

# Reversible time travel with freedom of choice

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General relativity predicts the existence of closed time-like curves, along which a material object could travel back in time and interact with its past self. The natural question is whether this possibility leads to inconsistencies: Could the object interact in such a way to prevent its own time travel? If this is the case, self-consistency should forbid certain initial conditions from ever happening, a possibility at odds with the local nature of dynamical laws. Here we consider the most general deterministic dynamics connecting classical degrees of freedom defined on a set of bounded space-time regions, requiring that it is compatible with *arbitrary* operations performed in the local regions. We find that any such dynamics can be realised through reversible interactions. We further find that consistency with local operations is compatible with non-trivial time travel: Three parties can interact in such a way to be all both *in the future and in the past* of each other, while being *free* to perform arbitrary local operations. We briefly discuss the quantum extension of the formalism.

## INTRODUCTION

One of the most baffling aspects of general relativity is that certain solutions to the Einstein equations contain closed time-like curves (CTCs) [1–7], where an event can be both in its own future and past. Although it is not known whether CTCs are actually possible in our universe [8–13], their mere logical possibility poses the challenge to understand what type of dynamics could be expected in their presence.

The first systematic studies of the subject concentrated on space-time geometries where CTCs only appear in the *future* of some space-like surface [14–16] (Fig. 1a). It is then meaningful to set initial conditions in the pre-CTCs era and look for corresponding solutions to the equations of motion. A prime case study is that of a billiard ball thrown in the direction of a wormhole. The trajectory is such that, if undisturbed, the ball will come out the second mouth of the wormhole in the past and kick its younger self off course, so it cannot reach the wormhole and kick itself. Classical physics is clearly at variance with such ‘inconsistent’ dynamics, so it seems that certain initial conditions are simply impossible.

The surprising result is that, in fact, self-consistent solutions exist for all cases studied. The ball does not enter the wormhole undisturbed: It is kicked softly, it comes out the wormhole at a slightly different angle than expected and gives its younger self just the right soft kick. Even including friction, exploding bombs, and the like, solutions for any considered initial condition were found [17, 18].

The existence of consistent solutions for every initial condition can be seen as a ‘*no new physics*’ principle [14]. After all, what would be the local mechanism preventing an experimenter from launching a billiard ball at any

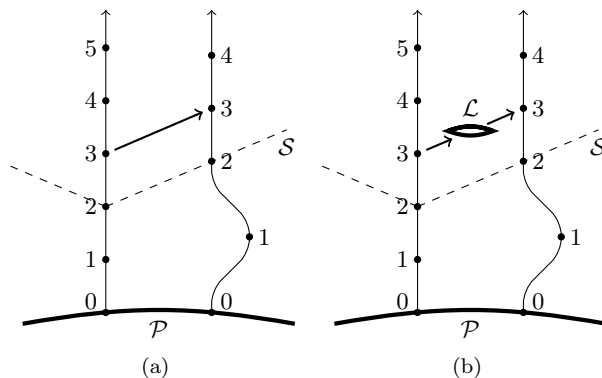


Figure 1. Wormhole space-time with closed time-like curves (CTCs) [8, 9]. (a) Events with equal proper times along the world lines of the two mouths of the wormhole are identified. Accelerating the right mouth produces time dilation, resulting in CTCs in the future of the surface  $S$ . An experimenter acting in the past of  $S$  should be able to prepare arbitrary initial states on a space-like surface  $\mathcal{P}$ . (b) An experimenter in a localised region  $\mathcal{L}$ , which does not contain but is traversed by CTCs, should be able to perform arbitrary local operations.

desired angle? It is then natural to require such a condition to be extended to experimenters acting in the region where CTCs are *already* present (Fig. 1b). Although in such a region there are no sufficiently regular space-like surfaces to set ‘global’ initial conditions, it should still be possible for an experimenter to perform *arbitrary* actions in a sufficiently localised region, as long as the region itself does not contain CTCs.

Here we pose the general question of whether CTCs can be compatible with local operations in arbitrary space-time regions. Rather than considering a specific type of system (billiard balls, fields, *etc.*), we develop a general framework to study the most general type of clas-

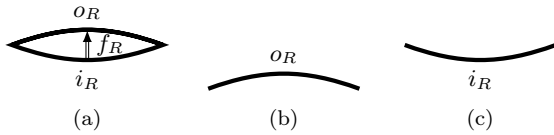


Figure 2. (a) A deterministic local operation  $f_R$  maps the classical input state  $i_R$  from the past boundary to the classical output state  $o_R$  at the future boundary of a local region  $R$ . (b) An ‘output only’ region (source). (c) An ‘input only’ region (sink).

sical, deterministic dynamics possible in the presence of CTCs. We provide an example of *reversible* deterministic dynamics where agents in three local regions can perform *arbitrary* operations, send information to each other—so they are effectively both in the future and in the past of one another—and *no contradiction* ever arises.

The formalism developed here is a deterministic version of the formalism of ‘classical correlations without causal order’ [19], which in turn is the classical limit of the quantum ‘process matrix’ formalism [20] (see also Refs. [21–31]).

## THE MODEL

The core assumption of our model is that any classical operation that is possible in an ordinary space-time should also be possible in the presence of CTCs, as long as the operation takes place in a localised region of space-time that does not contain CTCs. We thus consider  $N$  space-time regions (henceforth *local regions*) which, individually, cannot be distinguished from regions in ordinary, causal space-time. To simplify the analysis, we restrict to local regions that have only space-like boundaries, which we decompose into a past boundary and a future boundary. We assume that, for each local region, any time-like curve<sup>1</sup> that enters through the past (future) boundary exits through the future (past) boundary, and that the region contains no CTCs. We impose no restriction on the space-time in which the regions are embedded, except that it is a four-dimensional Lorentzian manifold fixed independently of any dynamical degree of freedom of interest. In fact, the formalism is largely independent on any detail of the geometry, which possibly would impose further constraints in a fully developed theory.

As we are interested in classical systems, we can assign a classical state spaces  $\mathcal{I}_R$  (input) and  $\mathcal{O}_R$  (output) respectively to the past and future boundaries of a local region  $R$ . States will be denoted as  $i_R \in \mathcal{I}_R$ ,

$o_R \in \mathcal{O}_R$ . A *deterministic local operation* in the local region is described by a function  $f_R$  from input to output space (Fig. 2a). We denote by  $\mathcal{D}_R := \{f_R : \mathcal{I}_R \rightarrow \mathcal{O}_R\}$  the set of all possible operations in region  $R$ . We drop the index to refer to collections of objects for all regions, as in  $i \equiv \{i_1, \dots, i_N\}$ ,  $\mathcal{I} \equiv \mathcal{I}_1 \times \dots \times \mathcal{I}_N$ ,  $\mathcal{D} \equiv \mathcal{D}_1 \times \dots \times \mathcal{D}_N$ , *etc.*<sup>2</sup> Local operations are not required to be reversible, *i.e.*, the local functions  $f_R$  need not be invertible. This corresponds to the assumption that the local experimenters and devices have the ability to erase information by accessing some reservoir, not included in the description of the physical degrees of freedom of interest. Furthermore, input and output state spaces need not be isomorphic, as degrees of freedom may be added or removed during the operation. We will also consider the special case in which either input or output state space is the empty set. An ‘output only’ region—called a *source*—can be identified with a space-like region on which an agent (acting somewhere in its past) can prepare an arbitrary state (Fig. 2b), while an ‘input only’ region—called a *sink*—can be identified with a space-like region where an agent (somewhere in its future) can only observe the state (Fig. 2c). Ordinary dynamics is concerned with the evolution from a source (state preparation on a space-like surface) to a sink (state observation on a space-like surface).

We want to define a generalised type of dynamics for an arbitrary number of regions—in which arbitrary classical operations can be performed—possibly embedded in a space-time with CTCs. The basic requirement of such a model is that it must be able to predict the state observed on the past boundary of each region, which in general can depend on all local operations. (In a CTC-free space-time, the input state on a space-like region would only depend on operations in its past light-cone). For a deterministic model, such a dependence is encoded in a function  $\omega \equiv \{\omega_1, \dots, \omega_N\} : \mathcal{D} \rightarrow \mathcal{I}$  that determines the state on the past boundary of each region as a function of all local operations. The function  $\omega$  represents all the background information necessary to make predictions: the space-time geometry, the type of degrees of freedom involved and the dynamical equations governing them, possible additional boundary conditions, *etc.*

The only condition we are going to impose on  $\omega$  is a weak form of locality. Locality implies that the observed input states should not depend on the details of the local operations but only on the output states on the future boundaries of each region. Formally, this means that there must exist a function  $w : \mathcal{O} \rightarrow \mathcal{I}$  such that the following consistency condition holds:

$$\omega(f) = w(f(\omega(f))) \quad \forall f \in \mathcal{D}. \quad (1)$$

<sup>1</sup> More rigorously, any reference to time-like curves should be replaced with causal curves, defined as either time-like or null. We keep the reference to time-like curves in accordance to the use in literature and to simplify the presentation.

<sup>2</sup> Note that  $\mathcal{D}$  is not the set of *all* functions  $\mathcal{I} \rightarrow \mathcal{O}$ , but rather of those of the form  $f(i) = \{f_1(i_1), \dots, f_N(i_N)\}$ .

We call  $\omega$  a *process* when condition (1) holds, and  $w$  a *process function*.

By writing  $\omega(f) = i$ , we see that condition (1) implies the following condition for  $w$ :

$$\forall f \in \mathcal{D}, \exists i \in \mathcal{I} \quad \text{such that} \quad w \circ f(i) = i. \quad (2)$$

In other words, if  $w$  is a process function, then  $w \circ f$  has a fixed point for every local operation  $f$ .

As it turns out, condition (2) is necessary and sufficient, that is, a function  $w$  satisfying (2) *uniquely* defines a process. This is because of the uniqueness of the fixed points:

**Theorem 1** (Unique fixed points). *Given a function  $w : \mathcal{O} \rightarrow \mathcal{I}$  that satisfies condition (2), the fixed point of  $w \circ f$  is unique for every set of local operations  $f = \{f_1, \dots, f_N\} \in \mathcal{D}$ .*

We prove this theorem in the appendix. As opposed to the analogous fixed-point theorem proved within the probabilistic version of the formalism [32], our proof also holds for continuous and not only discrete variables.

Because of Theorem 1, every function  $w : \mathcal{O} \rightarrow \mathcal{I}$  that satisfies condition (2) defines a unique function  $\omega : \mathcal{D} \rightarrow \mathcal{I}$ , with  $\omega(f)$  equal to the unique fixed point of  $w \circ f$ . It is furthermore easy to see that condition (2) implies the consistency condition (1). Therefore, we can identify a process with its process function  $w$ . The interpretation is that dynamics in the presence of CTCs is described by a function that maps the states on the future boundaries of all regions to states on the past boundaries of each region. Condition (2) imposes that such a dynamics is compatible with arbitrary operations in each region; Theorem 1 further guarantees that specifying the operations performed in each region is sufficient to predict a *unique* state on each of the past boundaries.

## REVERSIBILITY

Reversible dynamics is associated with invertible functions, such that the role of ‘preparation’ and ‘measurement’ can be swapped. Not all process functions are invertible; for example, the process function for a single ‘sink’ region (with trivial output) reduces to the specification of a state on that region and it is clearly not invertible. However, such a process function can be extended to a reversible one by introducing a ‘source’ region (with trivial input), in the past of the sink, so that the state on the sink can now be calculated as a function of the state prepared by the source, and this function can be invertible.

We can see that *every* process function can be extended to an invertible one (Fig. 3), as expressed by the following theorem, proved in the appendix.

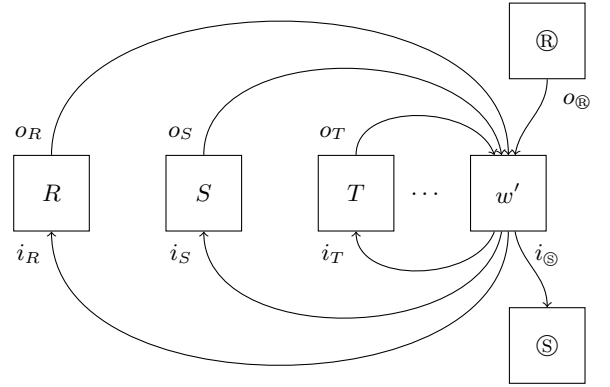


Figure 3. Reversible extension of a process function via the introduction of a source  $\mathbb{R}$  and a sink  $\mathbb{S}$ .

**Theorem 2** (Reversibility). *For every function  $w : \mathcal{O} \rightarrow \mathcal{I}$  that satisfies condition (2), there exists an invertible function  $w' : \mathcal{O} \times \mathcal{O}_{\mathbb{R}} \rightarrow \mathcal{I} \times \mathcal{I}_{\mathbb{S}}$ , where  $\mathcal{O}_{\mathbb{R}}$  is the output space of a region with trivial input (the ‘source’) and  $\mathcal{I}_{\mathbb{S}}$  is the input space of a region with trivial output (the ‘sink’), such that  $w'$  satisfies condition (2) and there exists  $\tilde{o}_{\mathbb{R}} \in \mathcal{O}_{\mathbb{R}}$  such that  $w'(o, \tilde{o}_{\mathbb{R}}) = \{w(o), g_{\mathbb{S}}(o)\}$  for some invertible function  $g_{\mathbb{S}}$ .*

This theorem shows that all process functions can be interpreted in terms of reversible dynamics: The source describes a space-like region ‘in the past’ of all other local regions, while the sink is a space-like region ‘in the future’ of all regions. The process determines the state of the sink as well as the states on the past boundaries of all local regions as a function of the states of the outputs of all regions and of the source. Because it is reversible, the process can be read in the opposite direction: Given the states on the sink and on the past boundaries of all local regions, it allows calculating the state on all future local boundaries, as well as the state of the source. The time-reversed process is then compatible with arbitrary reversed local operations that map local outputs to local inputs. In Ref. [33], it is further proven that the presence of a source and a sink is necessary in order to define a reversible process.

## CHARACTERIZATION OF PROCESS FUNCTIONS

Simple examples of process functions are causally ordered ones, namely those compatible with CTC-free dynamics. For example, for laboratories  $R, S, T, \dots$  a process function  $w \equiv \{w_R, w_S, w_T, \dots\}$  compatible with the causal order  $R \prec S \prec T \prec \dots$  is given by  $w_R(o_R) = \bar{i}_R$  (constant),  $w_S(o_R, o_S) = w_S(o_R)$ ,  $w_T(o_R, o_S, o_T) = w_S(o_R, o_S)$ , etc. It is easy to see that

condition (2) is satisfied in such cases, *i.e.*, a fixed point exists for every choice of local operations (it is given by  $i_R = \bar{i}_R$ ,  $i_S = w_S \circ f_R(\bar{i}_R)$ , and so on). The question we are concerned with is whether more general processes are possible, once CTCs are allowed. To answer this question, we will first give a complete characterisation of all process functions for up to three regions. The detailed proofs can be found in the appendix.

For a single local region, a process function has to be a constant:  $w(o) = \bar{i} \forall o$ . Thus, an observer acting in a localised region cannot send information back to herself; her observations are fully compatible with her region being embedded in a CTC-free space-time. A direct consequence is that, for an arbitrary number of regions, the input of each region  $R$  cannot depend on that region's output:

$$w_R(o) = w_R(o \setminus R),$$

where  $o \setminus R$  is the set of outputs of all regions except  $R$ .

Bipartite process functions are characterized by the following conditions:

- (i)  $w_R(o_R, o_S) = w_R(o_S)$ ,
- (ii)  $w_S(o_R, o_S) = w_S(o_R)$ ,
- (iii) at least one of  $w_R(o_S)$  or  $w_S(o_R)$  is constant.

In other words, deterministic process functions can only allow one-way signaling. Again, two observers in distinct localised regions would not be able to verify the presence of CTCs outside their regions. (Remarkably, this is not true for the quantum version of the framework [20].)

Consider now three regions  $R, S, T$ . For simplicity, we denote input and output variables as  $a \in \mathcal{A}$ ,  $b \in \mathcal{B}$ ,  $c \in \mathcal{C}$  and  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$ ,  $z \in \mathcal{Z}$ , respectively. A process function has then three component functions:  $a = w_R(y, z)$ ,  $b = w_S(x, z)$ ,  $c = w_T(x, y)$  (where we used the fact that the input of each region cannot depend on its own output, as seen above). We give a simple characterization of process functions as functions where the output variable of one region 'switches' the direction of causal influence between the two other parties.

**Theorem 3** (Tripartite process function). *Three functions  $w_R : \mathcal{Y} \times \mathcal{Z} \rightarrow \mathcal{A}$ ,  $w_S : \mathcal{X} \times \mathcal{Z} \rightarrow \mathcal{B}$ ,  $w_T : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{C}$  define a process function if and only if:*

1. for every  $z \in \mathcal{Z}$ ,  $w^z := \{w_R(\cdot, z), w_S(\cdot, z)\}$  is a bipartite process function;
2. for every  $x \in \mathcal{X}$ ,  $w^x := \{w_S(x, \cdot), w_T(x, \cdot)\}$  is a bipartite process function;
3. for every  $y \in \mathcal{Y}$ ,  $w^y := \{w_R(y, \cdot), w_T(\cdot, y)\}$  is a bipartite process function.

Recall that a bipartite process function is at most one-way signaling. The properties 1, 2, 3 above can thus be interpreted as a *one-way conditional-signaling condition* for  $w$ , in the sense that, for every fixed value for the outcome of one of the regions, only one-way signaling is possible between the other two. Theorem 3 shows that  $w$  is a tripartite process function if and only if it satisfies one-way conditional signaling. It is an open question whether a similar condition characterises arbitrary multipartite process functions.

## EXAMPLES

Given the above characterisation, it is simple to find process functions that cannot arise in ordinary, causal space time. Here we present an example, based on a similar process for 'bits,' first found by Araújo and Feix and published in Ref. [19]. Consider a tripartite scenario as above, where  $x, y, z, a, b, c \in \mathbb{R}$ . We define  $w : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  as

$$\begin{aligned} (x, y, z) &\mapsto (a, b, c), \quad \text{with} & (3) \\ a &= \Theta(-y)\Theta(z), \\ b &= \Theta(-z)\Theta(x), \\ c &= \Theta(-x)\Theta(y), \end{aligned}$$

where  $\Theta(t) = 1$  for  $t > 0$ ,  $\Theta(t) = 0$  for  $t \leq 0$ . In this process, the sign of the output of each region determines the direction of signaling between the other two. For example, for  $y \leq 0$  we have  $a = \Theta(z)$  ( $T$  can signal to  $R$ ) but  $c = 0$  ( $R$  cannot signal to  $T$ ), while for  $y > 0$  the opposite direction of signaling holds (and similarly for the other pairs of regions).

By Theorem 2, we can extend  $w$  to a *reversible* process function  $w'$ . To this end, we introduce source and sink spaces, both isomorphic to  $\mathbb{R}^3$ , with variables  $e_0, e_1, e_2$  and  $s_0, s_1, s_2$ , respectively. The extended process function  $w' : \mathbb{R}^6 \rightarrow \mathbb{R}^6$  is given by

$$\begin{aligned} (x, y, z, e_0, e_1, e_2) &\mapsto (a, b, c, s_0, s_1, s_2), \quad \text{with} \\ a &= \Theta(-y)\Theta(z) + e_0, \\ b &= \Theta(-z)\Theta(x) + e_1, \\ c &= \Theta(-x)\Theta(y) + e_2, \\ s_0 &= x, \\ s_1 &= y, \\ s_2 &= z. \end{aligned}$$

Given the process defined by the function above, three observers in regions  $R, S, T$  receive each a system from the respective past boundary and can perform *arbitrary* deterministic operations on it, sending the result out the respective future boundary. The outgoing systems then

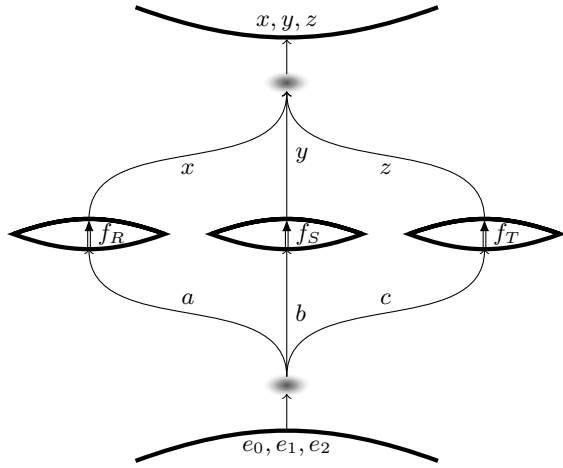


Figure 4. The output of three local regions fall into a CTC where they undergo a joint interaction with the state prepared by the source. The CTC outputs the input states to the three local regions and the sink.

enter the CTC region and undergo some reversible transformation, interacting with each other and with the output of the source  $\textcircled{R}$ , eventually determining the state in the past of each region and of the sink  $\textcircled{S}$  (Fig. 4). Crucially, the input state of each region depends non-trivially on the output state of the other two, thus each observer can communicate to every other. Thus, we have a situation where three observers can experimentally verify to be each *both in the past and in the future* of each other, they can perform *arbitrary* local operations, and *no contradiction* ever emerges.

## QUANTUM CLOSED TIME-LIKE CURVES

The above framework of classical, reversible dynamics can be extended to quantum systems. It is then interesting to compare the resulting model with existing quantum models for CTCs. We briefly present here the main results, and refer to Ref. [33] for a detailed analysis.

A classical system can be ‘quantised’ by associating to each state a distinct orthogonal state in a Hilbert space, with quantum superpositions represented by linear combinations. Thus, in the quantum version of the formalism, the boundary of each region is associated with a Hilbert space. A classical, reversible process defines a permutation of basis elements and can be extended by linearity to the entire Hilbert space, defining a unitary map from the future to the past boundaries. It is not *a priori* guaranteed that such a unitary defines a valid *quantum process*: Observers in the local regions should now be able to perform arbitrary *quantum* operations. The resulting constraints on quantum processes can be conveniently formalised using the *process matrix* formalism of Ref. [20]. Using the characterisation of tri-

partite quantum processes of Ref. [22], it is proven in Ref. [33] that the quantisation of a finite-dimensional version of Eq. (3) indeed defines a *valid* unitary quantum process.

The two most studied models of quantum systems in the presence of CTCs are the so-called post-selected CTC model (P-CTC) [34–39] and the Deutsch model (D-CTC) [40–44]. Both models assume that CTCs are only present in a limited portion of space-time. At some time before the CTCs, a chronology-respecting (CR) system is prepared. Then, the CR system interacts with a chronology-violating (CV) one, which travels along a CTC. The models prescribe how to calculate the state of the CR system obtained after interaction with the CV one. Within such frameworks, we can model the multi-region scenarios considered here by introducing a CR and a CV systems per region, and interpreting the interaction between each pair as our local operation in the corresponding local region. The CV systems then interact according to the unitary process and are later sent back in time, with the backward evolution described according to the specific model. We can then compare the evolution of the CV system predicted by each model.

As it turns out, the P-CTC model gives the same predictions as ours for any *valid* unitary process. The crucial difference is that the P-CTC model allows the CV system to evolve according to *arbitrary* unitaries, generically resulting in a *non-linear* evolution for the CR system and in a restriction on the local operations that can be performed. By contrast, our model imposes additional constraints, which effectively enforce the CR system to evolve *linearly*. The D-CTC model, on the other hand, allows arbitrary operations to be performed locally. However, it predicts *non-linear* evolution of the CR system, even when the CV system evolves according to a process subject to the constraints introduced here [45].

## CONCLUSIONS

We developed a framework for deterministic dynamics in the presence of CTCs. The framework extends the ordinary concept of time evolution—where a future state is calculated as a function of a past one—to the more general scenario of a number of space-time region, where the state on the past of each region is calculated as a function of the state in the future of all region. Insisting that *arbitrary* operations must be possible in each region imposes strong constraints on the allowed dynamics. Our main result is that it is possible to have *reversible* dynamics, compatible with *arbitrary* local operations, where the state observed in each region depends non-trivially on the states prepared in all other regions. Because such a functional relation is reversible, it is always possible to model it using some physical system subject to local dynamical laws, *e.g.*, in terms of a system of bouncing billiard

balls [46].

It is worth noting that, in general, there might be multiple ways to implement a reversible function in terms of physical interactions. This agrees with the classic CTC results [8, 14, 15]: The specification of boundary conditions in the local regions does *not* fix uniquely the dynamics in the exterior region. However, our model guarantees that, whenever an observation is made, a *unique* state is observed.

The main message of our result is that CTCs are not necessarily in conflict with local physics, nor with the ‘free will’ associated to the possibility of performing arbitrary operations. Importantly, quantum mechanics plays no particular role in the definition of the formalism, although a natural quantum extension is possible.

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## Properties of process functions

Here we derive a set of properties of process functions, which will be needed in later proofs. We will use the term *process function* to denote any function  $w$  that satisfies condition (2), namely that a fixed point of  $w \circ f$  exists for each  $f \in \mathcal{D}$ . This condition is equivalent to the main-text definition of process function, Eq. (1), thanks to Theorem 1, proved in the next section.

Let us first fix some notation. As in the main text, an object without index refers to a collection of objects:  $\mathcal{I} = \mathcal{I}_1 \times \dots \times \mathcal{I}_N$ , etc. We will also use the notation  $\mathcal{I}_{\setminus R} = \mathcal{I}_1 \times \dots \times \mathcal{I}_{R-1} \times \mathcal{I}_{R+1} \times \mathcal{I}_N$ ,  $i_{\setminus R} = \{i_1, \dots, i_{R-1}, i_{R+1}, \dots, i_N\}$ , etc., to denote collections with the component  $R$  removed. Appropriate reordering will be understood when joining variables, for example in expressions as  $i = i_R \cup i_{\setminus R}$ ,  $f(i) = f(i_R, i_{\setminus R})$ , and so on.

The first property we need is a necessary condition for process functions:

**Lemma 1.** *For a process function  $w$ , each component  $w_R : \mathcal{O} \rightarrow \mathcal{I}_R$  must be constant over  $\mathcal{O}_R$ :  $w_R(o) = w_R(o_{\setminus R})$  for  $R = 1, \dots, N$ .*

*Proof.* For some set of local operations  $f = \{f_R : \mathcal{I}_R \rightarrow \mathcal{O}_R\}_{R=1}^N$  and some fixed  $\bar{i}_{\setminus R}$ , let us define  $h_R : \mathcal{I}_R \rightarrow \mathcal{I}_R$  as  $h_R(i_R) = w_R \circ f(i_R, \bar{i}_{\setminus R})$ . Let us assume that  $h_R$  is not a constant, namely there exist  $i_R^1, i_R^2$  such that  $a_R^1 = h_R(i_R^1) \neq h_R(i_R^2) = a_R^2$ . We define then the function  $g_R : \mathcal{I}_R \rightarrow \mathcal{I}_R$  as

$$\begin{aligned} g_R(a_R^1) &= i_R^2, \\ g_R(i_R) &= i_R^1 \quad \forall i_R \neq a_R^1. \end{aligned}$$

It is then easy to see that  $h_R \circ g_R$  has no fixed point. Indeed,  $h_R \circ g_R(a_R^1) = h_R(i_R^2) = a_R^2 \neq a_R^1$ , while for any  $i_R \neq a_R^1$ ,  $h_R \circ g_R(i_R) = h_R(i_R^1) = a_R^1 \neq i_R$ . Thus, if  $w_R \circ f$  is not a constant over  $\mathcal{I}_R$ , then  $w_R \circ (f \circ g_R)$  has no fixed point and  $w_R$  is not a component of a process function. As this must hold for every set of local operations  $f$  and every  $\bar{i}_{\setminus R}$ , we conclude that each component  $w_R$  of a process function must be a constant over  $\mathcal{O}_R$ .  $\square$

Lemma 1 immediately implies a characterisation of single-region process functions:

**Corollary 1.** *Given a function  $w : \mathcal{O} \rightarrow \mathcal{I}$ ,  $w \circ f$  admits a fixed point for every  $f : \mathcal{I} \rightarrow \mathcal{O}$  if and only if  $w$  is a constant.*

*Proof.* As a consequence of Lemma 1, every single-partite process function must be a constant. On the other hand, for a constant function  $w$ ,  $i = w(o)$  is a fixed point of  $w \circ f$  for every  $f$ , thus every constant  $w$  is a process function.  $\square$

Intuitively, if we fix the operation performed in one of the  $N$ -th regions, we should obtain a process for the

remaining  $N - 1$  regions. This intuition will also play an important role in the proofs below. The first step to formalise such an intuition is the following definition:

**Definition 1.** *Consider a function  $w : \mathcal{O} \rightarrow \mathcal{I}$ , such that, for each region  $R = 1, \dots, N$ ,  $w_R(o) = w_R(o_{\setminus R})$ . For a given local operation  $f_R : \mathcal{I}_R \rightarrow \mathcal{O}_R$ , we define the **reduced function**  $w^{f_R} : \mathcal{O}_{\setminus R} \rightarrow \mathcal{I}_{\setminus R}$  on the remaining regions by composing  $w$  with  $f_R$ :*

$$w_S^{f_R}(o_{\setminus R}) := w_S(o_{\setminus R}, f_R(w_R(o_{\setminus R}))), \quad S \neq R. \quad (4)$$

We will need the following fact:

**Lemma 2.** *If  $i \in \mathcal{I}$  is a fixed point of  $w \circ f$ , then  $i_{\setminus R}$  is a fixed point of  $w^{f_R} \circ f_{\setminus R}$ .*

*Proof.* Since  $i$  is a fixed point of  $w \circ f$ , we have  $i_R = w_R(f_{\setminus R}(i_{\setminus R}))$ . Then, for  $S \neq R$ , Eq. (4) implies

$$\begin{aligned} w_S^{f_R} \circ f_{\setminus R}(i_{\setminus R}) &= w_S(f_{\setminus R}(i_{\setminus R}), f_R(i_R)) \\ &= w_S \circ f(i) = i_S. \end{aligned} \quad \square$$

We can now prove two crucial properties.

**Lemma 3.** *Given a function  $w : \mathcal{O} \rightarrow \mathcal{I}$ , such that, for each region  $R = 1, \dots, N$ ,  $w_R(o) = w_R(o_{\setminus R})$ , we have*

- (i) *If  $w$  is a process function, then  $w^{f_R}$  is also a process function for every region  $R$  and operation  $f_R$ .*
- (ii) *If there exists a region  $R$  such that, for every local operation  $f_R$ ,  $w^{f_R}$  is a process function, then  $w$  is also a process function.*

*Proof.* Point (i) is a direct consequence of Lemma 2: For every set of operations  $f_{\setminus R}$ , a fixed point of  $w_S^{f_R} \circ f_{\setminus R}$  is given by  $i_{\setminus R}$ , where  $i$  is a fixed point of  $f = f_R \cup f_{\setminus R}$ .

To prove (ii), we can set  $R = 1$  without loss of generality. We then have to prove that, if  $w^{f_1}$  is a process function for every  $f_1$ , it follows that  $w$  is also a process function, i.e., we have to find a fixed point of  $w \circ f$  for arbitrary  $f$ . As the reduced function  $w^{f_1}$  is a process function by assumption, there exists a fixed point  $i_{\setminus 1}$  of  $w^{f_1} \circ f_{\setminus 1}$ . Choosing  $i_1 = w_1 \circ f_{\setminus 1}(i_{\setminus 1})$  as input state for region 1, we see that the  $i := i_1 \cup i_{\setminus 1}$  is a fixed point of  $w \circ f$ . Indeed this is true, by definition of  $i_1$ , for the component  $w_1$ . For  $S > 1$ , the definition of reduced function, Eq. (4), gives

$$\begin{aligned} w_S \circ f(i) &= w_S(f_1(i_1), f_{\setminus 1}(i_{\setminus 1})) \\ &= w_S^{f_1} \circ f_{\setminus 1}(i_{\setminus 1}) = i_{\setminus 1}, \end{aligned}$$

where in the last equality we used the fact that  $i_{\setminus 1}$  is the fixed point of  $w_S^{f_1} \circ f_{\setminus 1}$ .  $\square$

Following the result of this lemma, we can call *reduced process function* the reduced function of a process.

### Uniqueness of the fixed point

Here we prove Theorem 1, namely the following  $N$ -dependent proposition.

$P[N]$ : Let  $w : \mathcal{O} \rightarrow \mathcal{I}$  be such that, for every collection of functions  $f = \{f_1, \dots, f_N\}$ ,  $f_R : \mathcal{I}_R \rightarrow \mathcal{O}_R$ , there exists at least one fixed point  $i \in \mathcal{I}$ ,  $w \circ f(i) = i$ . Then, the fixed point is unique for each  $f$ .

We prove this by induction, namely we first prove  $P[1]$  and then the implication  $P[N-1] \Rightarrow P[N]$  for  $N > 1$ .

$P[1]$  is a simple consequence of Corollary 1:  $w \circ f$  can have a fixed point for every  $f$  only if  $w$  is constant,  $w(o) = \bar{i}$  for every  $o$ . Then,  $\bar{i}$  is the unique fixed point of  $w \circ f$  for every  $f$ .

For  $N > 1$ , let us assume  $P[N]$  is false and let  $a = \{a_1, \dots, a_N\}$ ,  $b = \{b_1, \dots, b_N\}$  be two distinct fixed points of  $w \circ f$ , where  $w$  is an  $N$ -partite process function. Without loss of generality, we can assume that they differ in the first component,  $a_1 \neq b_1$ . This means that the reduced function  $w^{f_N} \circ f_{\setminus N}$  has two distinct fixed points,  $a_{\setminus N} \neq b_{\setminus N}$ . But this is in contradiction with  $P[N-1]$ , because, according to point (i) of Lemma 3,  $w^{f_N}$  is an  $N-1$ -partite process function and thus has a single fixed point for each  $f_{\setminus N}$ . This concludes the proof.

### Reversibility

Here we prove that every process function can be extended to an invertible process function (Theorem 2).

*Proof.* Given a process function  $w : \mathcal{O} \rightarrow \mathcal{I}$  over  $N$  local regions, we add two extra regions,  $\mathbb{R}$  and  $\mathbb{S}$ , a source and a sink, respectively. We take the output space of the source to be isomorphic with the entire input space of the  $N$  regions,  $e \in \mathcal{O}_{\mathbb{R}} \cong \mathcal{I}$ , while the input space of the sink is isomorphic to the output space of the regions,  $s \in \mathcal{I}_{\mathbb{S}} \cong \mathcal{O}$ . For each  $R = 1, \dots, N$  and each  $e_R \in \mathcal{I}_R$ , we introduce a function  $T_R^{e_R} : \mathcal{I}_R \rightarrow \mathcal{I}_R$  such that there exists  $\tilde{e}_R$  for which  $T_R^{e_R}(\tilde{e}_R) = i_R$  and, for each  $i_R \in \mathcal{I}_R$ ,  $T_R^{(\cdot)}(i_R)$  is invertible. We can take  $T_R^{e_R}(i_R) = i_R + e_R$  for concreteness<sup>3</sup>. We then extend the process function  $w : \mathcal{O} \rightarrow \mathcal{I}$  to a function  $w' \equiv (w^1, w^2) : \mathcal{O} \times \mathcal{O}_{\mathbb{R}} \rightarrow \mathcal{I} \times \mathcal{I}_{\mathbb{S}}$ , defined as

$$\begin{aligned} w_R^1(o, e) &= T_R^{e_R} \circ w_R(o) = w_R(o) + e_R \\ w_R^2(o, e) &= o_R. \end{aligned}$$

The function  $w'$  is invertible, with the inverse given by  $i_R = w_R(s) - e_R$ ,  $o_R = s_R$ . To show that it is a process function, we have to prove that its composition with arbitrary local operations has a fixed point, condition (2).

<sup>3</sup> If  $|\mathcal{I}_R| = c_R < \infty$ , we can use  $T_R^{e_R} = i_R \oplus e_R$ , where  $\oplus$  is addition modulo  $c_R$ .

Note that this condition is equivalent to the existence of output fixed points:  $f' \circ w'(o') = o'$ ,  $o' \equiv (o, e)$ . Since local operations for  $\mathbb{R}$  are functions  $\emptyset \rightarrow \mathcal{O}_{\mathbb{R}}$ , where  $\emptyset$  is the empty set, they can be identified with a state  $f(\emptyset) \equiv e \in \mathcal{O}_{\mathbb{R}}$ , interpreted as ‘state preparation.’ The fixed-point condition for the source components is then trivially satisfied by any  $e \in \mathcal{O}_{\mathbb{R}}$ . As the sink has no output space, the fixed-point condition reduces to

$$o_R = f_R \circ T_R^{e_R} \circ w_R(o),$$

which should be satisfied for every  $f \in \mathcal{D}$  and  $e \in \mathcal{O}_{\mathbb{R}}$ . This is true because  $f_R \circ T_R^{e_R}$  is a local operation and, as  $w$  is a process function, a fixed point  $o \in \mathcal{O}$  exists for every local operation.  $\square$

### Characterizations

Here we prove the characterisations of process functions for up to three local regions. The single-region characterisation is given by Corollary 1: all and only constants are single-region process functions.

#### Two regions

We relabel input and output of region  $R$  as  $i_R \rightarrow a \in \mathcal{A}$ ,  $o_R \rightarrow x \in \mathcal{X}$ , respectively, and inputs and outputs of region  $S$  as  $i_S \rightarrow b \in \mathcal{B}$ ,  $o_S \rightarrow y \in \mathcal{Y}$ . For a bipartite process function  $w = \{w_R, w_S\}$ , the single-party characterisation implies that  $w_R(x, y) = w_R(y)$ ,  $w_S(x, y) = w_S(x)$ . It is furthermore clear that, if at least one of the two components of a function  $w = \{w_R, w_S\}$  is a constant, then  $w$  is a process function. (Given  $w_R(y) = a_0$ , the fixed point  $i \equiv \{a, b\}$  is given by  $a = a_0$ ,  $b = w_S(f_R(a_0))$ .)

It remains to prove that if  $w$  is a process function, then at least one of the two components is constant.

*Proof.* The consistency condition (2) says that, for every local operation  $f_R, f_S$ , there exists  $a \in \mathcal{A}$ ,  $b \in \mathcal{B}$  such that

$$w_R(f_S(b)) = a, \quad w_S(f_R(a)) = b.$$

By plugging the second equation into the first we obtain

$$w_R \circ f_S \circ w_S \circ f_R(a) = a.$$

The single-party characterisation tells us that  $w_R \circ f_S \circ w_S$  must be a constant, and this must be true for all  $f_S$ . This is only possible if one of the two functions,  $w_R, w_S$ , is constant.  $\square$



Three regions

We prove here Theorem 3, which characterises tripartite process functions as those with one-way conditional signalling.

*Proof.* As in the main text, we consider three regions  $R$ ,  $S$ ,  $T$ , with input states  $a \in \mathcal{A}$ ,  $b \in \mathcal{B}$ ,  $c \in \mathcal{C}$  and output states  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$ ,  $z \in \mathcal{Z}$ . The function  $w^z : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{A} \times \mathcal{B}$ , defined as  $w^z(x, y) := \{w_R(y, z), w_S(x, z)\}$ , is the reduced function obtained from  $w$  by fixing the local operation of  $T$  to be the constant function with output  $z$ . If  $w$  is a process function, by point (i) of Lemma 3,  $w^z$  is a bipartite process function, and similarly for the analogously defined  $w^x$  and  $w^y$ . This proves the ‘easy’ direction of the theorem.

We want to prove the converse: If  $w^x$ ,  $w^y$ , and  $w^z$  are bipartite process functions for arbitrary  $x, y, z$ , then  $w$  is also a process function. Note that we cannot apply point (ii) of Lemma 3 directly, because that requires knowing that the reduced function is a process function for an *arbitrary* local operation, not just for the constant operation.

By assumption, we know that  $w^z$  is a bipartite process function. This means that, for every given  $z$ ,  $w^z$  is one-way signalling. In other words, at least one of the two components,  $w_R^z \equiv w_R(\cdot, z)$  or  $w_S^z \equiv w_S(\cdot, z)$ , must be a constant. We denote  $\mathcal{Z}_R \subset \mathcal{Z}$  the subset of outputs of  $T$  for which  $w_R^z$  is constant, and  $\mathcal{Z}_S \subset \mathcal{Z}$  the subset for which  $w_S^z$  is constant. Because at least one of the two components is constant, we have  $\mathcal{Z} = \mathcal{Z}_R \cup \mathcal{Z}_S$ . (The two subsets can have non-null intersection.) In a similar way, we write  $\mathcal{X} = \mathcal{X}_S \cup \mathcal{X}_T$  and  $\mathcal{Y} = \mathcal{Y}_R \cup \mathcal{Y}_T$ , where  $w_S^x$  is constant for  $x \in \mathcal{X}_S$ , and so on. Thus we have

$$\begin{aligned} w_R(y_R, z) &= w_R(y, z_R) = a_0 \\ &\quad \forall y_R \in \mathcal{Y}_R, y \in \mathcal{Y}, z_R \in \mathcal{Z}_R, z \in \mathcal{Z}; \\ w_S(x_S, z) &= w_S(x, z_S) = b_0 \\ &\quad \forall x_S \in \mathcal{X}_S, x \in \mathcal{X}, z_S \in \mathcal{Z}_S, z \in \mathcal{Z}; \\ w_T(x_T, y) &= w_T(x, y_T) = c_0 \\ &\quad \forall x_T \in \mathcal{X}_T, y \in \mathcal{Y}, y_T \in \mathcal{Y}_T, x \in \mathcal{X}. \end{aligned}$$

Now consider an arbitrary local operation  $f_R : \mathcal{A} \rightarrow \mathcal{X}$ . We want to show that the reduced function  $w^{f_R}$ , defined as in Eq. (4), is a bipartite process function. To this end, we need to prove:

- (i)  $w_S^{f_R}(y, z) = w_S^{f_R}(z)$ ,
- (ii)  $w_T^{f_R}(y, z) = w_T^{f_R}(y)$ ,
- (iii) At least one of the two components is constant.

Let us start with point (i). For  $z_S \in \mathcal{Z}_S$ , we have  $w_S^{f_R}(y, z_S) = b_0$ , independently of  $y$ . Let us then consider  $z_R \in \mathcal{Z}_R$ . By definition,  $w_S^{f_R}(y, z_R) =$

$w_S(f_R \circ w_R(y, z_R), z_R)$ . But  $w_R(y, z_R) = a_0$  for  $z_R \in \mathcal{Z}_R$ , thus  $w_S^{f_R}(y, z_R) = w_S(f_R(a_0), z_R)$ , which is again independent of  $y$ . By a similar argument,  $w_T^{f_R}(y, z)$  is independent of  $z$ .

We are thus left with proving point (iii). We shall prove the equivalent implications

$$\begin{aligned} w_S^{f_R} \text{ not constant} &\Rightarrow w_T^{f_R} \text{ constant}, \\ w_T^{f_R} \text{ not constant} &\Rightarrow w_S^{f_R} \text{ constant}. \end{aligned}$$

Say that  $w_S^{f_R}$  is not constant. Then, there is a  $z_R \in \mathcal{Z}_R$  such that  $w_S^{f_R}(z_R) \neq b_0$ . As we have seen above, for  $z_R \in \mathcal{Z}_R$ ,  $w_S^{f_R}(z_R) = w_S(f_R(a_0), z_R)$ , so we need  $f_R(a_0) \notin \mathcal{X}_S$  to have  $w_S^{f_R}(z_R) \neq b_0$ . This means that  $f_R(a_0) \in \mathcal{X}_T$ . But then, for  $y_R \in \mathcal{Y}_R$  we have  $w_T^{f_R}(y_R) = w_T(f_R(a_0), y_R) = c_0$ , and also  $w_T^{f_R}(y_T) = c_0$  for  $y_T \in \mathcal{Y}_T$ . This means that  $w_T^{f_R}$  is constant.

To recapitulate, we have proven that, if  $w^x$ ,  $w^y$ ,  $w^z$  are bipartite process functions for arbitrary  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$ ,  $z \in \mathcal{Z}$  (as per hypothesis), then  $w^{f_R}$  is a bipartite process function for an arbitrary operation  $f_R$ . Point (ii) of Lemma 3 finally implies that  $w$  is a process function, concluding the proof.  $\square$