WHAT ARE THE NUMBERS IN WHICH SPACETIME?

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ABSTRACT. Within an axiomatic framework, we investigate the possible structures of numbers (as physical quantities) in different theories of relativity.

1. INTRODUCTION

Basically, we would like to investigate the following metaphysical question:

What are the numbers in the physical world?

Without making this question more precise we can make the following two natural guesses which contradict each other:

- Obviously, the physical numbers are the real (or the complex) numbers since at least 99% of our physical theories are using these numbers.
- Obviously, the set of physical numbers is a subset of the rational numbers (or even the integers) since the outcomes of the measurements have finite decimal representations.

Clearly, this informal level is too naive to meaningfully investigate our question. However, that does not mean that it is impossible to scientifically investigate our question within some logical framework. In this paper, we are going to reformulate and investigate this question (restricted to spacetime theories) within a rigorous logical framework.

First of all, what do numbers have to do with the geometry of spacetime? The concepts related to numbers can be defined by the concepts of geometry by Hilbert's coordinatization, see, e.g., [12, pp.23-27]. Moreover, purely geometrical statements can correspond to statements about the structure of numbers. For example, in Cartesian planes over ordered fields, the statement *"every line which contains a point from the interior of a circle intersects the circle"* is equivalent to that *"every positive number has a square root,"* see, e.g., [13, Prop.16.2., p.144]. In the spirit of this example, here we investigate the question

"How are some properties of spacetime reflected on the structure of numbers?"

Among others, we will see axioms on observers also implying that positive numbers have square roots. Ordered fields in which positive numbers have square roots are called **Euclidean fields**, which got

their names after their role in Tarski's first-order logic axiomatization of Euclidean geometry [29].

Let Th be a theory of space-time which contains the concept of numbers (as physical quantities) together with some algebraic operations on them, such as addition (+), multiplication (·) (or at least these concepts are definable in Th.). In this case, we can introduce notation $Num(\mathsf{Th})$ for the class of the quantity parts (quantity structures) of the models of theory Th:

$$Num(\mathsf{Th}) = \{\text{The quantity parts of the models of } \mathsf{Th}\}.$$

We use the notation $\mathfrak{Q} \in Num(\mathsf{Th})$ for algebraic structure $\mathfrak{Q}$ the same way as the model theoretic notation $\mathfrak{Q} \in Mod(\mathsf{AxField})$, e.g., $\mathbb{Q} \in Num(\mathsf{Th})$ means that $\mathbb{Q}$, the field of rational numbers, can be the structure of quantities (numbers) in Th. Now we can scientifically investigate the question

"What are the numbers in physical theory Th?"

by studying what algebraic structures occur in $Num(\mathsf{Th})$.

In this paper, we investigate this question only in the case when Th is a theory of spacetimes. However, this question can be investigated in any other physical theory the same way.

We will see that the answer to our question often depends on the dimension of spacetime. Therefore, we will introduce notation $Num_n(\mathsf{Th})$ at page 8 for the class of the possible quantity structures of theory Th if all the investigated spacetimes are n-dimensional.

In the logic language of Section 2, we will introduce several theories and axioms of relativity theory. For example, our starting axiom system for special relativity (called SpecRel, see page 6) captures the kinematics of special relativity perfectly, see Theorem 3.1 and Corollary 3.4. Furthermore, without any extra assumptions SpecRel has a model over every ordered field, i.e.,

$$Num(\mathsf{SpecRel}) = \{\text{ ordered fields }\},$$

see Remark 3.7. Therefore, SpecRel has a model over $\mathbb{Q}$, too. However, if we assume that inertial observes can move with arbitrary speed less than that of light (in any direction every where), see AxThExp at page 9, then every positive number has to have a square root if $n \geq 3$ by Theorem 3.6, i.e.,

$$Num_n(\mathsf{SpecRel} + \mathsf{AxThExp}) = \{\text{ Euclidean fields }\}.$$

In particular, the number structure cannot be the field of rational numbers, but it can be the field of real algebraic numbers.

We will also see that our axiom system of special relativity has a model over $\mathbb{Q}$ if we assume axiom AxThExp only approximately (which is reasonable as we cannot be sure in anything perfectly accurately in physics), see Theorem 3.12, Corollary 3.13 and Conjecture 3.14.

It is interesting that, if the spacetime dimension is 3, then we do not need the symmetry axiom of SpecRel to prove that every positive number has a square root if AxThExp is assumed, see Theorem 3.8. However, in even dimensions, it is possible that some numbers do not have square roots, see Theorem 3.9 and Questions 3.10 and 3.11.

Moving toward general relativity we will see that our theory of accelerated observes (AccRel) requires the structure of quantities to be a real closed field, i.e., a Euclidean field in which every odd degree polynomial has a root, see Theorem 4.1. However, any real closed field, e.g., the field of real algebraic numbers, can be the quantity structure of AccRel.

If we extend AccRel by extra axiom Ax∃UnifOb stating that there are uniformly accelerated observers, then the field of real algebraic numbers cannot be the structure of quantities any more if $n \geq 3$, see Theorem 5.2. A surprising consequence of this result is that $Num_n(\mathsf{AccRel} + \mathsf{Ax\exists UnifOb})$ is not a first-order logic definable class of fields, see Remark 5.3.

In Section 6, we introduce an axiom system of general relativity GenRel and investigate our question a bit for GenRel.

2. The language of our theories

To investigate our reformulated question, we need an axiomatic theory of spacetimes. The first important decision in writing up an axiom system is to choose the set of basic symbols of our logic language, i.e., what objects and relations between them we will use as basic concepts.

Here we will use the following two-sorted[1] language of first-order logic (FOL) parametrized by a natural number $d \geq 2$ representing the dimension of spacetime:

$$\{\, B, Q \,;\, \mathsf{Ob}, \mathsf{IOb}, \mathsf{Ph}, +, \cdot, \leq, \mathsf{W} \,\},$$

where B (bodies[2]) and Q (quantities) are the two sorts, Ob (observers), IOb (inertial observers) and Ph (light signals) are one-place relation symbols of sort B, $+$ and $\cdot$ are two-place function symbols of sort Q, $\leq$ is a two-place relation symbol of sort Q, and W (the worldview relation) is a $d+2$-place relation symbol the first two arguments of which are of sort B and the rest are of sort Q.

Relations $\mathsf{Ob}(o)$, $\mathsf{IOb}(m)$ and $\mathsf{Ph}(p)$ are translated as *"o is an observer," "m is an inertial observer,"* and *"p is a light signal,"* respectively. To speak about coordinatization of observers, we translate relation $\mathsf{W}(k, b, x_1, x_2, \ldots, x_d)$ as *"body k coordinatizes body b at space-time*

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[1]That our theory is two-sorted means only that there are two types of basic objects (bodies and quantities) as opposed to, e.g., Zermelo–Fraenkel set theory where there is only one type of basic objects (sets).

[2]By bodies we mean anything which can move, e.g., test-particles, reference frames, electromagnetic waves, centers of mass, etc.

location $\langle x_1, x_2, \ldots, x_d \rangle$," (i.e., at space location $\langle x_2, \ldots, x_d \rangle$ and instant x_1).

Quantity terms are the variables of sort Q and what can be built from them by using the two-place operations $+$ and $\cdot$, **body terms** are only the variables of sort B. $\mathsf{IOb}(m)$, $\mathsf{Ph}(p, b)$, $\mathsf{W}(m, b, x_1, \ldots, x_d)$, $x = y$, and $x \leq y$ where m, p, b, x, y, x_1, $\ldots$, x_d are arbitrary terms of the respective sorts are so-called **atomic formulas** of our first-order logic language. The **formulas** are built up from these atomic formulas by using the logical connectives *not* ($\neg$), *and* ($\wedge$), *or* ($\vee$), *implies* ($\rightarrow$), *if-and-only-if* ($\leftrightarrow$) and the quantifiers *exists* ($\exists$) and *for all* ($\forall$).

To make them easier to read, we omit the outermost universal quantifiers from the formalizations of our axioms, i.e., all the free variables are universally quantified.

We use the notation Q^n for the set of all n-tuples of elements of Q. If $\bar{x} \in Q^n$, we assume that $\bar{x} = \langle x_1, \ldots, x_n \rangle$, i.e., x_i denotes the i-th component of the n-tuple $\bar{x}$. Specially, we write $\mathsf{W}(m, b, \bar{x})$ in place of $\mathsf{W}(m, b, x_1, \ldots, x_d)$, and we write $\forall \bar{x}$ in place of $\forall x_1 \ldots \forall x_d$, etc.

We use first-order logic set theory as a meta theory to speak about model theoretical terms, such as models, validity, etc. The **models** of this language are of the form

$$\mathfrak{M} = \langle B, Q; \mathsf{Ob}_\mathfrak{M}, \mathsf{IOb}_\mathfrak{M}, \mathsf{Ph}_\mathfrak{M}, +_\mathfrak{M}, \cdot_\mathfrak{M}, \leq_\mathfrak{M}, \mathsf{W}_\mathfrak{M} \rangle,$$

where B and Q are nonempty sets, $\mathsf{Ob}_\mathfrak{M}$, $\mathsf{IOb}_\mathfrak{M}$ and $\mathsf{Ph}_\mathfrak{M}$ are subsets of B, $+_\mathfrak{M}$ and $\cdot_\mathfrak{M}$ are binary functions and $\leq_\mathfrak{M}$ is a binary relation on Q, and $\mathsf{W}_\mathfrak{M}$ is a subset of $B \times B \times Q^d$. Formulas are interpreted in $\mathfrak{M}$ in the usual way. For the precise definition of the syntax and semantics of first-order logic, see, e.g., [7, §1.3], [10, §2.1, §2.2].

3. Numbers required by special relativity

In this section, we will investigate our main question within special relativity. To do so, first we formulate axioms for special relativity in the logic language of the previous section.

Since the language above contains the concept of quantities (and that of addition, multiplication and ordering), we can formulate statements about numbers directly. In our first axiom, we state some basic properties of addition, multiplication and ordering true for real numbers.[3]

AxOField**:** The quantity part $\langle Q, +, \cdot, \leq \rangle$ is an ordered field, i.e.,
- $\langle Q, +, \cdot \rangle$ is a field in the sense of abstract algebra; and
- the relation $\leq$ is a linear ordering on Q such that
  - i) $x \leq y \rightarrow x + z \leq y + z$ and
  - ii) $0 \leq x \wedge 0 \leq y \rightarrow 0 \leq xy$ holds.

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[3]Using axiom AxOFiled instead of assuming that the structure of quantities is the field of real numbers not just makes our theory more flexible, but also makes it possible to investigate our main question.

AxOField is a "mathematical" axiom in spirit. However, it has physical (even empirical) relevance. Its physical relevance is that we can add and multiply the outcomes of our measurements and some basic rules apply to these operations. Physicists use all properties of the real numbers tacitly, without stating explicitly which property is assumed and why. The two properties of real numbers which are the most difficult to defend from empirical point of view are the Archimedean property, see [22], [23, §3.1], [25],[24], and the supremum property,[4] see the remark after the introduction of CONT on page 13.

The rest of our axioms on special relativity will speak about the worldviews of inertial observers. To formulate them, we use the following concepts. The **time difference** of coordinate points $\bar{x}, \bar{y} \in Q^d$ is defined as:

$$\mathsf{time}(\bar{x}, \bar{y}) := x_1 - y_1.$$

To speak about the spatial distance of any two coordinate points, we have to use squared distance since it is possible that the distance of two points is not amongst the quantities. For example, the distance of points $\langle 0, 0 \rangle$ and $\langle 1, 1 \rangle$ is $\sqrt{2}$. So in the field of rational numbers, $\langle 0, 0 \rangle$ and $\langle 1, 1 \rangle$ do not have distance but they have squared distance. Therefore, we define the **squared spatial distance** of $\bar{x}, \bar{y} \in Q^d$ as:

$$\mathsf{space}^2(\bar{x}, \bar{y}) := (x_2 - y_2)^2 + \ldots + (x_d - y_d)^2.$$

We denote the **origin** of Q^n by $\bar{o}$, i.e., $\bar{o} := \langle 0, \ldots, 0 \rangle$.

The next axiom is the key axiom of our axiom system for special relativity, it has an immediate physical meaning. This axiom is the outcome of the Michelson-Morley experiment. It has been continuously tested ever since then. Nowadays it is tested by GPS technology.

AxPh: For any inertial observer, the speed of light is the same everywhere and in every direction (and it is finite). Furthermore, it is possible to send out a light signal in any direction (existing according to the coordinate system) everywhere:

$$\mathsf{IOb}(m) \rightarrow \exists c_m \Big[c_m > 0 \wedge \forall \bar{x}\bar{y}$$

$$\big(\exists p \big[\mathsf{Ph}(p) \wedge \mathsf{W}(m, p, \bar{x}) \wedge \mathsf{W}(m, p, \bar{y}) \big] \leftrightarrow \mathsf{space}^2(\bar{x}, \bar{y}) = c_m^2 \cdot \mathsf{time}(\bar{x}, \bar{y})^2 \big) \Big].$$

Let us note here that AxPh does not require (by itself) that the speed of light is the same for every inertial observer. It requires only that the speed of light according to a fixed inertial observer is a positive quantity which does not depend on the direction or the location.

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[4]The supremum property (i.e., every nonempty and bounded subset of the numbers has a least upper bound) implies the Archimedean property. So if we want to get ourselves free from the Archimedean property, we have to leave this one, too.

By AxPh, we can define the **speed of light** according to inertial observer m as the following binary relation:

$$\mathsf{c}(m,v) \quad \overset{def}{\Longleftrightarrow} \quad v > 0 \land \forall \bar{x}\bar{y}\big(\exists p\big[\mathsf{Ph}(p) \land \mathsf{W}(m,p,\bar{x}) \land \mathsf{W}(m,p,\bar{y})\big]$$
$$\rightarrow \mathsf{space}^2(\bar{x},\bar{y}) = v^2 \cdot \mathsf{time}(\bar{x},\bar{y})^2\big).$$

By AxPh, there is one and only one speed v for every inertial observer m such that $\mathsf{c}(m,v)$ holds. From now on, we will denote this unique speed by c_m.

Our next axiom connects the worldviews of different inertial observers by saying that all observers coordinatize the same "external" reality (the same set of events). By the **event** occurring for observer m at point $\bar{x}$, we mean the set of bodies m coordinatizes at $\bar{x}$:

$$\mathsf{ev}_m(\bar{x}) := \{b : \mathsf{W}(m,b,\bar{x})\}.$$

AxEv**:** All inertial observers coordinatize the same set of events:

$$\mathsf{IOb}(m) \land \mathsf{IOb}(k) \rightarrow \exists \bar{y}\, \forall b\big[\mathsf{W}(m,b,\bar{x}) \leftrightarrow \mathsf{W}(k,b,\bar{y})\big].$$

From now on, we will use $\mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{y})$ to abbreviate the subformula $\forall b[\mathsf{W}(m,b,\bar{x}) \leftrightarrow \mathsf{W}(k,b,\bar{y})]$ of AxEv. The next two axioms are only simplifying ones.

AxSelf**:** Any inertial observer is stationary relative to himself:

$$\mathsf{IOb}(m) \rightarrow \forall \bar{x}\big[\mathsf{W}(m,m,\bar{x}) \leftrightarrow x_2 = \ldots = x_d = 0\big].$$

Our last axiom on inertial observers is a symmetry axiom saying that they use the same units of measurement.

AxSymD**:** Any two inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them; furthermore, the speed of light is 1 for all observers:

$$\mathsf{IOb}(m) \land \mathsf{IOb}(k) \land x_1 = y_1 \land x_1' = y_1' \land \mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{x}')$$
$$\land\, \mathsf{ev}_m(\bar{y}) = \mathsf{ev}_k(\bar{y}') \rightarrow \mathsf{space}^2(\bar{x},\bar{y}) = \mathsf{space}^2(\bar{x}',\bar{y}'), \text{ and}$$
$$\mathsf{IOb}(m) \rightarrow \exists p\big[\mathsf{Ph}(p) \land \mathsf{W}(m,p,0,\ldots,0) \land \mathsf{W}(m,p,1,1,0,\ldots,0)\big].$$

Let us introduce an axiom system for special relativity as the collection of the axioms above, if $d \geq 3$:

$$\mathsf{SpecRel} := \{\mathsf{AxOField}, \mathsf{AxPh}, \mathsf{AxEv}, \mathsf{AxSelf}, \mathsf{AxSymD}\}.$$

In relativity theory, we are often interested in comparing the worldviews of two different observers. To do so, we introduce the worldview transformation between observers m and k (in symbols, w_{mk}) as the binary relation on Q^d connecting the coordinate points where m and k coordinatize the same (nonempty) events:

$$\mathsf{w}_{mk}(\bar{x},\bar{y}) \quad \overset{def}{\Longleftrightarrow} \quad \mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{y}) \neq \emptyset.$$

Map $P : Q^d \to Q^d$ is called a Poincaré transformation iff it is an affine bijection having the following property

$$\mathsf{time}(\bar{x}, \bar{y})^2 - \mathsf{space}^2(\bar{x}, \bar{y}) = \mathsf{time}(\bar{x}', \bar{y}')^2 - \mathsf{space}^2(\bar{x}', \bar{y}')$$

for all $\bar{x}, \bar{y}, \bar{x}', \bar{y}' \in Q^d$ for which $P(\bar{x}) = \bar{x}'$ and $P(\bar{y}) = \bar{y}'$.

Theorem 3.1 shows that our streamlined axiom system SpecRel perfectly captures the kinematics of special relativity since it implies that the worldview transformations between inertial observers are the same as in the standard non-axiomatic approaches.

Theorem 3.1. Let $d \geq 3$. Assume SpecRel. Then w_{mk} is a Poincaré transformation if m and k are inertial observers.[5]

We postpone the proof of Theorem 3.1 to Section 7, where we will prove a slightly stronger result, see Theorem 7.21. For a similar result over Euclidean fields, see, e.g., [3, Thms. 1.4 & 1.2], [4, Thm. 11.10], [26, Thm.3.1.4].

The so-called **worldline** of body b according to observer m is defined as follows:

$$\mathsf{wl}_m(b) := \{\bar{x} : \mathsf{W}(m, b, \bar{x})\}.$$

Corollary 3.2. Let $d \geq 3$. Assume SpecRel. The $\mathsf{wl}_m(k)$ is a straight line if m and k are inertial observers.[6]

Let m and k be inertial observers. The **squared speed** of k according to m is defined as follows:

$$\mathsf{speed}^2(m, k, v) \stackrel{def}{\Longleftrightarrow}$$
$$\exists \bar{x} \bar{y} \big[\bar{x} \neq \bar{y} \wedge \mathsf{W}(m, k, \bar{x}) \wedge \mathsf{W}(m, k, \bar{y}) \wedge \mathsf{space}^2(\bar{x}, \bar{y}) = v \cdot \mathsf{time}(\bar{x}, \bar{y})^2 \big].$$

By Corollary 3.2, SpecRel implies that, for each $m, k \in \mathsf{IOb}$, there is one and only one v such that $\mathsf{speed}^2(m, k, v)$ holds. From now on let us denote this unique v by $\mathsf{speed}_m^2(k)$.

Remark 3.3. Even if $\langle Q, +, \cdot, \leq \rangle$ is the ordered field of rational numbers, it is possible that the squared speed of an observer is 2. For example, $\mathsf{speed}_m^2(k) = 2$ if $d = 3$ and inertial observers k goes trough points $\langle 0, 0, 0 \rangle, \langle 1, 1, 1 \rangle \in \mathbb{Q}^3$ according to inertial observer m. However, some quantity cannot be the squared speed in some fields. For example, the squared speed cannot be 3 if $\langle Q, +, \cdot, \leq \rangle$ is the ordered field of rational numbers and $d = 3$. This is so, because the equation $x^2 + y^2 = 3z^2$ does not have a nonzero solution over the natural numbers (if x, y and z are

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[5]Actually, axioms AxOField, AxPh, AxEv, and AxSymD are enough to prove this statement, see Theorem 7.21.

[6]Axioms AxOField, AxPh, AxEv, and AxSelf are enough to prove this statement since, by Theorem 7.8, axioms AxOField, AxPh, and AxEv imply that the worldview transformations take lines to lines and $\mathsf{w}_m(k)$ is the w_{km} image of the time-axis by axiom AxSelf.

solutions, then x, y, and z are divisible by 3^n for all natural numbers n; hence $x = y = z = 0$). Consequently, it does not have a nonzero solution over the field of rational numbers.

Corollary 3.4 states basically that relatively moving inertial observers' clocks slow down by the Lorentz factor $\gamma = (1 - v^2/c^2)^{-1/2}$ where v is the relative speed of the observers.

Corollary 3.4. Let $d \geq 3$. Assume SpecRel. Let $m, k \in$ IOb and let $\bar{x}, \bar{y}, \bar{x}', \bar{y}' \in Q^d$ such that $\bar{x}, \bar{y} \in \mathsf{wl}_k(k)$, $\mathsf{w}_{km}(\bar{x}) = \bar{x}'$ and $\mathsf{w}_{km}(\bar{y}) = \bar{y}'$. Then

$$(1) \qquad \mathsf{time}(\bar{x}', \bar{y}')^2 = \frac{\mathsf{time}(\bar{x}, \bar{y})^2}{1 - \mathsf{speed}_m^2(k)}.$$

Proof. Formula (1) is always defined since $\mathsf{speed}_m^2(k)$ cannot be 1 by Theorem 3.1. The case $\bar{x} = \bar{y}$ is trivial since, in this case, both $\mathsf{time}(\bar{x}, \bar{y})$ and $\mathsf{time}(\bar{x}', \bar{y}')$ are 0. So let us assume that $\bar{x} \neq \bar{y}$. Since $\bar{x}, \bar{y} \in \mathsf{wl}_k(k)$, we have that $\mathsf{space}^2(\bar{x}, \bar{y}) = 0$ by AxSelf. By Theorem 3.1, w_{km} is a Poincaré transformation. Therefore,

$$\mathsf{time}(\bar{x}, \bar{y})^2 = \mathsf{time}(\bar{x}', \bar{y}')^2 - \mathsf{space}^2(\bar{x}', \bar{y}').$$

Consequently,

$$\mathsf{time}(\bar{x}, \bar{y})^2 = \mathsf{time}(\bar{x}', \bar{y}')^2 \left(1 - \frac{\mathsf{space}^2(\bar{x}', \bar{y}')}{\mathsf{time}(\bar{x}', \bar{y}')^2}\right).$$

Hence, by the definition of $\mathsf{speed}_m^2(k)$, we get

$$\mathsf{time}(\bar{x}, \bar{y})^2 = \mathsf{time}(\bar{x}', \bar{y}')^2 \left(1 - \mathsf{speed}_m^2(k)\right).$$

since $\mathsf{w}_{km}(\bar{x}) \neq \mathsf{w}_{km}(\bar{y})$ and $\mathsf{w}_{km}(\bar{x}), \mathsf{w}_{km}(\bar{y}) \in \mathsf{wl}_m(k)$. ∎

Theorem 3.1 and its consequences show that SpecRel captures special relativity well over every ordered field. It is a natural question to ask what happens with these theorems if we assume less about the quantities. This is one side of the question "what are the numbers?", which is a whole research direction:

Question 3.5 (Research direction). What remains from the theorems of SpecRel, if we replace ordered fields with other algebraic structures, e.g., with ordered rings?

Here we concentrate on the other side of our question; namely, "how can some physical assumptions implicitly enrich the structure of quantities?". To investigate this question, let us now introduce notation $Num_n(\mathsf{Th})$ for the class of the quantity parts of the models of theory Th if $d = n$:

$Num_n(\mathsf{Th}) = \{\text{The quantity parts}$

$$\langle Q, +, \cdot, \leq \rangle \text{ of the models of } \mathsf{Th} \text{ if } d = n\}.$$

The same way we use the notation $\mathfrak{Q} \in Num_n(\mathsf{Th})$ for ordered field $\mathfrak{Q}$ as the model theoretic notation $\mathfrak{Q} \in Mod(\mathsf{AxField})$.

AxThExp: Inertial observers can move along any straight line with any speed less than the speed of light:

$$\exists h \; \mathsf{IOb}(h) \wedge \big(\mathsf{IOb}(m) \wedge \mathsf{space}^2(\bar{x}, \bar{y}) < \mathsf{c}_m^2 \cdot \mathsf{time}(\bar{x}, \bar{y})^2$$
$$\to \exists k \big[\mathsf{IOb}(k) \wedge \mathsf{W}(m, k, \bar{x}) \wedge \mathsf{W}(m, k, \bar{y})\big]\big).$$

Theorem 3.6 below shows that axiom AxThExp implies that positive numbers have square roots if SpecRel is assumed.

Theorem 3.6. If $n \geq 3$,
$$Num_n(\mathsf{SpecRel} + \mathsf{AxThExp}) = \{\,\text{Euclidean fields}\,\}.$$

Proof. By Theorem 3.8.7 of [2], we have that SpecRel + AxThExp has a model over every Euclidean field. Consequently,
$$Num_n(\mathsf{SpecRel} + \mathsf{AxThExp}) \supseteq \{\,\text{Euclidean fields}\,\}.$$

To show the converse inclusion, we have to prove that every positive quantity has a square root in every model of SpecRel + AxThExp. To do so, let $x \in Q$ be a positive quantity. We have to show that x has a square root in Q.

First we will prove that $1 - v^2$ has a square root if $v \in Q$ and $0 \leq v < 1$. To do so, let $v \in Q$ for which $0 \leq v < 1$. Let $\bar{y} = \langle 1, v, 0, \ldots, 0 \rangle$. By AxTheExp there are inertial observers m and k such that $\bar{o}, \bar{y} \in \mathsf{wl}_m(k)$. By Corollary 3.2, $\mathsf{wl}_m(k)$ is a line. Thus $\mathsf{speed}_m^2(k) = v^2$. Therefore, there is a $z \in Q$ such that $1 - v^2 = z^2$ (i.e., $1 - v^2$ has a square root in Q) by AxField and Corollary 3.4.

From AxField, it is easy to show that
$$x = \left(\frac{x+1}{2}\right)^2 \cdot \left(1 - \left(\frac{x-1}{x+1}\right)^2\right)$$

for all $x \in Q$. There is a $z \in Q$ such that
$$1 - \left(\frac{x-1}{x+1}\right)^2 = z^2$$

since $0 \leq 1 - \left(\frac{x-1}{x+1}\right)^2 < 1$. So there is a quantity, namely $\frac{x+1}{2} \cdot z$, which is the square root of x; and that is what we wanted to prove. ∎

Remark 3.7. Axiom AxThExp cannot be omitted from Theorem 3.6 since SpecRel has a model over every ordered field, i.e.,
$$Num_n(\mathsf{SpecRel}) = \{\,\text{ordered fields}\,\}$$

for all $n \geq 2$. Moreover, it also has non trivial models in which there are several observers moving relative to each other. We conjecture that there is a model of SpecRel such that the possible speeds of observers

are dense in interval $[0, 1]$, see Corollary 3.13 and Conjecture 3.14 at pages 12 and 12.

In the proof of Theorem 3.6, axiom AxSymD is strongly used since SpecRel without AxSymD does not imply the exact ratio of the slowing down of moving clocks; SpecRel without AxSymD only implies that at least one of two relatively moving inertial observers' clocks run slow according to the other, see [2, §2.5]. So it is natural to investigate what remains of Theorem 3.6 if we leave the symmetry axiom out. It is surprising but, in the case of $d = 3$, Theorem 3.6 remains valid even if we assume only $c_m = 1$ from AxSymD, see Andréka–Madarász–Németi [2, Thm 3.6.17]. Now we will show that even the assumption $c_m = 1$ is not necessary. To do so, let us introduce the next axiom system

$$\text{SpecRel}_0 = \text{SpecRel} - \text{AxSymD}.$$

Theorem 3.8.

$$Num_3(\text{SpecRel}_0 + \text{AxThExp}) = \{\text{Euclidean fields}\}$$

Proof. By Theorem 3.6, $\text{SpecRel}_0 + \text{AxThExp}$ has a model over every Euclidean field since even $\text{SpecRel} + \text{AxThExp}$ has one. So

$$Num_3(\text{SpecRel}_0 + \text{AxThExp}) \supseteq \{\text{Euclidean fields}\}.$$

To prove the converse inclusion, we have to prove that the quantity structure of every model of $\text{SpecRel}_0 + \text{AxThExp}$ is a Euclidean field if $d = 3$. By Theorem 3.6.17 of [2], the quantity structures of the models of $\text{SpecRel}_0 + \text{AxThExp} + c_m = 1$ are Euclidean fields if $d = 3$. Therefore, it is enough to show that a model of $\text{SpecRel}_0 + \text{AxThExp} + c_m = 1$ can be constructed from every model of $\text{SpecRel}_0 + \text{AxThExp}$ without changing its quantity structure.

Let $\mathfrak{M}$ be an arbitrary 3 dimensional model of $\text{SpecRel}_0 + \text{AxThExp}$. Let $\mathfrak{M}^+$ be the model which is constructed from $\mathfrak{M}$ by rescaling the coordinatization of each inertial observer m of $\mathfrak{M}$ by the following map $\bar{x} \mapsto \langle c_m x_1, x_2, \ldots x_d \rangle$, i.e., rescaling the time of m by the factor c_m. It is clear that the speed of light becomes 1 according to m after the rescaling. So $c_m = 1$ holds in $\mathfrak{M}^+$. It is also easy to see that this rescaling does not change the validity of AxThExp and the other axioms of SpecRel_0. Therefore, $\mathfrak{M}^+$ is a model of axiom system $\text{SpecRel}_0 + \text{AxThExp} + c_m = 1$. By the construction, the quantity parts of $\mathfrak{M}^+$ and $\mathfrak{M}$ are the same. Consequently, the quantity part of $\mathfrak{M}$ is a Euclidean field. This completes our proof since $\mathfrak{M}$ was an arbitrary model of axiom system $\text{SpecRel}_0 + \text{AxThExp}$. ■

Until recently, it was unsolved whether Theorem 3.8 is valid or not in any higher dimension (see [2, Questions 3.6.17 and 3.6.19]) when Hajnal Andréka has provided counterexamples in the even dimensions, i.e., the following is true:

Theorem 3.9.

$$Num_{2k}(\mathsf{SpecRel_0} + \mathsf{AxThExp} + \mathsf{c}_m = 1) \supsetneq \{\text{Euclidean fields}\}$$

For the proof of Theorem 3.9, see [6].

The existence of models of $\mathsf{SpecRel_0} + \mathsf{AxThExp}$ over non Euclidean fields is a surprising result since it is natural to conjecture that a 3 dimensional model can be constructed from any $d \geq 4$ dimensional model of $\mathsf{SpecRel_0} + \mathsf{AxThExp}$ without changing its quantity structure (by "cutting out" a 3 dimensional part). Clearly, such a construction would imply Theorem 3.8 in any dimension higher than 3, too. It is interesting to note that this kind of construction works if the quantity structure is a Euclidean field.

Theorem 3.9 only shows that there are models of $\mathsf{SpecRel_0} + \mathsf{AxThExp}$ over some non-Euclidean fields. However, the question "what are the fields over which $\mathsf{SpecRel_0} + \mathsf{AxThExp}$ has a model?" is still unsolved even in 4 dimension:

Question 3.10. Exactly which ordered fields are the elements of the class $Num_n(\mathsf{SpecRel_0} + \mathsf{AxThExp})$ if $n \geq 4$.

Without adding extra axioms to $\mathsf{SpecRel} + \mathsf{AxThExp}$, it does not imply that the structure of numbers has to be a Euclidean field if $d = 2$. One of the reasons for this fact is that, if $d = 2$, the axioms of $\mathsf{SpecRel}$ do not imply that the world lines of inertial observers are straight lines. So we have to add it as an extra axiom stating this (AxLine). For a precise formulation of AxLine, see, e.g., [4, p.620]. Another reason is that, if $d = 2$, there are no two events which are simultaneous according to two relatively moving observers. Therefore, AxSymD states nothing if $d = 2$. So we have to change this axiom. For example, we may replace AxSymD with the statement "moving observers see each others clock the same way and $\mathsf{c}_m = 1$" (AxSymT). For a precise formulation of the first part of AxSymT, see, e.g., [3, p.8], [26, p.20]. Actually, AxSymT is equivalent to AxSymD if $\mathsf{SpecRel_0} + \mathsf{c}_m = 1$ is assumed and $d \geq 3$, see, e.g., [26, Thm.3.1.4].

Question 3.11. Does $\mathsf{SpecRel} + \mathsf{AxThExp} + \mathsf{AxLine} + \mathsf{AxSymT}$ imply that the quantities form a Euclidean field if $d = 2$? If not, what further natural axioms we have to assume to prove that the quantities form a Euclidean field?

Since our measurements have only finite accuracy, it is natural to assume $\mathsf{AxThExp}$ only approximately. To introduce an approximated version of $\mathsf{AxThExp}$, we need some definitions. The **space component** of coordinate point $\bar{x} \in Q^d$ is defined as $\bar{x}_s := \langle x_2, \ldots, x_d \rangle$. The **squared Euclidean distance** of $\bar{x}, \bar{y} \in Q^d$ is defined as

$$\mathsf{dist}^2(\bar{x}, \bar{y}) := (x_1 - y_1)^2 + \ldots + (x_d - y_d)^2$$

and the **difference of** $\bar{x}, \bar{y} \in Q^d$ is defined as

$$\bar{x} - \bar{y} := \langle x_1 - y_1, \ldots, x_d - y_d \rangle.$$

Let the **squared Euclidean length** of $\bar{x} \in Q^d$ be defined as

$$\mathsf{length}^2(\bar{x}) := x_1^2 + \ldots + x_d^2.$$

AxThExp⁻: Inertial observers can move roughly with any speed less than the speed of light roughly in any direction:

$$\exists h \; \mathsf{IOb}(h) \wedge \Big(\mathsf{IOb}(m) \wedge \varepsilon > 0 \wedge \mathsf{length}^2(\bar{v}_s) < \mathsf{c}_m^2$$

$$\wedge \, v_1 = 1 \rightarrow \exists \bar{w} \Big[\mathsf{dist}^2(\bar{w}, \bar{v}) < \varepsilon \wedge \forall \bar{x}\bar{y} \, \exists \lambda \big(\bar{x} - \bar{y} = \lambda \bar{w}$$

$$\rightarrow \exists k \big[\mathsf{IOb}(m) \wedge \mathsf{W}(m, k, \bar{y}) \wedge \mathsf{W}(m, k, \bar{y}) \big] \big) \Big] \Big).$$

A model of SpecRel + AxThExp⁻ can be constructed over the field of rational numbers, i.e., the following is true:

Theorem 3.12.

$$\mathbb{Q} \in Num_n(\mathsf{SpecRel} + \mathsf{AxThExp}^-)$$

For the proof of Theorem 3.12, see [16].

An ordered field is called **Archimedean ordered field** iff for all a, there is a natural number n such that

$$a < \underbrace{1 + \ldots + 1}_{n}$$

holds. By Pickert–Hion Theorem, every Archimedean ordered field is isomorphic to subfield of the field of real numbers, see, e.g., [11, §VIII], [18, C.44.2]. Consequently, the field of rational numbers is dense in any Archimedean ordered field since it is dense in the field of real numbers. Therefore, the following is a corollary of Theorem 3.12.

Corollary 3.13.

$$\{ \text{Archimedean ordered fields} \} \subsetneq Num_n(\mathsf{SpecRel} + \mathsf{AxThExp}^-)$$

By Lövenheim–Skolem Theorem it is clear that $Num_n(\mathsf{SpecRel} + \mathsf{AxThExp}^-)$ cannot be the class of Archimedean ordered fields since it has elements of arbitrarily large cardinality while Archimedean ordered fields are subsets of the field of real numbers by Pickert–Hion Theorem. The question "exactly which ordered fields can be the quantity structures of theory SpecRel + AxThExp⁻?" is open. We conjecture that there is a model of SpecRel + AxThExp⁻ over every ordered field, i.e.:

CONJECTURE 3.14.

$$Num_n(\mathsf{SpecRel} + \mathsf{AxThExp}^-) = \{ \text{ordered fields} \}$$

4. Numbers implied by accelerated observers

Now we are going to investigate what happens with the possible structures of quantities if we extend our theory SpecRel with accelerated observers. To do so, let us recall our first-order logic axiom system of accelerated observers AccRel. The key axiom of AccRel is the following:

AxCmv**:** At each moment of its worldline, each observer sees the nearby world for a short while as an inertial observer does.

For formalization of AxCmv, see [26]. In AccRel we will also use the following localized version of axioms AxEv and AxSelf of SpecRel.

AxEv$^-$**:** Observers coordinatize all the events in which they participate:

$$\mathsf{Ob}(k) \wedge \mathsf{W}(m, k, \bar{x}) \rightarrow \exists \bar{y} \ \mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{y}).$$

AxSelf$^-$**:** In his own worldview, the worldline of any observer is an interval of the time-axis containing all the coordinate points of the time-axis where the observer sees something:

$$\left[\mathsf{W}(m, m, \bar{x}) \rightarrow x_2 = \ldots = x_d = 0\right] \wedge$$
$$\left[\mathsf{W}(m, m, \bar{y}) \wedge \mathsf{W}(m, m, \bar{z}) \wedge x_1 < t < y_1 \rightarrow \mathsf{W}(m, m, t, 0, \ldots, 0)\right] \wedge$$
$$\exists b \left[\ \mathsf{W}(m, b, t, 0, \ldots, 0) \rightarrow \mathsf{W}(m, m, t, 0, \ldots, 0)\right].$$

Let us now introduce a promising theory of accelerated observers as SpecRel extended with the three axioms above.

$$\mathsf{AccRel}_0 = \mathsf{SpecRel} \cup \{\mathsf{AxCmv}, \mathsf{AxEv}^-, \mathsf{AxSf}^-\}$$

Since AxCmv ties the behavior of accelerated observers to the inertial ones and SpecRel captures the kinematics of special relativity perfectly by Theorem 3.1, it is quite natural to think that AccRel$_0$ is a strong enough theory of accelerated observers to prove the most fundamental results about accelerated observers. However, AccRel$_0$ does not even imply the most basic predictions about accelerated observers such as the twin paradox or that stationary observers measure the same time between two events [15], [26, §7]. Moreover, it can be proved that even if we add the whole firs-order logic theory of real numbers to AccRel$_0$ is not enough to get a theory that implies the twin paradox, see, e.g., [15], [26, §7].

In the models of AccRel$_0$ in which TwP is not true there are some definable gaps in the number line. Our next assumption is an axiom scheme excluding these gaps.

CONT**:** Every parametrically definable, bounded and nonempty subset of Q has a supremum (i.e., least upper bound) with respect to $\leq$.

In CONT "definable" means "definable in the language of AccRel, parametrically." For a precise formulation of CONT, see [15, p.692] or [26, §10.1].

That CONT requires the existence of supremum only for sets definable in the language of AccRel instead of every set is important because it makes this postulate closer to the physical/empirical level. This is true because CONT does not speak about "any fancy subset" of the quantities, but just about those "physically meaningful" sets which can be defined in the language of our (physical) theory.

Our axiom scheme of continuity (CONT) is a "mathematical axiom" in spirit. It is Tarski's first-order logic version of Hilbert's continuity axiom in his axiomatization of geometry, see [12, pp.161-162], fitted to the language of AccRel.

When Q is the ordered field of real numbers, CONT is automatically true. Let us introduce our axioms system AccRel as the extension of $AccRel_0$ by axiom scheme CONT.

$$AccRel = AccRel_0 + CONT$$

It can be proved that axiom system AccRel implies the twin paradox, see [15], [26, §7.2].

An ordered field is called **real closed field** if a first-order logic sentence of the language of ordered fields is true in it exactly when it is true in the field of real numbers, or equivalently if it is Euclidean and every polynomial of odd degree has a root in it, see, e.g., [28].

Theorem 4.1.

$$Num_n(AccRel) = \{\text{ real closed fields }\}$$

Proof. There is a model of AccRel over every real closed field since every model of SpecRel over a real closed field in which $B = Ph \cup IOb$ is a model of AccRel and SpecRel has a model even over every Euclidean ordered field by Theorem 3.6.

Axiom schema CONT is stronger than the whole first-order logic theory of real numbers, see, e.g., [26, Prop. 10.1.2]. Consequently, if axiom AxOField is assumed, CONT by itself implies that the quantities are real closed fields. ■

5. Numbers implied by uniformly accelerated observers

We have seen that assuming existence of observers can ensure the existence of numbers. So let us investigate another axiom of this kind.

The next axiom ensures the existence of uniformly accelerated observers. To introduce it, let us define the **life-curve** $lc_m(k)$ of observer k according to observer m as the worldline of k according to m *parametrized by the time measured by k*, formally:

$$lc_m(k) := \{\langle t, \bar{x}\rangle \in Q \times Q^d : \exists \bar{y} \ k \in ev_k(\bar{y}) = ev_m(\bar{x}) \wedge y_1 = t\}.$$

$\underline{\mathsf{Ax\exists UnifOb}}$: It is possible to accelerate an observer uniformly:[7]

$$\mathsf{IOb}(m) \to \exists k \Big[\mathsf{Ob}(k) \wedge \mathsf{Dom}\,\mathsf{lc}_m(k) = Q$$

$$\wedge\, \forall \bar{x} \big[\bar{x} \in \mathsf{Ran}\,\mathsf{lc}_m(k) \leftrightarrow x_2^2 - x_1^2 = a^2 \wedge x_3 = \ldots = x_d = 0 \big] \Big].$$

Theorem 5.1. Let $d \geq 3$. Assume AccRel and $\mathsf{Ax\exists UnifOb}$. Then there is a definable differentiable function $E : Q \to Q$ such that $\mathsf{Ran}\,E = Q^+ = [0, \infty)$, $\frac{dE}{dt} = E$ and $E(-t) = 1/E(t)$ for all $t \in Q$.

Let $\bar{\mathbb{Q}} \cap \mathbb{R}$ denote the ordered field of real algebraic numbers. Theorem 5.1 implies that the ordered field of algebraic real numbers cannot be the structure of quantities of theory $\mathsf{AccRel} + \mathsf{Ax\exists UnifOb}$:

Theorem 5.2. Let $n \geq 3$.
$$\bar{\mathbb{Q}} \cap \mathbb{R} \notin Num_n(\mathsf{AccRel} + \mathsf{Ax\exists UnifOb})$$

See [27] for proofs and more details of Theorems 5.1 and 5.2.

Remark 5.3. By Theorem 5.2, if $n \geq 3$, $Num_n(\mathsf{AccRel} + \mathsf{Ax\exists UnifOb})$ is not an elementary class of ordered fields, i.e., it is not a first-order logic axiomatizable class in the language of ordered fields. Of course, it is a pseudoelementary class, i.e., it is a reduct of an elementary class in a richer language.

By Theorem 5.2, we know that not every real closed field can be the quantity structure of $\mathsf{AccRel} + \mathsf{Ax\exists UnifOb}$. For example, the field of real algebraic numbers cannot be the quantity structure of $\mathsf{AccRel} + \mathsf{Ax\exists UnifOb}$. However, the problem that exactly which ordered fields can be the quantity structures of $\mathsf{AccRel} + \mathsf{Ax\exists UnifOb}$ is still open:

Question 5.4. Exactly which ordered fields are the elements of classes $Num_n(\mathsf{AccRel} + \mathsf{Ax\exists UnifOb})$ and $Num_n(\mathsf{AccRel_0} + \mathsf{Ax\exists UnifOb})$?

Theorem 5.1 suggests that the answer to Question 5.4 may have something to do with ordered exponential fields, see, e.g., [8, §4], [14].

6. Numbers required by general relativity

Let us now see some similar questions about the properties of numbers implied by axioms of general relativity. To do so, let us recall our axiom system GenRel of general relativity formulated in the same streamlined language as AccRel and $\mathsf{SpecRel}$. GenRel contains the localized versions of the axioms of $\mathsf{SpecRel}$ and the postulate that the worldview transformations between observers are differentiable maps, which is the localized version of the theorem of $\mathsf{SpecRel}$ stating that the worldview transformations between inertial observers are affine maps,

---

[7]In relativity theory, uniformly accelerated observers are moving along hyperbolas, see, e.g., [9, §3.8, pp.37-38], [19, §6], [20, §12.4, pp.267-272].

see Theorem 3.1. We have already introduced the localized versions of axioms AxEv and AxSelf, see AxEv$^-$ and AxSelf$^-$ at page 13. Now let us state the localized versions of AxPh and AxSymD.[8]

AxPh$^-$: The velocity of photons an observer "meets" is 1 when they meet, and it is possible to send out a photon in each direction where the observer stands.

AxSym$^-$: Meeting observers see each other's clocks slow down the same way.

AxDiff: The worldview transformations between observers are functions having linear approximations at each point of their domain (i.e., they are differentiable maps).

For a precise formulation of axioms AxPh$^-$, AxSym$^-$, and AxDiff, as well as a "derivation" of the axioms of GenRel from that of SpecRel, see, e.g., [5], [26, §9].

$$\mathsf{GenRel} := \{\mathsf{AxOFiled}, \mathsf{AxPh}^-, \mathsf{AxEv}^-, \mathsf{AxSelf}^-, \mathsf{AxSym}^-, \mathsf{AxDiff}\} \cup \mathsf{CONT}$$

Axiom system GenRel captures general relativity well since it is complete with respect the standard models of general relativity, i.e., with respect to Lorentzian manifolds, see, e.g., [5, Thm.4.1], [26, §9].

We call the worldline of observer m *timelike geodesic*, if each of its points has a neighborhood within which this observer "maximizes measured time" between any two encountered events, see Figure 1 for illustration and [5] for a formal definition of timelike geodesics in the language of GenRel.

According to the definition above, if there are only a few observers, then it is not a big deal that the worldline of m is a timelike geodesic (it is easy to be maximal if there are only a few to be compared to). To generate a real competition for the rank of having a timelike geodesic worldline, we postulate the existence of great many observers by the following axiom scheme of comprehension.

COMPR: For any parametrically definable timelike curve in any observer's worldview, there is another observer whose worldline is the range of this curve.

A precise formulation of COMPR can be obtained from that of its analogue in [4, p.679]. Let us now show that COMPR implies axiom Ax∃UnifOb, hence it requires at least as much properties of numbers.

Proposition 6.1.

$$Num_n(\mathsf{AccRel} + \mathsf{COMPR}) \subseteq Num_n(\mathsf{AccRel} + \mathsf{Ax\exists UnifOb})$$

On the proof. For all $a \in Q$, the hyperbola (line if $a = 0$)

$$(2) \qquad \{\bar{x} : x_2^2 - x_1^2 = a^2, x_3 = \ldots = x_d = 0\}$$

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[8]For technical reasons, in GenRel we use an equivalent version of AxSymD, and we introduce that the speed of light is 1 in AxPh instead of in AxSym$^-$.
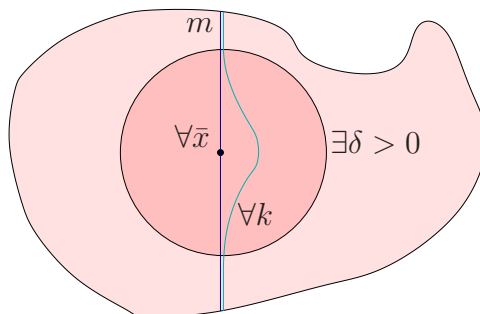
FIGURE 1. Illustration for the definition of timelike geodesic in GenRel

can be parametrized by the definable timelike curve

$$(3) \qquad \{\langle x_1, \bar{x} \rangle : x_2^2 - x_1^2 = a^2, x_3 = \ldots = x_d = 0\}.$$

So by COMPR, there is an observer whose worldline is this set. So COMPR implies Ax∃UnifOb. Therefore, every model of AccRel + COMPR is a model of of AccRel + Ax∃UnifOb. Hence the possible quantity structures of AccRel + COMPR is a subset of the possible quantity structures of AccRel + Ax∃UnifOb. ■

It is also quite easy to show that GenRel does not require more properties of numbers than AccRel.

Proposition 6.2.

$$Num_n(\mathsf{AccRel} + \mathsf{AxDiff}) \subseteqq Num_n(\mathsf{GenRel})$$

$$Num_n(\mathsf{AccRel} + \mathsf{AxDiff} + \mathsf{COMPR}) \subseteqq Num_n(\mathsf{GenRel} + \mathsf{COMPR})$$

On the proof. To prove this statement it is enough to show that the models of AccRel + AxDiff are also models of GenRel. Since AxPh⁻ and AxSym⁻ are the only two axioms of GenRel which are not also contained in AccRel + AxDiff, we only have to show that these two axioms are consequences of AccRel. Axioms AxPh⁻ and AxSym⁻ follow from AccRel since they are true for inertial observers in SpecRel and by AxCmv accelerated observers locally see the world the same way as their co-moving inertial observers. ■

Question 6.3. Exactly which ordered fields are the elements of classes $Num_n(\mathsf{AccRel} + \mathsf{COMPR})$ and $Num_n(\mathsf{GenRel} + \mathsf{COMPR})$?

Maybe the ordered field reducts of differentially closed fields of Abraham Robinson, see [17], [21], have to do something with the answer to the question above.

7. Proof of Theorem 3.1

In this section, we are going to prove Theorem 3.1. To do so, let us recall a version of Alexandrov–Zeeman theorem generalized over fields. To state this theorem, we need some concepts. Map $\mathsf{q} : Q^d \to Q$ is a **quadratic form** if

$$(4) \qquad \mathsf{q}(\lambda\bar{x}) = \lambda^2 \mathsf{q}(\bar{x})$$

for all $\lambda \in Q$ and $\bar{x} \in Q^d$, and

$$(5) \qquad (\bar{x}, \bar{y})_\mathsf{q} := \mathsf{q}(\bar{x} + \bar{y}) - \mathsf{q}(vx) - \mathsf{q}(\bar{y})$$

is a symmetric bilinear form. Quadratic form q is **non-degenerate** if

$$\forall \bar{x}\bar{a} \ (\bar{x}, \bar{a})_\mathsf{q} = 0 \wedge \mathsf{q}(\bar{a}) = 0 \to \bar{a} = \bar{o}.$$

A map $f : Q^d \to Q^d$ is called a **semilinear map** iff there is a field automorphism α such that

$$f(\bar{x} + \bar{y}) = f(\bar{x}) + f(\bar{y}) \quad \text{and} \quad f(\lambda\bar{x}) = \alpha(\lambda)f(\bar{x})$$

for all $\bar{x}, \bar{y} \in Q^d$ and $\lambda \in Q$. **Witt index** of quadratic form q is the maximal dimension of a subspace X of Q^d with the property $\mathsf{q}(\bar{x}) = 0$ for all $\bar{x} \in X$. q-**null cone** with vertex $\bar{a} \in Q^d$ is defined as

$$C(\bar{a}) = \{\bar{x} : \mathsf{q}(\bar{x} - \bar{a}) = 0\}.$$

Now we are ready to recall the version of Alexandrov-Zeeman theorem we need, see [30], [31]:

Theorem 7.1 (Vroegindewey). Let $\langle Q, +, \cdot \rangle$ be an commutative field. Let $d \geq 3$ and let q be a non-degenerate quadratic form with Witt index 1. Then every bijection of Q^d taking q-null cones to q-null cones is a composition of a translation and a semilinear map f with the property $\mathsf{q}(f(\bar{x})) = \lambda\alpha(\mathsf{q}(x))$ for some $\lambda \neq 0$ and field automorphism α.

We are going to apply Theorem 7.1 to the worldview transformations of inertial observers in SpecRel. To do so, we need several definitions and lemmas.

For all $c > 0$, let us define the c-**Minkowski quadratic form** as

$$\mu_c^2(\bar{x}) = c \cdot x_1^2 - x_2^2 - \ldots - x_d^2.$$

Lemma 7.2. Assume AxOField. Let $\bar{x} \in Q^d$ be such that $x_1 = 0$ and $\mu_c^2(\bar{x}) = 0$. Then $\bar{x} = \bar{o}$.

Proof. Since $x_1 = 0$ and $\mu_c^2(\bar{x}) = 0$, we have that $x_2^2 + \ldots + x_d^2 = 0$. This implies that $x_2 = \ldots = x_d = 0$ in ordered fields. Hence $\bar{x} = \bar{o}$ as stated. ■

Remark 7.3. Lemma 7.2 is not valid in every field. For example, in the field of complex numbers $\bar{x} = \langle 0, 1, i \rangle$ is a nonzero vector but $x_1 = 0$ and $\mu_1^2(\bar{x}) = 0$.

Lemma **7.4.** Assume AxOField. Let $c > 0$. Then Minkowski quadratic form μ_c^2 has Witt index 1.

Proof. Let $\bar{x}$ and $\bar{y}$ be vectors such that $\mu_c^2(\alpha\bar{x} + \beta\bar{y}) = 0$ for all $\alpha, \beta \in Q$. Let $\bar{z} = y_1\bar{x} - x_1\bar{y}$. Then $z_1 = 0$ and $\mu_c^2(\bar{z}) = 0$. Hence, by Lemma 7.2, $\bar{z} = \bar{o}$. So $y_1\bar{x} = x_1\bar{y}$. Therefore, the subspace spanned by $\bar{x}$ and $\bar{y}$ is 1 dimensional. Thus the Witt index of μ_c^2 is 1 as stated. ■

The **squared slope** of line l is defined as

$$\mathsf{slope}^2(l) = \frac{\mathsf{space}^2(\bar{x}, \bar{y})}{\mathsf{time}(\bar{x}, \bar{y})^2}$$

for all $\bar{x}, \bar{y} \in l$ for wihc $x_1 \neq y_1$.

Lemma **7.5.** Assume AxOField. Let $c > 0$. There is no non-degenerate triangle whose every side is of squared slope c.

Proof. Let $\bar{x}$, $\bar{y}$, and $\bar{z}$ be the vertices of a triangle whose sides are of squared slope c. Then $c \cdot \mathsf{time}(\bar{x}, \bar{y})^2 = \mathsf{space}^2(\bar{x}, \bar{y})$, $c \cdot \mathsf{time}(\bar{y}, \bar{z})^2 = \mathsf{space}^2(\bar{y}, \bar{z})$, and $c \cdot \mathsf{time}(\bar{z}, \bar{x})^2 = \mathsf{space}^2(\bar{z}, \bar{x})$. Let $\bar{p} = \bar{y} - \bar{x}$ and $\bar{q} = \bar{z} - \bar{y}$. Then

(6) $$cp_1^2 = p_2^2 + \ldots + p_d^2,$$

(7) $$cq_1^2 = q_2^2 + \ldots + q_d^2, \text{ and}$$

(8) $$c(p_1 + q_1)^2 = (p_2 + q_2)^2 + \ldots + (p_d + q_d)^2.$$

In other words $\mu_c^2(\bar{p}) = \mu_c^2(\bar{q}) = \mu_c^2(\bar{p}+\bar{q}) = 0$. By subtracting equations (6) and (7) from equation (8), we get

(9) $$2cp_1q_1 = 2p_2q_2 + \ldots + 2p_dq_d.$$

Let α and β be arbitrary elements of Q. Then

(10) $\mu_c^2(\alpha\bar{p} + \beta\bar{q})$
$$= \alpha^2\mu_c^2(\bar{p}) + 2\alpha\beta(cp_1q_1 - p_2q_2 - \ldots - p_dq_d) + \beta^2\mu_c^2(\bar{q})$$

for all $\alpha, \beta \in Q$. Therefore, $\mu_c^2(\alpha\bar{p} + \beta\bar{q}) = 0$. By Lemma 7.4, μ_c^2 has Witt index 1. So $\bar{p}$ and $\bar{q}$ are in the same 1 dimensional subspace of Q^d. Hence $\bar{x}$, $\bar{y} = \bar{x} + \bar{p}$, and $\bar{z} = \bar{x} + \bar{p} + \bar{q}$ are collinear. ■

The *f-**image** of set* H is defined as follows:

$$f[H] = \big\{ b : \exists a \big[a \in H \wedge f(a) = b\big] \big\}.$$

Proposition **7.6.** Assume AxOField, AxEv, and AxPh. Let $m, k \in$ IOb. Then w_{mk} is a bijection of Q^d taking lines of squared slope c_m^2 to lines of squared slope c_k^2.

Proof. Let $m \in \mathsf{IOb}$ and let $\bar{x}$ and $\bar{y}$ be two distinct coordinate points. Let $\bar{v} := \langle 1, \mathsf{c}_m, 0, \ldots, 0 \rangle$ and $\bar{u} := \langle 1, -\mathsf{c}_m, 0, \ldots, 0 \rangle$. By $\mathsf{AxOField}$, at most one of the lines

$$l_1 := \{\bar{x} + \lambda \cdot \bar{v} : \lambda \in Q\} \quad \text{and} \quad l_2 := \{\bar{x} + \lambda \cdot \bar{u} : \lambda \in Q\}$$

can contain $\bar{y}$ since $a\bar{v} = b\bar{u}$ implies $a = b = 0$. So, by AxPh, there is a light signal which is in $\mathsf{ev}_m(\bar{x})$ but not in $\mathsf{ev}_m(\bar{y})$ since $\mathsf{slope}^2(l_1) = \mathsf{slope}^2(l_2) = \mathsf{c}_m^2$. Thus inertial observers see different events at different coordinate points, i.e., $\mathsf{ev}_m(\bar{x}) = \mathsf{ev}_m(\bar{y})$ implies $\bar{x} = \bar{y}$. Therefore, binary relation w_{mk} is an injective function for all $m, k \in \mathsf{IOb}$.

Let $m, k \in \mathsf{IOb}$. By AxPh, every inertial observer sees a nonempty event in every coordinate point. By AxEv, inertial observers coordinatize the same events. Therefore, for all $\bar{x} \in Q^d$, there is a $\bar{y} \in Q^d$ such that $\mathsf{w}_{mk}(\bar{x}) = \bar{y}$. So $\mathsf{Dom}\, \mathsf{w}_{mk} = \mathsf{Ran}\, \mathsf{w}_{km} = Q^d$. Consequently, w_{mk} is a bijection of Q^d for all $m, k \in \mathsf{IOb}$.

Now we show that w_{mk} takes lines of squared slope c_m^2 to lines of squared slope c_k^2. To do so, let l be a line of squared slope c_m^2 and let $\bar{x}, \bar{y}, \bar{z}$ be three distinct points of l. By AxPh, there are light signals p_{xy}, p_{yz}, and p_{zx} such that $p_{xy}, p_{zx} \in \mathsf{ev}_m(\bar{x})$, $p_{yz}, p_{xy} \in \mathsf{ev}_m(\bar{y})$, and $p_{zx}, p_{yz} \in \mathsf{ev}_m(\bar{z})$. Since w_{mk} is a bijection, $\mathsf{w}_{mk}(\bar{x}), \mathsf{w}_{mk}(\bar{y})$, and $\mathsf{w}_{mk}(\bar{z})$ are also distinct points. By the definition of w_{mk}, we have $p_{xy}, p_{zx} \in \mathsf{ev}_k\big(\mathsf{w}_{mk}(\bar{x})\big)$, $p_{yz}, p_{xy} \in \mathsf{ev}_k\big(\mathsf{w}_{mk}(\bar{y})\big)$, and $p_{zx}, p_{yz} \in \mathsf{ev}_k\big(\mathsf{w}_{mk}(\bar{z})\big)$. So, by AxPh, coordinate points $\mathsf{w}_{mk}(\bar{x})$, $\mathsf{w}_{mk}(\bar{y})$, and $\mathsf{w}_{mk}(\bar{z})$ form a triangle such that all of its sides are of squared slope c_k^2. Therefore, by Lemma 7.5, they have to be on a line of squared slope c_k^2. So the w_{mk}-image of l is a subset of a line of c_k^2. Since m and k were arbitrary inertial observers we also have that the w_{km}-image of the line containing $\mathsf{w}_{mk}[l]$ is the subset of a line of squared slope c_m^2. Since w_{mk} is a bijection and its inverse is w_{km}, we have that $\mathsf{w}_{km}\big[\mathsf{w}_{mk}[l]\big] = l$. Consequently, $\mathsf{w}_{mk}[l]$ cannot be a proper subset of a line, but it has to be a whole line of squared slope c_k^2. This completes the proof of the proposition. ∎

Corollary 7.7. Assume $\mathsf{SpecRel}$. Let m and k be inertial observers. Then w_{mk} is a bijection of Q^d taking lines of squared slope 1 to lines of squared slope 1. ∎

Let us call a liner bijection of Q^d **almost Lorentz transformation** iff there is a $\lambda \neq 0$ such that $\mu_1^2\big(A(\bar{x})\big) = \lambda \mu_1^2(\bar{x})$ for all $\bar{x} \in Q^d$.

We think of functions as special binary relations. Hence we compose them as relations. The **composition** of binary relations R and S is defined as:

$$R \,\mathbin{\raise.3ex\hbox{$\scriptscriptstyle\circ$}}\, S := \{\langle a, c \rangle : \exists b \ R(a, b) \wedge S(b, c)\}.$$

So $(g \mathbin{;} f)(x) = f\big(g(x)\big)$ if f and g are functions. We will also use the notation $x \mathbin{;} g \mathbin{;} f$ for $(g \mathbin{;} f)(x)$ because the latter is easier to grasp. In

the same spirit, we will sometimes use the notation $x \, \mathbin{\fatsemi} \, f$ for $f(x)$. The **inverse** of R is defined as:

$$R^{-1} := \{\langle a, b \rangle : R(b, a)\}.$$

Let us introduce, for all $c > 0$, the **spatial distance and time rescaling maps** as

$$S_c(\bar{x}) = \langle x_1, cx_2, \ldots, cx_d \rangle \quad \text{and} \quad T_c(\bar{x}) = \langle cx_1, x_2, \ldots, x_d \rangle$$

for all $\bar{x} \in Q^d$. It is clear that $T_c^{-1} = T_{1/c}$ and $S_c^{-1} = S_{1/c}$.

Let α be an automorphism of field $\langle Q, \cdot, + \rangle$ and let $\tilde{\alpha}$ be the map $\tilde{\alpha}(\bar{x}) = \langle \alpha(x_1), \ldots, \alpha(x_d) \rangle$ for all $\bar{x} \in Q^d$. A map from Q^d to Q^d is called **automorphism-induced-map** if it is the form $\tilde{\alpha}$ for some automorphism α.[9]

Theorem 7.8. Let $d \geq 3$. Assume AxOField, AxEv, and AxPh. Let $m, k \in \mathsf{IOb}$. Then

- $\mathsf{w}_{mk} = S_{\mathsf{c}_m}^{-1} \, \mathbin{\fatsemi} \, A \, \mathbin{\fatsemi} \, \tilde{\alpha} \, \mathbin{\fatsemi} \, T \, \mathbin{\fatsemi} \, S_{\mathsf{c}_k}$ where T is a translation, A is an almost Lorentz transformation and α is field automorphism.
- $\mathsf{w}_{mk} = T_{\mathsf{c}_m} \, \mathbin{\fatsemi} \, A \, \mathbin{\fatsemi} \, \tilde{\alpha} \, \mathbin{\fatsemi} \, T \, \mathbin{\fatsemi} \, T_{\mathsf{c}_k}^{-1}$ where T is a translation, A is an almost Lorentz transformation and α is field automorphism.

Proof. By definitions, S_c and T_c^{-1} are linear bijections of Q^d taking lines of squared slope 1 to lines of squared slope c^2. Therefore, by Proposition 7.6, both maps $S_{\mathsf{c}_m} \, \mathbin{\fatsemi} \, \mathsf{w}_{mk} \, \mathbin{\fatsemi} \, S_{\mathsf{c}_k}^{-1}$ and $T_{\mathsf{c}_m}^{-1} \, \mathbin{\fatsemi} \, \mathsf{w}_{mk} \, \mathbin{\fatsemi} \, T_{\mathsf{c}_k}$ are bijections of Q^d taking lines of squared slope 1 to lines of squared slope 1. Since the μ_1^2-null cone $C(\bar{a})$ is the union of lines of squared slope 1 through $\bar{a}$, both $S_{\mathsf{c}_m} \, \mathbin{\fatsemi} \, \mathsf{w}_{mk} \, \mathbin{\fatsemi} \, S_{\mathsf{c}_k}^{-1}$ and $T_{\mathsf{c}_m}^{-1} \, \mathbin{\fatsemi} \, \mathsf{w}_{mk} \, \mathbin{\fatsemi} \, T_{\mathsf{c}_k}$ map μ_1^2-null cones to μ_1^2-null cones. Therefore, by Theorem 7.1 and Lemma 7.11, they are compositions of an almost Lorentz transformation A, a field-automorphism-induced map $\tilde{\alpha}$, and a translation T. ∎

Some of the following statements assume only that the quantity part is a field. Therefore, let us introduce the following axiom:

AxField: The quantity part $\langle Q, +, \cdot \rangle$ is a (commutative) field.

Lemma 7.9. Assume AxField and that $1 + 1 \neq 0$. Let α and β be two automorphisms of $\langle Q, +, \cdot \rangle$ such that $\alpha(a)^2 = \beta(a)^2$ for all $a \in Q$. Then $\alpha = \beta$.

Proof. For all $a \in Q$, we have that $\alpha(a) = \beta(a)$ or $\alpha(a) = -\beta(a)$. Let $a \in Q$ such that $\alpha(a) = -\beta(a)$. Then $\alpha(1 + a) = 1 + \alpha(a) = 1 - \beta(a)$. Also $\alpha(1+a) = \beta(1+a) = 1+\beta(a)$ or $\alpha(1+a) = -\beta(1+a) = -1-\beta(a)$. So $1 - \beta(a) = 1 + \beta(a)$ or $1 - \beta(a) = -1 - \beta(a)$. Therefore, $\beta(a) = 0$ since $1 + 1 \neq 0$. Hence $a = 0$. Thus $\alpha(a) = \beta(a)$ for all $a \in Q$. ∎

Let Id_H denote the **identity map** from $H \subseteq Q^d$ to H, i.e., $\mathsf{Id}_H(\bar{x}) = \bar{x}$ for all $\bar{x} \in H$.

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[9]Let us note that we have not required that α is order preserving.

Remark **7.10.** It is easy to see that Lemma 7.9 is not valid if the field has characteristic 2, i.e., if $1 + 1 = 0$. For example, the 4 element field has two automorphisms Id and α; and $\alpha^2 = \mathsf{Id}^2$, but $\alpha \neq \mathsf{Id}$.

Lemma **7.11.** Assume $\mathsf{AxField}$. Let $f : Q^d \to Q^d$ be a semilinear transformation having the property

$$(11) \qquad \mu_1^2\big(f(\bar{x})\big) = \lambda\alpha\big(\mu_1^2(\bar{x})\big)$$

for some $\lambda \neq 0$ and field automorphism α. Then there are almost Lorentz transformations A and A^* such that $f = \tilde{\alpha} \,\fatsemi\, A = A^* \,\fatsemi\, \tilde{\alpha}$.

Proof. Let A be $\tilde{\alpha}^{-1} \,\fatsemi\, f$, i.e.,

$$(12) \qquad A(\bar{x}) = f\big(\tilde{\alpha}^{-1}(\bar{x})\big)$$

for all $\bar{x} \in Q^d$. A is a bijection since both $\tilde{\alpha}^{-1}$ and f are so. A is additive, i.e., $A(\bar{x} + \bar{y}) = A(\bar{x}) + A(\bar{y})$ for all $\bar{x}, \bar{y} \in Q^d$, since $\tilde{\alpha}^{-1}$ and f are so.

Since f is semilinear, there is a automorphism β such that

$$(13) \qquad f(a\bar{x}) = \beta(a)f(\bar{x})$$

for all $\bar{x} \in Q^d$ and $a \in Q$. Consequently, we have

$$\mu_1^2\big(f(a\bar{x})\big) \overset{(13)}{=} \mu_1^2\big(\beta(a)f(\bar{x})\big) \overset{(4)}{=} \beta(a)^2\mu_1^2\big(f(\bar{x})\big) \overset{(11)}{=} \beta(a)^2\lambda\alpha\big(\mu_1^2(\bar{x})\big)$$

and

$$\mu_1^2\big(f(a\bar{x})\big) \overset{(11)}{=} \lambda\alpha\big(\mu_1^2(a\bar{x})\big) \overset{(4)}{=} \lambda\alpha\big(a^2\mu_1^2(\bar{x})\big) = \lambda\alpha(a)^2\alpha\big(\mu_1^2(\bar{x})\big).$$

for all $a \in Q$. Consequently, $\lambda\beta(a)^2\alpha\big(\mu_1^2(\bar{x})\big) = \lambda\alpha(a)^2\alpha\big(\mu_1^2(\bar{x})\big)$ for all $a \in Q$. So $\alpha(a)^2 = \beta(a)^2$ for all $a \in Q$. Therefore, by Lemma 7.9, $\alpha = \beta$. Consequently, equation (13) becomes

$$(14) \qquad f(a\bar{x}) = \alpha(a)f(\bar{x}).$$

Thus A is a linear bijection since

$$A(a\bar{x}) \overset{(12)}{=} f\big(\tilde{\alpha}^{-1}(a\bar{x})\big) \overset{(14)}{=} f\big(\alpha^{-1}(a)\tilde{\alpha}^{-1}(\bar{x})\big)$$

$$= \alpha\big(\alpha^{-1}(a)\big)f\big(\tilde{\alpha}^{-1}(\bar{x})\big) = af\big(\tilde{\alpha}^{-1}(\bar{x})\big) \overset{(12)}{=} aA(\bar{x})$$

for all $\bar{x} \in Q^d$ and $a \in Q$.

Now we are going to show that $\mu_1^2\big(A(\bar{x})\big) = \lambda\mu_1^2(\bar{x})$ for all $\bar{x} \in Q^d$. Let $\bar{x} \in Q^d$ and let $\bar{y} = \tilde{\alpha}^{-1}(\bar{x})$.

$$\mu_1^2\big(A(\bar{x})\big) \overset{(12)}{=} \mu_1^2\big(f\big(\tilde{\alpha}^{-1}(\bar{x})\big)\big) = \mu_1^2\big(f(\bar{y})\big)$$

$$\overset{(11)}{=} \lambda\alpha\big(\mu_1^2(\bar{y})\big) = \lambda\mu_1^2\big(\tilde{\alpha}(\bar{y})\big) = \lambda\mu_1^2(\bar{x}).$$

This proves that A is an almost Lorentz transformation; and $f = \tilde{\alpha} \,\fatsemi\, A$ by the definition of A.

We also have that $f = A^* \mathbin{\mathrm{\mathring{,}}} \tilde{\alpha}$ for almost Lorentz transformation $A^* = \tilde{\alpha} \mathbin{\mathrm{\mathring{,}}} A \mathbin{\mathrm{\mathring{,}}} \tilde{\alpha}^{-1}$. ■

Vectors $\bar{x}, \bar{y} \in Q^d$ are called **orthogonal** in the Euclidean sense, in symbols $\bar{x} \perp_e \bar{y}$, iff $x_1 y_1 + \ldots + x_d y_d = 0$.

Vectors $\bar{x}, \bar{y} \in Q^d$ are called **Minkowski orthogonal**, in symbols $\bar{x} \perp_\mu \bar{y}$, iff $(\bar{x}, \bar{y})_{\mu_1^2} = 0$, i.e., $x_1 y_1 = x_2 y_2 \ldots + x_d y_d$.

Lemma 7.12. Assume AxField. Let A be an almost Lorentz transformation. Then $\bar{x} \perp_\mu \bar{y}$ iff $A(\bar{x}) \perp_\mu A(\bar{y})$ for all $\bar{x}, \bar{y} \in Q^d$.

Proof. By definition, $\bar{x} \perp_\mu \bar{y}$ iff $(\bar{x}, \bar{y})_{\mu_1^2} = 0$. Also by definition $\big(A(\bar{x}), A(\bar{y})\big)_{\mu_1^2} = \mu_1^2\big(A(\bar{x}) + A(\bar{y})\big) - \mu_1^2\big(A(\bar{x})\big) - \mu^2\big(A(\bar{y})\big)$. Since A is an almost Lorentz transformation, $\big(A(\bar{x}), A(\bar{y})\big)_{\mu_1^2} = \lambda \cdot (\bar{x}, \bar{y})_{\mu_1^2}$ for some $\lambda \neq 0$. Therefore, $\big(A(\bar{x}), A(\bar{y})\big)_{\mu_1^2} = 0$ iff $(\bar{x}, \bar{y})_{\mu_1^2} = 0$; and this is what we wanted to prove. ■

Let us introduce the **time unit vector** as follows $\bar{1}_t := \langle 1, 0, \ldots, 0 \rangle$.

Proposition 7.13. Assume AxField. Let A be an almost Lorentz transformation. Then $y_1 = 0$ and $A(\bar{y})_1 = 0$ iff $A(\bar{1}_t) \perp_e A(\bar{y})$ and $A(\bar{y})_1 = 0$ for all $\bar{y} \in Q^d$.

Proof. Let $\bar{y} \in Q^d$. It is enough to show that $y_1 = 0$ is equivalent to $\bar{1}_t \perp_e A(\bar{y})$ assuming that $A(\bar{y})_1 = 0$. It is clear that $y_1 = 0$ iff $\bar{1}_t \perp_\mu \bar{y}$. By Lemma 7.12, $\bar{1}_t \perp_\mu \bar{y}$ iff $A(\bar{1}_t) \perp_\mu A(\bar{y})$. Since $A(\bar{y})_1 = 0$, we have $A(\bar{1}_t) \perp_\mu A(\bar{y})$ iff $A(\bar{1}_t) \perp_e A(\bar{y})$. Therefore, $y_1 = 0$ iff $\bar{1}_t \perp_e A(\bar{y})$ provided that $A(\bar{y})_1 = 0$. ■

Let m and k be inertial observers and let $\bar{x}, \bar{y} \in Q^d$. Events $\mathsf{ev}_m(\bar{x})$ and $\mathsf{ev}_m(\bar{y})$ are **simultaneous** for k iff $x_1' = y_1'$ for all $\bar{x}'$ and $\bar{y}'$ for which $\mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{x}')$ and $\mathsf{ev}_m(\bar{y}) = \mathsf{ev}_k(\bar{y}')$. Events $\mathsf{ev}_m(\bar{x})$ and $\mathsf{ev}_m(\bar{y})$ are **separated orthogonally to the plane of motion** of k according to m iff $x_1 = y_1$ and $(\bar{x} - \bar{y}) \perp_e \big(\mathsf{w}_{km}(\bar{1}_t) - \mathsf{w}_{km}(\bar{o})\big)$, see Figure 2.

Theorem 7.14. Let $d \geq 3$. Assume AxOField, AxPh, and AxEv. Let m and k be inertial observers and let $\bar{x}, \bar{y} \in Q^d$. Events $\mathsf{ev}_m(\bar{x})$ and $\mathsf{ev}_m(\bar{y})$ are simultaneous for both m and k iff $\mathsf{ev}_m(\bar{x})$ and $\mathsf{ev}_m(\bar{y})$ are separated orthogonally to the plane of motion of k according to m.[10]

Proof. Let $\bar{x}' = \mathsf{w}_{mk}(\bar{x})$, $\bar{y}' = \mathsf{w}_{mk}(\bar{y})$, and $\bar{v} = \bar{y} - \bar{x}$, see Figure 2. By Theorem 7.8, $\mathsf{w}_{km} = S_{\mathsf{c}_k}^{-1} \mathbin{\mathrm{\mathring{,}}} A \mathbin{\mathrm{\mathring{,}}} \tilde{\alpha} \mathbin{\mathrm{\mathring{,}}} T \mathbin{\mathrm{\mathring{,}}} S_{\mathsf{c}_m}$ for some field automorphism α, translation T and almost Lorentz transformation A. Maps S_c, $\tilde{\alpha}$ and T do not change the facts whether $\mathsf{ev}_m(\bar{x})$ and $\mathsf{ev}_m(\bar{y})$ are simultaneous for both m and k; and whether they are separated orthogonally to the plane of motion of k according to m. Therefore, we can assume, without

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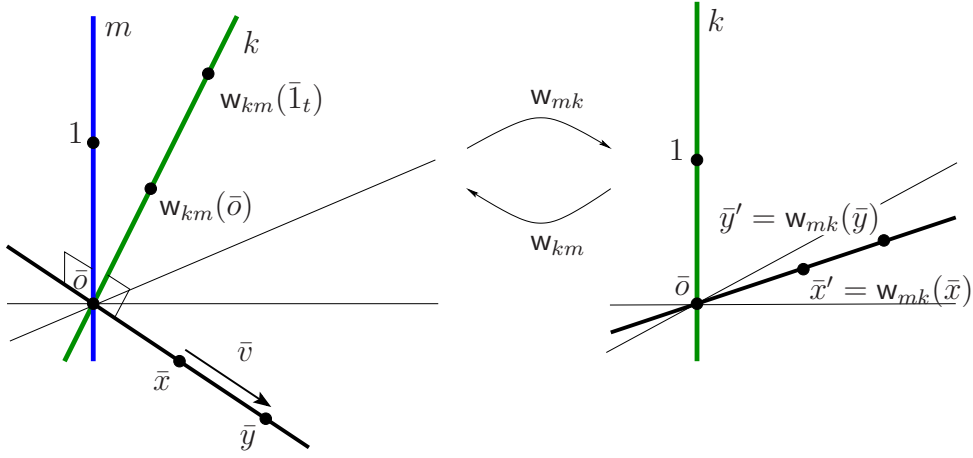[10]Specially, if $\mathsf{speed}_m^2(k) = 0$, the same events are simultaneous for m and k.

FIGURE 2. Illustration for the proof of Theorem 7.14

loss of generality, that w_{mk} is an almost Lorentz transformation. Then $\mathsf{w}_{km}(\bar{o}) = \bar{o}$. Therefore, events $\mathsf{ev}_m(\bar{x})$ and $\mathsf{ev}_m(\bar{y})$ are orthogonal to the plane of motion of k according to m iff $v_1 = 0$ and $\bar{v} \perp_e \mathsf{w}_{km}(\bar{1}_t)$. Let $\bar{v}' = \mathsf{w}_{mk}(\bar{v})$, then $\mathsf{ev}_m(\bar{x})$ and $\mathsf{ev}_m(\bar{y})$ are orthogonal to the plane of motion iff $\mathsf{w}_{km}(\bar{v}')_1 = 0$ and $\mathsf{w}_{km}(\bar{v}') \perp_e \mathsf{w}_{km}(\bar{1}_t)$. By Proposition 7.13, this is equivalent to $\mathsf{w}_{km}(\bar{v}')_1 = 0$ and $\bar{v}' = 0$. This means that $x_1 = y_1$ and $x_1' = y_1'$, i.e., that $\mathsf{ev}_m(\bar{x})$ and $\mathsf{ev}_m(\bar{y})$ are simultaneous both for m and k; and that is what we wanted to prove. ∎

Let $a \in Q$ such that $a \neq 0$. Let us introduce **dilation** D_a as the transformation mapping $\bar{x}$ to $a\bar{x}$ for all $\bar{x} \in Q^d$. It is clear that $D_a^{-1} = D_{1/a}$.

Lemma 7.15. Assume AxField. Let A be an almost Lorentz transformation such that $\mu_1^2(A(\bar{x})) = a^2\mu_1^2(\bar{x})$ for all $\bar{x} \in Q^d$. There are a unique Lorentz transformation L and a unique dilation D such that $A = D \,\fatsemi\, L = L \,\fatsemi\, D$.

Proof. Let L be $D_a^{-1} \,\fatsemi\, A$. L is a Lorentz transformation since

$$\mu_1^2\big(L(\bar{x})\big) = \mu_1^2\left(\frac{1}{a}A(\bar{x})\right) = \frac{1}{a^2}\mu_1^2\big(A(\bar{x})\big) = \frac{1}{a^2}a^2\mu_1^2(\bar{x}) = \mu_1^2(\bar{x}).$$

Therefore, $A = D_a \,\fatsemi\, L$ for Lorentz transformation L and dilation D_a. Since A is linear, $A = D_a^{-1} \,\fatsemi\, A \,\fatsemi\, D_a$. Thus $A = D_a^{-1} \,\fatsemi\, D_a \,\fatsemi\, L \,\fatsemi\, D_a = L \,\fatsemi\, D_a$.

If $A = D \,\fatsemi\, L$ for a Lorentz transformation L and dilation D, then D has to be D_a since $\mu_1^2\big(L(\bar{x})\big) = \mu_1^2(\bar{x})$ and $\mu_1^2\big(A(\bar{x})\big) = a^2\mu_1^2(\bar{x})$. Therefore, both D and L are unique in the decomposition of A. The same proof works when A is decomposed as $A = L \,\fatsemi\, D$. ∎

Lemma 7.16. Assume AxOField. Let $\bar{x}, \bar{y} \in Q^d$ such that $\mu_1^2(\bar{x}) > 0$ and $(\bar{x}, \bar{y})_{\mu_1^2} = 0$. Then $\mu_1^2(\bar{y}) < 0$.

Proof. Assume indirectly that $\mu_1^2(\bar{y}) \geq 0$, i.e., $y_1^2 \geq y_2^2 + \ldots + y_d^2$. Since $x_1^2 > x_2^2 + \ldots + x_d^2$, we have that $x_1^2 y_1^2 > (x_2^2 + \ldots x_d^2)(y_2^2 + \ldots + y_d^2)$. By Cauchy–Schwarz inequality[11] we have $(x_2^2 + \ldots x_d^2)(y_2^2 + \ldots + y_d^2) \geq (x_2 y_2 + \ldots + x_d y_d)^2$. Since $x_1 y_1 = x_2 y_2 + \ldots + x_d y_d$, we have that $x_1^2 y_1^2 > (x_1 y_1)^2$. This contradiction proves that $\mu_1^2(\bar{y}) < 0$. ∎

Proposition 7.17. Let $d \geq 3$. Assume AxOField. Let A be an almost Lorentz transformation. Then there is a $\lambda > 0$ such that $\mu_1^2\big(A(\bar{x})\big) = \lambda \mu_1^2(\bar{x})$ for all $\bar{x} \in Q^d$.

Proof. Since A is an almost Lorentz transformation there is a $\lambda \neq 0$ such that $\mu_1^2\big(A(\bar{x})\big) = \lambda \mu_1^2(\bar{x})$ for all $\bar{x} \in Q^d$. We are going to prove that this λ has to be positive. Assume indirectly that $\lambda < 0$. Let $\bar{y} = \langle 0, 1, 0, \ldots, 0 \rangle$ and $\bar{z} = \langle 0, 0, 1, 0, \ldots, 0 \rangle$. Then $\mu_1^2(\bar{y}) = \mu_1^2(\bar{z}) = -1$ and $(\bar{y}, \bar{z})_{\mu_1^2} = 0$. Let $\bar{y}' = A(\bar{y})$ and $\bar{z}' = A(\bar{z})$. Then $\mu_1^2(\bar{y}') > 0$ and $\mu_1^2(\bar{z}') > 0$ since $\lambda < 0$; and $(\bar{y}', \bar{z}')_{\mu_1^2} = 0$ by Lemma 7.12. These properties of $\bar{y}'$ and $\bar{z}'$ contradict Lemma 7.16. Therefore, $\lambda > 0$. ∎

Remark 7.18. Proposition 7.17 is not valid if $d = 2$ since reflection $\sigma_{tx} : \langle t, x \rangle \mapsto \langle x, t \rangle$ is an almost Lorentz transformation and $\mu_1^2\big(\sigma_{tx}(\bar{x})\big) = -\mu_1^2(\bar{x})$ for all $\bar{x} \in Q^2$.

Proposition 7.19. Let $d \geq 3$. Assume that $\langle Q, +, \cdot \rangle$ is a Euclidean field. Then every almost Lorentz transformation is a composition of a Lorentz transformation and a dilation. ∎

Proof. The statement follows from Lemma 7.15 and Proposition 7.17 since in Euclidean fields every positive number has a square root. ∎

Remark 7.20. Proposition 7.19 does not remain valid over arbitrary ordered fields. To construct a counterexample, let $d = 4$, $\langle Q, +, \cdot, \leq \rangle$ be the ordered field of rational numbers, and let A be the following linear map $A(\bar{x}) = \big\langle \frac{3x_1 + x_2}{2}, \frac{x_1 + 3x_2}{2}, x_3 - x_4, x_3 + x_4 \big\rangle$ for all $\bar{x} \in \mathbb{Q}^4$. It is straightforward to check that $\mu_1^2\big(A(\bar{x})\big) = 2\mu_1^2(\bar{x})$ for all $\bar{x} \in \mathbb{Q}^4$; so A is an almost Lorentz transformation. However, A cannot be the composition of a dilation D and a Lorentz transformation L over the field of rational numbers since then A would also be the composition of D and L over the field of real numbers; and, by Lemma 7.15, the dilation in the unique decomposition of A over the field of real numbers is $D_{\sqrt{2}}$, which does not map $\mathbb{Q}^4$ to $\mathbb{Q}^4$.

Now we are ready to prove Theorem 3.1. In Theorem 7.21 we prove a slightly stronger result since we will not use axiom AxSelf.

Theorem 7.21. Let $d \geq 3$. Assume AxOField, AxEv, AxPh, and AxSymD. Let $m, k \in \mathsf{IOb}$. Then w_{mk} is a Poincaré transformation.

[11]For a simple proof of Cauchy–Schwarz inequality that works also in ordered fields, see [1, §17].

Proof. Since, by AxSymD, the speed of light is 1 according to every inertial observer, w_{mk} is a composition of an almost Lorentz transformation A, a field-automorphism-induced map $\tilde{\alpha}$ and a translation T by Theorem 7.8. Specially, w_{mk} maps lines to lines.

By AxOField, there is a line l orthogonal to the plane of motion of k according to m. By Theorem 7.14, both l and $\mathsf{w}_{mk}[l]$ are horizontal. Therefore, by AxSymD, w_{mk} maps l to $\mathsf{w}_{mk}[l]$ preserving the squared Euclidean distances of the points of l. Let $\bar{v}$ be a direction vector of l.[12] Then, by axiom AxSymD, we have that

$$(15) \qquad \mathsf{length}^2(x\bar{v}) = \mathsf{length}^2\big(\tilde{\alpha}(A(x\bar{v}))\big)$$

for all $x \in Q$ since both $x\bar{v}$ and $\tilde{\alpha}\big(A(x\bar{v})\big)$ are horizontal vectors. Since both l and $\mathsf{w}_{mk}[l]$ are horizontal, we have that

$$(16) \qquad \mu_1^2(\bar{v}) = \mathsf{length}^2(\bar{v}) \quad \text{and} \quad \mu_1^2\big(\tilde{\alpha}(A(\bar{v}))\big) = \mathsf{length}^2\big(\tilde{\alpha}(A(\bar{v}))\big).$$

Since A is an almost Lorentz transformation, there is a $\lambda \neq 0$ such that

$$(17) \qquad \mu_1^2\big(A(\bar{x})\big) = \lambda \mu_1^2(\bar{x})$$

for all $\bar{x} \in Q^d$. Thus

$$(18) \quad \mathsf{length}^2\big(\tilde{\alpha}(A(\bar{v}))\big) \overset{(16)}{=} \mu_1^2\big(\tilde{\alpha}(A(\bar{v}))\big) = \alpha\big(\mu_1^2(A(\bar{v}))\big) \overset{(17)}{=} \alpha\big(\lambda\mu_1^2(\bar{v})\big)$$

$$= \alpha(\lambda)\alpha\big(\mu_1^2(\bar{v})\big) \overset{(16)}{=} \alpha(\lambda)\alpha\big(\mathsf{length}^2(\bar{v})\big)$$

Therefore, by the fact that that $\mathsf{length}^2(a\bar{y}) = a^2\mathsf{length}^2(\bar{y})$ and Equations (15) and (18), we get

$$(19) \qquad x^2\mathsf{length}^2(\bar{v}) = \alpha(\lambda)\alpha(x)^2\alpha\big(\mathsf{length}^2(\bar{v})\big)$$

for all $x \in Q$. Specially,

$$(20) \qquad \mathsf{length}^2(\bar{v}) = \alpha(\lambda)\alpha\big(\mathsf{length}^2(\bar{v})\big)$$

by choosing $x = 1$ in equation (19). Equations (19) and (20) imply that $x^2 = \alpha(x)^2$ for all $x \in Q$. Consequently, $\alpha = \mathsf{Id}_Q$ by Lemma 7.9. Thus $\tilde{\alpha} = \mathsf{Id}_{Q^d}$ and $1 = \alpha(\lambda)$ by equation (19). So $\lambda = 1$, i.e., A is a Lorentz transformation.

So $\tilde{\alpha}$ has to be the identity map and A has to be a Lorentz transformation. Thus w_{mk} is a composition of a Lorentz transformation and a translation, i.e., it is a Poincaré transformation as it was stated. ∎

8. Concluding remarks

We have seen that the possible structures of quantities strongly depend on the other axioms of spacetime. Typically, axioms requiring the existence of additional observers reduce the possible structures of quantities, see Theorems 3.6, 3.8, 5.1 and Proposition 6.1. We have proved several propositions about the connection between spacetime

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[12]That is, $\bar{v} = \bar{y} - \bar{x}$ for two distinct points $\bar{x}$ and $\bar{y}$ of l.

axioms and the possible structures of numbers. However, there are still great many open questions in this research area, see Questions 3.5, 3.10, 3.11, 5.4, 6.3 at pages 8, 11, 11, 15, 17, and Conjecture 3.14 at page 12.

9. Acknowledgments

References

[1] M. Aigner and G. M. Ziegler. *Proofs from The Book*. Springer-Verlag, Berlin, third edition, 2004. Including illustrations by Karl H. Hofmann.

[2] H. Andréka, J. X. Madarász, and I. Németi, with contributions from: A. Andai, G. Sági, I. Sain, and Cs. Tőke. *On the logical structure of relativity theories*. Research report, Alfréd Rényi Institute of Mathematics, Hungar. Acad. Sci., Budapest, 2002. http://www.math-inst.hu/pub/algebraic-logic/Contents.html.

[3] H. Andréka, J. X. Madarász, and I. Németi. Logical axiomatizations of spacetime. Samples from the literature. In A. Prékopa and E. Molnár, editors, *Non-Euclidean geometries*, pages 155–185. Springer-Verlag, New York, 2006.

[4] H. Andréka, J. X. Madarász, and I. Németi. Logic of space-time and relativity theory. In M. Aiello, I. Pratt-Hartmann, and J. van Benthem, editors, *Handbook of spatial logics*, pages 607–711. Springer-Verlag, Dordrecht, 2007.

[5] H. Andréka, J. X. Madarász, I. Németi, and G. Székely. A logic road from special relativity to general relativity. *Synthese*, pages Online–first: 1–17, 2011.

[6] H. Andréka, J. X. Madarász, I. Németi, and G. Székely. Square roots and special relativity, 2012. in preparation.

[7] C. C. Chang and H. J. Keisler. *Model theory*. North-Holland Publishing Co., Amsterdam, 1990.

[8] B. I. Dahn and H. Wolter. On the theory of exponential fields. *Z. Math. Logik Grundlag. Math.*, 29(5):465–480, 1983.

[9] R. d'Inverno. *Introducing Einstein's relativity*. Oxford University Press, New York, 1992.

[10] H. B. Enderton. *A mathematical introduction to logic*. Academic Press, New York, 1972.

[11] L. Fuchs. *Partially ordered algebraic systems*. Pergamon Press, Oxford, 1963.

[12] R. Goldblatt. *Orthogonality and spacetime geometry*. Springer-Verlag, New York, 1987.

[13] R. Hartshorne. *Geometry: Euclid and beyond*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2000.

[14] S. Kuhlmann. *Ordered exponential fields*, volume 12 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2000.

[15] J. X. Madarász, I. Németi, and G. Székely. Twin paradox and the logical foundation of relativity theory. *Found. Phys.*, 36(5):681–714, 2006.

[16] J. X. Madarász and G. Székely. Special relativity over the field of rational numbers, 2012. in preparation.

[17] D. Marker. Model theory of differential fields. In *Model theory, algebra, and geometry*, volume 39 of *Math. Sci. Res. Inst. Publ.*, pages 53–63. Cambridge Univ. Press, Cambridge, 2000.

[18] Alexander V. Mikhalev and Günter F. Pilz, editors. *The concise handbook of algebra*. Kluwer Academic Publishers, Dordrecht, 2002.

[19] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. W. H. Freeman and Co., San Francisco, 1973.

[20] W. Rindler. *Relativity. Special, General, and Cosmological*. Oxford University Press, New York, second edition, 2006.

[21] A. Robinson. On the concept of a differentially closed field. *Bull. Res. Council Israel Sect. F*, 8F:113–128 (1959), 1959.

[22] E. E. Rosinger. Two essays on the archimedean versus non-archimedean debate, 2008. arXiv:0809.4509v3.

[23] E. E. Rosinger. Special relativity in reduced power algebras, 2009. arXiv:0903.0296v1.

[24] E. E. Rosinger. Cosmic contact to be, or not to be archimedean. *Prespacetime Journal*, 2(2):234–248, 2011.

[25] E. E. Rosinger. How far should the principle of relativity go? *Prespacetime Journal*, 2(2):249–264, 2011.

[26] G. Székely. *First-Order Logic Investigation of Relativity Theory with an Emphasis on Accelerated Observers*. PhD thesis, Eötvös Loránd Univ., Budapest, 2009.

[27] G. Székely. What numbers we need to model accelerated observers in relativity?, 2012. in preparation.

[28] A. Tarski. *A decision method for elementary algebra and geometry*. University of California Press, Berkeley and Los Angeles, Calif., 1951.

[29] A. Tarski. What is elementary geometry? In *The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957-Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski)*, pages 16–29, Amsterdam, 1959. North-Holland Publishing Co.

[30] P. G. Vroegindewey. An algebraic generalization of a theorem of E. C. Zeeman. *Indag. Math.*, 36(1):77–81, 1974.

[31] P. G. Vroegindewey, V. Kreinovic, and O. M. Kosheleva. An extension of a theorem of A. D. Aleksandrov to a class of partially ordered fields. *Indag. Math.*, 41(3):363–376, 1979.