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DE SITTER INVARIANCE AND A POSSIBLE MECHANISM OF GRAVITY

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Abstract:

It is believed that gravity will be explained in the framework of the existing quantum theory when one succeeds in eliminating divergencies at large momenta or small distances (although the phenomenon of gravity has been observed only at nonrelativistic momenta and large distances). We consider a quantum-mechanical description of systems of two free particles in de Sitter invariant quantum theory (i.e. the paper contains nothing but the two-body de Sitter kinematics). In our pure algebraic approach the cosmological constant problem does not arise. It is shown that a system can be simultaneously quasiclassical in relative momentum and energy only if the cosmological constant is not anomalously small. We explicitly construct the relative distance operator. The corresponding eigenvectors differ from standard ones at both, large and small momenta. At large momenta they ensure fast convergence of quasiclassical wave functions. There also exists an anomalously large (but finite) contribution from small momenta, which is a consequence of the fact that the cosmological constant is finite. We argue that gravity might be a manifestation of this contribution.

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Chapter 1

Introduction

To Skiff Nikolaevich Sokolov

1.1 Motivation

Skiff Nikolaevich Sokolov has contributed to several areas of physics. I was very impressed by his method of packing operators in relativistic theory of direct interactions [1] and his proof with Shatny [2] that all the three Dirac's forms of relativistic dynamics are unitarily equivalent. And what is much more important, Skiff Nikolaevich is a man with extremely high moral standards.

Many years ago Skiff Nikolaevich told me about his idea that gravity might be a direct interaction and we discussed whether or not this might be in agreement with the results on the binary pulsar PSR B1913+16 (the present status of the problem is described e.g. in Refs. [3, 4]). If I remember Skiff Nikolaevich's arguments correctly, he said that until a direct and unique explanation of gravity is given, different possibilities should be investigated.

The results on binary pulsars are treated as a strong indirect indication of the existence of gravitons. In reality only some radiation is seen. Then one describes this radiation assuming that it belongs to a binary pulsar. One constructs models where the masses of the pulsars, their distances from each other etc. are adjustable parameters and it is assumed that the interaction of the pulsars with the interstellar matter is weak. Then by fitting the parameters, the Einstein quadrupole formula is reproduced with a high accuracy. This is indeed a strong argument in favor of gravitational waves. At the same time, this cannot be treated as a direct proof. In addition, the analysis performed in the above references is based only on classical General Relativity (GR) and quantum effects are not considered. Therefore, if the Einstein quadrupole formula is indeed valid, the fact that on quantum level the main contribution to the standard nonrelativistic gravity is the graviton exchange, is an additional assumption based on belief that gravity can be described in quantum field theory (QFT) or its generalizations (string theory, loop quantum gravity etc.).

One might wonder why this belief is so strong in spite of the fact that numerous efforts to construct quantum theory of gravity without infinities has not been successful yet. Probably this belief is based on the fact that in QED, electroweak theory and QCD the problem of infinities can be somehow circumvented and there are well known examples (especially in QED) of striking agreement between theory and experiment. This is considered as much more important than the fact that from the mathematical point of view, operators in QFT are not well defined. Therefore the usual statement is that the standard model describes almost everything and to include gravity one needs only to generalize the existing theory to Planck distances. The present paper will be of no interest for physicists having such a philosophy.

At the same time, some famous physicists are not so optimistic. For example, Weinberg believes [5] that the new theory may be 'centuries away'. Although he has contributed much to QFT and believes that it can be treated *the way it is*, he also believes that it is a *low energy approximation to a deeper theory that may not even be a field theory, but something different like a string theory* [6].

Dirac was probably the least optimistic famous physicist. In his opinion [7]: *The agreement with observation is presumably by coincidence, just like the original calculation of the hydrogen spectrum with Bohr orbits. Such coincidences are no reason for turning a blind eye to the faults of the theory. Quantum electrodynamics is rather like Klein-*

Gordon equation. It was built up from physical ideas that were not correctly incorporated into the theory and it has no sound mathematical foundation.

The main problem is the choice of strategy for constructing a new quantum theory. Since nobody knows for sure which strategy is the best one, different approaches should be investigated. Our philosophy is based on Dirac's advice given in Ref. [7]: *I learned to distrust all physical concepts as a basis for a theory. Instead one should put one's trust in a mathematical scheme, even if the scheme does not appear at first sight to be connected with physics. One should concentrate on getting an interesting mathematics.*

The main motivation for the present work was to investigate whether or not gravity might be simply a manifestation of the fact that nature is discrete and finite and therefore quantum theory should be based not on complex numbers but on a finite field. We did not succeed in giving a definite conclusion yet. However, we have realized that even in STANDARD de Sitter invariant quantum theory KINEMATICS of a free two-body system has several very interesting features, which seem to fully contradict our experience based on standard quantum theory.

One usually believes that there is no need to describe macroscopic systems (e.g. the Sun - Earth or the Earth - Moon systems) in the framework of quantum mechanics since quasiclassical approximation works for such systems with a very high accuracy. This belief is based mainly on our experience in standard nonrelativistic quantum mechanics. In this theory there exist well defined coordinate and momentum operators and it is easy to construct quasiclassical wave functions having a nice behavior when the relative distance or relative momentum go to infinity (there is a well known phrase that the Fourier transform of a Gaussian wave functions also is a Gaussian wave function). That's why nobody poses a question of what a real wave function of the Sun - Earth or Earth - Moon system is.

However, since 30th of the last century it is well known that, when quantum mechanics is combined with relativity, there is no operator satisfying all the properties of the spatial position operator (see e.g.

Ref. [8]). In other words, the coordinate cannot be exactly measured by itself even in situations when exact measurement is allowed by the nonrelativistic uncertainty principle.

In the introductory section of the well-known textbook [9] simple arguments are given that for a particle with the mass m the coordinate cannot be measured with the accuracy better than the Compton wave length $\hbar/mc$. Therefore exact measurement is possible only either in the nonrelativistic limit (when $c \to \infty$) or classical limit (when $\hbar \to 0$).

It has also been known for many years (see e.g. Ref. [10]) that there is no good operator, which can be related to time. In particular, we cannot construct a state which is the eigenvector of the time operator with the eigenvalue -5000 years BC or 2010 years AD.

In particular, the quantity x in the Lagrangian density $L(x)$ is only a parameter which becomes the spacetime coordinate in the nonrelativistic or classical limit. Note that even in the standard formulation of QFT, the Lagrangian is only an auxiliary tool for constructing Hilbert spaces and operators. After this construction has been done, one can safely forget about Lagrangian and concentrate his or her efforts on calculating different observables. As Rosenfeld writes in his memoirs about Bohr [11]: *His (Bohr) first remark ... was that field components taken at definite space-time points are used in the formalism as idealization without immediate physical meaning; the only meaningful statements of the theory concern averages of such fields components over finite space-time regions....Bohr certainly never showed any respect for the noble elegance of a Lagrangian principle.*

The above ideas became very popular in 60th (recall the famous Heisenberg S-matrix program). The authors of Ref. [9] claim that spacetime, Lagrangian and local quantum fields are rudimentary notions which will disappear in the ultimate quantum theory. Since that time, no arguments questioning those ideas have been given, but in view of the great success of gauge theories in 70th and 80th, these ideas are now almost forgotten.

The recent experimental data indicate (see e.g. Ref. [12]) that the cosmological constant is not exactly zero and is positive. It is quite natural to treat this situation as an indication that de Sitter invariance describes nature better than Poincare invariance. However, at present this symmetry is not popular for several reasons. Firstly, the works where de Sitter symmetry has been intensively investigated by group theoretical methods can be found only in journals published mainly in 60th and 70th of the last century. In particular, it is rather odd that the excellent book by Mensky [13] on a theory of induced representations for physicists has been published only in Russian. Secondly, from the point of view of QFT and its generalizations, de Sitter symmetry encounters serious difficulties and the case of anti de Sitter symmetry looks much more preferable (see e.g. Refs. [14, 15]). As noted by Witten [15] *'I don't know any clear-cut way to get de Sitter space from string theory or M-theory'*. In Sect. 2.1 this question is discussed in greater details. In any case, to the best of our knowledge, the problem of quasiclassical states in de Sitter invariant quantum theories has not been investigated at all.

One might say that since the Sun - Earth or Earth - Moon systems are classical and nonrelativistic, nonrelativistic quantum theory is a special case of relativistic one when $c \to \infty$ and relativistic theory is a special case of de Sitter invariant theory when $R \to \infty$, where R is the radius of the Universe, our experience can be applied to such systems and nothing unusual should be expected. However, in the spirit of the above Dirac's advice, it is always desirable not only to rely on physical intuition but to verify mathematically everything what can be verified. The results of the present paper show that the standard intuition does not work and anomalously large (but finite) contributions to quasiclassical wave functions arise not from the region of large momenta (as one might expect) but from the region of very small momenta. This might be an indication that the true quantum theory is based not on the field of complex numbers but on a finite field (or, in the spirit of Ref. [16], even on a finite ring). We suppose to consider this possibility in future publications, while in the present paper we consider in detail the construction of quasiclassical states only in standard de Sitter invariant quantum theory.

The paper is organized as follows. In the next section and in Chap. 2 we argue that gravity might be a special phenomenon, which is not described by analogy with the known local quantum field theories. We consider in detail a well known example of de Sitter antigravity and note that this is an example of true direct interaction (i.e. an interaction, which is not a consequence of exchange by intermediate particles). In Chap. 3 we develop a pure algebraic approach for describing free two-body systems in de Sitter invariant theory. In Chap. 4 the uncertainty relation between the de Sitter momentum and energy is discussed. It is shown that a system can be simultanenously quasiclassical in momentum and energy only if the cosmological constant is not anomalously small. The main results of the paper are obtained in Chap. 5. We explicitly construct a relative distance operator in dS theory and calculate its spectrum. The properties of the corresponding eigenstates considerably differ from ones in the standard theory at both, large and small momenta. At large momenta the falloff of the eigenstates is very fast and therefore quasiclassical wave functions rapidly decrease at such momenta. At the same time, there arises an anomalously large (but finite) contribution from the region of small momenta. This is a consequence of the fact that the cosmological constant is finite. There is no analog of such a behavior in standard theory. Finally, Chap. 6 is discussion.

1.2 Is gravity a special phenomenon or it should be combined with other interactions?

The main stream of the modern fundamental physics is that gravity should be treated by analogy with electromagnetic, weak and strong interactions and eventually all these interactions will be unified. In such an approach, gravitational interaction is treated as a consequence of the graviton exchange and in the nonrelativistic approximation the Newton gravitational law is obtained from the diagram of the one-graviton exchange in the same way as the Coulomb law is obtained from the diagram of the one-photon exchange. On the other hand, since in units where $\hbar = c = 1$ the gravitational constant has the dimension of length squared, a gravitational theory constructed by analogy with QED contains strong divergencies. Physicists adhering to QFT do not treat this fact as a major drawback of the theory but rather as a technical difficulty, which will be resolved by modifying the theory at small distances.

A rather unusual property of gravity is that it is the most known and, at the same time, the least understandable. In several aspects gravity is fully different from the electroweak and strong interactions. The phenomenon of gravity is known only at macroscopic level and only in nonrelativistic and post-Newtonian approximations. At this level gravity is universal, i.e. it applies to any bodies regardless of whether they have electric charges, magnetic moments etc. Therefore it is interesting to investigate a possibility that the mechanism of gravity is more fundamental (and more simple) than the mechanism of other interactions.

If one accepts that de Sitter invariance is a good (approximate or exact) symmetry then there exists another universal phenomenon, which is well known: the universal de Sitter antigravity. Consider a very simple example of two free particles. Then each particle is described by an irreducible representation (IR) of de Sitter (dS) algebra, and the fact that the particles are free means (by definition) that the two-body system is described by the tensor product of the respective IRs. However, when we describe such a system in Poincare invariant terms, it looks like the particle interact although no interaction have been introduced 'by hands'. On classical level the nature of this "interaction" is obvious since it is well known that in the dS space the force of repulsion between two particles is proportional (not inversely proportional!) to the distance between them and the relative acceleration of "free" particles in the dS space under no circumstances can be zero.

It is worth noting that the dS antigravity might be treated as a true direct interaction, i.e. it is not a manifestation of exchange by any virtual particles. It is rather strange that in the vast literature on de Sitter antigravity nobody pointed out to this fact. A typical objection against direct interactions is that any interaction should propagate with a final velocity. In particular, if a position of a particle is changed then the other particle should feel this change not instantly but only after some period of time. This seems to be not the case if the interaction depends only on the relative distance between the particles. However, the classical dS antigravity is not a typical interaction on the Poincare background but simply an inherent property of the dS space. There is no way to exclude this "interaction" by letting some interaction constant to be zero. The relative acceleration of two particles in the dS space is proportional to $1/R^2$ where R is the radius of the dS space (a detailed discussion is given in the next chapter). Therefore one might have a temptation to treat $1/R^2$ as an interaction constant. However, this "interaction constant" disapears only in the formal limit $R \to \infty$ when the dS space does not exist anymore (it becomes the Poincare one). In other words, it is reasonable to say that the universal de Sitter antigravity is not an interaction at all but simply an inherent property of de Sitter invariance.

The above example poses a problem of whether all the existing interactions in nature are in fact not true interactions but effective interactions arising as a result of transition from a higher symmetry to a lower one. This idea has something in common with ones in the string theory that the existing interactions represent a way how the extra dimensions are compactified. However, in our formulation we do not consider geometries on extremely small distances, since as noted above, they have no physical meaning.

Since in many aspects gravity is fully different from the electroweak and strong interactions (see the above discussion), it is interesting to investigate a possibility that gravity is not an interaction at all but simply an inherent property of nature, which cannot be excluded by letting some interaction constant to be zero. The results of the paper might be an indication that this is the case.

1.3 Elementary particles in quantum theory

Let us consider how one should define the notion of elementary particles. Although particles are observable and fields are not, in the spirit of QFT, fields are more fundamental than particles, and a possible definition is as follows [17]: *It is simply a particle whose field appears in the Lagrangian. It does not matter if it's stable, unstable, heavy, light — if its field appears in the Lagrangian then it's elementary, otherwise it's composite.*

Another approach has been developed by Wigner in his investigations of unitary irreducible representations (UIRs) of the Poincare group [18]. In view of this approach, one might postulate that a particle is elementary if the set of its wave functions is the space of a UIR of the symmetry group in the given theory.

Although in standard well-known theories (QED, electroweak theory and QCD) the above approaches are equivalent, the following problem arises. The symmetry group is usually chosen as a group of motions of some classical manifold. How does this agree with the above discussion that quantum theory in the operator formulation should not contain spacetime? A possible answer is as follows. One can notice that for calculating observables (e.g. the spectrum of the Hamiltonian) we need in fact not a representation of the group but a representation of its Lie algebra by Hermitian operators. After such a representation has been constructed, we have only operators acting in the Hilbert space and this is all we need in the operator approach. The representation operators of the group are needed only if it is necessary to calculate some macroscopic transformation, e.g. time evolution. In the approximation when classical time is a good approximate parameter, one can calculate evolution, but nothing guarantees that this is always the case (e.g. at the very early stage of the Universe). Let us also note that in the stationary formulation of scattering theory, the S-matrix can be defined without any mentioning of time (see e.g. Ref. [19]). For these reasons we can assume that on quantum level the symmetry algebra is more fundamental than the symmetry group. We will consider only representations of Lie algebras by Hermitian operators and IRs will imply irreducible representations.

Consider for illustration a well-known example of nonrelativistic quantum mechanics. Usually the existence of the Galilei spacetime is assumed from the beginning. Let $(\mathbf{r}, t)$ be the space-time coordinates of a particle in that spacetime. Then the particle momentum operator is $-i\partial/\partial\mathbf{r}$ and the Hamiltonian describes evolution by the Schroedinger equation. We believe that the notion of empty space is not physical since the existence of space without particles contradicts basic principles of quantum theory. Indeed, if we accept existence of a space without particles than nothing can be measured in such a space. In addition, the assumption that the notion of classical space R^3 can be applied on quantum level contradicts the existence of particles with half-integer spins. Indeed, in R^3 there can be no continuous functions acquiring the factor -1 after rotation by 2π. The only way to describe such particles in the standard approach is to replace the group SO(3) by its covering group SU(2) i.e. in fact to acknowledge that the notion of R^3 on quantum level is not applicable.

In our approach one starts from IR of the Galilei algebra. The momentum operator and the Hamiltonian represent four of ten generators of such a representation. If it is implemented in a space of functions $\psi(\mathbf{p})$ then the momentum operator is simply the operator of multiplication by $\mathbf{p}$. Then the position operator can be *defined* as $i\partial/\partial\mathbf{p}$ and time can be *defined* as an evolution parameter such that evolution is described by the Schroedinger equation with the given Hamiltonian. In this approach the problem with half-integer spins does not arise since the Lie algebras so(3) and su(2) are equivalent. Mathematically the both approaches are related to each other by the Fourier transform. However, the philosophies behind them are essentially different. In the second approach there is no empty spacetime and the spacetime coordinates have a physical meaning only if there are particles for which the coordinates can be measured.

Summarizing our discussion, we assume that, *by definition*, on quantum level a Lie algebra is the symmetry algebra if there exist physical observables such that their operators satisfy the commutation relations characterizing the algebra. Then, a particle is called elementary if the set of its wave functions is a space of IR of this algebra. Such an approach is in the spirit of that considered by Dirac in Ref. [20]. In the present paper we consider de Sitter invariant quantum theory (dS theory). Therefore elementary particles in this theory should be described by IRs of the de Sitter algebra so(1,4). In the next chapter we give a detailed description of such IRs.

Chapter 2

Basic properties of de Sitter invariant quantum theories

2.1 de Sitter invariance

As already mentioned, the original motivation for this work was to investigate whether the standard gravitational effects can be obtained in the framework of a free theory. In the standard nonrelativistic approximation gravity can be described by adding the term $-Gm_1m_2/r$ to the nonrelativistic Hamiltonian, where G is the gravitational constant, m_1 and m_2 are the particle masses and r is the distance between the particles. Since the kinetic energy is always positive, the free nonrelativistic Hamiltonian is positive definite and therefore there is no way to obtain gravity in the framework of the free theory. Analogously, in Poincare invariant theory the spectrum of the free two-body mass operator belongs to the interval $[m_1+m_2,\infty)$ while the existence of gravity necessarily requires that the spectrum should contain values less than m_1+m_2.

In theories where the invariance group is the anti de Sitter (AdS) group SO(2,3), the structure of IRs of the so(2,3) algebra is well known (see e.g. Ref. [21]). In particular, for positive energy IRs the AdS Hamiltonian has the spectrum in the interval $[m,\infty)$ and m has the meaning of the mass. Therefore the situation is pretty much analogous to that in Poincare invariant theories. In particular, the free two-body mass operator again has the spectrum in the interval $[m_1+m_2,\infty)$ and therefore there is no way to reproduce gravitational effects in the free AdS invariant theory.

Consider now the case when the dS algebra so(1,4) is chosen as the symmetry algebra. It is well known that in IRs of the dS algebra, the dS Hamiltonian is not positive definite and has the spectrum in the interval $(-\infty,+\infty)$ see e.g. Refs. [13, 22, 23, 24, 25, 26]). Note also that in contrast to the AdS algebra so(2,3), the dS one does not have a supersymmetric generalization. For this and other reasons it was believed that the dS group or algebra were not suitable for constructing elementary particle theory.

In the framework of QFT in curved spacetime (see e.g. Refs. [14, 15] and references therein) the choice of SO(1,4) as the symmetry group encounters serious difficulties. Our approach fully differs from that in Refs. [14, 15]. In particular, we do not require the existence of empty spacetime (see the discussion in Sect. 1.3). It has been also shown in Ref. [27], that a correct interpretation of IRs of the so(1,4) algebra requires that one IR should describe a particle and its antiparticle simultaneously. Then the theory with the dS symmetry become consistent (see Ref. [27] for details). As already noted, the existing experimental data indicate that the cosmological constant is positive. The nomenclature is such that the positive cosmological constant corresponds to the SO(1,4) symmetry and the negative one - to the SO(2,3) symmetry.

It is well known that the group SO(1,4) is the symmetry group of the four-dimensional manifold in the five-dimensional space, defined by the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0^2 = R^2 \tag{2.1}$$

where a constant R has the dimension of length. The quantity R^2 is often written as $R^2 = 3/\Lambda$ where Λ is the cosmological constant. Elements of a map of the point $(0,0,0,0,R)$ (or $(0,0,0,0,-R)$) can be parametrized by coordinates (x_0,x_1,x_2,x_3). If R is very large then such a map proceeds to Minkowski space and the action of the dS group on this map — to the action of the Poincare group.

In units $\hbar/2 = c = 1$, the spin of any particle is always an integer and for the normal relation between spin and statistics, the spin of fermions is odd and the spin of bosons is even. In this system of units the representation generators of the SO(1,4) group M^{ab} ($a, b = 0, 1, 2, 3, 4$, $M^{ab} = -M^{ba}$) should satisfy the commutation relations

$$[M^{ab}, M^{cd}] = -2i(\eta^{ac}M^{bd} + \eta^{bd}M^{as} - \eta^{ad}M^{bc} - \eta^{bc}M^{ad}) \qquad (2.2)$$

where η^{ab} is the diagonal metric tensor such that $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = -\eta^{44} = 1$.

An important observation is as follows. If we accept that the symmetry on quantum level means that proper commutation relations are satisfied (see Sect. 1.3) then Eq. (2.2) can be treated as the *definition* of the dS symmetry on that level. In this formulation the dS symmetry looks much more natural than the Poincare symmetry where six operators are angular momenta and the remaining four - linear momenta. Mirmovich has proposed a hypothesis [28] that only quantities having the dimension of the angular momentum can be fundamental. Our definition of the dS symmetry on quantum level does not involve the cosmological constant at all. It appears only if one is interested in interpreting results in terms of the dS spacetime or in the Poincare limit.

2.2 Comments on popular statements about fundamental physics

In view of our definition of the dS symmetry on quantum level, we would like to comment several popular statements widely discussed in the literature.

If one assumes that spacetime is fundamental then in the spirit of GR it is natural to think that the empty spacetime is flat, i.e. that the cosmological constant is equal to zero. This was the subject of the well-known dispute between Einstein and de Sitter. In QFT with the Poincare background, the cosmological constant is given by a contribution of vacuum diagrams, and the problem is to explain why it is so small. The theory is based on the existence of empty spacetime and on the assumption that the gravitational constant G is fundamental. Meanwhile, as noted in Ref. [29], *"Contrary to most of the other fundamental constants, as the precision of the measurements increased, the disparity between the measured values of G also increased. This led the CODATA in 1998 to raise the relative uncertainty for G from 0.013% to 0.15%".* Several well known physicists, including Dirac, discussed a possibility that G is time dependent. In view of the fact that only three digits in G are known for sure, its value has been measured only within a period of 210 years and the Universe exists at least thirteen billion years, such a possibility by no way can be excluded (this observation was pointed out to me by Volodia Netchitailo). However, physicists believing in QFT assume that G is fundamental, and as argued by the third author in Ref. [30], in the string theory the fundamental role is played by the string constant λ_s. In particular, there is a belief that quantum gravity will manifest itself at Planck distances. Since the pure gravitational theory based on QFT contains only the constant G, and Λ has the dimension $(length)^{-2}$, the value of Λ in this theory is of order G^{-2} and the discrepancy with observations is approximately 122 orders of magnitude. In the spirit of the modern philosophy that divergencies and mathematical inconsistencies of QFT do not represent fundamental drawbacks of the theory but only technical difficulties, a conclusion is made that the cosmological constant problem is a fundamental problem of modern physics. We believe, however, that this problem (sometimes it is called the old cosmological constant problem [31]) is fully artificial since it is not reasonable to draw conclusions from inconsistent theories.

If one accepts that symmetry on quantum level in our formulation is more fundamental, then the cosmological constant problem does not arise since Eq. (2.2) does not contain Λ at all. As noted above, it appears only if one is interested in interpreting results in terms of the dS spacetime or in the Poincare limit. Then one might wonder why nowadays Poincare symmetry is so good approximate symmetry. This is rather a problem of cosmology but not quantum physics. In particular, the cosmological constant might be not a constant at all.

If one accepts one of the main principles of quantum theory that any physical quantity is described by an operator, then, as noted in the preceding chapter, the notions of space and time are not fundamental on quantum level. Meanwhile, the success of gauge theories and new results in the string theory have revived the hope that Einstein's dream about geometrization of physics could be implemented. Einstein said that the left-hand-side of his equation of GR, containing the Ricci tensor, is made from gold while the right-hand-side containing the energy-momentum tensor of the matter is made from wood. Since that time a lot of efforts have been made to derive physics from geometry of spacetime. As already noted, the modern ideas in the superstring theory are such that quantum gravity comes into play at Plank distances and all the existing interactions can be described if one finds how the extra dimensions are compactified. These investigations involve very sophisticated methods of topology, algebraic geometry etc.

We believe that such investigations might be of mathematical interests and might give interesting results but cannot lead to ultimate quantum theory. It is rather obvious that geometrical and topological ideas originate from our macroscopic experience. For example, the water in the ocean seems to be continuous and is described with a good accuracy by equations of hydrodynamics. At the same time, we understand that this is only an approximation and in fact the water is discrete. The notion of spacetime at Planck distances does not have any physical significance. In particular, strings and manifolds at such distances have no physical meaning. Therefore methods involving geometry and topology at Planck distances cannot give a reasonable physics.

While the notion of spacetime coordinates for real bodies can be only a good approximation at some conditions, the notion of empty spacetime fully contradicts the basic principles of quantum theory that only measurable quantities can have a physical meaning. Indeed, coordinates of points which exist only in our imagination cannot be related to any measurement. Note that even in GR, which is a pure classical (i.e. non-quantum) theory, the meaning of reference frame is not quite clear. In standard textbooks (see e.g. Ref. [32]) the reference frame in GR is defined as a collection of weightless bodies, each of which is characterized by three numbers (coordinates) and is supplied by a clock. It is obvious that such a notion (which resembles ether) is not physical. Meanwhile the modern theories typically begin with the background empty spacetime. Many years ago, when quantum theory was not known, Mach proposed his famous principle, according to which the properties of space at a given point depend on the distribution of masses in the whole Universe. This principle is fully in the spirit of quantum theory.

The problem of choosing background spacetime is a subject of debates between adherents of loop quantum gravity and superstring theory. The former state that background independence is fundamental and the choice of the flat backgound in the string theory is not in spirit of GR. Many of those physicists believe that fundamental physical theory, where gravity is unified with other interactions, should resemble main features of GR but on quantum level.

There is no doubt that GR is a great achievement of theoretical physics and has achieved great successes in describing experimental data. At the same time, it is a pure classical theory fully based on classical spacetime. Therefore it is unrealistic to expect that successful quantum theory of gravity will be based on quantization of GR. The results of GR should follow from quantum theory of gravity only in situations when spacetime coordinates of real bodies is a good approximation while in general the formulation of quantum theory might not involve spacetime at all.

In the literature the following question is also widely discussed. How many independent dimensionful constants are needed for a complete description of nature? A recent paper [30] represents a trialogue between three well known scientists: M.J. Duff, L.B. Okun and G. Veneziano. The results of their discussions are summarized as follows: *LBO develops the traditional approach with three constants, GV argues in favor of at most two (within superstring theory), while MJD advocates zero.* According to Weinberg [33], a possible definition of a fundamental constant might be such that it cannot be calculated in the existing theory.

Below we would like to give additional arguments that the fundamental physical theory should not contain any dimensionful constants at all. The present status of fundamental constants is described in a wide literature (see e.g. Ref. [29]). The authors often note that even the term "constant" for them is not adequate since variations of those "constants" is not prohibited by our knowledge. Indeed, any dimensionful constant is related to a macroscopic measurement carried out at certain macroscopic conditions (at the present state of the Universe, in the given place of the Universe etc.). So there is no guarantee that any dimensionful constant is really constant.

Consider, for example the measurement of angular momentum. The result depends on the system of units. As noted above, in units $\hbar/2 = 1$ the result is given by an integer. But we can reverse the order of units and say that in units where the momentum is a dimensional integer l, its value in $g \cdot cm^2/sec$ is $(1.05457162 \cdot 10^{-27} \cdot l/2) g \cdot cm^2/sec$. Which of those two values has more physical significance? The meaning of l is clear: it shows how big the angular momentum is in comparison with the minimum nonzero value 1. At the same time, the measurement of the angular momentum in units $g \cdot cm^2/sec$ reflects only a historic fact that in macroscopic conditions on the Earth in the period between the 18th and 21st centuries people measured the angular momentum in such units. If tomorrow morning we decide to conduct a new measurement of $\hbar$ and realize that its value is now $2.0 \cdot 10^{-27} \cdot g \cdot cm^2/sec$ then the same experiment will give for the angular momentum the same value l in the dimensional units since the commutation relations between the angular momentum operators in the form

$$[M_x, M_y] = 2iM_z \quad [M_z, M_x] = 2iM_y \quad [M_y, M_z] = 2iM_x$$

are the same. On the other hand, in units $g \cdot cm^2/sec$ we will have a new value $1.0 \cdot 10^{-27} \cdot l \cdot g \cdot cm^2/sec$, which has any meaning only if we compare this value with other angular momenta measured in units $g \cdot cm^2/sec$.

As follows from Eq. (2.2), in our formulation the free de Sitter theory does not involve any arbitrary parameters (neither dimensionful nor dimesionless). One might say that commutation relations for the representation operators of the Poincare algebra also can be written without any arbitrary parameters (for example, in Planck units) and therefore Poincare invariance might be more fundamental than de Sitter one. As already noted, an argument in favor of de Sitter invariance is that the dS algebra is more symmetric than the Poincare algebra and, as already noted, the experimental data indicate that de Sitter invariance is more relevant than Poincare one. We believe that there also exists the following strong argument. If one accepts the idea that an ultimate quantum theory will be discrete and finite then the only possibility is that the theory will be based on Galois fields (or even Galois rings [34, 26, 16]). As argued in Ref. [34, 26], de Sitter invariant theory can be generalized to a theory based on a Galois field while Poincare invariant theory cannot. This is a consequence of the fact that de Sitter invariant theory can be formulated in such a basis that all physical quantities have only discrete spectrum while in Poincare invariant theories it is not possible to avoid quantities with continuous spectrum.

The "real" elements of Galois fields can be represented as 0,1,...$p-1$, where p is the characteristic of the field. In that case the theory depends on the number p and, as argued in Ref. [35], such a situation is natural. At the same time, such a theory cannot contain dimensionful quantities in principle and the meaning of each measurable quantity is clear by analogy with the above example with the angular momentum (this is also in the spirit of Mirmovich's idea [28] that only angular momenta are fundamental).

In summary, we agree with the first author of Ref. [30] that fundamental physics should not contain dimensionful quantities at all and its predictions should give only relations between dimensionless physical quantities. Dimensionful units arise only when we want to express those quantities in terms of macroscopic measurements on the Earth in the given historical period of time and in units we are comfortable with. We also believe that de Sitter invariance is more fundamental than Poincare invariance. There are no physical arguments in favor of treating Poincare invariance as superior over de Sitter invariance and de Sitter invariance - as a result of breaking down Poincare invariance as a consequence of nonzero vacuum energy in the theory with the Poincare background.

2.3 IRs of the so(1,4) algebra

As noted above, in our approach the description of elementary particles involves construction of IRs of the symmetry Lie algebra, not the Lie group. However, in some cases it is technically convenient to derive properties of IRs of Lie algebras from well known results on UIRs of the corresponding Lie groups. There exists a wide literature devoted to UIRs of the dS group and IRs of its Lie algebra (see e.g. Refs. [36, 37, 38, 39, 40, 13, 23, 22, 41, 42, 24, 25]). In particular the first complete mathematical classification of the UIRs has been given in Ref. [36], three well known implementations of the UIRs have been first considered in Ref. [37] and their physical context has been first discussed in Ref. [38].

It is well known that for classification of UIRs of the dS group, one should, strictly speaking, consider not the group SO(1,4) itself but its universal covering group. The investigation carried out in Refs. [36, 37, 38, 39, 23] has shown that this involves only replacement of the SO(3) group by its universal covering group SU(2). Since this procedure is well known then for illustrations we will work with the SO(1,4) group itself and follow a very elegant presentation for physicists in terms of induced representations, given in the book [13]. The elements of the SO(1,4) group can be described in the block form

$$g = \left\| \begin{array}{ccc} g_0^0 & \mathbf{a}^T & g_4^0 \\ \mathbf{b} & r & \mathbf{c} \\ g_0^4 & \mathbf{d}^T & g_4^4 \end{array} \right\| \tag{2.3}$$

where

$$\mathbf{a} = \left\| \begin{array}{c} a^1 \\ a^2 \\ a^3 \end{array} \right\| \quad \mathbf{b}^T = \| \; b_1 \;\; b_2 \;\; b_3 \; \| \quad r \in SO(3) \tag{2.4}$$

(the subscript ^{T} means a transposed vector).

UIRs of the SO(1,4) group are induced from UIRs of the subgroup H defined as follows [39, 13, 22]. Each element of H can be uniquely represented as a product of elements of the subgroups SO(3), A and $\mathbf{T}$: $h = r\tau_A \mathbf{a_T}$ where

$$\tau_A = \left\| \begin{array}{ccc} cosh(\tau) & 0 & sinh(\tau) \\ 0 & 1 & 0 \\ sinh(\tau) & 0 & cosh(\tau) \end{array} \right\| \quad \mathbf{a_T} = \left\| \begin{array}{ccc} 1+\mathbf{a}^2/2 & -\mathbf{a}^T & \mathbf{a}^2/2 \\ -\mathbf{a} & 1 & -\mathbf{a} \\ -\mathbf{a}^2/2 & \mathbf{a}^T & 1-\mathbf{a}^2/2 \end{array} \right\| \tag{2.5}$$

The subgroup A is one-dimensional and the three-dimensional group $\mathbf{T}$ is the dS analog of the conventional translation group (see e.g. Ref. [13]). We hope it should not cause misunderstandings when 1 is used in its usual meaning and when to denote the unit element of the SO(3) group. It should also be clear when r is a true element of the SO(3) group or belongs to the SO(3) subgroup of the SO(1,4) group.

Let $r \rightarrow \Delta(r;\mathbf{s})$ be a UIR of the group SO(3) with the spin $\mathbf{s}$ and $\tau_A \rightarrow exp(i\mu\tau)$ be a one-dimensional UIR of the group A, where μ is a real parameter. Then UIRs of the group H used for inducing to the SO(1,4) group, have the form

$$\Delta(r\tau_A\mathbf{a_T};\mu,\mathbf{s}) = exp(i\mu\tau)\Delta(r;\mathbf{s}) \tag{2.6}$$

We will see below that μ has the meaning of the dS mass and therefore UIRs of the SO(1,4) group are defined by the mass and spin, by analogy with UIRs in Poincare invariant theory.

Let G=SO(1,4) and $X = G/H$ be a factor space (or coset space) of G over H. The notion of the factor space is well known (see e.g. Refs. [22, 39, 13]). Each element $x \in X$ is a class containing the elements $x_G h$ where $h \in H$, and $x_G \in G$ is a representative of the class x. The choice of representatives is not unique since if x_G is a representative of the class $x \in G/H$ then $x_G h_0$, where h_0 is an arbitrary element from H, also is a representative of the same class. It is well known that X can be treated as a left G space. This means that if $x \in X$ then the action of the group G on X can be defined as follows: if $g \in G$ then gx is a class containing gx_G (it is easy to verify that such an action is correctly defined).

As noted above, although we can use well known facts about group representations, our final goal is the construction of the generators. The explicit form of the generators M^{ab} depends on the choice of representatives in the space G/H. As explained in several papers devoted to UIRs of the SO(1,4) group (see e.g. Ref. [13]), to obtain the possible closest analogy between UIRs of the SO(1,4) and Poincare groups, one should proceed as follows. Let $\mathbf{v}_L$ be a representative of the Lorentz group in the factor space SO(1,3)/SO(3) (strictly speaking, we should consider $SL(2,c)/SU(2)$). This space can be represented as the well known velocity hyperboloid with the Lorentz invariant measure

$$d\rho(\mathbf{v}) = d^3\mathbf{v}/v_0 \tag{2.7}$$

where $v_0 = (1+\mathbf{v}^2)^{1/2}$. Let $I \in SO(1,4)$ be a matrix which formally has the same form as the metric tensor η. One can show (see e.g. Ref. [13] for details) that $X = G/H$ can be represented as a union of three spaces, X_+, X_- and X_0 such that X_+ contains classes $\mathbf{v}_L h$, X_- contains classes $\mathbf{v}_L I h$ and X_0 is of no interest for UIRs describing elementary particles since it has measure zero relative to the spaces X_+ and X_-.

As a consequence, the space of IR of the so(1,4) algebra can be implemented as follows. If s is the spin of the particle under consideration, then we use $||...||$ to denote the norm in the space of IR of the su(2) algebra with the spin s. Then the space of IR in question is the space of functions $\{f_1(\mathbf{v}), f_2(\mathbf{v})\}$ on two Lorentz hyperboloids with the range in the space of IR of the su(2) algebra with the spin s and such that

$$\int [||f_1(\mathbf{v})||^2 + ||f_2(\mathbf{v})||^2] d\rho(\mathbf{v}) < \infty \tag{2.8}$$

We see that, in contrast with IRs of the Poincare algebra (and AdS one), where IRs are implemented on one Lorentz hyperboloid, IRs of the dS algebra can be implemented only on two Lorentz hyperboloids, X_+ and X_-. As shown in Ref. [27], this fact (which is well known) has a natural explanation if it is required that one IR should describe a particle and its antiparticle simultaneously.

In the case of the Poincare and AdS algebras, the positive energy IRs are implemented on an analog of X_+ and negative energy IRs - on an analog of X_-. Since the Poincare and AdS groups do not contain elements transforming these spaces to one another, the positive and negative energy IRs are fully independent. At the same time, the dS group contains such elements (e.g. I [13, 22, 41]) and for this reason its IRs cannot be implemented only on one hyperboloid.

In Ref. [27] we have described all the technical details needed for computing the explicit form of the generators M^{ab}. The action of the generators on functions with the supporter in X_+ has the form

$$\begin{aligned}
&\mathbf{M}^{(+)} = 2l(\mathbf{v}) + \mathbf{s}, \quad \mathbf{N}^{(+)} == -2iv_0\frac{\partial}{\partial \mathbf{v}} + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},\\
&\mathbf{B}^{(+)} = \mu\mathbf{v} + 2i[\frac{\partial}{\partial \mathbf{v}} + \mathbf{v}(\mathbf{v}\frac{\partial}{\partial \mathbf{v}}) + \frac{3}{2}\mathbf{v}] + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},\\
&M_{04}^{(+)} = \mu v_0 + 2iv_0(\mathbf{v}\frac{\partial}{\partial \mathbf{v}} + \frac{3}{2})
\end{aligned} \tag{2.9}$$

where $\mathbf{M} = \{M^{23}, M^{31}, M^{12}\}$, $\mathbf{N} = \{M^{01}, M^{02}, M^{03}\}$, $\mathbf{B} = -\{M^{14}, M^{24}, M^{34}\}$, $\mathbf{s}$ is the spin operator, and $\mathbf{l}(\mathbf{v}) = -i\mathbf{v}\times\partial/\partial\mathbf{v}$. At the same time, the action of the generators on functions with the supporter in X_- is given by

$$\begin{aligned}
&\mathbf{M}^{(-)} = 2l(\mathbf{v}) + \mathbf{s}, \quad \mathbf{N}^{(-)} == -2iv_0\frac{\partial}{\partial \mathbf{v}} + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},\\
&\mathbf{B}^{(-)} = -\mu\mathbf{v} - 2i[\frac{\partial}{\partial \mathbf{v}} + \mathbf{v}(\mathbf{v}\frac{\partial}{\partial \mathbf{v}}) + \frac{3}{2}\mathbf{v}] - \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},\\
&M_{04}^{(-)} = -\mu v_0 - 2iv_0(\mathbf{v}\frac{\partial}{\partial \mathbf{v}} + \frac{3}{2})
\end{aligned} \tag{2.10}$$

In view of the fact that SO(1,4)=SO(4)AT and H=SO(3)AT, there also exists a choice of representatives which is probably even more natural than that described above [13, 22, 23]. Namely, we can choose as representatives the elements from the coset space SO(4)/SO(3). Since the universal covering group for SO(4) is SU(2)×SU(2) and for SO(3) — SU(2), we can choose as representatives the elements of the first multiplier in the product SU(2)×SU(2). Elements of SU(2) can be represented by the points $u = (\mathbf{u}, u_4)$ of the three-dimensional sphere S^3 in the four-dimensional space as $u_4 + i\sigma\mathbf{u}$ where σ are the Pauli matrices and $u_4 = \pm(1-\mathbf{u}^2)^{1/2}$ for the upper and lower hemispheres, respectively. Then the calculation of the generators is similar to that described above and the results are as follows.

The Hilbert space is now the space of functions $\varphi(u)$ on S^3 with the range in the space of the IR of the su(2) algebra with the spin s and such that

$$\int ||\varphi(u)||^2 du < \infty \tag{2.11}$$

where du is the SO(4) invariant volume element on S^3. The explicit calculation shows that the generators for this realization have the form

$$\begin{aligned} &\mathbf{M} = 2l(\mathbf{u}) + \mathbf{s}, \quad \mathbf{B} = 2\imath u_4 \frac{\partial}{\partial \mathbf{u}} - \mathbf{s}, \\ &\mathbf{N} = -2\imath[\frac{\partial}{\partial \mathbf{u}} - \mathbf{u}(\mathbf{u}\frac{\partial}{\partial \mathbf{u}})] + (\mu + 3\imath)\mathbf{u} - \mathbf{u}\times\mathbf{s} + u_4\mathbf{s}, \\ &M_{04} = (\mu + 3\imath)u_4 + 2\imath u_4 \mathbf{u}\frac{\partial}{\partial \mathbf{u}} \end{aligned} \tag{2.12}$$

Since Eqs. (2.8-2.10) on the one hand and Eqs. (2.11) and (2.12) on the other are the different implementations of one and the same representation, there exists a unitary operator transforming functions $f(v)$ into $\varphi(u)$ and operators (2.9,2.10) into operators (2.12). For example in the spinless case the operators (2.9) and (2.12) are related to each other by a unitary transformation

$$\varphi(u) = exp(-\frac{\imath}{2}\,\mu\, lnv_0) v_0^{3/2} f(v) \tag{2.13}$$

where $\mathbf{u} = \mathbf{v}/v_0$.

2.4 Poincare limit

A general notion of contraction has been developed in Ref. [43]. In our case it can be performed as follows. Let us assume that $\mu > 0$ and denote $m = \mu/2R$, $\mathbf{P} = \mathbf{B}/2R$ and $E = M_{04}/2R$. Then, as follows from Eq. (2.9), in the limit when $R \to \infty$, $\mu \to \infty$ but μ/R is finite, one obtains a standard representation of the Poincare algebra for a particle with the mass m such that $\mathbf{P} = m\mathbf{v}$ is the particle momentum and $E = mv_0$ is the particle energy. In that case the generators of the Lorentz algebra have the same form for the Poincare and dS algebras. Analogously the operators given by Eq. (2.10) are contracted to ones describing negative energy IRs of the Poincare algebra.

In Sect. 2.2 we argued that fundamental physical theory should not contain dimensional parameters at all. In this connection it is interesting to note that the de Sitter mass μ has a clear meaning: it is a ratio of the radius of the Universe R (or $2R$) to the Compton wave length of the particle under consideration. Therefore even for elementary particles the de Sitter masses are very large. For example, if R is of order $10^{28}cm$ then the de Sitter masses of the electron, the Earth and the Sun are of order 10^{39}, 10^{93} and 10^{99}, respectively.

In the standard interpretation of IRs it is assumed that each element of the full representation space represents a possible physical state for the elementary particle in question. It is also well known (see e.g. Ref. [22, 13, 23, 41]) that the dS group contains elements (e.g. I) such that the corresponding representation operator transforms positive energy states to negative energy ones and *vice versa*. Are these properties compatible with the fact that in the Poincare limit there exist states with negative energies? This problem is discussed in detail in Ref. [27]. It has been shown that the interpretation of IRs of the so(1,4) algebra is consistent only we accept that one IR describes a particle and its antiparticle simultaneously. In this case the states with the negative energies can be converted to states with the positive energy by using the second quantization (by analogy with the interpretation of negative energy solutions of the Dirac equation as positrons).

At present the phenomenon of gravity has been observed only on macroscopic level, i.e. for particles which cannot be treated as elementary. Then the question arises whether they can be described by using the results on IRs. The usual assumption is as follows. In the approximation when it is possible to neglect the internal structure of the particles (e.g. when the distance between them is much greater than their sizes), the structure of the internal wave function is not important and one can consider only the part of the wave function describing the motion of the particle as the whole. This part is described by the same parameters as the wave function of the elementary particle. For this reason it is usually sufficient to describe the motion of the macroscopic system as a whole by using wave functions of IRs with zero spin.

2.5 de Sitter antigravity

Consider now the Poincare limit in the approximation when R is large but the first order corrections in $1/R$ to the conventional energy and momentum are taken into account. By using the definitions of the Poincare mass, energy and momentum from the preceding section and taking the dS generators in the form (2.9), one can obtain the expressions for the conventional energy and momentum in first order in $1/R$. For simplicity we assume that the particles are spinless and nonrelativistic. Consider a system of two free particles with the masses m_1 and m_2. Then the momentum and energy operators for each particle are given by

$$\mathbf{P}_j = \mathbf{p}_j + \frac{im_j}{R}\frac{\partial}{\partial \mathbf{p}_j}$$

$$E_j = m_j + \frac{\mathbf{p}_j^2}{2m_j} + \frac{i}{R}(\mathbf{p}_j\frac{\partial}{\partial \mathbf{p}_j} + \frac{3}{2}) \qquad (2.14)$$

where $\mathbf{p}_j = m\mathbf{v}_j$ and $j = 1, 2$.

The fact that the particle do not interact with each other implies (by definition) that the generators for the two-body system are equal to sums of the corresponding single-particle generators. Adding the corresponding operators and introducing the standard total and relative momenta

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \quad \mathbf{q} = (m_2\mathbf{p}_1 - m_1\mathbf{p}_2)/(m_1 + m_2) \tag{2.15}$$

one can obtain the expressions for the momentum $\mathbf{P}$ and energy E of the two-body system as a whole. Let M be the mass operator of the two-body system defined as $M^2 = E^2 - \mathbf{P}^2$. Then a simple calculation shows that in our approximation

$$M = m_1 + m_2 + \frac{\mathbf{q}^2}{2m_{12}} + \frac{i}{R}(\mathbf{q}\frac{\partial}{\partial \mathbf{q}} + \frac{3}{2}) \tag{2.16}$$

where $m_{12} = m_1m_2/(m_1 + m_2)$ is the reduced mass.

In spherical coordinates the nonrelativistic mass operator can be written as

$$M_{nr} = \frac{q^2}{2m_{12}} + V, \quad V = \frac{i}{R}(q\frac{\partial}{\partial q} + \frac{3}{2}) \tag{2.17}$$

where $q = |\mathbf{q}|$. Although this expression has been obtained in first order in $1/R$, let us consider for illustrative purposes the spectrum of this operator. It acts in the space of functions $\psi(q)$ such that

$$\int_0^\infty |\psi(q)|^2 q^2 dq < \infty \tag{2.18}$$

and the eigenfunction ψ_K of M_{nr} with the eigenvalue K satisfies the equation

$$q\frac{d\psi_K}{dq} = \frac{iRq^2}{2m_{12}}\psi_K - (\frac{3}{2} + iRK)\psi_K \tag{2.19}$$

The solution of this equation is

$$\psi_K = \sqrt{\frac{R}{2\pi}} q^{-3/2} exp(\frac{iRq^2}{4m_{12}} - iRK lnq) \tag{2.20}$$

and the normalization condition is

$$(\psi_K, \psi_{K'}) = \delta(K - K') \tag{2.21}$$

The spectrum of the operator M_{nr} obviously belongs to the interval $(-\infty, \infty)$ and one might think that this is unacceptable. Suppose however that $f(q)$ is a wave function of some state. As follows from Eq. (2.20), the probability to have the value of the kinetic energy K in this state is given by

$$c_K = \sqrt{\frac{R}{2\pi}} \int_0^\infty exp(-\frac{iRq^2}{4m_{12}} + iRKlnq) f(q)\sqrt{q}dq \tag{2.22}$$

If $f(q)$ does not on R and R is very large then c_K will practically be different from zero only if the integrand in Eq. (2.22) has a stationary point q_0. It is obvious that the stationary point is defined by the condition $K = q_0^2/2m_{12}$. Therefore, for negative K, when the stationary point is absent, the value of c_K will be very small.

We see that if one works only with a subset of wave functions not depending on R (which is typically the case), then the existence of the points of the spectrum of the two-body mass operator with the values less than $m_1 + m_2$ does not play an important role.

2.6 de Sitter antigravity in quasiclassical approximation

In conventional quantum mechanics the motion of a particle is quasiclassical if at each moment of time $t = t_0$ the particle wave function satisfies the following conditions (see e.g. Ref. [45]). In the coordinate representation the function has a sharp maximum at some $\mathbf{r} = \mathbf{r}_0$, and the uncertainty of the position $\Delta\mathbf{r}$ is much less than $\mathbf{r}_0$. At the same time, in the velocity representation it should have a sharp maximum at some $\mathbf{v} = \mathbf{v}_0$, and the uncertainty of the velocity $\Delta\mathbf{v}$ should be much less than $\mathbf{v}_0$. In particular, the particle cannot be quasiclassical if it is at rest, i.e. $\mathbf{v}_0 = 0$.

As follows from this definition, the notion of quasiclassical approximation necessarily implies that the position and velocity operators are well-defined and have a clear physical meaning. This is indeed the case in conventional nonrelativistic quantum mechanics. As already noted, in relativistic quantum theory there is no operator satisfying all the requirements for the position operator. In dS theories there exists an analogous problem. In particular, as seen from Eq. (2.9), the operator $\mathbf{v}$ by itself does not define the dS momentum (which is a physical operator) uniquely, and the operator $i\partial/(m\partial\mathbf{v})$, which in nonrelativistic quantum mechanics is the position operator in velocity representation, does not define the physical Lorentz boost operators uniquely. However, as noted in Sect. 2.4, when R is very large, the generators (2.9) can be contracted to standard generators of the IR of the Poincare algebra. In this case the momentum $\mathbf{P}$ is exactly proportional to $\mathbf{v}$ and the proportionality coefficient is the mass. Moreover, when the particle is nonrelativistic, then, as follows from Eq. (2.9), the Lorentz boost operators are proportional to the corresponding coordinate operators in velocity representation.

We see that, at least when R is large and $|\mathbf{v}| \ll 1$, there exists a well-defined quasiclassical approximation in the representation when the generators are given by Eq. (2.9). For example, the wave function can be chosen in the form

$$f(\mathbf{v}) = a(\mathbf{v})exp(-im\mathbf{v}\mathbf{r}_0/2) \tag{2.23}$$

where $a(\mathbf{v})$ has a sharp maximum at $\mathbf{v} = \mathbf{v}_0$ with a width $|\Delta\mathbf{v}| \ll |\mathbf{v}_0|$ and such that $|\partial a(\mathbf{v})/\partial\mathbf{v}| \ll m|\mathbf{r}_0|$. Note that the factor 1/2 in the exponent is a consequence of the fact that we are working with units where $\hbar/2 = 1$. A possible choice of $a(\mathbf{v})$ is

$$a(\mathbf{v}) = cv_0^{1/2}exp[-\frac{b^2}{2}(\mathbf{v}-\mathbf{v}_0)^2] \tag{2.24}$$

where the function is normalized to one (see Eq. (2.8)) if

$$c = \sqrt{2}b^{3/2}/\pi^{3/4}.$$

Then the condition $|\Delta\mathbf{v}| \ll |\mathbf{v}_0|$ is satisfied if $b|\mathbf{v}_0| \gg 1$ and the condition $|\Delta\mathbf{r}| \ll |\mathbf{r}_0|$ is satisfied if $b \ll m|\mathbf{r}_0|$. For macroscopic particles there exists a wide range of values b such that these conditions can be satisfied. Also, since the function (2.23) has the Gaussian form, the integral defining the mean values of the velocity and mass operators is rapidly convergent when $\mathbf{v}$ is far away from $\mathbf{v}_0$.

On classical level the effect of the additional term in Eq. (2.16) in comparison with the standard free nonrelativistic expression can be investigated as follows. We define the position operator $\mathbf{r}$ as $\mathbf{r} = 2i(\partial/\partial\mathbf{q})$. Then the classical Hamiltonian of the internal motion corresponding to the operator (2.16) is

$$H(\mathbf{r}, \mathbf{q}) = \frac{\mathbf{q}^2}{2m_{12}} + \frac{\mathbf{rq}}{R} \tag{2.25}$$

From classical equations of motion it follows that $d^2\mathbf{r}/dt^2 = \mathbf{r}/R^2$. It is well known that in classical dS space there exists a universal repulsion (antigravity) the force is which is proportional to the distance between particles. Therefore the operator V indeed corresponds to the dS antigravity.

In classical mechanics there exist transformations of the Hamiltonian, which do not change classical equations of motions. One can easily verify that classical equations of motion for the Hamiltonian

$$H(\mathbf{r}, \mathbf{q}) = \frac{\mathbf{q}^2}{2m_{12}} - \frac{m_{12}\mathbf{r}^2}{R^2} \tag{2.26}$$

are the same as for the Hamiltonian (2.25).

Although the example of the dS antigravity is extremely simple, we can draw the following very important conclusions.

The first conclusion is that the standard classical dS antigravity has been obtained from a quantum operator without introducing any classical background. When the position operator is defined as $\mathbf{r} = 2i(\partial/\partial\mathbf{q})$ and time is defined by the condition that the Hamiltonian is the evolution operator then one recovers the classical result obtained by considering a motion of particles in the classical dS spacetime. This is an illustration of the discussion in Chap. 1 and Sect. 2.2 about the difference between the standard approach, where the classical dS spacetime is introduced from the beginning, and our one.

The second conclusion is as follows. We have considered the particles as free, i.e. no interaction into the two-body system has been introduced. However, we have realized that when the two-body system in the dS theory is considered from the point of view of the Galilei invariant theory, the particles interact with each other. Although the reason of the effective interaction in our example is obvious, the existence of the dS antigravity poses the problem whether other interactions, e.g. gravity, can be treated as a result of transition from a higher symmetry to Poincare or Galilei one.

The third conclusion is that the dS antigravity is a true direct interaction since it is not a consequence of the exchange of virtual particles.

Finally, the fourth conclusion is that if in a free theory the spectrum of the mass operator has values less than $m_1 + m_2$, this does not necessarily mean that the theory is unphysical.

The above discussion shows that in dS theories our intuition based on nonrelativistic quantum mechanics still works at least when R is large and $\mathbf{v} \ll 1$ if decomposition in powers of $1/R$ and $\mathbf{v}$ is legitimate. In particular, the Lorentz boost operator can be treated as an operator proportional to the position operator. The problem arises how one should describe quasiclassical approximation without decomposition in powers of $1/R$ and $\mathbf{v}$. The only operators in our disposal are those defined by Eqs. (2.9) or (2.12). In quasiclassical approximation the action of the Lorentz boost operator (2.12) on the wave function (2.23) is given by

$$\mathbf{N}f(\mathbf{v}) = -m\sqrt{1+\mathbf{v}^2}\mathbf{r}_0 f(\mathbf{v}) \tag{2.27}$$

Therefore the position operator cannot be proportional anymore to the Lorentz boost operator. We will see below that it is possible to describe the two-body mass operator on pure algebraic level without using decomposition in powers of $\mathbf{v}$ and $1/R$.

Chapter 3

de Sitter invariant quantum theory in su(2)×su(2) basis

3.1 IRs in the su(2)×su(2) basis

Proceeding from the method of su(2)×su(2) shift operators, developed by Hughes [46] for constructing UIRs of the group SO(5), and following Ref. [26], we now give a pure algebraic description of IRs of the so(1,4) algebra. It will be convenient for us to deal with the set of operators $(\mathbf{J}', \mathbf{J}", R_{ij})$ $(i,j=1,2)$ instead of M^{ab}. Here $\mathbf{J}'$ and $\mathbf{J}"$ are two independent su(2) algebras (i.e. $[\mathbf{J}', \mathbf{J}"]=0$). In each of them one chooses as the basis the operators (J_+, J_-, J_3) such that $J_1 = J_+ + J_-$, $J_2 = -\imath(J_+ - J_-)$ and the commutation relations have the form

$$[J_3, J_+] = 2J_+, \quad [J_3, J_-] = -2J_-, \quad [J_+, J_-] = J_3 \tag{3.1}$$

The commutation relations of the operators $\mathbf{J}'$ and $\mathbf{J}"$ with R_{ij} have the form

$$\begin{aligned}
&[J_3', R_{1j}] = R_{1j}, \quad [J_3', R_{2j}] = -R_{2j}, \quad [J_3", R_{i1}] = R_{i1},\\
&[J_3", R_{i2}] = -R_{i2}, \quad [J_+', R_{2j}] = R_{1j}, \quad [J_+", R_{i2}] = R_{i1},\\
&[J_-', R_{1j}] = R_{2j}, \quad [J_-", R_{i1}] = R_{i2}, \quad [J_+', R_{1j}] =\\
&[J_+", R_{i1}] = [J_-', R_{2j}] = [J_-", R_{i2}] = 0,
\end{aligned} \tag{3.2}$$

and the commutation relations of the operators R_{ij} with each other have the form

$$
\begin{aligned}
&[R_{11}, R_{12}] = 2J'_{+}, \quad [R_{11}, R_{21}] = 2J_{+}", \\
&[R_{11}, R_{22}] = -(J'_3 + J_3"), \quad [R_{12}, R_{21}] = J'_3 - J_3" \\
&[R_{11}, R_{22}] = -2J_{-}", \quad [R_{21}, R_{22}] = -2J'_{-}
\end{aligned}
\tag{3.3}
$$

The relation between the sets $(\mathbf{J}', \mathbf{J}", R_{ij})$ and M^{ab} is given by

$$
\begin{aligned}
&\mathbf{M} = \mathbf{J}' + \mathbf{J}", \quad \mathbf{B} = \mathbf{J}' - \mathbf{J}", \quad M_{01} = \imath(R_{11} - R_{22}), \\
&M_{02} = R_{11} + R_{22}, \quad M_{03} = -i(R_{12} + R_{21}), \\
&M_{04} = R_{12} - R_{21}
\end{aligned}
\tag{3.4}
$$

Then it is easy to see that Eq. (2.2) follows from Eqs. (3.2-3.4) and *vice versa*.

Consider the space of maximal $su(2) \times su(2)$ vectors, i.e. such vectors x that $J'_{+}x = J_{+}"x = 0$. Then from Eqs. (3.2) and (3.3) it follows that the operators

$$
\begin{aligned}
&A^{++} = R_{11} \quad A^{+-} = R_{12}(J_3" + 1) - J_{-}"R_{11}, \\
&A^{-+} = R_{21}(J'_3 + 1) - J'_{-}R_{11} \\
&A^{--} = -R_{22}(J'_3 + 1)(J_3" + 1) + J_{-}"R_{21}(J'_3 + 1) + \\
&J'_{-}R_{12}(J_3" + 1) - J'_{-}J_{-}"R_{11}
\end{aligned}
\tag{3.5}
$$

act invariantly on this space. The notations are related to the property that if x^{kl} $(k, l > 0)$ is the maximal su(2)×su(2) vector and simultaneously the eigenvector of operators J'_3 and $J_3"$ with the eigenvalues k and l, respectively, then $A^{++}x^{kl}$ is the eigenvector of the same operators with the values $k+1$ and $l+1$, $A^{+-}x^{kl}$ - the eigenvector with the values $k+1$ and $l-1$, $A^{-+}x^{kl}$ - the eigenvector with the values $k-1$ and $l+1$ and $A^{--}x^{kl}$ - the eigenvector with the values $k-1$ and $l-1$.

As follows from Eq. (3.1), the vector $x^{kl}_{ij} = (J'_{-})^i(J_{-}")^j x^{kl}$ is the eigenvector of the operators J'_3 and $J_3"$ with the eigenvalues $k - 2i$ and $l - 2j$, respectively. Since

$$
\mathbf{J}^2 = J_3^2 - 2J_3 + 4J_{+}J_{-} = J_3^2 + 2J_3 + 4J_{-}J_{+}
$$

is the Casimir operator for the **J** algebra, and the Hermiticity condition can be written as $J_-^* = J_+$, it follows in addition that

$$\mathbf{J}'^2 x_{ij}^{kl} = k(k+2)x_{ij}^{kl}, \quad \mathbf{J}"^2 x_{ij}^{kl} = l(l+2)x_{ij}^{kl} \tag{3.6}$$

$$J'_+ x_{ij}^{kl} = i(k+1-i)x_{i-1,j}^{kl}, \quad J_+" x_{ij}^{kl} = j(l+1-j)x_{i,j-1}^{kl} \tag{3.7}$$

$$(x_{ij}^{kl}, x_{ij}^{kl})) = \frac{i!j!k!l!}{(k-i!)(l-j!)}(x^{kl}, x^{kl}) \tag{3.8}$$

where (...,...) is the scalar product in the representation space. From these formulas it follows that the action of the operators $\mathbf{J}'$ and $\mathbf{J}"$ on x^{kl} generates a space with the dimension $(k+1)(l+1)$ and the basis x_{ij}^{kl} $(i = 0, 1, ...k,\ j = 0, 1, ...l)$. Note that the vectors x_{ij}^{kl} are orthogonal but in this section we do not normalize them to one.

The Casimir operator of the second order for the algebra (2.2) can be written as

$$I_2 = -\frac{1}{2}\sum_{ab} M_{ab}M^{ab} =$$
$$4(R_{22}R_{11} - R_{21}R_{12} - J'_3) - 2(\mathbf{J}'^2 + \mathbf{J}"^2) \tag{3.9}$$

A direct calculation shows that for the generators given by Eqs. (2.9), (2.10) and (2.12), I_2 has the numerical value

$$I_2 = w - s(s+2) + 9 \tag{3.10}$$

where $w = \mu^2$. As noted in Sect. 2.4, $\mu = 2mR$ where m is the conventional mass. If $m \neq 0$ then μ is very large since R is very large. We conclude that for massive IRs the quantity I_2 is a large positive number.

The basis in the representation space can be explicitly constructed assuming that there exists a vector e^0 which is the maximal su(2)×su(2) vector such that

$$J'_3 e_0 = n_1 e_0 \quad J_3" e_0 = n_2 e_0 \tag{3.11}$$

and n_1 is the minimum possible eigenvalue of J'_3 in the space of the maximal vectors. Then e_0 should also satisfy the conditions

$$A^{--}e_0 = A^{-+}e_0 = 0 \tag{3.12}$$

We use $\tilde{I}$ to denote the operator $R_{22}R_{11}-R_{21}R_{12}$. Then as follows from Eqs. (3.2), (3.3), (3.5), (3.9), (3.11) and (3.12),

$$\tilde{I}n_1e_0 = 2n_1(n_1+1)e_0.$$

Therefore, if $n_1 \neq 0$ the vector e_0 is the eigenvector of the operator $\tilde{I}$ with the eigenvalue $2(n_1+1)$ and the eigenvector of the operator I_2 with the eigenvalue

$$-2[(n_1+2)(n_2-2)+n_2(n_2+2)].$$

The latter is obviously incompatible with Eq. (3.10) for massive IRs. Therefore the compatibility can be achieved only if $n_1 = 0$. In that case we use s to denote n_2 since it will be clear soon that the value of n_2 indeed has the meaning of spin. Then, as follows from Eqs. (3.10) and (3.11), the vector e_0 should satisfy the conditions

$$\begin{aligned} &\mathbf{J}'e^0 = J_+"e^0 = 0, \quad J_3"e^0 = se^0, \\ &I_2e^0 = [w-s(s+2)+9]e^0 \end{aligned} \tag{3.13}$$

where $w, s > 0$ and s is an integer.

Define the vectors

$$e^{nr} = (A^{++})^n(A^{+-})^re^0 \tag{3.14}$$

Then a direct calculation taking into account Eqs. (3.1)-(3.3), (3.5), (3.6), (3.9), (3.12) and (3.13) gives

$$A^{--}A^{++}e^{nr} = -\frac{1}{4}(n+1)(n+s+2)[w+(2n+s+3)^2]e^{nr} \tag{3.15}$$

$$A^{-+}A^{+-}e^{nr} = -\frac{1}{4}(r+1)(s-r)[w+1+(2r-s)(2r+2-s)]e^{nr} \tag{3.16}$$

$$(e^{n+1,r}, e^{n+1,r}) = \frac{(n+1)(n+s+2)[w+(2n+s+3)^2]}{4(n+r+2)(s-r+n+2)}(e^{nr}, e^{nr}) \tag{3.17}$$

$$\begin{aligned} (e^{n,r+1}, e^{n,r+1}) = &\frac{1}{4}(r+1)(s-r)[w+1+(2r-s)(2r+2-s)] \\ &\frac{s-r+n+1}{s-r+n+2}(e^{nr}, e^{nr}) \end{aligned} \tag{3.18}$$

As follows from Eqs. (3.15) and (3.17), the possible values of n are $n = 0, 1, 2, ...$ while, as follows from Eqs. (3.16) and (3.18), r can take only the values of $0, 1, ...s$ (and therefore s indeed has the meaning of the particle spin). Since e^{nr} is the maximal $su(2) \times su(2)$ vector with the eigenvalues of the operators $\mathbf{J}'$ and $\mathbf{J}$" equal to $n+r$ and $n+s-r$, respectively, then as a basis of the representation space one can take the vectors $e^{nr}_{ij} = (J'_-)^i (J_-")^j e^{nr}$ where, for the given n and s, the quantity i can take the values of $0, 1, ...n+r$ and j - the values of $0, 1, ...n+s-r$.

One can show [26] that the construction discussed in this section is is an implementation of the generators (2.12) but not (2.9) and (2.10).

Below we will discuss in detail a system of two spinless particles. If $s = 0$ then there exist only the maximal $su(2) \times su(2)$ vectors x^{kl} with $k = l$ and therefore the basis of the representation space is formed by the vectors $e^n_{\alpha\beta} \equiv e^{n0}_{\alpha\beta}$ where $n = 0, 1, 2, ...;\ \alpha, \beta = 0, 1, ...n$. The explicit expressions for the action of operators R_{ij} in this basis can be calculated by using Eq. (3.2), and the result is

$$
\begin{aligned}
&R_{11}e^n_{\alpha\beta} = \frac{(n+1-\alpha)(n+1-\beta)}{(n+1)^2}e^{n+1}_{\alpha\beta} + \\
&\frac{\alpha\beta n}{4(n+1)}[w + (2n+1)^2]e^{n-1}_{\alpha-1,\beta-1}, \\
&R_{12}e^n_{\alpha\beta} = \frac{n+1-\alpha}{(n+1)^2}e^{n+1}_{\alpha,\beta+1} - \frac{\alpha n}{4(n+1)}[w + (2n+1)^2]e^{n-1}_{\alpha-1,\beta}, \\
&R_{21}e^n_{\alpha\beta} = \frac{n+1-\beta}{(n+1)^2}e^{n+1}_{\alpha+1,\beta} - \frac{\beta n}{4(n+1)}[w + (2n+1)^2]e^{n-1}_{\alpha,\beta-1}, \\
&R_{22}e^n_{\alpha\beta} = \frac{1}{(n+1)^2}e^{n+1}_{\alpha+1,\beta+1} + \\
&\frac{n}{4(n+1)}[w + (2n+1)^2]e^{n-1}_{\alpha\beta}
\end{aligned}
\qquad (3.19)
$$

As follows from Eqs. (3.8) and (3.17)

$$
(e^n_{\alpha\beta}, e^n_{\alpha\beta}) = \frac{(n!)^2\alpha!\beta!}{4^n(n+1)(n-\alpha)!(n-\beta)!}\prod_{j=1}^{n}[w + (2j+1)^2] \qquad (3.20)
$$

if $(e_0, e_0) = 1$ and, as follows from Eqs. (3.4) and (3.6)

$$(\mathbf{B}^2 + \mathbf{M}^2)e^n_{\alpha\beta} = 4n(n+2)e^n_{\alpha\beta} \qquad (3.21)$$

As noted in Sect. 2.4, in Poincare limit $\mathbf{B}/2R$ becomes the momentum operator. Therefore in Poincare limit the eigenvalues of $\mathbf{B}^2$ are much greater than those for $\mathbf{M}^2$. For quasiclassical states the values of n are very large and therefore it follows from Eq. (3.21) that the quantum number n has a meaning that n/R becomes the magnitude of the momentum in Poincare limit.

3.2 Free two-body mass operator

We now consider a two-body system and assume that it is free, i.e. there is no interaction between the particles. How can one distinguish quantum-mechanical descriptions of free and interacting systems? In the literature (see e.g. Refs. [1, 48]) a system is called free if its generators are sums of the single-particle generators for the particles comprising the system. In our case this implies that $M_{ab} = M^{(1)}_{ab} + M^{(2)}_{ab}$ where $M^{(1)}_{ab}$ are the generators for the first particle and $M^{(2)}_{ab}$ - for the second one. Each generator acts over the variables of its "own" particle, as described in Sect. 2.3 or Sect. 3.1, and over the variables of another particle it acts as the identity operator. In other words, the representation describing the two-body system is the tensor product of single-particle IRs. Several authors (see e.g. Refs. [1, 2]) define a system as free if its S-matrix is an identical operator and interacting otherwise. In Poincare invariant theories the S-matrix is well defined but in de Sitter invariant theories it is probably not possible to define an analog of Poincare invariant S-matrix (but this does not mean that dS theories are unphysical). So we accept the first definition of free and interacting systems.

However, in this case the following problem arises. As noted in Sect. 2.3, the generators of IR are defined up to unitary equivalence. For example, the operators defined by Eqs. (2.9) and (2.10) on one hand and by Eq. (2.12) on the other, are unitarily equivalent and the unitary transformation implementing the equivalence is defined by Eq. (2.13). Suppose that the operators of IR are implemented in two forms, such that G_a and G_b denote the sets of operators in the form a and b, respectively. They are related by a unitary transformation U_{ba} such that $G_b = U_{ba}G_aU_{ba}^{-1}$. Let now $G_a^{(j)}, G_b^{(j)}, U_{ba}^{(j)}$ $(j = 1, 2)$ be the corresponding operators for particles 1 and 2, respectively and G_a, G_b be the two-body operators in the forms a and b, respectively. If we assume that the two-body operators are sums of the single-particle operators in the form a then $G_a = G_a^{(1)} + G_a^{(2)}$ while if we assume that the two-body operators are sums of the single-particle operators in the form b then

$$G_b = G_b^{(1)} + G_b^{(2)} = U_{ba}^{(1)}G_a^{(1)}(U_{ba}^{(1)})^{-1} + U_{ba}^{(2)}G_a^{(2)}(U_{ba}^{(2)})^{-1}$$

In this case the two-body operators G_a and G_b are not unitarily equivalent since the operators $U_{ba}^{(1)}$ and $U_{ba}^{(2)}$ are different.

We conclude that the notion of free or interacting system depends on the choice of the form of IR. With one choice the system can be treated as free but then with the other choice it will be treated as interacting. Therefore the problem arises, which form of IRs of the dS algebra is more physical. As noted in Sect. 2.4, the form of the operators defined by Eqs. (2.9) and (2.10) is convenient since in this case the Poincare limit is straightforward. On the other hand, the form of the operators defined by Eq. (2.12) has its own advantages (for example, the Hilbert space is the space of functions on S^3 rather than the space of functions on two Lorentz hyperboloids). We believe that the latter choice is more fundamental from the following considerations. As shown in the preceding section, this choice allows a pure algebraic description. If we believe that the ultimate quantum theory will be discrete and finite then we should be looking for such implementations of the standard theory, which can be generalized to a discrete and finite theory. As shown in Ref. [26], if the operators of IR are described as in the preceding section, then the theory can be directly generalized to a case when quantum theory is defined over a Galois field rather then the field of complex numbers. At the same time, such a generalization is not possible for operators defined by Eqs. (2.9) and (2.10) since the velocity operator has a pure continuous spectrum. In summary, we believe that the above arguments in favor of the representation described in the preceding section are much more important than the convenience of transition to the Poincare limit. We will see below that this representation does not contradict experiment but in some cases it leads to consequences, which are far from what one might expect from naive considerations.

Denote by μ_1 and μ_2 $(\mu_1, \mu_2 > 0)$ the dS masses of the corresponding particles and assume that they are spinless. Then, as follows from Eq. (3.10), $I_2^{(1)} = 2(w_1 + 9)$, $I_2^{(2)} = 2(w_2 + 9)$, where $w_1 = \mu_1^2$, $w_2 = \mu_2^2$. The tensor product of IRs can be decomposed into the direct integral of IRs and there exists a well elaborated general theory [44]. In terminology of the theory of induced UIRs, UIRs discussed in Sect. 2.3 belong to the principal series of UIRs. In general, the decomposition of the tensor product of UIRs belonging to the principal series may contain not only UIRs of the principal series (i.e. it may also contain UIRs not having "rest states" defined by Eq. (3.13)). We will consider only a part of the tensor product containing the "rest states" and show that even for this part the spectrum of the mass operator is not bounded below by the value of $(\mu_1 + \mu_2)^2$.

It is clear that only IRs with $s = 0, 2, 4...$ can enter the tensor product of two spinless representations. Therefore in order to find which values of w are possible for the given s one can act as follows. First define the two-body operators $\mathbf{J}'$ and $\mathbf{J}"$ as sums of the corresponding single-particle operators defined in the preceding section, i.e.

$$\mathbf{J}' = \mathbf{J}'^{(1)} + \mathbf{J}'^{(2)}, \quad \mathbf{J}" = \mathbf{J}"^{(1)} + \mathbf{J}"^{(2)}$$

Construct H_s — a space of elements x, satisfying the condition (compare with Eq. (3.13))

$$\mathbf{J}'x = J_+"x = 0, \quad \mathbf{J}"^2 x = s(s+2)x \tag{3.22}$$

We now use I_2 to denote the two-body operator constructed by analogy with Eq. (3.9). Since it commutes with all the two-body representation operators then H_s is invariant under the action of I_2. Since $I_{2s} =$

$2[W-s(s+2)+9]$, the operator W also can be reduced onto H_s and the spectrum of W in H_s defines possible values of w for the given s.

Note that although W is the dS analog of the mass operator squared in Poincare invariant theory, it is not a square of any operator. Therefore one cannot exclude a possibility that W has even a negative part of the spectrum. However, the part of W corresponding to the principle series IRs has only the positive spectrum.

To construct a basis in the space H_s, we have first to ascertain which linear combinations of the elements $e^{(1)n_1}_{\alpha_1\beta_1}e^{(2)n_2}_{\alpha_2\beta_2}$ belong to H_s. Since $e^{(1)n_1}_{\alpha_1\beta_1}$ is the spinor with the spin n_1 with respect to the algebra $\mathbf{J}'^{(1)}$ as well as to $\mathbf{J}"^{(1)}$, and analogously for $e^{(2)n_2}_{\alpha_2\beta_2}$, then zero eigenvalues of the operator $\mathbf{J}'$ can be obtained only if $n_1=n_2$, and the value s for the spin relative to the $\mathbf{J}"$ algebra can be obtained only if $n_1,n_2\geq s/2$. One can verify directly that the vectors

$$\begin{aligned}\Phi_n \;=\; & \frac{4^{n+j}(1+2j)!}{[(n+j)!]^2 j!}\sum_{\alpha=0}^{n+j}\sum_{\beta=0}^{n}(-1)^{\alpha+\beta}\times \\ & \frac{(j+\beta)!(n+j-\beta)!}{\beta!(n-\beta)!}e^{(1)n+j}_{\alpha\beta}e^{(2)n+j}_{n+j-\alpha,n-\beta} \qquad (3.23)\end{aligned}$$

where $j=s/2$ and $n=0,1,2...$, belong to H_s. The value of j is obviously equal to the spin of the two-body system in conventional units. The fact that the vectors $e^{(1)}$ and $e^{(2)}$ enter Eq. (3.23) with the same value of the quantum number n confirms the interpretation of n/R as the magnitude of the momentum (see the preceding section) since, by analogy with Poincare invariant theory, one would expect that the magnitudes of particle momenta in their common c.m. frame are equal to each other.

It is obvious that the vectors Φ_n with different n's are orthogonal to each other. The result of the calculation of the norm of the vector Φ_n (see Ref. [26] for details) is

$$(\Phi_n, \Phi_n) = \{\prod_{l=1}^{n+j}[w_1 + (2l+1)^2][w_2 + (2l+1)^2]\} \times \frac{(n+2j+1)!(1+2j)!}{(n+j+1)n!} \tag{3.24}$$

Our next goal is to find how the operator W acts in the basis $\{\Phi_n\}$. As follows from Eq. (3.9),

$$I_2^s\Phi_n = 8[(R_{22}^{(1)} + R_{22}^{(2)})(R_{11}^{(1)} + R_{11}^{(2)}) - (R_{21}^{(1)} + R_{21}^{(2)}) \times (R_{12}^{(1)} + R_{12}^{(2)})]\Phi^n - 4s(s+2)\Phi_n, \tag{3.25}$$

$$I_2^{(1)}e_{\alpha\beta}^{(1)n} = 8(R_{22}^{(1)}R_{11}^{(1)} - R_{21}^{(1)}R_{12}^{(1)})e_{\alpha\beta}^{(1)n} - 8[n(n+2) + (n-2\alpha)]e_{\alpha\beta}^{(1)n} = 2(w_1+9)e_{\alpha\beta}^{(1)n} \tag{3.26}$$

and an analogous formula takes place for $I_2^{(2)}e_{\alpha\beta}^{(2)n}$.

Taking into account Eqs. (3.19), (3.23), (3.25), (3.26) and the definition of the operator W, a direct calculation shows that

$$W\Phi_n = \sum_{l=0}^{\infty}\Phi_l W_{ln} \tag{3.27}$$

where the matrix $||W_{ln}||$ has only the following components different from zero:

$$W_{n+1,n}^s = \frac{n+1}{n+1+j}, \quad W_{nn}^s = w_1 + w_2 + 8(n+1)^2 + 2s(4n+3) + s^2 + 1, \quad W_{n,n+1}^s = \frac{n+2j+2}{n+j+2} \times [w_1 + (2n+2j+3)^2][w_2 + (2n+2j+3)^2] \tag{3.28}$$

Such a matrix is called three-diagonal and in fact, only the terms with $l = n-1, n, n+1$ contribute to the sum (3.27). Note that the operator W is certainly Hermitian, but since the basis elements are not normalized to one, the Hermiticity condition has not the usual form $W_{nl} = W_{ln}^*$, but $||\Phi_n||^2 W_{nl} = ||\Phi_l||^2 W_{ln}^*$.

The matrix of the operator $W - \lambda$ has the matrix elements $||W_{nl} - \lambda\delta_{nl}||$. We use $\Delta^n(\lambda)$ to denote the determinant of the matrix obtained from this one by taking into account only the rows and columns with the numbers $0, 1, ...n$. It is well known (and can be verified directly) that for the three-diagonal matrix the following relation is valid:

$$\Delta^{n+1}(\lambda) = (W_{n+1,n+1} - \lambda)\Delta^n(\lambda) - W_{n+1,n}W_{n,n+1}\Delta^{n-1}(\lambda) \quad (3.29)$$

where it is formally assumed that $\Delta^{-1}(\lambda) = 1$. Since $\Delta^0(\lambda) = W_{00} - \lambda$, Eqs. (3.28) and (3.29) make it possible to calculate $\Delta^n(\lambda)$ for any $n = 1, 2, 3....$ The result can be represented as follows. Denote

$$\lambda_l = [\mu_1 + \mu_2 + i(s + 4l + 3)]^2 \quad (l = 0, 1, 2...) \quad (3.30)$$

Then

$$\Delta^n(\lambda) = (-1)^{n+1} \sum_{k=0}^{n+1} C_{n+1}^k [\prod_{l=0}^{k-1} (\lambda - \lambda_l)] \prod_{l=k}^{n} \frac{s + k + l + 2}{j + l + 1}$$
$$[\mu_1 + i(s + 2l + 3)][\mu_2 + i(s + 2l + 3)] \quad (3.31)$$

Since we consider only representations of the principle series, we are interested only in the region of positive λ's. Therefore we can represent λ as $\lambda = (\mu_1 + \mu_2 + \sigma)^2$ where σ has the meaning of the dS kinetic energy. However as we will see below, not only $\sigma \geq 0$, but also $\sigma < 0$ is possible. Let $(a)_n = a(a + 1) \cdots (a + n - 1)$ be the Pochhammer symbol. Then the expression (3.31) can be written in terms of the hypergeometric series as

$$\Delta^n(\lambda) = 4^{n+1} \frac{(s + 2 + n)!j!}{(1 + s)!(j + n + 1)!} (\frac{s + 3 - i\mu_1}{2})_{n+1} (\frac{s + 3 - i\mu_2}{2})_{n+1}$$
$${}_4F_3(\frac{s + 3 + i\sigma}{4}, \frac{s + 3 - i(2\mu_1 + 2\mu_2 + \sigma)}{4}, s + n + 3, -(n + 1);$$
$$\frac{s + 3 - i\mu_1}{2}, \frac{s + 3 - i\mu_1}{2}, \frac{s + 3}{2}; 1) \quad (3.32)$$

Consider a vector

$$\chi(\lambda, N) = \sum_{n=0}^{N}(-1)^n\Delta^{n-1}(\lambda)[\prod_{l=0}^{n-1} W_{l,l+1}]^{-1}\Phi_n \tag{3.33}$$

where N is a natural number. Let $F(\lambda', \lambda, N) = (\chi(\lambda', N), \chi(\lambda, N))$ and $F(\lambda', \lambda)$ be the limit of $F(\lambda', \lambda, N)$ when $N \to \infty$. If the limit in Eq. (3.33) exists then, as follows from Eqs. (3.28) and (3.29), it is an eigenvector of the operator W with the eigenvalue λ. In this case λ is the true eigenvalue i.e. it belongs to the discrete spectrum of the operator W. There also exists a possibility that the limit does not represent a vector belonging to the Hilbert space but is a generalized eigenvector, i.e. $F(\lambda', \lambda)$ is proportional to $\delta(\lambda - \lambda')$. Then λ belongs to the continuous spectrum of the operator W. As follows from Eqs. (3.24), (3.32) and (3.33),

$$F(\lambda', \lambda, N) = \frac{[(1+s)!]^2}{(1+j)^2}\{\prod_{l=1}^{j}[w_1 + (2l+1)^2][w_1 + (2l+1)^2]\}$$
$$\sum_{n=0}^{N}(n+j+1)C_{n+s+1}^{1+s}\, {}_4F_3(\frac{s+3-i\sigma'}{4}, \frac{s+3+i(2\mu_1+2\mu_2+\sigma')}{4},$$
$$s+n+2, -n; \frac{s+3+i\mu_1}{2}, \frac{s+3+i\mu_1}{2}, \frac{s+3}{2}; 1)$$
$${}_4F_3(\frac{s+3+i\sigma}{4}, \frac{s+3-i(2\mu_1+2\mu_2+\sigma)}{4}, s+n+2, -n;$$
$$\frac{s+3+i\mu_1}{2}, \frac{s+3+i\mu_1}{2}, \frac{s+3}{2}; 1) \tag{3.34}$$

where $\lambda' = \mu_1 + \mu_2 + \sigma'$, and C_n^k is the binomial coefficient. By using the relations (see e.g. Ref. [47])

$${}_{q+1}F_q(\beta, \alpha_1, ...\alpha_q; \gamma, \rho_1, ...\rho_{q-1}; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}$$
$$\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\, {}_qF_{q-1}(\alpha_1, ...\alpha_q; \rho_1, ...\rho_{q-1}; tz)dt \tag{3.35}$$

$$n!C_n^{\lambda}(x) = (2\lambda)_n F(-n, n+2\lambda; \lambda+1/2; (1-x)/2) \tag{3.36}$$

where $C_n^\lambda(x)$ is the Gegenbauer polynomial and in the case of the hypergeometric function ${}_2F_1$ we write simply F, the expression (3.34) can be represented as

$$F(\lambda',\lambda,N)=\frac{[(1+s)!]^2}{(1+j)^2}\{\prod_{l=1}^{j}[w_1+(2l+1)^2][w_1+(2l+1)^2]\}$$
$$|\Gamma(\frac{s+3-i\mu_1}{2})\Gamma(\frac{s+3-i\mu_2}{2})|^2\{\Gamma(\frac{s+3}{2})^2\Gamma(\frac{s+3-i\sigma'}{4})$$
$$\Gamma(\frac{s+3+i\sigma}{4})\Gamma(\frac{s+3+i(2\mu_1+2\mu_2+\sigma')}{4})$$
$$\Gamma(\frac{s+3-i(2\mu_1+2\mu_2+\sigma)}{4})\}^{-1}\sum_{n=0}^{N}\frac{(n+j+1)}{C_{n+s+1}^{1+s}}\int_0^1 u^{(s-1-i\sigma')/4}$$
$$(1-u)^{(s+1)/2}C_n^{1+j}(1-2u)F(\frac{s+3-i(2\mu_1+\sigma')}{4},$$
$$\frac{s+3-i(2\mu_2+\sigma')}{4};\frac{s+3}{2};1-u)du\int_0^1 u^{(s-1+i\sigma)/4}(1-u)^{(s+1)/2}$$
$$C_n^{1+j}(1-2u)F(\frac{s+3+i(2\mu_1+\sigma')}{4},\frac{s+3+i(2\mu_2+\sigma)}{4};$$
$$\frac{s+3}{2};1-u)du \qquad (3.37)$$

To calculate the limit if this expression when $N\to\infty$ we use the fact that the system of Gegenbauer polynomials form a complete orthogonal set in the space of functions quadratically integrable on $[-1,1]$ with some weight (see e.g. Ref. [47]). As a result, the completeness relation for the Gegenbauer polynomials can be written as

$$\sum_{n=0}^{\infty}C_n^{1+j}(x)C_n^{1+j}(y)\frac{(n+1+j)\Gamma(1+j)}{\sqrt{\pi}C_{n+s+1}^{1+s}\Gamma((s+3)/2)}=\frac{\delta(x-y)}{(1-y^2)^{(s+1)/2}} \qquad (3.38)$$

and the limit of the expression (3.37) when $N \to \infty$ can be written as

$$F(\lambda',\lambda) = \frac{\sqrt{\pi}[(1+s)!]^2}{2^{s+2}(1+j)^2 j!}\{\prod_{l=1}^{j}[w_1+(2l+1)^2][w_1+(2l+1)^2]\}$$
$$|\Gamma(\frac{s+3-i\mu_1}{2})\Gamma(\frac{s+3-i\mu_2}{2})|^2\{\Gamma(\frac{s+3}{2})^2\Gamma(\frac{s+3-i\sigma'}{4})$$
$$\Gamma(\frac{s+3+i\sigma}{4})\Gamma(\frac{s+3+i(2\mu_1+2\mu_2+\sigma')}{4})$$
$$\Gamma(\frac{s+3-i(2\mu_1+2\mu_2+\sigma)}{4})\}^{-1}\int_0^1 u^{i(\sigma-\sigma')/4-1}(1-u)^{(s+1)/2}$$
$$F(\frac{s+3-i(2\mu_1+\sigma')}{4},\frac{s+3-i(2\mu_2+\sigma')}{4};\frac{s+3}{2};1-u)$$
$$F(\frac{s+3+i(2\mu_1+\sigma')}{4},\frac{s+3+i(2\mu_2+\sigma)}{4};\frac{s+3}{2};1-u)du \quad (3.39)$$

Since the integral diverges when $u \to 0$, we can set $u = 0$ in the nonsingular expressions in this intergal and the final result is

$$F(\lambda',\lambda) = \frac{\pi^{3/2}[(1+s)!]^2}{2^{s+2}(1+j)^2 j!}\{\prod_{l=1}^{j}[w_1+(2l+1)^2][w_1+(2l+1)^2]\}$$
$$|\frac{\Gamma(\frac{s+3+i\mu_1}{2})\Gamma(\frac{s+3+i\mu_2}{2})\Gamma(\frac{i}{2}(\mu_1+\mu+\sigma))}{\Gamma(\frac{s+3+i\sigma}{4})\Gamma(\frac{s+3+i(2\mu_1+\sigma)}{4})\Gamma(\frac{s+3+i(2\mu_2+\sigma)}{4})}|^2$$
$$\Gamma(\frac{s+3}{2})\delta(\sigma-\sigma') \quad (3.40)$$

Therefore the discrete spectrum is absent and continuous one fills all the interval $\lambda \in (0,\infty)$.

In Ref. [24], where the explicit expression for the two-body mass operator in the form of a differential operator in some space of functions has been has been found, a similar result has been obtained. The result (3.40), however is in fact algebraic since the operator W has been considered in the form of an infinite matrix. In Sect. 2.5 we have shown that the dS mass operator has the infinite spectrum in the range $(-\infty,\infty)$. However, this result has been obtained in the nonrelativistic approximation and in first order in $1/R$. On the contrary, no approximation has been assumed in deriving Eq. (3.40).

3.3 Mean value of the two-body mass operator

It is well known that in nonrelativistic quantum mechanics, dynamics of a two-body system is fully defined by the two-body mass operator, which acts only over the internal variables of the system. In a wide literature on Poincare invariant dynamics (see e.g. Ref. [1, 48]) the following problem has been widely discussed: how to define interactions in a many-body system such that the commutation relations of the Poincare algebra are preserved. It has been shown that it is possible to define internal and external variables in such a way that interactions are present only in the mass operator, which is unitarily equivalent to one acting only in the internal Hilbert space. Then the commutation relations are preserved if the mass operator commutes with spin operator of the system.

The problem arises what is the internal Hilbert space in dS theories. In the preceding section we have investigated the structure of the Hilbert space H_s, which might be treated as an analog of the internal Hilbert space in Poincare invariant theory if the spin of the systems equals s. Indeed, by analogy with the construction of IRs in Sect. 3.1, one can construct the full two-body Hilbert space by acting by the two-body operators R_{ij} $(i, j = 1, 2)$ on spaces H_s with all possible values of s. In this case, a subspace corresponding to a given s is a direct sum of spaces identical to H_s. Since the mass operator commutes with all the representation operators, it also commutes with all the projectors onto such spaces and its spectra in each space identical to H_s are the same. Therefore the properties of the dS mass operator is fully defined by its action in H_s for all possible values of s. Our first goal is to understand when the mean value of the dS mass operators corresponds to the well known mean value of the Poincare invariant mass operator.

Instead of the states Φ_n in H_s, we introduce the states

$$e_n = \Phi_n(\Phi_n, \Phi_n)^{-1/2} \tag{3.41}$$

Then the elements $\{e_n\}$ form the orthonormal basis in H_s. As follows from Eqs. (3.24) and (3.28), the nonzero matrix elements of the operator W in the basis $\{e_n\}$ are given by

$$W(n,j) = \{[w_1 + (2n+2j+3)^2][w_2 + (2n+2j+3)^2]$$
$$\frac{(n+2j+2)(n+1)}{(n+1+j)(n+2+j)}\}^{1/2} \quad W_{n+1,n} = W_{n,n+1} = W(n,j)$$
$$W_{nn} = w_1 + w_2 + 8(n+1)^2 + 4j(4n+3) + 4j^2 + 1 \qquad (3.42)$$

where $j = s/2$. Let

$$\Psi = \sum_{n=0}^{\infty} c_n e_n \quad (\sum_{n=0}^{\infty} |c_n|^2 = 1) \qquad (3.43)$$

be a state in H_s. Then the mean value of the operator W in the state Ψ is given by

$$(\Psi, W\Psi) = \sum_{n=0}^{\infty}[W_{nn}|c_n|^2 + 2W_{n,n+1}Re(c_{n+1}c_n^*)] \qquad (3.44)$$

where W_{nn} and $W_{n,n+1}$ are given by Eq. (3.42).

This result has been obtained without any approximation, so it is valid for any wave function c_n in the n-representation. If we are interested in comparing the results with Poincare invariant theory, the mass operator squared in Poincare terms should be defined as $M^2 = W/4R^2$ (see Sects. 2.4 and 3.2). We also take into account that typical values of n are very large since, as noted in the preceding section, n is of order qR where $\mathbf{q}$ is the relative momentum and $q = |\mathbf{q}|$. In particular, $n \gg j$ in the approximation when R is very large. Then taking into account the relation between the dS and Poincare masses (see Sect. 2.4), we get from Eq. (3.44)

$$(\Psi, M^2\Psi) = \sum_{n=0}^{\infty}\{[m_1^2 + m_2^2 + 2(n/R)^2]|c_n|^2 +$$
$$2[m_1^2 + (n/R)^2]^{1/2}[m_2^2 + (n/R)^2]^{1/2}Re(c_{n+1}c_n^*)\} \qquad (3.45)$$

i.e. the dependence on s formally dissapears.

One usually expects that the quasiclassical wave function in the relative momentum representation has a sharp maximum and therefore one might expect that in the n-representation the quasiclassical wave function has a sharp maximum at some value n_0 such that $q_0 = n_0/R$ has the meaning of relativistic relative momentum. Then, if we assume additinally that c_n does not change significantly when n is replaced by $n+1$, i.e. $c_n \approx c_{n+1}$, then Eq. (3.45) gives the standard relativistic result:

$$(\Psi, M^2\Psi) = [\epsilon_1(q) + \epsilon_2(q)]^2 \tag{3.46}$$

where $\epsilon_j(q) = \sqrt{m_j^2 + q^2}$ (j=1,2). This result is an additional argument in favor of calling n the de Sitter relative momentum. Therefore the set $\{c_n\}$ has the meaning of the internal wave function in de Sitter momentum representation. In contrast with the internal relativistic wave function in momentum representation $c(q)$, where q is a continuos variable in the range $[0, \infty)$, the quantity n takes the values 0,1,2,.., i.e. it is a nonnegative integer. In other words, the de Sitter relative momentum is quantized with the quantum equal to 1. If Poincare invariant theory is treated as a limit of de Sitter invariant theory when R is large, then the relativistic relative momentum becomes a quantity quantized with the quantum $1/R$. This is an extremely small value and therefore the discrete and continuous cases are practically indistinguishable. The result (3.46) is also an advancement in comparison with the resuls of Sect. 2.6 since a relation between Poincare and de Sitter invariant theories is obtained without assuming that the velocity is small. Nevertheless, we are still very far from our main goal - understanding quasiclassical approximation in de Sitter invariant theory without decomposition in powers of $1/R$. Our next task is to understand how the result (3.45) can recover de Sitter antigravity if we do not assume that the velocity is nonrelativistic. We first recall well known facts about quasiclassical approximation in the standard theory.

3.4 Discussion of standard quasiclassical approximation

In Sect. 2.6 we have discussed quasiclassical approximation in the nonrelativistic approximation assuming that the operators describing a particle are given by Eq. (2.9). Our goal in subsequent sections is to describe quasiclassical approximation in the framework of the algebraic approach developed in this chapter and not assuming that $|\mathbf{v}| \ll 1$. In this section we recall well known facts about quasiclassical approximation in the standard theory.

Let $\psi(x)$ be a one-dimensional wave function in the coordinate representation in nonrelativistic quantum mechanics. This function can be normalized as

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1 \tag{3.47}$$

If $\varphi(p)$ is the wave function in momentum representation then

$$\varphi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) exp(-ipx) dx \quad \int_{-\infty}^{\infty} |\varphi(p)|^2 dp = 1 \tag{3.48}$$

and in this section we are working with the usual system of units where $\hbar = 1$.

The usual assumptions about the wave function in coordinate representation are as follows. The function $\psi(x)$ has the form $\psi(x) = a(x) exp(ip_0 x)$ where p_0 is the classical momentum of the particle; the amplitude $a(x)$ is a real function, which has a sharp maximum at $x = x_0$, where x_0 is the classical coordinate of the particle; the width of the maximum, δx, is such that $\delta x \ll |x_0|$ and $|p_0|\delta x \gg 1$, such that the wave functions makes many oscillations on the interval δx. A well known example is the Gaussian wave function

$$\psi(x) = \frac{1}{\pi^{1/4}(\delta x)^{1/2}} exp[-\frac{(x-x_0)^2}{2(\delta x)^2} + ip_0 x] \tag{3.49}$$

Then, as follows from Eq. (3.48)

$$\varphi(p) = \frac{(\delta x)^{1/2}}{\pi^{1/4}} exp[-\frac{1}{2}(p-p_0)^2(\delta x)^2 + i(p_0 - p)x_0] \tag{3.50}$$

Eqs. (3.49) and (3.50) can be easily generalized to the three-dimensional case and to the internal wave function of the two-body system. If we assume for simplicity that the uncertainties in all the three relative coordinates are the same and equal δ then a possible generalization of Eq. (3.50) is

$$\varphi(\mathbf{q}) = \frac{\delta^{3/2}}{\pi^{3/4}} exp[-\frac{1}{2}(\mathbf{q}-\mathbf{q}_0)^2\delta^2 + i(\mathbf{q}_0-\mathbf{q})\mathbf{r}_0] \qquad (3.51)$$

where $\mathbf{q}_0$ and $\mathbf{r}_0$ are the classical relative momentum and the classical distance, respectively. Therefore one can conclude the following. The wave function in the relative momentum representation has a very sharp maximum at $\mathbf{q} = \mathbf{q}_0$ and therefore the mean value of any function $f(\mathbf{q})$, which does not rise exponentially at infinity, approximately equals $f(\mathbf{q}_0)$. In particular, the mean value of the nonrelativistic energy operator $H = \mathbf{q}^2/2m_{12}$ approximately equals $H = \mathbf{q}_0^2/2m_{12}$, the mean value of the free relativistic mass operator $\epsilon_1(q) + \epsilon_2(q)$ approximately equals $\epsilon_1(q_0) + \epsilon_2(q_0)$ etc. This is fully in agreement with our intuition in nonrelativistic quantum mechanics.

The above conclusion is essentially based on the fact that in standard theory all the spatial coordinates are in the range $(-\infty, \infty)$. As a consequence, any quasiclassical state contains an infinite number of angular momenta. However, as follows from Eq. (2.1), in the dS space all the spatial coordinates are finite. We also noted in the preceding section that the quantity $n = qR$ has the meaning of the dS relative momentum and, since the internal angular momentum is of order $l = qr$ or less, then $l \ll n$. Therefore one might expect that for the description of quasiclassical states in the dS theory, only a finite number of internal angular momentum states is important. For this reason we first recall how quasiclassical states with a fixed value of the internal angular momentum are treated in standard quantum mechanics.

Consider the internal wave function $\varphi(\mathbf{q}) = f(q)Y_{lm}(\mathbf{q}/q)$ where $q = |\mathbf{q}|$ and Y_{lm} is the spherical function. The normalization integral for this function is

$$||\varphi||^2 = \int_0^\infty q^2|f(q)|^2 dq = 1 \qquad (3.52)$$

The operator A of the relative distance squared acts on $\varphi(\mathbf{q})$ as

$$A\varphi(\mathbf{q}) = -\Delta\varphi(\mathbf{q}) = [-\frac{1}{q^2}\frac{\partial}{\partial q}(q^2\frac{\partial}{\partial q}) + \frac{l(l+1)}{q^2}]f(q)Y_{lm}(\mathbf{q}/q) \quad (3.53)$$

where $\Delta = \partial^2/\partial\mathbf{q}^2$ is the Laplacian. It is well known (and can be easily verified) that with the substitution $f(q) = \chi(q)/q$, the operator A acts as

$$A\chi(q) = [-\frac{d^2}{dq^2} + \frac{l(l+1)}{q^2}]\chi(q) \quad (3.54)$$

This operator is now defined in the Hilbert space of functions with the scalar product

$$(\chi_1, \chi_2) = \int_0^\infty \chi_1(q)^*\chi_2(q)dq \quad (3.55)$$

The term with $l(l+1)$ is called centrifugal one. We first consider the case $l = 0$ and the general case $l \neq 0$ will be discussed at the end of the section.

The physical wave function $\chi(q)$ should be twice differentiable inside the interval $(0, \infty)$ and, in view of Eq. (3.52) and the definition of $\chi(q)$, $\chi(0) = 0$. Let D be a set of such functions. The operator A with the domain D is positive definite, as it should be for the operator representing distance squared. It is well known that any positive definite self-adjoint operator in the Hilbert space has a positive definite self-adjoint square root. One might think that in the case $l = 0$, A is a square of a self-adjoint operator $i\partial/\partial q$, but this is not so. Indeed, let D_0 be a set of differentiable functions $\chi(q)$ such that $\chi(0) = 0$ and D_1 be a set of differentiable functions $\chi(q)$ such that $\chi(0) = 0$ is not required. Let B_0 be the operator $i\partial/\partial q$ with the domain D_0 and B_1 be the operator $i\partial/\partial q$ but with the domain D_1. If $\chi_1 \in D_0$ and $\chi_2 \in D_0$ then $(\chi_1, B_0\chi_2) = (B_0\chi_1, \chi_2)$, i.e. the operator B_0 is Hermitian. However, in infinite-dimensional Hilbert spaces the property of an operator to be Hermitian is necessary but not sufficient to be self-adjoint. If $\chi_1 \in D_1$ and $\chi_2 \in D_0$ then $(\chi_1, B_0\chi_2) = (B_1\chi_1, \chi_2)$. Therefore B_1 is adjoint to B_0 ($B_1 = B_0^*$), B_1 is an extention of B_0 and $B_1 \neq B_0$. Therefore B_0 is not self-adjoint (the operator T is self-adjoint if and only if $T^* = T$).

In particular, there is no guarantee that B_0 or B_1 have a spectral decomposition. The function $\chi_a(q) = exp(-iqa)$ is formally a generalized eigenfunction of B_1 with the eigenvalue a but, since $\chi_a(0) \neq 0$, this function does not satisfy $(\chi_a, B_1\chi_a) = (B_1\chi_1, \chi_2)$. At the same time, the operator A is self-adjoint since $(B\chi_1, \chi_2) = (\chi_1, A\chi_2)$ is possible only if $B = A$. The function $sin(qa)$ is the generalized eigenfunction of A with the eigenvalue a^2. From the physical point of view the fact that A is self-adjoint while B_0 and B_1 are not, is reasonable since B_0 and B_1 are not positive definite while an operator representing the distance or distance squared should be positive definite. In the above discussion of the operators B_0, B_1 and A we followed the book by Kato [19].

In standard quantum mechanics the relation between $\chi(q)$ and the radial wave function $\psi(r)$ is given by

$$\psi(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty \chi(q) sin(qr) dq, \quad \chi(q) = \sqrt{\frac{2}{\pi}} \int_0^\infty \psi(r) sin(qr) dr \tag{3.56}$$

The compatibility of these relations follows from the fact that

$$\int_0^\infty sin(qr) sin(qr') dq = (\pi/2)\delta(r - r')$$

By analogy with Eq. (3.49), consider a radial wave function

$$\psi(r) = const\, exp[-\frac{(r - r_0)^2}{2\delta^2}] sin(q_0 r) \tag{3.57}$$

where *const* can be found from the normalization condition. Here and below in this section the explicit values of constants will not be important for us. As follows from Eq. (3.56), the corresponding wave function in momentum space is given by

$$\chi(q) = const \int_0^\infty exp[-\frac{(r - r_0)^2}{2\delta^2}] sin(q_0 r) sin(qr) dr \tag{3.58}$$

We can represent this expression as

$$\chi(q) = const\, (I_1 - I_2) \tag{3.59}$$

where

$$I_1 = \int_{-\infty}^{\infty} exp[-\frac{(r-r_0)^2}{2\delta^2}]sin(q_0 r)sin(qr)dr$$
$$I_2 = \int_{-\infty}^{0} exp[-\frac{(r-r_0)^2}{2\delta^2}]sin(q_0 r)sin(qr)dr \qquad (3.60)$$

A simple calculation gives

$$I_1 = const\,\{cos[(q-q_0)r_0]exp[-\frac{1}{2}(q-q_0)^2\delta^2] -$$
$$cos[(q+q_0)r_0]exp[-\frac{1}{2}(q+q_0)^2\delta^2]\} \qquad (3.61)$$

Therefore, in contrast with the standard one-dimensional case discussed above (see Eq. (3.50)), the wave function contains not only a contribution depending on $(q-q_0)$ but also one depending on $(q+q_0)$. However, since we require that $q_0\delta \gg 1$, this contribution is exponentially small.

One might think that the term with I_2 in Eq. (3.60) is exponentially small since $r_0 \gg \delta$. Indeed, I_2 will always contain a factor, which is less than $exp(-r_0^2/2\delta^2)$. Nevertheless, we should investigate the dependence of I_2 on q when q is large. Since

$$sin(q_0 r)sin(qr) = -\frac{1}{4}\{exp[i(q+q_0)r] + exp[-i(q+q_0)r] -$$
$$exp[i(q-q_0)r] - exp[i(q_0-q)r]\} \qquad (3.62)$$

we can represent I_2 as a sum of four integrals containing imaginary exponents. In each of them we can integrate those exponents by parts. Integrating twice we get

$$I_2 = const\{\int_{-\infty}^{0} \frac{d^2}{dr^2}(exp[-\frac{(r-r_0)^2}{2\delta^2}])[\frac{cos((q+q_0)r)}{(q+q_0)^2} -$$
$$\frac{cos((q-q_0)r)}{(q-q_0)^2}]dr - exp(-\frac{r_0^2}{2\delta^2})[\frac{1}{(q+q_0)^2} - \frac{1}{(q-q_0)^2}]\} \qquad (3.63)$$

Therefore, when q is large, $I_2 < const\, exp(-r_0^2/2\delta^2)/q^3$. Although the falloff of I_2^2 at large q is not exponential, it is clear that the contribution of I_2 to the mean value of the mass operator is well defined (since $|I_2|^2 < const/q^6$) and small.

We conclude that the quasiclassical description of the state (3.57) is essentially the same as the description of the state (3.49). However, the following question arises: is the wave function (3.57) realistic for description of macroscopic bodies? For example, in the Sun - Earth system we know that the relative wave function in the coordinate space has a very sharp maximum at $r \approx 150 \cdot 10^6 km$. There is no doubt that the probability to find the Earth on the Venus or Mars orbits is extremely small. But is this probability exactly zero? One might think that this question is of academic interest only. However, this question is extremely important and the problem is as follows. If we accept the reduction of the wave function in accordance with the Copenhagen formulation of quantum theory, then after each measurement of the Sun - Earth distance, the function $\psi(r)$ for the Sun - Earth system is not equal to zero only in a very small vicinity of $r \approx 150 \cdot 10^6 km$. In the Copenhagen formulation, the measurement is treated as an interaction with a classical object. The theory of quantum measurements is not well understood yet. So it is not quite clear how often the quantities r and q in the Sun - Earth system are measured. If the reduction of the wave function applies to the coordinate wave function then the corresponding wave function in momentum space contains all possible momenta while if the reduction applies to the momentum wave function then the coordinate wave function contains all possible coordinates. Probably our intuition tells us that the first possibility takes place but at present this hypothesis cannot be proved.

On the other hand, at present the Everett multiworld interpretation of quantum theory is widely discussed. Adherents of this interpretation claim that the reduction postulate contradicts unitarity (see e.g. Ref. [49]). In this approach the reduction of the wave function is not accepted but it is assumed that different outcomes of measurement are observable in different worlds. Does it give any hint on the above problems? In any case, it is interesting to investigate a possibility when the coordinate wave function $\psi(r)$ is not equal to zero only if

$r \in [r_1, r_2]$.

For example, consider a model wave function

$$\psi(r) = \frac{1}{\sqrt{2d}} e(r) exp(iq_0 r) \quad (3.64)$$

where $d = (r_2 - r_1)/2$ and $e(r)$ is the characteristic function of the set $[r_1, r_2]$. If $r_1 > 0$ then the function satisfies the condition $\psi(0) = 0$. Such a function is not differentiable at $r = r_1$ and $r = r_2$ but it can be approximated (in the sense of the norm of the Hilbert space) with any desired accuracy by infinitely differentiable functions with the supporter in $[r_1, r_2]$. For example, as shown in standard textbooks on distributions (e.g. [50]), for any $\epsilon > 0$, $\epsilon < d$, it is possible to construct an infinitely differentiable function $e_\epsilon(r)$ such that $e_\epsilon(r) = 1$ at $r \in [r_1 + \epsilon, r_2 - \epsilon]$, $0 \leq e_\epsilon(r) \leq 1$ if $r \in [r_1, r_1 + \epsilon]$ and $r \in [r_2 - \epsilon, r_2]$ and $e_\epsilon(r) = 0$ if $r \leq r_1$ and $r \geq r_2$. Therefore one might expect that the wave function (3.64) describes a quasicassical state if $r_0 = (r_1 + r_2) \gg d$ and $q_0 d \gg 1$.

As follows from Eq. (3.56), the corresponding wave function in momentum space is

$$\chi(q) = \frac{i}{\sqrt{\pi d}} \{ \frac{exp[i(q + q_0)r_0]}{q + q_0} sin[(q + q_0)d] -$$
$$\frac{exp[i(q_0 - q)r_0]}{q - q_0} sin[(q - q_0)d] \} \quad (3.65)$$

This function is obviously normalized to one since it is a unitary transformation of the function given by Eq. (3.64). At the same time, since

$$\frac{1}{\pi d} \int_{-\infty}^{\infty} \{ \frac{sin[(q - q_0)d]}{q - q_0} \}^2 dq = 1 \quad (3.66)$$

it is clear that for macroscopic bodies the first term in Eq. (3.65) is negligible in comparison with the second one, and the wave function in momentum space is not small only if $|(q - q_0)| \leq 1/d$. Therefore one might think that Eqs. (3.64) and (3.65) give a good quasiclassical description. On the other hand, it is obvious that, since at large q, $|\chi(q)|^2 = O(1/q^2)$, the mean value of the relativistic mass operator

$\sqrt{m_1^2+q^2}+\sqrt{m_2^2+q^2}$ is divergent, to say nothing about the mean value of the nonrelativistic mass operator $q^2/2m_{12}$.

If $e(r)$ is replaced by $e_\epsilon(r)$ then the mean value of the mass operator becomes finite. This is clear from the fact that the mean value of $(q-q_0)^2$ over the Fourier transform is proportional to $\int_{r_1}^{r_2}[de_\epsilon(r)/dr]^2dr$. However, as follows from the Lebesgue theorem, if $\epsilon \to 0$, the mean value goes to infinity. Therefore, ϵ cannot be very small and we conclude that in standard quantum theory the wave function (3.64) is not realistic. One might investigate orther wave functions. For example, by analogy with a function discussed in textbooks on distributions, instead of $e_\epsilon(r)$ one might consider a function, which has a supporter $[r_1, r_2]$ and is proportional to $exp[-d^2/(d^2-(r-r_0)^2)]$ at $r \in [r_1, r_2]$. In any case, although quasiclassical approximation has been discussed in numerous works, we are not aware of examples where it has been explicitly shown that a wave function with the supporter $[r_1, r_2]$ such that $b \ll r_0$ is such that the uncertainty of momentum is much less than q_0 and the uncertainty of the kinetic energy is much less than $q_0^2/2m_{12}$. As it is clear from the example with the state (3.64), our intuition is not sufficient to make a conclusion without explicit calculations. An interesting pedagogical example has been proposed by Alik Makarov. Let E_n be the nth state of the harmonic oscilator ($n = 0, 1, ...$), $\omega(n)$ is the energy of the nth state and α is a number such that $1/2 < \alpha < 1$. Then the state $\sum_n E_n/\omega(n)^\alpha$ is normalizable but the mean value of the energy is infinite. We will see in Chap. 5 that in the dS theory an analog of (3.64) is a state with a finite energy.

Consider now the case $l \neq 0$. Let a^2 be an eigenvalue of the operator defined by Eq. (3.54). The corresponding eigenfunction is a solution of the equation

$$[-\frac{d^2}{dq^2}+\frac{l(l+1)}{q^2}]\chi(q) = a^2\chi(q) \quad (3.67)$$

It is well known (see e.g. Ref. [45]) that the solution regular at $q = 0$ can be written as

$$\chi_a(q) = \sqrt{aq}J_{l+1/2}(aq) \quad (3.68)$$

where $a > 0$ and $J_{l+1/2}$ is the Bessel function. Since we are interested in the behavior of quasiclassical wave functions at large momenta, it suffices to consider the solution at $aq \gg l$. The asymptotics of the Bessel function $J_{l+1/2}(z)$ at $z \gg l$ is given by [47]

$$J_{l+1/2}(z) = \sqrt{\frac{2}{\pi z}} sin(z - \frac{l\pi}{2}) \tag{3.69}$$

and therefore $\chi_a(q) \approx \sqrt{2/\pi} sin(aq - l\pi/2)$ at $aq \gg l$. This function depends on l only in the phase of the sine function while it is the eigenfunction of the operator $-d^2/dq^2$ with the eigenvalue a^2.

The fact that the centrifugal term is not important at $aq \gg l$ can also be understood as follows [45].The solution of Eq. (3.54) can be written as

$$\chi_a(q) = (-1)^l \frac{q^{l+1}}{a^l} (\frac{1}{q}\frac{d}{dq})^l (\frac{sin(aq)}{q}) \tag{3.70}$$

When $aq \gg l$, the main contribution arises from the differentiation of the sine and we again obtain a function proportional to $sin(aq - l\pi/2)$. This example shows that each differentiation gives a factor a. Therefore $-d^2/dq^2$ gives a factor a^2. This is much greater than the centrifugal term $l(l+1)/q^2$ if $qa \gg l$. It is also clear that if $aq \gg l$ then the centrifugal term in Eq. (3.67) is much smaller than the term with a^2, therefore at this condition the centrifugal term can be neglected.

In summary, if $l \neq 0$, we can draw the same conclusions about the behavior of quasiclassical wave functions at large momenta as in the case $l = 0$. In particular, the state (3.64) is not legitimate. It will be shown in Chap. 5 that in the dS theory the behavior of eigenstates of the relative distance operator at large momenta strongly depends on l: the greater is l, the faster is the falloff of the wave function. The behavior of the eigenstates at small momenta is also considerably different since there exist not only solutions with $J_{l+1/2}$ but also ones with $J_{-(l+1/2)}$.

The above discussion sheds some light on a possible form of a quasiclasical wave function in the dS theory. Since the relative distance operator in this theory has not been discussed yet, we cannot guarantee that r_0 is the relative distance. Suppose however that this is the case. As noted in the preceding section, the analog of relative momentum in the dS theory is a quantity n such that $q = n/R$. Therefore, by analogy with Eqs. (3.61)) and (3.65), one might expect that a quasiclassical wave function should contain a rapidly oscillating term $exp[i(n-n_0)r_0/R]$ or $cos[(n-n_0)r_0/R]$. Suppose, for example, that the quantities c_n in Eq. (3.45) have the form $c_n = a(n)cos[(n-n_0)r_0/R]$, where the amplitude $a(n)$ has a sharp maximum at $n = n_0$. In the spirit of quasiclassical approximation, we assume that when n changes by one, it suffices to consider only the change in the rapidly changing argument of $cos[(n-n_0)r_0/R]$. Then

$$c_{n+1}c_n* \approx |a(n)|^2\{(1 - r_0^2/2R^2)cos^2[(n-n_0)r_0/R] - (r_0/2R)sin[2(n-n_0)r_0/R]\} \qquad (3.71)$$

Since $sin[2(n-n_0)r_0/R]$ is a rapidly oscillating function, we assume that the contribution of the last term to the mean value in Eq. (3.45) is negligible. Therefore, if we take into account the correction of order r_0^2/R^2, then instead of Eq. (3.46), we get

$$(\Psi, M^2\Psi) = [\epsilon_1(q_0) + \epsilon_2(q_0)]^2 - \frac{r_0^2}{R^2}\epsilon_1(q_0)\epsilon_2(q_0) \qquad (3.72)$$

The same result can be obtained if $c_n = a(n)exp[i(n-n_0)r_0/R]$. Therefore, the correction of order r_0^2/R^2 to the mean value of the nonrelativistic mass operator is $-m_{12}r_0^2/2R^2$ in agreement with Eq. (2.26).

The results (3.46) and (3.72) give an additional argument that the quantum number n has the meaning of the dS relative momentum. At the same time, quasiclassical approximation implies that not only the mean value of the mass operator is in agreement with our expectations, but the wave function in the mass representation has a sharp maximum around the mean value. This problem is studied in the next chapter.

Chapter 4

Momentum - mass uncertainty relation

4.1 Spectral decomposition of the mass operator

In standard theory the free mass operator commutes with the momentum operator. Therefore these operators have a common set of eigenvectors. However, in dS theory the free mass operator does not commute with the momentum operator. Therefore if a quasiclassical wave function has a sharp maximum in the n-representations, this does not guarantee yet that it has a sharp maximum in the mass representation. Since the operators do not commute, there exists a momentum - mass uncertainty, i.e. the momentum and the mass cannot be measured simultaneously.

The decomposition of the eigenvector of the operator W with the eigenvalue λ over the eigenvectors of the dS momentum operator is given by Eq. (3.33), where in the standard dS theory based on complex numbers, the limit $N \to \infty$ should be taken. However, if for some reasons, only momenta $n = 0, 1, ...N$ are allowed, then, as follows from Eqs. (3.28) and (3.29), $\chi(\lambda, N)$ will be the eigenvector of W with the eigenvalue λ if $\Delta^N(\lambda) = 0$. Let $e_n = \Phi_n/||\Phi_n||$ be the normalized eigenvector of the momentum operator with the momentum n. Then, as follows from Eqs. (3.24), (3.32), (3.35) and (3.36)

$$\chi(\lambda, N) = \frac{(1+s)!}{(1+j)}\{\prod_{l=1}^{j}[w_1 + (2l+1)^2][w_1 + (2l+1)^2]\}^{1/2}$$
$$\frac{\Gamma((s+3-i\mu_1)/2)\Gamma((s+3-i\mu_2)/2)}{\Gamma((s+3+i\sigma)/4)\Gamma((s+3-i(2\mu_1+2\mu_2+\sigma))/4)\Gamma((s+3)/2)}$$
$$\sum_{n=0}^{N} e_n\{\prod_{l=0}^{n-1}\frac{[\mu_1+i(2l+s+3)][\mu_2+i(2l+s+3)]}{[w_1+(2l+s+3)^2]^{1/2}[w_2+(2l+s+3)^2]^{1/2}}\}$$
$$(\frac{n+j+1}{C_{n+s+1}^{1+s}})^{1/2}\{\int_0^1 (1-z)^{(s-1+i\sigma)/4} z^{(s+1)/2} C_n^{1+j}(2z-1)$$
$$F(\frac{s+3+i(2\mu_1+\sigma)}{4}, \frac{s+3+i(2\mu_2+\sigma)}{4}; \frac{s+3}{2}; z)dz\} \tag{4.1}$$

As noted in Sect. 2.4 and in the preceding chapter, the de Sitter masses of particles and the quantity n are typically very large. Therefore, the gamma functions depending on the de Sitter masses and kinetic energy σ can be calculated by using the Stirling formula and for the Gegenbauer polynomial we can take its asymptotic expression [47]

$$C_n^a(cos\theta) = 2\frac{(a)_n}{n!}\frac{cos[(n+a)\theta - a\pi/2]}{(2sin\theta)^a} \tag{4.2}$$

Our next task is to obtain the expression for the hypergeometric function in Eq. (4.1) using the fact that the de Sitter masses are very large. This function can be written as $F(a, b; c; z)$ where

$$a = \frac{s+3}{4} + i\alpha,\ b = \frac{s+3}{4} + i\beta,\ c = \frac{s+3}{2},\ \alpha = \frac{2\mu_1+\sigma}{4},\ \beta = \frac{2\mu_2+\sigma}{4} \tag{4.3}$$

We use the expression [47]

$$F(a, b; c; z) = \Gamma(c)[\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a, b; a+b+1-c; 1-z) +$$
$$\frac{\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}F(c-a, c-b; c+1-a-b; 1-z)] \tag{4.4}$$

Then by using the expression [47]

$$F(a,b;a+b+1-c;1-z) = \frac{\Gamma(a+b+1-c)}{\Gamma(b)\Gamma(a+1-c)} \cdot$$
$$\int_0^\infty t^{b-1}(1+t)^{c-b-1}(1+tz)^{-a}dt \tag{4.5}$$

Eqs. (4.3), (4.4) and the Stirling formula for the Gamma function, we have

$$F(a,b;c;z) = \frac{\Gamma(c)}{\pi|\beta|^{(s+1)/2}} exp[-\frac{i}{2}(\alpha+\beta)ln(1-z)]$$
$$Re\{exp[\frac{i}{2}(\alpha+\beta)ln(1-z)]\int_0^\infty \frac{[t(1+t)]^{(s-1)/4}}{(1+tz)^{(s+3)/4}} \cdot$$
$$exp[i\beta lnt - i\beta ln(1+t) - i\alpha ln(1+tz)]dt\} \tag{4.6}$$

Since the quantities α and β are very large, we can calculate the last integral by using the stationary phase method. The stationary point t should satisfy the equation

$$\beta(1+tz) = \alpha z t(1+t) \tag{4.7}$$

and the only possible solution is

$$t = \frac{1}{2\alpha}\{\beta - \alpha + [(\alpha-\beta)^2 + 4\alpha\beta/z]^{1/2}\} \tag{4.8}$$

As a consequence, the result for $F(a,b;c;z)$ can be described as follows. Let $\phi(\alpha,\beta,z)$ be a function such that

$$\frac{d\phi(\alpha,\beta,z)}{dz} = -\frac{1}{2(1-z)}[(\alpha-\beta)^2 + 4\alpha\beta/z]^{1/2}, \quad \phi(\alpha,\beta,0) = -\pi/4 \tag{4.9}$$

Then

$$F(a,b;c;z) = \sqrt{\frac{2}{\pi}}\Gamma(c)\frac{exp[-\frac{i}{2}(\alpha+\beta)ln(1-z)]cos[\phi(\alpha,\beta,z)]}{(\alpha\beta)^{(s+1)/4}z^{(s+2)/4}[4\alpha\beta + z(\alpha-\beta)^2]^{1/4}} \tag{4.10}$$

As follows from Eqs. (4.2) and (4.10), for calculating Eq. (4.1), we need to consider the integral

$$I = \int_0^1 cos[(n+j+1)\theta - \frac{\pi}{2}(j+1)]\frac{cos[\phi(\alpha,\beta,z)]}{[\lambda-(\mu_1-\mu_2)^2(1-z)]^{1/4}}$$
$$exp[-\frac{i}{4}(\mu_1+\mu_2)ln(1-z)]\frac{dz}{z^{1/2}(1-z)^{3/4}} \qquad (4.11)$$

where $cos\theta = 2z-1$. The product of two cosine functions in the integrand can be written as a sum of four imaginary exponents and then for calculating the integral we can apply the stationary phase method. Let us note that in Sect. 3.2 we represented λ as $(\mu_1+\mu_2+\sigma)^2$ only for convenience of calculations. We assumed only that $\lambda > 0$ and therefore we can write λ as $\lambda = \xi^2$ where $\xi > 0$. As follows from Eqs. (3.24), (3.28) and (3.33), the final result for the decomposition of $\chi(\lambda, N)$ should not depend on (μ_1, μ_2, σ) but on (w_1, w_2, λ) only. Therefore for performing calculations in Eq. (4.1) we can assume for definiteness that $\mu_1 > 0$, $\mu_2 > 0$, $\alpha > 0$ and $\beta > 0$. Then each contribution to the integral can be considered in four cases: $(\mu_1+\mu_2 > \xi,\ n^2 > \alpha\beta)$, $(\mu_1+\mu_2 < \xi,\ n^2 > \alpha\beta)$, $(\mu_1+\mu_2 > \xi,\ n^2 < \alpha\beta)$ and $(\mu_1+\mu_2 < \xi,\ n^2 < \alpha\beta)$.

The final result is as follows. Let

$$D = D(n,\lambda) = \{n^2\lambda - \frac{1}{16}[\lambda^2 - 2\lambda(w_1+w_2) + (w_1-w_2)^2]\}^{1/2}$$
$$tg\gamma(n,\lambda) = \frac{4D}{\lambda - w_1 - w_2 - 8n^2},\ 0 < \gamma(n,\lambda) < \pi \qquad (4.12)$$

and θ_0 is a constant, which does not depend on n and λ. As follows from the expression for $D(n,\lambda)$, for any fixed value of λ, n can be only in the range $[n_{min}, N]$ where

$$n_{min} = n_{min}(\lambda) = \frac{1}{4\xi}[\lambda^2 - 2\lambda(w_1+w_2) + (w_1-w_2)^2]^{1/2} \qquad (4.13)$$

Then

$$\chi(\lambda,N) = C(\lambda)\sum_{n=n_{min}}^{N}\frac{1}{D^{1/2}}cos[arg(n,\lambda)]e_n \qquad (4.14)$$

where $C(\lambda)$ can be found from the normalization condition (see below) and

$$arg(n,\lambda) = \frac{\xi}{4}ln|\frac{n\xi+D}{n\xi-D}| + \frac{\mu_1}{4}ln|\frac{n(\lambda+w_1-w_2)-2\mu_1 D}{n(\lambda+w_1-w_2)+2\mu_1 D}|$$
$$+\frac{\mu_2}{4}ln|\frac{n(\lambda+w_2-w_1)-2\mu_2 D}{n(\lambda+w_2-w_1)+2\mu_2 D}| - n\gamma + \theta_0 \qquad (4.15)$$

The function $arg(n,\lambda)$ indeed depends only on (w_1, w_2, λ) since it is invariant under the replacements $\mu_1 \to -\mu_1$, $\mu_2 \to -\mu_2$, $\xi \to -\xi$.

As follows from these expressions, the asymptotics of $arg(n,\lambda)$ at large values of n and $\xi > 0$ is

$$arg(n,\lambda) \approx \frac{\xi}{2}[ln(n\xi)+1+3ln2] - \frac{1}{4}(\xi+\mu_1+\mu_2)ln|\xi+\mu_1+\mu_2| -$$
$$\frac{1}{4}(\xi-\mu_1-\mu_2)ln|\xi-\mu_1-\mu_2| - \frac{1}{4}(\xi+\mu_1-\mu_2)ln|\xi+\mu_1-\mu_2| -$$
$$\frac{1}{4}(\xi-\mu_1+\mu_2)ln|\xi-\mu_1+\mu_2| + \theta_0 \qquad (4.16)$$

This result is obviously invariant under the replacements $\mu_1 \to -\mu_1$, $\mu_2 \to -\mu_2$. As noted above, if N is finite then the condition $cos[arg(N+1,\lambda)] = 0$ should be satisfied. Therefore $arg(N+1,\lambda) = (k+1/2)\pi$ $(k = \pm 1, \pm 2, ...)$. If N is finite then the spectrum of the operator W should be obviously discrete and, as follows from Eq. (4.16), the distance $\Delta\xi$ between neighboring eigenvalues is such that

$$\frac{\Delta\xi}{2}ln\frac{8N\xi}{|(\xi+\mu_1+\mu_2)(\xi-\mu_1-\mu_2)(\xi+\mu_1-\mu_2)(\xi-\mu_1+\mu_2)|^{1/2}} \approx \pi \qquad (4.17)$$

If $N \to \infty$ then $\Delta\xi \to 0$ as it should be since, as shown in Sect. 3.2, in the standard dS theory the spectrum of the operator W is continuous. Note that the spectrum is not equidistant and therefore $\Delta\xi$ is a function of ξ.

If N is finite then the function $\chi(\lambda, N)$ can be normalized to one, i.e.

$$C(\lambda)^2 \sum_{n=n_{min}}^{N} \frac{1+cos[2arg(n,\lambda)]}{D(n,\lambda)} = 1 \qquad (4.18)$$

Suppose again that with a good accuracy we can replace summation by integration over n. As follows from Eqs. (4.12), (4.13) and (4.17)

$$\int_{n_{min}}^{N} \frac{dn}{D(n,\lambda})) = \frac{2\pi}{\Delta\lambda} \tag{4.19}$$

where $\Delta\lambda = 2\xi\Delta\xi$. Consider now the contribution of $cos[2arg(n,\lambda)]$ to the integral in Eq. (4.18). If $arg(n,\lambda)$ is formally treated as a function of continuous variables then a direct calculation gives

$$\frac{\partial arg(n,\lambda)}{\partial n} = -\gamma(n,\lambda), \quad \frac{\partial arg(n,\lambda)}{\partial \xi} = \frac{1}{4}ln|\frac{n\xi + D}{n\xi - D}| \tag{4.20}$$

As follows from the first expression in these formulas and from Eq. (4.12), $cos[2arg(n,\lambda)]$ is a rapidly oscillating function in the integration range and therefore its contribution is negligible.

We conclude that, as follows from Eq. (4.14),

$$\chi(\lambda,N) = \sqrt{\frac{\Delta\lambda}{2\pi}} \sum_{n=n_{min}}^{N} \frac{1}{D^{1/2}} cos[arg(n,\lambda)]e_n$$
$$e_n = \sum_{\lambda=\lambda_{min}}^{\lambda_{max}} \sqrt{\frac{\Delta\lambda}{2\pi}} \frac{1}{D^{1/2}} cos[arg(n,\lambda)]\chi(\lambda,N) \tag{4.21}$$

where $\lambda_{min} = \lambda_{min}(n)$ and $\lambda_{max} = \lambda_{max}(n)$ are respectively the minimum and maximum values of λ at a fixed value of n. Note that since $\Delta\lambda$ is a function of λ, we cannot take $\sqrt{\Delta\lambda/2\pi}$ out of the summation sign in the second expression.

To show that Eq. (4.21) gives a one-to-one relation between the elements $\{e_n\}$ and $\{\chi(\lambda,N)\}$, we have to prove that the basis $\{\chi(\lambda,N)\}$ is complete, i.e. the norm of the second expression in Eq. (4.21) equals one. We denote

$$\omega_1(n) = \sqrt{w_1 + 4n^2}, \quad \omega_2(n) = \sqrt{w_2 + 4n^2} \tag{4.22}$$

It is clear from the consideration in Sect. 2.4 that these quantities are the standard particle energies in the Poincare invariant theory multiplied by $2R$. Then it follows from the expression for D in Eq. (4.12) that

$$\lambda_{min}(n) = |\omega_1(n) - \omega_2(n)|^2, \quad \lambda_{max}(n) = |\omega_1(n) + \omega_2(n)|^2$$
$$D(n,\lambda) = \frac{1}{4}[(\lambda - \lambda_{min}(n))(\lambda_{max}(n) - \lambda)]^{1/2} \qquad (4.23)$$

Note that when n increases, $\lambda_{min}(n)$ decreases and $\lambda_{max}(n)$ increases. If $N \gg \mu_1, \mu_2$ then the absolute minimum of λ_{min} is approximately zero and the absolute maximum of λ_{min} approximately equals $4N^2$.

In Poincare invariant theory the mass of the free two-body system equals a sum of the particle energies in the c.m. frame. For this reason one might expect that in the de Sitter invariant theory the operator W has a spectrum in a small vicinity of $|\omega_1(n) + \omega_2(n)|^2$. As shown in Chap. 2, the de Sitter antigravity reduces the value of the mass but the reduction is small since $r \ll R$. An important consequence of Eq. (4.23) is that $|\omega_1(n) + \omega_2(n)|^2$ *is the maximum value of* W while in general the values of λ considerably less than $|\omega_1(n) + \omega_2(n)|^2$ are possible. For example, if we define the mass operator as a square root from the positive definite operator $W/4R^2$ then *there is no law prohibiting a free nonrelativistic two-body system to be in a state where the mean value of the mass operator is* $\epsilon_1(p_0) + \epsilon_2(p_0) - Gm_1m_2/r_0$ *where* p_0 *is the mean value of the momentum in the c.m. frame and* r_0 *is the mean value of the distance between the particles.* The problem arises whether such a state can be quasiclassical and whether there are reasons making such a value of the mass operator more preferable than the others.

We are now in position to prove that the second expression in Eq. (4.21) is indeed compatible with the requirement $||e_n|| = 1$. As follows from this expression and Eq. (4.23),

$$||e_n||^2 = \frac{1}{\pi} \sum_{\lambda_{min}}^{\lambda_{max}} \frac{\{1 + cos[2arg(n,\lambda)]\}\Delta\lambda}{[(\lambda - \lambda_{min})(\lambda_{max} - \lambda)]^{1/2}} \qquad (4.24)$$

As follows from the second expression in Eq. (4.20), $cos[2arg(n,\lambda)]$ is a rapidly oscillating function in the summation region and therefore its contribution is negligible. With a good accuracy we can replace summation by integration and, as a result

$$||e_n||^2 = \frac{1}{\pi}\int_{\lambda_{min}}^{\lambda_{max}} \frac{d\lambda}{[(\lambda-\lambda_{min})(\lambda_{max}-\lambda)]^{1/2}} = 1 \qquad (4.25)$$

This completes the proof that there exists a one-to-one relation between the elements $\{e_n\}$ and $\{\chi(\lambda, N)\}$ and this relation is described by Eq. (4.21).

4.2 Quasiclassical wave functions in momentum and mass representations

As noted in Sect. 3.4, one might expect that the coefficients c_n representing a quasiclassical wave function in the n-representation, have the form $c_n = a(n)cos[(n-n_0)r_0/R]$ or $c_n = a(n)exp[-i(n-n_0)r_0/R]$, where the amplitude $a(n)$ has a sharp maximum at $n = n_0$. If the wave function can be written as $\sum_\lambda b(\lambda)\chi(\lambda, N)$ then the coefficients $b(\lambda)$ represent the wave function in the λ-representation. From standard quasiclassical experience one might expect that in quasiclassical approximation the function $b(\lambda)$ should have a sharp maximum at some λ_0 close to $|\omega_1(n_0)+\omega_2(n_0)|^2$. Let us investigate whether this is the case.

For simplicity we assume that $c_n = a(n)exp[-i(n-n_0)r_0/R]$. Then, as follows from Eq. (4.21)

$$b(\lambda) = \sqrt{\frac{\Delta\lambda}{2\pi}}\sum_{n_{min}}^{N} \frac{a(n)}{D(n,\lambda)^{1/2}} exp[-i(n-n_0)r_0/R]cos[arg(n,\lambda)] \quad (4.26)$$

We write *cos* as a sum of two imaginary exponents and assume that with a good accuracy summation can be replaced by an integration (see the discussion in the end of this section). Then

$$b(\lambda) = \sqrt{\frac{\Delta\lambda}{8\pi}} exp(in_0r_0/R)\int_{n_{min}}^{N} \frac{a(n)}{D(n,\lambda)^{1/2}}$$
$$exp\{i[arg(n,\lambda) - \frac{nr_0}{R}]\} + exp\{-i[arg(n,\lambda) + \frac{nr_0}{R}]\}dn \quad (4.27)$$

Since the exponents are rapidly oscillating, we can apply the stationary phase method. As follows from Eq. (4.20), the stationary point exists only in the second term and is defined by the condition

$$\gamma(n,\lambda) = \frac{r_0}{R} \tag{4.28}$$

Therefore the value of $\gamma(n,\lambda)$ in the stationary point is small but not zero. As follows from Eqs. (4.12) and (4.13), the standard relativistic relation $\lambda = |\omega_1(n)+\omega_2(n)|^2$ would imply $D=0$ and $n=n_{min}$ but this is impossible since the stationary point should necessarily be inside the integration region.

Since $\gamma(n,\lambda)$ is small, we can replace $tg\gamma$ by γ in Eq. (4.12). Then, as follows from Eqs. (4.12) and (4.23), the condition (4.28) can be written as

$$4D(n,\lambda) = [(\lambda-\lambda_{min})(\lambda_{max}-\lambda)]^{1/2} = \frac{r_0}{2R}(2\lambda-\lambda_{min}-\lambda_{max}) \tag{4.29}$$

Since $r_0 \ll R$, it is clear from this expression that only a solution with λ very close to λ_{max} is possible. The solution is given by

$$\lambda = \lambda_{max} - \frac{r_0^2}{R^2}\omega_1(n)\omega_2(n) = |\omega_1(n)+\omega_2(n)|^2 - \frac{r_0^2}{R^2}\omega_1(n)\omega_2(n) \tag{4.30}$$

This expression defines the position of the stationary point $n = n(\lambda)$ as a function of λ. It is easy to verify that in the nonrelativistic approximation Eq. (2.26) follows from Eq. (4.30).

For calculating integral in Eq. (4.27) in the stationary phase method, we need to know the second derivative of the phase in the stationary point. A direct calculation gives

$$-\frac{\partial^2 \arg(n,\lambda)}{\partial n^2} = \frac{\partial\gamma(n,\lambda)}{\partial n} = \frac{n[\lambda(\omega_1(n)^2+\omega_2(n)^2)-(w_1-w_2)^2]}{D(n,\lambda)\omega_1(n)^2\omega_2(n)^2} \tag{4.31}$$

As follows from Eq. (4.29), the quantity D in the stationary point is proportional to r_0/R. Therefore when λ satisfies Eq. (4.30) and $r_0 \ll R$, this condition can be written as

$$\frac{\partial\gamma(n,\lambda)}{\partial n} = \frac{8n\lambda_{max}}{D(n,\lambda)(\lambda_{max}-\lambda_{min})} = \frac{64nR\lambda_{max}}{r_0(\lambda_{max}-\lambda_{min})^2} \tag{4.32}$$

As a consequence

$$b(\lambda) = [\frac{\omega_1(n)\omega_2(n)\Delta\lambda}{8n\lambda_{max}(n)}]^{1/2}a(n)$$
$$exp[-i(arg(n,\lambda) + \frac{(n-n_0)r_0}{R} - \frac{\pi}{4})] \qquad (4.33)$$

where n should be understood as $n(\lambda)$. A check for consistency of this result is to verify that $b(\lambda)$ is normalized to one. We again replace a sum over λ by integration and note that, as follows from Eq. (4.30), in the approximation $r_0 \ll R$, $d\lambda = 8n\lambda_{max}(n)dn/[\omega_1(n)\omega_2(n)]$. Then, as follows from Eq. (4.33)

$$\int_0^{4N^2} |b(\lambda)|^2 d\lambda = \int_0^N |a(n)|^2 dn = 1 \qquad (4.34)$$

as it should be.

Consider now the following question. We used the method of stationary phase assuming that summation over n in Eq. (4.26) can be approximated by integration over n. A justification of such an approximation is not obvious. For example one might expect that integration could be a good approximation for summation if $arg(n,\lambda)$ does not change significantly when n changes by one. In other words, we should have $|\partial arg(n,\lambda)/\partial n| \ll 1$. At the same time, as follows from Eqs. (4.12) and (4.20), this is not the case for all values of n. However, if the stationary phase method can be used then the main contribution is given by a small vicinity of the stationary point where $|\gamma(n,\lambda)| \ll 1$ and normalization is conserved. Therefore we need to justify the usage of the stationary point method.

As already noted, the stationary point cannot be at the value of $n = n_{min}(\lambda)$ defined from the standard relativistic relation between the energy and momentum. However, if $r_0 \ll R$, the stationary point $n(\lambda)$ is close to that value. Let us estimate the difference $\Delta n = n(\lambda) - n_{min}(\lambda)$. We denote $n_r = n_{min}(\lambda)$ where the subscript r indicates that this quantity is obtained from the standard relativistic expression $\lambda =$

$|\omega_1(n_r) + \omega_2(n_r)|^2$. Then it follows from Eq. (4.30) that

$$\Delta n \approx \frac{[r_0\omega_1(n_r)\omega_2(n_r)]^2}{8n_r[R\omega_1(n_r)\omega_2(n_r)]^2} \qquad (4.35)$$

In the stationary point method the main contribution to the integral is given by a small vicinity of the stationary point defined by the second derivative of the exponent index. Let δn be the width of this vicinity. As follows from Eq. (4.31), the value of δn can be estimated as

$$\delta n \approx \frac{\omega_1(n_r)\omega_2(n_r)}{2[\omega_1(n_r) + \omega_2(n_r)]}\sqrt{\frac{r_0}{n_r R}} \qquad (4.36)$$

Then the stationary phase method can be used if the following conditions are satisfied: $\delta n \gg 1$, such that the vicinity of the stationary point giving the main contribution contains many values of n; $\delta n \ll \Delta n$ such that this vicinity is fully inside integration region.

For simplicity we consider the nonrelativistic approximation. Then $\omega_1 \approx 2Rm_1$, $\omega_2 \approx 2Rm_2$ and $n_r \approx Rm_{12}v$ where m_{12} is the reduced mass and v is the relative velocity of particles 1 and 2. We now can rewrite Eqs. (4.35) and (4.36) as follows

$$\Delta n \approx \frac{m_{12}r_0^2}{2Rv}, \quad \delta n = \frac{1}{2}\sqrt{\frac{m_{12}r_0}{v}} \qquad (4.37)$$

It is obvious that the conditions $\delta n \gg 1$ is always valid for macroscopic bodies since $v \ll 1$ and r_0 is much greater than their Compton wave lengths. The condition $\Delta n \gg \delta n$ is valid if

$$\frac{m_{12}r_0}{v}(\frac{r_0}{R})^2 \gg 1 \qquad (4.38)$$

One might conclude that this result is incorrect since in the Poincare limit $R \to \infty$ the stationary phase approximation should be undoubtedly valid for macroscopic particles. However, as noted in Sect. 3.2, we are working in the representation where there is no direct transition to the Poincare invariant theory and we argued that this representation is more physical than that described by Eqs. (2.9) and (2.10).

It is clear that the greater the values of m_{12} and r_0 are the higher is the accuracy of Eq. (4.38). Consider the Cavendish experiment where $m_1 = 348 pounds$, $m_2 = 1.61 pounds$ and $r_0 = 9 inches$. In this case $m_{12}r_0 = 4.8 \cdot 10^{41}$. The maximum acceleration of the small ball is $Gm_1/r_0^2 = 2.0 \cdot 10^{-7} m/s^{-2}$. Suppose that equilibrium of the torsion balance apparatus is reached within a time of order $1s$. Then v is of order $10^{-7} m/s$. Therefore in units where $c = 1$, $v = 6.7 \cdot 10^{-14}$ and $m_{12}r_0/v = 7.2 \cdot 10^{56}$. If, for example, $R = 10^{26} m$, then the l.h.s. of Eq. (4.38) is $3.8 \cdot 10^3$. In the literature it is discussed at what conditions deviations from the Newton law of gravitation could be expected. The above example shows that at the conditions of the Cavendish experiment the l.h.s. of Eq. (4.38) is not anomalously large. Breaking of Eq. (4.38) would mean that the wave function cannot be simultaneously quasiclassical in both momentum and mass representations, i.e. the state becomes essentially quantum. It is clear that this happens much earlier than one could expect from the standard quantum mechanical experience. In this case a measurement of the gravitational constant might give a value considerably different from the expected one.

Another consequence of Eq. (4.38) is as follows. If the requirement of the existence of quasiclassical states is such that the wave function should have a sharp maximum in momentum and mass representations simultaneously then the cosmological constant Λ, which is of order $1/R^2$, cannot be anomalously small. As noted in Chap. 1, at present the conclusion that $\Lambda \neq 0$ is based on astronomical data. On the other hand, Eq. (4.38) gives a restriction from the opposite side. If the wave function is simultaneously quasiclassical in momentum and mass representations then the values of R much greater than $10^{27} m$ are highly unlikely. As a consequence, the term "Poincare limit" should be understood not as a formal limit $R \to \infty$ but only as a situation when the distances in question are much smaller than R but R cannot be infinite.

One might treat the results of this chapter as a proof that standard quantum mechanical intuition works in the dS theory as well. Indeed, we have shown that not only the mean value of the dS mass operator is in agreement with the Poincare limit and dS antigravity but it is possible to construct wave functions which have a sharp maximum simultaneously in the momentum and mass representation. Our construction was based on the assumption that the r dependence of the quasiclassical wave function in momentum representation contains $exp[-i(n-n_0)r/R]$ in analogy with the nonrelativistic quantum mechanics. On the other hand, we have noted that even in relativistic theory there exists a serious problem with the position operator and, to the best of our knowledge, the problem of the position operator in the dS theory has not been discussed at all. Therefore the above assumption can be substantiated only if an operator having the meaning of the spatial coordinate is constructed in the framework of the dS theory.

Chapter 5

Relative distance operator in de Sitter invariant theory

5.1 Construction of relative distance operator

In Chaps. 1 and 2 we discussed problems in constructing a position operator in dS theory. Strictly speaking, a measurable quantity is not a position by itself but a relative position with respect to some other particle. In nonrelativistic quantum mechanics the position operators for particles 1 and 2 in momentum representation are defined as $\mathbf{r}_1 = i\partial/\partial\mathbf{p}_1$ and $\mathbf{r}_2 = i\partial/\partial\mathbf{p}_2$, respectively, and the relative position operator is defined as $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

In the representation (2.9) the form of the Lorentz boost operator is the same as in relativistic theory. For spinless particles the Lorentz boost operators in momentum representation can be written as

$$\mathbf{N}_r^{(1)} == -2i\epsilon_1(\mathbf{p}_1)\frac{\partial}{\partial\mathbf{p}_1} \quad \mathbf{N}_r^{(2)} == -2i\epsilon_2(\mathbf{p}_2)\frac{\partial}{\partial\mathbf{p}_2} \tag{5.1}$$

where the subscript r stands for "relativistic". Therefore, as already noted, the Lorentz boost operator is proportional to the standard position operator only in nonrelativistic approximation. Since the representation operators (2.9) are the only operators in our disposal, we conclude that in dS theory there are no physical operators proportional to $i\partial/\partial\mathbf{p}_1$ or $i\partial/\partial\mathbf{p}_2$. In relativistic theory this is a well known problem discussed in a wide literature. The most known solution is the Newton

- Wigner operator [8], but it is recognized that even this operator does not have all the required properties for the position operator.

The idea of the Newton - Wigner and other constructions is to have an operator, which satisfies some minimum requirements; in particular, this operator should be self-adjoint and proportional to the nonrelativistic position operator in the nonrelativistic limit. For example, one could define the relative position operator in relativistic theory as follows. Since the energy operator $E_r^{(1)}$ for particle 1 is the operator of multiplication by $\epsilon_1(\mathbf{p}_1)$ and analogously for $E_r^{(2)}$, a possible definition is

$$\mathbf{D}_r = E_r^{(1)}\mathbf{N}_r^{(2)} - E_r^{(2)}\mathbf{N}_r^{(1)} \tag{5.2}$$

This operator is indeed self-adjoint and proportional to the relative position operator in the nonrelativistic limit. It is obvious that $\mathbf{D}_r$ does not have the dimension of length. Since it should be treated as an operator related to the internal state of the two-body system, it should commute with the total momentum operator $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ and this is indeed the case.

Note that different components of $\mathbf{D}_r$ do not commute with each other. A possibility of noncommutative space-time coordinates has been first discussed by Snyder [51] and nowadays it is widely discussed in noncommutative field theory and string theory. In our approach, noncommutativity of different components of the operator $\mathbf{D}_r$ is not postulated from the beginning but follows from other requirements. As a consequence of noncommutativity, a part of the spectrum of the operator $\mathbf{D}_r$ might be not purely continuous and even the whole spectrum might be discrete. To the best of our knowledge, in the literature on relativistic two-body systems (see eg. Refs. [1, 48]) the operator $\mathbf{D}_r$ has not been investigated. Here a typical approach is that the relative momentum operator is constructed first and then, if the coordinate description is required, the relative position operator is defined as canonically conjugated with the relative momentum operator. Since in our approach there are no classical manifolds at all, the notion of canonically conjugated operator is not natural. Our goal is to construct a dS analog of the operator $\mathbf{D}_r$ and investigate the spectrum of the dS relative distance operator.

In Sect. 3.2 we discussed the construction of the operator W, which is a reduction of the two-body operator I_2 onto the space H_s. The reduction is defined by Eq. (3.22) and its meaning is as follows. The requirement $\mathbf{J}'x = 0$ means that we treat $\mathbf{J}'$ as a dS analog of the total momentum operator and consider a subspace of states with zero total momentum. The reduction of the operator $\mathbf{J}$" onto that subspace is treated as an internal dS angular momentum operator, i.e. as a spin operator for the two-body system. Then H_s is defined as a subspace of eigenvectors of the operator J_z" with the eigenvalue s and such that $J^{"+}x = 0$. Therefore we should define a dS relative position operator $\mathbf{D}$ in such a way that it commutes with $\mathbf{J}'$ and is a vector operator with respect to $\mathbf{J}$", i.e.

$$[J'^{\alpha}, D^{\beta}] = 0, \quad [J^{"\alpha}, D^{\beta}] = 2ie_{\alpha\beta\gamma}D^{\gamma} \tag{5.3}$$

where $(\alpha, \beta, \gamma) = 1, 2, 3$, $e_{\alpha\beta\gamma}$ is the absolutely antisymmetric tensor and a sum over repeated indices is assumed. By using the relations (3.2-3.4), one can directly verify that the operator

$$\mathbf{D} = E^{(1)}\mathbf{N}^{(2)} - E^{(2)}\mathbf{N}^{(1)} + \mathbf{N}^{(1)} \times \mathbf{N}^{(2)} \tag{5.4}$$

satisfies Eq. (5.3). As follows from the discussion in Sect. 2.4, the operator $\mathbf{D}/2R$ becomes $\mathbf{D}_r$ in the formal limit $R \to \infty$. The operator $\mathbf{D}$ is obviously dimensionless in agreement with our discussion in Sects. 2.1 and 2.2 that in dS theory there should be only dimensionless quantities. In the nonrelativistic approximation $\mathbf{D} = \mu_1\mu_2\mathbf{r}/2R = 2Rm_1m_2\mathbf{r}$ and therefore typically the components of $\mathbf{D}$ are huge numbers.

Different components of $\mathbf{D}$ do not commute with each other and the expressions for the commutators are rather cumbersome. However, since $[\mathbf{D}^2, \mathbf{J}"] = 0$, one can work in the representation where the operators $\mathbf{J}^{"2}$, J_z" and $\mathbf{D}^2$ are diagonal; in particular one can consider a reduction of $\mathbf{D}^2$ onto H_s. A direct calculation using Eqs. (3.2-3.4) gives

$$\mathbf{D}^2 = (E^{(1)2} + \mathbf{N}^{(1)2})(E^{(2)2} + \mathbf{N}^{(2)2}) - (E^{(1)}E^{(2)} + \mathbf{N}^{(1)}\mathbf{N}^{(2)})^2 - 16\mathbf{J}^{(1)'}\mathbf{J}^{(2)'} \tag{5.5}$$

We now consider a reduction of this operator onto the space H_s defined by Eq. (3.22). The elements Φ_n defined by Eq. (3.23) form a basis in this space. As follows from Eqs. (3.6), (3.22) and (3.23)

$$[\mathbf{J}^{(1)'}]^2\Phi_n = [\mathbf{J}^{(1)"}]^2\Phi_n = [\mathbf{J}^{(2)'}]^2\Phi_n = [\mathbf{J}^{(2)"}]^2\Phi_n = \\ (n+j)(n+j+2)\Phi_n \quad \mathbf{J}^{'}\Phi_n = 0 \quad \mathbf{J}^{"2}\Phi_n = s(s+2)\Phi_n \quad (5.6)$$

Now by using Eqs. (3.4), (3.9), (3.10), (3.25) and (5.5) one can show by a direct calculation that

$$\mathbf{D}^2\Phi_n = \{[w_1+9+4(n+j)(n+j+2)][w_2+9+4(n+j) \\ (n+j+2)] - \frac{1}{4}\tilde{W}^2 + 16(n+j)(n+j+2)\}\Phi_n \quad (5.7)$$

where $\tilde{W}$ is a nondiagonal part of W, i.e. the operator with the matrix elements $\tilde{W}_{nn} = 0$, $\tilde{W}_{n,n+1} = W^s_{n,n+1}$ and $\tilde{W}_{n+1,n} = W^s_{n+1,n}$.

Suppose there exist states which are quasiclassical in the momentum, mass and relative position representations simultaneously. If a wave function has a sharp maximum in the momentum representation at $n_0 = Rq_0$, a sharp maximum in the W representation at $w_0 = 4R^2M_0^2$ and a sharp maximum in the relative position representation at $\mathbf{D}_0$ then Eq. (5.7) makes it possible to write down a relation between q_0, M_0 and $\mathbf{D}_0$. We can neglect the commutators between n, W and $\mathbf{D}^2$, take into account Eq. (3.28) and the fact that in quasiclassical approximation $j \ll n$, $n \gg 1$ and $w_1, w_2 \gg 1$. The result is

$$M_0^2 = \epsilon_1(q)^2 + \epsilon_2(q)^2 + 2\epsilon_1(q)\epsilon_1(q)[1 - \frac{\mathbf{D}_0^2}{4R^2\epsilon_1(q)^2\epsilon_2(q)^2}]^{1/2} \quad (5.8)$$

This is a general relation between q_0, M_0 and $\mathbf{D}_0$ without assuming nonrelativistic approximation and that the distances are much less than R. Suppose now that these assumptions are valid. Then, as noted above, $\mathbf{D}_0$ can be written as $\mathbf{D}_0 = 2Rm_1m_2\mathbf{r}_0$ and it is easy to verify that Eq. (5.8) is compatible with Eq. (2.26).

We now use $\mathcal{D}$ to denote the operator, which in the normalized basis defined by Eq. (3.41) has the matrix elements $\mathcal{D}_{nn} = 0$, $\mathcal{D}_{n,n+1} = -iW(n,j)/2$ and $\mathcal{D}_{n+1,n} = iW(n,j)/2$ where $W(n,j)$ is defined in Eq.

(3.42). Then, as follows from Eqs. (3.28) and (4.22), Eq. (5.7) can be rewritten as

$$\mathbf{D}^2 e_n = \{\mathcal{D}^2 + (\frac{[w_1 + (2n+s+3)^2][w_2 + (2n+s+3)^2]}{n+j+2} + \frac{[w_1 + (2n+s+1)^2][w_2 + (2n+s+1)^2]}{n+j})\frac{j(j+1)}{2(n+j+1)} + 4[\omega_1(n+j+1)^2 + \omega_2(n+j+1)^2 + 2]\}e_n \quad (5.9)$$

If $j \ll n$ and $n \gg 1$, this expression can be simplified as

$$\mathbf{D}^2 e_n = \{\mathcal{D}^2 + \frac{j(j+1)}{n^2}\omega_1(n)^2\omega_2(n)^2 + 4[\omega_1(n)^2 + \omega_2(n)^2]\}e_n \quad (5.10)$$

and the operator $\mathcal{D}$ acts as

$$\mathcal{D}\sum_n c(n)e_n = \frac{i}{2}\sum_n [c(n+1)W(n) - c(n-1)W(n-1)]e_n \quad (5.11)$$

where $W(n) = \omega_1(n)\omega_2(n)$.

Let us compare Eq. (5.9) with the standard nonrelativistic expression (3.54). As follows from Eqs. (4.22) and (5.11), in the nonrelativistic approximation $\mathcal{D}$ is a finite difference version of the operator $i\mu_1\mu_2\partial/\partial n$. Since $n = Rq$ then, in view of the relation between $\mathbf{D}$ and $\mathbf{r}$ in the nonrelativistic approximation, we conclude that the term with $\mathcal{D}^2$ in Eq. (5.10) is in agreement with the first term in Eq. (3.54). Analogously, the term with $j(j+1)$ in Eq. (5.10) is in agreement with the centrifugal term in Eq. (3.54). Finally, in the nonrelativistic limit the last term in Eq. (5.10) is obviously negligible in comparison with the first and second terms.

The above results give grounds to conclude that the operator $\mathbf{D}$ can be treated as a relative position operator in dS theory and therefore $\mathbf{D}^2$ can be treated as a relative distance operator squared. For quasiclassical states the values of the internal angular momentum are typically very large and it is obvious that if $j \neq 0$, the centrifugal term is much greater than the last term in Eq. (5.10). Therefore in physically interesting cases this term can be neglected. We consider first a model when $j = 0$ and $\mathbf{D}^2 = \mathcal{D}^2$ and the case $j \neq 0$ will be discussed in Sect. 5.3.

5.2 Operators $\mathcal{D}$ and $\mathcal{D}^2$ in states with zero spin

As follows from the definition of the operator $\mathcal{D}$ and Eq. (3.42), the action of the operator $\mathcal{D}$ is defined by

$$\mathcal{D}_{n,n+1} = -\mathcal{D}_{n,n+1} = \frac{i}{2}\{[w_1 + (2n+3)^2][w_2 + (2n+3)^2]\}^{1/2} \quad (5.12)$$

while all the other matrix elements are equal to zero. We consider a case when the basis is formed by the elements $e_0, e_1, ...e_N$, so the case of the infinite dimensional space can be obtained in the limit $N \to \infty$. In view of the discussion in the preceding section and in Sect. 3.4, one might think that $\mathcal{D}$ can be treated as a dS analog of the operator id/dr. As noted in Sect. 3.4, the latter is not a well defined self-adjoint operator. At the same time, $\mathcal{D}$ is not an operator defined on a space of functions of continuous variable. It is obviously well defined when N is finite but a problem arises whether $\mathcal{D}$ is a good physical operator when $N \to \infty$. We will see below that in the latter case only $\mathcal{D}^2$ is the operator with good physical properties while $\mathcal{D}$ is not.

By analogy with Sect. 3.2, in this chapter we use $\Delta_k^n(\eta)$ to denote the determinant of the matrix of the operator $(\mathcal{D}-\eta)$ truncated to a subspace with the basis $e_k, e_{k+1}, ...e_n$. We believe the usage of the same notation should not lead to misunderstanding. As follows from Eq. (5.12), an analog of Eq. (3.29) is

$$\Delta^{n+1}(\eta) = -\eta\Delta^n(\eta) - a_n\Delta^{n-1}(\eta), \quad a_n = \mathcal{D}_{n,n+1}\mathcal{D}_{n+1,n} = \frac{1}{4}[w_1 + (2n+3)^2][w_2 + (2n+3)^2] \quad (5.13)$$

where $\Delta^n(\eta) \equiv \Delta_0^n(\eta)$. As follows from Eqs. (5.12) and (5.13)

$$E(\eta) = \sum_{n=0}^{N}(-1)^n\Delta^{n-1}(\eta)e_n / \prod_{k=0}^{n-1}\mathcal{D}_{k,k+1} \quad (5.14)$$

is the eigenvector of the operator $\mathcal{D}$ with the eigenvalue η if

$$\Delta^N(\eta) = 0 \quad (5.15)$$

Our next goal is to find solutions of this equation.

It is obvious that $\Delta^0(\eta) = -\eta$ and $\Delta^1(\eta) = \eta^2 - a_0$. Therefore Eq. (5.13) makes it possible to calculate $\Delta^n(\eta)$ for any n. In particular, it is clear that the expression for $\Delta^n(\eta)$ contains only odd powers of η if n is even and only even powers of η if n is odd. Therefore

$$\Delta^n(\eta) = \sum_{l=0}^{n+1} b(l,n)\eta^l \tag{5.16}$$

where the coefficients $b(l,n)$ are such that $b(l,n) = 0$ if $n+l$ is even. If n is odd, it is easy to show by induction that

$$b(0,n) = (-1)^{(n+1)/2} a_0 a_2 \cdots a_{n-1} \tag{5.17}$$

and, as follows from Eq. (5.13)

$$b(0,n) = (-1)^{(n+1)/2} 4^{3(n+1)/2} \cdot \\ |\frac{\Gamma((-i\mu_1 + 2n + 5)/4)\Gamma((-i\mu_2 + 2n + 5)/4)}{\Gamma((-i\mu_1 + 3)/4)\Gamma((-i\mu_2 + 3)/4)}|^2 \tag{5.18}$$

As follows from the Jacobi formula for the derivative of the determinant

$$b(l,n) = \frac{1}{l!}\frac{d^l}{d\eta^l}[\Delta^n(0)] \tag{5.19}$$

Therefore by using the fact that the matrix of the operator $\mathcal{D}$ is three-diagonal and all the diagonal matrix elements of the the operator $\mathcal{D} - \eta$ are equal to $-\eta$, one obtains

$$b(l,n) = \sum_{i_1 i_2 \ldots i_l} \Delta_0^{i_1 - 1}(0)\Delta_{i_1+1}^{i2-1}(0) \cdots \Delta_{i_l+1}^{n}(0) \tag{5.20}$$

where $0 \le i_1 < i_2 ... < i_l \le n$.

Consider, for example, a case when n is odd. Since $\Delta_l^k(0) = 0$ if $k - l$ is even, the nonzero coefficients can be written as

$$b(2l,n) = \sum_{j_1 j_2 \ldots j_{2l}} \Delta_0^{2j_1-1}(0)\Delta_{2j_1+1}^{2j_2}(0)\Delta_{2j_2+2}^{2j_3-1}(0) \cdots \Delta_{2j_{2l}+2}^{n}(0) \tag{5.21}$$

where $0 \leq j_1 \leq j_2 < j_3 \leq j_4 < j_5 ... \leq j_{2l} \leq (n-1)/2$. By analogy with Eq. (5.18), one can show that

$$\Delta_{2j_1+1}^{2j_2}(0) = (-1)^{j_2-j_1} 4^{3(j_2-j_1)} \cdot$$
$$|\frac{\Gamma((-i\mu_1+4j_2+5)/4)\Gamma((-i\mu_2+4j_2+5)/4)}{\Gamma((-i\mu_1+4j_1+5)/4)\Gamma((-i\mu_2+4j_1+5)/4)}|^2$$
$$\Delta_{2j_2+2}^{2j_3-1}(0) = (-1)^{j_2-j_1+1} 4^{3(j_2-j_1-1)} \cdot$$
$$|\frac{\Gamma((-i\mu_1+4j_3+3)/4)\Gamma((-i\mu_2+4j_3+3)/4)}{\Gamma((-i\mu_1+4j_2+7)/4)\Gamma((-i\mu_2+4j_2+7)/4)}|^2 \quad (5.22)$$

and, as a consequence

$$b(2l,n) = (-1)^{\frac{n+1}{2}-l} 4^{3(\frac{n+1}{2}-l)} \cdot$$
$$|\frac{\Gamma((-i\mu_1+2n+5)/4)\Gamma((-i\mu_2+2n+5)/4)}{\Gamma((-i\mu_1+3)/4)\Gamma((-i\mu_2+3)/4)}|^2$$
$$\sum_{j_1 j_2 ... j_{2l}} A(j_1)B(j_2)\cdots A(j_{2l-1})B(j_{2l}) \quad (5.23)$$

where

$$A(j) = |\frac{\Gamma((-i\mu_1+4j+3)/4)\Gamma((-i\mu_2+4j+3)/4)}{\Gamma((-i\mu_1+4j+5)/4)\Gamma((-i\mu_2+4j+5)/4)}|^2$$
$$B(j) = |\frac{\Gamma((-i\mu_1+4j+5)/4)\Gamma((-i\mu_2+4j+5)/4)}{\Gamma((-i\mu_1+4j+7)/4)\Gamma((-i\mu_2+4j+7)/4)}|^2 \quad (5.24)$$

So far, no approximation has been made and the general expression (5.23) is rather complicated. It can be simplified as follows. We can use the fact that if z is large then $\Gamma(z+1/2) \approx \Gamma(z)z^{1/2}$ and take into account that $A(j)$ and $B(j)$ typically enter Eq. (5.23) with large values of j. Therefore

$$A(j) \approx B(j) \approx 4[(w_1+16j^2)(w_2+16j^2)]^{-1/2} \quad (5.25)$$

The next approximation is as follows. If $l \ll n$, the contribution to the sum in Eq. (5.23) of the terms where some of the $j's$ are equal to each other, is much smaller than the contribution of the terms with $j_1 < j_2 < ... j_{2l}$. Therefore

$$\sum_{j_1 j_2 ... j_{2l}} A(j_1)B(j_2)\cdots A(j_{2l-1})B(j_{2l}) \approx \frac{1}{(2l)!} u(\frac{n-1}{2})^{2l} \quad (5.26)$$

where the function $u(k)$ is defined as

$$u(k) = \sum_{j=0}^{k} [(w_1 + 16j^2)(w_2 + 16j^2)]^{-1/2} \tag{5.27}$$

Since $w_1, w_2 \gg 1$, with a good accuracy $u(k)$ can be treated as a function of a continuous variable k defined as

$$u(k) = \int_0^k [(w_1 + 16x^2)(w_2 + 16x^2)]^{-1/2} dx \tag{5.28}$$

With these approximations, the expression for $\Delta^n(\eta)$ when n is odd, is proportional to the sum

$$1 - \frac{1}{2!}(2\eta u(K))^2 + \frac{1}{4!}(2\eta u(K))^4 \pm \frac{1}{(2K)!}(2\eta u(K))^{2K}$$

where $K = (n-1)/2$. A question arises whether this sum can be replaced by $cos(2\eta u(K))$. The matter is that the argument of the cosine is typically very large. For example, in the nonrelativistic approximation, $u(K) = n/2\mu_1\mu_2$, $\eta = 2\mu_1\mu_2 r/$ and therefore $2\eta u(K) = rn/R = rq$ as expected. For macroscopic bodies this quantity is indeed very large. On the other hand, if x is a large positive value, the Taylor series for $cos(x)$ gives a good approximation if the number of terms in the series is of order x. Therefore the above sum can be replaced by $cos(2\eta u(K))$ if $r \ll R$.

The case when n is even can be considered analogously and the final result is

$$\Delta^n(\eta) = (-1)^{(n+1)/2} 4^{3(n+1)/2} cos(2\eta u(\frac{n-1}{2})) \cdot$$
$$|\frac{\Gamma((-i\mu_1 + 2n + 5)/4)\Gamma((-i\mu_2 + 2n + 5)/4)}{\Gamma((-i\mu_1 + 3)/4)\Gamma((-i\mu_2 + 3)/4)}|^2$$
$$\Delta^n(\eta) = (-1)^{n/2+1} 4^{3(n+1)/2} sin(2\eta u(\frac{n-1}{2})) \cdot$$
$$|\frac{\Gamma((-i\mu_1 + 2n + 5)/4)\Gamma((-i\mu_2 + 2n + 5)/4)}{\Gamma((-i\mu_1 + 3)/4)\Gamma((-i\mu_2 + 3)/4)}|^2 \tag{5.29}$$

if n is odd and even, respectively. Therefore solutions of Eq. (5.15) can be written as

$$\eta_1(N) = \pm(l+\frac{1}{2})\pi/2u(\frac{N-1}{2}), \quad \eta_2(N) = \pm l\pi/2u(\frac{N}{2}) \qquad (5.30)$$

where $l = 0, 1, 2...$ and N is odd and even, respectively. As follows from Eqs. (5.12), (5.14) and (5.29), the eigenvector of the operator $\mathcal{D}$ with the eigenvalue η corresponding to $r \ll R$ can be written as

$$E(\eta) = E_1(\eta) + E_2(\eta)$$
$$E_1(\eta) = const \sum_{k=0}^{[N/2]} \frac{cos[2\eta u(k-1)]e_{2k}}{[(w_1+16k^2)(w_2+16k^2)]^{1/4}}$$
$$E_2(\eta) = const \sum_{k=0}^{[(N-1)/2]} \frac{sin[2\eta u(k)]e_{2k+1}}{[(w_1+16k^2)(w_2+16k^2)]^{1/4}} \qquad (5.31)$$

where $[N/2]$ is the integer part of $N/2$ and the values of *const* should be determined from the normalization condition. Since we have calculated the coefficients assuming that $n \gg 1$, then strictly speaking, the lower limit in these sums should be not zero but some quantity $k_0 \gg 1$.

We see that the spectrum of the operator $\mathcal{D}$ is pure discrete, at least in the region corresponding to $r \ll R$. Let us estimate the quantum of the distance. If $N \gg \mu_1$ and $N \gg \mu_2$, the upper limit in Eq. (5.28) can be replaced by ∞ and the integral becomes the complete elliptic integral of the first kind [47]

$$u(\infty) \equiv u_\infty = \frac{\pi}{8\mu_1}F(\frac{1}{2},\frac{1}{2};1;\frac{\mu_1^2-\mu_2^2}{\mu_1^2}) \qquad (5.32)$$

Therefore, if μ_1 and μ_2 are of the same order, u_∞ is of order $\pi/8\mu$, where μ is of order μ_1 and μ_2. If $\mu_1 \gg \mu_2$, the hypergeometric function can be estimated by using its integral representation and the result is

$$u_\infty = \frac{1}{4\mu_1}ln\frac{2\mu_1^2}{\mu_2^2} \qquad (5.33)$$

In the nonrelativistic approximation $\eta = \mu_1\mu_2 r/R$. Therefore if the masses are of the same order, the quantum of r is $R/\mu = 1/m$ while if $\mu_1 \gg \mu_2$, the quantum of r is of order $1/[m_2 ln(m_1/m_2)]$. This is in agreement with what was noted about the position operator in Chaps. 1 and 2. On the other hand, as already noted, in dS theory any operator with the dimension of length is artificial and we treat η as the dS analog of the length.

As follows from Eq. (5.30), if $\eta \neq 0$ belongs to the spectrum of the operator $\mathcal{D}$ then $-\eta$ also belongs to the spectrum. This does not mean yet that the operator $\mathcal{D}$ is unphysical since it is $\mathcal{D}^2$, which is treated as the operator of the distance squared. However, if one is working with the operator $\mathcal{D}$, the following problems arise. First of all, if one requires that the basis is formed by the eigenvectors of the operator $\mathcal{D}^2$ which are simultaneously the eigenvectors of $\mathcal{D}$, then each eigenvalue of the operator $\mathcal{D}^2$ is doubly degenerated and therefore such eigenvalues do not represent a full set of quantum numbers. Another problem is related to the behavior of the coefficients in Eq. (5.31) when $k \to \infty$. This limit is not defined uniquely since, depending on whether N is odd or even, in the limit $N \to \infty$ one obtains two different sets of eigenvalues. Suppose, for example, that $N \to \infty$ and N is odd. Then, if $E(\eta) = \sum(c_{2k}e_{2k} + c_{2k+1}e_{2k+1})$, it follows from Eqs. (5.30) and (5.31) that at large k, $c_{2k} = O(1/k)$ while $c_{2k+1} = O(1/k^2)$. Analogously, if N is even then $c_{2k} = O(1/k^2)$ and $c_{2k+1} = O(1/k)$. Such a behavior of the coefficients is unphysical since it would mean that at large momenta a half of the coefficients can be neglected. In addition, we will see below that the falloff of the coefficients as $O(1/k)$ leads to infinite mean value of the mass operator.

The vectors $E_1(\eta) \pm E_2(\eta)$ are the eigenvectors of the operator $\mathcal{D}$ with the eigenvalues $\pm\eta$, respectively but $E_1(\eta)$ and $E_2(\eta)$ are not the eigenvectors of $\mathcal{D}$. One can represent the Hilbert space H as a direct sum $H = H_1 + H_2$ where H_1 is the space with the basis $\{e_{2k}\}$ and H_2 is the space with the basis $\{e_{2k+1}\}$. As follows from Eq. (5.12), $\mathcal{D}$ has nonzero matrix elements only for transitions between H_1 and H_2 and therefore each subspace, H_1 and H_2, is invariant under the action of $\mathcal{D}^2$. In particular, $E_1(\eta)$ and $E_2(\eta)$ are the eigenvectors of $\mathcal{D}^2$ with the eigenvalue η^2. If $x_1 \in H_1$ and $x_2 \in H_2$ are nonzero vectors then $x_1 \pm x_2$

can be the eigenvector of $\mathcal{D}^2$ only if x_1 and x_2 are the eigenvectors of $\mathcal{D}^2$ with the same eigenvalue. However, if we do not require that each eigenvector of $\mathcal{D}^2$ should be the eigenvector of $\mathcal{D}$, we can seek eigenvectors of $\mathcal{D}^2$ in the subspaces H_1 and H_2 independently. Suppose, for example, that N is even. Then $E_1(\eta_1(N))$ is the eigenvector of $\mathcal{D}^2$ with the eigenvalue $\eta_1(N)^2$ and $E_2(\eta_2(N-1))$ is the eigenvector of $\mathcal{D}^2$ with the eigenvalue $\eta_2(N-1)^2$. These eigenvalues are different and therefore the set of eigenvalues of the operator $\mathcal{D}^2$ is a full set of quantum numbers. When $n \to \infty$, the decomposition coefficients of the both vectors, E_1 and E_2 over the basis $\{e_n\}$ have the behavior $O(1/n^2)$. Therefore indeed, $\mathcal{D}^2$ has good physical properties while $\mathcal{D}$ has not.

If $E_1(\eta)$ and $E_2(\eta)$ are treated as independent eigenvectors of the operator $\mathcal{D}^2$ then $||E_1(\eta)|| = ||E_2(\eta)|| = 1$ and the value of *const* in Eq. (5.31) can be calculated as follows. Since only large values of k are important, one can use Eq. (5.28) and $u(k-1) \approx u(k)$. Then

$$||E_{1,2}||^2 = \frac{1}{2}const^2 \int_0^{u(N/2)} [1 \pm cos(4\eta u)]du \approx \frac{1}{2}const^2 u(N/2) \quad (5.34)$$

since the contributin of $cos(4\eta u)$ is negligible. Therefore

$$E_1(\eta) = [\frac{2}{u(N/2)}]^{1/2} \sum_{k=0}^{[N/2]} \frac{cos[2\eta u(k)]e_{2k}}{[(w_1+16k^2)(w_2+16k^2)]^{1/4}}$$
$$E_2(\eta) = [\frac{2}{u(N/2)}]^{1/2} \sum_{k=0}^{[(N-1)/2]} \frac{sin[2\eta u(k)]e_{2k+1}}{[(w_1+16k^2)(w_2+16k^2)]^{1/4}} \quad (5.35)$$

The vectors $E_1(\eta)$ and $E_2(\eta)$ are orthogonal since they belong to orthogonal subspaces, H_1 and H_2, respectively. Since $\mathcal{D}^2 E_{1,2}(\eta) = \eta^2 E_{1,2}(\eta)$,

$$(E_1(\eta_1), E_1(\eta_2)) = (E_1(\eta_1), E_1(\eta_2)) = 0$$

if $\eta_1^2 \neq \eta_2^2$. The orthogonality can also be proved explicitly since, as follows from Eqs. (5.28), (5.35) and (5.30)

$$(E_{1,2}(\eta_1), E_{1,2}(\eta_2)) = \frac{2}{u(N/2)} \int_0^{u(N/2)} [cos(2(\eta_1-\eta_2)u] \pm$$
$$cos(2(\eta_1+\eta_2)u)du = 0 \quad (5.36)$$

Let us note an important difference between discrete and continuous problems. In the eigenvalue problem for a differential operator defined on a space of functions $f(x)$, $x \in [a,b]$, one should require $f(a) = f(b) = 0$. In particular, as noted in Chap. 3, the radial wave function should be equal to zero both, at $r = 0$ and $r \to \infty$. In the discrete case the coefficients $c(n)$ defining the vectors $E_1(\eta)$ and $E_2(\eta)$, are such that $c(n) \to 0$ when $n \to \infty$. However, in this case there is no requirement $c(n) \to 0$ when $n \to 0$. As seen from Eq. (5.35), this is the case only for coefficients defining $E_2(\eta)$. One might think that physical states should be described only by vectors belonging to H_2. However, as noted in Sect. 3.3, the standard expression for the mean value of the mass operator can be obtained only when $c(n) \approx c(n-1)$. This observation will be important in the next sections where we consider a case of nonzero spin.

5.3 Operator $\mathbf{D}^2$ with centrifugal contribution

As follows from Eqs. (3.28), (3.42), (4.22) and (5.7), without any approximations the action of the operator $\mathbf{D}^2$ can be written as

$$\begin{aligned}
&\mathbf{D}^2 e_n = -\frac{1}{4}\{[w_1 + (2\tilde{n}+5)^2][w_2 + (2\tilde{n}+5)^2][w_1 + (2\tilde{n}+3)^2]\\
&[w_2 + (2\tilde{n}+3)^2][1 - \frac{j(j+1)}{(\tilde{n}+2)(\tilde{n}+3)}][1 - \frac{j(j+1)}{(\tilde{n}+1)(\tilde{n}+2)}]\}^{1/2} e_{n+2}\\
&-\frac{1}{4}\{[w_1 + (2\tilde{n}+1)^2][w_2 + (2\tilde{n}+1)^2][w_1 + (2\tilde{n}-1)^2]\\
&[w_2 + (2\tilde{n}-1)^2][1 - \frac{j(j+1)}{\tilde{n}(\tilde{n}+1)}][1 - \frac{j(j+1)}{\tilde{n}(\tilde{n}-1)}]\}^{1/2} e_{n-2}\\
&+\frac{1}{4}\{[w_1 + (2\tilde{n}+3)^2][w_2 + (2\tilde{n}+3)^2][1 + \frac{j(j+1)}{(\tilde{n}+1)(\tilde{n}+2)}] +\\
&[w_1 + (2\tilde{n}+1)^2][w_2 + (2\tilde{n}+1)^2][1 + \frac{j(j+1)}{\tilde{n}(\tilde{n}+1)}] +\\
&16[w_1 + w_2 + 8(\tilde{n}+1)^2]\}e_n
\end{aligned} \tag{5.37}$$

where $\tilde{n} = n + j$. In macroscopic systems the value of j is very large (for example, in the Earth - Moon system j is of order 10^{68}) and $n \gg 1$. Therefore, if $E(\eta) = \sum_n c(n)e_n$ is an eigenvector of the operator $\mathbf{D}^2$ with an eigenvalue η^2 then, with a high accuracy

$$-\frac{1}{4}(w_1 + 4\tilde{n}^2)(w_2 + 4\tilde{n}^2)[1 - \frac{j(j+1)}{\tilde{n}^2}]\{[c(n+2) + c(n-2) - 2c(n)] + 8\tilde{n}[(\frac{1}{w_1 + 4\tilde{n}^2} + \frac{1}{w_2 + 4\tilde{n}^2}][c(n+2) - c(n-2)]\} + (w_1 + 4\tilde{n}^2)(w_2 + 4\tilde{n}^2)\frac{j(j+1)}{\tilde{n}^2}c(n) = \eta^2 c(n) \quad (5.38)$$

Here we take into account that if $j \gg 1$ then $j(j+1)(w_1 + 4\tilde{n}^2)(w_1 + 4\tilde{n}^2)/\tilde{n}^2 \gg w_1, w_2, \tilde{n}^2$. As already noted, in macroscopic systems $n \gg j$. Therefore if $\tilde{n}$ is formally treated as a continuos variable and $c(\tilde{n})$ does not change significantly when $\tilde{n}$ changes by one, the above finite difference equation can be replaced by

$$(w_1 + 4\tilde{n}^2)(w_1 + 4\tilde{n}^2)[-\frac{d^2c(\tilde{n})}{d\tilde{n}^2} - 8\tilde{n}(\frac{1}{w_1 + 4\tilde{n}^2} + \frac{1}{w_2 + 4\tilde{n}^2})\frac{dc(\tilde{n})}{d\tilde{n}} + \frac{j(j+1)}{\tilde{n}^2}c(n)] = \eta^2 c(\tilde{n}) \quad (5.39)$$

Here comes an important difference between the standard and dS theories. Compare Eqs. (3.67) and (5.39). In the former the centrifugal contribution at large momenta is negligible while in the latter it is much greater than the r.h.s. As a consequence, as follows from Eq. (5.39), if at large $\tilde{n}$, $c(\tilde{n}) = O(\tilde{n}^k)$ and $c(\tilde{n}) \to 0$ then $k = -(j+2)$, i.e. $c(\tilde{n})$ is a function rapidly decreasing at infinity. Below this question is discussed in greater details. We will also see that at small momenta the behavior of wave functions in standard and dS theories is essentially different as well.

If $c(\tilde{n}) = b(\tilde{n})[(w_1 + 4\tilde{n}^2)(w_2 + 4\tilde{n}^2)]^{-1/4}$ then Eq. (5.39) can be written as

$$-\frac{1}{4}\frac{d^2b(u)}{du^2} + (w_1 + 4\tilde{n}^2)(w_2 + 4\tilde{n}^2)\frac{j(j+1)}{\tilde{n}^2}b(u) = \eta^2 b(u) \quad (5.40)$$

where u is now the independent variable and $\tilde{n}$ is treated as a function of u defined from $u = u(\tilde{n}/2)$ (see Eq. (5.28)). Here we again assume that $j \gg 1$.

Eq. (5.40) can be considerably simplified if $\mu_1 = \mu_2 = \mu$ and we first consider this case. In terms of the variable $\theta = 8\mu u$ we have $\tilde{n} = (\mu/2)tan^{-1}(\theta/2)$ and Eq. (5.40) can be written as

$$-\frac{d^2b(\theta)}{d\theta^2} + \frac{j(j+1)}{sin^2\theta}b(\theta) = \frac{\eta^2}{16\mu^2}b(\theta) \qquad (5.41)$$

This is a special case of the associated Mathieu equation, which can be written as an equation for the spheroidal function $f(\theta) = b(\theta)/(sin\theta)^{1/2}$:

$$\frac{d^2f(\theta)}{d\theta^2} + cotan\theta\frac{df(\theta)}{d\theta} + [(\frac{\eta^2}{16\mu^2} - \frac{1}{4}) - \frac{(j+1/2)^2}{sin^2\theta}]f(\theta) = 0 \qquad (5.42)$$

In summary, one can consider an approximation when in the case of equal masses the variable $\tilde{n}$ is replaced by a continuous variable θ such that the normalization condition for the wave function is given by

$$\sum_n |c(n)|^2 = \frac{1}{4\mu}\int_{\theta_{min}}^{\pi} |f(\theta)|^2 sin\theta d\theta \qquad (5.43)$$

where $f(\theta)$ is a solution of Eq. (5.42) and the value of θ_{min} will be discussed later.

In the literature Eq. (5.42) is written in the form

$$\frac{d^2f(\theta)}{d\theta^2} + cotan\theta\frac{df(\theta)}{d\theta} + [\nu(\nu+1) - \frac{\mu^2}{sin^2\theta}]f(\theta) = 0 \qquad (5.44)$$

and its two linearly independent solutions are the associated Legendre functions $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ where $x = cos\theta$. If ν or μ are integers, they are written as n and m, respectively. When both ν and μ are integers, $P_n^m(x)$ is the associated Legendre polynomial, which describes the θ dependence of the spherical wave function $Y_{nm}(\theta,\varphi)$. Since it is an eigenfunction of a differential operator defined on a space of functions $f(\theta)$, $\theta \in [0,\pi]$, it should satisfy the condition $P_n^m(0) = P_n^m(\pi) = 0$ if $n \neq 0$ (see the remark at the end of the preceding section). Such a polynomial can be constructed only if the magnetic quantum number m is an integer.

Since we reserve μ for other purposes, we will write solutions of Eq. (5.42) as $P_l^k(x)$ and $Q_l^k(x)$ where $k = j + 1/2$ and $\eta^2 = 16\mu^2(l + 1/2)^2$. One can immediately notice that, in contrast with the standard case, the magnetic quantum number k is now not an integer but a half-integer. Therefore one might wonder whether it is possible to find solutions which becomes zero at $\theta = 0$ and $\theta = \pi$. However, as already noted, since we consider Eq. (5.42) as an approximation for a discrete problem, it is not necessary to require that the solution should be zero at $\theta = 0$. At the same time, $\theta = \pi$ corresponds to $n \to \infty$ and therefore the requirements $P_l^k(x) \to 0$ and $Q_l^k(x) \to 0$ when $x \to -1$ are necessary.

Since (see e.g. Ref. [47])

$$P_l^k(x) = \frac{1}{\Gamma(1-k)}(\frac{1+x}{1-x})^{k/2}F(-l, l+1; 1-k; \frac{1-x}{2}) \tag{5.45}$$

and k is a half-integer, the above expression can be regular at $x \to -1$ only if l or $-(l+1)$ is a positive integer and these cases are essentially the same. So we will assume that l is a positive integer. Then the series for the hypergeometric function becomes a finite polynomial and therefore $P_l^k(x) = O((1-x)^{k/2})$ when $x \to -1$. It is easy to see that this is in agreement with the above remark that $c(n) = O(1/n^{j+2})$ when $n \to \infty$. At the same time, in contrast with the standard case, the function $P_l^k(x)$ is very singular when $\theta \to 0$ and it is not possible to avoid the singularity when k is a half-integer. However, as noted above, the continuous approximation can be valid only if θ is not anomalously small.

Consider now a case when $Q_l^k(x)$ is chosen as a solution. When k is a half-integer (see e.g. Ref. [47]),

$$Q_l^k(x) = \frac{\Gamma(1+l+k)\Gamma(-k)}{2\Gamma(1+l-k)}(\frac{1-x}{1+x})^{k/2}F(-l, l+1; 1+k; \frac{1-x}{2}) \tag{5.46}$$

We can use the relation (see e.g. Ref. [47])

$$F(a, b; c; z) = (1-z)^{c-a-b}F(c-a, c-b; c; z) \tag{5.47}$$

and rewrite Eq. (5.46) as

$$Q_l^k(x) = \frac{\Gamma(1+l+k)\Gamma(-k)}{2^{1+j}\Gamma(1+l-k)}[(1-x)(1+x)]^{k/2}$$
$$F(1+k+l, k-l; 1+k; \frac{1-x}{2}) \tag{5.48}$$

This expression can be regular at $x \to -1$ only when the hypergeometric function becomes a finite polynomial. This happens when $l-k$ or $-(1+l+k)$ are positive integers. We will assume that l is a positive half-integer and such that $l-k \geq 0$. In this case the gamma functions are well defined and $Q_l^k(x)$ is regular at both, $x \to 1$ and $x \to -1$. So $Q_l^k(cos\theta)$ is a true eigenfunction of the differential operator defined by Eq. (5.42) if l and k are positive half-integers.

Since $\eta^2 = 16\mu^2(l+1/2)^2$ and η^2 is the eigenvalue of the operator $\mathbf{D}^2$, we cannot choose a solution, which is a linear combination of $P_l^k(x)$ and $Q_l^k(x)$ since the values of η^2 are different when the quantities l are integers or half-integers. So the only possible solutions are either $P_l^k(x)$ or $Q_l^k(x)$. What is a correct choice?

As noted in the preceding section, the Hilbert space in question is a direct sum of H_1 and H_2 which are invariant subspaces of the operator $\mathbf{D}^2$. We can seek eigenfunctions of $\mathbf{D}^2$ in H_1 and H_2 independently and the above discussion can be applied in the both cases. Also it has been noted that if the eigenvalues of $\mathbf{D}^2$ in H_1 and H_2 are different, they represent a full set of quantum numbers. Therefore in view of the above discussion we have only two possibilities: either the solution $P_l^k(x)$ is chosen in H_1 and $Q_l^k(x)$ in H_2 or vice versa. To choose a correct possibility one might consider the eigenvalue problem beyond the continuous approximation. However, a simpler way is as follows. In the region where the centrifugal contribution is much less than the term with η^2 in Eq. (5.40), we can use the results of the preceding section that the solution for $b(\theta)$ in Eq. (5.40) should be proportional to $cos(2\eta u)$ in H_1 and to $sin(2\eta u)$ in H_2. On the other hand, we can use the asymptotics of Legendre functions when l is very large [47]:

$$P_l^k(cos\theta) = const[\frac{1}{sin\theta}]^{1/2} sin[(l+\frac{1}{2})\theta + \frac{(2k+1)\pi}{4}]$$
$$Q_l^k(cos\theta) = const[\frac{1}{sin\theta}]^{1/2} cos[(l+\frac{1}{2})\theta + \frac{(2k+1)\pi}{4}] \quad (5.49)$$

Therefore if j is even, one should choose the solution P_l^k in H_1 and Q_l^k in H_2 and vice versa if j is odd.

Consider now a case when the masses of the particles are not necessarily equal to each other. Suppose for simplicity that the system is nonrelativistic. This implies that if $\mathbf{v}_0$ is the quasiclassical value of the relative velocity and $v_0 = |\mathbf{v}_0|$ then $v_0 \ll 1$. Let $\mathbf{r}_0$ be the quasiclassical value of the relative distance and $r_0 = |\mathbf{r}_0|$. Then j is of order $m_{12}|\mathbf{r}_0 \times \mathbf{v}_0|$ where m_{12} is the reduced mass. Let us compare the contributions of the second and last terms in Eq. (5.40). Suppose that j is of order $m_{12}r_0v_0$ and η is of order $\mu_1\mu_2 r_0/R$. Then one can make the following conclusions. When $\tilde{n} \ll m_{12}Rv_0$, the centrifugal term is much greater than the one with η^2. When $\tilde{n}$ increases and becomes of order $m_{12}Rv_0$, the terms become of the same order and then the one with η^2 becomes much greater. They again become of the same order in the ultrarelativistic region where $\tilde{n}$ is of order $(m_1+m_2)R/v_0$ and finally the centrifugal term becomes much greater if $\tilde{n} \gg (m_1+m_2)R/v_0$. As already noted, there is no analog of the last case in the standard theory.

If the centrifugal contribution is small, we can use the results of the preceding section. Then, as noted above, $b(u)$ should be proportional to $cos(2\eta u)$ in H_1 and to $sin(2\eta u)$ in H_2. Consider first the case when $\tilde{n}$ is of order $(m_1+m_2)R/v_0$ or greater. Then, as follows from Eq. (5.28), $u_\infty - u = y = 1/8\tilde{n}$ and Eq. (5.40) can be rewritten as

$$y^2\frac{d^2b(y)}{dy^2} + [4\eta^2y^2 - j(j+1)]b(y) = 0 \quad (5.50)$$

This is the Riccati - Bessel equation, which has two solutions. Since a necessary requirement is $b(u) \to 0$ when $\tilde{n} \to \infty$, the only possible choice is

$$b(y) = const\,(2\eta y)^{1/2}J_k(2\eta y) \quad (5.51)$$

where J_k is the Bessel function of the order $k = j + 1/2$. This solution should be compatible with the one in the region where the centrifugal contribution is small. Suppose that in this region we can use the asymptotic expression for the Bessel function and thus

$$b(u) = const\, cos[2\eta u_\infty - \frac{(j+1)\pi}{2} - 2\eta u] \tag{5.52}$$

If j is even then this solution is proportional to $cos(2\eta u)$ if $2\eta u_\infty = (l+\frac{1}{2})\pi$ and to $sin(2\eta u)$ if $2\eta u_\infty = l\pi$, where $l = 0, 1, 2, ...$ (compare with Eq. (5.30)). Analogously, if j is odd, the solution is proportional to $sin(2\eta u)$ if $2\eta u_\infty = l\pi$ and to $cos(2\eta u)$ if $2\eta u_\infty = (l+\frac{1}{2})\pi$. Therefore if j is even, one should choose the solution $2\eta u_\infty = (l+\frac{1}{2})\pi$ in H_1, $2\eta u_\infty = l\pi$ in H_2 and vice versa if j is odd. Hence the spectrum of the operator $\mathbf{D}^2$ is the same as the spectrum of the operator $\mathcal{D}^2$ discussed in the preceding section.

Consider now a case when $\tilde{n}$ is of order $m_{12}Rv_0$ or less but the continuos approximation is still possible. This is a nonrelativistic region and, as follows from Eq. (5.28), $u \approx \tilde{n}/(2\mu_1\mu_2)$. Therefore Eq. (5.40) can be represented as

$$u^2\frac{d^2b(u)}{du^2} + [4\eta^2u^2 - j(j+1)]b(u) = 0 \tag{5.53}$$

It is interesting to note that in dS theory the both opposite cases, when $\tilde{n}$ is very small and very large, are described by the Riccati - Bessel equation. Eq. (5.53) has two solutions

$$b(u) = const\,(2\eta u)^{1/2}J_{\pm k}(2\eta u) \tag{5.54}$$

(compare with Eq. (5.51)) and, by analogy with the above discussion of the case with equal masses, we cannot conclude that only the first case is possible. Eq. (5.54) should be compatible with the result in the region where the centrifugal contribution is small. By using the asymptotic expression of the Bessel function, we obtain that

$$(2\eta u)^{1/2}J_k(2\eta u) \approx const\, cos[2\eta u - \frac{(j+1)\pi}{2}]$$
$$(2\eta u)^{1/2}J_{-k}(2\eta u) \approx const\, cos[2\eta u + \frac{j\pi}{2}] \tag{5.55}$$

Therefore if j is even, one has to choose the solution with J_{-k} in H_1 and with J_k in H_2, while if j is odd, the opposite choice should be made. It is easy to see that this conclusion is compatible with the above result for equal masses.

The above results can be summarized as follows. For analogy with the case of equal masses, we denote $\theta = \pi u/u_\infty$. Let θ_1 be a value of θ when the continuous approximation is still valid. As noted above, this value should correspond to $\tilde{n}$, which is still greater than j. Therefore $\theta_1 \geq \pi j/(\mu_1\mu_2 u_\infty)$. This is a very small value. For example, when μ_1 and μ_2 are of the same order, θ_1 is of order $v_0 r_0$/R. Let θ_2 be a value of θ corresponding to $\tilde{n}$ of order $(\mu_1+\mu_2)/v_0$. Then the centrifugal contribution is dominant when $\theta_2 \leq \theta \leq \pi$. The value of $\pi - \theta_2$ is of order $v_0/[(\mu_1+\mu_2)u_\infty]$. When μ_1 and μ_2 are of the same order, $\pi-\theta_2$ is of order v_0. Let θ_3 b such that $\theta_1 < \theta_3 < \theta_2$ and at $\theta = \theta_3$ the centrifugal contribution can be neglected. Then the solutions at $\theta_1 \leq \theta \leq \theta_3$ and $\theta_3 \leq \theta \leq \theta_2$ can be joined as follows. Consider first the case when j is even. Then the eigenvalues η^2 of the operator $\mathbf{D}^2$ in H_1 are such that $2\eta u_\infty = (l+\frac{1}{2})\pi$ $(l = 0, 1...)$ and the corresponding eigenfunctions are given by

$$b_l(\theta) = (-1)^{j/2}[\frac{\pi}{2}(l+\frac{1}{2})\theta]^{1/2} J_{-(j+1/2)}((l+\frac{1}{2})\theta) \quad (\theta_1 \leq \theta \leq \theta_3)$$
$$b_l(\theta) = (-1)^{l+j/2}[\frac{\pi}{2}(l+\frac{1}{2})(\pi-\theta)]^{1/2} J_{(j+1/2)}((l+\frac{1}{2})(\pi-\theta))$$
$$(\theta_3 \leq \theta \leq \pi) \tag{5.56}$$

In this case $b_l(\theta) = cos((l+1/2)\theta)$ in the region where the centrifugal contribution can be neglected. The eigenvalues η^2 in H_2 are such that $2\eta u_\infty = l\pi$ $(l = 0, 1...)$ and the corresponding eigenfunctions are given by

$$b_l(\theta) = (-1)^{j/2}[\frac{\pi}{2}l\theta]^{1/2} J_{(j+1/2)}(l\theta) \quad (\theta_1 \leq \theta \leq \theta_3)$$
$$b_l(\theta) = (-1)^{l+j/2}[\frac{\pi}{2}l(\pi-\theta)]^{1/2} J_{(j+1/2)}(l(\pi-\theta))$$
$$(\theta_3 \leq \theta \leq \pi) \tag{5.57}$$

In this case $b_l(\theta) = sin(l\theta)$ in the region where the centrifugal contribution can be neglected. Finally, when j is odd, the solutions in H_1 and H_2 should be interchanged.

The following important remarks are in order. While in the case of equal masses the solutions (5.49) describe the eigenfunctions in the region $\theta_1 \leq \theta \leq \pi$, in the general case we did not succeed in finding solutions describing the eigenfunctions at all those values of θ. We tried to join the solutions in different regions assuming that if $\eta u \gg j$ or $\eta(u_\infty - u) \gg j$ (such that the terms with η in Eqs. (5.50) and (5.53) are much greater than the centrifugal contribution) then one can take asymptotic expressions for the corresponding Bessel functions. This is the case when j is not very large. However, when both, z and ν are large, the condition $z \gg \nu$ is not sufficient for guaranteeing that for the Bessel functions $J_\nu(z)$ and $J_{-\nu}(z)$ one can take their asymptotic expressions. This can be guaranteed only if $z \gg \nu^2$. Since as already noted, in real situations the value of j is very large, the problem of finding solutions in the case of unequal masses requires a further study.

Another very important feature of the above results is that it is not possible to avoid solutions, which are anomalously large at small values of θ. When j is very large, the functions $P_l^{(j+1/2)}(\theta)$ and $J_{-(j+1/2)}((l+1/2)\theta)$ are very singular at $\theta \to 0$. The existence of these solutions is a consequence of the fact that the cosmological constant is finite and therefore there are no analogs of such solutions in the standard theory. As already noted, such solutions are valid only in the continuous approximation when $\theta \geq \theta_1$, and the very problem of finding eigenvalues and eigenvectors of the operator $\mathbf{D}^2$ is such that singularities are not possible. Nevertheless a question arises whether the contribution of $\theta \leq \theta_1$ can be neglected. If $\theta \ll 1$ then $n \ll \mu_1, \mu_2$. We also assume that $j \ll \mu_1, \mu_2$. Then, if $E(\eta) = \sum_n a(n)e_n$ is the eigenvector of the operator $\mathbf{D}^2$ with the eigenvalue η^2, it follows from

Eqs. (3.42) and (5.7) that

$$-\frac{a(n+2)}{4(n+j+2)}[\frac{(n+1)(n+2)(n+2j+2)(n+2j+3)}{(n+j+1)(n+j+3)}]^{1/2}-$$
$$-\frac{a(n-2)}{4(n+j)}[\frac{n(n-1)(n+2j)(n+2j+1)}{(n+j-1)(n+j+1)}]^{1/2}+a(n)[1-\frac{\eta^2}{w_1w_2}-$$
$$\frac{(n+1)(n+2j+2)}{4(n+j+1)(n+j+2)}-\frac{n(n+2j+1)}{4(n+j)(n+j+1)}]=0 \quad (5.58)$$

When n is of order j or less, one can neglect the term with η since η^2/w_1w_2 is of order r_0^2/R^2. If $j \gg 1$ and $n \ll j$ then it follows from this expression that

$$a(n+2) \approx \frac{2j}{[(n+1)(n+2)]^{1/2}}a(n) \quad (5.59)$$

Therefore at these conditions the coefficients $a(n)$ sharply increase when n increases. The speed of the increase becomes slower when n increases but nevertheless, when n becomes of order j, the values of $a(n)/a(0)$ in H_1 and $a(n)/a(1)$ in H_2 are huge numbers of order $[2j]^{j/2}/\sqrt{j!} \approx (2e)^{j/2}/(2\pi j)^{1/4}$. This might be an indication that the region $n \leq j$ is important. It is clear that in the standard theory the relation $n \leq j$ is impossible since j does not depend on R, n is proportional to R and the standard theory corresponds to the case $R \to \infty$.

5.4 A possible mechanism of gravity

Let us consider the internal Hilbert space at fixed j as a direct sum $H = H_1 + H_2$, where the basis in H_1 is $(e_0, e_2, ...e_{2k}...)$ and the basis in H_2 is $(e_1, e_3, ...e_{2k+1}...)$. Consider the mean value of the mass operator over the state

$$\Psi = \sum_{k=0}^{\infty}[a(k)e_{2k} + b(k)e_{2k+1}]$$

We assume that the main contribution is given by $k \gg 1$ and the coefficients $a(k)$ and $b(k)$ do not change significantly when k changes by one. Then Eq. (3.45) can be written in the form

$$(\Psi, M^2\Psi) = \sum_{k=0}^{\infty}\{[m_1^2 + m_2^2 + 8(k/R)^2][|a(k)|^2 + |b(k)|^2] +$$
$$4[m_1^2 + 4(k/R)^2]^{1/2}[m_2^2 + 4(k/R)^2]^{1/2}Re[a(k)b(k)^*]\} \qquad (5.60)$$

As explained in Chap. 3, the value of $2k/R$ has the meaning of the standard relative momentum q and, if the coefficients c_n in Eq. (3.45) do not change significantly when n changes by one, this equation results in the standard expression for the mean value of the mass operator $(\Psi, M\Psi) = M_0(q_0) = \epsilon_1(q_0) + \epsilon_2(q_0)$ where $\epsilon_1(q) = \sqrt{m_1^2 + q^2}$, $\epsilon_2(q) = \sqrt{m_2^2 + q^2}$. The effect that the phase of c_n slightly changes when n changes by one results in the correction of order r_0^2/R^2, which is the dS antigravity.

The last term in Eq. (5.60) shows that the result depends on interference between the wave functions in H_1 and H_2. It is not possible to obtain the standard expression for the mean value if Ψ belongs only to H_1 or H_2. Indeed, Eq. (5.60) can be rewritten as

$$(\Psi, M^2\Psi) = \sum_{k=0}^{\infty}\{M_0(2k/R)^2[|a(k)|^2 + |b(k)|^2] -$$
$$2\epsilon_1(2k/R)\epsilon_2(2k/R)|a(k) - b(k)|^2\} \qquad (5.61)$$

This expression clearly demonstrates that the standard result can be obtained only in approximation $a(k) \approx b(k)$. Meanwhile, the results of this chapter give strong indications that the functions $a(k)$ and $b(k)$ might be considerably different.

Consider first the case $j = 0$. In view of the results of Sects. 5.2 and 5.3, we will use $E_1(l)$ to denote the eigenvector of the operator $\mathcal{D}^2$ in H_1 with the eigenvalue η^2 such that $2\eta u_\infty = (l+1/2)\pi$ and $E_2(l)$ to denote the eigenvector of the operator $\mathcal{D}^2$ in H_2 with the eigenvalue η^2 such that $2\eta u_\infty = l\pi$. Then if $\theta(k) = \pi u(k)/u_\infty$, Eq. (5.35) at

$N \to \infty$ can be written as

$$E_1(l) = [\frac{2}{u_\infty}]^{1/2} \sum_{k=0}^{\infty} \frac{cos[(l+1/2)\theta(k)]e_{2k}}{[(w_1+16k^2)(w_2+16k^2)]^{1/4}}$$
$$E_2(l) = [\frac{2}{u_\infty}]^{1/2} \sum_{k=0}^{\infty} \frac{sin[l\theta(k)]e_{2k+1}}{[(w_1+16k^2)(w_2+16k^2)]^{1/4}} \qquad (5.62)$$

If $\Psi = \sum_l [A(l)E_1(l) + B(l)E_2(l)]$ then the functions $A(l)$ and $B(l)$ can be called the coordinate wave functions in H_1 and H_2, respectively, while $a(k)$ and $b(k)$ can be called the momentum wave functions in H_1 and H_2, respectively. It is clear that there exists a one to one correspondence between the functions $A(l)$ and $a(k)$ and, analogously, a one to one correspondence between the functions $B(l)$ and $b(k)$.

As noted in Sect. 3.4, in view of the wave function reduction principle, it is natural to believe that the coordinate wave function of a quasiclassical state has a finite supporter. For this reason we will assume that the functions $A(l)$ and $B(l)$ are not equal to zero only if $l \in [L_1, L_2 - 1]$. Then as follows from Eq. (5.62)

$$a(k) = \frac{[2/u_\infty]^{1/2}}{[(w_1+16k^2)(w_2+16k^2)]^{1/4}} \sum_{l=L_1}^{L_2-1} A(l)cos[(l+1/2)\theta(k)]$$
$$b(k) = \frac{[2/u_\infty]^{1/2}}{[(w_1+16k^2)(w_2+16k^2)]^{1/4}} \sum_{l=L_1}^{L_2-1} B(l)sin[l\theta(k)] \qquad (5.63)$$

These expressions are analogs of the relations between the coordinate and momentum wave functions in the standard theory. It is clear from Eq. (5.63) that, in contrast with the standard theory, it is not possible to obtain an arbitrary falloff of the momentum wave function at $k \to \infty$ since $\theta(k) \to \pi$ when $k \to \infty$. As follows from Eqs. (5.27) and (5.63), when $k \to \infty$, $a(k) = O(1/k^2)$ and $b(k) = O(1/k^2)$.

In view of the above remarks, it is important to understand for which coordinate wave functions the momentum wave functions $a(k)$ and $b(k)$ are as close to each other as possible. In the standard theory the coordinate wave function should be differentiable and therefore it is natural to assume that the functions $A(l)$ and $B(l)$ should not significantly differ from each other. As an analog of the wave function (3.64), we consider

$$A(l) = \frac{exp[i(l+1/2)\theta(k_0)]}{[2(L_2-L_1)]^{1/2}} \quad B(l) = -i\frac{exp[il\theta(k_0)]}{[2(L_2-L_1)]^{1/2}} \tag{5.64}$$

if $l \in [L_1, L_2-1]$ and $A(l) = B(l) = 0$ otherwise. Then the wave function is normalized to one and we will see below that such a choice of the phase factors is needed to obtain as close dependence of $a(k)$ and $b(k)$ on k as possible. The meaning of k_0 is that $2k_0/R = q_0$ where q_0 is the classical relative momentum of the two-body system.

As follows from Eqs. (5.63) and (5.64)

$$a(k) = \frac{1}{2}[u_\infty(L_2-L_1)]^{-1/2}\frac{exp[\frac{i}{2}(L_1+L_2)\theta(k_0)]}{[(w_1+16k^2)(w_2+16k^2)]^{1/4}}$$
$$\{exp[\frac{-i}{2}(L_1+L_2)\theta(k)]\frac{sin[\frac{L_2-L_1}{2}(\theta(k)-\theta(k_0))]}{sin[\frac{1}{2}(\theta(k)-\theta(k_0))]} +$$
$$\{exp[\frac{i}{2}(L_1+L_2)\theta(k)]\frac{sin[\frac{L_2-L_1}{2}(\theta(k)+\theta(k_0))]}{sin[\frac{1}{2}(\theta(k)+\theta(k_0))]}\}$$
$$b(k) = \frac{1}{2}[u_\infty(L_2-L_1)]^{-1/2}\frac{exp[\frac{i}{2}(L_1+L_2-1)\theta(k_0)]}{[(w_1+16k^2)(w_2+16k^2)]^{-1/4}}$$
$$\{exp[\frac{-i}{2}(L_1+L_2-1)\theta(k)]\frac{sin[\frac{L_2-L_1}{2}(\theta(k)-\theta(k_0))]}{sin[\frac{1}{2}(\theta(k)-\theta(k_0))]} -$$
$$\{exp[\frac{i}{2}(L_1+L_2-1)\theta(k)]\frac{sin[\frac{L_2-L_1}{2}(\theta(k)+\theta(k_0))]}{sin[\frac{1}{2}(\theta(k)+\theta(k_0))]}\} \tag{5.65}$$

Since our coordinate wave function is normalized to 1/2 in H_1 and H_2 (and the total probability equals 1) it is clear that

$$\sum_{k=0}^{\infty}|a(k)|^2 = \sum_{k=0}^{\infty}|b(k)|^2 = 1/2$$

by construction.

Consider the normalization of $a(k)$ if only the first term in Eq. (5.65) is taken into account. By using Eq. (5.28) and the definition of the variable θ, we can replace summation over k by integration over θ:

$$\sum_{k=0}^{\infty}|a(k)|^2 = \frac{1}{4\pi(L_2-L_1)}\int_0^{\pi}|\frac{sin[\frac{L_2-L_1}{2}(\theta-\theta_0)]}{sin[\frac{1}{2}(\theta-\theta_0)]}|^2 d\theta \qquad (5.66)$$

where $\theta_0=\theta(k_0)$. Let us assume that only a small vicinity of θ near θ_0 gives an important contribution. Then we can replace $sin[(\theta-\theta_0)/2]$ by $(\theta-\theta_0)/2$ and formally integrate over θ from $-\infty$ to ∞. Then by using Eq. (3.66) we obtain that indeed $a(k)$ is normalized to 1/2. This result justifies our assumption and shows that the main contribution is given by the values of θ such that $|\theta-\theta_0|\leq 1/(L_2-L_1)$. If η is of order $\mu_1\mu_2 r/R$ and the masses are of order m then l is of order mr and L_2-L_1 is of order $m\Delta r$ where Δr is the uncertainty of r. For macroscopic bodies this value is very large.

A simple explanation that the main contribution is given by $|\theta-\theta_0|\leq 1/(L_2-L_1)$ is as follows. At such values of θ, $|sin[(L_2-L_1)(\theta-\theta_0)/2]/sin[(\theta-\theta_0)/2]|^2$ is of order $(L_2-L_1)^2$, the factor $1/(L_2-L_1)$ arises from the fact that $|\theta-\theta_0|\leq 1/(L_2-L_1)$ and therefore the overall contribution is of order unity. The contribution of such values of θ where $sin[(L_2-L_1)(\theta-\theta_0)/2]$ cannot be replaced by $(L_2-L_1)(\theta-\theta_0)/2$ is order $1/(L_2-L_1)$ since the integral is of order unity. This explanation also makes it clear that for macroscopic bodies the main contributions to $a(k)$ and $b(k)$ are given by the first terms in the corresponding expressions in Eq. (5.65) and therefore $a(k)\approx b(k)$ as expected. In addition, it follows from Eq. (5.27) that at least in the nonrelativistic approximation the condition $|\theta-\theta_0|\leq 1/(L_2-L_1)$ is equivalent to the standard uncertainty relation $\Delta q\Delta r\geq 1$.

Consider now whether the mean value of the mass operator is close to $M(q_0)$ where $q_0=2k_0/R$. It is clear from the above discussion that the contribution of $k's$ not close to k_0 is suppressed by a factor $1/(L_2-L_1)$. The question arises whether the extent of suppression is sufficient to exclude a contribution of large values of k. As already noted, when k is large, $a(k)=O(1/k^2)$ and $b(k)=O(1/k^2)$. Therefore the sum for the mean value of the mass operator converges, in contrast with the situation discussed in Sect. 3.4. However, one can notice that each of two terms for $a(k)$ and $b(k)$ in Eq. (5.65) behaves as $O(1/k)$. The terms with $O(1/k)$ cancel out only at $k \geq (L_2 - L_1)/u_\infty$. This is an extremely large value. For example, if m_1 and m_2 are of the same order m then $(L_2 - L_1)/u_\infty$ is of order $mR(L_2 - L_1)$. Therefore the mean value of the mass operator is anomalously large.

As noted in the preceding section, for quasiclassical particles the value $j = 0$ is not realistic and in practice the value of j is very large. It has been also noted that at large values of k the eigenstates of the operator $\mathbf{D}^2$ drop as $O(k^{-(j+2)})$ beginning from k's of order $(m_1 + m_2)/v_0$. Therefore in this case the factor $1/(L_2 - L_1)$ is sufficent to suppress the contribution of large k's. At the same time, a problem with an anomalously large contribution from small values of k arises. Let us estimate this contribution. Suppose that j is even. Then, as follows from Eqs. (5.56) and (5.57), the contribution of small k's to the eigenvectors of the operator $\mathbf{D}^2$ can be written as

$$E_1(l) = const \sum_k [(l+1/2)\theta(k)]^{1/2} N_{(j+1/2)}((l+1/2)\theta(k))e_{2k}$$
$$E_2(l) = const \sum_k [l\theta(k)]^{1/2} J_{(j+1/2)}(l\theta(k))e_{2k+1} \quad (5.67)$$

where $N_{(j+1/2)} = (-1)^{j+1} J_{-(j+1/2)}$ is the Bessel function of the second kind (the Neumann function). As an analog of Eq. (5.64) for the coordinate wave functions one can take

$$A(l) = const[(l+1/2)\theta(k_0)]^{1/2} H^{(1)}_{(j+1/2)}((l+1/2)\theta(k_0))$$
$$B(l) = const[l\theta(k_0)]^{1/2} H^{(1)}_{(j+1/2)}(l\theta(k_0)) \quad (5.68)$$

if $l \in [L_1, L_2 - 1]$, where $H^{(1)}_{(j+1/2)}$ is the Bessel function of the third kind (the Hankel function). Indeed, in situations when for the Hankel functions one can take their asymptotic expressions [47], Eq. (5.68) is compatible with Eq. (5.64). As follows from Eqs. (5.67) and (5.68),

$$a(k) = const[\theta(k)\theta(k_0)]^{1/2}\sum_{l=L_1}^{L_2-1}(l+1/2)N_{(j+1/2)}((l+1/2)\theta(k))$$
$$H^{(1)}_{(j+1/2)}((l+1/2)\theta(k_0))$$
$$b(k) = const[\theta(k)\theta(k_0)]^{1/2}\sum_{l=L_1}^{L_2-1} lJ_{(j+1/2)}(l\theta(k))$$
$$H^{(1)}_{(j+1/2)}(l\theta(k_0)) \tag{5.69}$$

Suppose that the sums in these expressions can be replaced by integrals. Then by using the formulas for the integrals involving Bessel functions of different kinds [47] we obtain

$$a(k) = const\frac{[\theta(k)\theta(k_0)]^{1/2}}{\theta(k)^2-\theta(k_0)^2}$$
$$[(l+1/2)\theta(k)N_{(j+3/2)}((l+1/2)\theta(k))H^{(1)}_{(j+1/2)}((l+1/2)\theta(k_0)) -$$
$$(l+1/2)\theta(k_0)N_{(j+1/2)}((l+1/2)\theta(k))H^{(1)}_{(j+3/2)}((l+1/2)\theta(k_0))]_{l=L_1}^{l=L_2-1}$$
$$b(k) = const\frac{[\theta(k)\theta(k_0)]^{1/2}}{\theta(k)^2-\theta(k_0)^2}[l\theta(k)J_{(j+3/2)}(l\theta(k))H^{(1)}_{(j+1/2)}(l\theta(k_0)) -$$
$$l\theta(k_0)J_{(j+1/2)}(l\theta(k))H^{(1)}_{(j+3/2)}(l\theta(k_0))]_{l=L_1}^{l=L_2-1} \tag{5.70}$$

In situations when the Bessel functions can be replaced by their asymptotic expressions, these expressions are compatible with Eq. (5.65). However, the expression for $a(k)$ is formally divergent when $k \to 0$ and it is not possible to avoid the divergency if the coordinate wave function has a finite supporter. As noted in the preceding section, in fact there is no infinite contribution to $a(k)$ since the continuous approximation is not valid when $k \leq j$.

Let us summarize the discussion in this chapter. Since the internal two-body space H can be represented as a direct sum $H = H_1 + H_2$ such that H_1 and H_2 are invariant subspaces of the operator $\mathbf{D}^2$, the internal two-body state can be represented as a direct sum of the states in H_1 and H_2. If $a(k)$ and $b(k)$ are the momentum wave functions in H_1 and H_2, respectively, then the standard expression for the mean value of the mass operator can be obtained only in the approximation $a(k) \approx b(k)$. At the same time, the above discussion shows that the behavior of $a(k)$ and $b(k)$ at small values of k is essentially different. As argued in the preceding section, one might expect that the continuous approximation is not sufficient at small k's and the contribution of the region $k \leq j$ might be very important. As noted in the preceding section, there is no analog of such a contribution in the standard theory. The investigation of the contribution of small k's requires further study but in any case it seems extremely unlikely that $a(k) \approx b(k)$ and thus the contribution of the last term in Eq. (5.61) is negligible. Any nonnegligible contribution to the mean value of the mass operator implies that on classical level there exists an effective interaction. If one accepts that important effective interaction does arise then the only reasonable assumption is that this effective interaction is just gravity.

Chapter 6

Discussion

We postulate that on quantum level the dS invariance implies that the expressions in Eq. (2.2) are valid. They do not contain any free parameters and, in particular, they do not contain the cosmological constant. Therefore in such a formulation the cosmological constant problem does not arise. We argue in Chap. 2 that the cosmological constant arises only if one wishes to describe the results in terms of Poincare invariant theories or classical dS space. We also mention the discussion about fundamental constants in Ref. [30] and give additional arguments in favor of the statement of the first author that a fundamental physical theory should not contain dimensionful constants at all. In particular, we argue that the gravitational and cosmological constants are not fundamental and that there are no physical arguments in favor of an approach where Poincare invariance is treated as fundamental and de Sitter invariance - as a result of nonzero vacuum energy in the theory with the Poincare background.

In Chap. 2 we also discuss in detail the well known example of de Sitter antigravity. Although the de Sitter antigravity has been discussed by numerous authors, it seems rather strange that important features of this phenomenon have not been discussed. Although on classical level the dS antigravity means a repulsion between particles in the dS space, a problem arises whether the dS antigravity can be treated as an interaction at all. The phenomenon of the dS antigravity does not obey the rule of QFT that any interaction is a result of exchange of some virtual particles. The classical dS antigravity is even more universal than gravity since the relative acceleration (or rather retardation) of particles in the dS space does not depend on their masses and depends only on the radius of the de Sitter space R. However, the fact that the acceleration is proportional to $1/R^2$ does not mean that $1/R^2$ can be treated as an interaction constant. The dS antigravity is simply a consequence of dS invariance and it can be called an interaction only if we accept by definition that relative acceleration implies interaction. However, such a notion about interactions is based only on our experience in Poincare invariant theories.

The phenomenon of the dS antigravity also poses the problem whether our understanding of other interactions should be revisited. Indeed, at first glance the dS antigravity might be important only at cosmological distances and therefore it is not important for quantum physics. However, consider, for example, any interacting two-body system in dS theory. Since at large distances the dS antigravity is much greater than all the other known interactions, the dS Hamiltonian cannot contain bound states, its spectrum is the same as the spectrum of the free Hamiltonian and the free and interacting Hamiltonians are unitarily equivalent [27]. In Poincare invariant theory it is also a possible situation when the bound states are absent and therefore the free and interacting Hamiltonians are unitarily equivalent. However, in this case the criterion of interaction is whether the S matrix is an identity operator or not. In dS theory there is no S matrix (or the notion of the S matrix can be only approximate) and therefore the problem arises whether unitarily equivalent representations are equivalent physically.

Also a question arises whether, by analogy with the dS antigravity, all interactions in nature manifest themselves only as interactions in Poincare terms while in fact they are simply a consequence of a higher symmetry. We believe that gravity might be the first candidate on the role of such an "interaction". The phenomenon of gravity is known only on macroscopic level at nonrelativistic momenta and large distances. So is very natural to consider two-body quasiclassical wave functions in dS theory, trying to understand whether gravity might be simply a consequence of de Sitter invariance. Although such a problem statement seems to be extremely simple and natural, we are not aware of the literature where such a possibility has been discussed.

In Chap. 3 we note that although quasiclassical approximation in standard quantum mechanics has been discussed in a wide literature, it is not quite clear yet what are typical quasiclassical wave functions for macroscopic bodies. A possible definition of quasiclassical states might be as follows. Let A be an operator of a physical quantity and a system is in a state where the mean value of A is A_0 and the uncertainty of A is ΔA. Then the system is quasiclassical in A if $|\Delta A| \ll |A_0|$. The system is typically meant to be quasiclassical if it is simultaneously quasiclassical in all the coordinates, momenta and energy. A typical example when this is the case is when the system wave function is Gaussian. Since the Fourier transform of the Gaussian wave function is again a Gaussian wave function, such a wave function has a good behavior at infinity in both, the coordinate and momentum representations. The question arises whether Gaussian wave functions are realistic quasiclassical wave functions. Indeed, if we accept the reduction wave function postulate in the framework of the Copenhagen formulation then after each measurement of the coordinate the wave function is not equal to zero only in a small range of coordinates.

Consider, for example, the Sun - Earth system. We know that the relative wave function in the coordinate space has a very sharp maximum at $r \approx 150 \cdot 10^6 km$. There is no doubt that the probability to find the Earth on the Venus or Mars orbits is extremely small. But is this probability exactly zero? One might think that this question is of academic interest only. However, if we accept the Copenhagen formulation, then after each measurement of the Sun - Earth distance, the wave function of the Sun - Earth system is not equal to zero only in a very small vicinity of $r \approx 150 \cdot 10^6 km$. In the Copenhagen formulation, the measurement is treated as an interaction with a classical object. So it is not quite clear how often the relative distance in the Sun - Earth system is measured.

This example indicates that realistic wave functions of macroscopic bodies might have only a small finite supporter. But then, as noted in Sect. 3.4, it is not obvious what types of wave functions ensure that the system is quasiclassical in all the coordinates, momenta and energy. In particular, it is not difficult to construct wave functions where uncertainties of coordinates and momenta are such that $\Delta r \Delta p$ is of order $\hbar$, but we should also guarantee that such wave functions are quasiclassical in energy.

In dS theory the operators of de Sitter energy and de Sitter momenta do not commute with each other. Therefore there is the energy-momentum uncertainty relation and the problem arises when a system is simultaneously quasiclassical in energy and momentum. This problem is discussed in Chap. 4. We derive a relation (4.38), which shows that a system can be simultaneously quasiclassical in dS energy and momentum only if the cosmological constant is not anomalously small. We also estimate the quantities entering this expression for the case of the famous Cavendish experiment. At the same time, our conclusions are based on the assumption that the coordinate dependence of quasiclassical wave functions in dS theory is similar to the standard one. To understand whether this is the case, one needs to explicitly construct the relative distance operator in dS theory.

Such a construction is carried out in Chap. 5, which contains the main results of the paper. It is shown that when the relative momentum is not asymptotically small or asymptotically large, our standard intuition works. However, the behavior of eigenstates at very large and very small momenta considerably differs from the standard one. At large momenta the falloff of the eigenstates of the relative distance operator is much faster than in standard theory. The modern approaches to gravity assume that on quantum level gravity manifests itself at Planck distances, which are associated with very large momenta. On the other hand, as already noted, the phenomenon of gravity has been observed so far only at nonrelativistic momenta and large distances. So the question arises whether indeed quantum gravity is sought where it really is. Since the approaches based on QFT, string theory, loop quantum gravity etc. are now dominant, this question usually is not posed. On the other hand, our result poses the question whether the region of large momenta is indeed relevant for explaining quantum gravity. Let us stress that we consider only two-body de Sitter kinematics and no dynamics is assumed. The only uncertainty in our consideration is whether our relative distance operator is indeed the physical relative distance operator. Although we argue that it is, a problem arises what is the uncertainty in choosing the form of the relative distance operator in dS theory. In any case our result seems to be more natural than the standard assumption.

Our calculations also show that an anomalously large (but finite) contribution arises from the region of extremely small momenta. There is no analog of such a contribution in the standard theory. The region in question is such that the de Sitter momentum n, which is dimensionless, is less or smaller than the spin of the two-body system j. Indeed, the quantity n is related to the standard relative momentum q as $n = qR$, where R is the radius of the de Sitter world. Since the standard theory corresponds to the formal limit $R \to \infty$, the relation $n \leq j$ in the standard theory is impossible. In orther words, the contribution of the region $n \leq j$ is possible only in the case when the cosmological constant is finite. It is clear that in this region the standard intuition does not work. As noted in Sect. 5.3, the contribution of this region might be large since the decomposition coefficients describing the eigenstates of the relative distance operator rise exponentially when n is rising but is still in the region $n \leq j$. Since typically small momenta are related to large distances, these results is an additional argument (see the above discussion) that the phenomenon of the dS antigravity is important not only at cosmological distances.

In Sect. 5.4 we argue that the existence of anomalously large contribution from the region of small momenta implies that some effective interaction arises. The nature of this interaction has to be investigated but a natural assumption is that it is just gravity. In other words, the dS antigravity and gravity might be only different aspects of dS invariance. The problem of investigating the region of small momenta is that here the continuous approximation does not work and discreteness of the de Sitter momentum becomes important. The fact that the decomposition coefficients rise exponentially and become huge numbers might be an indication that gravity might be explained as a manifestation of the fact that the ultimate quantum theory is in fact based not on the field of complex numbers but on a Galois field. This possibility has been proposed in our Refs. [34, 26, 35] and a further study is required.

We would like to conclude the paper with the following remark. Although quantum theory and de Sitter invariance are already known for many years, such a seemingly simple problem as de Sitter invariant quantum mechanics of the free two-body system is far from being understood. The above paper gives a strong indication that the investigation of this problem might considerably change our understanding of quantum theory.

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