Reconstructing the metric in group field theory

Steffen Gielen[*] and Lisa Mickel[†]
School of Mathematics and Statistics, University of Sheffield,
Hicks Building, Hounsfield Road, Sheffield S3 7RH, United Kingdom
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We study a group field theory (GFT) for quantum gravity coupled to four massless scalar fields, using these matter fields to define a (relational) coordinate system. We exploit symmetries of the GFT action, in particular under shifts in the values of the scalar fields, to derive a set of classically conserved currents, and show that the same conservation laws hold exactly at the quantum level regardless of the choice of state. We propose a natural interpretation of the conserved currents which implies that the matter fields always satisfy the Klein–Gordon equation in GFT. We then observe that in our matter reference frame, the same conserved currents can be used to extract all components of an effective GFT spacetime metric. Finally, we apply this construction to the simple example of a spatially flat homogeneous and isotropic universe. Our proposal goes substantially beyond the GFT literature in which only specific geometric quantities such as the total volume or volume perturbations could be defined, opening up the possibility to study more general geometries as emerging from GFT.

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[*] s.c.gielen@sheffield.ac.uk
[†] lmickel1@sheffield.ac.uk

I. INTRODUCTION

The conceptual foundation of classical general relativity starts from the notion of a spacetime metric, from which all relevant geometric properties of spacetime, as well as physical effects related to gravity, can be derived. While the dynamical equations of general relativity are formulated in diffeomorphism-covariant terms – they take the same form no matter what coordinate system is used – the tensorial quantities in them, most prominently the metric itself, are not diffeomorphism-invariant and therefore depend on the coordinate system. This means that the metric or curvature tensors cannot directly be observable. Constructing interesting (diffeomorphism-invariant) observables from the metric is in general a highly nontrivial task [1–4].

The implementation of diffeomorphism symmetry at the quantum level is often seen as one of the most formidable obstacles in the construction of a full theory of quantum gravity. Discrete approaches somewhat circumvent this problem since they no longer work with a differentiable manifold on which diffeomorphisms act, but directly with quantities such as lengths, areas or finite parallel transports which are to an extent diffeomorphism-invariant. However, such approaches then face at least two important basic challenges: one is the recovery of a continuum limit in which differentiable structures, and with them the usual freedom to choose coordinates emerge [5, 6]; the other is the extraction of relevant observables, given that there is no useful way of directly defining tensorial objects, such as curvature invariants, in the discrete setting (see, e.g., [7] for discussion in the setting of causal dynamical triangulations).

A common strategy to construct useful observables is to focus on relational observables [4], particularly those built from using matter fields as coordinates. A prime example of this is homogeneous cosmology, where a massless scalar field can serve as a good clock, and the expansion of the universe can be characterised by stating the evolution of the scale factor relative to the value of the scalar field [8]. Such a characterisation is indeed invariant under time reparametrisations. More generally, suitably chosen matter fields allow the construction of a relational coordinate system, such that the coordinates are now physical degrees of freedom rather than arbitrary gauge structure. This idea has been employed particularly in dust models (see [9–12] for some of the vast literature) and gravity coupled to scalar fields [13, 14].

In this paper we focus on the group field theory (GFT) approach to quantum gravity [15–17], which is closely related to loop quantum gravity, matrix and tensor models. GFT is a fundamentally discrete (and background-independent) setting for quantum gravity; one does not work with fields on a manifold but with combinatorial structures from which spacetime, and all continuum matter fields, are supposed to emerge in a continuum limit. Because of this, GFT shares with other discrete approaches the issues of defining observables and in particular an analogue of a spacetime metric, which would be important in order to connect to classical gravitational theories (general relativity or extensions) or any type of phenomenology. Gravitational observables that have been constructed so far in GFT are defined in analogy with simple geometric operators in loop quantum gravity; in particular one can define a GFT volume operator, and from this a relational volume observable representing the total spatial volume at a given instant of relational time, here again given by a massless scalar field. This observable has been used to derive an effective Friedmann equation [18, 19], very similar to the one in loop quantum cosmology. By coupling additional massless scalar fields that are used as "rods" or spatial coordinates, one can turn this global volume into a local volume element (now dependent on the spatial coordinates as well) and try and use this to define an effective cosmological perturbation theory in GFT [20–24]. Such a formalism can then be used to gain initial insights on dynamics of volume perturbations around an effective homogeneous universe characterised by the Friedmann equation obtained before, but given that volume perturbations are not gauge-invariant, translating the results into the usual textbook discussions in terms of gauge-invariant quantities requires some effort. There is also no direct way of accessing tensor fluctuations which do not affect the local volume element.

In the following we will propose a new approach towards tackling this issue, based on the idea of locally conserved currents associated to GFT symmetries. All previous work on GFT models for quantum gravity with massless scalar fields, starting from [18, 19], starts by identifying symmetries of the corresponding classical theory, and requiring that these are represented as symmetries in GFT. In the case of cosmological models based on a single matter field, the most important symmetry is with respect to shifts in the field; this symmetry of a classical free, massless scalar field justifies its use as a clock. In this relational coordinate interpretation, it can be seen as a time-translation symmetry. The conserved quantity associated to this symmetry is the momentum conjugate to the scalar field, whose conservation gives the Klein–Gordon equation in the homogeneous approximation. Hence, the matter dynamics obtained from GFT are consistent with classical expectations, and the scalar field is a good clock also in GFT.

Our proposal is to extend this line of argument to four free, massless scalar fields, now used as coordinates for space and time. There are now four independent translational symmetries, leading to four conserved currents which form the analogue of the energy-momentum tensor in standard quantum field theory. At the classical level, each massless scalar field has its own Klein–Gordon current, whose conservation gives the classical field equations. By showing that the GFT energy-momentum tensor arising from the translational invariance of the GFT action with respect to the relational fields is conserved classically and quantum-mechanically, we immediately obtain the Klein–Gordon equations for all matter fields, now no longer restricted to the spatially homogeneous setting. Thus, as a first major insight we show that these matter fields always satisfy the same dynamical equations in GFT as they do in standard spacetime field theory. Within the approximations we use regarding the dynamics of GFT, this is true for any state, and does not require any semiclassical approximations as are often used in the literature. We then use the fact that classically the Klein–Gordon currents depend explicitly on the metric, and so in a matter reference frame can be used to read off all components of the metric. Given that the GFT energy-momentum tensor represents the same physical quantities, we show how to define the analogue of a spacetime metric in full GFT, again valid for any state and without any approximations. This result is the main achievement of our work, given the previous severe limitations in defining relational observables. We illustrate our new formalism in the case of homogeneous, isotropic cosmology, finding some familiar results regarding a Friedmann equation and bounce, but also some puzzling results regarding the role of the new spatial coordinate fields. These results deserve further attention, but our formalism also suggests applications to inhomogeneous cosmology, black holes or other spacetimes of interest, which will be explored in future work.

In section II we show how in general relativity the classical shift symmetry of four free massless scalar fields leads to conserved currents that encode the metric in a (relational) coordinate system given by these scalar fields. Section III gives a short review of the canonical quantisation of GFT, leading to the definition of an energy-momentum tensor. We explicitly show that the energy-momentum tensor is conserved as an operator. Section IV illustrates the general idea in a simple cosmological example. By choosing an appropriate coherent state, we find an effective metric corresponding to a spatially flat homogeneous and isotropic universe. We also derive the effective

Friedmann equation, which is similar to equations derived by other methods in the literature. At late times, this Friedmann equation reduces to the classical equation expected for a single scalar field only, leading to a discussion of why the other fields do not contribute. We conclude in section V.

II. SPACETIME METRIC AS A CONSERVED CURRENT

In standard field theory, a free massless scalar field χ on a curved background (with Lorentzian metric $g_{\mu\nu}$) can be defined in terms of the action

$$S = -\frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi\,, \tag{1}$$

which is invariant under constant shifts in the field $\chi \mapsto \chi + \epsilon$, where ϵ is a constant. By Noether's theorem this symmetry implies a conservation law

$$\partial_\mu j^\mu = 0\,, \quad j^\mu = -\sqrt{-g}\,g^{\mu\nu}\partial_\nu\chi\,. \tag{2}$$

This conservation law is of course nothing but the Klein–Gordon equation $\Box\chi = 0$.

Let us now identify the scalar field with a spacetime coordinate x^A (i.e., surfaces of constant x^A are taken to be surfaces of constant χ). In this case, by definition, we have $\partial_\mu\chi = \delta^A_\mu$ and hence

$$(j^\mu)^A = -\sqrt{-g}\,g^{\mu A}\,. \tag{3}$$

We can use four such free massless scalar fields $\chi^A,\ A = 0,\ldots,3$ to define an entire relational coordinate system by identifying each spacetime point with the values of all χ^A taken at that point. The resulting relational coordinate system is locally well-defined as long as we assume the non-degeneracy condition (with respect to an arbitrary well-defined coordinate system) $\det(\partial_\mu\chi^A) \neq 0$.[1] In this coordinate system, where the gradients of the scalar fields are mere numbers and thereby dimensionless, the metric components have units of length^4, $[g_{AB}] = L^4$, and the conserved current has $[(j^\mu)^A] = L^4$ (with $\hbar = c = 1$). In principle, we could have inserted an arbitrary dimensionful proportionality factor ξ when fixing the coordinate system, $\partial_\mu\chi^A = \xi\delta^A_\mu$, but as ξ just represents a unit convention without physical significance, the simplest choice is $\xi = 1$.

In this coordinate system, knowledge of the currents on the left-hand side of (3) can be used to define a symmetric matrix field $j^{AB} = \left(j^A\right)^B$ which defines the inverse metric (in this coordinate system) as

$$g^{AB} = -\left(-\det(j^{AB})\right)^{-1/2} j^{AB}\,. \tag{4}$$

The positioning of the capital Latin index on the left-hand side of (3) is initially conventional, given that it merely corresponds to a label for the different matter fields. However, once we define j^{AB} and establish its relation to the inverse metric, the notation becomes more intuitive if we think of A, B as contravariant indices.

There are two minus signs in the relation (4); the one inside the brackets comes from the assumption that $g_{\mu\nu}$, and hence also j^{AB}, has negative determinant (coming from a Lorentzian

[1] For a single scalar field to be used as clock, the equivalent condition is $\partial_t\chi \neq 0$, which is (for a free massless scalar field) almost always the case in homogeneous cosmology. Outside of spatial homogeneity and for four scalars, it is less straightforward to say in general where this condition is satisfied.

signature). The overall minus sign can be traced back to the minus sign in the action (1), which comes from the assumption that the metric $g_{\mu\nu}$ has one negative and three positive eigenvalues (the "East Coast" signature convention). Of course, classically these assumptions are reasonable, but in a quantum gravity setting it may not be a priori clear whether we can fix the metric signature or the number of positive and negative eigenvalues; see [25] for a recent discussion of a general classical framework in which all possible signatures may co-exist. Adopting the "West Coast" signature convention throughout would change the signature of $g^{\mu\nu}$ but also add an additional minus sign in (3), leading to a $(j^\mu)^A$ of the same signature $(+---)$. We will come back to a discussion of this point when looking at concrete examples.

When coupling these matter fields to general relativity, we obtain a diffeomorphism-invariant theory in which $g_{\mu\nu}$ becomes dynamical. The condition $\partial_\mu \chi^A = \delta^A_\mu$ is a local gauge-fixing of this gauge symmetry, which can be seen as a specific case of the harmonic gauge condition $\Box x^\mu = 0$ whose use has a long history in classical general relativity [26–28]. Whereas the harmonic gauge condition in general does not uniquely fix the gauge (there are many solutions to it for a general $g_{\mu\nu}$), the "scalar field gauge" we are adopting here does fix it completely, assuming it is well-defined. For free massless scalar fields, this gauge does not determine which of the coordinate directions given by χ^A are timelike, spacelike or null, unlike for dust constructions such as [9] which are more suited to a $(3+1)$ splitting in which spacelike and timelike directions need to be separated.

The central result of this discussion is (4), which tells us how to compute all components of the inverse metric from the symmetric j^{AB}. In a quantum theory in which j^{AB} can be defined as an operator, one can define an effective g^{AB}, e.g., from expectation values of j^{AB} in a semiclassical state by using (4). In the following we will see that a GFT coupled to four scalar fields does have an operator analogue of j^{AB} (given by the GFT energy-momentum tensor) and hence an effective spacetime metric can be written down unambiguously.

III. GROUP FIELD THEORY

GFT can be seen as a "quantum field theory not on, but of spacetime". The basic object in any GFT is a (typically real or complex bosonic) group field $\varphi(g_i, \chi^A)$, where the g_i are elements of some group and the χ^A real valued, as discussed further below. The arguments of this field do not represent coordinates on a spacetime manifold; instead, the "particles" associated with excitations of this group field are seen as elementary building blocks of spacetime geometry and matter, not living in a pre-defined spacetime. Concretely, such an elementary building block is most commonly identified with a tetrahedron seen as the basic unit of simplicial geometry, or equivalently with a four-valent spin network vertex as the basic structure forming loop quantum gravity spin networks [17]. In a Fock space picture, a macroscopic geometry can then only emerge from a large number of such excitations over the initial vacuum.

The discussions of this paper are applicable to a large class of models, but for concreteness we can choose to use SU(2) variables to represent parallel transports in the GFT discrete geometry, in analogy with the basic variables of canonical loop quantum gravity. More importantly, we include four $\mathbb{R}$-valued arguments χ^A, which represent scalar matter degrees of freedom. We also restrict ourselves to the simpler case of a real group field; generalisation to a complex field should be entirely straightforward but does not seem necessary for our purposes. We thus have $\varphi : \mathrm{SU}(2)^2 \times \mathbb{R}^4 \to \mathbb{R}$.

As in loop quantum gravity, it is useful to expand the group field in modes associated to SU(2) representation data,

$$\varphi(g_i, \chi^A) = \sum_J \varphi_J(\chi^A)\, D_J(g_i)\,, \tag{5}$$

where $D_J(g_i)$ represent suitable combinations of Wigner D-matrices and $J = (\vec{j},\, \vec{m},\, \iota)$ is a multi-index representing SU(2) irreducible representations $\vec{j}$, magnetic indices $\vec{m}$, and intertwiners ι (for more details see, e.g., [29]). In the following, we will only need to use the existence of such an expansion, and no details of SU(2) representation theory. This means that our results will apply to any choice of (at least compact) group for the g_i variables.

In constructing possible actions, one starting point is then to demand that the GFT action is invariant under symmetries representing symmetries of the matter fields one wants to include [18, 19]. As discussed above, the spacetime scalar field action (1) is invariant under shifts in the field χ. It is also invariant under reflections $\chi \mapsto -\chi$. Demanding the same symmetries in GFT means that a GFT action local in χ^A cannot depend explicitly on the χ^A, and can only include derivatives of even order. If all χ^A represent physically indistinguishable matter fields, we might also require symmetry under rotations $\chi^A \mapsto R^A{}_B\chi^B$ [30]. Assuming that the action is also local in the g_i gives the general form

$$S[\varphi] = \int \mathrm{d}^4\chi \left(\frac{1}{2} \sum_J \sum_{n=0}^{\infty} \mathcal{K}_J^{(2n)}\, \varphi_J(\chi^A)\Delta^n\varphi_J(\chi^A) - V(\varphi) \right)\,, \tag{6}$$

where the $\mathcal{K}_J^{(2n)}$ are arbitrary couplings and $\Delta = \sum_A \left(\frac{\partial}{\partial\chi^A}\right)^2$ is the Laplacian on $\mathbb{R}^4$. Here we have written the quadratic part of the action explicitly and moved all higher-order terms into the potential $V(\varphi)$.

In practice, the sum over higher derivatives in the quadratic part needs to terminate at some finite n, and indeed in most models studied in detail in the literature [31–33] only the terms $n = 0$ and $n = 1$ corresponding to a mass term and a single Laplacian appear. After an integration by parts of the term involving the Laplacian, one obtains an action that only depends on φ and first derivatives, and is hence amenable to straightforward canonical quantisation. (A theory with higher derivatives would have to be treated with more involved methods, see, e.g., [34].)

Here we have not specified the precise form of the interaction terms making up the potential $V(\varphi)$. This form can be chosen by requiring that Feynman amplitudes of the resulting interacting GFT match those of spin foam models [35, 36] and/or by again using symmetry arguments to write down a number of possible terms, which are constrained by renormalisability [37]. In our work here, we will neglect the effect of interactions. Our discussion of classical GFT extends to interacting models but their quantum analysis will be more involved in general. Neglecting interactions is a common assumption in applications of GFT to cosmology [18, 19, 24, 38], since these are expected to be subdominant in the very early universe. In general, the range of applicability of this approximation will be limited.

The symmetry of (6) under translations $\chi^A \mapsto \chi^A + \epsilon^A$ for arbitrary constant ϵ^A leads to a conserved current, the GFT energy-momentum tensor

$$T^{AB} := -\frac{\partial\mathcal{L}}{\partial(\partial_A\varphi)}\partial_B\varphi + \delta^{AB}\,\mathcal{L} = \sum_J \left(\mathcal{K}_J^{(2)}\partial_A\varphi_J\,\partial_B\varphi_J\right) + \delta^{AB}\,\mathcal{L} \tag{7}$$

with a Lagrangian density

$$\mathcal{L} = \sum_J \left(\frac{1}{2}\mathcal{K}_J^{(0)}\varphi_J - \frac{1}{2}\mathcal{K}_J^{(2)}(\partial_A\varphi_J)^2 \right) - V(\varphi)\,, \tag{8}$$

in which we now assume that only the terms $n = 0$ and $n = 1$ are present in (6), and we have performed the integration by parts discarding a boundary term. In these expressions, we do not need to worry too much about the positioning of $A, B, \ldots$ indices; due to the $E(4)$ symmetry of the GFT action these can be raised and lowered with the Kronecker delta δ_{AB}. In the identification of these GFT quantities with classical spacetime tensors, we need to be more careful, as discussed above and further below.

Naturally, the GFT energy-momentum tensor satisfies $\partial_A T^{AB} = 0$. The fact that translations in the scalar field variables χ^A represent constant shifts in the fields that span the relational coordinate system as discussed in section II now suggests identifying the conserved GFT quantity T^{AB}, or rather its expectation value in a suitable semiclassical state, with the classically conserved current j^{AB}. This leads to an effective spacetime metric g^{AB} via (4). In the following, we construct the operators corresponding to T^{AB}; we then show a concrete example of such an identification in section IV.

A. Canonical quantisation

We can now implement a canonical quantisation procedure for a theory defined by (8). This "deparametrised" approach to quantisation, in which a scalar field is used as a clock from the beginning, was introduced in [39], and an extension of this procedure to the case of a GFT action with four scalar fields was proposed in [30]. In the latter case, which we summarise in this section, one needs to single out a clock field to construct the Hamiltonian, which breaks the $E(4)$ symmetry between the fields. In the following we will denote the clock field with χ^0 and the other "spatial" fields as χ^a or χ^b, where $a, b = 1, 2, 3$. The Hamiltonian associated to (8) reads

$$H = \int \mathrm{d}^3\chi \sum_J \frac{\mathcal{K}_J^{(2)}}{2} \left(-\frac{\pi_J^2}{|\mathcal{K}_J^{(2)}|^2} + m_J^2\varphi_J^2 + \sum_b (\partial_b\varphi_J)^2 \right) + V(\varphi)\,, \tag{9}$$

where we introduced the canonical momentum $\pi_J = -\mathcal{K}_J^{(2)}\partial_0\varphi_J$ and the shorthand $m_J^2 = -\frac{\mathcal{K}_J^{(0)}}{\mathcal{K}_J^{(2)}}$.

Restricting to the free theory with $V(\varphi) = 0$ from now on, we then carry out a Fourier decomposition of the above, defining $\omega_{J,k}^2 = m_J^2 + \vec{k}^2$:

$$H = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \sum_J \frac{\mathcal{K}_J^{(2)}}{2} \left(-\frac{1}{|\mathcal{K}_J^{(2)}|^2}\pi_{J,-k}(\chi^0)\pi_{J,k}(\chi^0) + \omega_{J,k}^2\,\varphi_{J,-k}(\chi^0)\varphi_{J,k}(\chi^0) \right)\,. \tag{10}$$

For the (Heisenberg picture) quantisation we promote the Fourier modes of the group field φ_J and its conjugate momentum π_J to operators satisfying

$$[\varphi_{J,k}(\chi^0), \pi_{J',k'}(\chi^0)] = \mathrm{i}\,\delta_{JJ'}(2\pi)^3\delta(\vec{k} + \vec{k}')\,, \tag{11}$$

so that these operators evolve in time according to

$$\partial_0\pi_{J,k} = -\mathrm{i}\;[\pi_{J,k}, H] = -\mathcal{K}_J^{(2)}\omega_{J,k}^2\varphi_{J,k}\,, \qquad \partial_0\varphi_{J,k} = -\frac{\pi_{J,k}}{\mathcal{K}^{(2)}}\,, \tag{12}$$

just as the classical field modes would.

It is then useful to introduce time-dependent creation and annihilation operators $A_{J,k}(\chi^0)$, $A^\dagger_{J,k}(\chi^0)$, which satisfy the equal-time commutation relations

$$[A_{J,k}(\chi^0), A^\dagger_{J',k'}(\chi^0)] = \delta_{JJ'}(2\pi)^3\delta(\vec{k}-\vec{k'})\,, \tag{13}$$

with all other commutators vanishing. These operators are defined from the field operators via

$$\pi_{J,k}(\chi^0) = -\mathrm{i}\alpha_{J,k}(A_{J,k} - A^\dagger_{J,-k})\,,\quad \varphi_{J,k}(\chi^0) = \frac{1}{2\alpha_{J,k}}(A_{J,k} + A^\dagger_{J,-k})\,;\quad \alpha_{J,k} = \sqrt{\frac{|\omega_{J,k}||\mathcal{K}^{(2)}|}{2}}\,. \tag{14}$$

In the following, we will also use time-independent (or Schrödinger picture) creation and annihilation operators $a_{J,k}$, $a^\dagger_{J,k}$, defined by $a_{J,k} = A_{J,k}(0)$ and $a^\dagger_{J,k} = A^\dagger_{J,k}(0)$.

As discussed in detail in [30, 39], when written in terms of ladder operators the Hamiltonian takes on a different form depending on the sign of $\omega^2_{J,k}$, which in general depends both on the sign of m^2_J and the value of $\vec{k}$. Writing the total Hamiltonian (10) as $H = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\sum_J H_{J,k}$, for a mode with $\omega^2_{J,k} < 0$ one finds the Hamiltonian of a harmonic oscillator

$$H_{J,k} = -\,\mathrm{sgn}(\mathcal{K}^{(2)}_J)\frac{|\omega_{J,k}|}{2}\left(a_{J,-k}a^\dagger_{J,-k} + a^\dagger_{J,k}a_{J,k}\right)\,, \tag{15}$$

whereas for a mode with $\omega^2_{J,k} > 0$ we obtain a squeezing Hamiltonian

$$H_{J,k} = \mathrm{sgn}(\mathcal{K}^{(2)}_J)\frac{\omega_{J,k}}{2}\left(a_{J,k}a_{J,-k} + a^\dagger_{J,k}a^\dagger_{J,-k}\right)\,. \tag{16}$$

The modes with squeezing Hamiltonian are of particular interest for cosmology, given that cosmological time evolution can be interpreted as squeezing [40], or in other words, given that the action of such a Hamiltonian leads to an exponentially growing number of quanta of geometry which one can interpret as an expanding universe. In contrast, a harmonic oscillator Hamiltonian leads to a conserved particle number for the given mode, more akin to a static cosmology.

A particularly natural choice for $\mathcal{K}^{(2)}_J$ and $\mathcal{K}^{(0)}_J$, which is often considered in the literature (see also our discussion below (6)), is obtained from a GFT action whose kinetic term includes a mass term and a Laplace–Beltrami operator on $\mathrm{SU}(2)^4\times\mathbb{R}^4$. In this case, one may set

$$\mathcal{K}^{(0)}_J = \mu - \sum_{i=1}^4 j_i(j_i+1), \quad \mathcal{K}^{(2)}_J = \tau\,, \tag{17}$$

for some constants μ and τ, and where the j_i are the irreducible representations associated to an elementary "quantum of space" (for isotropic tetrahedra these all take on the same value). We then have $m^2_J = \frac{\sum_i j_i(j_i+1)-\mu}{\tau}$ and, if $\mu > 0$, there is a sign change in m^2_J for large j_i values. Choosing $\tau < 0$ would imply that only small j modes have $m^2_J > 0$ [41].

For the harmonic oscillator Hamiltonian (15) the explicit expressions for time-dependent ladder operators are

$$A_{J,k} = a_{J,k}e^{-\mathrm{i}|\omega_{J,k}|\chi^0}\,,\quad A^\dagger_{J,k} = a^\dagger_{J,k}e^{\mathrm{i}|\omega_{J,k}|\chi^0}\,, \tag{18}$$

whereas for a squeezing Hamiltonian of the form of (16), the operator dynamics are solved by

$$\begin{aligned} A_{J,k} &= a_{J,k}\cosh\left(|\omega_{J,k}|\chi^0\right) - \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})a^\dagger_{J,-k}\sinh\left(|\omega_{J,k}|\chi^0\right)\,,\\ A^\dagger_{J,k} &= a^\dagger_{J,k}\cosh\left(|\omega_{J,k}|\chi^0\right) + \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})a_{J,-k}\sinh\left(|\omega_{J,k}|\chi^0\right)\,. \end{aligned} \tag{19}$$

These expressions can be used to write down the time-dependent expression for all operators constructed out of $A_{J,k}(\chi^0)$, $A^\dagger_{J,k}(\chi^0)$; within the approximation to the free theory, dynamics can be solved exactly in the Heisenberg picture. We will use this fact in the new results obtained below.

B. Energy-momentum tensor

We now proceed to quantise the components of the GFT energy-momentum tensor T^{AB} (7), thus introducing a novel set of operators. From hereon we restrict to a single J mode and therefore omit the J label; the single mode assumption is often used when considering GFT cosmology, as the mode with the largest $|m_J|$ will dominate at late times [41]. Here we are not yet interested in a specific physical scenario, but from the form of the Hamiltonian (10) it is apparent that the construction we carry out below can be conducted for any of the J modes, and the total operators would be a sum over all these modes. It would thus be straightforward to extend our discussion to multiple modes, which could be interesting, e.g., for phenomenological applications such as [38, 42, 43].

First, we insert the expression for π into (7), where the expressions below depend on the clock as well as the spatial fields $T^{AB} = T^{AB}(\chi^0, \vec{\chi})$. We find

$$\begin{aligned} T^{00} &= \frac{\pi^2}{2\mathcal{K}^{(2)}} - \frac{\mathcal{K}^{(2)}}{2}\left(m^2\varphi^2 + \sum_b (\partial_b\varphi)^2\right), \\ T^{0b} &= -\pi\partial_b\varphi\,, \qquad T^{a\neq b} = \mathcal{K}^{(2)}\partial_a\varphi\partial_b\varphi\,, \\ T^{aa} &= -\frac{\pi^2}{2\mathcal{K}^{(2)}} - \frac{\mathcal{K}^{(2)}}{2}\left(m^2\varphi^2 - (\partial_a\varphi)^2 + \sum_{b\neq a}(\partial_b\varphi)^2\right) \quad \text{(no sum over } a). \end{aligned} \tag{20}$$

The Fourier transforms $T^{AB}_k = T^{AB}_k(\chi^0)$ of the above functions, where $\varphi_k = \varphi_k(\chi^0)$ and $\pi_k = \pi_k(\chi^0)$ denote the Fourier transforms of $\varphi(\chi^0, \vec{\chi})$ and $\pi(\chi^0, \vec{\chi})$, respectively, are given by the following convolutions:

$$\begin{aligned} T^{00}_k &= \frac{1}{2}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\left[\frac{\pi_\gamma\pi_{k-\gamma}}{\mathcal{K}^{(2)}} - \mathcal{K}^{(2)}\left(m^2 - \vec{\gamma}\cdot(\vec{k}-\vec{\gamma})\right)\varphi_\gamma\varphi_{k-\gamma}\right], \\ T^{0b}_k &= -\mathrm{i}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\gamma_b\pi_{k-\gamma}\varphi_\gamma\,, \qquad T^{a\neq b}_k = -\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\mathcal{K}^{(2)}\gamma_a(k_b-\gamma_b)\varphi_\gamma\varphi_{k-\gamma}\,, \\ T^{aa}_k &= \frac{1}{2}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\left[\mathcal{K}^{(2)}\left(-\gamma_a(k_a-\gamma_a) + \sum_{b\neq a}\gamma_b(k_b-\gamma_b) - m^2\right)\varphi_\gamma\varphi_{k-\gamma} - \frac{\pi_\gamma\pi_{k-\gamma}}{\mathcal{K}^{(2)}}\right]. \end{aligned} \tag{21}$$

Initially, we will think of the corresponding operators $\mathcal{T}^{AB}_k$ as defined simply by replacing the Fourier modes φ_k and π_k by operators, with $\pi_{k-\gamma}$ appearing to the left of φ_γ in $\mathcal{T}^{0b}_k$. (We discuss normal ordering below.)

The classical energy-momentum tensor satisfies $\partial_0 T^{0B}_k + \mathrm{i}\sum_a k_a T^{aB}_k = 0$. One would expect this conservation law to also hold at the level of operators: any alterations could only arise from commutators (11) that appear from necessary operator reordering. However, such terms are always proportional to $\delta(k)$ and time-independent, so that we obtain $\partial_0 \mathcal{T}_k^{0B} + \mathrm{i}\sum_a k_a \mathcal{T}_k^{aB} = \partial_0 \xi^{(0)}\delta(k) + \mathrm{i}\sum_a \xi^{(a)} k_a \delta(k) = 0$, where $\xi^{(A)}$ are independent of χ^0. Explicitly, we find

$$\begin{aligned}
\partial_0 \mathcal{T}_k^{00} &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{1}{2}\left[\left(\frac{\partial_0\pi_\gamma}{\mathcal{K}^{(2)}} + \left(m^2 - \vec{\gamma}\cdot(\vec{k}-\vec{\gamma})\right)\varphi_\gamma\right)\pi_{k-\gamma} + \pi_{k-\gamma}\left(\frac{\partial_0\pi_\gamma}{\mathcal{K}^{(2)}} + \left(m^2 - \vec{\gamma}\cdot(\vec{k}-\vec{\gamma})\right)\varphi_\gamma\right)\right], \\
\mathrm{i}\sum_a k_a \mathcal{T}_k^{0a} &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{1}{2}\vec{k}\cdot\vec{\gamma}\left(\pi_{k-\gamma}\varphi_\gamma + \varphi_\gamma\pi_{k-\gamma} + [\pi_{k-\gamma},\varphi_\gamma]\right),
\end{aligned} \tag{22}$$

where we made use of $\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} f(\vec{\gamma})g(\vec{k}-\vec{\gamma}) = \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} f(\vec{k}-\vec{\gamma})g(\vec{\gamma})$.

Combining the above terms, we then obtain

$$\begin{aligned}
\partial_0 \mathcal{T}_k^{00} &+ \mathrm{i}\sum_a k_a \mathcal{T}_k^{0a} \\
&= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{1}{2}\left[\left(\frac{\partial_0\pi_\gamma}{\mathcal{K}^{(2)}} + \omega_\gamma^2\varphi_\gamma\right)\pi_{k-\gamma} + \pi_{k-\gamma}\left(\frac{\partial_0\pi_\gamma}{\mathcal{K}^{(2)}} + \omega_\gamma^2\varphi_\gamma\right) + \vec{k}\cdot\vec{\gamma}[\pi_{k-\gamma},\varphi_\gamma]\right] \\
&= 0
\end{aligned} \tag{23}$$

from (12) and $[\pi_{k-\gamma},\varphi_\gamma] \propto \delta(k)$. As anticipated, the extra term coming from operator ordering does not contribute.

In the case of $\partial_0 \mathcal{T}_k^{0b} + \mathrm{i}\sum_a k_a \mathcal{T}_k^{ab} = 0$ we have

$$\begin{aligned}
\partial_0 \mathcal{T}_k^{0b} &= \mathrm{i}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\gamma_b\left(\mathcal{K}^{(2)}\omega_{k-\gamma}^2\varphi_{k-\gamma}\varphi_\gamma + \frac{\pi_{k-\gamma}\pi_\gamma}{\mathcal{K}^{(2)}}\right), \\
\mathrm{i}\sum_a k_a \mathcal{T}_k^{ab} &= \mathrm{i}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\left[\left(-(k_b-\gamma_b)\vec{k}\cdot\vec{\gamma} + \frac{k_b}{2}(\vec{k}-\vec{\gamma})\cdot\vec{\gamma} - \frac{m^2}{2}k_b\right)\mathcal{K}^{(2)}\varphi_\gamma\varphi_{k-\gamma} - \frac{k_b}{2}\frac{\pi_\gamma\pi_{k-\gamma}}{\mathcal{K}^{(2)}}\right].
\end{aligned} \tag{24}$$

In this case there are no mixed $\pi\varphi$ terms, so there can be no nontrivial commutators and the conservation law should follow as for the classical fields. To see this, in the expression for $\partial_0 \mathcal{T}_k^{0b}$ change integration variables $\vec{\gamma} \to \vec{k} - \vec{\gamma}$, add to the original expression and divide by 2 to obtain

$$\begin{aligned}
\partial_0 \mathcal{T}_k^{0b} &= \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\left[\left(\gamma_b\omega_{k-\gamma}^2 + (k_b-\gamma_b)\omega_\gamma^2\right)\mathcal{K}^{(2)}\varphi_{k-\gamma}\varphi_\gamma + k_b\frac{\pi_{k-\gamma}\pi_\gamma}{\mathcal{K}^{(2)}}\right] \\
&= \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\left[\left(\gamma_b(\vec{k}-\vec{\gamma})^2 + (k_b-\gamma_b)\vec{\gamma}^2 + k_b m^2\right)\mathcal{K}^{(2)}\varphi_{k-\gamma}\varphi_\gamma + k_b\frac{\pi_{k-\gamma}\pi_\gamma}{\mathcal{K}^{(2)}}\right].
\end{aligned} \tag{25}$$

We then obtain

$$\partial_0 \mathcal{T}_k^{0b} + \mathrm{i}\sum_a k_a \mathcal{T}_k^{ab} = \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\left[\left(\gamma_b\vec{k}^2 - k_b(\vec{k}\cdot\vec{\gamma})\right)\mathcal{K}^{(2)}\varphi_{k-\gamma}\varphi_\gamma\right] = 0\,, \tag{26}$$

since the integral can be transformed into minus itself under a variable change $\vec{\gamma} \to \vec{k} - \vec{\gamma}$.

We have hence shown explicitly that the operators representing the Fourier modes of the GFT energy-momentum tensor for a single mode satisfy the conservation law $\partial_0 \mathcal{T}_k^{0B} + \mathrm{i}\sum_a k_a \mathcal{T}_k^{aB} = 0$. We should stress again that this is true already for the operators, and would in particular hold for the corresponding expectation values in <u>any</u> state.

Here we have defined the energy-momentum tensor without applying a normal ordering prescription. One might then be worried that there are divergences in this definition of $\mathcal{T}_k^{AB}$, and wonder whether the same conservation law applies to a normal-ordered definition (which we denote by $:\mathcal{T}_k^{AB}:$). Normal ordering can be implemented in terms of the time-dependent operators $A_k(\chi^0)$ and $A_k^\dagger(\chi^0)$, i.e., moving all $A_k(\chi^0)$ to the right of any $A_k^\dagger(\chi^0)$, if one wants the procedure to be equivalent to standard normal ordering in the Schrödinger picture. However, this is not enough to cancel divergences coming from the unstable squeezed modes with Hamiltonian (16). In [30], normal ordering for time-dependent operators was followed by a regularisation in which vacuum expectation values were subtracted. Both steps can be implemented as one by imposing normal ordering at the level of the a_k and $a_k^\dagger$ operators, leading to the definition

$$
\begin{aligned}
:\mathcal{T}_k^{00}: &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{4\sqrt{|\omega_\gamma||\omega_{k-\gamma}|}}\left[2\beta^+_{k,\gamma} : A^\dagger_{-\gamma}A_{k-\gamma}: + \beta^-_{k,\gamma}\left(:A^\dagger_{-\gamma}A^\dagger_{\gamma-k}: + :A_\gamma A_{k-\gamma}:\right)\right],\\
:\mathcal{T}_k^{0b}: &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{1}{2}\sqrt{\frac{|\omega_{k-\gamma}|}{|\omega_\gamma|}}\gamma_b\left(:A^\dagger_{\gamma-k}A_\gamma: - :A^\dagger_{-\gamma}A_{k-\gamma}: - :A_{k-\gamma}A_\gamma: + :A^\dagger_{\gamma-k}A^\dagger_{-\gamma}:\right),\\
:\mathcal{T}_k^{a\neq b}: &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{2\sqrt{|\omega_\gamma||\omega_{k-\gamma}|}}\gamma_a\left(\gamma_b - k_b\right)\left(:A^\dagger_{-\gamma}A_{k-\gamma}: + :A^\dagger_{\gamma-k}A_\gamma: + :A^\dagger_{-\gamma}A^\dagger_{\gamma-k}: + :A_\gamma A_{k-\gamma}:\right),\\
:\mathcal{T}_k^{aa}: &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{4\sqrt{|\omega_\gamma||\omega_{k-\gamma}|}}\Big[2(\beta^-_{k,\gamma} - 2\gamma_a(k_a-\gamma_a)) : A^\dagger_{-\gamma}A_{k-\gamma}:\\
&\qquad + (\beta^+_{k,\gamma} - 2\gamma_a(k_a-\gamma_a))\left(:A^\dagger_{-\gamma}A^\dagger_{\gamma-k}: + :A_\gamma A_{k-\gamma}:\right)\Big],
\end{aligned}
\tag{27}
$$

where we defined $\beta^\pm_{k,\gamma} = -m^2 + \vec{\gamma}\cdot(\vec{k}-\vec{\gamma}) \pm |\omega_\gamma||\omega_{k-\gamma}|$.

Using (18) and (19), one can now write the Fourier modes of the energy-momentum tensor in terms of time-dependent functions of ladder operators a_k and $a_k^\dagger$, and implement normal ordering. Notice that in general only four independent combinations of ladder operators are needed to define all components of $\mathcal{T}^{AB}$. However, given that in general there are two different types of modes (squeezed, $\omega_k^2 > 0$, and oscillating, $\omega_k^2 < 0$, ones), these explicit expressions depend on what types of mode contribute, and hence on the value of m^2. If $m^2 < 0$ and for $\vec{\gamma}^2 < |m^2|$, A_γ and $A_\gamma^\dagger$ operators have the dynamics of oscillating modes (18); for all other cases they follow the dynamics of squeezing modes (19). Hence, in the operator products appearing in the expressions for $\mathcal{T}^{AB}$ both operators can be of squeezing type, one can be of squeezing and one of oscillating type, or both can be oscillating modes. The mixed case occurs for all $\vec{k} \neq 0$ if $m^2 < 0$, but in the following we will only be interested in $\vec{k} = 0$ expressions, and therefore only provide the expressions for operator pairs of the same type; it is straightforward to derive the mixed cases from (18) and (19).

For pairs of operators associated to two oscillating modes ($\omega_\gamma^2 < 0$ and $\omega_{k-\gamma}^2 < 0$) we obtain the simple form

$$
\begin{aligned}
:A^\dagger_{-\gamma}A_{k-\gamma}: &= a^\dagger_{-\gamma}a_{k-\gamma}, & :A^\dagger_{-\gamma}A^\dagger_{\gamma-k}: &= a^\dagger_{-\gamma}a^\dagger_{\gamma-k}e^{\mathrm{i}(|\omega_{-\gamma}|+|\omega_{\gamma-k}|)\chi^0},\\
:A^\dagger_{\gamma-k}A_\gamma: &= a^\dagger_{\gamma-k}a_\gamma, & :A_\gamma A_{k-\gamma}: &= a_\gamma a_{k-\gamma}e^{-\mathrm{i}(|\omega_\gamma|+|\omega_{k-\gamma}|)\chi^0}.
\end{aligned}
\tag{28}
$$

Normal-ordered expressions for pairs of A and $A^\dagger$ operators associated to two squeezed modes

($\omega_\gamma^2 > 0$ and $\omega_{k-\gamma}^2 > 0$) are given by

$$
\begin{aligned}
:A^\dagger_{-\gamma}A_{k-\gamma}: &= a^\dagger_{\gamma-k}a_\gamma \sinh\left(\omega_{-\gamma}\chi^0\right)\sinh\left(\omega_{k-\gamma}\chi^0\right) + a^\dagger_{-\gamma}a_{k-\gamma}\cosh\left(\omega_{-\gamma}\chi^0\right)\cosh\left(\omega_{k-\gamma}\chi^0\right) \\
&\quad + \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})\left(a_\gamma a_{k-\gamma}\sinh\left(\omega_{-\gamma}\chi^0\right)\cosh\left(\omega_{k-\gamma}\chi^0\right) - a^\dagger_{-\gamma}a^\dagger_{\gamma-k}\cosh\left(\omega_{-\gamma}\chi^0\right)\sinh\left(\omega_{k-\gamma}\chi^0\right)\right) \\
&= A^\dagger_{-\gamma}A_{k-\gamma} - \sinh(\omega_\gamma\chi^0)^2(2\pi)^3\delta(\vec{k})\,, \\
:A^\dagger_{\gamma-k}A_\gamma: &= a^\dagger_{-\gamma}a_{k-\gamma}\sinh\left(\omega_\gamma\chi^0\right)\sinh\left(\omega_{\gamma-k}\chi^0\right) + a^\dagger_{\gamma-k}a_\gamma\cosh\left(\omega_\gamma\chi^0\right)\cosh\left(\omega_{\gamma-k}\chi^0\right) \\
&\quad + \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})\left(-a^\dagger_{\gamma-k}a^\dagger_{-\gamma}\sinh\left(\omega_\gamma\chi^0\right)\cosh\left(\omega_{\gamma-k}\chi^0\right) + a_{k-\gamma}a_\gamma\cosh\left(\omega_\gamma\chi^0\right)\sinh\left(\omega_{\gamma-k}\chi^0\right)\right) \\
&= A^\dagger_{-\gamma}A_{k-\gamma} - \sinh(\omega_\gamma\chi^0)^2(2\pi)^3\delta(\vec{k})\,, \\
:A^\dagger_{-\gamma}A^\dagger_{\gamma-k}: &= a^\dagger_{-\gamma}a^\dagger_{\gamma-k}\cosh\left(\omega_{-\gamma}\chi^0\right)\cosh\left(\omega_{\gamma-k}\chi^0\right) - a_\gamma a_{k-\gamma}\sinh\left(\omega_{-\gamma}\chi^0\right)\sinh\left(\omega_{\gamma-k}\chi^0\right) \\
&\quad + \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})\left(a^\dagger_{-\gamma}a_{k-\gamma}\cosh\left(\omega_{-\gamma}\chi^0\right)\sinh\left(\omega_{\gamma-k}\chi^0\right) + a^\dagger_{\gamma-k}a_\gamma\sinh\left(\omega_{-\gamma}\chi^0\right)\cosh\left(\omega_{\gamma-k}\chi^0\right)\right) \\
&= A^\dagger_{-\gamma}A^\dagger_{\gamma-k} - \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})\sinh(\omega_\gamma\chi^0)\cosh(\omega_\gamma\chi^0)(2\pi)^3\delta(\vec{k})\,, \\
:A_\gamma A_{k-\gamma}: &= -a^\dagger_{-\gamma}a^\dagger_{\gamma-k}\sinh\left(\omega_\gamma\chi^0\right)\sinh\left(\omega_{k-\gamma}\chi^0\right) + a_\gamma a_{k-\gamma}\cosh\left(\omega_\gamma\chi^0\right)\cosh\left(\omega_{k-\gamma}\chi^0\right) \\
&\quad - \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})\left(a^\dagger_{-\gamma}a_{k-\gamma}\sinh\left(\omega_\gamma\chi^0\right)\cosh\left(\omega_{k-\gamma}\chi^0\right) + a^\dagger_{\gamma-k}a_\gamma\cosh\left(\omega_\gamma\right)\sinh\left(\omega_{k-\gamma}\chi^0\right)\right) \\
&= A_\gamma A_{k-\gamma} + \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})\sinh(\omega_\gamma\chi^0)\cosh(\omega_\gamma\chi^0)(2\pi)^3\delta(\vec{k})\,.
\end{aligned}
\tag{29}
$$

The normal ordering procedure only affects the form of the $\mathcal{T}^{00}$ and $\mathcal{T}^{aa}$ operators (the terms arising from operator reordering vanish for the other components), and the difference depends on the sign of ω_γ^2 and hence on the type of mode:

$$
:\mathcal{T}^{00}_k: = \mathcal{T}^{00}_k - \delta(\vec{k})\frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{4}\int \mathrm{d}^3\gamma\,|\omega_\gamma|(1-\mathrm{sgn}(\omega_\gamma^2))\,, \tag{30}
$$

$$
\begin{aligned}
:\mathcal{T}^{aa}_k: = \mathcal{T}^{aa}_k &+ \delta(\vec{k})\frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{4}\int \mathrm{d}^3\gamma\left(|\omega_\gamma|(1+\mathrm{sgn}(\omega_\gamma^2)) - \frac{2\gamma_a^2}{|\omega_\gamma|}\right) \\
&+ \delta(\vec{k})\,\mathrm{sgn}(\mathcal{K}^{(2)})\int \mathrm{d}^3\gamma\;\Theta(\omega_\gamma^2)\left(|\omega_\gamma| - \frac{\gamma_a^2}{|\omega_\gamma|}\right)\sinh^2(\omega_\gamma\,\chi^0)\,.
\end{aligned}
\tag{31}
$$

Given that $\omega_\gamma^2 \sim \vec{\gamma}^2$ at large $|\vec{\gamma}|$, the integrals appearing as the difference between the normal-ordered $:\mathcal{T}^{aa}_k:$ and the previous definition (21) are divergent and require regularisation, e.g., by a cutoff. The time-dependent integral in the last line (which involves the Heaviside function $\Theta(\omega_\gamma^2)$ since it only comes from squeezed modes), is particularly badly divergent with an integrand growing exponentially at large $|\vec{\gamma}|$ for any $\chi^0 \neq 0$. The integral appearing in $:\mathcal{T}^{00}_k:$ is only relevant for $m^2 < 0$ and finite, given that there is always a finite R such that $\omega_\gamma^2 > 0$ for $|\vec{\gamma}| > R$, and modes with $\omega_\gamma^2 > 0$ have a vanishing "vacuum energy" already before normal ordering. Even this finite integral is however still multiplied by a delta distribution. If $m^2 > 0$, this integral vanishes entirely. The additional term appearing in the normal ordered $:\mathcal{T}^{00}_k:$ is time-independent, whereas the terms in $:\mathcal{T}^{aa}_k:$ are multiplied by $\delta(\vec{k})$. Hence, none of these terms contribute to the conservation law and $\partial_0 :\mathcal{T}^{0B}_k: + \mathrm{i}\sum_a k_a :\mathcal{T}^{aB}_k: = 0$ also for the normal-ordered definition, as expected.

A particular case of the conservation law applies to the zero modes of $\mathcal{T}^{0B}$, which satisfy $\partial_0\mathcal{T}^{00}_0 = \partial_0\mathcal{T}^{0a}_0 = 0$. These are the usual global conserved quantities corresponding to the total energy and total momentum respectively, which were discussed already in [44]. We will encounter them again in our explicit example below.

This concludes the discussion of the GFT energy-momentum tensor $\mathcal{T}^{AB}$. We considered its conservation law, which holds irrespective of operator ordering, and obtained explicit forms in terms of ladder operators whose dynamics depend on the type of mode we consider: using (18) and (19) one can write down the explicit time-dependent form of (27). Through our earlier identification of $\mathcal{T}^{AB}$ with the classical current j^{AB}, $\langle \mathcal{T}^{AB} \rangle = j^{AB}$, and hence the spacetime metric via (4), these solutions can be used to define an effective metric from any GFT state. In what follows we examine the implications of this generic construction for a specific example.

IV. SIMPLE COSMOLOGY EXAMPLE

As a first consistency check, we apply the above construction to a scenario widely studied in existing GFT literature: the flat Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime. We begin by recalling the classical scenario, which differs from standard cosmological models due to the four massless scalar fields required to construct the relational coordinate system (see also [24] for a discussion of cosmology with massless scalar fields used as coordinates).

We then take expectation values of the operators (27) in a highly peaked Gaussian state, which we argue is a good candidate for the physical scenario we wish to consider. These can then be compared to the components of the classical current (3) and their dynamics.

A. Classical dynamics

We first recall the line element of a flat FLRW universe

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{32}$$

where $N(t)$ is the lapse function and $a(t)$ the scale factor. As described in section II, we construct the quantum theory in a relational coordinate system characterised by $\partial_\mu \chi^A = \delta^A_\mu$ and any identification of classical quantities with operator expectation values holds only for this choice of coordinates. The units of the lapse and scale factor are $[a] = [N] = L^2$ and $[\partial_\mu \chi^A] = L^0$. The lapse for this choice of coordinate system can be obtained from the definition of the canonical momentum π_0 conjugate to χ^0, namely $\pi_0 = \frac{a^3}{N}(\partial_t \chi^0) \Rightarrow N = \frac{a^3}{\pi_0}$. (The momenta of the spatial fields give the shift vector as $N^a = -\frac{\pi_a}{\pi_0}$, which vanishes for (32).) The Klein–Gordon equation for χ^0 with FLRW symmetry is equivalent to the statement $\partial_t \pi_0 = 0$.

In a flat FLRW universe, the conserved currents (3) thus take the form

$$j^{AB} = \begin{pmatrix} |\pi_0| & 0 \\ 0 & -\frac{a^4}{|\pi_0|}\delta^{ab} \end{pmatrix} . \tag{33}$$

Notice that all components have a fixed sign, determined by the Lorentzian signature implemented in (32). For Euclidean signature, all entries would be positive.

The classical energy-momentum tensor of the four scalar fields, which we will denote as ${}^{(\chi)}T^\mu{}_\nu$ to avoid confusion with the GFT energy-momentum tensor T^{AB}, differs from that of the single matter field case. It contains contributions $\propto a^{-2}$ from the spatial fields, as their derivatives are non-vanishing at background level:

$$-{}^{(\chi)}T^0{}_0 = \rho = \frac{\pi_0^2}{2a^6} + \frac{3}{2a^2}\,, \qquad {}^{(\chi)}T^a{}_a = P = \frac{\pi_0^2}{2a^6} - \frac{1}{2a^2}\,. \tag{34}$$

The gradient energy coming from the spatial coordinate fields appears as an additional term that would be equivalent to negative spatial curvature.

The resulting first and second Friedmann equations read (with $\kappa = 8\pi G$, and $'$ denoting derivatives with respect to χ^0)

$$H^2 = \left(\frac{a'}{a}\right)^2 = \frac{\kappa}{6}\left(1 + 3\frac{a^4}{\pi_0^2}\right), \qquad \frac{a''}{a} = \frac{\kappa}{6}\left(1 + 9\frac{a^4}{\pi_0^2}\right), \tag{35}$$

where, again, the terms proportional to $\frac{a^4}{\pi_0^2}$ would not appear in the case of a single (clock) scalar field. This also implies that we no longer have an equation of state parameter $w = \frac{P}{\rho}$ exactly equal to one, but instead $w < 1$. The contribution of the spatial fields to the energy density and pressure becomes negligible in the limit where $\frac{\pi_0^2}{a^4} \gg 1$, effectively recovering the standard cosmological background scenario with a massless scalar field. Similarly, for the sound speed we find $c_s^2 = \frac{P'}{\rho'} = \frac{3\pi_0^2 - a^4}{3\pi_0 + 3a^4}$, and thus $c_s^2 \approx 1$ if we again assume $\frac{\pi_0^2}{a^4} \gg 1$. This limit can be achieved for sufficiently early times, depending on the value of π_0, but at late times the additional contributions from the spatial fields will dominate.

The scenario we have described here is rather unusual from the viewpoint of conventional homogeneous cosmology, where one would require the fields χ^A to be spatially homogeneous, such that there can be no contributions from gradient energy. Of course, such a symmetry requirement is incompatible with the condition that the χ^a can be spatial coordinates, $\partial_i \chi^a = \delta_i^a$. Here we adopted the view that only observables such as the energy-momentum tensor need to be spatially homogeneous, and given that the field values themselves never enter any observables for free massless scalar fields, this is classically consistent with nonvanishing gradient energy. In the context of GFT, these assumptions may indeed be mutually incompatible, as we discuss shortly.

B. Effective FLRW metric from GFT energy-momentum tensor

In order to extract an effective metric, we need to determine a state which reflects the physical scenario we are interested in, as well as being sufficiently semiclassical. As shown in [44] for models with a single massless scalar field, Fock coherent states form a suitable class of semiclassical states (see also [45] for a more in-depth analysis). We therefore use a coherent state $|\sigma\rangle$ which is an eigenstate of the (time-independent) annihilation operator $a_J(k)\,|\sigma\rangle = \sigma_J(k)\,|\sigma\rangle$:

$$|\sigma\rangle = e^{-||\sigma||^2/2} \exp\left(\sum_J \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sigma_J(\vec{k}) a_J^\dagger(\vec{k})\right) |0\rangle\,, \tag{36}$$

where $|0\rangle$ is the GFT Fock vacuum and $||\sigma||^2 = \sum_J \int \frac{\mathrm{d}^3 k}{(2\pi)^3} |\sigma_J(\vec{k})|^2$.

We are interested in flat FLRW cosmology, where all quantities, in particular the components of the classical current j^{AB}, are homogeneous. A cosmological quantum state should reflect this homogeneity. We thus choose a Gaussian as a sharply peaked mean field,

$$\sigma_J(\vec{k}) = \delta_{J,J_0} \frac{\mathcal{A} + \mathrm{i}\mathcal{B}}{c_\sigma}\; e^{-\frac{(\vec{k}-\vec{k}_0)^2}{2s^2}}\,, \tag{37}$$

where $\mathcal{A}$, $\mathcal{B} \in \mathbb{R}$, s determines the peakedness of the state and $\vec{k}_0$ is the initially dominantly excited Fourier mode. In what follows, we assume only one Peter–Weyl mode with $J = J_0$ is excited. (In general, if multiple modes are included, the initial conditions $\mathcal{A}$, $\mathcal{B}$ and s could of course be J dependent.) We have fixed the normalisation factor $c_\sigma = \left(\frac{s}{2\sqrt{\pi}}\right)^{3/2}$ for convenience regarding later calculations. As $\vec{k} = 0$ corresponds to the homogeneous mode, a strictly homogeneous state would correspond to an infinitely peaked state around $\vec{k}_0 = 0$, i.e., the limit of $s \to 0$. To avoid any divergences, we choose a state with small, but finite s. This introduces a conceptual discrepancy to the classical case: in the quantum theory it is not possible to excite solely the homogeneous background, but inhomogeneous modes will always be excited to some degree, which clearly differs from the standard distinction between background and perturbations in cosmology. In the following we set $\vec{k}_0 = 0$ and consider the dynamics of the homogeneous, $\vec{k} = 0$ mode; all other modes should be identified with perturbations on the homogeneous background (which we will study in detail in an upcoming article).

The single mode J_0 we consider can either have $m^2_{J_0} = m^2 > 0$ or $m^2 < 0$ (we ignore the fine-tuned special case $m^2 = 0$ which needs to be analysed separately [46]). Since the mean field is sharply peaked at $\vec{k} = 0$, the expectation value of the energy-momentum tensor will be determined by (low $|\vec{k}|$) squeezed modes for $m^2 > 0$ and by (low $|\vec{k}|$) oscillating modes for $m^2 < 0$.

For $m^2 > 0$ we obtain the following expectation values:

$$\begin{aligned}
\langle\sigma|\, \mathcal{T}_0^{00}\, |\sigma\rangle &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} \mathrm{sgn}(\mathcal{K}^{(2)})|\omega_\gamma|(\mathcal{B}^2 - \mathcal{A}^2)\frac{e^{-\gamma^2/s^2}}{c_\sigma^2} \approx \mathrm{sgn}(\mathcal{K}^{(2)})|m|(\mathcal{B}^2 - \mathcal{A}^2)\,, \\
\langle\sigma|\, \mathcal{T}_0^{0b}\, |\sigma\rangle &= 0\,, \qquad \langle\sigma|\, \mathcal{T}_0^{a\neq b}\, |\sigma\rangle = 0\,, \\
\langle\sigma|\, \mathcal{T}_0^{aa}\, |\sigma\rangle &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} \mathrm{sgn}(\mathcal{K}^{(2)})\frac{e^{-\gamma^2/s^2}}{c_\sigma^2} \\
&\quad \times \left(\left(-|\omega_\gamma| + \frac{\gamma_a^2}{|\omega_\gamma|}\right)\left((\mathcal{A}^2 + \mathcal{B}^2)\cosh(2\omega_\gamma\chi^0) - 2\,\mathrm{sgn}(\mathcal{K}^{(2)})\mathcal{A}\mathcal{B}\sinh(2\omega_\gamma\chi^0)\right) + \frac{\gamma_a^2}{|\omega_\gamma|}(\mathcal{A}^2 - \mathcal{B}^2)\right) \\
&\approx -\,\mathrm{sgn}(\mathcal{K}^{(2)})|m|\left((\mathcal{A}^2 + \mathcal{B}^2)\cosh(2|m|\chi^0) - 2\,\mathrm{sgn}(\mathcal{K}^{(2)})\mathcal{A}\mathcal{B}\sinh(2|m|\chi^0)\right)\,.
\end{aligned} \tag{38}$$

To simplify the integrals in the expressions for $\langle\sigma|\, \mathcal{T}_0^{00}\, |\sigma\rangle$ and $\langle\sigma|\, \mathcal{T}_0^{aa}\, |\sigma\rangle^2$, we used the saddle-point approximation

$$\int \mathrm{d}^3x\; e^{-\frac{(\vec{x}-\vec{\mu})^2}{s^2}} f(\vec{x}) \approx f(\vec{\mu}) \int \mathrm{d}^3x\; e^{-\frac{(\vec{x}-\vec{\mu})^2}{s^2}} = f(\vec{\mu})(\sqrt{\pi}s)^3, \tag{39}$$

which holds for sharply peaked Gaussians such that $f(\vec{x})$ can be considered approximately constant in the region $|\vec{x} - \vec{\mu}| \leq s$. In effect, this approximation corresponds to the idealised limit of $s \to 0$ given that we are ignoring all finite s contributions. The approximation will break down at late times for $\langle\sigma|\, \mathcal{T}_0^{aa}\, |\sigma\rangle$, as in the similar discussion of [30]. The integrals for $\langle\sigma|\, \mathcal{T}_0^{0b}\, |\sigma\rangle$ and $\langle\sigma|\, \mathcal{T}_0^{a\neq b}\, |\sigma\rangle$ vanish due to antisymmetry.

For $m^2 < 0$, the integral over $\vec{\gamma}$ will contain both squeezed and oscillating modes, but for a very sharply peaked mean field, only the region near $\vec{\gamma} = 0$, which consists of oscillating modes only,

[2] The integral for $\langle\sigma|\, \mathcal{T}_0^{00}\, |\sigma\rangle$ can be carried out analytically to give a Tricomi confluent hypergeometric function, but as this does not add value to the results we present, we omit this result.

can contribute. We can hence write, using again the saddle-point approximation,

$$
\begin{aligned}
\langle\sigma|\,\mathcal{T}_0^{00}\,|\sigma\rangle &\approx \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\mathrm{sgn}(\mathcal{K}^{(2)})\frac{e^{-\gamma^2/s^2}}{c_\sigma^2}|\omega_\gamma|(\mathcal{A}^2+\mathcal{B}^2) \approx \mathrm{sgn}(\mathcal{K}^{(2)})|m|(\mathcal{A}^2+\mathcal{B}^2)\,,\\
\langle\sigma|\,\mathcal{T}_0^{0b}\,|\sigma\rangle &= 0\,, \qquad \langle\sigma|\,\mathcal{T}_0^{a\neq b}\,|\sigma\rangle = 0\,,\\
\langle\sigma|\,\mathcal{T}_0^{aa}\,|\sigma\rangle &\approx \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{|\omega_\gamma|}\frac{e^{-\gamma^2/s^2}}{c_\sigma^2}\\
&\qquad \times\left(\gamma_a^2(\mathcal{A}^2+\mathcal{B}^2)+(|\omega_\gamma|^2+\gamma_a^2)\left((\mathcal{A}^2-\mathcal{B}^2)\cos(2|\omega_\gamma|\chi^0)+2\mathcal{A}\mathcal{B}\sin(2|\omega_\gamma|\chi^0)\right)\right)\\
&\qquad \approx \mathrm{sgn}(\mathcal{K}^{(2)})|m|\left((\mathcal{A}^2-\mathcal{B}^2)\cos(2|m|\chi^0)+2\mathcal{A}\mathcal{B}\sin(2|m|\chi^0)\right)\,.
\end{aligned}
\tag{40}
$$

Before we can proceed to use the identification $j^{AB} = \langle\sigma|\,\mathcal{T}^{AB}\,|\sigma\rangle$ to extract an effective metric from the expressions (33) and (38), a discussion of signs is in order. The signs of the conserved current given in (3) are determined by the choice of metric signature; in particular, for a Euclidean metric we would expect the entries of j^{AB} to all be positive. The signature of the effective metric is so far just an assumption which has not been derived from the quantum theory. Given that $\mathrm{sgn}(\langle\sigma|\,\mathcal{T}_0^{aa}\,|\sigma\rangle) = -\mathrm{sgn}(\mathcal{K}^{(2)})$ and $\mathrm{sgn}(\langle\sigma|\,\mathcal{T}_0^{00}\,|\sigma\rangle) = \mathrm{sgn}(\mathcal{K}^{(2)})\mathrm{sgn}(\mathcal{B}^2-\mathcal{A}^2)$, the initial conditions $\mathcal{A}$, $\mathcal{B}$ determine whether the effective metric we reconstruct is Euclidean or Lorentzian. To fix the signature, one could choose a preferred $\mathrm{sgn}(\mathcal{K}^{(2)})$: identification with the classical discussion of section II would require $\mathrm{sgn}(\mathcal{K}^{(2)}) = 1$. However, for $\mathrm{sgn}(\mathcal{K}^{(2)}) = -1$ we could also simply identify j^{AB} with $-\langle\sigma|\,\mathcal{T}^{AB}\,|\sigma\rangle$, given that the symmetry arguments used to make this identification would be compatible with any constant rescaling, and already our definition (7) of the energy-momentum tensor could equally well be replaced by a definition with the opposite sign.

Restricting to the Lorentzian case $\mathcal{B}^2 > \mathcal{A}^2$, we then find the following effective expressions for the momentum of the clock field and the scale factor in the case of squeezing modes (38):

$$
|\pi_0| = \langle\sigma|\,\mathcal{T}_0^{00}\,|\sigma\rangle = |m|(\mathcal{B}^2-\mathcal{A}^2), \tag{41}
$$

$$
a^4 = -\,|\pi_0|\,\langle\sigma|\,\mathcal{T}_0^{aa}\,|\sigma\rangle = m^2(\mathcal{B}^2-\mathcal{A}^2)\left((\mathcal{A}^2+\mathcal{B}^2)\cosh(2|m|\chi^0)-2\mathcal{A}\mathcal{B}\sinh(2|m|\chi^0)\right)\,, \tag{42}
$$

from which we can calculate an effective Friedmann equation

$$
H^2 = \frac{1}{4}m^2\left(1-\frac{4(\mathcal{A}^2-\mathcal{B}^2)^2}{((\mathcal{A}-\mathcal{B})^2e^{2m\chi^0}+(\mathcal{A}+\mathcal{B})^2e^{-2m\chi^0})^2}\right) = \frac{1}{4}m^2\left(1-\frac{|\pi_0|^4}{a^8}\right) \quad \xrightarrow[\text{late times}]{} \quad \frac{1}{4}m^2. \tag{43}
$$

We thus obtain a bounce at $a^4 = |\pi_0|$, or equivalently, $\langle\sigma|\,\mathcal{T}_0^{aa}\,|\sigma\rangle^2 = 1$.

The late-time limit – H^2 going to a constant – agrees with all Friedmann equations previously obtained for GFT models with a single clock (matter) field (see, e.g., [44]) as well as with the case of general relativity with a single massless scalar field where $H^2 = \frac{\kappa}{6}$, see (35). This is an interesting observation given that in previous works effective Friedmann equations were always obtained by studying the evolution of the number operator $N = A^\dagger A$, whose dynamics for the squeezing case are given by $\langle\sigma|\,N\,|\sigma\rangle = (\mathcal{A}^2+\mathcal{B}^2)\cosh(2|m|\chi^0) - 2\mathcal{A}\mathcal{B}\sinh(2|m|\chi^0)$. In the case of a single J mode, the number operator is proportional to the GFT volume operator V, whose expectation value is identified with the classical volume element, hence in previous literature $\langle\sigma|\,N\,|\sigma\rangle \propto a^3$. The expectation value of the energy-momentum tensor component $\mathcal{T}_0^{aa}$, leading to (42), is in fact proportional to the expectation value of the number operator in this case; hence our expression for a^4 is proportional to the one derived in previous literature for a^3. Agreement with general relativity (with one massless scalar field) at late times only requires that $H = (\log a)'$ should go to a constant, which is compatible with identifying $\langle\sigma|\, N\,|\sigma\rangle$ with any power of a. (The GFT Friedmann equation could also have an additional $\frac{1}{\langle\sigma|N|\sigma\rangle}$ term as in [30], which disappears here due to the normal ordering procedure we impose.) In our case, m can be fixed to $m^2 = \frac{2}{3}\kappa$ to reproduce the Friedmann equation of general relativity with a single massless scalar field in the large volume limit, which is consistent with all previous literature. The only change is in the numerical factor needed in the identification of m and κ.

We saw, however, that the classical Friedmann equation for a model with four massless scalar coordinate fields (35) differs from that with a single clock field. The additional contributions coming from spatial coordinate fields become dominant when $\frac{a^4}{\pi_0^2} \sim 1$, which is exactly when the bounce appears in (43). There is no early-time regime where these fields would not yet dominate classically, but we are still far away from the bounce in the GFT model. Hence, the GFT Friedmann equation does not match the Friedmann equation for a classical scenario with four massless scalar fields.

For an oscillating mode with expectation values of $\mathcal{T}^{AB}$ given in (40), we have the identification

$$|\pi_0| = \mathrm{sgn}(\mathcal{K}^{(2)})|m|(\mathcal{A}^2 + \mathcal{B}^2) \tag{44}$$

$$a^4 = -\,|m|^2(\mathcal{A}^2 + \mathcal{B}^2)\left((\mathcal{A}^2 - \mathcal{B}^2)\cos(2m\chi^0) + 2\mathcal{A}\mathcal{B}\sin(2m\chi^0)\right)\,. \tag{45}$$

If we use the same convention as above with $\mathrm{sgn}(\mathcal{K}^{(2)}) = 1$, we again find a positive $|\pi_0|$, now for arbitrary initial conditions in terms of $\mathcal{A}$ and $\mathcal{B}$. The quantity a^4, however, can change sign throughout the evolution due to its oscillatory behaviour. This is rather different to what one finds in the case with $a^3 \propto \langle\sigma|\, N\,|\sigma\rangle$, where the number operator remains constant for oscillating modes (with an expectation value that is always positive). It is clear that a single oscillating mode cannot lead to a realistic cosmology, however, if one includes both mode types, squeezing modes would lead to an expanding universe and oscillating modes would constitute an additional modulation in the evolution of the scale factor, whose relative effect is reduced with the expansion.

Finally, we would like to point out that $\langle\sigma|\, \mathcal{T}_0^{00}\,|\sigma\rangle$ is time-independent already prior to the saddle-point approximation, implying that in this cosmological model $|\pi_0|$ is always constant. This is an important consistency check, given that constancy of π_0 corresponds to the Klein–Gordon equation for the classical FLRW model. We have of course shown above that the energy-momentum tensor is always conserved at the quantum level, so that we are guaranteed to obtain the exact Klein–Gordon equation $\partial_A j^{AB} = 0$; however, what is nontrivial is to show that our quantum state can be consistently interpreted as an FLRW universe. This interpretation has been substantially strengthened in our new approach, given that the effective metric we recover explicitly gives a flat FLRW spacetime.

V. CONCLUSION

In this article we proposed a novel set of operators for GFT which we used to reconstruct an effective metric directly from the quantum theory. This proposal goes beyond the entire previous GFT literature in which only a limited set of geometric observables, usually derived from the volume (or defining effective anisotropies [38]), were used. Working in a (deparametrised) Hamiltonian setting, we established a relational coordinate system by coupling four massless scalar fields to the GFT action, where one of these fields is singled out as a clock field and the others serve as spatial coordinates. The GFT action remains unchanged upon translation of the scalar fields, leading to a conserved GFT energy-momentum tensor according to Noether's theorem. This translational symmetry represents the spacetime symmetry of constant shifts in the matter fields, which in that context leads to a conserved current directly related to the metric in the relational coordinate system. Hence, we proposed to identify the expectation values of the GFT energy-momentum tensor with components of the classically conserved current, obtaining an effective metric. We showed that the classical conservation law for the GFT energy-momentum tensor arising from the symmetry holds also at operator level, for different possible choices of operator ordering. In particular, this applies to the most relevant case of a normal-ordering prescription which removes divergent contributions to the GFT energy-momentum tensor. In general, the free GFT action is decomposed into two different types of field modes, squeezed and oscillating modes, whose dynamics differ and which appear in various combinations in the expressions for the GFT energy-momentum tensor. The conservation law holds regardless of the choice of state, so that the matter fields always satisfy the classical Klein–Gordon equation exactly.

While our proposal is completely general, in the sense that an effective metric can be associated in principle to any state, the particular choice of the latter governs the specific form of the former and thereby the physical scenario the effective metric might be associated with. We tested our proposition in a simple example: homogeneous cosmology as described by a flat FLRW metric. Here our choice of state was based on a few physical requirements. Firstly, the state should be sufficiently semiclassical in order to be able to consider operator expectation values as effective quantities; we chose Fock coherent states as commonly done in the GFT literature. Secondly, the spacetime we are reconstructing should be spatially homogeneous, which is why the coherent state is peaked around the homogeneous $\vec{k} = 0$ mode.

We found that the canonically conjugate momentum of the clock field is conserved in time, as it must be from the Klein–Gordon equation. Interestingly, only half of the possible space of initial conditions leads to an effective Lorentzian metric, with the other half leading to a Euclidean signature metric instead. This observation seems compatible with the fact that spacetime signature is not built into the GFT formalism at the fundamental level, but should be considered to be emergent. Moreover, the metric signature would be invisible if (as in almost all of the past work) one only has access to a Friedmann equation, which can take a very similar form in Euclidean or Lorentzian signature. (At the level of perturbations, the spacetime signature can of course be inferred from the equations of motion; here the work of [23, 24] similarly finds that the effective signature depends on initial conditions.) We also found an effective Friedmann equation that agrees with the previous GFT literature for models with a single massless scalar field; it agrees with general relativity (coupled to a single scalar field) at late times, and resolves the singularity by a bounce. We find a relation between the number operator and the effective scale factor that differs from the literature, namely $\langle\sigma| N |\sigma\rangle \propto a^4$ instead of $\langle\sigma| N |\sigma\rangle \propto a^3$. Oscillating modes would give a contribution to the effective scale factor for which a^4 can take negative values. Therefore, it seems that such modes can only appear in conjunction with one or more squeezed modes.

While we find agreement with the standard GFT scenario for homogeneous cosmology, the agreement with the classical Friedmann equation is not entirely satisfactory: given that the spatial coordinate fields need to have nonvanishing gradient energy, they would be expected to contribute additional terms in the energy density that can only be neglected when $\frac{\pi_0^2}{a^4} \gg 1$. Such terms are not seen in the effective GFT dynamics, so that at best one might expect a matching between GFT and classical cosmology for early times, where these terms do not yet dominate. However, we found that the bounce already occurs at $\frac{\pi_0^2}{a^4} = 1$, so that there is no such early-time regime in the GFT setting. The bounce scale is simply a dimensionless ratio determined by initial conditions, unrelated to quantities like the ratio of the energy density to the Planck density. This seems to imply that the bounce could happen at low curvatures, which needs to be clarified further.

One might think that the two Friedmann equations can be brought into agreement if one deviates from the assumption of a flat FLRW universe on the classical side; indeed, a positive curvature term could cancel the contribution from the spatial matter fields. In previous work on GFT cosmology, where one only had access to the Friedmann equation, such an interpretation would have been viable. However, in our scenario we have access to all metric components, and since $\langle\sigma|\,\mathcal{T}_0^{ab}\,|\sigma\rangle \propto \delta^{ab}$, our choice of state clearly corresponds to a flat metric. The mismatch seems to arise from a more basic conceptual clash between the assumption of spatial homogeneity and the use of additional fields as spatial coordinates. As we have seen, spatial homogeneity means that the mean field should be peaked around $\vec{k} = 0$; indeed we have neglected all finite $\vec{k}$ contributions in our saddle-point approximation. But this means that the spatial coordinate fields are not excited at the quantum level; in GFT cosmology any physical effects coming from spatial coordinate fields can only be in the inhomogeneous modes, completely unlike what happens in classical cosmology (where the number and physical nature of matter fields and the assumption of spatial homogeneity are independent). It is then intuitively clear that we cannot recover the dynamics of FLRW cosmology with multiple scalar fields, as demanding homogeneity is linked to recovering the classical Friedmann equation of a flat FLRW universe with a single massless scalar field at late times. This result is consistent with previous literature, in which the spatial coordinate fields are either simply assumed to be negligible at background level [23, 24], or effective Friedmann dynamics based on a state peaked around $\vec{k}_0 \neq 0$ show additional terms that can to some extent be associated to additional matter fields, with the precise interpretation remaining unclear [30].

A possible alternative to the model we studied would be to introduce an additional scalar matter field as in [23, 24]. This fifth field is not interpreted as a relational coordinate and has its own independent initial conditions; if this field dominates over the coordinate fields at some initial time, such a scenario might yield an intermediate regime where the effective Friedmann equation matches that of general relativity, before spatial gradient terms would be expected to dominate. Whether this is realised in our new approach needs to be studied in more detail. Another immediate question for future work would be how different types of coherent state, such as those with a mean field peaked around $\vec{k}_0 \neq 0$, can be interpreted in terms of the effective metric introduced in this paper. More generally, the constructions shown here can be extend to small perturbations of an FLRW universe or to situations such as spherical symmetry, where they could describe the dynamics of an effective GFT black hole.

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