

# Relativity of arithmetics as a fundamental symmetry of physics

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Arithmetic operations can be defined in various ways, even if one assumes commutativity and associativity of addition and multiplication, and distributivity of multiplication with respect to addition. In consequence, whenever one encounters ‘plus’ or ‘times’ one has certain freedom of interpreting this operation. This leads to some freedom in definitions of derivatives, integrals and, thus, practically all equations occurring in natural sciences. A change of realization of arithmetics, without altering the remaining structures of a given equation, plays the same role as a symmetry transformation. An appropriate construction of arithmetics turns out to be particularly important for dynamical systems in fractal space-times. Simple examples from classical and quantum, relativistic and nonrelativistic physics are discussed.

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Symmetries of physical systems can be rather obvious or very abstract. Lorentz transformations, discovered as a formal symmetry of Maxwell’s equations, seemed abstract until their physical meaning was understood by Einstein. Theory of group representations, the cornerstone of quantum mechanics and field theory, had its roots in Lie’s studies of abstract symmetries of differential equations.

Einstein’s relativity, gauge invariance, Noether’s theorems, or supersymmetry are prominent examples of symmetry principles in physics. Theory of group representations has taught us that differences in mathematical realizations of a symmetry may directly reflect physical differences. Here we discuss a new type of symmetry, occurring in any physical theory: The symmetry of mathematical equations under modifications of arithmetic operations, the induced modifications of derivatives and integrals included. Similarly to other physical symmetries, the symmetry maintains the form of relevant equations, but may possess different mathematical realizations. Fractal structures provide nontrivial examples. A generalized arithmetics can lead to nontrivial continuous dynamics in sets of measure zero, invisible from the point of view of quantum mechanics. It opens a new room for phenomena ‘coming out from nowhere’, such as dark energy.

To begin with, let us consider a bijection  $f : X \rightarrow Y \subset \mathbb{R}$ , where  $X$  is some set. The map  $f$  allows us to define addition, multiplication, subtraction, and division in  $X$ ,

$$x \oplus y = f^{-1}(f(x) + f(y)), \quad (1)$$

$$x \ominus y = f^{-1}(f(x) - f(y)), \quad (2)$$

$$x \odot y = f^{-1}(f(x)f(y)), \quad (3)$$

$$x \oslash y = f^{-1}(f(x)/f(y)). \quad (4)$$

One easily verifies the standard properties [1]: (1) associativity  $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ ,  $(x \odot y) \odot z = x \odot (y \odot z)$ , (2) commutativity  $x \oplus y = y \oplus x$ ,  $x \odot y = y \odot x$ , (3) distributivity  $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ . Elements  $0, 1 \in X$  are defined by  $0 \oplus x = x$ ,  $1 \otimes x = x$ , which

implies  $f(0) = 0$ ,  $f(1) = 1$ . One further finds  $x \ominus x = 0$ ,  $x \oslash x = 1$ , as expected [2]. In general, it is better to define subtraction independently of addition since it may happen that  $f(-x)$  is undefined (think of the important example [3–5] of the Cantor function  $f$ , where  $X$  is the Cantor subset of  $[0, 1]$  and  $Y = [0, 1]$ ). If  $0 \ominus x$  exists, one can denote it by  $\ominus x$ .

Practically the only difference between  $\oplus$ ,  $\ominus$ ,  $\odot$ ,  $\oslash$  and  $+$ ,  $-$ ,  $\cdot$ , and  $/$  is that in general multiplication is not just a repeated addition:  $x \oplus x \neq 2 \odot x$ . Multiplication and addition are now truly independent.

Having all these arithmetic operations one can define a derivative of a function  $A : X \rightarrow X$ ,

$$\frac{d_f A(x)}{d_f x} = \lim_{h \rightarrow 0} \left( A(x \oplus h) \ominus A(x) \right) \oslash h, \quad (5)$$

satisfying

$$\frac{d_f A(x) \odot B(x)}{d_f x} = \frac{d_f A(x)}{d_f x} \odot B(x) \oplus A(x) \odot \frac{d_f B(x)}{d_f x}, \quad (6)$$

$$\frac{d_f A(x) \oplus B(x)}{d_f x} = \frac{d_f A(x)}{d_f x} \oplus \frac{d_f B(x)}{d_f x}, \quad (7)$$

$$\frac{d_f A[B(x)]}{d_f x} = \frac{d_f A[B(x)]}{d_f B(x)} \odot \frac{d_f B(x)}{d_f x}. \quad (8)$$

Now consider functions  $F : Y \rightarrow Y$  and  $F_f : X \rightarrow X$  related by

$$F_f(x) = f^{-1}\left(F(f(x))\right). \quad (9)$$

Employing (5) and the fact that  $f(0) = 0$  one finds

$$\frac{d_f F_f(x)}{d_f x} = f^{-1}\left(F'(f(x))\right), \quad (10)$$

where  $F'(y) = dF/dy$  is the usual derivative in  $Y$ , defined in terms of  $+$ ,  $-$ ,  $\cdot$ , and  $/$ . It is extremely important to note that (10) has been derived with no need of differentiability of  $f$ .  $f(0) = 0$  is enough to obtain a well defined

derivative. (10) is not the standard formula known for composite functions since no derivatives of  $f$  occur. To understand why functions of the form (9) are so essential let us solve the differential equation

$$\frac{d_f A(x)}{d_f x} = A(x), \quad A(0) = 1 \quad (11)$$

by assuming that  $A(x) = \oplus_{n=0}^{\infty} a_n \odot x^{\odot n}$ , where  $x^{\odot n} = x \odot \cdots \odot x$  ( $n$  times). Then, comparing term by term, one finds the unique solution

$$A(x) = f^{-1}(e^{f(x)}) = \exp_f(x), \quad (12)$$

fulfilling

$$\exp_f(x \oplus y) = \exp_f x \odot \exp_f y. \quad (13)$$

Its inverse is

$$\ln_f x = f^{-1}(\ln f(x)), \quad (14)$$

$$\ln_f(x \odot y) = \ln_f x \oplus \ln_f y. \quad (15)$$

As our next example consider a classical harmonic oscillator

$$\frac{d_f^2 x(t)}{d_f^2 t} = \frac{d_f}{d_f t} \frac{d_f x(t)}{d_f t} = \ominus \omega^{\odot 2} \odot x(t) \quad (16)$$

where  $\omega^{\odot 2} = \omega \odot \omega$ . The minus sign has to have a precise meaning so here we assume that  $-f(x) = f(-x)$ . Setting  $x(t) = \oplus_{n=0}^{\infty} a_n \odot t^{\odot n}$ , one obtains

$$x(t) = C_1 \odot \sin_f(\omega \odot t) \oplus C_2 \odot \cos_f(\omega \odot t) \quad (17)$$

where

$$\sin_f x = f^{-1}(\sin f(x)), \quad \cos_f x = f^{-1}(\cos f(x)), \quad (18)$$

and  $C_1, C_2$  are constants. An instructive exercise is to plot phase-space trajectories of the harmonic oscillator corresponding to various choices of  $f$ . Fig. 1 shows the trajectories for the Cantor function (extended to all reals by  $f(-x) = -f(x)$ ), and  $f(x) = x^n$ , with  $n = 1, 3, 5$ . All these trajectories represent a classical harmonic oscillator that satisfies the usual law of ‘force oppositely proportional to displacement’, with conserved energy ‘ $\dot{x}^2 + \omega^2 x^2$ ’, but with different meanings of ‘plus’ and ‘times’.

One might still have the impression that what we do is just standard physics in nonstandard coordinates. So, consider the problem of a fractal Universe. Assume that we live in a Universe of dimension  $4 - \epsilon$ , for example a Cartesian product of fractals of the Cantor set variety. Our physical equations have to be formulated in terms of notions that are intrinsic to the Universe, but what should be meant by a velocity, say? We have to subtract positions and divide by time, but we have to do it in a way that is intrinsic to the Universe we live in. Moreover, from our perspective positions and flow of time seem continuous even if they would appear discontinuous from an

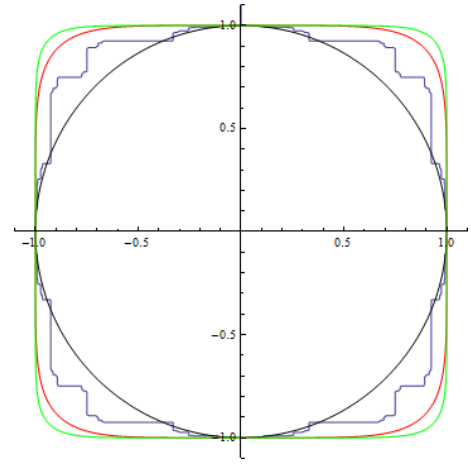


FIG. 1: [Color online] Phase-space trajectories of the harmonic oscillator with  $\omega = 1$  and  $f(x) = x$  (black),  $f(x) = x^3$  (red),  $f(x) = x^5$  (green), and the Cantor function of the Cantor line (blue). Taking  $f(x) = x^n$  with sufficiently large  $n$  we would find a dynamics looking like a motion along a square.

exactly 4-dimensional perspective. We should not make the usual step and turn to fractional derivatives [6], since for inhabitants of the Cantorian  $(4 - \epsilon)$ -dimensional Universe the velocity is just the first derivative of position with respect to time, and not some derivative of order  $0 < \alpha < 1$ .

Let us concentrate on the triadic Cantor set. In what follows we first extend the Cantor set from  $[0, 1]$  to the Cantor half line or the Cantor line. The corresponding  $f$ s will be collected under the name of the Cantor function. Now take any point  $x = \sum_{k=-\infty}^n a_k 3^k$ ,  $a_k = 0, 2$  (i.e. whose ternary expansion contains only 0s and 2s), replace all 2s by 1s and treat the resulting sequence of bits as a binary number,

$$f(x) = f\left(\sum_{k=-\infty}^n a_k 3^k\right) = \frac{1}{2} \sum_{k=-\infty}^n a_k 2^k, \quad (19)$$

$$f^{-1}(y) = f^{-1}\left(\sum_{k=-\infty}^n b_k 2^k\right) = 2 \sum_{k=-\infty}^n b_k 3^k. \quad (20)$$

The inverse image  $f^{-1}(\mathbb{R}_+) = C_+$  defines the Cantor half-line. Extending  $f^{-1}$  to the whole of  $\mathbb{R}$  by  $f^{-1}(-x) = -f^{-1}(x)$  one obtains the Cantor line  $f^{-1}(\mathbb{R}) = C$ . Fig. 2 illustrates Cantorian  $\sin_f$  and  $\cos_f$ .

An integral is defined so that the fundamental law of calculus,

$$\int_a^b \frac{d_f A(x)}{d_f x} \odot d_f x = A(b) \ominus A(a), \quad (21)$$

holds true. Its explicit form reads

$$\int_a^b F_f(x) \odot d_f x = f^{-1}\left(\int_{f(a)}^{f(b)} F(y) dy\right), \quad (22)$$

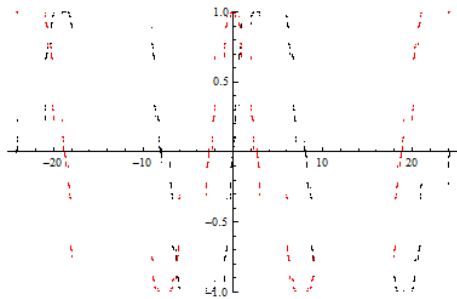


FIG. 2: [Color online] Cantor-world oscillation in Cantorian time.  $\sin_f(t)$  (black) and  $\cos_f(t)$  (red) for the Cantor function  $f$ . Inhabitants of the Cantor-dust space-time would experience this as a continuous process.

where  $\int F(y)dy$  is the standard (say, Lebesgue) integral in  $\mathbb{R}$ .

The integral so defined is not equivalent to the fractal measure. Indeed, fractal measure of the Cantor set embedded in an interval of length  $L$  is  $L^D$ , where  $D = \log_3 2$ . Thus, for  $L = 1/3$  one finds  $L^D = 1/2$ . Since segments  $[0, 1/3]$  and  $[2/3, 1]$  both have  $L = 1/3$  they both have the same  $D$ -dimensional volume equal  $1/2$ . Taking  $F_f(x) = 1$  we find

$$\int_a^b d_f x = \int_a^b \frac{d_f x}{d_f x} \odot d_f x = f^{-1}(f(b) - f(a)), \quad (23)$$

and  $\int_0^{1/3} d_f x = 1/3$ , but  $\int_{2/3}^1 d_f x = 2/3$ . Hence, the third  $1/3$  of the Cantor set is twice ‘longer’ than the first one even though they look the same as subsets of  $[0, 1]$ . The middle  $1/3$  has measure 0 since  $f(0.2_3) - f(0.0(2)_3) = 0.1_2 - 0.0(1)_2 = 0$  and  $f^{-1}(0) = 0$ .

Now let us switch to higher dimensional examples. First consider the plane, i.e. the Cartesian product of two lines. One checks that  $\sin_f^{\odot 2} x \oplus \cos_f^{\odot 2} x = 1$ ,  $\cosh_f^{\odot 2} x \ominus \sinh_f^{\odot 2} x = 1$ . Moreover,  $\sin_f$ ,  $\cos_f$ ,  $\sinh_f$ ,  $\cosh_f$ , functions satisfy the basic standard formulas such as

$$\sin_f(a \oplus b) = \sin_f a \odot \cos_f b \oplus \cos_f a \odot \sin_f b \quad (24)$$

and the like. Accordingly,

$$x' = x \odot \cos_f \alpha \oplus y \odot \sin_f \alpha, \quad (25)$$

$$y' = y \odot \cos_f \alpha \ominus x \odot \sin_f \alpha, \quad (26)$$

defines a rotation. The rotation satisfies the usual group composition rule, a fact immediately implying that one can work with generalized-arithmetics matrix equations. In an analogous way one arrives at Lorentz transformations in Cantorian Minkowski space, the Cartesian product of four Cantor lines with the invariant form  $x_0^{\odot 2} \oplus x_1^{\odot 2} \oplus x_2^{\odot 2} \oplus x_3^{\odot 2}$ . Fig. 3 shows three proper-time hyperbolas in 1+1 dimensional Cantorian Minkowski space. This is a Cantorian analogue of the empty-universe limit of the Friedmann-Lemaitre-Robertson-Walker space-time. The

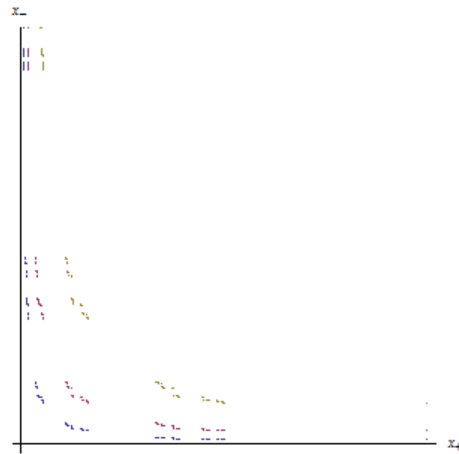


FIG. 3: [Color online] Three proper-time hyperbolas from Cantorian foliation of interior of the future light cone in 1+1 dimensional Cantorian Minkowski space. Generators of the light-cone are given by Cantorian half-lines

axes correspond to  $x_+ = x_0 \oplus x_1$  and  $x_- = x_0 \ominus x_1$  light-cone coordinates. In order to obtain the time coordinate  $x_0$  one first computes

$$x_+ \oplus x_- = x_0 \oplus x_0 = f^{-1}(2f(x_0)) = f^{-1}(2) \odot x_0 \quad (27)$$

and then  $\odot$ -divides by  $f^{-1}(2) \in C_+$ .

Arithmetics of complex numbers requires some care. One should not just take  $f : \mathbb{C} \rightarrow \mathbb{C}$  due to the typical multi-valuedness of  $f^{-1}$  and the resulting ill-definiteness of  $\oplus$  and  $\odot$ . Definition of  $i$  as a  $\pi/2$  rotation also does not properly work since one cannot guarantee a correct behaviour of  $i^{\odot n}$  if  $f$  is nonlinear. The correct solution is the simplest one: One should treat complex numbers as pairs of reals satisfying the following arithmetics

$$(x, y) \oplus (x', y') = (x \oplus x', y \oplus y'), \quad (28)$$

$$(x, y) \odot (x', y') = (x \odot x' \ominus y \odot y', y \odot x' \oplus x \odot y') \quad (29)$$

$$i = (0, 1), \quad (30)$$

supplemented by conjugation  $(x, y)^* = (x, -y)$ . As stressed in [7], the resulting complex structure is just the standard one, but no mysterious ‘imaginary number’ is employed.

In this way we have arrived at quantum mechanics. As our final example let us solve the eigenvalue problem for a 1-dimensional harmonic oscillator. Consider

$$\begin{aligned} \hat{H}_f \psi_f(x) &= -\alpha^{\odot 2} \odot \frac{d_f^2 \psi_f(x)}{d_f^2 x} \oplus \beta^{\odot 2} \odot x^{\odot 2} \odot \psi_f(x) \\ &= E_f \odot \psi_f(x), \end{aligned} \quad (31)$$

where  $\alpha, \beta$  are parameters. The normalized ground state is

$$\psi_{0f}(x) = f^{-1} \left( \left( \frac{f(\beta)}{\pi f(\alpha)} \right)^{1/4} e^{-\frac{f(\beta)f(x)^2}{2f(\alpha)}} \right), \quad (32)$$

with the eigenvalue  $E_{0f} = \alpha \odot \beta$ . The excited states can be derived in the usual way.

There are two peculiarities of the resulting quantum mechanics one should be aware of. First of all, if  $f$  is a Cantor-like function representing a fractal whose dimension is less than 1, then the real-line Lebesgue measure of the fractal is zero. Keeping in mind that states in quantum mechanics are represented by equivalence classes of wave functions that are identical up to sets of measure zero, we can remove the Cantor set, from  $\mathbb{R}$  without altering standard quantum mechanics. Having removed the Cantor line  $C$  from  $\mathbb{R}$  we still can do ordinary quantum mechanics on  $\mathbb{R} \setminus C$ , whereas  $C$  itself can become a universe for its own, Cantorian theory. Removing  $C$  from  $\mathbb{R}$  does not mean that we impose some fractal-like boundary conditions or that we consider a Schrödinger equation with a delta-peaked potential of Cantor-set support [8]. We just use the freedom to modify wave functions on sets of measure zero. So we can keep the standard Gaussian  $f(x) = x$  ground state on  $\mathbb{R} \setminus C$ , and employ the Cantorian  $\psi_{0f}(x)$  on  $C$ . According to quantum mechanics the resulting wave function belongs to the same equivalence class as the usual Gaussian, and thus represents the same state. However, now the energy is  $\hbar\omega/2 + \alpha \odot \beta$ , with  $\alpha \odot \beta$  ‘appearing from nowhere’. The analogy to dark energy is evident. The additional energy is a real number so it can be added to  $\hbar\omega/2$ , similarly to many other energies that occur in physics and are additive in spite of unrelated origins.

The second subtlety concerns physical dimensions of various quantities occurring in  $f$ -generalized arithmetics. Even the simple case of  $\omega \odot t$  may imply a necessity of dimensionless  $\omega$  and  $t$  if  $f$  is sufficiently nontrivial. In general we have to work with dimensionless variables  $x$  in order to make  $f(x)$  meaningful. It is thus simplest to begin with reformulating all the ‘standard’ theories in dimensionless forms, similarly to  $c = 1$  and  $\hbar = 1$  conventions often employed in relativity and quantum theory.

Quantum mechanics has brought us to the issue of probability. An appropriate normalization is  $\oplus_k p_k = 1$  which, in virtue of  $f(1) = 1$ , implies  $\sum_k f(p_k) = \sum_k P_k = 1$ . We automatically obtain two coexisting but inequivalent sets of probabilities, in close analogy

to probabilities  $P_k$  and escort probabilities  $p_k = P_k^q$  occurring in generalized statistics and multifractal theory [9, 10]. Averages

$$\langle a \rangle_f = \oplus_k p_k \odot a_k = f^{-1} \left( \sum_k P_k f(a_k) \right), \quad (33)$$

have the form of Kolmogorov-Nagumo averages [10], which implies the usual bounds  $a_{\min} \leq \langle a \rangle_f \leq a_{\max}$ . From the point of view of modified arithmetics the constraints one should impose on escort probabilities and Kolmogorov-Nagumo averages are, though, completely different from those employed in nonextensive statistics and Rényi’s information theory [11], provided instead of additivity one has  $\oplus$ -additivity in mind. The Rényi’s linear or exponential  $f$  now can be replaced by a much wider class of  $f$ s, and the analogue of CHSH-Bell inequality is

$$| \langle AB \rangle \oplus \langle AB' \rangle \oplus \langle A'B \rangle \oplus \langle A'B' \rangle | \leq f^{-1}(2), \quad (34)$$

with  $f^{-1}(2) = 6$  for the Cantor function.

The modified calculus is as simple as the one one knows from undergraduate education. What may be nontrivial is to find  $f$  if  $X$  is a sufficiently ‘strange’ object. The case of the Cantor set was quite obvious, but the choice of  $f$  may be much less evident if  $X$  is a multifractal or a higher-dimensional fractal.

In order to conclude, let us return to Fig. 1. All the phase-space trajectories represent the *same* physical system: A harmonic oscillator satisfying the Newton equation  $d^2x/dt^2 = -\omega^2x$ , with the same physical parameters for each of the trajectories. So how come the trajectories are different? The answer is: Because the very form of Newton’s equation does not tell us what should be meant by ‘plus’ or ‘times’. This observation extends to any theory that employs arithmetics of real numbers. It would not be very surprising if some alternative arithmetics proved essential for Planck-scale physics, where fractal space-time is expected, or to biological modeling where fractal structures are ubiquitous.

I am indebted to D. Aerts, M. Kuna and J. Naudts for discussions.

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- [1] All proofs are given in the Supplemental Material.
  - [2] Keeping the same symbols for  $0, 1 \in X$  and  $0, 1 \in \mathbb{R}$  will not lead to confusion.
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## I. SUPPLEMENTAL MATERIAL

### A. Arithmetic operations

Explicit cross-checks:

$$\begin{aligned}
(x \oplus y) \oplus z &= f^{-1} [f(x \oplus y) + f(z)] \\
&= f^{-1} \left[ f \left[ f^{-1} (f(x) + f(y)) \right] + f(z) \right] \\
&= f^{-1} [f(x) + f(y) + f(z)] \\
&= f^{-1} \left[ f(x) + f \left[ f^{-1} (f(y) + f(z)) \right] \right] \\
&= f^{-1} [f(x) + f(y \oplus z)] \\
&= x \oplus (y \oplus z)
\end{aligned} \tag{35}$$

$$\begin{aligned}
(x \oplus y) \ominus z &= f^{-1} [f(x \oplus y) - f(z)] \\
&= f^{-1} \left[ f \left[ f^{-1} (f(x) + f(y)) \right] - f(z) \right] \\
&= f^{-1} [f(x) + f(y) - f(z)] \\
&= f^{-1} \left[ f(x) + f \left[ f^{-1} (f(y) - f(z)) \right] \right] \\
&= f^{-1} [f(x) + f(y \ominus z)] \\
&= x \oplus (y \ominus z)
\end{aligned} \tag{36}$$

$$\begin{aligned}
(x \odot y) \odot z &= f^{-1} [f(x \odot y)f(z)] \\
&= f^{-1} \left[ f \left[ f^{-1} (f(x)f(y)) \right] f(z) \right] \\
&= f^{-1} [f(x)f(y)f(z)] \\
&= f^{-1} \left[ f(x)f \left[ f^{-1} (f(y)f(z)) \right] \right] \\
&= f^{-1} [f(x)f(y \odot z)] \\
&= x \odot (y \odot z)
\end{aligned} \tag{37}$$

$$\begin{aligned}
(x \odot y) \oslash z &= f^{-1} [f(x \odot y)/f(z)] \\
&= f^{-1} \left[ f \left[ f^{-1} (f(x)f(y)) \right] / f(z) \right] \\
&= f^{-1} [f(x)f(y)/f(z)] \\
&= f^{-1} \left[ f(x)f \left[ f^{-1} (f(y)/f(z)) \right] \right] \\
&= f^{-1} [f(x)f(y \oslash z)] \\
&= x \odot (y \oslash z) \\
&= (x \oslash z) \odot y
\end{aligned} \tag{38}$$

Similarly one proves

$$(x \odot y) \oslash z = (x \oslash z) \odot y \tag{39}$$

Distributivity

$$\begin{aligned}
(x \oplus y) \odot z &= f^{-1} [f(x \oplus y)f(z)] \\
&= f^{-1} \left[ f \left[ f^{-1} (f(x) + f(y)) \right] f(z) \right] \\
&= f^{-1} [[f(x) + f(y)]f(z)] \\
&= f^{-1} [f(x)f(z) + f(y)f(z)] \\
&= f^{-1} [f(f^{-1}(f(x)f(z))) + f(f^{-1}(f(y)f(z)))] \\
&= f^{-1} [f(x \odot z) + f(y \odot z)] \\
&= (x \odot z) \oplus (y \odot z)
\end{aligned} \tag{40}$$

$$\begin{aligned}
(x \oplus y) \oslash z &= f^{-1} [f(x \oplus y)/f(z)] \\
&= f^{-1} \left[ f \left[ f^{-1} (f(x) + f(y)) \right] / f(z) \right] \\
&= f^{-1} [[f(x) + f(y)]/f(z)] \\
&= f^{-1} [f(x)/f(z) + f(y)/f(z)] \\
&= f^{-1} [f(f^{-1}(f(x)/f(z))) + f(f^{-1}(f(y)/f(z)))] \\
&= f^{-1} [f(x \oslash z) + f(y \oslash z)] \\
&= (x \oslash z) \oplus (y \oslash z)
\end{aligned} \tag{41}$$

## B. Derivatives

Definition

$$F^\circ(x) = \lim_{h \rightarrow 0} \left( F(x \oplus h) \ominus F(x) \right) \oslash h \tag{42}$$

Derivative of a sum

$$\begin{aligned}
[F \oplus G]^\circ(x) &= \lim_{h \rightarrow 0} \left( F(x \oplus h) \oplus G(x \oplus h) \ominus F(x) \ominus G(x) \right) \oslash h \\
&= \lim_{h \rightarrow 0} \left( F(x \oplus h) \ominus F(x) \oplus G(x \oplus h) \ominus G(x) \right) \oslash h \\
&= \lim_{h \rightarrow 0} [F(x \oplus h) \ominus F(x)] \oslash h \oplus [G(x \oplus h) \ominus G(x)] \oslash h \\
&= F^\circ(x) \oplus G^\circ(x)
\end{aligned} \tag{43}$$

The Leibnitz rule

$$\begin{aligned}
[F \odot G]^\circ(x) &= \lim_{h \rightarrow 0} \left( F(x \oplus h) \odot G(x \oplus h) \ominus F(x) \odot G(x) \right) \oslash h \\
&= \lim_{h \rightarrow 0} \left( F(x \oplus h) \odot G(x \oplus h) \ominus F(x) \odot G(x \oplus h) \oplus F(x) \odot G(x \oplus h) \ominus F(x) \odot G(x) \right) \oslash h \\
&= \lim_{h \rightarrow 0} [F(x \oplus h) \ominus F(x)] \oslash h \odot G(x \oplus h) \oplus F(x) \odot [G(x \oplus h) \ominus G(x)] \oslash h \\
&= F^\circ(x) \odot G(x) \oplus F(x) \odot G^\circ(x)
\end{aligned} \tag{44}$$

The chain rule

$$F[G]^\circ(x) = \lim_{h \rightarrow 0} \left( F[G(x \oplus h)] \ominus F[G(x)] \right) \oslash h \tag{45}$$

$$= \lim_{h \rightarrow 0} \left( F[G(x \oplus h)] \ominus F[G(x)] \right) \oslash [G(x \oplus h) \ominus G(x)] \odot [G(x \oplus h) \ominus G(x)] \oslash h \tag{46}$$

Denote  $g = G(x \oplus h) \ominus G(x)$ , so that

$$G(x \oplus h) = G(x) \oplus G(x \oplus h) \ominus G(x) = G(x) \oplus g \tag{47}$$

and

$$F[G(x)]^\circ = \lim_{g \rightarrow 0} \left( F[G(x) \oplus g] \ominus F[G(x)] \right) \oslash g \odot \lim_{h \rightarrow 0} [G(x \oplus h) \ominus G(x)] \oslash h \tag{48}$$

$$= F^\circ[G(x)] \odot G^\circ[x] \tag{49}$$

### C. Alternative proofs for derivatives

Consider

$$F_f(x) = f^{-1}\left(F(f(x))\right), \quad (50)$$

$$G_f(x) = f^{-1}\left(G(f(x))\right), \quad (51)$$

$$F_f \oplus G_f(x) = f^{-1}\left(f[F_f(x)] + f[G_f(x)]\right) \quad (52)$$

$$= f^{-1}\left(F(f(x)) + G(f(x))\right) \quad (53)$$

$$= f^{-1}\left((F + G)(f(x))\right), \quad (54)$$

$$F_f \odot G_f(x) = f^{-1}\left(f[F_f(x)]f[G_f(x)]\right) \quad (55)$$

$$= f^{-1}\left(F(f(x))G(f(x))\right) \quad (56)$$

$$= f^{-1}\left(FG(f(x))\right) \quad (57)$$

The derivative

$$F_f^\circ(x) = \lim_{h \rightarrow 0} \left( f^{-1}\left(F[f(x \oplus h)]\right) \ominus f^{-1}\left(F[f(x)]\right) \right) \oslash h \quad (58)$$

$$= \lim_{h \rightarrow 0} f^{-1}\left(f\left[f^{-1}\left(F[f(x \oplus h)]\right)\right] - f\left[f^{-1}\left(F[f(x)]\right)\right]\right) \oslash h \quad (59)$$

$$= \lim_{h \rightarrow 0} f^{-1}\left(F[f(x \oplus h)] - F[f(x)]\right) \oslash h \quad (60)$$

$$= \lim_{h \rightarrow 0} f^{-1}\left(F[f(x) + f(h)] - F[f(x)]\right) \oslash h \quad (61)$$

$$= \lim_{h \rightarrow 0} f^{-1}\left(f\left[f^{-1}\left(F[f(x) + f(h)] - F[f(x)]\right)\right] / f(h)\right) \quad (62)$$

$$= \lim_{h \rightarrow 0} f^{-1}\left(\left(F[f(x) + f(h)] - F[f(x)]\right) / f(h)\right) \quad (63)$$

$$= f^{-1}\left(\lim_{h \rightarrow 0} \left(F[f(x) + h] - F[f(x)]\right) / h\right) \quad (64)$$

$$= f^{-1}\left(F'[f(x)]\right) \quad (65)$$

Accordingly

$$[F_f \oplus G_f]^\circ(x) = f^{-1}\left((F' + G')(f(x))\right) \quad (66)$$

$$= f^{-1}\left(F'(f(x)) + G'(f(x))\right) \quad (67)$$

$$= f^{-1}\left(f[F_f^\circ(x)] + f[G_f^\circ(x)]\right) \quad (68)$$

$$= F_f^\circ \oplus G_f^\circ(x) \quad (69)$$

$$[F_f \odot G_f]^\circ(x) = f^{-1}\left((FG)'(f(x))\right) \quad (70)$$

$$= f^{-1}\left((F'G + FG')(f(x))\right) \quad (71)$$

$$= f^{-1}\left(F'(f(x))G(f(x)) + F(f(x))G'(f(x))\right) \quad (72)$$

$$= f^{-1}\left(f\left[f^{-1}[F'(f(x))G(f(x))]\right] + f\left[f^{-1}[F(f(x))G'(f(x))]\right]\right) \quad (73)$$

$$= f^{-1}\left[f^{-1}[F'(f(x))G(f(x))] \oplus f^{-1}[F(f(x))G'(f(x))]\right] \quad (74)$$

$$= f^{-1}\left[f\left[f^{-1}[F'(f(x))]\right]f\left[f^{-1}[G(f(x))]\right] \oplus f^{-1}\left[f\left[f^{-1}[F(f(x))]\right]f\left[f^{-1}[G'(f(x))]\right]\right] \quad (75)$$

$$= f^{-1}\left[f\left[F_f^\circ(x)\right]f\left[G_f(x)\right] \oplus f^{-1}\left[f\left[F_f(x)\right]f\left[G_f^\circ(x)\right]\right] \quad (76)$$

$$= F_f^\circ \odot G_f(x) \oplus F_f \odot G_f^\circ(x) \quad (77)$$

Composition of functions

$$F_f \circ G_f(x) = f^{-1}\left(F\left[f\left(f^{-1}\left(G(f(x))\right)\right)\right]\right) \quad (78)$$

$$= f^{-1}\left(F\left[G(f(x))\right]\right) \quad (79)$$

$$= f^{-1}\left(F \circ G(f(x))\right) \quad (80)$$

and its derivative

$$[F_f \circ G_f]^\circ(x) = f^{-1}\left((F \circ G)'(f(x))\right) \quad (81)$$

$$= f^{-1}\left(F'[G(f(x))]G'(f(x))\right) \quad (82)$$

$$= f^{-1}\left(f\left[f^{-1}\left(F'[G(f(x))]\right)\right]f\left[f^{-1}\left(G'(f(x))\right)\right]\right) \quad (83)$$

$$= f^{-1}\left(f\left[f^{-1}\left(F'f\left[f^{-1}\left([G(f(x))]\right)\right]\right)\right]f\left[f^{-1}\left(G'(f(x))\right)\right]\right) \quad (84)$$

$$= F_f^\circ\left(G_f(x)\right) \odot G_f^\circ(x) \quad (85)$$

#### D. Some explicit derivatives

Begin with  $F(x) = x$ . Then

$$F^\circ(x) = \lim_{h \rightarrow 0} \left(F(x \oplus h) \ominus F(x)\right) \odot h, \quad (86)$$

$$= \lim_{h \rightarrow 0} (x \oplus h \ominus x) \odot h, \quad (87)$$

$$= \lim_{h \rightarrow 0} h \odot h = 1 \quad (88)$$

Now  $F(x) = x^{\odot n}$ . Using the Leibnitz rule

$$\begin{aligned} F^\circ(x) &= \oplus_{k=1}^n x^{\odot(n-1)} = f^{-1}\left(\sum_{k=1}^n f(x^{\odot(n-1)})\right) \\ &= f^{-1}\left(nf(x)^{n-1}\right) = f^{-1}\left(f[f^{-1}(n)]f(x)^{n-1}\right) = f^{-1}(n) \odot x^{\odot(n-1)} \end{aligned} \quad (89)$$



### E. Exponent

Now solve  $F^\circ = F$ . Let

$$F(x) = \bigoplus_{n=0}^{\infty} a_n \odot x^{\odot n} \quad (90)$$

$$F^\circ(x) = \bigoplus_{n=1}^{\infty} a_n \odot \left( \bigoplus_{k=1}^n x^{\odot(n-k)} \right) \quad (91)$$

$$= \bigoplus_{n=1}^{\infty} \bigoplus_{k=1}^n a_n \odot x^{\odot(n-k)} \quad (92)$$

$$= a_1 \oplus \underbrace{a_2 \odot x \oplus a_2 \odot x}_{\oplus} \oplus \underbrace{a_3 \odot x^{\odot 2} \oplus a_3 \odot x^{\odot 2} \oplus a_3 \odot x^{\odot 2}}_{\oplus} \oplus \dots \quad (93)$$

$$= a_0 \oplus \underbrace{a_1 \odot x}_{\oplus} \oplus \underbrace{a_2 \odot x^{\odot 2}}_{\oplus} \oplus \dots \quad (94)$$

Therefore

$$a_0 = a_1, \quad (95)$$

$$a_1 \odot x = a_2 \odot x \oplus a_2 \odot x, \quad (96)$$

$$a_2 \odot x^{\odot 2} = a_3 \odot x^{\odot 2} \oplus a_3 \odot x^{\odot 2} \oplus a_3 \odot x^{\odot 2} \quad (97)$$

$$(98)$$

and so on. Explicitly,

$$f^{-1}\left(f(a_1)f(x)\right) = f^{-1}\left(f(a_2 \odot x) + f(a_2 \odot x)\right) = f^{-1}\left(2f[f^{-1}(f(a_2)f(x))]\right) = f^{-1}\left(2f(a_2)f(x)\right) \quad (99)$$

$$f(a_1)f(x) = 2f(a_2)f(x) \quad (100)$$

Let us start with  $a_0 = 1 = a_1$ . Since  $f(1) = 1$ , we get

$$a_0 = a_1 = 2f(a_2) = 1, \quad (101)$$

$$a_2 = f^{-1}(1/2) = 2/3 \quad (102)$$

Next consider  $a_2 \odot x^{\odot 2} = a_3 \odot x^{\odot 2} \oplus a_3 \odot x^{\odot 2} \oplus a_3 \odot x^{\odot 2}$

$$f^{-1}\left(f(a_2)f(x^{\odot 2})\right) = f^{-1}\left(3f(a_3 \odot x^{\odot 2})\right) = f^{-1}\left(3f(a_3)f(x^{\odot 2})\right) \quad (103)$$

$$f(a_2) = 3f(a_3) = 1/2 \quad (104)$$

$$a_3 = f^{-1}(1/3!) \quad (105)$$

Similarly  $a_n = f^{-1}(1/n!)$ . So

$$F(x) = \bigoplus_{n=0}^{\infty} a_n \odot x^{\odot n} \quad (106)$$

$$= \bigoplus_{n=0}^{\infty} f^{-1}(1/n!) \odot x^{\odot n} \quad (107)$$

$$= \bigoplus_{n=0}^{\infty} f^{-1}\left(f(x^{\odot n})/n!\right) \quad (108)$$

$$= \bigoplus_{n=0}^{\infty} f^{-1}\left(f(x)^n/n!\right) \quad (109)$$

$$= f^{-1}\left(\sum_{n=0}^{\infty} f(x)^n/n!\right) \quad (110)$$

$$= f^{-1}\left(e^{f(x)}\right) \quad (111)$$

### F. Harmonic oscillator

Consider

$$F^{\circ\circ}(x) = \ominus \omega^{\odot 2} \odot F(x) \quad (112)$$

with  $f(-x) = -f(x)$ , so that  $\ominus x = -x$ . Recall

$$\begin{aligned} (x^{\odot n})^{\circ} &= \oplus_{k=1}^n x^{\odot(n-1)} = f^{-1}\left(nf(x^{\odot(n-1)})\right) = f^{-1}\left(nf(x)^{n-1}\right) = f^{-1}(n) \odot x^{\odot(n-1)} \\ (x^{\odot n})^{\circ\circ} &= f^{-1}(n) \odot f^{-1}(n-1) \odot x^{\odot(n-2)} = f^{-1}(n(n-1)) \odot x^{\odot(n-2)} = f^{-1}\left(n(n-1)f(x)^{n-2}\right) \end{aligned} \quad (113)$$

Let

$$F(x) = \oplus_{n=0}^{\infty} a_n \odot x^{\odot n} \quad (114)$$

$$F^{\circ\circ}(x) = \oplus_{n=2}^{\infty} a_n \odot f^{-1}\left(n(n-1)f(x)^{n-2}\right) \quad (115)$$

$$= \oplus_{n=2}^{\infty} f^{-1}\left(f(a_n)n(n-1)f(x)^{n-2}\right) \quad (116)$$

$$= f^{-1}\left(\sum_{n=2}^{\infty} f(a_n)n(n-1)f(x)^{n-2}\right) \quad (117)$$

$$= -\omega^{\odot 2} \odot F(x) \quad (118)$$

$$= f^{-1}\left(-f(\omega)^2 f(F(x))\right) \quad (119)$$

$$= f^{-1}\left(-f(\omega)^2 \sum_{n=0}^{\infty} f(a_n)f(x)^n\right) \quad (120)$$

Then

$$-f(\omega)^2 \sum_{n=0}^{\infty} f(a_n)f(x)^n = \sum_{n=2}^{\infty} f(a_n)n(n-1)f(x)^{n-2} \quad (121)$$

$$= \sum_{m=0}^{\infty} f(a_{m+2})(m+2)(m+1)f(x)^m \quad (122)$$

$$= \sum_{n=0}^{\infty} f(a_{n+2})(n+2)(n+1)f(x)^n \quad (123)$$

and

$$-f(\omega)^2 f(a_n) = f(a_{n+2})(n+2)(n+1) \quad (124)$$

$$-f(\omega)^2 f(a_0) = f(a_2)2 \quad (125)$$

$$-f(\omega)^2 f(a_1) = f(a_3)3 \times 2 \quad (126)$$

$$-f(\omega)^2 f(a_2) = f(a_4)4 \times 3 \quad (127)$$

$$-f(\omega)^2 f(a_3) = f(a_5)5 \times 4 \quad (128)$$

$$-f(\omega)^2 f(a_4) = f(a_6)6 \times 5 \quad (129)$$

$$-f(\omega)^2 f(a_5) = f(a_7)7 \times 6 \quad (130)$$

$$(131)$$

Finally

$$f(a_7) = -\frac{f(\omega)^2}{7 \times 6} f(a_5) = \frac{f(\omega)^4}{7 \times 6 \times 5 \times 4} f(a_3) = (-)^3 \frac{f(\omega)^6}{7!} f(a_1) \quad (132)$$

$$f(a_6) = -\frac{f(\omega)^2}{6 \times 5} f(a_4) = \frac{f(\omega)^4}{6 \times 5 \times 4 \times 3} f(a_2) = (-)^3 \frac{f(\omega)^6}{6!} f(a_0) \quad (133)$$

$$f(a_5) = -\frac{f(\omega)^2}{5 \times 4} f(a_3) = (-)^2 \frac{f(\omega)^4}{5!} f(a_1) \quad (134)$$

$$f(a_4) = -\frac{f(\omega)^2}{4 \times 3} f(a_2) = (-)^2 \frac{f(\omega)^4}{4!} f(a_0) \quad (135)$$

$$f(a_3) = (-)^1 \frac{f(\omega)^2}{3!} f(a_1) \quad (136)$$

$$f(a_2) = (-)^1 \frac{f(\omega)^2}{2!} f(a_0) \quad (137)$$

and

$$F(x) = f^{-1}\left(\sum_{n=0}^{\infty} f(a_n)f(x)^n\right) \quad (138)$$

$$= f^{-1}\left(f(a_0) - \frac{f(\omega)^2}{2!}f(a_0)f(x)^2 + \frac{f(\omega)^4}{4!}f(a_0)f(x)^4 + \dots\right. \\ \left. + f(a_1)f(x) - \frac{f(\omega)^2}{3!}f(a_1)f(x)^3 + \frac{f(\omega)^4}{5!}f(a_1)f(x)^5 + \dots\right) \quad (139)$$

$$= f^{-1}\left(f(a_0)\left(1 - \frac{f(\omega)^2f(x)^2}{2!} + \frac{f(\omega)^4f(x)^4}{4!} + \dots\right)\right. \\ \left. + \frac{f(a_1)}{f(\omega)}\left(f(\omega)f(x) - \frac{f(\omega)^3f(x)^3}{3!} + \frac{f(\omega)^5f(x)^5}{5!} + \dots\right)\right) \quad (140)$$

$$= f^{-1}\left(f(a_0)\cos[f(\omega)f(x)] + \frac{f(a_1)}{f(\omega)}\sin[f(\omega)f(x)]\right) \quad (141)$$

$$= f^{-1}\left(f(a_0)\cos f(f^{-1}[f(\omega)f(x)]) + \frac{f(a_1)}{f(\omega)}\sin f(f^{-1}[f(\omega)f(x)])\right) \quad (142)$$

$$= f^{-1}\left(f(a_0)\cos f(\omega \odot x) + \frac{f(a_1)}{f(\omega)}\sin f(\omega \odot x)\right) \quad (143)$$

$$= f^{-1}\left(f\left(f^{-1}\left[f(a_0)\cos f(\omega \odot x)\right]\right) + f\left(f^{-1}\left[\frac{f(a_1)}{f(\omega)}\sin f(\omega \odot x)\right]\right)\right) \quad (144)$$

$$= f^{-1}\left(f\left(f^{-1}\left[f(a_0)\underbrace{f\left(f^{-1}[\cos f(\omega \odot x)]\right)}_{\cos_f(\omega \odot x)}\right]\right) + f\left(f^{-1}\left[\frac{f(a_1)}{f(\omega)}\underbrace{f\left(f^{-1}[\sin f(\omega \odot x)]\right)}_{\sin_f(\omega \odot x)}\right]\right)\right) \quad (145)$$

$$= a_0 \odot \cos_f(\omega \odot x) \oplus a_1 \otimes \omega \odot \sin_f(\omega \odot x) \quad (146)$$

### G. Properties of trigonometric and hyperbolic functions

Values at 0:

$$\sin_f(0) = f^{-1}(\sin f(0)) = f^{-1}(\sin 0) = f^{-1}(0) = 0, \quad (147)$$

$$\cos_f(0) = f^{-1}(\cos f(0)) = f^{-1}(\cos 0) = f^{-1}(1) = 1, \quad (148)$$

$$\sinh_f(0) = f^{-1}(\sinh f(0)) = f^{-1}(\sinh 0) = f^{-1}(0) = 0, \quad (149)$$

$$\cosh_f(0) = f^{-1}(\cosh f(0)) = f^{-1}(\cosh 0) = f^{-1}(1) = 1. \quad (150)$$

Pitagorean identity

$$\sin_f^{\odot 2} x \oplus \cos_f^{\odot 2} x = f^{-1}\left(f(\sin_f^{\odot 2} x) + f(\cos_f^{\odot 2} x)\right) \quad (151)$$

$$= f^{-1}\left(f(\sin_f x)^2 + f(\cos_f x)^2\right) \quad (152)$$

$$= f^{-1}\left(\sin^2 f(x) + \cos^2 f(x)\right) = f^{-1}(1) = 1 \quad (153)$$

Hyperbolic identity

$$\cosh_f^{\odot 2} x \ominus \sinh_f^{\odot 2} x = f^{-1}\left(f(\cosh_f^{\odot 2} x) - f(\sinh_f^{\odot 2} x)\right) \quad (154)$$

$$= f^{-1}\left(f(\cosh_f x)^2 - f(\sinh_f x)^2\right) \quad (155)$$

$$= f^{-1}\left(\cosh^2 f(x) - \sinh^2 f(x)\right) = f^{-1}(1) = 1 \quad (156)$$

## Formulas for sums of arguments

$$\sin_f(a \oplus b) = f^{-1} \left( \sin f \left( f^{-1}(f(a) + f(b)) \right) \right) \quad (157)$$

$$= f^{-1} \left( \sin (f(a) + f(b)) \right) \quad (158)$$

$$= f^{-1} \left( \sin f(a) \cos f(b) + \cos f(a) \sin f(b) \right) \quad (159)$$

$$= f^{-1} \left( f \left( f^{-1}(\sin f(a) \cos f(b)) \right) + f \left( f^{-1}(\cos f(a) \sin f(b)) \right) \right) \quad (160)$$

$$= f^{-1} (\sin f(a) \cos f(b) \oplus f^{-1}(\cos f(a) \sin f(b))) \quad (161)$$

$$= f^{-1} \left( f \left( f^{-1}(\sin f(a)) \right) f \left( f^{-1}(\cos f(b)) \right) \right) \oplus f^{-1} \left( f \left( f^{-1}(\cos f(a)) \right) f \left( f^{-1}(\sin f(b)) \right) \right) \quad (162)$$

$$= f^{-1} \left( f(\sin_f a) f(\cos_f b) \right) \oplus f^{-1} \left( f(\cos_f a) f(\sin_f b) \right) \quad (163)$$

$$= \sin_f a \odot \cos_f b \oplus \cos_f a \odot \sin_f b \quad (164)$$

$$\sin_f(a \ominus b) = f^{-1} \left( \sin f \left( f^{-1}(f(a) - f(b)) \right) \right) \quad (165)$$

$$= f^{-1} \left( \sin (f(a) - f(b)) \right) \quad (166)$$

$$= f^{-1} \left( \sin f(a) \cos f(b) - \cos f(a) \sin f(b) \right) \quad (167)$$

$$= f^{-1} \left( f \left( f^{-1}(\sin f(a) \cos f(b)) \right) - f \left( f^{-1}(\cos f(a) \sin f(b)) \right) \right) \quad (168)$$

$$= f^{-1} (\sin f(a) \cos f(b) \ominus f^{-1}(\cos f(a) \sin f(b))) \quad (169)$$

$$= f^{-1} \left( f \left( f^{-1}(\sin f(a)) \right) f \left( f^{-1}(\cos f(b)) \right) \right) \ominus f^{-1} \left( f \left( f^{-1}(\cos f(a)) \right) f \left( f^{-1}(\sin f(b)) \right) \right) \quad (170)$$

$$= f^{-1} \left( f(\sin_f a) f(\cos_f b) \right) \ominus f^{-1} \left( f(\cos_f a) f(\sin_f b) \right) \quad (171)$$

$$= \sin_f a \odot \cos_f b \ominus \cos_f a \odot \sin_f b \quad (172)$$

$$\cos_f(a \oplus b) = f^{-1} \left( \cos f \left( f^{-1}(f(a) + f(b)) \right) \right) \quad (173)$$

$$= f^{-1} \left( \cos (f(a) + f(b)) \right) \quad (174)$$

$$= f^{-1} \left( \cos f(a) \cos f(b) - \sin f(a) \sin f(b) \right) \quad (175)$$

$$= f^{-1} \left( f \left( f^{-1}(\cos f(a) \cos f(b)) \right) - f \left( f^{-1}(\sin f(a) \sin f(b)) \right) \right) \quad (176)$$

$$= f^{-1}(\cos f(a) \cos f(b)) \ominus f^{-1}(\sin f(a) \sin f(b)) \quad (177)$$

$$= f^{-1} \left( f \left( f^{-1}(\cos f(a)) \right) f \left( f^{-1}(\cos f(b)) \right) \right) \ominus f^{-1} \left( f \left( f^{-1}(\sin f(a)) \right) f \left( f^{-1}(\sin f(b)) \right) \right) \quad (178)$$

$$= f^{-1} \left( f(\cos_f a) f(\cos_f b) \right) \ominus f^{-1} \left( f(\sin_f a) f(\sin_f b) \right) \quad (179)$$

$$= \cos_f a \odot \cos_f b \ominus \sin_f a \odot \sin_f b \quad (180)$$

and so on.

## H. Integral

Assume integrals are defined as inverses of the derivative

$$\int F_f^\circ(x) \odot dx = F_f(x) \oplus \text{const} \quad (181)$$

$$\int_a^b F_f^\circ(x) \odot dx = F_f(b) \ominus F_f(a) \quad (182)$$

For

$$F_f^\circ(x) = f^{-1} \left( F'(f(x)) \right) \quad (183)$$

we get

$$\int_a^b F_f^\circ(x) \odot dx = f^{-1} \left( F(f(b)) \right) \ominus f^{-1} \left( F(f(a)) \right) \quad (184)$$

$$= f^{-1} \left( F(f(b)) - F(f(a)) \right) \quad (185)$$

$$= f^{-1} \left( \int_{f(a)}^{f(b)} F'(x) dx \right) \quad (186)$$

So, for a general

$$F_f(x) = f^{-1} \left( F(f(x)) \right) \quad (187)$$

we define

$$\int_a^b F_f(x) \odot dx = f^{-1} \left( \int_{f(a)}^{f(b)} F(x) dx \right) \quad (188)$$

Let us cross-check

$$\left( \int_a^x F_f(y) \odot dy \right)^\circ = \left[ f^{-1} \left( \int_{f(a)}^{f(x)} F(y) dy \right) \right]^\circ \quad (189)$$

$$= \lim_{h \rightarrow 0} \left[ f^{-1} \left( \int_{f(a)}^{f(x \oplus h)} F(y) dy \right) \ominus f^{-1} \left( \int_{f(a)}^{f(x)} F(y) dy \right) \right] \oslash h \quad (190)$$

$$= \lim_{h \rightarrow 0} \left[ f^{-1} \left( \int_{f(a)}^{f(x)+f(h)} F(y) dy - \int_{f(a)}^{f(x)} F(y) dy \right) \right] \oslash h \quad (191)$$

$$= f^{-1} \left[ \lim_{h \rightarrow 0} \left( \int_{f(a)}^{f(x)+f(h)} F(y) dy - \int_{f(a)}^{f(x)} F(y) dy \right) / f(h) \right] \quad (192)$$

$$= f^{-1} \left( F(f(x)) \right) = F_f(x) \quad (193)$$

In particular

$$\int_a^b (x)^\circ \odot dx = \int_a^b 1 \odot dx = b \ominus a, \quad (194)$$

$$\int_a^b F_f^\circ(x) \odot dx \oplus \int_b^c F_f^\circ(x) \odot dx = F_f(b) \ominus F_f(a) \oplus F_f(c) \ominus F_f(b) = F_f(c) \ominus F_f(a) \quad (195)$$

On the Cantor set

$$\int_0^{1/3} 1 \odot dx = 0.0(2)_3 \ominus 0 = f^{-1}(f[0.0(2)_3] - f(0)) = f^{-1}(0.0(1)_2) = 0.0(2)_3 = 1/3, \quad (196)$$

$$\begin{aligned} \int_{1/3}^{2/3} 1 \odot dx &= 0.2_3 \ominus 0.0(2)_3 = f^{-1}(f[0.2_3] - f[0.0(2)_3]) \\ &= f^{-1}(0.1_2 - 0.1_2) = 0 \end{aligned} \quad (197)$$

$$\begin{aligned} \int_{2/3}^1 1 \odot dx &= 0.(2)_3 \ominus 0.2_3 = f^{-1}(f[0.(2)_3] - f[0.2_3]) \\ &= f^{-1}(0.(1)_2 - 0.1_2) = f^{-1}(1 - 1/2) = f^{-1}(1/2) = f^{-1}(0.1_2) = 0.2_3 = 2/3 \end{aligned} \quad (198)$$

### I. Bell inequality

For any Kolmogorov-Nagumo average

$$\langle a \rangle_f = f^{-1} \left( \sum_k P_k f(a_k) \right), \quad (199)$$

one finds the standard bounds

$$a_{\min} \leq \langle a \rangle_f \leq a_{\max} \quad (200)$$

since  $f$  is strictly monotonic.

Let

$$\begin{aligned} p_j &= N_j \oslash \left( \oplus_{k=1}^K N_k \right) \\ &= f^{-1} \left( \frac{f(N_j)}{\sum_{k=1}^K f(N_k)} \right), \end{aligned} \quad (201)$$

$$\oplus_{j=1}^K p_j = \oplus_{j=1}^K N_j \oslash \left( \oplus_{k=1}^K N_k \right) = 1 \quad (202)$$

If all  $p_j$  are equal then

$$\begin{aligned} p_j &= f^{-1} \left( \frac{f(N_j)}{\sum_{k=1}^K f(N_k)} \right) \\ &= f^{-1} \left( \frac{1}{K} \right) = f^{-1} \left( f(1)/f[f^{-1}(K)] \right) = 1 \odot f^{-1}(K) \end{aligned} \quad (203)$$

Recall that  $f(1) = 1$ . Assume that  $f(-x) = -f(x)$ , so that  $f(-1) = -1$ . Consider

$$\langle AB \rangle_f = \oplus_{kl} p_{kl} \odot a_k \odot b_l \quad (204)$$

$$-1 = \min\{a_k \odot b_l\} \leq \langle AB \rangle_f \leq \max\{a_k \odot b_l\} = 1 \quad (205)$$

$$\begin{aligned} \langle C \rangle_f &= \langle AB \rangle_f \oplus \langle AB' \rangle_f \oplus \langle A'B \rangle_f \oplus \langle A'B' \rangle_f \\ &= \oplus_{kl} p_{kl} \odot \left( a_k \odot b_l \oplus a_k \odot b'_l \oplus a'_k \odot b_l \oplus a'_k \odot b'_l \right) \\ &= \oplus_{kl} p_{kl} \odot \left( a_k \odot (b_l \oplus b'_l) \oplus a'_k \odot (b_l \oplus b'_l) \right) \\ &= \oplus_{kl} p_{kl} \odot c_{kl} \end{aligned} \quad (206)$$

So

$$\min\{c_{kl}\} \leq \langle C \rangle_f \leq \max\{c_{kl}\} \quad (207)$$

$$\begin{aligned} c_{kl} &= a_k \odot (b_l \oplus b'_l) \oplus a'_k \odot (b_l \oplus b'_l) \\ &= f^{-1} \left( f \left( a_k \odot (b_l \oplus b'_l) \right) + f \left( a'_k \odot (b_l \oplus b'_l) \right) \right) \\ &= f^{-1} \left( f \left( f^{-1} \left[ f(a_k) f(b_l \oplus b'_l) \right] \right) + f \left( f^{-1} \left[ f(a'_k) f(b_l \oplus b'_l) \right] \right) \right) \\ &= f^{-1} \left( f(a_k) f(b_l \oplus b'_l) + f(a'_k) f(b_l \oplus b'_l) \right) \\ &= f^{-1} \left( f(a_k) f \left( f^{-1} \left[ f(b_l) + f(b'_l) \right] \right) + f(a'_k) f \left( f^{-1} \left[ f(b_l) - f(b'_l) \right] \right) \right) \\ &= f^{-1} \left( f(a_k) (f(b_l) + f(b'_l)) + f(a'_k) (f(b_l) - f(b'_l)) \right) \\ &= f^{-1} \left( a_k (b_l + b'_l) + a'_k (b_l - b'_l) \right) \end{aligned} \quad (208)$$

$$-f^{-1}(2) \leq c_{kl} \leq f^{-1}(2) \quad (209)$$

$$\begin{aligned} \langle C \rangle &= \oplus_{kl} p_{kl} \odot f^{-1} \left( a_k (b_l + b'_l) + a'_k (b_l - b'_l) \right) \\ &= f^{-1} \left( \sum_{kl} f(p_{kl}) (a_k (b_l + b'_l) + a'_k (b_l - b'_l)) \right) \\ &= f^{-1} \left( \sum_{kl} P_{kl} (a_k (b_l + b'_l) + a'_k (b_l - b'_l)) \right) \end{aligned}$$

Finally

$$-f^{-1}(2) \leq \langle C \rangle \leq f^{-1}(2) \quad (210)$$

For the Cantor set  $C = f^{-1}(\mathbb{R})$

$$f^{-1}(2) = f^{-1}(2^1) = f^{-1}(10_2) = 20_3 = 2 \times 3^1 = 6 \quad (211)$$

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