A non-SUSY extension of the Poincaré group

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In this paper a nontrivial extension of the Poincaré group is presented, circumventing the Coleman-Mandula no-go theorem in a very different way than that of SUSY. The extended part can be identified with a gauge group, containing a U(1) component along with a nilpotent normal subgroup. The physical interpretation of the nilpotent normal subgroup is given in terms of a quantum field theory toy model. The presented mechanism could provide a unification of Standard Model gauge groups, as well as relating the gauge symmetries to spacetime symmetries.

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Introduction.—The non-trivialness of the problematics of unification of internal symmetries to spacetime symmetries is a well-known fact in particle field theory. The Coleman-Mandula no-go theorem [1] forbids the most simple unification scenarios. Namely, any finite dimensional connected Lie group, satisfying a set of plausible properties required by a particle field theory context, and containing the Poincaré group as a subgroup as well as a "gauge" symmetry group with a positive definite non-degenerate invariant scalar product on its Lie algebra, must be of the trivial form: gauge group × Poincaré group. Also, the earlier theorem of McGlinn [2] in a mathematically slightly simpler setting concluded in the same direction. The classification result of O'Raifeartaigh [3] on Poincaré group extensions is also usually interpreted in a similar manner. After the discovery of these results, the simple unification attempts of internal (gauge) symmetries with spacetime symmetries were not pursued further. Instead, a large amount of research was carried out along the question: can the Poincaré Lie algebra be extended at all in at least by means of some mathematically generalized manner? The answer was positive, as stated by the result of Haag, Lopuszanski and Sohnius [4], and hence the era of supersymmetry (SUSY) was born.

By studying the details of the proof of Coleman-Mandula or McGlinn theorems, as presented e.g. in the review in [5], one finds that the assumption of presence of a positive definite non-degenerate invariant scalar product on the Lie algebra of the gauge group is essential in the argumentation. As such, the gauge group is assumed to be direct product of copies of U(1) and a semisimple compact Lie group. The motivations behind this requirement are threefold. (i) Group theoretical convenience: the classification of semisimple Lie groups is well understood. (ii) Experimental justification: the Standard Model has a gauge group U(1) × SU(2) × SU(3), which satisfies the pertinent requirement. (iii) Positive energy condition, or unitarity: the energy density expression of a Yang-Mills (gauge) field involves the invariant scalar product on the Lie algebra of the gauge group, and that is required to be positive definite.

Traditionally, non-semisimple gauge groups, i.e. the ones with a solvable normal subgroup, apart from a possible U(1) contribution, is considered unphysical. If that is postulated, indeed there seems to be no other Poincaré extensions than that of SUSY. However, if that requirement is relaxed, and only the existence of a positive *semidefinite* invariant inner product is required on the Lie algebra of the gauge group, then it is possible to show quite natural Poincaré group extensions. In terms of the group structure this would mean that only the so called *Levi factor* of the gauge group needs to be compact semisimple, not the entire gauge group itself. Our example shall be of this nature. We argue that this property is in accordance with the positive energy condition, or equivalently with the unitarity: the energy density expression of such a Yang-Mills field becomes positive *semidefinite*, however this does not pose a major problem, as also the kinetic Lagrange density vanishes in the degenerate directions and thus these do not become part of the propagating degrees of freedom [6].

Overview on Levi decomposition.—The powerful tool of Levi decomposition theorem of finite dimensional connected Lie groups states that a Lie group always has the structure $\mathcal{R} \rtimes \mathcal{L}$, $\mathcal{R}$ being a solvable normal subgroup called the *radical* and $\mathcal{L}$ being a semisimple subgroup called the *Levi factor*, and the symbol $\rtimes$ denotes semi-direct product. Whenever $\mathcal{L}$ is also a normal subgroup, then we have simply a direct product decomposition $\mathcal{R}\rtimes\mathcal{L} = \mathcal{R}\times\mathcal{L}$. The *semisimpleness* of $\mathcal{L}$ means that the *Killing form* $(x, y) \mapsto \mathrm{Tr}(\mathrm{ad}_x\, \mathrm{ad}_y)$ is non-degenerate on the Lie algebra of $\mathcal{L}$, using the symbol $\mathrm{ad}_x(\cdot) := [x, \cdot]$ for any Lie algebra element x. The *solvability* of $\mathcal{R}$ means that it represents the degenerate directions of the Killing form. It may also be formulated in terms of an equivalent property: for the Lie algebra r of $\mathcal{R}$ the sequence $r^0 := r$, $r^1 := [r^0, r^0]$, $r^2 := [r^1, r^1]$, $\ldots$, $r^k := [r^{k-1}, r^{k-1}]$, $\ldots$, arrives at the trivial Lie algebra in finite iterations, i.e. $r^k = \{0\}$ for some finite k. A special case is when the radical $\mathcal{R}$ is said to be *nilpotent*: there exists a finite k for which for all $x_1, \ldots, x_k \in r$ one has $\mathrm{ad}_{x_1} \ldots \mathrm{ad}_{x_k} = 0$. An even more special case is when the radical $\mathcal{R}$ is *abelian*: for all $x \in r$, one has $\mathrm{ad}_x = 0$. The Levi decomposition theorem is a consequence of *Jordan's decomposition theorem* in finite dimensional complex linear algebra, stating that a linear operator can be always be uniquely split to a diagonalizable part and to a nilpotent part.

A nice demonstration of a typical Levi decomposition is the (proper) Poincaré group itself: it can be written in the form $\mathcal{T} \rtimes \mathcal{L}$, where $\mathcal{T}$ is the abelian normal subgroup consisting of spacetime translations, being the radical, and where $\mathcal{L}$ is the semisimple subgroup consisting of the (proper) homogeneous Lorentz transformations, being the Levi factor. The group of spacetime translations, quite naturally, act as

$$x^a \mapsto x^a + d^a \tag{1}$$

in terms of affine spacetime coordinates.

Overview on supersymmetry group.—The Levi decomposition theorem also sheds a light on the group structure of supersymmetry transformations. That has the Levi decomposition of the form $\mathcal{S} \rtimes \mathcal{L}$, where $\mathcal{S}$ is the nilpotent normal subgroup consisting of *supertranslations*, being the radical, and where $\mathcal{L}$ is the semisimple subgroup consisting of the (proper) homogeneous Lorentz transformations, being the Levi factor. The supertranslations are defined as transformations on the vector bundle of superfields [7–9]: they are of the form

$$\begin{aligned}&\theta^A \mapsto \theta^A + \epsilon^A,\\ &x^a \mapsto x^a + d^a + \sigma^a_{AA'}\mathrm{i}\big(\theta^A\bar{\epsilon}^{A'} - \epsilon^A\bar{\theta}^{A'}\big)\end{aligned} \tag{2}$$

in terms of "supercoordinates" (Grassmann valued two-spinors) and affine spacetime coordinates. From Eq.(2) it is seen that although the pure spacetime translations $\mathcal{T}$ form an abelian normal subgroup inside $\mathcal{S}$, but $\mathcal{S}$ cannot be further split in the form of $\mathcal{T} \rtimes \{\text{some other subgroup}\}$, and thus such splitting is not applicable for the entire supersymmetry group. The above structural observations imply that the supersymmetry group is of type (iii) in the classification theorem of O'Raifeartaigh [3] on Poincaré group extensions: its Levi factor is the (proper) homogeneous Lorentz group, and the extension is on the side of the radical, where the pure spacetime translations form a normal subgroup of the radical. Our Poincaré group extension presented in this paper shall also be of such type, but the radical will be a split extension: it shall be of the form $\mathcal{T} \times \{\text{some other subgroup}\}$, both components coupling to the homogeneous Lorentz part. Thus, our Poincaré group extension shall have the form $\mathcal{T} \rtimes \{\text{some group acting at points of spacetime}\}$, which is not the case for the supersymmetry group. Note that in the paper of O'Raifeartaigh the pertinent type (iii) Poincaré group extension is regarded as "unphysical", however it is seen that in fact, the SUSY mechanism is just making use of that group theoretical possibility as well.

As a closing note about the context of SUSY we would like to reflect on a common way of presentation of SUSY algebra. In the traditional view, also used by [4], the SUSY algebra is not a Lie algebra, but rather a so called *graded Lie algebra*, being a mathematically slightly generalized version of a Lie algebra with even and odd sectors. However, as summarized in [7–9] and Eq.(2), it may as well be also regarded as an ordinary Lie algebra and corresponding Lie group, as done in the present paper.

The alternative Lorentz group extension.—We start by defining the group action of our extended group at a given point of spacetime. It shall be an extension of the (proper) homogeneous conformal Lorentz group.

Let A be a finite dimensional complex unital associative algebra, with its unit denoted by $\mathbb{1}$. Whenever A is also equipped with a conjugate-linear involution $(\cdot)^+ : A \to A$ such that for all $x, y \in A$ one has $(xy)^+ = x^+y^+$, then it shall be called a $^+$*-algebra* [10]. Let now A be a finite dimensional complex associative algebra with unit, being also $^+$-algebra, and possessing a minimal generator system (e_1, e_2, e_3, e_4) obeying the identity

$$\begin{aligned}
&e_ie_j + e_je_i = 0 \quad (i,j \in \{1,2\} \text{ or } i,j \in \{3,4\}),\\
&e_ie_j - e_je_i = 0 \quad (i \in \{1,2\} \text{ and } j \in \{3,4\}),\\
&e_3 = e_1^+,\\
&e_4 = e_2^+,\\
&e_{i_1}e_{i_2}\dots e_{i_k} \quad (1 \leq i_1 < i_2 < \dots < i_k \leq 4,\ 0 \leq k \leq 4)\\
&\qquad\text{are linearly independent.}
\end{aligned} \tag{3}$$

Then we call A *spin algebra*, and we call a minimal generator system obeying Eq.(3) a *canonical generator system*, whereas the $^+$-operation is called *charge conjugation*. That is, spin algebra is a freely generated unital complex associative algebra with four generators, and the generators admit two sectors within which the generators anticommute, whereas the two sectors commute with each-other, and are charge conjugate to each-other. It is easy to check that if S^* is a complex two dimensional vector space (called the *cospinor space*), and $\bar{S}^*$ is its complex conjugate vector space, then $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ naturally becomes spin algebra, where $\Lambda(\cdot)$ denotes the exterior algebra of its argument. It is also seen that any spin algebra is isomorphic (not naturally) to this algebra, i.e. they all have the same structure, but there is a freedom in matching the canonical generators. Some properties of the pertinent mathematical structure is listed in [11]. In terms of a formal quantum field theory (QFT) analogy, the spin algebra can be regarded as a creation operator algebra of a fermion particle with two internal degrees of freedom along with its antiparticle, at a fixed point of spacetime, or equivalently, at a fixed point of momentum space. It is important to understand, however, that in this construction the creation operators of antiparticles are not yet identified with the annihilation operators of particles, i.e. it is not a canonical anticommutation relation (CAR) algebra.

Given a canonical generator system (e_1, e_2, e_1^+, e_2^+) of A, one can define the following subspaces: $\Lambda_{\bar{p}q}$ are the linear subspaces of p, q-forms, i.e. the polynomials consisting of p powers of $\{e_1, e_2\}$ and q powers of $\{e_1^+, e_2^+\}$ ($p, q \in \{0, 1, 2\}$), and one has $A = \overset{2}{\underset{p,q=0}{\oplus}} \Lambda_{\bar{p}q}$, called to be the $\mathbb{Z}\times\mathbb{Z}$-*grading* of A. Then, there are the linear subspaces of k-forms, Λ_k, i.e. the polynomials consisting of k powers of $\{e_1, e_2, e_1^+, e_2^+\}$ ($k \in \{0, 1, 2, 3, 4\}$), and one has $A = \overset{4}{\underset{k=0}{\oplus}} \Lambda_k$, called to be the $\mathbb{Z}$-*grading* of A. Finally, there are the subspaces Λ_{ev} and Λ_{od} being the even and odd polynomials of $\{e_1, e_2, e_1^+, e_2^+\}$, and one has $A = \Lambda_{\mathrm{ev}} \oplus \Lambda_{\mathrm{od}}$, called to be the $\mathbb{Z}_2$-*grading* of A. The subspace $B := \Lambda_{\bar{0}0} = \mathbb{C}\,\mathbb{1}$ of zero-forms and the subspace $M := \overset{4}{\underset{k=1}{\oplus}} \Lambda_k$ of at-least-1-forms shall play an important role as well, and one has $A = B \oplus M$. B is a one-dimensional unital associative subalgebra of A, spanned by the unity and called the *unit algebra*, whereas M is the so called *maximal ideal* of A. An other important subspace is $Z = \Lambda_{\bar{0}0} \oplus \Lambda_{\bar{2}0} \oplus \Lambda_{\bar{0}2} \oplus \Lambda_{\bar{2}2}$, the *center* of A, being the largest unital associative subalgebra in A commuting with all elements of A.

Our extension of the homogeneous conformal Lorentz group shall be nothing but $\mathrm{Aut}(A)$, the *automorphism group* of the spin algebra A. That consists of those invertible $A \to A$ linear transformations, which preserve the algebraic product as well as the charge conjugation operation. It is seen that an element of $\mathrm{Aut}(A)$ maps a canonical generator system to a canonical generator system, and that an element of $\mathrm{Aut}(A)$ can be uniquely characterized by its group action on an arbitrary preferred canonical generator system. Let us take such a system (e_1, e_2, e_1^+, e_2^+), with occasional notation $e_3 = e_1^+$, $e_4 = e_2^+$. The group structure of $\mathrm{Aut}(A)$ can then be characterized with the following four subgroups. (i) Let $\mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A)$ be the group of $\mathbb{Z}\times\mathbb{Z}$-grading preserving automorphisms: they act on the canonical generators as $e_i \mapsto \sum_{j=1}^{2} \alpha_{ij} e_j$ and $e_i^+ \mapsto \sum_{j=1}^{2} \bar{\alpha}_{ij} e_j^+$ ($i \in \{1, 2\}$), the bar $\bar{(\cdot)}$ meaning complex conjugation and the 2×2 complex matrix $(\alpha_{ij})_{i,j\in\{1,2\}}$ being invertible. (ii) Let $\mathcal{J} := \{I, J\}$ be the two element subgroup of $\mathbb{Z}$-grading preserving automorphisms, I being the identity and J being the involutive complex-linear operator of *particle-antiparticle label exchanging* acting as $e_1 \mapsto e_3$, $e_2 \mapsto e_4$, $e_3 \mapsto e_1$, $e_4 \mapsto e_2$. (iii) Let $\tilde{N}_{\mathrm{ev}}$ be a subgroup of the $\mathbb{Z}_2$-grading preserving automorphisms defined by the relations $e_i \mapsto e_i + b_i$ and $e_i^+ \mapsto e_i^+ + b_i^+$ with uniquely determined parameters $b_i \in \Lambda_{\bar{1}2}$ ($i \in \{1, 2\}$). (iv) Let $\mathrm{InAut}(A)$ be the subgroup of inner automorphisms, i.e. the ones of the form $\exp(a)(\cdot)\exp(a)^{-1}$ with some $a \in \mathrm{Re}(A)$. These are of the form $e_i \mapsto e_i + [a, e_i] + \frac{1}{2}[a, [a, e_i]]$ ($i \in \{1, 2, 3, 4\}$) with uniquely determined parameter $a \in \mathrm{Re}(\Lambda_{\bar{1}0} \oplus \Lambda_{\bar{0}1} \oplus \Lambda_{\bar{1}1} \oplus \Lambda_{\bar{2}1} \oplus \Lambda_{\bar{1}2})$. With these, the semi-direct product splitting

$$\mathrm{Aut}(A) = \underbrace{\mathrm{InAut}(A) \rtimes \tilde{N}_{\mathrm{ev}}}_{=:N} \rtimes \underbrace{\mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A) \rtimes \mathcal{J}}_{=\mathrm{Aut}_{\mathbb{Z}}(A)} \tag{4}$$

holds. It is seen that a $\mathbb{Z}$-grading almost determines the underlying $\mathbb{Z}\times\mathbb{Z}$-grading: only the two-element discrete group of label exchanging transformations $\mathcal{J}$ introduces an ambiguity. The subgroup N shall be called the group of *dressing transformations*, being a nilpotent normal subgroup of $\mathrm{Aut}(A)$. These transformations are mixing higher forms to lower forms, i.e. do not preserve the $\mathbb{Z}$ and $\mathbb{Z}_2$-grading defined by our preferred canonical generator system: they map a system of canonical generators like $e_i \mapsto e_i + \beta_i$, the elements β_i residing in the space of at-least-2-forms M^2 ($i \in \{1, 2, 3, 4\}$), deforming the original $\mathbb{Z}$ and $\mathbb{Z}_2$-grading to an other one. By direct substitution it is seen that the transformations (i)–(iv) indeed define subgroups of the automorphisms of A, however the proof of decomposition theorem Eq.(4) needs a bit more complex mathematical apparatus [12]. The principle of the proof is motivated by [13], studying the automorphism group of ordinary finite dimensional complex Grassmann (exterior) algebras.

By scrutinizing the subgroups, it is seen that the group $\mathcal{J}$ of label exchanging transformations has the structure of $\mathbb{Z}_2$. On the other hand, one has

$$\mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A) \equiv \mathrm{GL}(2,\mathbb{C}) \equiv \mathrm{U}(1)\times\mathrm{D}(1)\times\mathrm{SL}(2,\mathbb{C}), \tag{5}$$

where $\mathrm{D}(1)$ is the dilatation group, i.e. $\mathbb{R}^+$ with the real multiplication. Note that $\mathrm{D}(1)\times\mathrm{SL}(2,\mathbb{C})$ is nothing but the universal covering group of the (proper) homogeneous conformal Lorentz group. As far as a fixed $\mathbb{Z}\times\mathbb{Z}$-grading is taken, A can be always be represented via ordinary two-spinor calculus, and the algebra identification $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ can greatly ease the calculations due to well-known identities in that formalism [14, 15]. The group of dressing transformations N, however, does not fit automatically into that framework: it needs the proper apparatus of the introduced spin algebra formalism, or care is needed when represented in terms of two-spinors.

Due to the presence of the nilpotent normal subgroup N, $\mathrm{Aut}(A)$ is not semisimple. As a consequence, there can be nontrivial invariant subspaces even in the defining representation, i.e. when $\mathrm{Aut}(A)$ acts on A. However, for the same reason, the existence of an invariant subspace in a representation of $\mathrm{Aut}(A)$ does not imply the existence of an invariant complement. The list of indecomposable $\mathrm{Aut}(A)$-invariant subspaces of A are the followings: B, M^l ($l \in \{1, 2, 3, 4\}$), $M^2 \cap Z$, $V := \Lambda_{\bar{1}0} \oplus \Lambda_{\bar{0}1} \oplus \Lambda_{\bar{2}0} \oplus \Lambda_{\bar{0}2} \oplus \Lambda_{\bar{2}1} \oplus \Lambda_{\bar{1}2} \oplus \Lambda_{\bar{2}2}$, $U := \Lambda_{\bar{2}0} \oplus \Lambda_{\bar{0}2} \oplus \Lambda_{\bar{2}1} \oplus \Lambda_{\bar{1}2} \oplus \Lambda_{\bar{2}2}$, $W := \Lambda_{\bar{1}1} \oplus \Lambda_{\bar{2}1} \oplus \Lambda_{\bar{1}2} \oplus \Lambda_{\bar{2}2}$. This is seen via the orbit of $\Lambda_{\bar{p}q}$ ($p, q \in \{0, 1, 2\}$) by $\mathcal{J}$ and N. The existence of the trivial invariant splitting $A = B \oplus M$ is observed, which holds in any polynomial algebra. The dual vector space A^* of A has the following indecomposable $\mathrm{Aut}(A)$-invariant subspaces: $\mathrm{Ann}(M)$, $\mathrm{Ann}(B)$, $\mathrm{Ann}(B \oplus M^l)$ ($l \in \{2, 4\}$), $\mathrm{Ann}(Z)$, $\mathrm{Ann}(B \oplus W)$, $\mathrm{Ann}(B \oplus V)$, where $\mathrm{Aut}(A)$ is understood to act on A via the transpose group action. Here $\mathrm{Ann}(X) \subset A^*$ denotes annulator subspace of the subspace $X \subset A$, i.e. the subspace in A^* which maps X to $\{0\}$. It is seen that the $\mathrm{Aut}(A)$-invariant subspace $\mathrm{Ann}(B \oplus V) \equiv \Lambda^*_{\bar{1}1}$ is nothing but a four vector multiplet of $\mathrm{Aut}(A)$, on which $\mathrm{Aut}(A)$ acts as the homogeneous conformal Lorentz group. In the two-spinor representation $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ one has simply $\Lambda^*_{\bar{1}1} \equiv \bar{S} \otimes S$. The kernel of the corresponding homomorphism of $\mathrm{Aut}(A)$ onto the homogeneous conformal Lorentz group is said to be the *gauge group*, having the structure $N \rtimes \mathrm{U}(1)$. Given a four dimensional real vector space T, any injection $T \to \mathrm{Re}(\Lambda^*_{\bar{1}1})$ is called a *Pauli injection*, which is the analogue of the "soldering form" in the traditional two-spinor calculus [14, 15], extending the group action of $\mathrm{Aut}(A)$ onto the real four dimensional vector space T. In the usual Penrose abstract index notation that is nothing but the usual mapping $\sigma_a^{AA'}$ between spacetime vectors T and hermitian mixed spinor-tensors $\mathrm{Re}(\bar{S} \otimes S)$. It is seen that the group of dressing transformations N respects this basic relation of two-spinor calculus and hence realizes the group action of $\mathrm{Aut}(A)$ on the spacetime vectors T as the homogeneous conformal Lorentz group.

Adding the translation group.—Adding translations to the presented homogeneous conformal Lorentz group extension is trivial. One simply takes a four dimensional real affine space $\mathcal{M}$ as the model of the flat spacetime manifold, with underlying vector space ("tangent space") T. One takes in addition the spin algebra A, and constructs the trivial vector bundle $\mathcal{M} \times A$. The algebraic product on A extends to the sections of this vector bundle (i.e. to the A-valued fields) pointwise, being translationally invariant. Given a Pauli injection (soldering form) between T and $\mathrm{Re}(\Lambda^*_{\bar{1}1})$, $\mathrm{Aut}(A)$ acts on T as the homogeneous conformal Lorentz group. The vector bundle automorphisms of $\mathcal{M} \times A$ preserving the algebraic product of fields as well as preserving the Pauli injection shall have the desired group structure including both the spacetime translations and $\mathrm{Aut}(A)$ in a semi-direct product, acting on $\mathcal{M}$ as the Poincaré group combined with global metric rescalings. This also implies a causal structure on $\mathcal{M}$.

The "gauging" of $\mathrm{Aut}(A)$, i.e. making $\mathrm{Aut}(A)$ a local symmetry is also trivial. Let $\mathcal{M}$ be a four dimensional real manifold modeling the spacetime manifold, with tangent bundle $T(\mathcal{M})$. Take in addition a vector bundle $A(\mathcal{M})$ whose fiber in each point is spin algebra. Take also a pointwise Pauli injection between $T(\mathcal{M})$ and $\mathrm{Re}(\Lambda^*_{\bar{1}1})(\mathcal{M})$. The gauged version of $\mathrm{Aut}(A)$ shall be nothing but the product preserving vector bundle automorphisms of $A(\mathcal{M})$, and they act on $T(\mathcal{M})$ as the combined group of diffeomorphisms and pointwise spacetime metric conformal rescalings, being the symmetries of (conformal) general relativity.

Discussion.—The presented approach can be used for GUT attempts. Assume that a solvable normal subgroup N is allowed in the symmetry group of matter fields at a point of spacetime, then one could search for a unified group having the structure

$$\underbrace{\underbrace{N \rtimes \Big(\underbrace{\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)}_{\text{SM}-\text{internal}}}_{\text{full gauge group}} \times \underbrace{\mathrm{D}(1) \times \mathrm{SL}(2, \mathbb{C})}_{\text{spacetime related}} \Big)}_{\text{symmetries of matter fields at a spacetime point or mom. space}} \tag{6}$$

which is indecomposable. In this mechanism, the solvable normal subgroup N glues together the otherwise independent gauge group to the spacetime related symmetry group, and as a byproduct the Standard Model gauge groups are glued to each-other as well. The unital connected component of the group presented in this paper provides a simplified example for the

$$\underbrace{\underbrace{N \rtimes \Big(\underbrace{\mathrm{U}(1)}_{\text{internal}}}_{\text{full gauge group}} \times \underbrace{\mathrm{D}(1) \times \mathrm{SL}(2, \mathbb{C})}_{\text{spacetime related}} \Big)}_{\text{symmetries of } A-\text{fields at a spacetime point or mom. space}} \tag{7}$$

case. When adding the translations as well, all this falls into the type (iii) Poincaré group extension of O'Raifeartaigh classification, and circumvents McGlinn's theorem as well as the Coleman-Mandula no-go theorem because of the presence of the solvable normal subgroup N. As mentioned, the presence of N shall not mean a problem from the point of view of positivity of energy, or equivalently, from the point of view of unitarity: the Yang-Mills degrees of freedom in the direction of N will not be propagating ones, since the corresponding Yang-Mills kinetic Lagrangian also vanishes due to the degeneration of the Killing form in the directions of N.

In the presented example the physical meaning of N could be understood as the "dressing" of pure one-particle states of a formal QFT model at a fixed momentum. Note, that spin algebra differs from a CAR algebra of QFT with the fact that the antiparticle creation operators are not yet identified with particle annihilation operators. It can be shown however [12], that an $\mathrm{Aut}(A)$-covariant family of self-dual CAR algebras can be associated to the spin algebra A, and vice-versa. Here, the self-dual CAR algebra is a mathematical structure, introduced by Araki [16], formally describing the algebraic behavior of quantum field operators. With the use of this relation, the spin algebra looks like a convenient reparametrization of the quantum field algebra of a QFT at a fixed point of spacetime or momentum space. The details of this correspondence is, however, out of the scope of the present paper mainly focusing on Poincaré group extensions.

Conclusions.—In this paper a non-SUSY extension of the Poincaré group was presented. The extended symmetry group is an affine group, i.e. is a semi-direct product of the group of spacetime translations and a subgroup acting on fields at the points of spacetime. The latter group is a semi-direct product of internal (gauge) symmetries and of the covering group of the homogeneous conformal Lorentz group $\mathrm{D}(1) \times \mathrm{SL}(2, \mathbb{C})$. The normal subgroup accounting for gauge symmetries has a semi-direct product structure of a nilpotent normal subgroup and of $\mathrm{U}(1)$. The presented example circumvents the Coleman-Mandula theorem, because the invariant inner product of the gauge group is required only to be positive semidefinite, or equivalently, only the Levi factor of the gauge group is required to be compact semisimple, not the entire gauge group. The nilpotent part glues together the otherwise unrelated compact gauge group $\mathrm{U}(1)$ and the group accounting for the spacetime symmetries. This mechanism could be used for unification of Standard Model gauge groups along with relating the gauge symmetries to spacetime symmetries.

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