motion for this system are

$$\frac{\partial H}{\partial \xi^i} = \omega_{ji} \dot{\xi}^j \quad . \quad (10.46)$$

Suppose now that ω is exact. By definition, then, there exists a globally defined one-form $f = f_i(\xi)d\xi^i$ such that

$$\omega = df \quad . \quad (10.47)$$

The equations of motion in this case follow from the global Lagrangian

$$L = f_i(\xi)\dot{\xi}^i - H(\xi) \quad . \quad (10.48)$$

In familiar situations where Q admits global canonical coordinates, we see from Eq.(10.46) that the variational principle associated with L is just Hamilton's variational principle.

If ω is not exact as for the charge-monopole system, then a global f does not exist. Thus we have to modify the above procedure for finding L . One such procedure was described in the previous section. We now point out an alternative approach.

The first step in the modification is to change the configuration space from Q to the space of paths PQ over Q . It is defined as follows. Let ξ_0 be a fixed reference point in Q . This point may be chosen at will. Then a point of PQ is a path γ from ξ_0 to some point ξ :

$$\gamma = \{ \gamma(\sigma) \mid 0 \leq \sigma \leq 1 , \gamma(0) = \xi_0 , \gamma(1) = \xi \} \quad . \quad (10.49)$$

These paths are defined at a given time. We denote the time-dependent paths by

$$\gamma(\sigma, t) \quad [\gamma(\sigma, 0) = \xi_0] \quad . \quad (10.50)$$

We now show that we can always write an action principle with configuration space as PQ . The procedure, of course, works also when ω is exact. We illustrate it in this context first.

The Hamiltonian H can first be promoted to a functional \tilde{H} on paths at a given time:

$$\int_0^1 d\sigma \tilde{H}[\gamma(\sigma, t)] = H[\gamma(1, t)] . \quad (10.51)$$

Consider next a family of paths $\gamma(\sigma, t)$ with

$$\gamma(1, t) = \xi(t) . \quad (10.52)$$

Thus as σ and t vary, $\gamma(\sigma, t)$ sweeps out a surface Δ in Q with the boundary

$$\partial\Delta = \partial\Delta_1 \cup \partial\Delta_2 \cup \partial\Delta_3 , \quad (10.53)$$

where

$$\begin{aligned} \partial\Delta_1 &= \{ \xi(t) \mid t_1 \leq t \leq t_2 \} , \\ \partial\Delta_2 &= \{ \gamma(\sigma, t_1) \mid 1 \leq \sigma \leq 1 \} , \\ \partial\Delta_3 &= \{ \gamma(\sigma, t_2) \mid 1 \leq \sigma \leq 1 \} . \end{aligned} \quad (10.54)$$

By applying Stokes' theorem, we can write the action S as

$$S = \int_{t_1}^{t_2} dt [f_i(\xi)\dot{\xi}^i - H(\xi)] = \int_{\partial\Delta_1} [f_i(\xi)d\xi^i - H(\xi)dt] , \quad (10.55)$$

as

$$\begin{aligned} S = \int_{\Delta} \left[\frac{1}{2} \omega_{ij}[\gamma(\sigma, t)] d\gamma^i(\sigma, t) \wedge d\gamma^j(\sigma, t) - \tilde{H}[\gamma(\sigma, t)] d\sigma \wedge dt \right] + \\ \left\{ \int_{\partial\Delta_3} f_i[\gamma(\sigma, t)] d\gamma^i(\sigma, t) - \int_{\partial\Delta_2} f_i[\gamma(\sigma, t)] d\gamma^i(\sigma, t) \right\} . \end{aligned} \quad (10.56)$$

Since we shall not vary the initial and final paths $\gamma(\sigma, t_1)$ and $\gamma(\sigma, t_2)$ in the variational principle, the expression in the "script bracket" above will not contribute to the equations of motion. The action on the space of paths PQ can thus be taken to be

$$S = \int_{\Delta} \left[\omega[\gamma(\sigma, t)] - \tilde{H}[\gamma(\sigma, t)] d\sigma \wedge dt \right] . \quad (10.57)$$

It involves only the symplectic form ω and not the one-form f . It appears to define a field theory in one "space" and one time.

The action S as given by Eq.(10.57) was derived in the case that ω was exact. However, it involves only ω and thus is expected to be valid even if ω is not exact. This expectation is correct as may be shown by varying

$$S = \frac{1}{2} \int_{\Delta} \omega_{ij} \frac{\partial \gamma^i}{\partial \sigma^a} \frac{\partial \gamma^j}{\partial \sigma^b} \varepsilon^{ab} d\sigma \wedge dt - \int_{\partial \Delta_1} H dt , \quad (10.58)$$

where $\sigma^0 = t$, $\sigma^1 = \sigma$, and $\varepsilon^{01} = -\varepsilon^{10} = 1$. We find, upon using $d\omega = 0$, i.e., $\partial_i \omega_{jk} + \partial_j \omega_{ki} + \partial_k \omega_{ij} = 0$, and regrouping terms,

$$\delta \left(\frac{1}{2} \int_{\Delta} \omega_{ij} \frac{\partial \gamma^i}{\partial \sigma^a} \frac{\partial \gamma^j}{\partial \sigma^b} \varepsilon^{ab} d\sigma \wedge dt \right) = - \int_{\Delta} d [\omega_{ij} d\gamma^i \delta \gamma^j] = - \int_{\partial \Delta_1} \omega_{ij} d\gamma^i \delta \gamma^j , \quad (10.59)$$

since $\delta \gamma^j = 0$ on $\partial \Delta_2 \cup \partial \Delta_3$. Also

$$\delta \left(\int_{\partial \Delta_1} H dt \right) = \int_{\partial \Delta_1} \frac{\partial H}{\partial \xi^j} \delta \gamma^j dt . \quad (10.60)$$

Thus the equations of motion (Cf. Eq.(10.46) is recovered.

For a charge-monopole system the preceding technique can be directly applied to the conventional local Lagrangian:

$$L = L_0 + L_I , \quad L_0 = \frac{1}{2} m \dot{x}_i^2 , \quad L_I = e A_i \dot{x}_i , \quad (10.61)$$

in order to find the global Lagrangian. Here (Cf. Eq.(9.12)),

$$\partial_i A_j - \partial_j A_i = -\varepsilon_{ijk} \frac{g x_k}{4\pi r^3} , \quad (10.62)$$

is the monopole magnetic field. The latter is globally defined but, of course, the potential A_i is not. The space Q is $\mathbf{R}^3 - \{0\} = S^2 \times \mathbf{R}_+$, where

$$S^2 = \left\{ \hat{x}_i = \frac{x_i}{\sqrt{x_i x_i}} \right\} , \quad \mathbf{R}_+ = \{ r = \sqrt{x_i x_i} \mid x_i x_i > 0 \} . \quad (10.63)$$

Let the reference point be $\xi_0 = (1, 0, 0)$. Then the space PQ is the space of paths γ radiating from ξ_0 . The globally defined action and Lagrangian are

$$S = \int \tilde{L}(\sigma, t) d\sigma \wedge dt , \quad \tilde{L}(\sigma, t) = \tilde{L}_0(\sigma, t) + \tilde{L}_I(\sigma, t) ,$$

$$S = \int \tilde{L}(\sigma, t) d\sigma \wedge dt , \quad (10.64)$$

$$\tilde{L}_0(\sigma, t) = -\frac{eg}{8\pi} \varepsilon_{ijk} \varepsilon^{ab} \hat{\gamma}_i(\sigma, t) \frac{\partial \hat{\gamma}_j}{\partial \sigma^a}(\sigma, t) \frac{\partial \hat{\gamma}_k}{\partial \sigma^b}(\sigma, t) .$$

Here

$$\hat{\gamma}_i(\sigma, t) = \frac{\gamma_i(\sigma, t)}{\sqrt{\gamma_j(\sigma, t)\gamma_j(\sigma, t)}} . \quad (10.65)$$

and we identify $\gamma_i(1, t)$ with $x_i(t)$.

Finally we make contact with the fibre bundle approach as follows. If ω fulfills a quantization condition of the form Eq.(10.21), we know that there is a $U(1)$ bundle E over Q on which ω becomes exact:

$$\omega = d\chi \text{ on } E . \quad (10.66)$$

The action S can be thought of as defined on PE , the path space for E . Thus we now regard Δ as a surface in E . Now

$$\int_{\Delta} \omega = \int_{\partial\Delta_1} \chi , \quad (10.67)$$

plus terms which are not varied and may be discarded. In this way, we have a globally defined action on E :

$$S = \int_{\partial\Delta_1} (\chi - Hdt) . \quad (10.68)$$

Here ,as in the treatment of the kinetic energy term for the charge-monopole system, we regard H as being defined on E . Note that this procedure does not work without the quantization condition.

The quantization condition allows us to reduce the path space PE to the $U(1) \times U(1) \times \dots \times U(1)$ - bundle E over Q with k factors of $U(1)$ when there are k two-cycles S_1, S_2, \dots, S_k , such that

$$\int_{S_j} d\omega = a_j \text{ and } \frac{a_i}{a_j} \text{ is rational} . \quad (10.69)$$

For further details, see Ref.[75, 1].

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