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The number of fundamental constants from a spacetime-based perspective

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Abstract: We revisit Duff, Okun, and Veneziano's divergent views on the number of fundamental constants and argue that the issue can be set to rest by having spacetime as the starting point. This procedure disentangles the resolution in what depends on the assumed spacetime (whether relativistic or not) from the theories built over it. By defining that the number of fundamental constants equals the minimal number of independent standards necessary to express all observables, as assumed by Duff, Okun, and Veneziano, it is shown that the same units fixed by the apparatuses used to construct the spacetimes are enough to express all observables of the physical laws defined over them. As a result, the number of fundamental constants equals two in Galilei spacetime and one in relativistic spacetimes.

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Contents

1 Introduction

In 2002, three eminent physicists published in this journal an intriguing paper exposing their divergent views on the number of fundamental constants [1]: Okun argues in favor of the most traditional view that the answer is three, Veneziano favors two (based on superstring theory), and Duff advocates that the response is zero. This article has attracted considerable attention. Compared to other publications in the same field, it has received 54 times more citations than average, as seen on the paper's site at JHEP. Despite this, to our knowledge, no clear-cut resolution of this issue has been presented yet. Although science progresses in the middle of unresolved controversies, it is imperative to step back from time to time to tie up the loose ends left in the way before proceeding. This is our goal here.

Firstly, we must concord what the quest for the "number of fundamental constants" is all about. Concerning it, let us combine the definition given by Okun and Veneziano, who identify it with the number of units necessary to express all observables (see abstracts in Okun and Veneziano's parts in ref. [1]), and Duff's operational definition: *"let us ask whether there are any experiments that can be performed which would tell us whether the alien's universe has the same or different constants of nature as ours"* (see Sec. 4 of Duff's part in ref. [1]). By this token, the question we will answer to resolve Duff, Okun, and Veneziano's (DOV's) puzzle is:

> *What is the minimum number of apparatuses (or standards if one prefers) that a cosmic factory must build and distribute all over the Universe to allow distinct labs to compare the values of the observables?*

Answering this complies with DOV's understanding of "fundamental constants" and ratifies Duff's statement that: *"It seems to me that this issue of what is fundamental will continue to go round and around until we can all agree on an operational definition of 'fundamental constants' "*. Moreover, this is endorsed by modern metrology since, after the 2019 revision, each unit of SI was defined by fixing the exact number of one constant of nature [2].

Our strategy to answer the question above is to start from the spacetimes (whether relativistic or not), over which all other theories are built. To construct a spacetime, some apparatuses are required. Minkowski and other relativistic spacetimes only demand the existence of bona fide clocks, while Galilei spacetime also needs rulers. The units fixed by these apparatuses account for expressing spacetime observables, posing a lower bound for the number of dimensional units. Interestingly, we show that these units are also sufficient to express the entire set of observables of the physical laws constructed in the corresponding spacetimes. As a result, we will show that the number of fundamental constants equals two in Galilei spacetime and one in relativistic spacetimes.

The paper is organized as follows. In section 2, Galilei and Minkowski spacetimes are revisited, emphasizing that the definition of Galilei spacetime demands "bona fide" rulers *and* clocks while for Minkowski spacetime (and other relativistic ones), bona fide clocks suffice. Those apparatuses provide the space and time units that account for expressing all spacetime observables. In section 3, we run history in reverse and recall how the SI units can be reduced to the MKS system, which suffices to express all observables of the physical laws. Although DOV do not dispute that MKS is enough to express all observables, we have included this section for completeness and further discussion. In section 4, observables of the physical laws in Galilei spacetime are shown to be expressible solely in terms of space and time units. In section 5, observables defined in Minkowski spacetime are shown to be expressible in terms of units of time only. In particular, a simple procedure that allows the evaluation of distances with three clocks is presented. In section 6, we fulfill the program and connect the single unit necessary to express all observables in relativistic spacetime with one "fundamental" constant. Our closing remarks are in section 7.

2 Galilei and Minkowski spacetimes

Galilei and Minkowski spacetimes are sets of *events* satisfying certain conditions. They are both four-dimensional, homogeneous, spatially isotropic, and rigid (meaning here that they have no dynamical degrees of freedom). For every event O, let us define the

- *past* of O as the subset of events $\mathcal{P}$ that O can be reached from ($\mathcal{P} \prec O$),
- *future* of O as the subset of events $\mathcal{F}$ that can be reached from O ($\mathcal{F} \succ O$).

Moreover, Galilei and Minkowski spacetimes are time oriented in the sense that

$$\mathcal{P} \prec \mathcal{F} \;\Rightarrow\; \mathcal{F} \nprec \mathcal{P}.$$

Galilei *and* Minkowski spacetimes demand the existence of "bona fide" clocks. **Bona fide clocks** are pointlike apparatuses that ascribe the *same* real number (time interval) to any given *arbitrarily-close*-causally-connected pair of events they visit regardless of the state of motion and past history of the clocks [3]. Next, the properties that characterize each spacetime separately are summarized.
Galilei spacetime: It follows from above that

$$O_1 \sim O_2 \Leftrightarrow O_2 \sim O_1, \tag{2.1}$$

where "∼" was used as a shortcut for $\not\lessgtr$ (neither precedes nor succeeds). Equation (2.1) supplied with the *nontrivial* property of Galilei spacetime,

$$(O_1 \sim O_2 \quad \text{and} \quad O_2 \sim O_3) \Rightarrow O_1 \sim O_3, \tag{2.2}$$

allows the foliation of Galilei spacetime in equivalence classes Σ_t ($t \in \mathbb{R}$) of events that are neither to the future nor to the past of each other, and every event will belong to one, and only one, Σ_t. Thus, for every event O, let us define the

- *present* of O as the subset of *simultaneous* events $\mathcal{S}$ that belong neither to the past nor to the future of O ($\mathcal{S} \sim O$).

Each Σ_t of Galilei spacetime is a 3-dimensional Euclidean space, $(\mathbb{R}^3, \delta)$, where δ stands for the Euclidean metric. To make sense of the Euclidean spaces $\Sigma_t = (\mathbb{R}^3, \delta)$, Galilei spacetime must be endowed with bona fide rulers. **Bona fide rulers** are identical one-dimensional straight segments as defined by Euclid's axioms, assigning the same real number to every given pair of simultaneous events they visit regardless of their states of motion and past histories.

In order to calibrate the rulers' unit to be the same in different Σ_t, one can use *congruences of inertial observers.* A congruence of observers is a set of observers covering the spacetime such that each event is visited by one, and only one, observer. A congruence of inertial observers is composed of freely-moving observers as witnessed by comoving "*inertiometers*." An inertiometer may be thought of as a small cubic box with a mass at the center held by six identical springs attached to the cube faces. An observer is inertial if, and only if, the mass of the inertiometer lies at rest in the center. It is a property of Galilei spacetime that observers of a given inertial congruence lie still from each other at a constant distance.

Finally, bona fide clocks are necessary to make sense of the following nontrivial property of Galilei spacetime: the time interval, $t_2 - t_1$, as measured by *any* (inertial or non-inertial) observers between *any* events $\mathcal{P} \in \Sigma_{t_1}$ and $\mathcal{F} \in \Sigma_{t_2}$ ($\mathcal{P} \prec \mathcal{F}$) is the same. Once some hypersurface is chosen to be Σ_0, the other ones, Σ_t, are labeled accordingly.

In practice, apparatuses able to accurately count 9 192 631 770 oscillations of the radiation emitted in the transition between two hyperfine ground states of cesium-133 fulfill all the requirements to be considered bona fide clocks for all actual purposes, and the corresponding time-lapse is defined to be 1 s. Similarly, space segments traced by light rays along 1/299 792 458 s can be considered bona fide rulers, and the corresponding size measures 1 m. (Whether nature realizes perfect bona fide clocks and rulers is a separate issue, which we shall briefly touch on at the end.)

So far, we have required bona fide rulers and clocks to build up Galilei spacetime. Let us move on and show that bona fide clocks suffice to define Minkowski spacetime.

Minkowski spacetime: Although Minkowski spacetime complies with eq. (2.1) it does not with eq. (2.2) (see figure 1). This precludes us from sorting out its events in equivalence classes of simultaneous events as in Galilei's case. Instead, Minkowski spacetime holds on distinct presumptions:

- *Galilei's principle of relativity,* namely, that identical experiments conducted by distinct congruences of inertial observers lead to equivalent results.

- *Worldlines of light rays in the vacuum are absolute (meaning they do not depend on the emitter's worldline).*

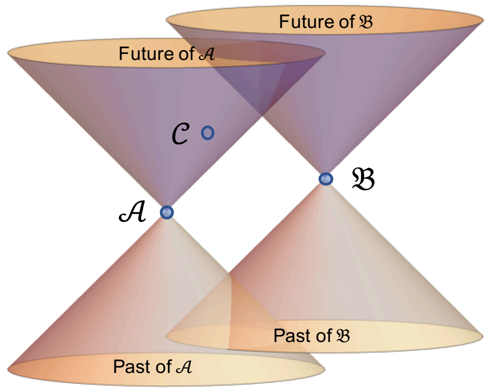

Figure 1. The figure illustrates three events in Minkowski spacetime. Event $\mathcal{B}$ is neither in the past nor in the future of $\mathcal{A}$, $\mathcal{A} \sim \mathcal{B}$, and event C is neither in the past nor in the future of $\mathcal{B}$, $\mathcal{B} \sim C$. Despite this, $C \nsim \mathcal{A}$. Indeed, C is in the future of $\mathcal{A}$: $C > \mathcal{A}$.

As a consequence, the structure of Minkowski spacetime turns out to be $(\mathbb{R}^4, \eta)$, where η stands for the spacetime metric. To make sense of it, bona fide clocks defined above are all one needs. To see it, let it be two arbitrary events, $\mathcal{P}$ and $\mathcal{Q}$, and some inertial congruence C — see figure 2. Now, suppose that some observer O of congruence C passes through $\mathcal{P}$. This observer emits a light ray at event $\mathcal{R}$ to hit event $\mathcal{Q}$. The ray is then reflected back and received by O in event $\mathcal{S}$. Observer O uses bona fide clocks to measure the proper time intervals $\Delta\tau_{\mathcal{RP}}$ and $\Delta\tau_{\mathcal{PS}}$ between events $\mathcal{R}$-$\mathcal{P}$ and $\mathcal{P}$-$\mathcal{S}$, respectively. Naturally, some other inertial observer O' of some other congruence C' going through $\mathcal{P}$ and repeating the same procedure will obtain, in general, other values $\Delta\tau_{\mathcal{R'P}}$ and $\Delta\tau_{\mathcal{PS'}}$. However, Minkowski spacetime is characterized by the fact that the *product of these time intervals is an invariant*:

$$\Delta\tau_{\mathcal{RP}}\Delta\tau_{\mathcal{PS}} = \Delta\tau_{\mathcal{R'P}}\Delta\tau_{\mathcal{PS'}}. \tag{2.3}$$

In order to make contact of it with the usual Minkowski *line element* in Cartesian coordinates, let us define the spatial distance between $\mathcal{P}$ and $\mathcal{Q}$ with respect to observer O (in light-seconds) as

$$\Delta\ell^O \equiv (\Delta\tau_{\mathcal{RP}} + \Delta\tau_{\mathcal{PS}})/2. \tag{2.4}$$

Let us also define a time interval between $\mathcal{P}$ and $\mathcal{Q}$ as the time interval between $\mathcal{P}$ and the (arbitrarily defined) simultaneous-with-respect-to-O event, here denoted by $\mathcal{Q}_s$, located at the middle point between events $\mathcal{R}$ and $\mathcal{S}$ (on the worldline of observer O):

$$\Delta t^O \equiv (-\Delta\tau_{\mathcal{RP}} + \Delta\tau_{\mathcal{PS}})/2. \tag{2.5}$$

Analogous definitions hold to O'. Using them to rewrite eq. (2.3), one obtains

$$-(\Delta t^O)^2 + (\Delta\ell^O)^2 = -(\Delta t^{O'})^2 + (\Delta\ell^{O'})^2,$$

which reflects the invariance of the line element of Minkowski spacetime in Cartesian coordinates $\{t, x, y, z\}$:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.6}$$

Definitions (2.4)-(2.5) might seem artificial, but they only reflect the arbitrariness in the choice of the Cartesian coordinates. ($\Delta\ell^O$, as well as x, y, z, are defined to have time units, say, seconds, although it is usual to add the unnecessary "light-" prefix to them.)

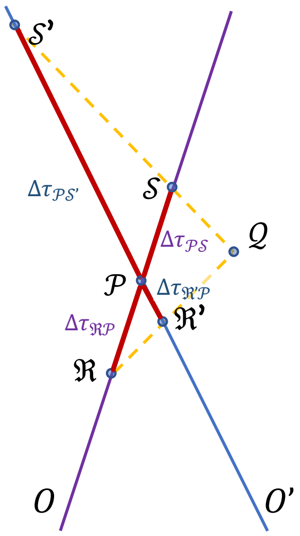

Figure 2. Let a pair of events $\mathcal{P}$ and $\mathcal{Q}$ and two arbitrary inertial observers O and O' passing through $\mathcal{P}$. Here $\mathcal{P} \sim \mathcal{Q}$ but we could have chosen $\mathcal{P} \nsim \mathcal{Q}$, as well. Observers emit light rays at event $\mathcal{R}$ and $\mathcal{R}'$, respectively, to be received at $\mathcal{Q}$, where they are reflected back reaching the corresponding observers in $\mathcal{S}$ and $\mathcal{S}'$. In this illustration, $\mathcal{R} \prec \mathcal{P}$ and $\mathcal{P} \prec \mathcal{S}$, and, hence, both time intervals, $\Delta\tau_{\mathcal{R}\mathcal{P}}$ and $\Delta\tau_{\mathcal{P}\mathcal{S}}$, are positive definite and, similarly, for $\Delta\tau_{\mathcal{R}'\mathcal{P}}$ and $\Delta\tau_{\mathcal{P}\mathcal{S}'}$.

Thus, by embracing Galilei and Minkowski spacetimes one must automatically presuppose the existence of the apparatuses needed to define them. *We will show that the units fixed by these apparatuses are enough to express all observables.*

3 Recuperating the MKS system from the SI

The International System (SI) has seven basic units: meter, second, kilogram, kelvin, ampere, candela, and mol. Its birth can be traced back to 1960. After the last revision in 2019 [4], the units of SI were defined by fixing the exact numerical values of seven constants [5–7]: the speed of light in vacuum c, the transition frequency between two hyperfine ground states of cesium-133 $\Delta\nu_{\text{Cs}}$, the Planck constant h, the elementary charge e, the Boltzmann constant k_B, the Avogadro constant N_A, and the luminous efficacy K_{cd}. If all seven SI units were needed to express the observables of nature, then, according to the criterion stated in the Introduction, the number of fundamental constants would be seven, but this is not so. Next, we run history in reverse and recall how the SI units can be reduced to the MKS system. Although DOV do not dispute that MKS is enough to express all observables, this section was included for completeness and further reference.

The mol equals a natural number and will not concern us here. By the same token, 1 cd $\equiv$ $(1/683)$ kg $\cdot$ m^2 $\cdot$ s^{-3}/sr is simply a unit of power per solid angle for a green-light source emitting at a frequency of 540×10^{12} s^{-1} (which approximates the frequency of maximum sensitivity of the human eye). Thus, let us move on and focus on the kelvin and ampere units.

After the 2019 revision, the kelvin was defined by fixing the exact value of the Boltzmann constant. This also clarifies the role played by the Boltzmann constant as an energy-to-temperature conversion factor:

$$1.380\,649 \times 10^{-23}\ \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2} \quad \xrightarrow{\times k_B^{-1}} \quad 1\ \mathrm{K}.$$

Had the scale of thermometers been fixed in units of energy from the start, the Boltzmann constant would have been needless. That said, one can eliminate the kelvin unit by rewriting the physical laws in terms of $k_B T, S/k_B, \ldots$ rather than temperature T, entropy S, …, respectively; i.e., k_B should escort the thermodynamic variables to convert their units into MKS. Clearly, this would not impact the physical content of the four laws of thermodynamics and, consequently, the derived ones. For example, the Clapeyron equation for perfect gases would read $PV = N(k_B T)$, where P, V, N, and $k_B T$ would stand for pressure, volume, number of molecules, and temperature expressed now in units of energy, respectively.

The situation is quite analogous if one replaces the kelvin with the ampere and the Boltzmann constant with the Coulomb one. After the 2019 revision, the fundamental charge e was fixed to have an exact magnitude, while the Coulomb constant k_e was determined experimentally [8]. Although convenient, this is conceptually as good as the practice adopted before 2019 when the exact value of k_e was defined, while the value of e was experimentally determined. Thus, similarly to the previous case, the Coulomb constant can be seen as an MKS unit-to-ampere conversion factor:

$$9.480\,270 \times 10^{4}\ \mathrm{kg}^{1/2} \cdot \mathrm{m}^{3/2} \cdot \mathrm{s}^{-2} \quad \xrightarrow{\times k_e^{-1/2}} \quad 1\ \mathrm{A}.$$

Conversely, any quantities that involve the ampere unit may be combined with the Coulomb constant to be written in terms of MKS units only. For instance, the value of the fundamental charge in MKS units is

$$k_e^{1/2}\, e = 1.518\,907 \times 10^{-14}\ \mathrm{kg}^{1/2} \cdot \mathrm{m}^{3/2} \cdot \mathrm{s}^{-1}$$

and the well-known electrostatic force between two charges Q_1 and Q_2 (expressed in Coulomb units) set far apart by a distance $L = \text{const}$ (expressed in meter) would be written as

$$F = \mathcal{Q}_1\, \mathcal{Q}_2 / L^2, \tag{3.1}$$

where $\mathcal{Q}_i \equiv k_e^{1/2} Q_i$ $(i = 1, 2)$ is the charge written in MKS units: kg$^{1/2}$ $\cdot$ m$^{3/2}$ $\cdot$ s^{-1}. Equation (3.1) is how the electrostatic force reads in Gaussian–centimeter-gram-second (G-CGS) units, wherein the G-CGS system the unit of electric charge is statcoulomb $\equiv$ g$^{1/2}$ $\cdot$ cm$^{3/2}$ $\cdot$ s^{-1} [9].

So far, we have recalled that the MKS system would suffice to express all observables of the physical laws. Next, let us show that all observables of our theories defined in Galilei spacetime can be solely expressed in terms of time and space units of the bona fide clocks and rulers needed to define the spacetime itself.

4 Two units to rule us all in Galilei spacetime

Before 2019, the kilogram was defined by a cylinder of platinum-iridium in the custody of the International Metrology Center at Sèvres in France. After 2019, the kilogram was defined by fixing the exact value of the Planck constant: $h = 6.626\ 070\ 15 \times 10^{-34}\mathrm{J} \cdot \mathrm{s}$ while the value of the Newtonian constant G was measured. Although this is understandable from a metrological perspective, it would be conceptually better if the kilogram was defined by fixing the exact value of G (see ref. [10] for CODATA's recommended value for G).

Let us note that h gives the spin scale of elementary particles while G alone gives no scale at all. The physical quantity that is responsible for the gravitational attraction between bodies is GM, which has units of $\mathrm{m}^3 \cdot \mathrm{s}^{-2}$, being measured, hence, with clocks and rulers. Apparently, this common knowledge faded out in the last 150 years. Let us quote Maxwell's 1873 masterpiece [11] (in Preliminaries):

> *"... the unit of mass is deduced from the units of time and length, combined with the fact of universal gravitation. The astronomical unit of mass is that mass which attracts another body placed at the unit of distance so as to produce in that body the unit of acceleration... If, as in the astronomical system the unit of mass is defined with respect to its attractive power, the dimensions of M are L^3T^{-2}."*

Maxwell's statement just says that the attractive power of a mass is $GM = aL^2$ and can be determined by measuring the acceleration a of a test mass resting at a distance L from (the center of mass of) the mass (where the equivalence between "gravitational" and "inertial" masses was taken for granted, as well as, in DOV trialogue).

Interestingly enough, the gravitational constant was only introduced in 1873, the same year Maxwell published ref. [11], to convert a mass of $GM = 6.674 \times 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{s}^{-2}$ into 1 kg (defined during the French Revolution as the mass corresponding to 1 liter of water):

$$6.674 \times 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{s}^{-2} \xrightarrow{\times G^{-1}} 1\ \mathrm{kg}.$$

As noted by Quinn and Speake [12]:

> *"Newton did not express his law of gravitation in a way that explicitly included a constant G, its presence was implied as if it had a value equal to 1. It was not until 1873 that Cornu and Bailey explicitly introduced a symbol for the coupling constant in Newton's law of gravity, in fact, they called it f. (The current designation G for the gravitational constant was only introduced sometime in the 1890s.)"*

Thus, had G not been introduced, all observables would be expressed in units of distance and time, (MS) system, and all equations of physics would remain the same with the MKS observables

$$O_i = \Omega_i\ \mathrm{m}^{\alpha_i} \cdot \mathrm{s}^{\beta_i} \cdot \mathrm{kg}^{\gamma_i}, \quad \Omega_i, \alpha_i, \beta_i, \gamma_i \in \mathbb{R} \tag{4.1}$$

being replaced by MS observables O_i^{MS}:

$$\begin{aligned} O_i \xrightarrow{\times G^{\gamma_i}} O_i^{\mathrm{MS}} &\equiv G^{\gamma_i} \times O_i \\ &= \Omega_i^{\mathrm{MS}}\ \mathrm{m}^{\alpha_i + 3\gamma_i} \cdot \mathrm{s}^{\beta_i - 2\gamma_i}, \end{aligned} \tag{4.2}$$

where $\Omega_i^{\rm MS} = (6.674 \times 10^{-11})^{\gamma_i} \times \Omega_i$. (The index i gives information not only on the physical quantity (energy, spin, ...) but also on the state of the system, no matter how the theory chooses to describe it. As a result, Ω_i, for given i, is a real number instead of a real-valued function.) In the MS system, Newton's constant becomes $G \to G^{\rm MS} = 1$, as well as, other conversion factors: $k_B = k_e = 1$.

Additionally, the posterior unveiling of non-relativistic quantum mechanics does not affect the conclusion above, even though one cannot use $M^{\rm MS} = aL^2$ to evaluate the mass of a quantum particle because of the Heisenberg uncertainty principle. The masses of quantum particles are suitably evaluated through their wave properties. The Schrödinger equation for a free particle can be cast as

$$i\frac{\partial\phi}{\partial t} + \frac{c\lambda\!\!\!^{-}}{2}\nabla^2\phi = 0, \tag{4.3}$$

where, in Galilei spacetime, c refers to the speed of light with respect to some assumed ether. In the Schrödinger equation (as well as in its relativistic counterparts: Klein-Gordon and Dirac equations) the complete information on the particle mass is codified in the reduced Compton wavelength $\lambda\!\!\!^{-} = \hbar/mc$. For electrons, the $m/\hbar$ ratio can be determined from the measurement of the Rydberg constant via hydrogen spectroscopy, leading to $\lambda\!\!\!^{-}_e = 3.862 \times 10^{-13}$ m. Hence, in the MS system, $m_e^{MS} = \hbar^{\rm MS}/c\lambda\!\!\!^{-}_e = 6.080 \times 10^{-41}\ {\rm m}^3 \cdot {\rm s}^{-2}$ (where the independently-measured value of $\hbar^{\rm MS} \equiv G\hbar = 7.039 \times 10^{-45}\ {\rm m}^5 \cdot {\rm s}^{-3}$ was used).

We close this section by answering the question posed in the Introduction:

> *Assuming Galilei spacetime, the minimum number of apparatuses that a cosmic factory must build to allow distinct labs to compare the values of the observables is two.*

Let us see what changes if we assume Minkowski spacetime (or any other relativistic spacetime).

5 Time to rule us all in relativistic spacetimes

Let us focus on the Minkowski spacetime for simplicity since the conclusions follow the same for any other relativistic one. In Minkowski spacetime, space and time are connected and, hence, it is natural to expect that observables can be expressed in terms of one single unit. We shall avoid using light rays since the Minkowski spacetime does not depend on the existence of physical worldlines evolving on the causality cone. Thus, let us present an elegant protocol due to William Unruh (private communication) according to which distances are measured with three bona fide clocks, implying that *all observables in relativistic spacetimes can be simply expressed in units of time.*

Let us assume that a rod with proper length D lies at rest with a congruence of inertial observers evolving in Minkowski spacetime as shown in figure 3. Now, let us assume that an inertial clock C1 takes a proper time interval τ_1 to move from the left end to the right end of the rod with some speed $v_1 = $ const. Immediately after C1 reaches the right end, some clock, C2, is sent back with speed $v_2 = $ const, taking a proper time interval τ_2 for its return trip. Finally, a third clock, C3, stays at rest at the left end, registering a total proper time τ between the departure of C1 and the return of C2. Then, one can write the rod's proper length in terms of τ_1,τ_2, and τ as

$$D = \frac{[(\tau^2 - \tau_1^2 - \tau_2^2)^2 - 4\tau_1^2\tau_2^2]^{1/2}}{2\tau}. \tag{5.1}$$

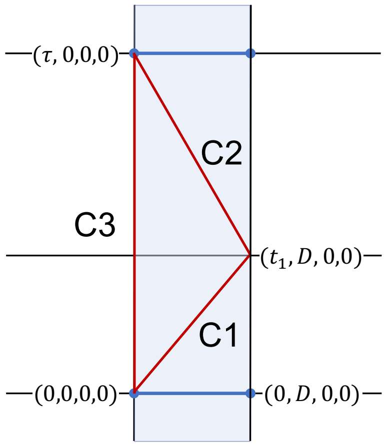

Figure 3. (Color online) The (blue) rectangle represents the worldsheet of an inertial rod with proper length D. The rod lies at rest in a frame defined by a congruence of observers at x, y, z = const who are represented by vertical (black) lines. The graph also shows the worldlines of three clocks. Clock C3 stays at rest at the rod's left end and measures the round trip from the departure of C1 to the return of C2. Note that clock C2 departs as C1 arrives at the rod's right end. All clocks are inertial.

Let us also note that in arbitrary relativistic spacetimes, a sound prescription of distance for arbitrarily close events lying on the simultaneity hypersurface of some local congruence of observers can be given, rendering eq. (5.1) valid in the proper limit. Let us also stress that the relativistic nature of the spacetime guarantees that eq. (5.1) does not depend on v_1 and v_2. Indeed, this can be used as a test of the relativistic nature of the spacetime.

Now, under the reader's discretion, the result above for D given in seconds can be converted into the usual meter unit using the conversion rule:

$$(299\ 792\ 458)^{-1}\ \mathrm{s} \xrightarrow{\times c} 1\ \mathrm{m},$$

making it clear that c is a conversion factor in relativistic spacetimes, and all observables can be expressed in the time units of the bona fide clocks, which must equip Minkowski spacetime to be defined. To move from the MS system, where observables take the form

$$O_i^{\mathrm{MS}} = \Omega_i^{\mathrm{MS}}\ \mathrm{m}^{\alpha_i} \cdot \mathrm{s}^{\beta_i}, \tag{5.2}$$

to the S system, where observables are measured only in units of time, one must replace O_i^{MS} by O_i^{S} in all equations:

$$\begin{aligned} O_i^{\mathrm{MS}} \xrightarrow{\times c^{-\alpha_i}} O_i^{\mathrm{S}} &\equiv c^{-\alpha_i} \times O_i^{\mathrm{MS}} \\ &= \Omega_i^{\mathrm{S}}\ \mathrm{s}^{\alpha_i+\beta_i} \end{aligned} \tag{5.3}$$

where $\Omega_i^{\mathrm{S}} = (299\ 792\ 458)^{-\alpha_i} \times \Omega_i^{\mathrm{MS}}$. Eventually, the S system corresponds to the *geometrized system of units*, where $k_B = k_e = G = c = 1$. Thus, the answer to the question posed in the Introduction is:

Assuming relativistic spacetimes, the minimum number of apparatuses that a cosmic factory must build to allow distinct labs to compare the values of the observables is one.

6 One fundamental constant

We have shown so far that in relativistic spacetimes, k_B, k_e, G, and c can be harmlessly dispensed for being conversion factors, and that all observables can be expressed in units of time. As a result, our cosmic factory is released from producing all but *one* time standard associated with a bona fide clock. The transition frequency between two states of cesium-133 is seen to satisfy the conditions required for bona fide clocks under state-of-the-art technology.

Now, instead of demanding the cosmic factory to produce and deliver bona fide clocks, let us follow the International Committee for Weights and Measures and fix the exact value of one second as *the time that elapses during* 9 192 631 770 *cycles of the radiation produced by the transition between the two hyperfine ground levels of the cesium-133 atom.* Following DOV's understanding of "fundamental," this would be a fundamental constant associated with the 1 s standard (to be used to express all observables).

Clearly, any physical quantity tested constant by bona fide clocks can be used as a standard to express the observables. Once $\hbar$ is tested constant, as verified by the International Committee for Weights and Measures (see section 2.2.1 of ref. [2]), one may move, e.g., from the geometrized system (S) of sec. 5, where observables read

$$O_i^{\rm S} = \Omega_i^{\rm S} {\rm s}^{\alpha_i}, \tag{6.1}$$

to the Planck system (P), commonly used in high-energy physics, by replacing $O_i^{\rm S}$ by $O_i^{\rm P}$ in the equations:

$$\begin{aligned} O_i^{\rm S} \stackrel{\times(\hbar^{\rm S})^{-\alpha_i/2}}{\longrightarrow} O_i^{\rm P} &\equiv (\hbar^{\rm S})^{-\alpha_i/2} \times O_i^{\rm S} \\ &= \Omega_i^{\rm P}, \end{aligned} \tag{6.2}$$

where $\hbar^S = 2.907 \times 10^{-87}$ s^2, and $\Omega_i^{\rm P} = (2.907 \times 10^{-87})^{-\alpha_i/2} \times \Omega_i^{\rm S}$. In Planck units, thus, $\hbar = k_B = k_e = G = c = 1$ (where the P labels are omitted for simplicity), and all observables are dimensionless. Nevertheless, we stress again that this does not change the demand for the existence of bona fide clocks, which are still necessary for testing whether or not $\hbar$ is constant.

7 Closing remarks

We have shown that observables of the physical laws defined in Minkowski spacetime can be solely expressed in units of time defined by bona fide clocks that are demanded in the first place to construct the spacetime. The transition frequency between two states of cesium-133 satisfies the conditions required for bona fide clocks under present state-of-the-art technology. Thus, the answer to the core question posed in the Introduction about the minimum number of apparatuses that a cosmic factory must produce to allow distinct labs to compare the values of observables is *one* (assuming relativistic spacetimes as DOV do).

Instead of demanding the cosmic factory of producing and delivering bona fide clocks, we may follow the International Committee for Weights and Measures and define one second as being *the time that elapses during* 9 192 631 770 *cycles of the radiation produced by the transition between* *the two hyperfine ground levels of the cesium-133 atom.* Following DOV's understanding of "fundamental," the 9 192 631 770 cycles of cesium-133 would be a fundamental constant associated with the 1 s standard, which should be communicated to the alien labs to allow for the comparison of all physical observables.

Finally, let us comment that although some atomic clocks [13] have reached a precision better than 10^{-17} s, being excellent realizations of bona fide clocks for all present practical purposes, *quantum mechanics tells us that there are no arbitrarily good clocks* [14]. This jeopardizes the very concept of relativistic spacetime, $(\mathcal{M}, g)$, as a smooth manifold $\mathcal{M}$ endowed with a metric g. Most probably when our technology reaches the Planck scale (being able to measure time intervals with a precision of order 10^{-44} s), we will be urged to replace our spacetime concept with something else. How many dimensional units (if any) will be granted by the quantum gravity spacetime (to express observables of the new laws of physics) we do not know.

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