Spacetime and fields

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1 Spacetime

Einstein's principle of general covariance states that all physical laws do not change their form (are covariant) under continuous coordinate transformations in four-dimensional spacetime.

1.1 Tensors

1.1.1 Vectors

Consider a coordinate transformation from old (unprimed) to new (primed) *coordinates* in a four-dimensional manifold:

$$x^i \to x'^j(x^i), \tag{1.1.1}$$

where x'^j are differentiable and nondegenerate functions of x^i and the index i can be 0,1,2,3. Thus the matrix $\frac{\partial x'^j}{\partial x^i}$ has the nonzero determinant $|\frac{\partial x'^j}{\partial x^i}| \neq 0$, so x^i are differentiable and nondegenerate functions of x'^j. The matrix $\frac{\partial x^i}{\partial x'^j}$ is the inverse of $\frac{\partial x'^j}{\partial x^i}$:

$$\sum_i \frac{\partial x'^i}{\partial x^j}\frac{\partial x^k}{\partial x'^i} = \delta^k_j, \tag{1.1.2}$$

where

$$\delta^i_k = \left\{ \begin{array}{cc} 1 & i = k \\ 0 & i \neq k \end{array} \right\}. \tag{1.1.3}$$

The differentials and derivatives transform according to

$$dx'^j = \frac{\partial x'^j}{\partial x^i}dx^i, \tag{1.1.4}$$

$$\frac{\partial}{\partial x' j} = \frac{\partial x^i}{\partial x'^j}\frac{\partial}{\partial x^i}. \tag{1.1.5}$$

A *scalar* (invariant) is defined as a quantity that does not change:

$$\phi' = \phi. \tag{1.1.6}$$

A *contravariant vector* is defined as a quantity that transform like a differential:

$$A'^j = \frac{\partial x'^j}{\partial x^i}A^i. \tag{1.1.7}$$

A *covariant vector* is defined as a quantity that transforms like a derivative:

$$B_{'j} = \frac{\partial x^i}{\partial x'^j}B_i. \tag{1.1.8}$$

Therefore a derivative of a scalar is a covariant vector. The coordinates x^i do not form a vector.

1.1.2 Tensors

A product of several vectors transforms such that each coordinate index transforms separately:

$$A'^i B'^j \dots C_{'k} D_{'l} \dots = \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\frac{\partial x^p}{\partial x'^k}\frac{\partial x^q}{\partial x'^l} A^m B^n \dots C_p D_q \dots. \tag{1.1.9}$$

A *tensor* is defined as a quantity that transforms like a product of vectors:

$$T'^{ij\dots}_{kl\dots} = \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\frac{\partial x^p}{\partial x'^k}\frac{\partial x^q}{\partial x'^l} T'^{mn\dots}_{pq\dots}. \tag{1.1.10}$$

A tensor is of rank (k,l) if it has k contravariant and l covariant indices. A scalar is a tensor of rank (0,0), a contravariant vector is a tensor of rank (1,0), and a covariant vector is a tensor of rank (0,1). A linear combination of two tensors of rank (k,l) is a tensor of rank (k,l). The product of two tensors of ranks (k_1,l_1) and (k_2,l_2) is a tensor of rank (k_1+k_2, l_1+l_2). Tensor indices (all contravariant or all covariant) can be *symmetrized*:

$$T_{(ij\dots k)} = \frac{1}{n!}\sum_{\text{permutations}} T_{\{ij\dots k\}}, \tag{1.1.11}$$

or *antisymmetrized*:

$$T_{[ij\dots k]} = \frac{1}{n!}\sum_{\text{permutations}} T_{\{ij\dots k\}}(-1)^m, \tag{1.1.12}$$

where n is the number of symmetrized or antisymmetrized indices and m is the number of permutations that bring $T_{ij\dots k}$ into $T_{\{ij\dots k\}}$. For example, for two indices: $T_{(ik)} = \frac{1}{2}(T_{ik}+T_{ki})$ and $T_{[ik]} = \frac{1}{2}(T_{ik}-T_{ki})$, and for three indices: $T_{[ijk]} = \frac{1}{3}(T_{ijk} + T_{jki} + T_{kij})$. If $n > 4$ then $T_{[ij\dots k]} = 0$. Symmetrized and antisymmetrized tensors or rank (k,l) are tensors of rank (k,l). Symmetrization of an antisymmetric tensor or antisymmetrization of a symmetric tensor bring these tensors to zero. Any tensor of rank (0,2) is the sum of its symmetric and antisymmetric part,

$$T_{(ik)} + T_{[ik]} = T_{ik}. \tag{1.1.13}$$

The number 0 can be regarded as a tensor of arbitrary rank. Therefore all covariant equations of classical physics must be represented in the tensor form: $T^{ij\dots}_{kl\dots} = 0$.

1.1.3 Densities

The element of volume in four-dimensional spacetime transforms according to

$$d^4x' = \left|\frac{\partial x'^i}{\partial x^k}\right| d^4x. \tag{1.1.14}$$

A *scalar density* is defined as a quantity that transforms such that its product with the element of volume is a scalar, $\mathfrak{l}' d^4x' = \mathfrak{l} d^4x$:

$$\mathfrak{l}' = \left|\frac{\partial x^i}{\partial x'^k}\right| \mathfrak{l}. \tag{1.1.15}$$

A *tensor density*, which includes a contravariant and covariant vector density, is defined as a quantity that transforms like a product of a tensor and a scalar density:

$$\mathfrak{T}'^{ij\ldots}_{kl\ldots} = \left|\frac{\partial x^i}{\partial x'^k}\right| \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\frac{\partial x^p}{\partial x'^k}\frac{\partial x^q}{\partial x'^l}\mathfrak{T}'^{mn\ldots}_{pq\ldots}. \tag{1.1.16}$$

For example, the square root of the determinant of a tensor of rank $(0,2)$ is a scalar density of weight 1:

$$\sqrt{|T'_{ik}|} = \sqrt{|\frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}T_{lm}|} = \sqrt{|\frac{\partial x^j}{\partial x'^n}|^2|T_{ik}} = |\frac{\partial x^j}{\partial x'^n}|\sqrt{|T_{ik}|}. \tag{1.1.17}$$

The above densities are said to be of weight 1. One can generalize this definition of densities by introducing densitites of weight w, which transform like normal densities except that $|\frac{\partial x^i}{\partial x'^k}|$ is replaced by $|\frac{\partial x^i}{\partial x'^k}|^w$. For example, d^4x is a scalar density of weight -1. A linear combination of two densities of weight w is a density of weight w. The product of two densities of weights w_1 and w_2 is a density of weight w_1+w_2. Symmetrized and antisymmetrized densities of weight w are densities of weight w. Densities of weight 1 are simply referred to as densities. Tensors are densities of weight 0.

1.1.4 Contraction

We adopt Einstein's convention: if the same coordinate index i appears twice (as a contravariant index and covariant index) then we perform the summation $\sum_i$ over a given tensor or density. Such a tensor or density is said to be *contracted* over index i. A contracted tensor of rank (k,l) transforms like a tensor of rank $(k-1,l-1)$:

$$T'^{ij\ldots}_{il\ldots} = \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\frac{\partial x^p}{\partial x'^i}\frac{\partial x^q}{\partial x'^l}T'^{mn\ldots}_{pq\ldots} = \frac{\partial x'^j}{\partial x^n}\frac{\partial x^q}{\partial x'^l}\delta^p_m T'^{mn\ldots}_{pq\ldots} = \frac{\partial x'^j}{\partial x^n}\frac{\partial x^q}{\partial x'^l}T'^{mn\ldots}_{mq\ldots}. \tag{1.1.18}$$

For example, the contraction of a contravariant and covariant vector A^iB_i is a scalar (*scalar product*). A contracted tensor density of rank (k,l) and weight w transforms like a tensor density of rank $(k-1,l-1)$ and weight w:

$$\begin{aligned}\mathfrak{T}'^{ij\ldots}_{il\ldots} &= \left|\frac{\partial x^i}{\partial x'^k}\right|^w \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\frac{\partial x^p}{\partial x'^i}\frac{\partial x^q}{\partial x'^l}\mathfrak{T}'^{mn\ldots}_{pq\ldots} = \left|\frac{\partial x^i}{\partial x'^k}\right|^w \frac{\partial x'^j}{\partial x^n}\frac{\partial x^q}{\partial x'^l}\delta^p_m\mathfrak{T}'^{mn\ldots}_{pq\ldots} \\ &= \left|\frac{\partial x^i}{\partial x'^k}\right|^w \frac{\partial x'^j}{\partial x^n}\frac{\partial x^q}{\partial x'^l}\mathfrak{T}'^{mn\ldots}_{mq\ldots}.\end{aligned} \tag{1.1.19}$$

Contraction of a symmetric tensor with an antisymmetric tensor (over indices with respect to which these tensors are symmetric or antisymmetric) gives zero. If contraction of two tensors gives zero, these tensors are said to be *orthogonal*. Two orthogonal vectors (one contravariant and one covariant) are said to be *perpendicular*.

1.1.5 Kronecker and Levi-Civita symbols

The *Kronecker symbol* δ^i_k (1.1.3) is a tensor with constant components:

$$\delta'^i_k = \frac{\partial x'^i}{\partial x^j}\frac{\partial x^l}{\partial x'^k}\delta^j_l = \frac{\partial x'^i}{\partial x^j}\frac{\partial x^j}{\partial x'^k} = \delta^i_k. \tag{1.1.20}$$

A totally antisymmetric tensor of rank $(4,0)$, $T^{ijkl} = T^{[ijkl]}$ has 1 independent component T: $T^{ijkl} = T\epsilon^{ijkl}$, where ϵ^{ijkl} is the totally antisymmetric, contravariant permutation *Levi-Civita symbol*:

$$\epsilon^{0123} = 1, \;\; \epsilon^{ijkl} = (-1)^m, \tag{1.1.21}$$

and m is the number of permutations that bring ϵ^{ijkl} into ϵ^{0123}. The determinant of a matrix S^i_k is defined through the permutation symbol as

$$|S^r_s|\epsilon^{ijkl} = S^i_m S^j_n S^k_p S^l_q \epsilon^{mnpq}. \tag{1.1.22}$$

Taking $S^i_k = \frac{\partial x'^i}{\partial k}$ gives

$$\epsilon^{ijkl} = \left|\frac{\partial x^r}{\partial x'^s}\right| \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\frac{\partial x'^k}{\partial x^p}\frac{\partial x'^l}{\partial x^q}\epsilon^{mnpq}. \tag{1.1.23}$$

This equation looks like a transformation law for a tensor density with constant components: $\epsilon'^{ijkl} = \epsilon^{ijkl}$. Accordingly, T is a scalar density of weight -1. We also introduce the covariant Levi-Civita symbol ϵ_{ijkl} through:

$$\epsilon^{ijkl}\epsilon_{mnpq} = -\begin{vmatrix} \delta^i_m & \delta^i_n & \delta^i_p & \delta^i_q \\ \delta^j_m & \delta^j_n & \delta^j_p & \delta^j_q \\ \delta^k_m & \delta^k_n & \delta^k_p & \delta^k_q \\ \delta^l_m & \delta^l_n & \delta^l_p & \delta^l_q \end{vmatrix}. \tag{1.1.24}$$

Thus the covariant Levi-Civita symbol is a tensor density of weight -1 and its product with a scalar density is a tensor. The covariant Levi-Civita symbol is given by

$$\epsilon_{0123} = -1, \;\; \epsilon_{ijkl} = (-1)^m, \tag{1.1.25}$$

where m is the number of permutations that bring ϵ_{ijkl} into ϵ_{0123}, and satisfies

$$|S^r_s|\epsilon_{ijkl} = S^m_i S^n_j S^p_k S^q_l \epsilon_{mnpq}. \tag{1.1.26}$$

Contracting (1.1.24) gives the following relations:

$$\begin{aligned} \epsilon^{ijkl}\epsilon_{mnpl} &= -\begin{vmatrix} \delta^i_m & \delta^i_n & \delta^i_p \\ \delta^j_m & \delta^j_n & \delta^j_p \\ \delta^k_m & \delta^k_n & \delta^k_p \end{vmatrix}, \\ \epsilon^{ijkl}\epsilon_{mnkl} &= -2(\delta^i_m\delta^j_n - \delta^i_n\delta^j_m), \\ \epsilon^{ijkl}\epsilon_{mjkl} &= -6\delta^i_m, \\ \epsilon^{ijkl}\epsilon_{ijkl} &= -24. \end{aligned} \tag{1.1.27}$$

1.1.6 Dual densities

A contracted product of a covariant tensor and the contravariant Levi-Civita symbol gives a *dual* contravariant tensor density:

$$\epsilon^{iklm}A_m = \mathfrak{A}^{ikl}, \;\; \epsilon^{iklm}B_{lm} = \mathfrak{B}^{ik}, \;\; \epsilon^{iklm}C_{klm} = \mathfrak{C}^{i}. \tag{1.1.28}$$

A contracted product of a contravariant tensor and the covariant Levi-Civita symbol gives a dual covariant tensor density:

$$\epsilon_{iklm}A^m = \mathfrak{A}_{ikl}, \;\; \epsilon_{iklm}B^{lm} = \mathfrak{B}_{ik}, \;\; \epsilon_{iklm}C^{klm} = \mathfrak{C}_{i}. \tag{1.1.29}$$

Therefore there exists an algebraic correspondence between covariant tensors and contravariant densities, and between contravariant tensors and covariant densities.

1.1.7 Covariant integrals

A covariant line integral is an integral of a tensor contracted with the line differential dx^i: $\int T^{j\dots}_{i\dots}dx^i$. A covariant surface integral is an integral of a tensor contracted with the surface differential $df^{ik} = dx^i dx'^k - dx^k dx'^i$ (which can be geometrically represented as a parallelogram spanned by the vectors dx^i and dx'^i): $\int T^{j\dots}_{ik\dots}df^{ik}$. A covariant hypersurface (volume) integral is an integral of a tensor contracted with the volume differential $dS^{ikl} = \begin{vmatrix} dx^i & dx'^i & dx''^i \\ dx^k & dx'^k & dx''^k \\ dx^l & dx'^l & dx''^l \end{vmatrix}$ (which can be geometrically represented as a parallelepiped spanned by the vectors dx^i, dx'^i) and dx''^i: $\int T^{j\dots}_{ikl\dots}dS^{ikl}$. A covariant four-volume integral is an integral of a tensor contracted with the four-volume differential dS^{ijkl}, defined analogously to dS^{ikl}. The dual density corresponding to the surface element is given by

$$df^{\star}_{ik} = \frac{1}{2}\epsilon_{lmik}df^{lm}. \tag{1.1.30}$$

The dual density corresponding to the hypersurface element is given by

$$dS_i = \frac{1}{6}\epsilon_{klmi}dS^{klm}. \tag{1.1.31}$$

The dual density corresponding to the four-volume element is given by

$$d\Omega = \frac{1}{24}\epsilon_{iklm}dS^{iklm} = dx^0dx^1dx^2dx^3. \tag{1.1.32}$$

Covariant integrands that include the above dual densities of weight -1 must be multiplied by a scalar density, for example, by the square root of the determinant of a tensor of rank $(0,2)$. According to Gauß' and Stokes' theorems, there exists relations between integrals over different elements:

$$dx^i \leftrightarrow df^{ik}\frac{\partial}{\partial x^k}, \tag{1.1.33}$$

$$df^\star_{ik} \leftrightarrow dS_i\frac{\partial}{\partial x^k} - dS_k\frac{\partial}{\partial x^i}, \tag{1.1.34}$$

$$dS_i \leftrightarrow d\Omega\frac{\partial}{\partial x^i}. \tag{1.1.35}$$

1.1.8 Derivatives

A derivative of a covariant vector does not transform like a tensor:

$$\frac{\partial A'_k}{\partial x'^i} = \frac{\partial x^l}{\partial x'^i}\frac{\partial}{\partial x^l}\Big(\frac{\partial x^m}{\partial x'^k}A_m\Big) = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}\frac{\partial A_m}{\partial x^l} + \frac{\partial^2 x^m}{\partial x'^i \partial x'^k}A_m, \tag{1.1.36}$$

because of the second term which is linear and homogeneous in A_i, unless x^i are linear functions of x'^j. This term is symmetric in the indices i, k so the antisymmetric part of $\frac{\partial A_k}{\partial x^i}$ with respect to these indices is a tensor:

$$\partial_{[i}A'_{k]} = \frac{\partial x^l}{\partial x'^{[i}}\frac{\partial x^m}{\partial x'^{k]}}\partial_l A_m = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}\partial_{[l}A_{m]}, \tag{1.1.37}$$

where we denote $\partial_i = \frac{\partial}{\partial x^i}$. The *curl* of a covariant vector A_i is defined as twice the antisymmetric part of $\partial_i A_k$: $\partial_i A_k - \partial_k A_i$, and is a tensor. We will also use $_{,i} = \frac{\partial}{\partial x^i}$ to denote a partial derivative with respect to x^i. Similarly, totally antisymmetrized derivatives of tensors of rank $(0,2)$ and $(0,3)$, $\partial_{[i}B_{kl]}$ and $\partial_{[i}C_{klm]}$, are tensors. If $B_{kl} = A_{[k,l]}$ then $\partial_{[i}B_{kl]} = 0$, or conversely, if $\partial_{[i}B_{kl]} = 0$ then there exists a vector A_i such that $B_{kl} = A_{[k,l]}$. The *divergence* of a tensor (or density) is a contracted derivative of this tensor (density): $\partial_i T^{\cdot\cdot il\dots}_{\ jk\dots}$. Because of the correspondence between tensors and dual densities, divergences of (totally antisymmetric if more than 1 index) contravariant densities are densities, dual to totally antisymmetrized derivatives of tensors:

$$\partial_i\mathfrak{C}^i = \epsilon^{iklm}\partial_{[i}C_{klm]}, \ \ \partial_k\mathfrak{B}^{ik} = \epsilon^{iklm}\partial_{[k}B_{lm]}, \ \ \partial_l\mathfrak{A}^{ikl} = \epsilon^{iklm}\partial_{[l}A_{m]}. \tag{1.1.38}$$

For example, the equations $\mathfrak{F}^{ik}{}_{,i} = \mathfrak{j}^k$ and $F_{[ik,l]} = 0$, that describe Maxwell's electrodynamics, are tensorial.
References: [1, 2].

1.2 Affine connection

1.2.1 Covariant differentiation of tensors

An ordinary derivative of a covariant vector A_i is not a tensor, because its coordinate transformation law contains an additional noncovariant term, linear and homogeneous in A_i. Consider the expression

$$A_{i;k} = A_{i,k} - \Gamma^{\ l}_{i\,k}A_l, \tag{1.2.1}$$

where the quantity $\Gamma^{\,l}_{i\,k}$ (in the second term which is linear and homogeneous in A_i) transforms such that $A_{i;k}$ is a tensor:

$$A'_{i;k} = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}A_{l;m} = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}(A_{l,m} - \Gamma^{\,n}_{l\,m}A_n). \tag{1.2.2}$$

On the other hand (1.1.36) gives

$$A'_{i;k} = A'_{i,k} - \Gamma'^{\,l}_{i\,k}A'_l = \frac{\partial x^m}{\partial x'^k}\frac{\partial x^l}{\partial x'^i}A_{l,m} + \frac{\partial^2 x^n}{\partial x'^k \partial x'^i}A_n - \frac{\partial x^n}{\partial x'^l}\Gamma'^{\,l}_{i\,k}A_n, \tag{1.2.3}$$

so we obtain

$$\frac{\partial x^n}{\partial x'^l}\Gamma'^{\,l}_{i\,k} = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}\Gamma^{\,n}_{l\,m} + \frac{\partial^2 x^n}{\partial x'^k \partial x'^i}. \tag{1.2.4}$$

Multiplying this equation by $\frac{\partial x'^j}{\partial x^n}$ gives the transformation law for $\Gamma^{\,l}_{i\,k}$:

$$\Gamma'^{\,j}_{i\,k} = \frac{\partial x'^j}{\partial x^n}\frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}\Gamma^{\,n}_{l\,m} + \frac{\partial x'^j}{\partial x^n}\frac{\partial^2 x^n}{\partial x'^k \partial x'^i}. \tag{1.2.5}$$

The algebraic object $\Gamma^{\,l}_{i\,k}$, which equips spacetime in order to covariantize a derivative of a vector, is referred to as the *affine connection*, affinity or simply connection. The connection has generally 64 independent components. The tensor $A_{i;k}$ is the *covariant derivative* of a vector A_i with respect to x^i. We will also use $\nabla_i =_{;i}$ to denote a covariant derivative. The contracted affine connection transforms according to

$$\Gamma'^{\,i}_{i\,k} = \frac{\partial x^m}{\partial x'^k}\Gamma^{\,l}_{l\,m} + \frac{\partial x'^i}{\partial x^n}\frac{\partial^2 x^n}{\partial x'^k \partial x'^i}. \tag{1.2.6}$$

The affine connection is not a tensor because of the second term on the right-hand side of (1.2.5).

A derivative of a scalar is a covariant vector. Therefore a covariant derivative of a scalar is equal to an ordinary derivative:

$$\phi_{;i} = \phi_{,i}. \tag{1.2.7}$$

If we also assume that a covariant derivative of the product of two tensors obeys the same chain rule as an ordinary derivative:

$$(TU)_{;i} = T_{;i}U + TU_{;i}, \tag{1.2.8}$$

then

$$(A_k B^k)_{,i} = (A_k B^k)_{;i} = A_{k;i}B^k + A_k B^k_{\ ;i} = A_{k,i}B^k - \Gamma^{\,k}_{l\,i}A_k B^l + A_k B^k_{\ ;i}. \tag{1.2.9}$$

Therefore we obtain a covariant derivative of a contravariant vector:

$$B^k_{\ ;i} = B^k_{\ ,i} + \Gamma^{\,k}_{l\,i}B^l. \tag{1.2.10}$$

The chain rule (1.2.8) also implies that a covariant derivative of a tensor is equal to the sum of the corresponding ordinary derivative of this tensor and terms with the affine connection that covariantize each index:

$$T^{ij\dots}_{kl\dots;m} = T^{ij\dots}_{kl\dots,m} + \Gamma^{\,i}_{n\,m}T^{nj\dots}_{kl\dots} + \Gamma^{\,j}_{n\,m}T^{in\dots}_{kl\dots} + \dots - \Gamma^{\,n}_{k\,m}T^{ij\dots}_{nl\dots} - \Gamma^{\,n}_{l\,m}T^{ij\dots}_{kn\dots} - \dots . \tag{1.2.11}$$

A covariant derivative of the Kronecker symbol vanishes:

$$\delta^k_{l;i} = \Gamma^{\,k}_{j\,i}\delta^j_l - \Gamma^{\,j}_{l\,i}\delta^k_j = 0. \tag{1.2.12}$$

The second term on the right-hand side of (1.2.5) does not depend on the affine connection, but only on the coordinate transformation. Therefore the difference between two different connections transforms like a tensor of rank (1,2). Consequently, the variation $\delta\Gamma^{\,j}_{i\,k}$, which is an infinitesimal difference between two connections, is a tensor of rank (1,2).

1.2.2 Parallel transport

Consider two infinitesimally separated points in spacetime, $P(x^i)$ and $Q(x^i + dx^i)$, and a vector field A which takes the value A^k at P and $A^k + dA^k$ at Q. Because $dA^k = A^k_{,i}dx^i$ and $A^k_{,i}$ is not a tensor, the difference dA^k is not a vector, which is related to subtracting of two vectors at two points with different coordinate transformation laws. In order to calculate the covariant difference between two vectors at two different points, we must bring these vectors to the same point. Instead of subtracting from the vector $A^k + dA^k$ at Q the vector A^k at P, we must subtract a vector $A^k + \delta A^k$ at Q that corresponds to A^k at P, so the resulting difference (covariant differential) $DA^k = dA^k - \delta A^k$ is a vector. The vector $A^k + \delta A^k$ is the *parallel-transported* or parallel-translated A^k from P to Q. A parallel-transported linear combination of vectors must be equal to the same linear combination of parallel-transported vectors. Therefore δA^k is a linear and homogeneous function of A^k. It is also on the order of a differential, thus a linear and homogeneous function of dx^i. The most general form of δA^k is

$$\delta A^k = -\Gamma^{\,k}_{l\,i}A^l dx^i, \tag{1.2.13}$$

so

$$DA^k = dA^k + \Gamma^{\,k}_{l\,i}A^l dx^i = A^k_{;i}dx^i. \tag{1.2.14}$$

Because δA^k is not a vector, $\Gamma^{\,k}_{l\,i}$ is not a tensor. Because DA^k is a vector, $A^k_{;i}$ is a tensor. The expressions for covariant derivatives of a covariant vector and tensors result from

$$\delta\phi = 0, \;\; \delta(TU) = \delta TU + T\delta U. \tag{1.2.15}$$

1.2.3 Torsion tensor

The second term on the right-hand side of (1.2.5) is symmetric in the indices i, k so the antisymmetric part of the connection with respect to these indices, $S^j_{ik} = \Gamma^{\,\,j}_{[i\,k]}$, is a tensor:

$$S'^j_{ik} = \frac{\partial x'^j}{\partial x^n}\frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}S^n_{lm}. \tag{1.2.16}$$

This tensor is called the Cartan *torsion tensor*. The torsion tensor has generally 24 independent components. The contracted torsion tensor,

$$S^k_{\ ik} = S_i, \tag{1.2.17}$$

is the torsion trace vector.

1.2.4 Covariant differentiation of densities

A derivative of a scalar density of weight w, $\mathfrak{l}$, does not transform like a covariant vector density:

$$\begin{aligned}
\partial_i\mathfrak{l}' &= \frac{\partial x^l}{\partial x'^i}\partial_l\Big(\Big|\frac{\partial x^j}{\partial x'^k}\Big|^w\mathfrak{l}\Big) = \frac{\partial x^l}{\partial x'^i}\Big|\frac{\partial x^j}{\partial x'^k}\Big|^w\partial_l\mathfrak{l} + w\frac{\partial x^l}{\partial x'^i}\Big|\frac{\partial x^j}{\partial x'^k}\Big|^{w-1}\partial_l\Big|\frac{\partial x^r}{\partial x'^s}\Big|\mathfrak{l} \\
&= \frac{\partial x^l}{\partial x'^i}\Big|\frac{\partial x^j}{\partial x'^k}\Big|^w\partial_l\mathfrak{l} + w\frac{\partial x^l}{\partial x'^i}\Big|\frac{\partial x^j}{\partial x'^k}\Big|^{w-1}\Big|\frac{\partial x^r}{\partial x'^s}\Big|\frac{\partial x'^n}{\partial x^m}\frac{\partial}{\partial x^l}\frac{\partial x^m}{\partial x'^n}\mathfrak{l} \\
&= \frac{\partial x^l}{\partial x'^i}\Big|\frac{\partial x^j}{\partial x'^k}\Big|^w\partial_l\mathfrak{l} + w\Big|\frac{\partial x^j}{\partial x'^k}\Big|^w\frac{\partial x'^n}{\partial x^m}\frac{\partial^2 x^m}{\partial x'^n\partial x'^i}\mathfrak{l}.
\end{aligned} \tag{1.2.18}$$

Consider the expression

$$\mathfrak{l}_{;i} = \mathfrak{l}_{,i} - w\Gamma_i\mathfrak{l}, \tag{1.2.19}$$

where the quantity Γ_i transforms such that $\mathfrak{l}_{;i}$ is a vector density of weight w:

$$\mathfrak{l}'_{;i} = \frac{\partial x^l}{\partial x'^i}\Big|\frac{\partial x^j}{\partial x'^k}\Big|^w\mathfrak{l}_{;l} = \frac{\partial x^l}{\partial x'^i}\Big|\frac{\partial x^j}{\partial x'^k}\Big|^w(\mathfrak{l}_{,l} - w\Gamma_l\mathfrak{l}). \tag{1.2.20}$$

On the other hand (1.2.18) gives

$$\mathfrak{l}'_{;i} = \mathfrak{l}'_{,i} - w\Gamma'_i\mathfrak{l}' = \frac{\partial x^l}{\partial x'^i}\Big|\frac{\partial x^j}{\partial x'^k}\Big|^w\partial_l\mathfrak{l} + w\Big|\frac{\partial x^j}{\partial x'^k}\Big|^w\frac{\partial x'^n}{\partial x^m}\frac{\partial^2 x^m}{\partial x'^n\partial x'^i}\mathfrak{l} - w\Big|\frac{\partial x^j}{\partial x'^k}\Big|^w\Gamma'_i\mathfrak{l}, \tag{1.2.21}$$

so we obtain the transformation law for Γ_i:

$$\Gamma'_i = \frac{\partial x^l}{\partial x'^i}\Gamma_l + \frac{\partial x'^n}{\partial x^m}\frac{\partial^2 x^m}{\partial x'^n\partial x'^i}, \tag{1.2.22}$$

which is the same as the transformation law for $\Gamma^{\,k}_{k\,i}$ (1.2.6). Therefore the difference $\Gamma_i - \Gamma^{\,k}_{k\,i}$ is some covariant vector V_i.

If we assume that parallel transport of the product of a scalar density of any weight and a tensor obeys the chain rule:

$$\delta(\mathfrak{l}T) = \delta\mathfrak{l}T + \mathfrak{l}\delta T, \tag{1.2.23}$$

so a covariant derivative of such product behaves like an ordinary derivative:

$$(\mathfrak{l}T)_{;i} = \mathfrak{l}_{;i}T + \mathfrak{l}T_{;i}, \tag{1.2.24}$$

then a covariant derivative of a tensor density of weight w is equal to the sum of the corresponding ordinary derivative of this tensor, terms with the affine connection that covariantize each index, and the term with Γ_i:

$$\begin{aligned}\mathfrak{T}^{ij\dots}_{kl\dots;m} &= \mathfrak{T}^{ij\dots}_{kl\dots,m} + \Gamma^{\,i}_{n\,m}\mathfrak{T}^{nj\dots}_{kl\dots} + \Gamma^{\,j}_{n\,m}\mathfrak{T}^{in\dots}_{kl\dots} + \dots \\ &-\Gamma^{\,n}_{k\,m}\mathfrak{T}^{ij\dots}_{nl\dots} - \Gamma^{\,n}_{l\,m}\mathfrak{T}^{ij\dots}_{kn\dots} - \dots - w\Gamma_m\mathfrak{T}^{ij\dots}_{kl\dots}.\end{aligned} \tag{1.2.25}$$

A covariant derivative of the contravariant Levi-Civita density is

$$\epsilon^{ijkl}_{;m} = \Gamma^{\,i}_{n\,m}\epsilon^{njkl} + \Gamma^{\,j}_{n\,m}\epsilon^{inkl} + \Gamma^{\,k}_{n\,m}\epsilon^{ijnl} + \Gamma^{\,l}_{n\,m}\epsilon^{ijkn} - \Gamma_m\epsilon^{ijkl}. \tag{1.2.26}$$

In the summations over n only one term does not vanish for each term on the right-hand side of (1.2.26), so

$$\begin{aligned}&\epsilon^{ijkl}_{;m} = \Gamma^{\,i}_{n=i|m}\epsilon^{n=i|jkl} + \Gamma^{\,j}_{n=j|m}\epsilon^{i|n=j|kl} + \Gamma^{\,k}_{n=k|m}\epsilon^{ij|n=k|l} + \Gamma^{\,l}_{n=l|m}\epsilon^{ijk|n=l} - \Gamma_m\epsilon^{ijkl} \\ &= (\Gamma^{\,n}_{n\,m} - \Gamma_m)\epsilon^{ijkl} = -V_m\epsilon^{ijkl}.\end{aligned} \tag{1.2.27}$$

The Levi-Civita symbol is a tensor density with constant components, so it does not change under a parallel transport, $\delta\epsilon = 0$. Therefore $\epsilon^{ijkl}_{;m} = 0$, so $V_i = 0$ and

$$\Gamma_i = \Gamma^{\,k}_{k\,i}. \tag{1.2.28}$$

1.2.5 Covariant derivatives

Totally antisymmetrized ordinary derivatives of covariant tensors, $A_{[i;k]}$, $B_{[ik;l]}$ and $C_{[ikl;m]}$, are tensors because of antisymmetrization. Totally antisymmetrized covariant derivatives of tensors are clearly tensors because ∇_i is a covariant operation, and are given by direct calculation using the definition of a covariant derivative:

$$A_{[i;k]} = A_{[i,k]} - S^l_{ik}A_l, \ \ B_{[ik;l]} = B_{[ik,l]} - 2S^m_{[ik}B_{l]m}. \tag{1.2.29}$$

Divergences of (totally antisymmetric if more than 1 index) contravariant densities, $\mathfrak{C}^i_{,i}$, $\mathfrak{B}^{ik}_{,i}$ and $\mathfrak{A}^{ikl}_{,i}$, are densities because of the correspondence between tensors and dual densities. Covariant divergences of contravariant densities are clearly densities, and are given by direct calculation:

$$\mathfrak{C}^i_{;i} = \mathfrak{C}^i_{,i} + 2S_i\mathfrak{C}^i, \ \ \mathfrak{B}^{ik}_{;i} = \mathfrak{B}^{ik}_{,i} - S^k_{il}\mathfrak{B}^{il} + 2S_i\mathfrak{B}^{ik}. \tag{1.2.30}$$

1.2.6 Partial integration

If the product of two quantities (tensors or densities) TU is a contravariant density $\mathfrak{C}^k$ then

$$\int TU_{;k}d\Omega = \int (TU)_{;k}d\Omega - \int T_{;k}Ud\Omega = \int (TU)_{,k}d\Omega + 2\int S_kTUd\Omega - \int T_{;k}Ud\Omega. \tag{1.2.31}$$

The first term on the right-hand side can be transformed into a hypersurface integral $\int TU dS_k$. If the region of integration extends to infinity and $\mathfrak{C}^k$ corresponds to some physical quantity then the boundary integral $\int TU dS_k$ vanishes, giving

$$\int TU_{;k} d\Omega = 2\int S_k TU d\Omega - \int T_{;k} U d\Omega. \tag{1.2.32}$$

If $T = \delta^k_i$ then $U = \mathfrak{C}^i$ and

$$\int \mathfrak{C}^i{}_{;i} d\Omega = 2\int S_i \mathfrak{C}^i d\Omega. \tag{1.2.33}$$

1.2.7 Geodesic frame of reference

Consider a coordinate transformation

$$x^k = x'^k + \frac{1}{2} a^k{}_{lm} x'^l x'^m, \tag{1.2.34}$$

where $a^k{}_{lm}$ is symmetric in the indices l, m. Substituting this transformation to (1.2.5) and calculating it at $x^k = x'^k = 0$ gives

$$\frac{\partial x^i}{\partial x'^k} = \delta^i_k \tag{1.2.35}$$

and

$$\Gamma'^{\,j}_{i\,k} = \Gamma^{\,j}_{i\,k} + a^j{}_{ik}. \tag{1.2.36}$$

Putting

$$a^j{}_{ik} = -\Gamma^{\,j}_{(i\,k)}|_{x^l=0} \tag{1.2.37}$$

gives

$$\Gamma'^{\,j}_{(i\,k)} = 0. \tag{1.2.38}$$

Therefore there always exists a coordinate frame of reference in which the symmetric part of the connection vanishes locally (at one point). If the affine connection is symmetric in the covariant indices, $\Gamma^{\,j}_{i\,k} = \Gamma^{\,j}_{k\,i}$ (the torsion tensor vanishes) then (1.2.38) gives

$$\Gamma'^{\,j}_{i\,k} = 0. \tag{1.2.39}$$

The coordinate frame of reference in which the connection vanishes (locally) is referred to as *geodesic*.

1.2.8 Affine geodesics and four-velocity

Consider a point in spacetime $P(x^k)$ and a vector dx^k at this point. Construct a point $P'(x^k + dx^k)$ and find the vector $d'x^k$ which is the parallel-transported dx^k from P to P'. Then construct a point $P''(x^k + dx^k + d'x^k)$ and find the vector $d''x^k$ which is the parallel-transported $d'x^k$ from P' to P''. The next point is $P'''(x^k + dx^k + d'x^k + d''x^k)$ etc. Repeating this step constructs a polygonal line which in the limit $dx^k \to 0$ becomes a curve such that the vector $\frac{dx^k}{d\lambda}$ (where λ is a parameter along the curve) tangent to it at any point, when parallely translated to another point on this curve, coincides with the tangent vector there. Such curve is referred to as an autoparallel curve or *affine geodesic*. Affine geodesics can be attributed with the concept of length, which, for the polygonal curve, is proportional to the number of parallel-transport steps described above.

The condition that parallel transport of a tangent vector be a tangent vector is

$$\frac{dx^i}{d\lambda}+\delta\Big(\frac{dx^i}{d\lambda}\Big)=\frac{dx^i}{d\lambda}-\Gamma^{\,i}_{k\,l}\frac{dx^k}{d\lambda}dx^l=M\Big(\frac{dx^i}{d\lambda}+\frac{d^2x^i}{d\lambda^2}d\lambda\Big), \tag{1.2.40}$$

where the proportionality factor M is some function of λ, or

$$M\frac{d^2x^i}{d\lambda^2}+\Gamma^{\,i}_{k\,l}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda}=\frac{1-M}{d\lambda}\frac{dx^i}{d\lambda}, \tag{1.2.41}$$

from which it follows that M must differ from 1 by the order of $d\lambda$. In the first term on the left-hand side of (1.2.41) we can therefore put $M=1$, and we denote $1-M$ by $\phi(\lambda)d\lambda$, so

$$\frac{d^2x^i}{d\lambda^2}+\Gamma^{\,i}_{k\,l}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda}=\phi(\lambda)\frac{dx^i}{d\lambda}. \tag{1.2.42}$$

If we replace λ by a new variable $s(\lambda)$ then (1.2.42) becomes

$$\frac{d^2x^i}{ds^2}+\Gamma^{\,i}_{k\,l}\frac{dx^k}{ds}\frac{dx^l}{ds}=\frac{\phi s'-s''}{s'^2}\frac{dx^i}{ds}, \tag{1.2.43}$$

where the prime denotes differentiation with respect to λ. Requiring $\phi s'-s''=0$, which has a general solution $s=\int^\lambda d\lambda\exp[-\int^\lambda\phi(x)dx]$, brings (1.2.43) into

$$\frac{d^2x^i}{ds^2}+\Gamma^{\,i}_{k\,l}\frac{dx^k}{ds}\frac{dx^l}{ds}=0, \tag{1.2.44}$$

where the scalar variable s is the affine parameter. The autoparallel equation (1.2.44) is invariant under linear transformations $s\rightarrow as+b$ since the two lower limits of integration in the expression for $s(\lambda)$ are arbitrary. Defining the *four-velocity* vector

$$u^i=\frac{dx^i}{ds} \tag{1.2.45}$$

brings (1.2.14) into

$$\frac{DA^k}{ds}=A^k_{\ ;i}u^i,\ \ \frac{dA^k}{ds}=A^k_{\ ,i}u^i, \tag{1.2.46}$$

so

$$\frac{Du^i}{ds}=\frac{du^i}{ds}+\Gamma^{\,i}_{k\,l}u^ku^l=u^i_{\ ;j}u^j=0. \tag{1.2.47}$$

The relations (1.2.46) can be generalized to any tensor density T:

$$\frac{DT}{ds}=T_{;i}u^i,\ \ \frac{dT}{ds}=T_{,i}u^i, \tag{1.2.48}$$

The vector $\frac{dx^i}{ds}|_Q$ is a parallel translation of $\frac{dx^i}{ds}|_P$. Because ds is a scalar, it is invariant under parallel transport, $ds|_Q = ds|_P$. Therefore the vector $dx^i|_Q$ is a parallel translation of $dx^i|_P$, so ds measures the length of an infinitesimal section of an affine geodesic.

Only the symmetric part $\Gamma^{\,i}_{(k\,l)}$ of the connection enters the autoparallel equation (1.2.44) because of the symmetry of $\frac{dx^k}{ds}\frac{dx^l}{ds}$ with respect to the indices k, l; affine geodesics do not depend on torsion. At any point, a coordinate transformation to the geodesic frame (1.2.34) brings all the components $\Gamma^{\,i}_{(k\,l)}$ to zero, so the autoparallel equation becomes $\frac{du^i}{ds} = 0$. The autoparallel equation is also invariant under a *projective* transformation

$$\Gamma^{\,i}_{k\,l} \rightarrow \Gamma^{\,i}_{k\,l} + \delta^i_k A_l, \tag{1.2.49}$$

where A_i is an arbitrary vector. Substituting this transformation to (1.2.47) gives

$$\frac{du^i}{ds} + \Gamma^{\,i}_{k\,l} u^k u^l = -u^i u^k A_k. \tag{1.2.50}$$

If we replace s by a new variable $\tilde{s}(s)$ then (1.2.50) becomes

$$\frac{dU^i}{d\tilde{s}} + \Gamma^{\,i}_{k\,l} U^k U^l = -\frac{u^k A_k \tilde{s}' + \tilde{s}''}{\tilde{s}'^2}\frac{dx^i}{d\tilde{s}}, \tag{1.2.51}$$

where

$$U^i = \frac{dx^i}{d\tilde{s}} \tag{1.2.52}$$

and the prime denotes differentiation with respect to s. Requiring $u^k A_k \tilde{s}' + \tilde{s}'' = 0$, which has a general solution $\tilde{s} = -\int^s ds \exp[\int^s A_k u^k(x) dx]$, brings (1.2.51) into

$$\frac{dU^i}{d\tilde{s}} + \Gamma^{\,i}_{k\,l} U^k U^l = 0. \tag{1.2.53}$$

1.2.9 Infinitesimal coordinate transformations

Consider a coordinate transformation

$$x'^i = x^i + \xi^i, \tag{1.2.54}$$

where $\xi^i = \delta x^i$ is an infinitesimal vector (variation of x^i). For a tensor or density T define

$$\delta T = T'(x'^i) - T(x^i), \tag{1.2.55}$$
$$\bar{\delta} T = T'(x^i) - T(x^i) = \delta T - \xi^k T_{,k}. \tag{1.2.56}$$

For a scalar we find

$$\delta\phi = 0, \ \ \bar{\delta}\phi = -\xi^k \phi_{,k}. \tag{1.2.57}$$

For a covariant vector

$$\delta A_i = \frac{\partial x^k}{\partial x'^i} A_k - A_i \approx -\xi^k_{\ ,i} A_k, \tag{1.2.58}$$
$$\bar{\delta} A_i \approx -\xi^k_{\ ,i} A_k - \xi^k A_{i,k}. \tag{1.2.59}$$

The variation (1.2.58) is not a tensor, but (1.2.59) is:

$$\bar{\delta}A_i = -\xi^k{}_{;i}A_k - \xi^k A_{i;k} - 2S^j{}_{ik}\xi^k A_j. \tag{1.2.60}$$

We refer to $-\bar{\delta}T$ as a *Lie derivative* of T, $\mathcal{L}_\xi T$. For a contravariant vector

$$\delta B^i = \frac{\partial x'^i}{\partial x^k}B^k - B^i \approx \xi^i{}_{,k}B^k, \tag{1.2.61}$$

$$\bar{\delta}B^i \approx \xi^i{}_{,k}B^k - \xi^k B^i{}_{,k} = \xi^i{}_{;k}B^k - \xi^k B^i{}_{;k} + 2S^i{}_{jk}\xi^k B^j. \tag{1.2.62}$$

For a scalar density

$$\delta\mathfrak{l} = \left(\left|\frac{\partial x^i}{\partial x'^i}\right| - 1\right)\mathfrak{l} \approx -\xi^i{}_{,i}\mathfrak{l}, \tag{1.2.63}$$

$$\bar{\delta}\mathfrak{l} \approx -\xi^i{}_{,i}\mathfrak{l} - \xi^k\mathfrak{l}_{,k} = -\xi^i{}_{;i}\mathfrak{l} - \xi^k\mathfrak{l}_{;k} + 2S_i\xi^i\mathfrak{l}. \tag{1.2.64}$$

The chain rule for δ implies that for a tensor density of weight w (which includes tensors as densities of weight 0)

$$\delta\mathfrak{T}^{ij\dots}_{kl\dots} \approx \xi^i{}_{,m}\mathfrak{T}^{mj\dots}_{kl\dots} + \xi^j{}_{,m}\mathfrak{T}^{im\dots}_{kl\dots} + \dots - \xi^m{}_{,k}\mathfrak{T}^{ij\dots}_{ml\dots} - \xi^m{}_{,l}\mathfrak{T}^{ij\dots}_{km\dots} - \dots$$
$$-w\xi^m{}_{,m}\mathfrak{T}^{ij\dots}_{kl\dots}, \tag{1.2.65}$$

$$\bar{\delta}\mathfrak{T}^{ij\dots}_{kl\dots} \approx \xi^i{}_{;m}\mathfrak{T}^{mj\dots}_{kl\dots} + \xi^j{}_{;m}\mathfrak{T}^{im\dots}_{kl\dots} + \dots - \xi^m{}_{;k}\mathfrak{T}^{ij\dots}_{ml\dots} - \xi^m{}_{;l}\mathfrak{T}^{ij\dots}_{km\dots} - \dots$$
$$-w\xi^m{}_{;m}\mathfrak{T}^{ij\dots}_{kl\dots} - \xi^m\mathfrak{T}^{ij\dots}_{kl\dots;m} + 2S^i{}_{nm}\xi^m\mathfrak{T}^{nj\dots}_{kl\dots} + 2S^j{}_{nm}\xi^m\mathfrak{T}^{in\dots}_{kl\dots} + \dots$$
$$-2S^n{}_{km}\xi^m\mathfrak{T}^{ij\dots}_{nl\dots} - 2S^n{}_{lm}\xi^m\mathfrak{T}^{ij\dots}_{kn\dots} - \dots + 2wS_m\xi^m\mathfrak{T}^{ij\dots}_{kl\dots}. \tag{1.2.66}$$

A Lie derivative of a tensor density of rank (k,l) and weight w is a tensor density of rank (k,l) and weight w.

The formula for a covariant derivative of T can be written as

$$T_{;k} = T_{,k} + \Gamma^{\,j}_{i\,k}\hat{C}^i_j T, \tag{1.2.67}$$

where $\hat{C}$ is an operator acting on tensor densities:

$$\hat{C}^i_j\phi = 0,\ \ \hat{C}^i_j A_k = -\delta^i_k A_j,\ \ \hat{C}^i_j B^k = \delta^k_j B^i,\ \ \hat{C}^i_j\mathfrak{l} = -\delta^i_j\mathfrak{l}, \tag{1.2.68}$$

or generally

$$\hat{C}^m_n\mathfrak{T}^{ij\dots}_{kl\dots} = \delta^i_n\mathfrak{T}^{mj\dots}_{kl\dots} + \delta^j_n\mathfrak{T}^{im\dots}_{kl\dots} + \dots - \delta^m_k\mathfrak{T}^{ij\dots}_{nl\dots} - \delta^m_l\mathfrak{T}^{ij\dots}_{kn\dots} - \dots - w\delta^m_n\mathfrak{T}^{ij\dots}_{kl\dots}. \tag{1.2.69}$$

Such defined operator also enters the formula for δT:

$$\delta T = \hat{C}^k_i T\xi^i{}_{,k}. \tag{1.2.70}$$

1.2.10 Killing vectors

A vector ζ_i that satisfies

$$\zeta_{(i;k)} = 0 \tag{1.2.71}$$

is referred to as a *Killing vector*. Along an affine geodesic

$$\frac{D}{ds}(u^i\zeta_i) = u^k(u^i\zeta_i)_{;k} = u^i u^k \zeta_{i;k} + \zeta_i u^k u^i_{\ ;k} = 0. \tag{1.2.72}$$

The first term in the sum in (1.2.72) vanishes because of the definition of ζ_i and the second term vanishes because of the affine geodesic equation. Therefore, to each Killing vector ζ_i there corresponds a quantity $u^i\zeta_i$ which is constant along an affine geodesic.
References: [1, 2, 3].

1.3 Curvature

1.3.1 Curvature tensor

The commutator of covariant derivatives of a contravariant vector is a tensor:

$$\begin{aligned}
[\nabla_j, \nabla_k]B^i &= 2\nabla_{[j}\nabla_{k]}B^i = 2\partial_{[j}\nabla_{k]}B^i - 2\Gamma^{\ l}_{[k\,j]}\nabla_l B^i + 2\Gamma^{\ i}_{l\,[j}\nabla_{k]}B^l \\
&= 2\partial_{[j}(\Gamma^{\ i}_{|m|\,k]}B^m) + 2S^l_{\ jk}\nabla_l B^i + 2\Gamma^{\ i}_{l\,[j}\partial_{k]}B^l + 2\Gamma^{\ i}_{l\,[j}\Gamma^{\ l}_{|m|\,k]}B^m \\
&= 2(\partial_{[j}\Gamma^{\ i}_{|m|\,k]} + \Gamma^{\ i}_{l\,[j}\Gamma^{\ l}_{|m|\,k]})B^m + 2S^l_{\ jk}\nabla_l B^i = R^i_{\ mjk}B^m + 2S^l_{\ jk}\nabla_l B^i,
\end{aligned} \tag{1.3.1}$$

where || an index which is excluded from symmetrization or antisymmetrization. Therefore $R^i_{\ mjk}$, defined as

$$R^i_{\ mjk} = \partial_j\Gamma^{\ i}_{m\,k} - \partial_k\Gamma^{\ i}_{m\,j} + \Gamma^{\ i}_{l\,j}\Gamma^{\ l}_{m\,k} - \Gamma^{\ i}_{l\,k}\Gamma^{\ l}_{m\,j}, \tag{1.3.2}$$

is a tensor, referred to as the *curvature tensor*. The curvature tensor $R^i_{\ mjk}$ is antisymmetric in the indices j, k and has generally 96 independent components. The commutator of covariant derivatives of a covariant vector is

$$[\nabla_j, \nabla_k]A_i = -R^m_{\ \ ijk}A_m + 2S^l_{\ jk}\nabla_l A_i, \tag{1.3.3}$$

and the commutator of covariant derivatives of a tensor is

$$\begin{aligned}
[\nabla_j, \nabla_k]T^{in\dots}_{\ \ \ lp\dots} &= R^i_{\ mjk}T^{mn\dots}_{\ \ \ \ lp\dots} + R^n_{\ mjk}T^{im\dots}_{\ \ \ \ lp\dots} + \dots - R^m_{\ \ ljk}T^{in\dots}_{\ \ \ mp\dots} - R^m_{\ \ pjk}T^{in\dots}_{\ \ \ lm\dots} \\
&- \dots + 2S^l_{\ jk}\nabla_l T^{in\dots}_{\ \ \ lp\dots}.
\end{aligned} \tag{1.3.4}$$

A change in the connection

$$\Gamma^{\ i}_{j\,k} \to \Gamma^{\ i}_{j\,k} + T^i_{\ jk}, \tag{1.3.5}$$

where $T^i_{\ jk}$ is a tensor, results in the following change of the curvature tensor:

$$R^i_{\ klm} \to R^i_{\ klm} + T^i_{\ km;l} - T^i_{\ kl;m} + T^j_{\ km}T^i_{\ jl} - T^j_{\ kl}T^i_{\ jm}. \tag{1.3.6}$$

For a projective transformation (1.2.49), $T^i_{\ jk} = \delta^i_j A_k$, so

$$R^i_{\ klm} \to R^i_{\ klm} + \delta^i_k (A_{m;l} - A_{l;m}). \tag{1.3.7}$$

The variation of the curvature tensor is

$$\begin{aligned}
&\delta R^i_{\ klm} = (\delta\Gamma^{\,i}_{k\,m})_{,l} - (\delta\Gamma^{\,i}_{k\,l})_{,m} + \delta\Gamma^{\,i}_{j\,l}\Gamma^{\,j}_{k\,m} + \Gamma^{\,i}_{j\,l}\delta\Gamma^{\,j}_{k\,m} - \delta\Gamma^{\,i}_{j\,m}\Gamma^{\,j}_{k\,l} - \Gamma^{\,i}_{j\,m}\delta\Gamma^{\,j}_{k\,l} \\
&= (\delta\Gamma^{\,i}_{k\,m})_{;l} - \Gamma^{\,i}_{j\,l}\delta\Gamma^{\,j}_{k\,m} + \Gamma^{\,j}_{k\,l}\delta\Gamma^{\,i}_{j\,m} + \Gamma^{\,j}_{m\,l}\delta\Gamma^{\,i}_{k\,j} - (\delta\Gamma^{\,i}_{k\,l})_{;m} + \Gamma^{\,i}_{j\,m}\delta\Gamma^{\,j}_{k\,l} - \Gamma^{\,j}_{k\,m}\delta\Gamma^{\,i}_{j\,l} \\
&-\Gamma^{\,j}_{l\,m}\delta\Gamma^{\,i}_{k\,j} + \delta\Gamma^{\,i}_{j\,l}\Gamma^{\,j}_{k\,m} + \Gamma^{\,i}_{j\,l}\delta\Gamma^{\,j}_{k\,m} - \delta\Gamma^{\,i}_{j\,m}\Gamma^{\,j}_{k\,l} - \Gamma^{\,i}_{j\,m}\delta\Gamma^{\,j}_{k\,l} \\
&= (\delta\Gamma^{\,i}_{k\,m})_{;l} - (\delta\Gamma^{\,i}_{k\,l})_{;m} - 2S^n_{\ lm}\delta\Gamma^{\,i}_{k\,n}.
\end{aligned} \tag{1.3.8}$$

1.3.2 Integrability of connection

The affine connection is *integrable* if parallel transport of a vector from point P to point Q is independent of a path along which this vector is parallelly translated, or equivalently, parallel transport of a vector around a closed curve does not change this vector. For an integrable connection, we can uniquely translate parallelly a given vector h^i at point P to all points in spacetime:

$$\delta h^i = dh^i, \tag{1.3.9}$$

or

$$h^i_{\ ,k} = -\Gamma^{\,i}_{j\,k} h^j. \tag{1.3.10}$$

Therefore

$$(\Gamma^{\,i}_{j\,k}h^j)_{,l} - (\Gamma^{\,i}_{j\,l}h^j)_{,k} = \Gamma^{\,i}_{j\,k,l}h^j - \Gamma^{\,i}_{j\,k}\Gamma^{\,j}_{m\,l}h^m - \Gamma^{\,i}_{j\,l,k}h^j + \Gamma^{\,i}_{j\,l}\Gamma^{\,j}_{m\,k}h^m = R^i_{\ jlk}h^j = 0, \tag{1.3.11}$$

so, because h^i is arbitrary,

$$R^i_{\ klm} = 0. \tag{1.3.12}$$

Spacetime with a vanishing curvature tensor $R^i_{\ klm} = 0$ is *flat*. Consider 4 linearly independent vectors h^i_a, where a is 1,2,3,4, and vectors inverse to h^i_a:

$$\sum_a h^i_a h_{ka} = \delta^i_k. \tag{1.3.13}$$

If the affine connection is integrable then (1.3.10) becomes

$$h^i_{a,k} = -\Gamma^{\,i}_{l\,k} h^l_a. \tag{1.3.14}$$

Multiplying (1.3.14) by h_{ja} gives

$$\Gamma^{\,i}_{j\,k} = -h_{ja}h^i_{a,k} = h_{ja,k}h^i_a. \tag{1.3.15}$$

An integrable connection has thus 16 independent components. If the connection is also symmetric, $S^i_{\ jk} = 0$, then

$$h_{ja,k} - h_{ka,j} = 0, \tag{1.3.16}$$

which is the condition for the independence of the coordinates

$$y_a = \int_P^Q h_{ia} dx^i \tag{1.3.17}$$

of the path of integration PQ. Adopting y_a as the new coordinates (with point $P = (0, 0, 0, 0)$ in the center) gives

$$\frac{\partial y_a}{\partial x^i} = h_{ia}, \quad \frac{\partial x^i}{\partial y_a} = h^i_a, \tag{1.3.18}$$

so (1.3.15) becomes

$$\Gamma^{\,i}_{j\,k}(x^i) = \frac{\partial x^i}{\partial y_a}\frac{\partial^2 y_a}{\partial x^k \partial x^j}. \tag{1.3.19}$$

The transformation law for the connection (1.2.5) gives (with y_a corresponding to x'^j)

$$\Gamma^{\,i}_{j\,k}(y_a) = 0. \tag{1.3.20}$$

A torsionless integrable connection can be thus transformed to zero; one can always find a system of coordinates which is geodesic everywhere. If a connection is symmetric but nonintegrable then a geodesic frame of reference can be constructed only at a given point (or along a given world line).

1.3.3 Parallel transport along closed curve

Consider parallel transport of a covariant vector around an infinitesimal closed curve. Such transport changes this vector by

$$\begin{aligned}\Delta A_k &= \oint \delta A_k = \oint \Gamma^{\,i}_{k\,l} A_i dx^l = \frac{1}{2}\int\Big(\frac{\partial(\Gamma^{\,i}_{k\,m}A_i)}{\partial x^l} - \frac{\partial(\Gamma^{\,i}_{k\,l}A_i)}{\partial x^m}\Big)df^{lm} \\ &\approx \frac{1}{2}\int\Big[\Big(\frac{\partial\Gamma^{\,i}_{k\,m}}{\partial x^l} - \frac{\partial\Gamma^{\,i}_{k\,l}}{\partial x^m}\Big)A_i + (\Gamma^{\,i}_{k\,m}\Gamma^{\,n}_{i\,l} - \Gamma^{\,i}_{k\,l}\Gamma^{\,n}_{i\,m})A_n\Big]df^{lm} \\ &= \frac{1}{2}R^i_{\ klm}A_i\Delta f^{lm},\end{aligned} \tag{1.3.21}$$

where we use Stokes' theorem (1.1.33) and $A_{k,l} = \Gamma^{\,i}_{k\,l}A_i$ which is valid along the curve and thus is approximately valid (to terms of first order in Δf^{lm}) inside this curve. The change of a contravariant vector due to parallel transport around an infinitesimal closed curve results from $\Delta(A_k B^k) = 0$:

$$\delta B^k = -\frac{1}{2}R^k_{\ ilm}B^i\Delta f^{lm}, \tag{1.3.22}$$

and the corresponding change of a tensor results from the chain rule for parallel transport:

$$\delta T^{ik\dots}_{\ \ \ np\dots} = -\frac{1}{2}(R^i_{\ jlm}T^{jk\dots}_{\ \ \ np\dots} + R^k_{\ jlm}T^{ij\dots}_{\ \ \ np\dots} + \dots - R^j_{\ nlm}T^{ik\dots}_{\ \ \ jp\dots} - R^j_{\ plm}T^{ik\dots}_{\ \ \ nj\dots} - \dots)\Delta f^{lm}. \tag{1.3.23}$$

1.3.4 Bianchi identities

Consider

$$\nabla_j\nabla_{[k}\nabla_{l]}B^i = \frac{1}{2}\nabla_j(R^i{}_{mkl}B^m) + \nabla_j(S^m{}_{kl}\nabla_m B^i) \tag{1.3.24}$$

and

$$\nabla_{[j}\nabla_{k]}\nabla_l B^i = -\frac{1}{2}R^m{}_{ljk}\nabla_m B^i + \frac{1}{2}R^i{}_{mjk}\nabla_l B^m + S^m{}_{jk}\nabla_m\nabla_l B^i = -\frac{1}{2}R^m{}_{ljk}\nabla_m B^i$$
$$+\frac{1}{2}R^i{}_{mjk}\nabla_l B^m + S^m{}_{jk}\nabla_l\nabla_m B^i + S^m{}_{jk}R^i{}_{nml}B^n + 2S^m{}_{jk}S^n{}_{ml}\nabla_n B^i. \tag{1.3.25}$$

Total antisymmetrization of the indices j, k, l in (1.3.24) and (1.3.25) gives

$$\nabla_{[j}\nabla_k\nabla_{l]}B^i = \frac{1}{2}\nabla_{[j}R^i{}_{|m|kl]}B^m + \frac{1}{2}R^i{}_{m[kl]}\nabla_{j]}B^m + \nabla_{[j}S^m{}_{kl]}\nabla_m B^i + S^m{}_{kl}\nabla_{j]}\nabla_m B^i \tag{1.3.26}$$

and

$$\nabla_{[j}\nabla_k\nabla_{l]}B^i = -\frac{1}{2}R^m{}_{[ljk]}\nabla_m B^i + \frac{1}{2}R^i{}_{m[jk}\nabla_{l]}B^m + S^m{}_{[jk}\nabla_{l]}\nabla_m B^i$$
$$+S^m{}_{[jk}R^i{}_{|nm|l]}B^n + 2S^m{}_{[jk}S^n{}_{|m|l]}\nabla_n B^i, \tag{1.3.27}$$

so

$$\frac{1}{2}\nabla_{[j}R^i{}_{|m|kl]}B^m + \nabla_{[j}S^m{}_{kl]}\nabla_m B^i = -\frac{1}{2}R^m{}_{[ljk]}\nabla_m B^i + S^m{}_{[jk}R^i{}_{|nm|l]}B^n$$
$$+2S^m{}_{[jk}S^n{}_{|m|l]}\nabla_n B^i. \tag{1.3.28}$$

Comparing terms in (1.3.28) with B^i gives the first Bianchi identity or simply *Bianchi identity*:

$$R^i{}_{n[jk;l]} = 2R^i{}_{nm[j}S^m{}_{kl]}, \tag{1.3.29}$$

while comparing terms with $\nabla_k B^i$ gives the second Bianchi identity or *cyclic identity*:

$$R^m{}_{[jkl]} = -2S^m{}_{[jk;l]} + 4S^m{}_{n[j}S^n{}_{kl]}. \tag{1.3.30}$$

For a symmetric connection, $S^i{}_{jk} = 0$, these identities reduce to

$$R^i{}_{n[jk;l]} = 0, \tag{1.3.31}$$
$$R^m{}_{[jkl]} = 0. \tag{1.3.32}$$

The cyclic identity (1.3.32) imposes 16 constraints on the curvature tensor, so the curvature tensor with a vanishing torsion has 80 independent components.

1.3.5 Ricci tensor

Contraction of the curvature tensor with respect to the contravariant index and the second covariant index gives the *Ricci tensor*:

$$R_{ik} = R^j{}_{ijk} = \Gamma^{\,j}_{i\,k,j} - \Gamma^{\,j}_{i\,j,k} + \Gamma^{\,l}_{i\,k}\Gamma^{\,j}_{l\,j} - \Gamma^{\,l}_{i\,j}\Gamma^{\,j}_{l\,k}. \tag{1.3.33}$$

Contraction of the curvature tensor with respect to the contravariant index and the third covariant index gives the Ricci tensor with the opposite sign due to the antisymmetry of the curvature tensor with respect to its last indices. Contraction of the curvature tensor with respect to the contravariant index and the first covariant index gives the homothetic or *segmental curvature* tensor:

$$Q_{ik} = R^j_{\ jik} = \Gamma^{\,j}_{j\,k,i} - \Gamma^{\,j}_{j\,i,k}, \tag{1.3.34}$$

which is a curl. A change in the connection (1.3.5) results in the following changes of the Ricci tensor and segmental curvature tensor:

$$R_{ik} \rightarrow R_{ik} + T^l_{\ ik;l} - T^l_{\ il;k} + T^j_{\ ik}T^l_{\ jl} - T^j_{\ il}T^l_{\ jk}, \tag{1.3.35}$$

$$Q_{ik} \rightarrow Q_{ik} + T^j_{\ jk,i} - T^j_{\ ji,k}. \tag{1.3.36}$$

For a projective transformation (1.2.49)

$$R_{ik} \rightarrow R_{ik} + A_{k;i} - A_{i;k}, \tag{1.3.37}$$

$$Q_{ik} \rightarrow Q_{ik} + 4(A_{k,i} - A_{i,k}). \tag{1.3.38}$$

Therefore the symmetric part of the Ricci tensor is invariant under projective transformations. The variation of the Ricci tensor is

$$\delta R_{ik} = (\delta\Gamma^{\,l}_{i\,k})_{;l} - (\delta\Gamma^{\,l}_{i\,l})_{;k} - 2S^j_{\ lk}\delta\Gamma^{\,l}_{i\,j}, \tag{1.3.39}$$

while the variation of the segmental curvature tensor is

$$\delta Q_{ik} = (\delta\Gamma^{\,j}_{j\,k})_{,i} - (\delta\Gamma^{\,j}_{j\,i})_{,k}. \tag{1.3.40}$$

1.3.6 Geodesic deviation

Consider a family of affine geodesics characterized by the affine parameter s and distinguished by a scalar parameter t. Define the separation vector

$$v^i = \frac{dx^i}{dt}, \tag{1.3.41}$$

so

$$v^i_{\ ;k}u^k - u^i_{\ ;k}v^k = v^i_{\ ,k}u^k - u^i_{\ ,k}v^k - 2S^i_{\ kl}u^k v^l = \frac{du^i}{dt} - \frac{dv^i}{ds} - 2S^i_{\ kl}u^k v^l = -2S^i_{\ kl}u^k v^l. \tag{1.3.42}$$

Therefore

$$\begin{aligned}
&\frac{D^2v^i}{ds^2} = (v^i_{\ ;j}u^j)_{;k}u^k = (u^i_{\ ;j}v^j)_{;k}u^k - 2(S^i_{\ kl}u^k v^l)_{;j}u^j \\
&= u^i_{\ ;jk}v^j u^k + u^i_{\ ;j}v^j_{\ ;k}u^k - 2(S^i_{\ kl}u^k v^l)_{;j}u^j \\
&= u^i_{\ ;kj}v^j u^k - R^i_{\ ljk}u^l v^j u^k - 2S^l_{\ jk}u^i_{\ ;l}v^j u^k + u^i_{\ ;j}v^j_{\ ;k}u^k - 2(S^i_{\ kl}u^k v^l)_{;j}u^j \\
&= u^i_{\ ;kj}v^j u^k - R^i_{\ ljk}u^l v^j u^k - 2S^l_{\ jk}u^i_{\ ;l}v^j u^k + u^i_{\ ;j}(u^j_{\ ;k}v^k - 2S^j_{\ kl}u^k v^l) \\
&-2(S^i_{\ kl}u^k v^l)_{;j}u^j = (u^i_{\ ;k}u^k)_{;j}v^j + R^i_{\ jkl}u^j u^k v^l - 2(S^i_{\ kl}u^k v^l)_{;j}u^j \\
&= R^i_{\ jkl}u^j u^k v^l - 2\frac{D}{ds}(S^i_{\ kl}u^k v^l)
\end{aligned} \tag{1.3.43}$$

or

$$\frac{D}{ds}\Big(\frac{Dv^i}{ds} + 2S^i_{\ kl}u^k v^l\Big) = R^i_{\ jkl}u^j u^k v^l. \tag{1.3.44}$$

This is the equation of *geodesic deviation*. If we replace affine geodesics by arbitrary curves then $u^i_{\ ;k}u^k \neq 0$ and (1.3.44) becomes

$$\frac{D}{ds}\Big(\frac{Dv^i}{ds} + 2S^i_{\ kl}u^k v^l\Big) = R^i_{\ jkl}u^j u^k v^l + (u^i_{\ ;k}u^k)_{;j}v^j. \tag{1.3.45}$$

References: [1, 2, 3, 4].

1.4 Metric

1.4.1 Metric tensor

An affine parameter s is a measure of the length only along an affine geodesic. In order to extend the concept of length to all points in spacetime, we equip spacetime with an algebraic object g_{ik}, referred to as the covariant *metric tensor* and defined as

$$ds^2 = g_{ik}dx^i dx^k. \tag{1.4.1}$$

The metric tensor is a symmetric tensor of rank (0,2):

$$g_{ik} = g_{ki}. \tag{1.4.2}$$

The affine parameter s, whose differential is given by (1.4.1), is referred to as the *interval*. Because ds does not change under parallel transport along an affine geodesic from point $P(x^i)$ to point $Q(x^i + dx^i)$, $ds|_Q = ds|_P$, and $dx^i|_Q$ is a parallel translation of $dx^i|_P$, $g_{ik}|_Q = g_{ik}|_P + g_{ik,j}dx^j$ is a parallel translation of $g_{ik}|_P$:

$$g_{ik}|_Q = g_{ik}|_P + \delta g_{ik}, \tag{1.4.3}$$

so

$$Dg_{ik} = g_{ik;j}dx^j = dg_{ik} - \delta g_{ik} = g_{ik,j}dx^j - \delta g_{ik} = 0. \tag{1.4.4}$$

Therefore a covariant derivative of the covariant metric tensor vanishes:

$$N_{jik} = -g_{ik;j} = 0 \tag{1.4.5}$$

or

$$g_{ik,j} - \Gamma^{\ l}_{i\ j}g_{lk} - \Gamma^{\ l}_{k\ j}g_{il} = 0, \tag{1.4.6}$$

where N_{ijk} is the *nonmetricity tensor*. The symmetric contravariant metric tensor $g^{ik} = g^{ki}$ is defined as the inverse of g_{ik}:

$$g_{ij}g^{ik} = \delta^k_j. \tag{1.4.7}$$

A covariant derivative of the contravariant metric tensor also vanishes:

$$g^{ik}_{\ \ ;j} = 0. \tag{1.4.8}$$

The metric tensor allows to associate covariant and contravariant vectors:

$$A^i = g^{ik} A_k, \tag{1.4.9}$$
$$B_i = g_{ik} B^k, \tag{1.4.10}$$

because such association works for the covariant differentials of these vectors which are vectors:

$$DA^i = D(g^{ik} A_k) = g^{ik} DA_i, \;\; DB_i = D(g_{ik} B^k) = g_{ik} DB^k \tag{1.4.11}$$

(raising and lowering of coordinate indices commutes with covariant differentiation with respect to $\Gamma^{\,\rho}_{\mu\,\nu}$). For covariant and contravariant indices of tensors and densities this association is

$$g_{im} \mathfrak{T}^{ij\dots}_{kl\dots} = \mathfrak{T}_m^{j\dots}{}_{kl\dots}, \tag{1.4.12}$$
$$g^{km} \mathfrak{T}^{ij\dots}_{kl\dots} = \mathfrak{T}^{ijm\dots}_{l\dots}. \tag{1.4.13}$$

The square root of the absolute value of the determinant

$$\mathfrak{g} = |g_{ik}| \tag{1.4.14}$$

of the metric tensor is a scalar density, which we can use to multiply covariant integrands that contain dual densities of weight -1, since

$$e_{iklm} = \sqrt{|\mathfrak{g}|} \epsilon_{iklm}, \;\; e^{iklm} = \frac{1}{\sqrt{|\mathfrak{g}|}} \epsilon^{iklm} \tag{1.4.15}$$

are tensors. Thus the relations (1.1.27) are also valid if we replace ϵ by e. The variation of the determinant of the metric tensor is

$$\delta \mathfrak{g} = \mathfrak{g} g^{ik} \delta g_{ik} = -\mathfrak{g} g_{ik} \delta g^{ik}. \tag{1.4.16}$$

A covariant derivative of the determinant of the metric tensor vanishes:

$$\mathfrak{g}_{;j} = 0. \tag{1.4.17}$$

A Lie derivative of the metric tensor is

$$\mathcal{L}_\xi g^{ik} = -2\xi^{(i;k)} - 4S^{(ik)}{}_l \xi^l, \tag{1.4.18}$$

where $^{;i} =_{;k} g^{ik}$. The four-velocity vector (1.2.45) is normalized due to (1.4.1):

$$u^i u_i = 1, \tag{1.4.19}$$

thus having 3 independent components.

The commutator of covariant derivatives (1.3.4) of the metric tensor gives

$$R^{(ij)}{}_{kl} = -N_{[k}{}^{ij}{}_{;l]} - S^m{}_{kl} N_m{}^{ij} = -N_{[k}{}^{ij}{}_{,l]}, \tag{1.4.20}$$

so the segmental curvature tensor (1.3.34) is

$$Q_{kl} = -N_{[k\ \ ,l]}^{\ \ ij} g_{ij}. \tag{1.4.21}$$

Because the nonmetricity tensor (1.4.5) vanishes, the curvature tensor is antisymmetric in its first two indices:

$$R_{ijkl} = -R_{jikl}. \tag{1.4.22}$$

Thus the segmental curvature tensor also vanishes, and

$$R_{ijkl} g^{jl} = R_{ik}, \tag{1.4.23}$$

so there is only one independent way to contract the curvature tensor, which gives the Ricci tensor up to the sign.

1.4.2 Christoffel symbols

The condition (1.4.5) is referred to as metricity or *metric compatibility* of the affine connection, and imposes 40 constraints on the connection:

$$\begin{aligned} g_{ik;j} + g_{kj;i} - g_{ji;k} &= g_{ik,j} - \Gamma^{\,l}_{i\,j} g_{lk} - \Gamma^{\,l}_{k\,j} g_{il} + g_{kj,i} - \Gamma^{\,l}_{k\,i} g_{lj} - \Gamma^{\,l}_{j\,i} g_{kl} - g_{ji,k} + \Gamma^{\,l}_{j\,k} g_{li} \\ +\Gamma^{\,l}_{i\,k} g_{jl} &= g_{ik,j} + g_{kj,i} - g_{ji,k} - 2\Gamma^{\,l}_{(i\,j)} g_{kl} - 2S^l_{\ kj} g_{il} - 2S^l_{\ ki} g_{jl} = 0. \end{aligned} \tag{1.4.24}$$

Multiplying (1.4.24) by g^{km} gives

$$\Gamma^{\,m}_{(i\,j)} = \{^{\,m}_{i\,j}\} + 2S_{(ij)}^{\ \ \ \ m}, \tag{1.4.25}$$

where

$$\{^{\,m}_{i\,j}\} = \frac{1}{2} g^{mk} (g_{ki,j} + g_{kj,i} - g_{ij,k}) \tag{1.4.26}$$

are the *Christoffel symbols*, symmetric in their covariant indices:

$$\{^{\,k}_{i\,j}\} = \{^{\,k}_{j\,i}\}. \tag{1.4.27}$$

Because $\Gamma^{\,k}_{i\,j} = \Gamma^{\,k}_{(i\,j)} + S^k_{\ ij}$, the metric-compatible affine connection equals

$$\Gamma^{\,k}_{i\,j} = \{^{\,k}_{i\,j}\} + C^k_{\ ij}, \tag{1.4.28}$$

where

$$C^k_{\ ij} = 2S_{(ij)}^{\ \ \ \ k} + S^k_{\ ij} \tag{1.4.29}$$

is the *contortion tensor*, antisymmetric in its first two indices:

$$C_{ijk} = -C_{jik}. \tag{1.4.30}$$

The inverse relation between the torsion and contortion tensor is

$$S^i_{\ jk} = C^i_{\ [jk]}. \tag{1.4.31}$$

The difference between two affine connections is a tensor, so the sum of a connection and a tensor of rank (1,2) is a connection. Therefore the Christoffel symbols form a connection, referred to as the *Levi-Civita connection*. Define the covariant derivative with respect to the Levi-Civita connection analogously to (1.2.11), with $\Gamma^{\,k}_{i\,j}$ replaced by $\{^{\,k}_{i\,j}\}$, and denote it $_{:i}$ instead of $_{;i}$, or $\nabla^{\{\}}_i$ instead of ∇_i. A covariant derivative with respect to the Levi-Civita connection of the metric tensor vanishes due to the definition of the Christoffel symbols:

$$g_{ik:j} = g_{ik,j} - \{^{\,l}_{i\,j}\}g_{lk} - \{^{\,l}_{k\,j}\}g_{il} = 0, \tag{1.4.32}$$

which gives the inverse relation between ordinary derivatives of the metric tensor and the Christoffel symbols. The variation of the Levi-Civita connection is a tensor:

$$\delta\{^{\,k}_{i\,j}\} = \frac{1}{2}g^{kl}((\delta g_{li})_{:j} + (\delta g_{lj})_{:i} - (\delta g_{ij})_{:l}). \tag{1.4.33}$$

The covariant derivative over s of a tensor density with respect to the Levi-Civita connection is, analogously to (1.2.48),

$$\frac{D^{\{\}}T}{ds} = T_{:i}u^i. \tag{1.4.34}$$

One can show that the following formulae hold:

$$\{^{\,k}_{k\,i}\} = (\ln\sqrt{|\mathfrak{g}|})_{,i}, \tag{1.4.35}$$

$$\{^{\,k}_{i\,j}\}g^{ij} = -\frac{1}{\sqrt{|\mathfrak{g}|}}(\sqrt{|\mathfrak{g}|}g^{ik})_{,i}, \tag{1.4.36}$$

$$B^i_{\ :i} = \frac{1}{\sqrt{|\mathfrak{g}|}}(\sqrt{|\mathfrak{g}|}B^i)_{,i}, \tag{1.4.37}$$

$$F^{ik}_{\ \ :i} = \frac{1}{\sqrt{|\mathfrak{g}|}}(\sqrt{|\mathfrak{g}|}F^{ik})_{,i}, \tag{1.4.38}$$

$$A_{i:k} - A_{k:i} = A_{i,k} - A_{k,i}, \tag{1.4.39}$$

$$\oint B^i\sqrt{|\mathfrak{g}|}dS_i = \int B^i_{\ :i}\sqrt{|\mathfrak{g}|}d\Omega, \tag{1.4.40}$$

where $F^{ik} = -F^{ki}$. The Christoffel symbols satisfy all formulae that are satisfied by $\Gamma^{\,k}_{i\,j}$ in which $S^i_{\ jk} = 0$. Because the Levi-Civita connection is a symmetric connection, it can be brought to zero by transforming the coordinates to a geodesic frame. In a geodesic frame, the covariant derivative with respect to the Levi-Civita connection, $\nabla^{\{\}}_i$, coincides with the ordinary derivative ∂_i. A Lie derivative of the metric tensor (1.4.18) can be written as

$$\mathcal{L}_\xi g^{ik} = -2\xi^{(i:k)}, \ \ \mathcal{L}_\xi g_{ik} = 2\xi_{(i:k)}, \tag{1.4.41}$$

where $^{:i} =_{:k} g^{ik}$. A Killing vector (1.2.71) for the Levi-Civita connection satisfies

$$\zeta_{(i:k)} = 0, \tag{1.4.42}$$

thus becomes a generator of *isometries*, transformations that do not change the metric tensor.

If the nonmetricity tensor does not vanish, the general formula for the affine connection (1.4.28) is

$$\Gamma^{\,k}_{i\,j} = \{^{\,k}_{i\,j}\} + C^k_{\ ij} - N^k_{\ ij} + \frac{1}{2}N_{(i\ \ j)}^{\ \ k}. \tag{1.4.43}$$

1.4.3 Riemann curvature tensor

The curvature tensor constructed from the Levi-Civita connection is referred to as the *Riemann tensor*:

$$P^i_{\ mjk} = \partial_j\{^{\,i}_{m\,k}\} - \partial_k\{^{\,i}_{m\,j}\} + \{^{\,i}_{l\,j}\}\{^{\,l}_{m\,k}\} - \{^{\,i}_{l\,k}\}\{^{\,l}_{m\,j}\}. \tag{1.4.44}$$

The commutator of covariant derivatives of the metric tensor vanishes:

$$[\nabla^{\{\}}_j, \nabla^{\{\}}_k]g_{lp} = -P^m_{\ \ ljk}g_{mp} - P^m_{\ \ pjk}g_{lm} = 0, \tag{1.4.45}$$

so the covariant Riemann tensor P_{imjk} is also antisymmetric in the indices i, m. Substituting (1.4.26) in (1.4.44) gives

$$P_{iklm} = \frac{1}{2}(g_{im,kl} + g_{kl,im} - g_{il,km} - g_{km,il}) + g_{jn}(\{^{\,j}_{i\,m}\}\{^{\,n}_{k\,l}\} - \{^{\,j}_{i\,l}\}\{^{\,n}_{k\,m}\}), \tag{1.4.46}$$

which explicitly shows the following symmetry and antisymmetry properties:

$$P_{iklm} = -P_{ikml}, \tag{1.4.47}$$
$$P_{iklm} = -P_{kilm}, \tag{1.4.48}$$
$$P_{iklm} = P_{lmik}. \tag{1.4.49}$$

Accordingly, the Riemannian Ricci tensor is symmetric:

$$P_{ik} = P^j_{\ ijk} = P_{ki}. \tag{1.4.50}$$

Substituting (1.4.28) in (1.3.5) and (1.3.6) gives the relation between the curvature and Riemann tensors:

$$R^i_{\ klm} = P^i_{\ klm} + C^i_{\ km:l} - C^i_{\ kl:m} + C^j_{\ km}C^i_{\ jl} - C^j_{\ kl}C^i_{\ jm}. \tag{1.4.51}$$

Contracting (1.4.51) with respect to in the indices i, l gives

$$R_{km} = P_{km} + C^i_{\ km:i} - C^i_{\ ki:m} + C^j_{\ km}C^i_{\ ji} - C^j_{\ ki}C^i_{\ jm}. \tag{1.4.52}$$

Consequently, the Ricci or *curvature scalar*,

$$R = R_{ik}g^{ik}, \tag{1.4.53}$$

is given by

$$R = P - g^{ik}(2C^l_{\ il:k} + C^j_{\ ij}C^l_{\ kl} - C^l_{\ im}C^m_{\ \ kl}), \tag{1.4.54}$$

where P is the Riemannian curvature scalar,

$$P = P_{ik}g^{ik}. \tag{1.4.55}$$

The variation of the Riemann tensor is, analogously to (1.3.8),

$$\delta P^i_{\ klm} = (\delta\{^{\,i}_{k\,m}\})_{:l} - (\delta\{^{\,i}_{k\,l}\})_{:m}, \tag{1.4.56}$$

and the variation of the Riemannian Ricci tensor is

$$\delta P_{ik} = (\delta\{^{\,l}_{i\,k}\})_{:l} - (\delta\{^{\,l}_{i\,l}\})_{:k}. \tag{1.4.57}$$

The Bianchi identities (1.3.29) and (1.3.30) contracted with respect to one contravariant and one covariant index give

$$R^i_{\ n[ik;l]} = 2R^i_{\ nm[i}S^m_{\ kl]}, \tag{1.4.58}$$
$$R^k_{\ [jkl]} = -2S^k_{\ [jk;l]} + 4S^k_{\ n[j}S^n_{\ kl]}. \tag{1.4.59}$$

Contracting these equations with the metric tensor gives

$$R_{nk;l} - R_{nl;k} + R^i_{\ nkl;i} = -2R_{nm}S^m_{\ kl} - 2R^i_{\ nmk}S^m_{\ il} + 2R^i_{\ nml}S^m_{\ ik} \tag{1.4.60}$$

and the *contracted cyclic identity*:

$$R_{jl} - R_{lj} = -2S_{j;l} + 2S_{l;j} - 2S^k_{\ lj;k} + 4S_nS^n_{\ lj}. \tag{1.4.61}$$

Further contraction of (1.4.60) with the metric tensor gives the *contracted Bianchi identity*:

$$R^i_{\ l;i} - \frac{1}{2}R_{;l} = 2R_{km}S^{mk}_{\ \ \ l} - R^{ik}_{\ \ ml}S^m_{\ ik}. \tag{1.4.62}$$

The Bianchi identities (1.3.31) and (1.3.32) for the Riemann tensor are

$$P^i_{\ n[jk:l]} = 0, \tag{1.4.63}$$
$$P^m_{\ [jkl]} = 0. \tag{1.4.64}$$

Contracting these equations with the metric tensor gives

$$P_{nk:l} + P^i_{\ nkl:i} - P_{nl:k} = 0, \tag{1.4.65}$$
$$P_{jl} - P_{lj} = 0, \tag{1.4.66}$$

in agreement with (1.4.50). Further contraction of (1.4.65) with the metric tensor gives the covariant conservation,

$$G^i_{\ k:i} = 0, \tag{1.4.67}$$

of the symmetric *Einstein tensor*,

$$G_{ik} = P_{ik} - \frac{1}{2}Pg_{ik}. \tag{1.4.68}$$

1.4.4 Properties of Riemann tensor

In two dimensions there is only 1 independent component of the Riemann tensor, P_{1212}. The Riemann scalar is

$$P = \frac{2P_{1212}}{\mathfrak{t}}, \tag{1.4.69}$$

where $\mathfrak{t}$ is the determinant of the two-dimensional metric tensor γ_{ik}:

$$\mathfrak{t} = |\gamma_{ik}| = \gamma_{11}\gamma_{22} - \gamma_{12}^2. \tag{1.4.70}$$

A surface near point $x = 0, y = 0$ is given by

$$z = \frac{x^2}{2\rho_1} + \frac{y^2}{2\rho_2}, \tag{1.4.71}$$

where ρ_1 and ρ_2 are the radii of curvature. Substituting (1.4.71) to

$$dl^2 = dx^2 + dy^2 + dz^2 = \gamma_{ik} dx^i dx^k \tag{1.4.72}$$

gives $\gamma_{ik}(x, y)$, which then gives

$$\frac{P}{2}\bigg|_{x=y=0} = K = \frac{1}{\rho_1 \rho_2}, \tag{1.4.73}$$

where K is the Gaußcurvature.

In three dimensions there are 3 independent pairs, 12, 23, and 31, so the Riemann tensor has 6 independent components: 3 with identical pairs and $\frac{3\cdot 2}{2} = 3$ with different pairs (the cyclic identity does not reduce the number of independent components). The Ricci tensor has also 6 components, which are related to the components of the Riemann tensor by

$$P_{\alpha\beta\gamma\delta} = P_{\alpha\gamma}\gamma_{\beta\delta} - P_{\alpha\delta}\gamma_{\beta\gamma} + P_{\beta\delta}\gamma_{\alpha\gamma} - P_{\beta\gamma}\gamma_{\alpha\delta} + \frac{P}{2}(\gamma_{\alpha\delta}\gamma_{\beta\gamma} - \gamma_{\alpha\gamma}\gamma_{\beta\delta}). \tag{1.4.74}$$

Choosing the *Cartesian coordinates* at a given point, defined by the condition

$$g_{\alpha\beta} = \mathrm{diag}(1, 1, 1), \tag{1.4.75}$$

and diagonalizing $P_{\alpha\beta}$, which is equivalent to 3 rotations, brings $P_{\alpha\beta}$ to the canonical form with $6 - 3 = 3$ independent components. Consequently, the Riemann tensor in three dimensions has 3 physically independent components. The Gaußcurvature of a surface perpendicular to the x^3 axis is given by

$$K = \frac{P_{1212}}{\gamma_{11}\gamma_{22} - \gamma_{12}^2}. \tag{1.4.76}$$

In four dimensions there are 6 independent pairs, 01, 02, 03, 12, 23, and 31, so there are 6 components with identical pairs and $\frac{6\cdot 5}{2} = 15$ with different pairs. The cyclic identity reduces the number of independent components by 1, so the Riemann tensor in four dimensions has generally 20 independent components. Choosing the Cartesian coordinates at a given point and applying 6 rotations brings P_{ijkl} to the canonical form with $20 - 6 = 14$ physically independent components.

1.4.5 Weyl tensor

In four dimensions the *Weyl tensor* is defined as

$$W_{iklm} = P_{iklm} - \frac{1}{2}(P_{il}g_{km} + P_{km}g_{il} - P_{im}g_{kl} - P_{kl}g_{im}) + \frac{1}{6}P(g_{il}g_{km} - g_{im}g_{kl}). \quad (1.4.77)$$

This tensor has all the symmetry and antisymmetry properties of the Riemann tensor, and is also traceless (any contraction of the Weyl tensor vanishes).

1.4.6 Metric geodesics

Consider two points in spacetime, P and Q. Among curves that connect these points, one curve has the minimal value of the interval $s = \int ds$, and is referred to as a *metric geodesic*. The equation of a metric geodesic is given by the condition that $\int ds$ be an extremum with the endpoints of the curve fixed:

$$\delta \int ds = \delta \int (g_{ik}dx^i dx^k)^{1/2} = \int \frac{\delta dx^i g_{ij} dx^j}{ds} + \frac{1}{2}\int \frac{\delta g_{ij} dx^i dx^j}{ds} = \int g_{ij}u^j \delta dx^i$$
$$+\frac{1}{2}\int g_{ij,k}\delta x^k u^i u^j ds = \int d(u_i \delta x^i) - \int du_i \delta x^i + \frac{1}{2}\int g_{ij,k}\delta x^k u^i u^j ds$$
$$= -\int \frac{du_i}{ds}\delta x^i ds + \frac{1}{2}\int g_{jk,i}\delta x^i u^j u^k ds = 0, \quad (1.4.78)$$

where we omit the total differential term $\int d(u_i \delta x^i)$ because $\delta x^i = 0$ at the endpoints. Since δx^i is arbitrary, we obtain

$$\frac{d}{ds}(g_{ij}u^j) - \frac{1}{2}\int g_{jk,i}u^j u^k ds = g_{ij}\frac{du^j}{ds} + u^k g_{ij,k}u^j - \frac{1}{2}\int g_{jk,i}u^j u^k ds$$
$$= g_{ij}\frac{du^j}{ds} + \left\{{}^{\,m}_{j\,k}\right\} g_{im}u^j u^k = 0 \quad (1.4.79)$$

or, after multiplying (1.4.79) by g^{il}:

$$\frac{D^{\{\}}u^l}{ds} = \frac{du^l}{ds} + \left\{{}^{\,l}_{j\,k}\right\} u^j u^k = 0. \quad (1.4.80)$$

The metric geodesic equation (1.4.80) can be written as

$$\frac{d^2x^i}{ds^2} + \left\{{}^{\,i}_{k\,l}\right\}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0. \quad (1.4.81)$$

Using (1.4.28) and (1.4.29), the affine geodesic equation (1.2.44) can be written as

$$\frac{d^2x^i}{ds^2} + \left\{{}^{\,i}_{k\,l}\right\}\frac{dx^k}{ds}\frac{dx^l}{ds} + 2S_{kl}{}^{i}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0. \quad (1.4.82)$$

If the torsion tensor is completely antisymmetric then the last term in (1.4.82) vanishes and the affine geodesic equation coincides with the metric geodesic equation. The equation of geodesic deviation with respect to the Levi-Civita connection is, analogously to (1.3.44),

$$\frac{D^{\{\}2}v^i}{ds^2} = P^i{}_{jkl}u^j u^k v^l. \quad (1.4.83)$$

1.4.7 Galilean frame of reference and Minkowski tensor

At a given point, the nondegenerate ($\mathfrak{g} \neq 0$) metric tensor can be brought to a diagonal (canonical) form $g_{ik} = \mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1)$. *Physical* systems are described by the metric tensor with $\mathfrak{g} < 0$. Without loss of generality, we assume that the canonical form of the metric tensor is

$$g_{ik} = \eta_{ik} = \mathrm{diag}(1, -1, -1, -1), \quad g^{ik} = \eta^{ik} = \mathrm{diag}(1, -1, -1, -1). \tag{1.4.84}$$

A frame of reference in which g_{ik} has the canonical form is referred to as *Galilean*. The transformation (1.2.34) with (1.2.37) brings a symmetric affine connection, thus the Christoffel symbols, to zero at a given point without changing the components of the metric tensor because of (1.2.35). Therefore a frame of reference can be both geodesic and Galilean. In such *locally inertial* frame first derivatives of the metric tensor vanish because of (1.4.32). The corresponding metric tensor (1.4.84) is referred to as the *Minkowski tensor*. In a locally inertial frame the coordinates x^i, not only the differentials dx^i, are components of a contravariant vector.

In the absence of torsion, spacetime with a vanishing Riemann tensor $P^i_{klm} = 0$ is flat. In the new coordinates y_a (1.3.17), (1.3.18) gives

$$g^{ab}(y) = g_{ik}(x) \frac{\partial x^i}{\partial y_a} \frac{\partial x^k}{\partial y_b} = g_{ik}(x) h^{ia} h^{kb} = \eta^{ab}. \tag{1.4.85}$$

Therefore in a flat spacetime without torsion one can always find a system of coordinates which is Galilean everywhere.

1.4.8 Intervals, proper time and distances

The form of the Minkowski tensor distinguishes the coordinate x^0 from the rest of the coordinates x^α, where the index α can be 1,2,3. The *temporal* coordinate $x^0 = ct$, where t is referred to as *time* and c is referred to as the *velocity of propagation of interaction*. The coordinates x^α are *spatial* and span *space*. The set of 4 coordinates x^i describe an *event* and span *spacetime*. The curve $x^i(\lambda)$, where λ is a parameter, is referred to as a *world line* of a given point. The quantitites

$$v^\alpha = \frac{dx^\alpha}{dt} \tag{1.4.86}$$

are the components of a three-dimensional vector, the *velocity* of this point. An infinitesimal interval ds is *timelike* if $ds^2 > 0$, *spacelike* if $ds^2 < 0$, and *null* if $ds^2 = 0$. In the Galilean frame, the interval between two infinitesimally separated points (events) is

$$ds^2 = \eta_{ik} dx^i dx^k = c^2 dt^2 - dx^\alpha dx^\alpha, \tag{1.4.87}$$

where dx^i are infinitesimal coordinate differences between the two points. The interval between two finitely separated points is

$$\Delta s^2 = \eta_{ik} \Delta x^i \Delta x^k = c^2 \Delta t^2 - \Delta x^\alpha \Delta x^\alpha, \tag{1.4.88}$$

where Δx^i are finite coordinate differences between the two points. If Δs is timelike, one can always find a frame of reference in which the two events occur at the same place, $\Delta x^\alpha = 0$. A frame of reference in which $dx^\alpha = 0$ describes a point at rest and is referred to as the *rest frame*. In this frame $t = \tau$,

$$ds^2 = c^2 d\tau^2, \tag{1.4.89}$$

where τ is the *proper time*. If $dx^\alpha \neq 0$ along a world line then the point moves. The proper time for a moving point is equal to the time measured by a clock moving with this point. If Δs is spacelike, one can always find a frame of reference in which the two events occur at the same time (are *synchronous*), $\Delta x^0 = 0$. If $ds = 0$ along a world line, this world line describes the propagation of a signal (*interaction*), with $v = (v^\alpha v^\alpha)^{1/2} = c$. Equations (1.4.87) and (1.4.89) give

$$d\tau^2 = dt^2 - \frac{1}{c} dx^\alpha dx^\alpha, \tag{1.4.90}$$

so the proper time τ goes more slowly than the coordinate time t.

In the rest frame $dx^\alpha = 0$ gives $u^\alpha = 0$. At each point in space, the condition $dx^\alpha = 0$ gives the relation between the proper time and the coordinate time:

$$d\tau = \frac{1}{c}\sqrt{g_{00}}dx^0, \tag{1.4.91}$$

which requires

$$g_{00} \geq 0. \tag{1.4.92}$$

The relation (1.4.19) gives

$$u^0 = (g_{00})^{-1/2}. \tag{1.4.93}$$

The distance between two infinitesimally separated points cannot be obtained by imposing dx^0 because x^0 transforms differently at these points. Instead, consider a signal that leaves point $B(x^\alpha + dx^\alpha)$ at $x^0 + dx^0_-$, reaching point $A(x^\alpha)$ at x^0 and coming back to point B at $x^0 + dx^0_+$. Therefore

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0\alpha}dx^0 dx^\alpha + g_{\alpha\beta}dx^\alpha dx^\beta = 0 \tag{1.4.94}$$

gives

$$dx^0_\pm = \frac{1}{g_{00}}(-g_{0\alpha}dx^\alpha \pm \sqrt{(g_{0\alpha}g_{0\beta} - g_{00}g_{\alpha\beta})dx^\alpha dx^\beta}). \tag{1.4.95}$$

The difference in the time coordinate between emitting and receiving the signal at point B is equal to the difference between dx^0_+ and dx^0_- times $\sqrt{g_{00}}/c$, and the distance dl between points A and B is equal to this difference times $c/2$:

$$dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta, \tag{1.4.96}$$

where

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}} \tag{1.4.97}$$

is the symmetric *spatial metric tensor* of spacetime. This tensor is used to raise and lower spatial indices of quantities written in a three-dimensional (spatial) form:

$$A^\alpha = \gamma^{\alpha\beta} A_\beta, \tag{1.4.98}$$

$$B_\alpha = \gamma_{\alpha\beta} B^\beta, \tag{1.4.99}$$

where $\gamma^{\alpha\beta}$ is the inverse of $\gamma_{\alpha\beta}$:

$$\gamma^{\alpha\delta}\gamma_{\beta\delta} = \delta^\alpha_\beta, \tag{1.4.100}$$

One can show that the following formulae hold:

$$\gamma^{\alpha\beta} = -g^{\alpha\beta}, \tag{1.4.101}$$

$$\mathfrak{g} = -g_{00}\mathfrak{l}, \tag{1.4.102}$$

$$g^\alpha = -g^{0\alpha}, \tag{1.4.103}$$

$$g^{00} = \frac{1}{g_{00}} - g_\alpha g^\alpha, \tag{1.4.104}$$

where

$$\mathfrak{l} = \det\gamma_{\alpha\beta}, \tag{1.4.105}$$

$$g_\alpha = -\frac{g_{0\alpha}}{g_{00}}. \tag{1.4.106}$$

The components g_α form a three-dimensional vector $\mathbf{g}$.

The event at point A at x^0 is *synchronized* with the event at point B at the arithmetic mean of the time coordinates of emitting and receiving the signal, i.e. at

$$x^0 + \frac{1}{2}(dx^0_- + dx^0_+) = x^0 + g_\alpha dx^\alpha. \tag{1.4.107}$$

Therefore

$$\delta x^0 = g_\alpha \delta x^\alpha, \tag{1.4.108}$$

which is equivalent to $\delta x_0 = 0$, is the difference in x^0 between two synchronized infinitesimally separated points.

1.4.9 Spatial vectors

The spatial components of a contravariant vector A^i form a three-dimensional vector $\mathbf{A}$:

$$A^i = (A^0, A^\alpha) = (A^0, \mathbf{A}). \tag{1.4.109}$$

The spatial components of a covariant-vector operator ∂_i form a spatial *gradient* operator grad $= \boldsymbol{\nabla}$:

$$\partial_i = \Big(\frac{\partial}{c\partial t}, \frac{\partial}{c\partial x^\alpha}\Big) = \Big(\frac{\partial}{c\partial t}, \boldsymbol{\nabla}\Big). \tag{1.4.110}$$

The *scalar product* of two spatial vectors is

$$\mathbf{A}\cdot\mathbf{B} = \gamma_{\alpha\beta} A^\alpha B^\beta. \tag{1.4.111}$$

The *square* of a spatial vector $\mathbf{A}$ is

$$A^2 = \mathbf{A} \cdot \mathbf{A}. \tag{1.4.112}$$

In three-dimensional space, the *permutation symbol* is defined as

$$\epsilon_{\alpha\beta\gamma} = -\epsilon_{0\alpha\beta\gamma}. \tag{1.4.113}$$

The three-dimensional equivalent of (1.4.15) is

$$e_{\alpha\beta\gamma} = \sqrt{\mathfrak{l}}\epsilon_{\alpha\beta\gamma}, \;\; e^{\alpha\beta\gamma} = \frac{1}{\sqrt{\mathfrak{l}}}\epsilon^{\alpha\beta\gamma}. \tag{1.4.114}$$

The *cross product* of two three-dimensional vectors $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is

$$C_\alpha = e_{\alpha\beta\gamma}A^\beta B^\gamma, \;\; C^\alpha = e^{\alpha\beta\gamma}A_\beta B_\gamma. \tag{1.4.115}$$

The three-dimensional *divergence* of a spatial vector $\mathbf{A}$ is, in analogy to (1.4.37,

$$\mathrm{div}\mathbf{A} = \boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{\sqrt{\mathfrak{l}}}(\sqrt{\mathfrak{l}}A^\alpha)_{,\alpha}. \tag{1.4.116}$$

The three-dimensional *curl* of a spatial vector is

$$(\mathrm{curl}\mathbf{A})^\alpha = (\boldsymbol{\nabla} \times \mathbf{A})^\alpha = e^{\alpha\beta\gamma}A_{\gamma,\beta}. \tag{1.4.117}$$

The *Laplacian* operator is the divergence of the gradient,

$$\triangle = \nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}. \tag{1.4.118}$$

In a locally galilean frame of reference, the covariant and contravariant three-dimensional components of a vector are identical, because

$$\gamma_{\alpha\beta} = \delta_{\alpha\beta}, \tag{1.4.119}$$

where $\delta_{\alpha\beta}$ is the *Cartesian metric tensor*,

$$\delta_{\alpha\beta} = \mathrm{diag}(1,1,1), \;\; \delta^{\alpha\beta} = \mathrm{diag}(1,1,1). \tag{1.4.120}$$

In this frame we refer to the coordinates x^1, x^2, x^3, which are Cartesian, as x, y, z.

The permutation symbol (1.4.113) satisfies

$$\epsilon_{\alpha\beta\gamma}\epsilon^{\alpha}{}_{\delta\zeta} = \delta_{\beta\delta}\delta_{\gamma\zeta} - \delta_{\beta\zeta}\delta_{\gamma\delta}, \tag{1.4.121}$$

$$\epsilon_{\alpha\beta\gamma}\epsilon^{\alpha\beta}{}_{\delta} = 2\delta_{\gamma\delta}, \tag{1.4.122}$$

$$\epsilon_{\alpha\beta\gamma}\epsilon^{\alpha\beta\gamma} = 6. \tag{1.4.123}$$

One can show that the following formulae hold:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \tag{1.4.124}$$
$$\operatorname{curl} \operatorname{grad}\phi = 0, \tag{1.4.125}$$
$$\operatorname{div} \operatorname{curl} \mathbf{A} = 0, \tag{1.4.126}$$
$$\operatorname{grad}(\phi\psi) = \operatorname{grad}\phi\, \psi + \phi \operatorname{grad}\psi, \tag{1.4.127}$$
$$\operatorname{grad}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} + \mathbf{A} \times \operatorname{curl} \mathbf{B} + \mathbf{B} \times \operatorname{curl} \mathbf{A}, \tag{1.4.128}$$
$$\operatorname{div}(\phi\mathbf{A}) = \operatorname{grad}\phi \cdot \mathbf{A} + \phi \operatorname{div} \mathbf{A}, \tag{1.4.129}$$
$$\operatorname{curl}(\phi\mathbf{A}) = \operatorname{grad}\phi \times \mathbf{A} + \phi \operatorname{curl} \mathbf{A}, \tag{1.4.130}$$
$$\operatorname{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \operatorname{curl} \mathbf{A} - \mathbf{A} \cdot \operatorname{curl} \mathbf{B}, \tag{1.4.131}$$
$$\operatorname{curl}(\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + \mathbf{A} \operatorname{div} \mathbf{B} - \mathbf{B} \operatorname{div} \mathbf{A}, \tag{1.4.132}$$
$$\operatorname{curl} \operatorname{curl} \mathbf{A} = \operatorname{grad} \operatorname{div} \mathbf{A} - \triangle\mathbf{A}, \tag{1.4.133}$$

where

$$(\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} = A^{\alpha}\partial_{\alpha}\mathbf{B}. \tag{1.4.134}$$

References: [1, 2, 3, 4].

1.5 Tetrad and spin connection

1.5.1 Tetrad

In addition to the coordinate systems, at each spacetime point we set up four linearly independent vectors e^i_a such that

$$e^i_a e_{ib} = \eta_{ab}, \tag{1.5.1}$$

where $a, b = 0, 1, 2, 3$ are *Lorentz indices* and $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$ is the coordinate-invariant Minkowski metric tensor in a locally geodesic frame of reference at this point. This set of four vectors is referred to as a *tetrad*. The inverse tetrad e^{ai} satisfies

$$e^i_a e^b_i = \delta^b_a, \tag{1.5.2}$$
$$e^i_a e^a_k = \delta^i_k. \tag{1.5.3}$$

The coordinate metric tensor g_{ik} is related to the Minkowski metric tensor through the tetrad:

$$g_{ik} = e^a_i e^b_k \eta_{ab}. \tag{1.5.4}$$

Accordingly, the determinant $\mathfrak{g}$ of the metric tensor g_{ik} is related to the determinant of the tetrad $\mathfrak{e} = |e^a_i|$ by

$$\sqrt{|\mathfrak{g}|} = \mathfrak{e}. \tag{1.5.5}$$

Any vector V can be specified by its components V^i with respect to the coordinate system or by the coordinate-invariant projections V^a of the vector onto the tetrad field:

$$V^a = e^a_i V^i, \;\; V_a = e^i_a V_i, \tag{1.5.6}$$
$$V^i = e^i_a V^a, \;\; V_i = e^a_i V_a, \tag{1.5.7}$$

and similarly for tensors and densities with more indices. We can use η_{ab} and its inverse η^{ab} to lower and raise Lorentz indices, as we use g_{ik} and its inverse g^{ik} to lower and raise coordinate indices.

1.5.2 Lorentz transformation

The relation (1.5.4) imposes 10 constraints on the 16 components of the tetrad, leaving 6 components arbitrary. If we change from one tetrad e^i_a to another, $\tilde{e}^i_b$, then the vectors of the new tetrad are linear combinations of the vectors of the old tetrad:

$$\tilde{e}^i_a = \Lambda^b_{\ a} e^i_b. \tag{1.5.8}$$

The relation (1.5.4) applied to the tetrad field $\tilde{e}^i_b$,

$$g_{ik} = \tilde{e}^a_i \tilde{e}^b_k \eta_{ab}, \tag{1.5.9}$$

imposes on the matrix $\Lambda^b_{\ a}$ the orthogonality condition:

$$\Lambda^c_{\ a} \Lambda^d_{\ b} \eta_{cd} = \eta_{ab}. \tag{1.5.10}$$

We refer to $\Lambda^b_{\ a}$ as a *Lorentz matrix*, and to a transformation of form (1.5.8) as the *Lorentz transformation*.

1.5.3 Tetrad transport

A natural choice for the zeroth component of a tetrad at a given point is

$$e^i_0 = u^i. \tag{1.5.11}$$

Along a world line this tetrad should be transported such that the zeroth component always coincides with the four-velocity. The *Fermi-Walker transport* of a tetrad is defined as

$$\frac{\nabla e^i_a}{ds} = -u^i e^j_a \frac{Du_j}{ds} + \frac{Du^i}{ds} e^j_a u_j. \tag{1.5.12}$$

Putting $a = 0$ in (1.5.12) gives

$$\frac{\nabla u^i}{ds} = \frac{Du^i}{ds}, \tag{1.5.13}$$

so the Fermi-Walker transport of the four-velocity is equivalent to its covariant change and thus (1.5.11) is valid at all points. This transport does not change the orthogonality relation for tetrads (1.5.1) since (1.5.12) gives

$$\frac{\nabla}{ds}(e^i_a e_{ib}) = 0. \tag{1.5.14}$$

1.5.4 Spin connection

Define

$$\omega^i_{\ ak} = e^i_{a;k} = e^i_{a,k} + \Gamma^{\ i}_{j\ k} e^j_a. \tag{1.5.15}$$

The quantities

$$\omega^a_{\ bi} = e^a_j \omega^j_{\ bi} \tag{1.5.16}$$

transform like vectors under coordinate transformations. We can extend the notion of covariant differentiation to quantities with Lorentz coordinate-invariant indices by regarding $\omega^{ab}_{\ \ i}$ as a connection, referred to as Lorentz or *spin connection*. For a contravariant Lorentz vector

$$V^a_{\ |i} = V^a_{\ ,i} + \omega^a_{\ bi} V^b, \tag{1.5.17}$$

where $_{|i}$ is a covariant derivative of such a quantity with respect to x^i. The covariant derivative of a scalar $V^a W_a$ coincides with its ordinary derivative:

$$(V^a W_a)_{|i} = (V^a W_a)_{,i}, \tag{1.5.18}$$

which gives a covariant derivative of a covariant Lorentz vector:

$$W_{a|i} = W_{a,i} - \omega^b_{\ ai} W_b. \tag{1.5.19}$$

The chain rule implies that a covariant derivative of a Lorentz tensor is equal to the sum of the corresponding ordinary derivative of this tensor and terms with spin connection corresponding to each Lorentz index:

$$T^{ab\dots}_{\ \ \ \ cd\dots|i} = T^{ab\dots}_{\ \ \ \ cd\dots,i} + \omega^a_{\ ei} T^{eb\dots}_{\ \ \ \ cd\dots} + \omega^b_{\ ei} T^{ae\dots}_{\ \ \ \ cd\dots} + \dots - \omega^e_{\ ci} T^{ab\dots}_{\ \ \ \ ed\dots} - \omega^e_{\ di} T^{ab\dots}_{\ \ \ \ ce\dots} - \dots. \tag{1.5.20}$$

We assume that the covariant derivative $_{|i}$ is total, that is, also recognizes coordinate indices, acting on them like $_{;i}$. For a tensor with both coordinate and Lorentz indices

$$T^{aj\dots}_{\ \ \ \ bk\dots|i} = T^{aj\dots}_{\ \ \ \ bk\dots,i} + \omega^a_{\ ei} T^{ej\dots}_{\ \ \ \ bk\dots} + \Gamma^{\ j}_{l\ i} T^{al\dots}_{\ \ \ \ bk\dots} + \dots - \omega^e_{\ bi} T^{aj\dots}_{\ \ \ \ ek\dots} - \Gamma^{\ l}_{k\ i} T^{aj\dots}_{\ \ \ \ bl\dots} - \dots. \tag{1.5.21}$$

A total covariant derivative of a tetrad is

$$e^i_{a|k} = e^i_{a,k} + \Gamma^{\ i}_{j\ k} e^j_a - \omega^b_{\ ak} e^i_b = 0, \tag{1.5.22}$$

due to (1.5.15). Therefore total covariant differentiation commutes with converting between coordinate and Lorentz indices. Equation (1.5.22) also determines the spin connection $\omega^a_{\ bi}$ in terms of the affine connection, tetrad and its ordinary derivatives:

$$\omega^a_{\ bi} = e^a_k (e^k_{b,i} + \Gamma^{\ k}_{j\ i} e^j_b). \tag{1.5.23}$$

Conversely, the affine connection is determined by the spin connection, tetrad and its derivatives:

$$\Gamma^{\ j}_{i\ k} = \omega^j_{\ ik} + e^a_{i,k} e^j_a. \tag{1.5.24}$$

The torsion tensor is then

$$S^j_{\ ik} = \omega^j_{\ [ik]} + e^a_{[i,k]} e^j_a, \tag{1.5.25}$$

and the torsion vector is

$$S_i = \omega^k{}_{[ik]} + e^a_{[i,k]}e^k_a. \tag{1.5.26}$$

Metric compatibility of the affine connection leads to

$$g_{ik;j} = g_{ik|j} = e^a_i e^b_k \eta_{ab|j} = -e^a_i e^b_k(\omega^c{}_{aj}\eta_{cb} + \omega^c{}_{bj}\eta_{ac}) = -(\omega_{kij} + \omega_{ikj}) = 0, \tag{1.5.27}$$

so the spin connection is antisymmetric in its first two indices:

$$\omega^a{}_{bi} = -\omega_b{}^a{}_i. \tag{1.5.28}$$

Accordingly, the spin connection has 24 independent components. The contortion tensor is

$$C_{ijk} = \omega_{ijk} + \Delta_{ijk}, \tag{1.5.29}$$

where

$$\Delta_{ijk} = e_{ia}e^a_{[j,k]} - e_{ja}e^a_{[i,k]} - e_{ka}e^a_{[i,j]} \tag{1.5.30}$$

are the *Ricci rotation coefficients*. The first term on the right-hand side in (1.5.29) is expected because both the contortion tensor and spin connection are antisymmetric in their first two indices. The quantities

$$\varpi^i{}_{ak} = e^i_{a:k} = e^i_{a,k} + \{{}^{\,i}_{j\,k}\}e^j_a \tag{1.5.31}$$

form the *Levi-Civita spin connection* and are related to the Ricci rotation coefficients by (1.5.29) with $C_{ijk} = 0$,

$$\varpi_{ijk} = -\Delta_{ijk}, \tag{1.5.32}$$

so

$$C_{ijk} = \omega_{ijk} - \varpi_{ijk}. \tag{1.5.33}$$

1.5.5 Tetrad representation of curvature tensor

The commutator of the covariant derivatives of a tetrad with respect to the affine connection is

$$2e^k_{a;[ji]} = R^k{}_{lij}e^\sigma_a + 2S^l{}_{ij}e^k_{a;l}. \tag{1.5.34}$$

This commutator can also be expressed in terms of the spin connection:

$$\begin{aligned} e^k_{a;[ji]} &= \omega^k{}_{a[j;i]} = (e^k_b\omega^b{}_{a[j})_{;i]} = \omega_{ba[j}\omega^{kb}{}_{i]} + \omega^b{}_{a[j;i]}e^k_b \\ &= \omega_{ba[j}\omega^{kb}{}_{i]} + \omega^b{}_{a[j,i]}e^k_b + S^l{}_{ij}\omega^k{}_{al}. \end{aligned} \tag{1.5.35}$$

Consequently, the curvature tensor with two Lorentz and two coordinate indices depends only on the spin connection and its ordinary derivatives:

$$R^a{}_{bij} = \omega^a{}_{bj,i} - \omega^a{}_{bi,j} + \omega^a{}_{ci}\omega^c{}_{bj} - \omega^a{}_{cj}\omega^c{}_{bi}. \tag{1.5.36}$$

Because the spin connection is antisymmetric in its first two indices, the tensor (1.5.36) is antisymmetric in its first two (Lorentz) indices, like the Riemann tensor. The contraction of the curvature tensor (1.5.36) with a tetrad gives the Ricci tensor with one Lorentz and one coordinate index:

$$R_{bj} = R^a{}_{bij}e^i_a. \tag{1.5.37}$$

The contraction of the tensor $R^a{}_i$ with a tetrad gives the Ricci scalar,

$$R = R^a{}_i e^i_a = R^{ab}{}_{ij} e^i_a e^j_b. \tag{1.5.38}$$

The Riemann tensor with two Lorentz and two coordinate indices depends on the Levi-Civita connection (1.5.31) the same way the curvature tensor depends on the affine connection:

$$P^a{}_{bij} = \varpi^a{}_{bj,i} - \varpi^a{}_{bi,j} + \varpi^a{}_{ci}\varpi^c{}_{bj} - \varpi^a{}_{cj}\varpi^c{}_{bi}. \tag{1.5.39}$$

The contraction of (1.5.39) with a tetrad gives the Riemannian Ricci tensor with one Lorentz and one coordinate index:

$$P_{bj} = P^a{}_{bij} e^i_a. \tag{1.5.40}$$

The contraction of the tensor P^a_i with a tetrad gives the Riemann scalar,

$$P = P^a_i e^i_a = P^{ab}{}_{ij} e^i_a e^j_b. \tag{1.5.41}$$

References: [3, 5, 6, 7, 8].

1.6 Lorentz group

1.6.1 Subgroups of Lorentz group and principle of relativity

A composition of two Lorentz transformations Λ_1 and Λ_2,

$$\Lambda^a{}_b = \Lambda^a_{(1)c}\Lambda^c_{(2)b}, \tag{1.6.1}$$

satisfies (1.5.10), so it is a Lorentz transformation. The Kronecker symbol δ^a_b also satisfies (1.5.10), so it can be regarded as the identity Lorentz transformation. Therefore Lorentz transformations form a group, referred to as the *Lorentz group*. Taking the determinant of the relation (1.5.10) gives

$$|\Lambda^a{}_b| = \pm 1. \tag{1.6.2}$$

A Lorentz transformation with $|\Lambda^a{}_b| = 1$ is *proper* and with $|\Lambda^a{}_b| = -1$ is *improper*. Proper Lorentz transformations form a group because the determinant of the product of two proper Lorentz transformations is 1. Improper Lorentz transformations include the *parity* transformation P

$$\Lambda^a{}_b(P) = \mathrm{diag}(1, -1, -1, -1), \tag{1.6.3}$$

and the *time reversal* T

$$\Lambda^a{}_b(T) = \mathrm{diag}(-1, 1, 1, 1). \tag{1.6.4}$$

The relation (1.5.10) gives $\Lambda^0{}_0\Lambda^0{}_0 - \Lambda^0{}_\alpha\Lambda^0{}_\alpha = 1$, so

$$|\Lambda^0{}_0| \geq 1. \tag{1.6.5}$$

Lorentz transformations with ${\Lambda^0}_0 \geq 1$ are *orthochronous* and form a group. If x^i is a timelike vector, $x^i x_i > 0$, then for an orthochronous transformation $x'^0 = {\Lambda^0}_0 x^0 + {\Lambda^0}_\alpha x^\alpha$,

$$|{\Lambda^0}_\alpha x^\alpha| \leq \sqrt{{\Lambda^0}_\alpha {\Lambda^0}_\alpha x^\beta x^\beta} < \sqrt{({\Lambda^0}_0)^2 (x^0)^2} = |{\Lambda^0}_0 x^0|. \tag{1.6.6}$$

Thus the time component of a timelike vector does not change the sign under orthochronous transformations. Einstein's *principle of relativity* states that all physical laws are invariant under transformations within the orthochronous proper subgroup of the Lorentz group.

Under the parity transformation, the spatial components of contravariant and covariant vectors (three-dimensional vectors) change the sign, while the spatial components of dual vectors (cross products) do not change the sign. Similarly, the scalar contraction of the Levi-Civita symbol and a tensor changes the sign, while a scalar does not. Quantities that transform under proper Lorentz transformations like vectors and do not change the sign in their spatial components under parity are referred to as *axial* vectors or *pseudovectors*. Quantities that transform under proper Lorentz transformations like scalars and change the sign under parity are referred to as *pseudoscalars*.

1.6.2 Infinitesimal Lorentz transformations

Consider an infinitesimal Lorentz transformation

$${\Lambda^\mu}_\nu = \delta^\mu_\nu + {\epsilon^\mu}_\nu, \tag{1.6.7}$$

where ${\epsilon^\mu}_\nu$ are infinitesimal quantities. The relation (1.5.10) gives

$$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}, \tag{1.6.8}$$

where the indices are raised and lowered using the Minkowski metric tensor. Therefore Lorentz transformations are given by 6 independent antisymmetric parameters $\epsilon_{\mu\nu}$. The corresponding transformation of a contravariant vector A^μ is

$$A'^\mu = A^\mu + {\epsilon^\mu}_\nu A^\nu = A^\mu + \frac{1}{2}\epsilon^{\rho\sigma}(\delta^\mu_\rho \eta_{\sigma\nu} - \delta^\mu_\sigma \eta_{\rho\nu})A^\nu = A^\mu + \frac{1}{2}\epsilon^{\rho\sigma} J^\mu_{\nu\rho\sigma} A^\nu, \tag{1.6.9}$$

where

$$J^\mu_{\nu\rho\sigma} = \delta^\mu_\rho \eta_{\sigma\nu} - \delta^\mu_\sigma \eta_{\rho\nu}. \tag{1.6.10}$$

Define matrices $J_{\rho\sigma}$ such that

$$(J_{\rho\sigma})^\mu_\nu = J^\mu_{\nu\rho\sigma}. \tag{1.6.11}$$

Therefore, in the matrix notation (with A^μ treated as a column),

$$A' = \Big(1 + \frac{1}{2}\epsilon^{\rho\sigma} J_{\rho\sigma}\Big)A. \tag{1.6.12}$$

The 6 matrices $J_{\rho\sigma}$ are the infinitesimal *generators* of the *vector representation* of the Lorentz group. The explicit form of the generators of the Lorentz group in the vector representation is

$$J_{01} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_{02} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$J_{03} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad J_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$J_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{1.6.13}$$

1.6.3 Generators and Lie algebra of Lorentz group

The commutator of the generators of the Lorentz group in the vector representation is, using (1.6.10) and (1.6.11),

$$[J_{\kappa\tau}, J_{\rho\sigma}]^{\mu}_{\nu} = (J_{\kappa\tau})^{\mu}_{\lambda}(J_{\rho\sigma})^{\lambda}_{\nu} - (J_{\rho\sigma})^{\mu}_{\lambda}(J_{\kappa\tau})^{\lambda}_{\nu} = (-J_{\kappa\rho}\eta_{\tau\sigma} - J_{\tau\sigma}\eta_{\kappa\rho} + J_{\kappa\sigma}\eta_{\tau\rho} + J_{\tau\rho}\eta_{\kappa\sigma})^{\mu}_{\nu}, \tag{1.6.14}$$

so

$$[J_{\kappa\tau}, J_{\rho\sigma}] = -J_{\kappa\rho}\eta_{\tau\sigma} - J_{\tau\sigma}\eta_{\kappa\rho} + J_{\kappa\sigma}\eta_{\tau\rho} + J_{\tau\rho}\eta_{\kappa\sigma}. \tag{1.6.15}$$

The relation (1.6.15) constitutes the *Lie algebra* of the *Lorentz group*. If a set of qantitites ϕ transforms under a Lorentz transformation Λ with a matrix $D(\Lambda)$

$$\phi \to D(\lambda)\phi, \tag{1.6.16}$$

then D is a representation of the Lorentz group if

$$D(I) = I, \;\; D(\Lambda_1\Lambda_2) = D(\Lambda_1)D(\Lambda_2), \tag{1.6.17}$$

where I denotes the identity transformation, and Λ_1 and Λ_2 are two Lorentz transformations. Therefore

$$D(\Lambda^{-1}) = D^{-1}(\Lambda), \tag{1.6.18}$$

where Λ^{-1} is the Lorentz transformation to Λ: $\Lambda\Lambda^{-1} = I$. For an infinitesimal Lorentz transformation in any representation,

$$D(\Lambda) = I + \frac{1}{2}\epsilon^{\rho\sigma}J_{\rho\sigma}, \tag{1.6.19}$$

according to (1.6.12). The relation

$$D(\Lambda_1\Lambda_2\Lambda_1^{-1}) = D(\Lambda_1)D(\Lambda_2)D^{-1}(\Lambda_1) \tag{1.6.20}$$

gives (1.6.15), valid for any representation of the Lorentz group.

If Λ_1 and Λ_2 are two group transformations then $\Lambda_3 = \Lambda_1\Lambda_2\Lambda_1^{-1}$ is a group transformation. If $\Lambda_2 = I + \epsilon_2 G_2$ is an infinitesimal group transformation with generator G_2 then $\Lambda_3 = I + \epsilon_2\Lambda_1 G_2\Lambda_1^{-1}$ is an infinitesimal group transformation with generator $G_3 = \Lambda_1 G_2\Lambda_1^{-1}$. If $\Lambda_1 = I + \epsilon_1 G_1$ is an infinitesimal group transformation with generator G_1 then, neglecting terms in ϵ_1 of higher order, $G_3 = G_2 + \epsilon_1[G_1, G_2]$, so $[G_1, G_2]$ is a generator. For a finite number N of linearly independent generators, a general infinitesimal group transformation is $\Lambda = I + \Sigma_{a=1}^N \epsilon_a G_a$. Because $[G_a, G_b]$ is a generator, it is a linear combination of the N generators: $[G_a, G_b] = \Sigma_{c=1}^N f_{abc} G_c$, where f_{abc} are *structure constants* of the Lie algebra of the given group. For the Lorentz group, $\epsilon_a G_a = D(\Lambda) - I$, where $D(\Lambda)$ is given by (1.6.19).

1.6.4 Rotations and boosts

Rotations are proper orthochronous Lorentz transformations with

$$\Lambda^0{}_\alpha = \Lambda^\alpha{}_0 = 0, \;\; \Lambda^0{}_0 = 1. \tag{1.6.21}$$

Rotations act only on the spatial coordinates x^α and form a group, referred to as the *rotation group*. *Boosts* are proper orthochronous Lorentz transformations with

$$\Lambda^\alpha{}_\beta = 0. \tag{1.6.22}$$

Define

$$J_\alpha = \frac{1}{2} e_{\alpha\beta\gamma} J^{\beta\gamma}, \tag{1.6.23}$$

$$K_\alpha = J_{0\alpha}, \tag{1.6.24}$$

and

$$\vartheta_\alpha = \frac{1}{2} e_{\alpha\beta\gamma} \epsilon^{\beta\gamma}, \tag{1.6.25}$$

$$\eta_\alpha = \epsilon_{0\alpha} \tag{1.6.26}$$

(for the Lorentz group $\mathfrak{g} = 1$, so the tensors e and densities ϵ are numerically identical). The explicit form of the generators of the rotation group J_α in the vector representation is

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \;\; J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \;\; J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.6.27}$$

For an infinitesimal Lorentz transformation (1.6.19)

$$D = 1 + \boldsymbol{\vartheta} \cdot \mathbf{J} + \boldsymbol{\eta} \cdot \mathbf{K}. \tag{1.6.28}$$

A finite Lorentz transformation can be regarded as a composition of successive identical infinitesimal Lorentz transformations:

$$D = \lim_{n\to\infty}(1 + \boldsymbol{\theta} \cdot \mathbf{J}/n + \boldsymbol{\eta} \cdot \mathbf{K}/n)^n = e^{\boldsymbol{\theta}\cdot\mathbf{J}+\boldsymbol{\eta}\cdot\mathbf{K}}. \tag{1.6.29}$$

The finite parameters $\boldsymbol{\theta}$, $\boldsymbol{\eta}$ are the *canonical parameters* for a given Lorentz transformations. For a finite Lorentz transformation, (1.6.19) gives

$$D(\Lambda) = e^{\frac{1}{2}\epsilon^{\rho\sigma}J_{\rho\sigma}}, \tag{1.6.30}$$

so

$$J_{\mu\nu} = \frac{\partial D(\Lambda)}{\partial \epsilon^{\mu\nu}}\Big|_{\Lambda=I}. \tag{1.6.31}$$

The explicit form of a finite Lorentz transformation in the vector representation is

$$R_1 = e^{\theta J_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_2 = e^{\theta J_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix},$$
$$R_3 = e^{\theta J_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_1 = e^{\eta K_1} = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$B_2 = e^{\eta K_2} = \begin{pmatrix} \cosh\eta & 0 & \sinh\eta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\eta & 0 & \cosh\eta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_3 = e^{\eta K_3} = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}, \tag{1.6.32}$$

where R_α denotes a rotation about the x^α-axis and B_α denotes a boost along this axis. The canonical parameters $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are referred to as the *angle of rotation* and *rapidity*, respectively. The explicit form of a finite rotation in the three-dimensional vector representation is

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_2(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix},$$
$$R_3(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.6.33}$$

For instance,

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \to \begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = R_3 \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x\cos\theta - V_y\sin\theta \\ V_x\sin\theta + V_y\cos\theta \\ V_z \end{pmatrix}. \tag{1.6.34}$$

The relation (1.6.31) gives

$$J_\alpha = \frac{\partial R_\alpha(\theta)}{\partial \theta}\Big|_{\theta=0}. \tag{1.6.35}$$

The commutation relation (1.6.15) gives

$$[J_\alpha, J_\beta] = e_{\alpha\beta\gamma} J_\gamma, \tag{1.6.36}$$
$$[J_\alpha, K_\beta] = e_{\alpha\beta\gamma} K_\gamma, \tag{1.6.37}$$
$$[K_\alpha, K_\beta] = -e_{\alpha\beta\gamma} J_\gamma. \tag{1.6.38}$$

Therefore rotations do not commute and form a nonabelian group, rotations and boosts do not commute, and boosts do not commute - changing the order of two nonparallel boosts is equivalent to applying a rotation, referred to as the *Thomas-Wigner rotation*. The structure constants of the Lie algebra of the rotation group are $f_{abc} = e_{abc}$. Moreover, the square of the generators of rotation,

$$J^2 = J_\alpha J_\alpha, \tag{1.6.39}$$

commutes with J_α:

$$[J^2, J_\beta] = [J_\alpha, J_\beta] J_\alpha + J_\alpha [J_\alpha, J_\beta] = e_{\alpha\beta\gamma}(J_\gamma J_\alpha + J_\alpha J_\gamma) = 0. \tag{1.6.40}$$

Defining

$$\mathbf{L} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}), \tag{1.6.41}$$
$$\mathbf{Q} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}), \tag{1.6.42}$$

gives

$$[L_\alpha, L_\beta] = e_{\alpha\beta\gamma} L_\gamma, \tag{1.6.43}$$
$$[Q_\alpha, Q_\beta] = e_{\alpha\beta\gamma} Q_\gamma, \tag{1.6.44}$$
$$[L_\alpha, Q_\beta] = 0, \tag{1.6.45}$$

so the Lorentz group is isomorphic with the product of two complex rotation groups. Accordingly, the Lorentz group can be regarded as the group of four-dimensional rotations in the Minkowski space, or the group of *tetrad rotations*.

1.6.5 Poincaré group

Under the infinitesimal coordinate transformation (1.2.54) in a locally flat spacetime, (1.4.41) gives

$$\eta_{ik} \rightarrow \eta_{ik} - \xi_{i,k} - \xi_{k,i}. \tag{1.6.46}$$

Thus the tensor η_{ik} is invariant under (1.2.54) (isometric) if ξ^i is a Killing vector,

$$\xi_{(i,k)} = 0, \tag{1.6.47}$$

which has the solution

$$\xi^i = \epsilon^{ik} x_k + \epsilon^i, \tag{1.6.48}$$

where ϵ^{ik} and ϵ^{i} are constant. The first term on the right-hand side of (1.6.48) corresponds to a Lorentz rotation described by 6 parameters ϵ^{ik}. The second term on the right-hand side of (1.6.48) corresponds to a *translation*. A combination of two translations does not change if their order is reversed, so translations commute:

$$[T_\mu, T_\nu] = 0, \tag{1.6.49}$$

where T_μ is the generator of translation. The relations (1.6.36) and (1.6.37) mean that J^α and K^α are spatial vectors under rotations. Spatial translations are spatial vectors under rotations, while a time translation is a scalar:

$$[J_\alpha, T_\beta] = e_{\alpha\beta\gamma} T_\gamma, \tag{1.6.50}$$
$$[J_\alpha, T_0] = 0. \tag{1.6.51}$$

The last relation indicates that the generators of rotations, like generators of spatial translation, correspond to *conserved* quantities, which are quantities that do not change in time. The covariant generalization of (1.6.50) and (1.6.51) is

$$[J_{\mu\nu}, T_\rho] = T_\mu \eta_{\nu\rho} - T_\nu \eta_{\mu\rho}. \tag{1.6.52}$$

The relations (1.6.15), (1.6.49) and (1.6.52) constitute the Lie algebra of the inhomogeneous Lorentz or *Poincaré group*. In particular,

$$[K_\alpha, T_\beta] = -T_0 \delta_{\alpha\beta}, \tag{1.6.53}$$
$$[K_\alpha, T_0] = -T_\alpha. \tag{1.6.54}$$

The last relation indicates that the generators of boosts do not correspond to conserved quantities.

For an infinitesimal rotation about the z-axis,

$$(1 + \theta J_z) f(ct, \mathbf{x}) = D(R_z(\theta)) f(ct, \mathbf{x}) = f(ct, R_z(\theta)\mathbf{x}) \approx f(ct, x - \theta y, \theta x + y, z)$$
$$= f(ct, \mathbf{x}) - \theta y \frac{\partial f}{\partial x} + \theta x \frac{\partial f}{\partial y}, \tag{1.6.55}$$

or

$$J_z = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \tag{1.6.56}$$

which gives the differential representation of rotations:

$$J_\alpha = e_{\alpha\beta\gamma} x_\beta \partial_\gamma. \tag{1.6.57}$$

For an infinitesimal boost along the z-axis,

$$(1 + \eta K_z) f(ct, \mathbf{x}) = D(B_z(\eta)) f(ct, \mathbf{x}) = f(B_z(\eta)(ct, \mathbf{x})) \approx f(ct + \eta z, y, z + \eta ct)$$
$$= f(ct, \mathbf{x}) + \eta z \frac{\partial f}{c\partial t} + \eta ct \frac{\partial f}{\partial z}, \tag{1.6.58}$$

or

$$K_z = z \frac{\partial}{c\partial t} + ct \frac{\partial}{\partial z}, \tag{1.6.59}$$

which gives the differential representation of boosts:

$$K_\alpha = x_\alpha \frac{\partial}{c\partial t} + ct \frac{\partial}{\partial x^\alpha}. \tag{1.6.60}$$

The relation for an infinitesimal translation, analogous to (1.6.19), is

$$D(t) = I + \epsilon^\mu T_\mu, \tag{1.6.61}$$

so a finite translation is given by

$$D(t) = e^{\epsilon^\mu T_\mu}. \tag{1.6.62}$$

Translation in (1.6.48) can also be written as

$$t_\mu(\epsilon)x^\nu = x^\nu + \epsilon\delta^\nu_\mu. \tag{1.6.63}$$

The relation analogous to (1.6.35) is

$$T_\mu = \frac{\partial t_\mu(\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{1.6.64}$$

The differential representation of a translation is thus

$$T_\mu = \frac{\partial}{\partial x^\mu}. \tag{1.6.65}$$

1.6.6 Casimir operators of Lorentz and Poincaré group

Analogously to (1.6.40),

$$[L^2, L_\beta] = 0, \tag{1.6.66}$$
$$[Q^2, Q_\beta] = 0, \tag{1.6.67}$$

so L^2 and Q^2 commute with all 6 generators of the Lorentz group. Consequently, $J^2 + K^2$ and $\mathbf{J}\cdot\mathbf{K}$ commute with all generators of the Lorentz group, that is, are the invariants or *Casimir operators* of the Lorentz group. The Casimir operators of Lorentz group do not commute with the generators of translation T_μ, so they are not the invariants of the Poincaré group. Instead, the *mass operator*

$$m^2 = -T^\mu T_\mu \tag{1.6.68}$$

and

$$W^2 = W^\mu W_\mu, \tag{1.6.69}$$

where W^μ is the *Pauli-Lubański pseudovector*

$$W^\mu = \frac{1}{2}e^{\mu\nu\rho\sigma} J_{\rho\sigma} T_\nu, \tag{1.6.70}$$

commute with all generators of the Poincaré group, so they are the Casimir operators of the Poincaré group. The Pauli-Lubański pseudovector obeys the commutation relations

$$
\begin{aligned}
&[T_\mu, W_\nu] = 0, && (1.6.71)\\
&[J_{\mu\nu}, W_\rho] = W_\mu \eta_{\nu\rho} - W_\nu \eta_{\mu\rho}, && (1.6.72)\\
&[W^\mu, W^\nu] = e^{\mu\nu\rho\sigma} W_\rho T_\sigma. && (1.6.73)
\end{aligned}
$$

The relation (1.6.72) is analogous to (1.6.52) because W^μ behaves like a vector under proper Lorentz transformations.

Define the *four-momentum operator*

$$P_\mu = iT_\mu, \tag{1.6.74}$$

whose time component is the *energy operator* $P_0 = iT_0$ and spatial components form the *momentum operator* $P_\alpha = iT_\alpha$. Define the *angular four-momentum operator*

$$M_{\mu\nu} = iJ_{\mu\nu}, \tag{1.6.75}$$

whose spatial components form the *angular momentum operator*

$$M_\alpha = iJ_\alpha. \tag{1.6.76}$$

Therefore the following relations are satisfied:

$$
\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = -i(M_{\mu\rho}\eta_{\nu\sigma} + M_{\nu\sigma}\eta_{\mu\rho} - M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\rho}\eta_{\mu\sigma}), && (1.6.77)\\
&[P_\mu, P_\nu] = 0, && (1.6.78)\\
&[M_{\mu\nu}, P_\rho] = i(P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho}), && (1.6.79)\\
&m^2 = P^\mu P_\mu, && (1.6.80)\\
&W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\rho\sigma} P_\nu, && (1.6.81)\\
&[P_\mu, W_\nu] = 0, && (1.6.82)\\
&[M_{\mu\nu}, W_\rho] = i(W_\mu \eta_{\nu\rho} - W_\nu \eta_{\mu\rho}), && (1.6.83)\\
&[W^\mu, W^\nu] = -ie^{\mu\nu\rho\sigma} W_\rho P_\sigma, && (1.6.84)\\
&[M_\alpha, M_\beta] = ie_{\alpha\beta\gamma} M_\gamma. && (1.6.85)
\end{aligned}
$$

1.6.7 Relativistic kinematics

Consider a boost in the direction of the z-axis

$$x'^i = e^{-\eta K_3} x^i, \tag{1.6.86}$$

where x^i and x'^i have a form of a column (4×1 matrix), and $e^{\eta K_3}$ is given by (1.6.32). Therefore the coordinates in an inertial K-system (unprimed) are related to the coordinates in an inertial K'-system (primed) by

$$
\begin{aligned}
&ct = ct' \mathrm{cosh}\eta + z' \mathrm{sinh}\eta,\\
&x = x', \;\; y = y',\\
&z = z' \mathrm{cosh}\eta + ct' \mathrm{sinh}\eta. && (1.6.87)
\end{aligned}
$$

Consider the origin of the K'-system, $x' = y' = z' = 0$, in the K-system. Therefore

$$\begin{aligned} ct &= ct'\cosh\eta, \\ z &= ct'\sinh\eta, \end{aligned} \tag{1.6.88}$$

which gives the relation between the rapidity η and velocity $V = \frac{dz}{dt}$ of K' relative to K:

$$\tanh\eta = \beta, \tag{1.6.89}$$

where

$$\beta = \frac{V}{c}. \tag{1.6.90}$$

Accordingly, $\cosh\eta = \gamma$ and $\sinh\eta = \beta\gamma$, where

$$\gamma = \Big(1 - \frac{V^2}{c^2}\Big)^{-1/2}. \tag{1.6.91}$$

The relations (1.6.87) become

$$\begin{aligned} t &= \gamma\Big(t' + \frac{V}{c^2}z'\Big), \\ x &= x', \ \ y = y', \\ z &= \gamma(z' + Vt'), \end{aligned} \tag{1.6.92}$$

and are referred to as a *special Lorentz transformation* in the z-direction. The reverse transformation is

$$\begin{aligned} t' &= \gamma\Big(t - \frac{V}{c^2}z\Big), \\ x' &= x, \ \ y' = y, \\ z' &= \gamma(z - Vt). \end{aligned} \tag{1.6.93}$$

For a boost along an arbitrary direction, the spatial vector $\mathbf{x} = (x, y, z)$ transforms such that its component parallel to the velocity $\mathbf{V} = c\boldsymbol{\beta}$ of K' relative to K, $\mathbf{x}_{\parallel} = (\mathbf{x}\cdot\mathbf{V})\mathbf{V}/V^2$ (similarly for primed), behaves like z in (1.6.92) and its component perpendicular to $\mathbf{V}$, $\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{x}_{\parallel}$, behaves like x in (1.6.92):

$$\begin{aligned} t &= \gamma\Big(t' + \frac{\mathbf{V}\cdot\mathbf{x}'}{c^2}\Big), \\ \mathbf{x}_{\perp} &= \mathbf{x}'_{\perp}, \\ \mathbf{x}_{\parallel} &= \gamma(\mathbf{x}'_{\parallel} + \mathbf{V}t'), \end{aligned} \tag{1.6.94}$$

so

$$\mathbf{x} = \gamma(\mathbf{x}'_{\parallel} + \mathbf{V}t') + \mathbf{x}'_{\perp} = \gamma\mathbf{V}t' + \mathbf{x}' + \frac{(\gamma-1)(\mathbf{V}\cdot\mathbf{x}')\mathbf{V}}{V^2}. \tag{1.6.95}$$

Therefore the transformation law for the coordinates in two inertial frames of reference is

$$\begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta} \\ \gamma\boldsymbol{\beta} & 1 + \frac{(\gamma-1)\boldsymbol{\beta}}{\beta^2}\boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} ct' \\ \mathbf{x}' \end{pmatrix}, \tag{1.6.96}$$

or equivalently

$$\begin{pmatrix} ct' \\ \mathbf{x}' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta} \\ -\gamma\boldsymbol{\beta} & 1 + \frac{(\gamma-1)\boldsymbol{\beta}}{\beta^2}\boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}. \tag{1.6.97}$$

The matrix in (1.6.97) is called a *boost matrix*. In the local Minkowski spacetime, contravariant vectors transform like x^i, according to (1.6.96),

$$\begin{pmatrix} W^0 \\ \mathbf{W} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta} \\ \gamma\boldsymbol{\beta} & 1 + \frac{(\gamma-1)\boldsymbol{\beta}}{\beta^2}\boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} V'^0 \\ \mathbf{W}' \end{pmatrix}, \tag{1.6.98}$$

covariant vectors transform such that they remain related to contravariant vectors by the Minkowski metric tensor, and tensors transform like products of vectors. For example, if $\mathbf{V} = c\beta\hat{z}$ is parallel to the z-axis, a tensor of rank (0,2) transforms according to

$$\begin{aligned}
&T_{00} = \gamma(T_{00'} + \beta T_{03'}) = \gamma^2(T_{0'0'} + \beta T_{3'0'} + \beta T_{0'3'} + \beta^2 T_{3'3'}), \\
&T_{0\perp} = \gamma(T_{0'\perp'} + \beta T_{3'\perp'}), \\
&T_{03} = \gamma(T_{03'} + \beta T_{00'}) = \gamma^2(T_{0'3'} + \beta T_{3'3'} + \beta T_{0'0'} + \beta^2 T_{3'0'}), \\
&T_{\perp\perp} = T_{\perp'\perp'}, \\
&T_{3\perp} = \gamma(T_{3'\perp'} + \beta T_{0'\perp'}), \\
&T_{33} = \gamma(T_{33'} + \beta T_{30'}) = \gamma^2(T_{3'3'} + \beta T_{0'3'} + \beta T_{3'0'} + \beta^2 T_{0'0'}),
\end{aligned} \tag{1.6.99}$$

where the index $\perp$ denotes either 1 or 2, and the transposed components $T^T_{ik} = T_{ki}$ transform like the transpositions of the right-hand sides in (1.6.99). If T_{ik} is antisymmetric then $T_{03} = T_{0'3'}$.

The relations (1.6.92) can be written as

$$\begin{aligned}
dt &= \gamma\Big(dt' + \frac{V}{c^2}dz'\Big), \\
dx &= dx', \;\; dy = dy', \\
dz &= \gamma(dz' + V dt'),
\end{aligned} \tag{1.6.100}$$

which gives

$$\begin{aligned}
v_x &= \frac{v'_x}{\gamma(1 + V v'_z/c^2)}, \\
v_y &= \frac{v'_y}{\gamma(1 + V v'_z/c^2)}, \\
v_z &= \frac{v'_z + V}{1 + V v'_z/c^2},
\end{aligned} \tag{1.6.101}$$

where

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}, \;\; \mathbf{v}' = \frac{d\mathbf{x}'}{dt'}. \tag{1.6.102}$$

Two special Lorentz transformations in the same direction commute because of (1.6.38). If a Lorentz transformation from K' to K has parameters β_1 and γ_1, and a Lorentz transformation from K'' to K' has parameters β_2 and γ_2, then a Lorentz transformation from K'' to K has parameters β_3 and γ_3 such that

$$\beta_3 = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}, \ \ \gamma_3 = \gamma_1\gamma_2(1 + \beta_1\beta_2). \tag{1.6.103}$$

For a boost along an arbitrary direction, (1.6.96) gives the Lorentz transformation of velocities:

$$\mathbf{v} = \frac{\mathbf{v'} + \gamma\mathbf{V} + (\gamma - 1)(\mathbf{v'} \cdot \mathbf{V})\mathbf{V}/V^2}{\gamma(1 + \mathbf{v'} \cdot \mathbf{V}/c^2)}. \tag{1.6.104}$$

If $v' = |\mathbf{V'}| = c$ then $v = |\mathbf{V}| = c$, in agreement with the constancy of the velocity of propagation of interaction.

Consider two points at rest in the inertial frame of reference K with positions z_1 and z_2, so the distance between them is $\Delta z = z_2 - z_1$. In the inertial frame K', moving relative to K in the z-direction with velocity V, $z_1 = \gamma(z'_1 + Vt'_1)$ and $z_2 = \gamma(z'_2 + Vt'_2)$, so if $t'_1 = t'_2$ is the time at which we measure (simultaneously) the positions of the two points then $\Delta z = \gamma(z'_2 - z'_1) = \gamma\Delta z'$. Therefore the length of an object in K', whose length in the rest frame K is l (*proper length*), is

$$l' = \frac{l}{\gamma} < l, \tag{1.6.105}$$

which is referred to as the *Lorentz-FitzGerald contraction*. The volume of an object in K', whose volume in the rest frame K is V (*proper volume*), is

$$V' = \frac{V}{\gamma}. \tag{1.6.106}$$

Suppose that there are two rods of equal lengths, moving parallel relative to each other. From the point of view of an observer moving with the first rod, the second one is shorter, and from the point of view of an observer moving with the second rod, the first one is shorter. There is no contradiction in this statement because the positions of both ends of a rod must be measured simultaneously and the simultaneity is not invariant: from the transformation law (1.6.92) it follows that if $\delta t = 0$ then $\delta t' \neq 0$ and if $\delta t' = 0$ then $\delta t \neq 0$.

Consider a clock (any mechanism with a periodic or evolutionary behavior) at rest in K' with position z'; the time difference between two events with t'_1 and t'_2, as measured by this clock, is $\Delta t' = t'_2 - t'_1$. In the frame K, $t_1 = \gamma(t'_1 + Vz'/c^2)$ and $t_2 = \gamma(t'_2 + Vz'/c^2)$, so

$$\Delta t = t_2 - t_1 = \gamma\Delta t' > \Delta t'. \tag{1.6.107}$$

Thus the rate of time is slower for moving clocks than those at rest (*time dilation*), in agreement with (1.4.90) and (1.4.96), from which $c^2 d\tau^2 = c^2 dt^2 - dl^2$ and

$$d\tau = \frac{1}{\gamma}dt. \tag{1.6.108}$$

Suppose that there are two clocks linked to the inertial frames K and K', and that when the clock in K passes by the clock in K' the readings of the two clocks coincide.

From the point of view of an observer in K clocks in K' go more slowly, and from the point of view of an observer in K' clocks in K go more slowly. There is no contradiction in this statement because to compare the rates of the two clocks in K and K' we must compare the readings of the same moving clock in K' with different clocks in K; we require several clocks in one frame and one in the other, thus the measurement process is not symmetric with respect to the two frames of reference. The clock that goes more slowly is the one which is being compared with different clocks in the other frame. The time interval measured by a clock is equal to the integral

$$\Delta t = \frac{1}{c}\int ds \tag{1.6.109}$$

along its world line. Since the world line is a straight line for a clock at rest and a curved line for a clock moving such that it returns to the starting point, the integral $\int ds$ taken between two world points has its maximum value if it is taken along the straight line connecting these two points.

For a Lorentz transformation with velocity $V = |\mathbf{V}|$, (1.6.104) gives

$$\tan\theta = \frac{v'\sin\theta'}{\gamma(v'\cos\theta' + V)}, \tag{1.6.110}$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{V}$, and θ' is the angle between $\mathbf{v}'$ and $\mathbf{V}$. If $v = v' = c$ then

$$\cos\theta = \frac{\cos\theta' + \frac{V}{c}}{1 + \frac{V}{c}\cos\theta'}, \tag{1.6.111}$$

which is referred to as the *aberration* of a signal. Suppose an observer in frame K measures a periodic signal with period T, frequency $\nu = \frac{1}{T}$ and wavelength $\lambda = \frac{c}{\nu}$, propagating in the $-z$ direction; the number of pulses in time dt is $n = \nu dt$. A second observer in frame K', moving in the z direction with velocity V relative to the first one, travels a distance Vdt and measures $\frac{Vdt}{\lambda}$ more pulses: $n' = \nu(1+\frac{V}{c})dt$. Because the time interval dt with respect to K' is $dt' = \frac{dt}{\gamma}$, the frequency of the signal in K' is $\nu' = \gamma\nu(1+\frac{V}{c})$ or

$$\nu' = e^{\eta}\nu. \tag{1.6.112}$$

This dependence of the frequency of a signal on a frame of reference is referred to as the *Doppler effect*.

When $c \to \infty$ (at which $\gamma \to 1$) the above formulae, referring to *relativistic kinematics*, reduce to their *nonrelativistic* limit. The Lorentz transformation (1.6.96) reduces to the *Galileo transformation*,

$$\begin{aligned} t &= t', \\ \mathbf{x} &= \mathbf{x}' + \mathbf{V}t', \end{aligned} \tag{1.6.113}$$

so the time is an absolute (invariant) quantity in nonrelativistic (*Newtonian*) physics. Any two Galileo transformations commute. The transformation law for velocities (1.6.104) reduces to the simple addition of vectors,

$$\mathbf{v} = \mathbf{v}' + \mathbf{V}. \tag{1.6.114}$$

1.6.8 Four-acceleration

In a locally inertial frame of reference, the four-velocity is

$$u^i = \Big(\gamma, \gamma\frac{\mathbf{v}}{c}\Big), \;\; u_i = \Big(\gamma, -\gamma\frac{\mathbf{v}}{c}\Big), \tag{1.6.115}$$

where $\mathbf{v}$ is the velocity and $\gamma = (1 - \frac{v^2}{c^2})^{-1/2}$. Define the *four-acceleration*

$$w^i = \frac{Du^i}{ds} = \frac{D^2x^i}{ds^2}, \tag{1.6.116}$$

which is orthogonal to u^i because of (1.4.19):

$$w^i u_i = 0, \tag{1.6.117}$$

thus having 3 independent components. In a locally inertial frame of reference, the four-acceleration is

$$w^i = \frac{du^i}{ds} = \frac{d^2x^i}{ds^2} = c^{-2}\Big(\gamma^4\frac{\mathbf{v}\cdot\mathbf{a}}{c}, \gamma^2\mathbf{a} + \gamma^4\frac{(\mathbf{v}\cdot\mathbf{a})\mathbf{v}}{c^2}\Big), \tag{1.6.118}$$

where $\mathbf{a}$ is the three-dimensional *acceleration* vector

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}. \tag{1.6.119}$$

The invariant square of the four-acceleration is thus

$$w^i w_i = -\frac{\gamma^4}{c^4}\Big(\mathbf{a}^2 + \frac{\gamma^2}{c^2}(\mathbf{v}\cdot\mathbf{a})^2\Big). \tag{1.6.120}$$

If $\mathbf{v} = 0$ at a given instant of time, the corresponding frame of reference is referred to as the *instantaneous rest frame*. In this frame

$$w^i w_i = -\frac{a^2}{c^4}, \tag{1.6.121}$$

so

$$a_0 = c^2\sqrt{-w^i w_i} \tag{1.6.122}$$

is the absolute value of the acceleration in the instantaneous rest frame.
References: [2, 3].

1.7 Spinors

1.7.1 Spinor representation of Lorentz group

Let γ^a be the coordinate-invariant 4×4 *Dirac matrices* defined as

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}I, \tag{1.7.1}$$

where I is the unit 4×4 matrix (4 is the lowest dimension for which (1.7.1) has solutions). Accordingly, the spacetime-dependent Dirac matrices, $\gamma^i = e^i_a \gamma^a$, satisfy

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2g^{ij} I. \tag{1.7.2}$$

Under a tetrad rotation, (1.5.8) gives

$$\tilde{\gamma}^a = \Lambda^a{}_b \gamma^b. \tag{1.7.3}$$

Let L be a 4×4 matrix such that

$$\gamma^a = \Lambda^a{}_b L \gamma^b L^{-1} = L \tilde{\gamma}^a L^{-1}, \tag{1.7.4}$$

where L^{-1} is the matrix inverse to L: $LL^{-1} = L^{-1}L = I$. The condition (1.7.4) represents the constancy of the Dirac matrices γ^a under the combined tetrad rotation and transformation $\gamma \rightarrow L\gamma L^{-1}$. We refer to L as the *spinor representation* of the Lorentz group. The relation (1.7.4) gives the matrix L as a function of the Lorentz matrix $\Lambda^a{}_b$. For an infinitesimal Lorentz transformation (1.6.7), the solution for L is

$$L = I + \frac{1}{2}\epsilon_{ab} G^{ab}, \ \ L^{-1} = I - \frac{1}{2}\epsilon_{ab} G^{ab}, \tag{1.7.5}$$

where G^{ab} are the *generators* of the spinor representation of the Lorentz group:

$$G^{ab} = \frac{1}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a). \tag{1.7.6}$$

A *spinor* ψ is defined as a quantity that, under tetrad rotations, transforms according to

$$\tilde{\psi} = L\psi. \tag{1.7.7}$$

An *adjoint spinor* $\bar{\psi}$ is defined as a quantity that transforms according to

$$\tilde{\bar{\psi}} = \bar{\psi} L^{-1}, \tag{1.7.8}$$

so the product $\bar{\psi}\psi$ is a scalar:

$$\tilde{\bar{\psi}}\tilde{\psi} = \bar{\psi}\psi. \tag{1.7.9}$$

The indices of the γ^a and L that are implicit in the 4×4 matrix multiplication in (1.7.1), (1.7.2) and (1.7.4) are spinor indices. The relation (1.7.4) implies that the Dirac matrices γ^a can be regarded as quantities that have, in addition to the invariant index a, one spinor index and one adjoint-spinor index. The product $\psi\bar{\psi}$ transforms like the Dirac matrices:

$$\tilde{\psi}\tilde{\bar{\psi}} = L\psi\bar{\psi}L^{-1}. \tag{1.7.10}$$

The spinors ψ and $\bar{\psi}$ can be used to construct tensors. For example, $\bar{\psi}\gamma^a\psi$ transforms like a contravariant Lorentz vector:

$$\bar{\psi}\gamma^a\psi \rightarrow \bar{\psi}L^{-1}\Lambda^a{}_b L\gamma^b L^{-1} L\psi = \Lambda^a{}_b \bar{\psi}\gamma^b\psi. \tag{1.7.11}$$

1.7.2 Spinor connection

The derivative of a spinor does not transform like a spinor:

$$\tilde{\psi}_{,i} = L\psi_{,i} + L_{,i}\psi. \tag{1.7.12}$$

If we introduce the *spinor connection* Γ_i that transforms according to

$$\tilde{\Gamma}_i = L\Gamma_i L^{-1} + L_{,i}L^{-1}, \tag{1.7.13}$$

then a *covariant derivative* of a spinor,

$$\psi_{;i} = \psi_{,i} - \Gamma_i\psi, \tag{1.7.14}$$

is a spinor:

$$\tilde{\psi}_{;i} = \tilde{\psi}_{,i} - \tilde{\Gamma}_i\tilde{\psi} = L\psi_{,i} + L_{,i}\psi - (L\Gamma_i L^{-1} + L_{,i}L^{-1})L\psi = L\psi_{;i}. \tag{1.7.15}$$

Because $\bar{\psi}\psi$ is a scalar,

$$(\bar{\psi}\psi)_{;i} = (\bar{\psi}\psi)_{,i}, \tag{1.7.16}$$

the chain rule for covariant differentiation gives a covariant derivative of an adjoint spinor

$$\bar{\psi}_{;i} = \bar{\psi}_{,i} + \bar{\psi}\Gamma_i. \tag{1.7.17}$$

Also

$$\psi_{|i} = \psi_{;i}, \;\; \bar{\psi}_{|i} = \bar{\psi}_{;i}. \tag{1.7.18}$$

The Dirac matrices γ^a transform like $\psi\bar{\psi}$, whose covariant derivative is

$$(\psi\bar{\psi})_{;i} = \psi_{;i}\bar{\psi} + \psi\bar{\psi}_{;i} = (\psi\bar{\psi})_{,i} - \Gamma_i\psi\bar{\psi} + \psi\bar{\psi}\Gamma_i = (\psi\bar{\psi})_{,i} - [\Gamma_i, \psi\bar{\psi}]. \tag{1.7.19}$$

Therefore a covariant derivative of the Dirac matrices is

$$\gamma^a{}_{;i} = \gamma^a{}_{,i} - [\Gamma_i, \gamma^a] = -[\Gamma_i, \gamma^a], \tag{1.7.20}$$

so

$$\gamma^j{}_{;i} = \gamma^j{}_{|i} = \gamma^j{}_{,i} + \Gamma^{\,j}_{k,i}\gamma^k - [\Gamma_i, \gamma^j]. \tag{1.7.21}$$

Accordingly

$$\gamma^a{}_{|i} = \omega^a{}_{bi}\gamma^b - [\Gamma_i, \gamma^a]. \tag{1.7.22}$$

The quantity $\bar{\psi}\gamma^i\psi_{|i}$ transforms under Lorentz rotations like a scalar:

$$\bar{\psi}\gamma^i\psi_{|i} \to \bar{\psi}L^{-1}L\gamma^i L^{-1}L\psi_{|i} = \bar{\psi}\gamma^i\psi_{|i}. \tag{1.7.23}$$

The relation $\eta_{ab|i} = 0$ implies that

$$\gamma^a{}_{|i} = 0, \tag{1.7.24}$$

because the Dirac matrices γ^a only depend on η_{ab}. Multiplying both sides of (1.7.22) by γ_a from the left gives

$$\omega_{abi}\gamma^a\gamma^b - \gamma_a\Gamma_i\gamma^a + 4\Gamma_i = 0. \tag{1.7.25}$$

We seek the solution of (1.7.25) in the form

$$\Gamma_i = -\frac{1}{4}\omega_{abi}\gamma^a\gamma^b - A_i, \tag{1.7.26}$$

where A_i is a spinor-tensor quantity with one vector index. Substituting (1.7.26) to (1.7.25), together with the identity $\gamma_c\gamma^a\gamma^b\gamma^c = 4\eta^{ab}$, gives

$$-\gamma_a A_i\gamma^a + 4A_i = 0, \tag{1.7.27}$$

so A_i is an arbitrary vector multiple of I. Therefore the spinor connection Γ_i is given, up to the addition of an arbitrary vector multiple of I, by the *Fock-Ivanenko coefficients*:

$$\Gamma_i = -\frac{1}{4}\omega_{abi}\gamma^a\gamma^b = -\frac{1}{2}\omega_{abi}G^{ab}. \tag{1.7.28}$$

Using the definition (1.5.15), we can also write (1.7.28) as

$$\Gamma_i = -\frac{1}{8}e^j_{c;i}[\gamma_j,\gamma^c] = \frac{1}{8}[\gamma^j{}_{;i},\gamma_j]. \tag{1.7.29}$$

1.7.3 Curvature spinor

The commutator of total covariant derivatives of a spinor is

$$\begin{aligned}\psi_{|ji} - \psi_{|ij} &= (\psi_{|j})_{,i} - \Gamma_i\psi_{|j} - \Gamma^{\,k}_{j\,i}\psi_{|k} - (\psi_{|i})_{,j} + \Gamma_j\psi_{|i} + \Gamma^{\,k}_{i\,j}\psi_{|k} \\ &= -\Gamma_{j,i}\psi + \Gamma_i\Gamma_j\psi + \Gamma_{i,j}\psi - \Gamma_j\Gamma_i\psi + 2S^k{}_{ij}\psi_{|k} = K_{ij}\psi + 2S^k{}_{ij}\psi_{|k},\end{aligned} \tag{1.7.30}$$

where $K_{ij} = -K_{ji}$ is defined as

$$K_{ij} = \Gamma_{i,j} - \Gamma_{j,i} + [\Gamma_i,\Gamma_j]. \tag{1.7.31}$$

Substituting (1.7.13) to (1.7.31) gives

$$\tilde{K}_{ij} = \tilde{\Gamma}_{i,j} - \tilde{\Gamma}_{j,i} + [\tilde{\Gamma}_i,\tilde{\Gamma}_j] = L(\Gamma_{i,j} - \Gamma_{j,i} + [\Gamma_i,\Gamma_j])L^{-1} = LK_{ij}L^{-1}, \tag{1.7.32}$$

so K_{ij} transforms under tetrad rotations like the Dirac matrices γ^a, that is, K_{ij} is a spinor with one spinor index and one adjoint-spinor index. We refer to K_{ij} as the *curvature spinor*.

The relation (1.7.24) leads to

$$\gamma^k{}_{|i} = 0. \tag{1.7.33}$$

Thus the commutator of covariant derivatives of the spacetime-dependent Dirac matrices vanishes:

$$2\gamma^k{}_{|[ji]} = R^k{}_{lij}\gamma^l + 2S^l{}_{ij}\gamma^k{}_{|l} + [K_{ij},\gamma^k] = R^k{}_{lij}\gamma^l + [K_{ij},\gamma^k] = 0. \tag{1.7.34}$$

Multiplying both sides of (1.7.34) by γ_k from the left gives

$$R_{klij}\gamma^k\gamma^l + \gamma_k K_{ij}\gamma^k - 4K_{ij} = 0. \tag{1.7.35}$$

We seek the solution of (1.7.35) in the form

$$K_{ij} = \frac{1}{4} R_{klij} \gamma^k \gamma^l + B_{ij}, \tag{1.7.36}$$

where B_{ij} is a spinor-tensor quantity with two vector indices. Substituting (1.7.36) to (1.7.35) gives

$$\gamma_k B_{ij} \gamma^k - 4B_{ij} = 0, \tag{1.7.37}$$

so B_{ij} is an antisymmetric-tensor multiple of I. The tensor B_{ij} is related to the vector A_i in (1.7.26) by

$$B_{ij} = A_{j,i} - A_{i,j} + [A_i, A_j]. \tag{1.7.38}$$

Because ψ has no indices other than spinor indices, A_i is a vector and $[A_i, A_j] = 0$. The invariance of (1.7.35) under the addition of an antisymmetric-tensor multiple B_{ij} of the unit matrix to the curvature spinor is related to the invariance of (1.7.25) under the addition of a vector multiple A_i of the unit matrix to the spinor connection. Setting $A_i = 0$, which corresponds to the Fock-Ivanenko spinor connection, gives $B_{ij} = 0$. Therefore the curvature spinor K_{ij} is given, up to the addition of an arbitrary antisymmetric-tensor multiple of I, by

$$K_{ij} = \frac{1}{4} R_{klij} \gamma^k \gamma^l = \frac{1}{2} R_{klij} G^{kl}. \tag{1.7.39}$$

References: [3, 4].

2 Fields

2.1 Principle of least action

The most general formulation of the law that governs the dynamics of classical systems is Hamilton's *principle of least action*, according to which every classical system is characterized by a definite scalar-density function $\mathfrak{L}$, and the dynamics of the system is such that a certain condition is satisfied. Let $\phi_A(x^i)$ be a set of physical *fields*, being differentiable functions of the coordinates, and let $\mathfrak{L}$ be a Lorentz covariant quantity constructed from the ϕ_A and their derivatives. Consider a scalar quantity

$$S = \frac{1}{c}\int \mathfrak{L}d\Omega, \tag{2.1.1}$$

where the integration is over some region in locally Minkowski spacetime. Let $\delta\phi_A$ be arbitrary small changes in ϕ_A (regarded as a dynamical variable) over the region of integration, which vanish on the boundary. Then the change in S can be written as

$$\delta S = \frac{1}{c}\int F^A\delta\phi_A d\Omega. \tag{2.1.2}$$

The principle of least action states that the dynamics of a physical system is given by the condition the scalar S be a local minimum. Therefore any infinitesimal change in the dynamics of the system does not alter the value of S:

$$\delta S = 0 \tag{2.1.3}$$

(S is a local extremum). If $\mathfrak{L}$ is covariant and ϕ_A transform covariantly under the Lorentz group, the variational condition (2.1.3) gives the Lorentz covariant equations

$$\phi_A = 0. \tag{2.1.4}$$

These equations are also invariant for any other transformations (internal symmetries) for which $\mathfrak{L}$ is invariant. $\mathfrak{L}$ is referred to as the *Lagrangian density*, S is the *action* functional, $\delta S = 0$ is the principle of least action, and $F^A = 0$ are the *field equations*. The field equations of a physical system are the result of the action being a local extremum. The condition that the action be a local minimum imposes additional restrictions on possible choices for S. The number of independent field equations for a given system is referred to as the number of the *degrees of freedom* representing this system.

In most cases $\mathfrak{L}$ contains only ϕ_A and their first derivatives (the Lagrangian density for the gravitational field contains second derivatives). A Lagrangian density containing higher derivatives can always be written in terms of first derivatives by increasing the number of the components of ϕ_A. Consider a physical system in the galilean frame of reference. If $\mathfrak{L}$ depends only on ϕ_A and $\partial_i\phi_A$, $\mathfrak{L} = \mathfrak{L}(\phi, \phi_{,i})$, then

$$\begin{aligned}\delta S &= \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\delta(\phi_{,i})\Big)d\Omega = \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}(\delta\phi)_{,i}\Big)d\Omega \\ &= \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}}{\partial\phi}\delta\phi - \partial_i\Big(\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\Big)\delta\phi + \partial_i\Big(\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\delta\phi\Big)\Big)d\Omega.\end{aligned} \tag{2.1.5}$$

The last term in the second line of (2.1.5) is a divergence, which, after integration, can be transformed into a hypersurface integral over the boundary of integration region, where $\delta\phi = 0$ on the boundary, so this term does not contribute to the action variation:

$$\delta S = \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}}{\partial\phi} - \partial_i\Big(\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\Big)\Big)\delta\phi d\Omega + \int\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\delta\phi dS_i = \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}}{\partial\phi} - \partial_i\Big(\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\Big)\Big)\delta\phi d\Omega. \tag{2.1.6}$$

If $\delta S = 0$ for arbitrary variations $\delta\phi$ that vanish on the boundary then

$$\frac{\partial\mathfrak{L}}{\partial\phi} - \partial_i\Big(\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\Big) = 0, \tag{2.1.7}$$

or

$$\frac{\delta\mathfrak{L}}{\delta\phi} = 0, \tag{2.1.8}$$

where

$$\frac{\delta\mathfrak{L}}{\delta\phi} = \frac{\partial\mathfrak{L}}{\partial\phi} - \partial_i\Big(\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\Big) \tag{2.1.9}$$

is a *variational derivative* of $\mathfrak{L}$ with respect to ϕ. This set of equations, for each component ϕ_A, is referred to as the *Lagrange equations*. Generalizing the Lagrange equations to an arbitrary coordinate frame gives

$$\frac{\partial\mathfrak{L}}{\partial\phi} - \nabla_i^{\{\}}\Big(\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\Big) = 0. \tag{2.1.10}$$

There is some arbitrariness in the choice of $\mathfrak{L}$; adding to it the divergence of an arbitrary vector density or multiplying it by a constant produces the same field equations. If a system consists of two noninteracting parts A and B, with corresponding Lagrangian densitites $\mathfrak{L}_A(\phi_A, \partial\phi_A)$ and $\mathfrak{L}_B(\phi_B, \partial\phi_B)$, then the Lagrangian for this system is the sum $\mathfrak{L}_A + \mathfrak{L}_B$. This additivity of the Lagrangian density express the fact that the field equations for either of the two parts do not involve quantities pertaining to the other part. If $\mathfrak{L}_A$ also depends on ϕ_B and/or $\partial\phi_B$, and/or $\mathfrak{L}_B$ depends on ϕ_A and/or $\partial\phi_A$, then the subsystems A and B interact.
References: [1, 2, 3].

2.2 Action for gravitational field

Consider a Lagrangian density that depends on the affine (or spin) connection and its first derivatives. Such Lagrangian density can be decomposed into the covariant part that contains derivatives of the affine/spin connection, which is referred to as the Lagrangian density for the *gravitational field*, and the covariant part that does not contain these derivatives, which is referred to as the Lagrangian density for *matter*. The simplest covariant scalar that can be constructed from the affine/spin connection and its first derivatives is the Ricci scalar R. The corresponding Lagrangian density for the gravitational field is proportional to the product of R and the scalar density $\sqrt{-\mathfrak{g}}$:

$$\mathfrak{L}_g = -\frac{1}{2\kappa}\sqrt{-\mathfrak{g}}R, \tag{2.2.1}$$

where κ is *Einstein's gravitational constant.* There exist two variational principles in the theory of the gravitational field. The *metric* variational principle regards the metric tensor or tetrad as a dynamical variable and assumes the affine connection to be the Levi-Civita connection. The *metric-affine* variational principle regards both the metric tensor (or tetrad) and the metric-compatible affine connection (or spin connection) as dynamical variables.

In the metric variational formulation, the Lagrangian density for the gravitational field is proportional to the Riemann scalar P:

$$\mathfrak{L}_g = -\frac{1}{2\kappa}\sqrt{-\mathfrak{g}}P. \tag{2.2.2}$$

Because P is linear in derivatives of $\{^{\,i}_{k\,l}\}$:

$$\begin{aligned}\sqrt{-\mathfrak{g}}P &= \sqrt{-\mathfrak{g}}g^{ik}(\{^{\,l}_{i\,k}\}_{,l} - \{^{\,l}_{i\,l}\}_{,k} + \{^{\,m}_{i\,k}\}\{^{\,l}_{m\,l}\} - \{^{\,m}_{i\,l}\}\{^{\,l}_{m\,k}\}) \\ &= (\sqrt{-\mathfrak{g}}g^{ik}\{^{\,l}_{i\,k}\})_{,l} - \{^{\,l}_{i\,k}\}(\sqrt{-\mathfrak{g}}g^{ik})_{,l} - (\sqrt{-\mathfrak{g}}g^{ik}\{^{\,l}_{i\,l}\})_{,k} + \{^{\,l}_{i\,l}\}(\sqrt{-\mathfrak{g}}g^{ik})_{,k} \\ &+\sqrt{-\mathfrak{g}}g^{ik}(\{^{\,m}_{i\,k}\}\{^{\,l}_{m\,l}\} - \{^{\,m}_{i\,l}\}\{^{\,l}_{m\,k}\}),\end{aligned} \tag{2.2.3}$$

we can subtract from $\sqrt{-\mathfrak{g}}P$ total derivatives without altering the field equations, replacing P by a noncovariant quantity G:

$$\begin{aligned}\sqrt{-\mathfrak{g}}G &= \{^{\,l}_{i\,l}\}(\sqrt{-\mathfrak{g}}g^{ik})_{,k} - \{^{\,l}_{i\,k}\}(\sqrt{-\mathfrak{g}}g^{ik})_{,l} + \sqrt{-\mathfrak{g}}g^{ik}(\{^{\,m}_{i\,k}\}\{^{\,l}_{m\,l}\} - \{^{\,m}_{i\,l}\}\{^{\,l}_{m\,k}\}) \\ &= \{^{\,l}_{i\,l}\}((\sqrt{-\mathfrak{g}}g^{ik})_{:k} + \{^{\,j}_{j\,k}\}\sqrt{-\mathfrak{g}}g^{ik} - \sqrt{-\mathfrak{g}}\{^{\,i}_{j\,k}\}g^{jk} - \sqrt{-\mathfrak{g}}\{^{\,k}_{j\,k}\}g^{ij}) \\ &-\{^{\,l}_{i\,k}\}((\sqrt{-\mathfrak{g}}g^{ik})_{:l} + \{^{\,j}_{j\,l}\}\sqrt{-\mathfrak{g}}g^{ik} - \sqrt{-\mathfrak{g}}\{^{\,i}_{j\,l}\}g^{jk} - \sqrt{-\mathfrak{g}}\{^{\,k}_{j\,l}\}g^{ij}) \\ &+\sqrt{-\mathfrak{g}}g^{ik}(\{^{\,m}_{i\,k}\}\{^{\,l}_{m\,l}\} - \{^{\,m}_{i\,l}\}\{^{\,l}_{m\,k}\}) = \{^{\,l}_{i\,l}\}(\{^{\,j}_{j\,k}\}\sqrt{-\mathfrak{g}}g^{ik} - \sqrt{-\mathfrak{g}}\{^{\,i}_{j\,k}\}g^{jk} \\ &-\sqrt{-\mathfrak{g}}\{^{\,k}_{j\,k}\}g^{ij}) - \{^{\,l}_{i\,k}\}(\{^{\,j}_{j\,l}\}\sqrt{-\mathfrak{g}}g^{ik} - \sqrt{-\mathfrak{g}}\{^{\,i}_{j\,l}\}g^{jk} - \sqrt{-\mathfrak{g}}\{^{\,k}_{j\,l}\}g^{ij}) \\ &+\sqrt{-\mathfrak{g}}g^{ik}(\{^{\,m}_{i\,k}\}\{^{\,l}_{m\,l}\} - \{^{\,m}_{i\,l}\}\{^{\,l}_{m\,k}\}) = \sqrt{-\mathfrak{g}}g^{ik}(\{^{\,m}_{i\,l}\}\{^{\,l}_{m\,k}\} - \{^{\,m}_{i\,k}\}\{^{\,l}_{m\,l}\}).\end{aligned} \tag{2.2.4}$$

Therefore

$$G = g^{ik}(\{^{\,m}_{i\,l}\}\{^{\,l}_{m\,k}\} - \{^{\,m}_{i\,k}\}\{^{\,l}_{m\,l}\}), \tag{2.2.5}$$

and the Lagrangian density for the gravitational field is

$$\mathfrak{L}_g = -\frac{1}{2\kappa}\sqrt{-\mathfrak{g}}G. \tag{2.2.6}$$

.

Any coordinate transformation results in variations of g^{ik}, so S is not necessarily a minimum with respect to these variations (only an extremum) because not all δg^{ik} correspond to actual variations of the gravitational field. In order to exclude the variations δg^{ik} resulting from changing the coordinates, we must impose on the metric tensor 4 arbitrary constraints. If we choose

$$g_{0\alpha} = 0,\ \ |g_{\alpha\beta}| = \text{const}, \tag{2.2.7}$$

then G becomes

$$G = -\frac{1}{4}g^{00}g^{\alpha\beta}g^{\gamma\delta}g_{\alpha\gamma,0}g_{\beta\delta,0}. \tag{2.2.8}$$

In the locally galilean frame $g_{\alpha\beta} = -\delta_{\alpha\beta}$, so

$$G = -\frac{1}{4}g^{00}(g_{\alpha\beta,0})^2. \tag{2.2.9}$$

For physical systems $g^{00} > 0$. Therefore in order for S to have a minimum, κ must be positive, otherwise an arbitrarily rapid change of $g_{\alpha\beta}$ in time would result in an arbitrarily low value of S and there would be no minimum.
References: [2, 3].

2.3 Matter

2.3.1 Metric dynamical energy-momentum density

The variation of the matter action $S_m = \int \mathfrak{L}_m d\Omega$ with respect to the metric tensor,

$$\delta S_m = \frac{1}{2c}\int \mathcal{T}_{ij}\delta g^{ij}d\Omega = -\frac{1}{2c}\int \mathcal{T}^{ij}\delta g_{ij}d\Omega, \tag{2.3.1}$$

defines the *metric dynamical energy-momentum density* $\mathcal{T}_{ij}$, which is symmetric:

$$\mathcal{T}_{ij} = \mathcal{T}_{ji}. \tag{2.3.2}$$

Equivalently

$$\mathcal{T}_{ij} = 2\frac{\delta\mathfrak{L}_m}{\delta g^{ij}} = 2\Big(\frac{\partial\mathfrak{L}_m}{\partial g^{ij}} - \partial_k(\frac{\partial\mathfrak{L}_m}{\partial g^{ij}}\Big). \tag{2.3.3}$$

The *metric dynamical energy-momentum tensor* T_{ij} is defined as

$$T_{ij} = \frac{\mathcal{T}_{ij}}{\sqrt{-\mathfrak{g}}}. \tag{2.3.4}$$

2.3.2 Tetrad dynamical energy-momentum density

The variation of the matter action S_m with respect to the tetrad,

$$\delta S_m = \frac{1}{c}\int \mathfrak{T}_i{}^a \delta e^i_a d\Omega, \tag{2.3.5}$$

defines the *tetrad dynamical energy-momentum* density $\mathfrak{T}_i{}^a$. Equivalently

$$\delta\mathfrak{L}_m = \mathfrak{T}_i{}^a\delta e^i_a \tag{2.3.6}$$

or

$$\mathfrak{T}_i{}^a = \frac{\delta\mathfrak{L}_m}{\delta e^i_a}. \tag{2.3.7}$$

If $\mathfrak{L}_m$ depends only on tensor matter fields expressed in terms of the coordinate indices and it depends on neither derivatives of the metric tensor nor derivatives of the tetrad then the tetrad enters $\mathfrak{L}_m$ only through the metric tensor, in a combination $g^{ij} = \eta^{ab}e^i_a e^j_b$. Thus

$$\delta e^i_a = \frac{1}{2}e_{aj}\delta g^{ij}. \tag{2.3.8}$$

Substituting (2.3.8) to (2.3.5) gives

$$\delta S_m = \frac{1}{2c} \int \mathfrak{T}_{ij} \delta g^{ij} d\Omega, \tag{2.3.9}$$

where

$$\mathfrak{T}_{ij} = e_{aj} \mathfrak{T}_i^{\ a}. \tag{2.3.10}$$

The tensor $\mathfrak{T}_{ij}$ is generally not symmetric. Comparing (2.3.9) with (2.3.1) gives the relation between the tetrad dynamical energy-momentum density and the metric dynamical energy-momentum density for tensor matter fields:

$$\mathfrak{T}_{(ij)} = \mathcal{T}_{ij}. \tag{2.3.11}$$

2.3.3 Canonical energy-momentum density

If we express the matter Lagrangian density $\mathfrak{L}_m$, depending on matter fields ϕ and their first derivatives $\phi_{,i}$, only in terms of Lorentz and spinor indices, then the tetrad appears in $\mathfrak{L}_m$ only through a derivative of ϕ, in a covariant combination $e^i_a \phi_{|i}$. Since $\mathfrak{L}_m = \mathfrak{e}L$, where L is a scalar, we obtain

$$\delta\mathfrak{L}_m = \mathfrak{e}\delta L - \mathfrak{e}e^a_i L\delta e^i_a = \mathfrak{e}\frac{\partial L}{\partial\phi_{|a}}\phi_{|i}\delta e^i_a - \mathfrak{L}_m e^a_i \delta e^i_a = \Big(\frac{\partial\mathfrak{L}_m}{\partial\phi_{|a}}\phi_{|i} - \mathfrak{L}_m e^a_i\Big)\delta e^i_a$$
$$= \Big(\frac{\partial\mathfrak{L}_m}{\partial\phi_{,a}}\phi_{|i} - \mathfrak{L}_m e^a_i\Big)\delta e^i_a. \tag{2.3.12}$$

The last term in (2.3.12),

$$\Theta_i^{\ a} = \frac{\partial\mathfrak{L}_m}{\partial\phi_{,a}}\phi_{|i} - e^a_i\mathfrak{L}_m, \tag{2.3.13}$$

is referred to as the *canonical energy-momentum density.* Accordingly

$$\Theta_j^{\ i} = \frac{\partial\mathfrak{L}_m}{\partial\phi_{,i}}\phi_{|j} - \delta^i_j\mathfrak{L}_m. \tag{2.3.14}$$

Comparing (2.3.12) with (2.3.6) shows that the canonical energy-momentum density is identical with the dynamical tetrad energy-momentum density:

$$\Theta_i^{\ a} = \mathfrak{T}_i^{\ a}. \tag{2.3.15}$$

2.3.4 Spin density

The variation of the matter action S_m with respect to the spin connection,

$$\delta S_m = \frac{1}{2c}\mathfrak{S}_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i}d\Omega, \tag{2.3.16}$$

defines the *dynamical spin density* $\mathfrak{S}_{ab}^{\ \ i}$:

$$\mathfrak{S}_{ab}^{\ \ i} = 2\frac{\delta\mathfrak{L}_m}{\delta\omega^{ab}_{\ \ i}}, \tag{2.3.17}$$

which is antisymmetric in the Lorentz indices:

$$\mathfrak{S}_{ab}{}^{i} = -\mathfrak{S}_{ba}{}^{i}. \tag{2.3.18}$$

In the metric variational formulation of gravity, the variations $\delta\omega^{ab}{}_{i} = \delta\omega^{\{\}ab}{}_{i}$ are functions of the variations δe^{i}_{a} and their derivatives, so the spin density is a function of the energy-momentum density. In the metric-affine variational formulation of gravity, the variations $\delta\omega^{ab}{}_{i}$ are independent of δe^{i}_{a} and their derivatives. The relation (1.5.29) indicates that the spin density is generated by the contortion tensor:

$$\mathfrak{S}_{ij}{}^{k} = 2\frac{\delta\mathfrak{L}_m}{\delta C^{ij}{}_{k}}. \tag{2.3.19}$$

Accordingly, the variation of $\mathfrak{L}_m$ with respect to the torsion tensor,

$$\tau_{i}{}^{jk} = 2\frac{\delta\mathfrak{L}_m}{\delta S^{i}{}_{jk}}, \tag{2.3.20}$$

is a homogeneous linear function of the spin connection because of (1.4.29):

$$\begin{aligned}\tau_{ijk} &= 2\frac{\delta\mathfrak{L}_m}{\delta S^{ijk}} = 2\frac{\delta\mathfrak{L}_m}{\delta C^{lmn}}\frac{\partial C^{lmn}}{\partial S^{ijk}} = \mathfrak{S}_{lmn}(\delta^{l}_{i}\delta^{m}_{[j}\delta^{n}_{k]} + \delta^{m}_{i}\delta^{n}_{[j}\delta^{l}_{k]} + \delta^{n}_{i}\delta^{m}_{[j}\delta^{l}_{k]}) \\ &= \mathfrak{S}_{ijk} - \mathfrak{S}_{jki} + \mathfrak{S}_{kij}, \qquad (2.3.21)\\ \mathfrak{S}_{ijk} &= \tau_{[ij]k}, \qquad (2.3.22)\end{aligned}$$

antisymmetric in the last two indices:

$$\tau_{ijk} = -\tau_{ikj}. \tag{2.3.23}$$

The variation of $\mathfrak{L}_m$ with respect to the metric-compatible affine connection in the metric-affine variational formulation of gravity is equivalent to the variation with respect to the torsion (or contortion) tensor. The spin connection $\omega^{ab}{}_{i}$ enters $\mathfrak{L}_m$ only through derivatives of ϕ, in a combination $-\frac{\partial\mathfrak{L}}{\partial\phi_{,i}}\Gamma_i\phi$, where Γ_i is the covariant derivative acting on ϕ:

$$\Gamma_i = -\frac{1}{2}\omega_{abi}G^{ab}. \tag{2.3.24}$$

Consequently, the dynamical spin density $\mathfrak{S}_{ab}{}^{i}$ is identical with

$$\Sigma_{ab}{}^{i} = \frac{\partial\mathfrak{L}_m}{\partial\phi_{,i}}G_{ab}\phi, \tag{2.3.25}$$

referred to as the *canonical spin density*. *Spin tensor* is defined as

$$s_{ijk} = \frac{\mathfrak{S}_{ijk}}{\sqrt{-\mathfrak{g}}}. \tag{2.3.26}$$

2.3.5 Belinfante-Rosenfeld relation

The total variation of the matter action with respect to geometrical variables is either

$$\delta S_m = \frac{1}{c}\int d\Omega \mathfrak{T}_i^{\ a}\delta e^i_a + \frac{1}{2c}\int d\Omega \mathfrak{S}_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i} \tag{2.3.27}$$

or

$$\delta S_m = \frac{1}{2c}\int d\Omega \mathcal{T}_{ik}\delta g^{ik} + \frac{1}{2c}\int d\Omega \tau_j^{\ ik}\delta S^j_{\ ik}. \tag{2.3.28}$$

Equation (1.5.25) gives

$$\begin{aligned}
&\frac{1}{2}\int d\Omega \tau_j^{\ ik}\delta S^j_{\ ik} = \frac{1}{2}\int d\Omega \tau_j^{\ ik}\Big(\delta(e^j_a e_{ib}\omega^{ab}_{\ \ k}) + \delta e^a_{i,k}e^j_a + e^a_{i,k}\delta e^j_a\Big) \\
&= \frac{1}{2}\int d\Omega\Big(\tau_j^{\ li}\delta(e^j_a e_{lb})\omega^{ab}_{\ \ i} + \tau_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i} + (\tau_j^{\ ik}e^j_a\delta e^a_i)_{,k} - (\tau_j^{\ ik}e^j_a)_{,k}\delta e^a_i + \tau_j^{\ ik}e^a_{i,k}\delta e^j_a\Big) \\
&= \frac{1}{2}\int d\Omega\Big(\tau_j^{\ lk}\omega^{cb}_{\ \ k}e_{lb}\delta e^j_c + \tau_j^{\ li}\omega^{ab}_{\ \ i}e^j_a\delta e_{lb} + \tau_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i} - (\tau_j^{\ ik}e^j_a)_{,k}\delta e^a_i + \tau_j^{\ lm}e^b_{l,m}\delta e^j_b\Big) \\
&+\frac{1}{2}\int dS_k\tau_j^{\ ik}e^j_a\delta e^a_i = \frac{1}{2}\int d\Omega\Big(-\tau_j^{\ lk}\omega^{cb}_{\ \ k}e_{lb}e^i_c e^j_a\delta e^a_i + \tau_j^{\ il}\omega^b_{\ al}e^j_b\delta e^a_i + \tau_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i} \\
&-(\tau_a^{\ ik}{}_{|k} - S^i_{\ jk}\tau_a^{\ jk} - 2S_j\tau_a^{\ ij} + \omega^b_{\ ak}\tau_b^{\ ik})\delta e^a_i - \tau_j^{\ lm}e^b_{l,m}e^i_b e^j_a\delta e^a_i\Big) \\
&= \frac{1}{2}\int d\Omega\Big(\tau_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i} - \tau_j^{\ ik}{}_{;k}e^j_a\delta e^a_i + 2S_j\tau_a^{\ ij}\delta e^a_i\Big),
\end{aligned} \tag{2.3.29}$$

so comparing of (2.3.27) with (2.3.28) leads to

$$\begin{aligned}
&\int d\Omega \mathfrak{T}_i^{\ a}\delta e^i_a + \frac{1}{2}\int d\Omega \mathfrak{S}_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i} = \frac{1}{2}\int d\Omega \mathcal{T}_{ik}\delta g^{ik} + \frac{1}{2}\int d\Omega\Big(\tau_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i} - \tau_j^{\ ik}{}_{;k}e^j_a\delta e^a_i \\
&+2S_j\tau_a^{\ ij}\delta e^a_i\Big) = \int d\Omega \mathcal{T}_{ik}e^{ka}\delta e^i_a + \frac{1}{2}\int d\Omega \tau_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i} + \frac{1}{2}\int d\Omega \tau_i^{\ jk}{}_{;k}e^a_j\delta e^i_a \\
&-\int d\Omega S_j\tau_b^{\ kj}e^a_k e^b_i\delta e^i_a.
\end{aligned} \tag{2.3.30}$$

The terms with $\delta\omega^{ab}_{\ \ i}$ give (2.3.22), while the terms with δe^i_a give

$$\mathfrak{T}_i^{\ a} = \mathcal{T}_{ik}e^{ka} + \frac{1}{2}\tau_i^{\ jk}{}_{;k}e^a_j - S_j\tau_i^{\ aj} \tag{2.3.31}$$

or

$$\mathcal{T}_{ik} = \mathfrak{T}_{ik} - \frac{1}{2}\nabla_j(\mathfrak{S}_{ik}^{\ \ j} - \mathfrak{S}_k^{\ j}{}_i + \mathfrak{S}^j_{\ ik}) + S_j(\mathfrak{S}_{ik}^{\ \ j} - \mathfrak{S}_k^{\ j}{}_i + \mathfrak{S}^j_{\ ik}). \tag{2.3.32}$$

Equation (2.3.32) is referred to as the *Belinfante-Rosenfeld relation* between the dynamical metric and dynamical tetrad (canonical) energy-momentum densites. In the absence of torsion, (2.3.32) is consistent with (2.3.11). The Belinfante-Rosenfeld relation can be written as

$$T_{ik} = \frac{1}{\sqrt{-\mathfrak{g}}}\mathfrak{T}_{ik} - \frac{1}{2}\nabla^*_j(s_{ik}^{\ \ j} - s_k^{\ j}{}_i + s^j_{\ ik}), \tag{2.3.33}$$

where

$$\nabla^*_i = \nabla_i - 2S_i \tag{2.3.34}$$

is the *modified covariant derivative.*
References: [2, 3, 5, 6, 7].

2.4 Symmetries and conservation laws

2.4.1 Noether theorem

Consider a physical system in the galilean frame of reference, described by the Lagrangian density $\mathfrak{L}$ that depends on matter fields ϕ_A, their first derivatives $\phi_{,i}$, and the coordinates x^i. The change of the Lagrangian density $\delta\mathfrak{L}$ under an infinitesimal coordinate transformation (1.2.54) is thus

$$\delta\mathfrak{L} = \frac{\partial\mathfrak{L}}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}}{\partial\phi_{,i}}\delta(\phi_{,i}) + \frac{\bar{\partial}\mathfrak{L}}{\partial x^i}\xi^i, \tag{2.4.1}$$

where the changes $\delta\phi$ and $\delta(\phi_{,i})$ are brought by the transformation (1.2.54) and $\bar{\partial}$ denotes partial differentiation with respect to x^i at constant ϕ and $\phi_{,i}$. The variation $\delta\mathfrak{L}$ under this transformation is also given by (1.2.63):

$$\delta\mathfrak{L} = \xi^i_{\ ,i}\mathfrak{L}. \tag{2.4.2}$$

Using the Lagrange equations (2.1.7) and the identities

$$\mathfrak{L}_{,i} = \frac{\bar{\partial}\mathfrak{L}}{\partial x^i} + \frac{\partial\mathfrak{L}}{\partial\phi}\phi_{,i} + \frac{\partial\mathfrak{L}}{\partial\phi_{,j}}\phi_{,ji}, \tag{2.4.3}$$

$$\delta(\phi_{,i}) = (\delta\phi)_{,i} - \xi^j_{\ ,i}\phi_{,j}, \tag{2.4.4}$$

we bring (2.4.1) to

$$\delta\mathfrak{L} = \xi^i\mathfrak{L}_{,i} + \Big(\frac{\partial\mathfrak{L}}{\partial\phi_{,i}}(\delta\phi - \xi^j\phi_{,j})\Big)_{,i}. \tag{2.4.5}$$

Combining (2.4.2) and (2.4.5) gives the conservation law,

$$\mathfrak{J}^i_{\ ,i} = 0, \tag{2.4.6}$$

for the current

$$\mathfrak{J}^i = \xi^i\mathfrak{L} + \frac{\partial\mathfrak{L}}{\partial\phi_{,i}}(\delta\phi - \xi^j\phi_{,j}) = \xi^i\mathfrak{L} + \frac{\partial\mathfrak{L}}{\partial\phi_{,i}}\bar{\delta}\phi. \tag{2.4.7}$$

Equations (2.4.6) and (2.4.7) represent the *Noether theorem*, which states that to each continuous symmetry of a Lagrangian density there corresponds a conservation law. Generalizing (2.4.6) to an arbitrary coordinate frame gives

$$\mathfrak{J}^i_{\ :i} = 0. \tag{2.4.8}$$

2.4.2 Conservation of spin

The Lorentz group is the group of tetrad rotations. Since a physical matter Lagrangian density $\mathfrak{L}_m(\phi, \phi_{,i})$ is invariant under local, proper Lorentz transformations, it is invariant under tetrad rotations:

$$\delta\mathfrak{L}_m = \frac{\partial\mathfrak{L}_m}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}_m}{\partial\phi_{,i}}\delta(\phi_{,i}) + \mathfrak{T}_i^{\ a}\delta e^i_a + \frac{1}{2}\mathfrak{S}_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i} = 0, \tag{2.4.9}$$

where the changes δ correspond to a tetrad rotation. Under integration of (2.4.9) over spacetime, the first two terms vanish because of the Lagrange equations for ϕ (2.1.7):

$$\int\Big(\mathfrak{T}_i^{\,a}\delta e^i_a+\frac{1}{2}\mathfrak{S}_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i}\Big)d^4x=0. \tag{2.4.10}$$

For an infinitesimal Lorentz transformation (1.6.7), the tetrad e^a_i changes by

$$\delta e^a_i=\tilde{e}^a_i-e^a_i=\Lambda^a_{\ b}e^b_i-e^a_i=\epsilon^a_{\ i}, \tag{2.4.11}$$

and the tetrad e^i_a, because of the identity $\delta(e^a_i e^j_a)=0$, according to

$$\delta e^i_a=-\epsilon^i_{\ a}. \tag{2.4.12}$$

The spin connection changes by

$$\delta\omega^{ab}_{\ \ i}=\delta(e^a_j\omega^{jb}_{\ \ i})=\epsilon^a_{\ j}\omega^{jb}_{\ \ i}-e^a_j\epsilon^{jb}_{\ \ ;i}=\epsilon^a_{\ c}\omega^{cb}_{\ \ i}-e^a_j\epsilon^{jb}_{\ \ |i}+\epsilon^a_{\ c}\omega^{bc}_{\ \ i}=-\epsilon^{ab}_{\ \ |i}. \tag{2.4.13}$$

Substituting (2.4.12) and (2.4.13) to (2.4.10), together with partial integration (1.2.33), gives

$$\begin{aligned}&-\int\Big(\mathfrak{T}_i^{\,a}\epsilon^i_{\ a}+\frac{1}{2}\mathfrak{S}_{ab}^{\ \ i}\epsilon^{ab}_{\ \ |i}\Big)d^4x=-\int\Big(\mathfrak{T}_{ij}\epsilon^{ij}+\frac{1}{2}\mathfrak{S}_{ij}^{\ \ k}\epsilon^{ij}_{\ \ |k}\Big)d^4x\\&=\int\Big(-\mathfrak{T}_{[ij]}-S_k\mathfrak{S}_{ij}^{\ \ k}+\frac{1}{2}\mathfrak{S}_{ij\ \ ;k}^{\ \ k}\Big)\epsilon^{ij}d^4x.\end{aligned} \tag{2.4.14}$$

Since the infinitesimal Lorentz rotation ϵ^{ij} is arbitrary, we obtain the covariant *conservation law for the spin density*:

$$\mathfrak{S}_{ij\ \ ;k}^{\ \ k}=\mathfrak{T}_{ij}-\mathfrak{T}_{ji}+2S_k\mathfrak{S}_{ij}^{\ \ k} \tag{2.4.15}$$

or

$$\nabla^*_k s_{ij}^{\ \ k}=\frac{1}{\sqrt{-\mathfrak{g}}}(\mathfrak{T}_{ij}-\mathfrak{T}_{ji}). \tag{2.4.16}$$

The conservation law for the spin density (2.4.16) also results from antisymmetrizing the Belinfante-Rosenfeld relation (2.3.33) with respect to the indices i,k. If we use the metric-compatible affine connection $\Gamma^{\,k}_{i\,j}$, which is invariant under tetrad rotations, instead of the spin connection $\omega^{ab}_{\ \ i}$ as a variable in $\mathfrak{L}_m$ then we must replace the term with $\delta\omega^{ab}_{\ \ i}$ in (2.4.9) by a term with $\delta(e^i_{a,j})$.

2.4.3 Conservation of metric energy-momentum

Consider the metric variational formulation of gravity. Under an infinitesimal coordinate transformation (1.2.54), the matter Lagrangian density $\mathfrak{L}_m(\phi,\phi_{,i})$ changes according to

$$\delta\mathfrak{L}_m=\frac{\partial\mathfrak{L}_m}{\partial\phi}\delta\phi+\frac{\partial\mathfrak{L}_m}{\partial\phi_{,i}}\delta(\phi_{,i})+\frac{\partial\mathfrak{L}_m}{\partial g^{ik}}\delta g^{ik}+\frac{\partial\mathfrak{L}_m}{\partial g^{ik}_{\ \ ,l}}\delta(g^{ik}_{\ \ ,l}). \tag{2.4.17}$$

The matter action $S_m = \frac{1}{c}\int \mathfrak{L}_m(\phi, \phi_{,i})d\Omega$ is a scalar, so it does not change under this transformation:

$$\delta S_m = \frac{1}{c}\int \Big(\frac{\partial \mathfrak{L}_m}{\partial \phi}\delta\phi + \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}}\delta(\phi_{,i}) + \frac{\partial \mathfrak{L}_m}{\partial g^{ik}}\delta g^{ik} + \frac{\partial \mathfrak{L}_m}{\partial {g^{ik}}_{,l}}\delta({g^{ik}}_{,l})\Big)d\Omega = 0. \tag{2.4.18}$$

The first two terms in (2.4.18) vanish because of the Lagrange equations for ϕ (2.1.7), so

$$\delta S_m = \frac{1}{c}\int \Big(\frac{\partial \mathfrak{L}_m}{\partial g^{ik}} - \partial_l \frac{\partial \mathfrak{L}_m}{\partial {g^{ik}}_{,l}}\Big)\delta g^{ik} d\Omega = \frac{1}{c}\int \frac{\delta \mathfrak{L}_m}{\delta g^{ik}}\delta g^{ik} d\Omega = \frac{1}{2c}\int \mathcal{T}_{ij}\delta g^{ij} d\Omega = 0. \tag{2.4.19}$$

If the components of the metric tensor change because of an infinitesimal coordinate transformation (1.2.54) then the corresponding variation of the metric tensor is given by (1.4.41):

$$\delta g_{ij} = \bar{\delta} g_{ij} = -2\xi_{(i:j)}, \tag{2.4.20}$$

so

$$\begin{aligned}\delta S_m = \bar{\delta} S_m &= -\frac{1}{2c}\int \mathcal{T}^{ij}\bar{\delta} g_{ij} d\Omega = -\frac{1}{c}\int \mathcal{T}^{ij}\xi_{(i:j)} d\Omega = -\frac{1}{c}\int \mathcal{T}^{ij}\xi_{i:j} d\Omega \\ &= -\frac{1}{c}\int (\mathcal{T}^{ij}\xi_i)_{:j} d\Omega + \frac{1}{c}\int {\mathcal{T}^{ij}}_{:j}\xi_i d\Omega = -\frac{1}{c}\int (\mathcal{T}^{ij}\xi_i)_{,j} d\Omega + \frac{1}{c}\int {\mathcal{T}^{ij}}_{:j}\xi_i d\Omega \\ &= -\frac{1}{c}\int \mathcal{T}^{ij}\xi_i dS_j + \frac{1}{c}\int {\mathcal{T}^{ij}}_{:j}\xi_i d\Omega = 0. \end{aligned} \tag{2.4.21}$$

If the variation of the coordinates ξ^i vanishes on the boundary of the region of integration then

$$\int {\mathcal{T}^{ij}}_{:j}\xi_i d\Omega = 0, \tag{2.4.22}$$

which, for arbitrary variations ξ^i gives the covariant conservation of the metric energy-momentum density (4 equations):

$$ {\mathcal{T}^{ij}}_{:j} = 0. \tag{2.4.23}$$

Equivalently

$$ {T^{ij}}_{:j} = 0. \tag{2.4.24}$$

Note that vanishing of $\int \mathcal{T}^{ij}\bar{\delta} g_{ij} d\Omega$ in (2.4.21) does not imply $\mathcal{T}^{ij} = 0$, because 10 variations $\bar{\delta} g_{ij}$ are functions of 4 variations ξ^i and thus not independent.

2.4.4 Conservation of tetrad energy-momentum

The matter Lagrangian density $\mathfrak{L}_m$ is invariant under infinitesimal translations of the coordinate system (1.2.54). The corresponding changes of the tetrad and spin connection are given by Lie derivatives

$$\bar{\delta} e^i_a = -\mathcal{L}_\xi e^i_a = {\xi^i}_{,j} e^j_a - \xi^j e^i_{a,j}, \tag{2.4.25}$$

$$\bar{\delta} {\omega^{ab}}_i = -\mathcal{L}_\xi {\omega^{ab}}_i = -{\xi^j}_{,i} {\omega^{ab}}_j - \xi^j {\omega^{ab}}_{i,j}. \tag{2.4.26}$$

Equation (2.4.10) becomes now

$$\int\Big(\mathfrak{T}_i^{\ a}\bar{\delta}e^i_a + \frac{1}{2}\mathfrak{S}_{ab}^{\ \ i}\bar{\delta}\omega^{ab}_{\ \ i}\Big)d^4x = 0. \tag{2.4.27}$$

Substituting (2.4.25) and (2.4.26) into (2.4.27) gives

$$\begin{aligned}&\int\Big(\mathfrak{T}_i^{\ a}\xi^i_{\ ,j}e^j_a - \mathfrak{T}_i^{\ a}\xi^j e^i_{a,j} - \frac{1}{2}\mathfrak{S}_{ab}^{\ \ i}\xi^j_{\ ,i}\omega^{ab}_{\ \ j} - \frac{1}{2}\mathfrak{S}_{ab}^{\ \ i}\xi^j\omega^{ab}_{\ \ i,j}\Big)d^4x \\ &= \int\Big(-\mathfrak{T}_{i\ ,j}^{\ j} - \mathfrak{T}_j^{\ a}e^j_{a,i} + \frac{1}{2}(\mathfrak{S}_{ab}^{\ \ j}\omega^{ab}_{\ \ i})_{,j} - \frac{1}{2}\mathfrak{S}_{ab}^{\ \ j}\omega^{ab}_{\ \ j,i}\Big)\xi^i d^4x = 0.\end{aligned} \tag{2.4.28}$$

This equation holds for an arbitrary vector ξ^i, so we obtain

$$\begin{aligned}&\mathfrak{S}_{ab\ ,j}^{\ \ j}\omega^{ab}_{\ \ i} + \mathfrak{S}_{ab}^{\ \ j}(\omega^{ab}_{\ \ i,j} - \omega^{ab}_{\ \ j,i}) - 2\mathfrak{T}_{i\ ,j}^{\ j} - 2\mathfrak{T}_j^{\ a}e^j_{a,i} \\ &= (\mathfrak{S}_{ab\ |j}^{\ \ j} - 2S_k\mathfrak{S}_{ab}^{\ \ k} + \mathfrak{S}_{cb}^{\ \ j}\omega^c_{\ aj} + \mathfrak{S}_{ac}^{\ \ j}\omega^c_{\ bj})\omega^{ab}_{\ \ i} - 2\mathfrak{T}_{i\ ,j}^{\ j} - 2\mathfrak{T}_j^{\ a}e^j_{a,i} \\ &+\mathfrak{S}_{ab}^{\ \ j}(-R^{ab}_{\ \ ij} + \omega^a_{\ ci}\omega^{cb}_{\ \ j} - \omega^a_{\ cj}\omega^{cb}_{\ \ i}) = 0,\end{aligned} \tag{2.4.29}$$

which reduces to

$$\begin{aligned}&(\mathfrak{S}_{ab\ |j}^{\ \ j} - 2S_k\mathfrak{S}_{ab}^{\ \ k})\omega^{ab}_{\ \ i} - R^{ab}_{\ \ ij}\mathfrak{S}_{ab}^{\ \ j} - 2\mathfrak{T}_{i\ ;j}^{\ j} + 4S_j\mathfrak{T}_i^{\ j} - 2\mathfrak{T}_{jk}\omega^{jk}_{\ \ i} + 4S^{jk}_{\ \ i}\mathfrak{T}_{jk} \\ &= (\mathfrak{S}_{jl\ ;k}^{\ \ k} - 2S_k\mathfrak{S}_{jl}^{\ \ k})\omega^{jl}_{\ \ i} - R^{kl}_{\ \ ij}\mathfrak{S}_{kl}^{\ \ j} - 2\mathfrak{T}_{i\ ;j}^{\ j} + 4S_j\mathfrak{T}_i^{\ j} - 2\mathfrak{T}_{jk}\omega^{jk}_{\ \ i} \\ &+4S^{jk}_{\ \ i}\mathfrak{T}_{jk} = 0.\end{aligned} \tag{2.4.30}$$

The conservation law for the spin density (2.4.15) brings (2.4.30) to the covariant *conservation law for the energy-momentum density*:

$$\mathfrak{T}_{i\ ;j}^{\ j} = 2S_j\mathfrak{T}_i^{\ j} + 2S^j_{\ ki}\mathfrak{T}_j^{\ k} + \frac{1}{2}\mathfrak{S}_{kl}^{\ \ j}R^{kl}_{\ \ ji} \tag{2.4.31}$$

or

$$\mathfrak{T}^{ij}_{\ \ :j} = C_{jk}^{\ \ i}\mathfrak{T}^{jk} + \frac{1}{2}\mathfrak{S}_{klj}R^{klji}. \tag{2.4.32}$$

2.4.5 Conservation laws for Lorentz group

Consider a matter Lagrangian $\mathfrak{L}_m$ for a physical system in the galilean and geodesic frame of reference, that depends on the coordinates only through a field ϕ and its first derivatives $\phi_{,i}$. Therefore

$$\partial_i\mathfrak{L}_m = \frac{\mathfrak{L}_m}{\partial\phi}\phi_{,i} + \frac{\partial\mathfrak{L}_m}{\partial\phi_{,j}}\phi_{,ji} = \partial_j\Big(\frac{\partial\mathfrak{L}_m}{\partial\phi_{,j}}\Big)\phi_{,i} + \frac{\partial\mathfrak{L}_m}{\partial\phi_{,j}}\phi_{,ji} = \partial_j\Big(\frac{\mathfrak{L}_m}{\partial\phi_{,j}}\phi_{,i}\Big), \tag{2.4.33}$$

where we use the Lagrange equations (2.1.7), from which we obtain the conservation law,

$$\theta_{i\ ,j}^{\ j} = 0, \tag{2.4.34}$$

for

$$\theta_i^{\ j} = \frac{\partial\mathfrak{L}_m}{\partial\phi_{,j}}\phi_{,i} - \delta^j_i\mathfrak{L}_m. \tag{2.4.35}$$

The conservation law (2.4.34) is a special case of (2.4.31) in the absence of torsion and spin, expressed in the Galilean and geodesic frame. The quantity (2.4.35) is a special case of the canonical energy-momentum density (2.3.14) in the absence of torsion and spin, expressed in the galilean and geodesic frame.

If x^i are Cartesian coordinates then for translations, $\xi^i = \epsilon^i$ =const and $\delta\phi = 0$, the current (2.4.7) is

$$\mathfrak{J}^i = \epsilon^i \mathfrak{L}_m - \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} \epsilon^j \phi_{,j}. \tag{2.4.36}$$

The conservation law (2.4.6) gives

$$\epsilon^j \theta_{j\ ,i}^{\ i} = 0, \tag{2.4.37}$$

which gives (2.4.34) because ϵ^i are arbitrary. For Lorentz rotations, $\xi^i = \epsilon^i_{\ j} x^j$ and $\phi = \frac{1}{2}\epsilon_{ij} G^{ij}\phi$, where G^{ij} are the generators of the Lorentz group, the current (2.4.7) is

$$\mathfrak{J}^i = \epsilon^{ij} x_j \mathfrak{L}_m + \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} \Big(\frac{1}{2} \epsilon^{kl} G_{kl}\phi - \epsilon^{jk} x_k j \phi_{,j} \Big) = \epsilon^{kl} \Big(x_k \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} \phi_{,l} - x_k \delta^i_l \mathfrak{L}_m + \frac{1}{2} \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} G_{kl}\phi \Big). \tag{2.4.38}$$

The conservation law (2.4.6) gives

$$\epsilon^{kl} \Big(\frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} \phi_{,[l} x_{k]} - \delta^i_{[l} x_{k]} \mathfrak{L}_m + \frac{1}{2} \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} G_{kl}\phi \Big)_{,i}, \tag{2.4.39}$$

which, because ϵ^{kl} are arbitrary, gives

$$\mathfrak{M}_{kl\ ,i}^{\ \ i} = 0, \tag{2.4.40}$$

where

$$\mathfrak{M}_{kl}^{\ \ i} = x_k \theta_l^{\ i} - x_l \theta_k^{\ i} + \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} G_{kl}\phi. \tag{2.4.41}$$

The quantity $\mathfrak{M}_{kl}^{\ \ i}$ is referred to as the *angular momentum density*, and is the sum,

$$\mathfrak{M}_{kl}^{\ \ i} = \Lambda_{kl}^{\ \ i} + \Sigma_{kl}^{\ \ i}, \tag{2.4.42}$$

of two densities: the *orbital angular momentum density*,

$$\Lambda_{kl}^{\ \ i} = x_k \theta_l^{\ i} - x_l \theta_k^{\ i}, \tag{2.4.43}$$

and the canonical spin density (2.3.25).

The conservation law (2.4.40) for the angular momentum density is equivalent to

$$\theta_{kl} - \theta_{lk} - \Sigma_{kl\ ,i}^{\ \ i} = 0, \tag{2.4.44}$$

which is a special case of the conservation law for the spin density (2.4.15) in the absence of torsion, expressed in the galilean and geodesic frame. The canonical energy-momentum density θ_{ik} is not symmetric. However, the quantity

$$\tau_{ik} = \theta_{ik} - \frac{1}{2}\partial_j(\Sigma_{ik}^{\ \ j} - \Sigma_{k\ i}^{\ j} + \Sigma^j_{\ ik}), \tag{2.4.45}$$

is symmetric, which follows from (2.4.44), and conserved:

$$\tau_{ik} = \tau_{ki}, \tag{2.4.46}$$
$$\tau_{k\ ,i}^{\ i} = 0. \tag{2.4.47}$$

The symmetric energy-momentum density τ_{ik} corresponds to the metric dynamical energy-momentum density (2.3.3), expressed in the galilean and geodesic frame. Equation (2.4.45) is a special case of the Belinfante-Rosenfeld relation (2.3.32) in the absence of torsion, expressed in the Galilean and geodesic frame. The second term on the right-hand side of (2.4.45) has the form $\partial_j \psi^{ikj}$, where $\psi^{ikj} = -\psi^{ijk}$. Adding such term to θ^{ik} preserves the conservation law (2.4.34) and brings θ^{ik} to a symmetric form.

2.4.6 Components of energy-momentum tensor

Integrating the conservation law (2.4.34), valid in the galilean and geodesic frame of reference, over a hypersurface enclosing matter represented by τ^{ik} and using the Gauß-Stokes theorem gives

$$\oint \tau^{ik} dS_k = 0, \tag{2.4.48}$$

which gives the conservation of the *four-momentum vector*

$$P^i = \frac{1}{c}\int \tau^{ik} dS_k = \text{const.} \tag{2.4.49}$$

Choosing the volume hypersurface $dV = dS_0$ gives

$$P^i = \frac{1}{c}\int \tau^{i0} dV, \tag{2.4.50}$$

so the components $\frac{1}{c}\tau^{i0}$ form the *four-momentum density*. The component τ^{00}, referred to as the *energy density*,

$$W = \tau^{00} = \dot{\phi}\frac{\partial \mathfrak{L}_m}{\partial \dot{\phi}} - \mathfrak{L}_m, \tag{2.4.51}$$

integrated over the volume gives the time component of the four-momentum, the *energy*

$$E = cP_0,\ \ cP^0 = \int \tau^{00} dV = \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L, \tag{2.4.52}$$

where

$$L = \int \mathfrak{L}_m dV \tag{2.4.53}$$

is the Lagrange function or *Lagrangian*. Hereinafter, a dot above any quantity ϕ denotes the partial derivative of ϕ with respect to time, $\dot{\phi} = \frac{d\phi}{dt}$, and two dots above ϕ denote the second derivative of ϕ with respect to time, $\ddot{\phi} = \frac{d^2\phi}{dt^2}$. Consequently, the action of a physical system is the time integral of the Lagrangian,

$$S = \int L dt. \tag{2.4.54}$$

The components $\frac{1}{c}\tau^{\alpha 0}$, referred to as the *momentum density*, integrated over the volume give the spatial components of the four-momentum, the *momentum* vector

$$\mathbf{P}: \ P^{\alpha} = \frac{1}{c}\int \tau^{\alpha 0} dV. \tag{2.4.55}$$

Adding a total divergence $\partial_j \psi^{ikj}$ to τ^{ik} does not alter the definition of the four-momentum vector (2.4.49).

The symmetry of τ^{ik} can be written as

$$\partial_l (x^i \tau^{kl} - x^k \tau^{il}) = 0, \tag{2.4.56}$$

which upon the integration over a hypersurface enclosing matter represented by τ^{ik} and using the Gauß-Stokes theorem gives

$$\oint (x^i \tau^{kl} - x^k \tau^{il}) dS_l = 0, \tag{2.4.57}$$

which gives the conservation of the *angular momentum tensor*

$$M^{ik} = \int (x^i dP^k - x^k dP^i) = \frac{1}{c}\int (x^i \tau^{kl} - x^k \tau^{il}) dS_l = \text{const}. \tag{2.4.58}$$

Choosing the volume hypersurface $dV = dS_0$ gives

$$M^{ik} = \frac{1}{c}\int (x^i \tau^{k0} - x^k \tau^{i0}) dV. \tag{2.4.59}$$

The conservation of $M^{0\alpha}$,

$$M^{\alpha 0} = \frac{1}{c}\Big(\int x^{\alpha} \tau^{00} dV - x^0 \int \tau^{\alpha 0} dV\Big) = \frac{1}{c}\int x^{\alpha}\tau^{00} dV - ctP^{\alpha} = \text{const}, \tag{2.4.60}$$

divided by the conservation of P^0 (2.4.50), $P^0 = \text{const}$, gives a uniform motion,

$$X^{\alpha} = V^{\alpha} t + \text{const}, \tag{2.4.61}$$

with velocity

$$V^{\alpha} = \frac{cP^{\alpha}}{P^0}, \tag{2.4.62}$$

of the *center of inertia* with the coordinates X^{α},

$$X^{\alpha} = \frac{\int x^{\alpha} \tau^{00} dV}{\int \tau^{00} dV}. \tag{2.4.63}$$

The coordinates of the center of inertia (2.4.63) are not the spatial components of a four-dimensional vector.

The conservation law (2.4.34) can be written as

$$\frac{1}{c}\frac{\partial \tau^{00}}{\partial t} + \frac{\partial \tau^{0\alpha}}{\partial x^{\alpha}} = 0, \tag{2.4.64}$$

$$\frac{1}{c}\frac{\partial \tau^{\alpha 0}}{\partial t} + \frac{\partial \tau^{\alpha\beta}}{\partial x^{\beta}} = 0. \tag{2.4.65}$$

Integrating these equations over the volume hypersurface and using the Gauß-Stokes theorem gives

$$\frac{\partial}{\partial t}\int \tau^{00} dV = -c \oint \tau^{0\alpha} df_\alpha, \tag{2.4.66}$$

$$\frac{\partial}{\partial t}\int \frac{1}{c}\tau^{\alpha 0} dV = -\oint \tau^{\alpha\beta} df_\beta, \tag{2.4.67}$$

where

$$df_\alpha = df^\star_{0\alpha} \tag{2.4.68}$$

is the spatial surface element (1.1.30). The integral of a three-dimensional vector V^α over the two-dimensional surface element df_α, $\oint V^\alpha df_\alpha$, is referred to as the *flux* of this vector. Therefore the components

$$S^\alpha = c\tau^{0\alpha} \tag{2.4.69}$$

of the *energy current* **S** form, upon integrating over df_α, the *energy flux*. The components $\tau^{\alpha\beta}$ represent the *momentum current* and give, upon integrating over df_α, the *momentum flux*. The *stress tensor* is defined as

$$\sigma_{\alpha\beta} = -\tau_{\alpha\beta}. \tag{2.4.70}$$

The components of the energy-momentum tensor form the matrix

$$\tau^{ik} = \begin{pmatrix} W & \frac{\mathbf{S}}{c} \\ \frac{\mathbf{S}}{c} & -\sigma_{\alpha\beta} \end{pmatrix}. \tag{2.4.71}$$

Define the spatial *surface force* vector,

$$F^\alpha = \oint \sigma^{\alpha\beta} df_\beta. \tag{2.4.72}$$

The relations (2.4.55), (2.4.67), (2.4.70) and (2.4.72) equal the time derivative of the momentum P^α to the surface force F^α,

$$\dot{P}^\alpha = F^\alpha. \tag{2.4.73}$$

In an arbitrary frame of reference, the metric dynamical energy-momentum tensor $\mathcal{T}_{ik}$ describing *isotropic* matter (without a preferred direction in its rest frame) can be decomposed into the part proportional to $u_i u_k$, the part proportional to the *projection tensor*,

$$h_{ik} = g_{ik} - u_i u_k, \tag{2.4.74}$$

which is orthogonal to u^i,

$$h_{ik} u^k = 0, \tag{2.4.75}$$

and parts containing covariant derivatives of u^i. The projection tensor satisfies

$$h_i^{\ j} h_j^{\ k} = h_i^{\ k}. \tag{2.4.76}$$

Assume that $\mathcal{T}_{ik}$ does not depend on derivatives of u^i. Therefore

$$T_{ik} = \epsilon u_i u_k - p h_{ik}, \tag{2.4.77}$$

where a scalar ϵ is equal to the energy density W in the locally Galilean rest frame and a scalar p is the *pressure*. In this frame $T^{ik} = \mathrm{diag}(\epsilon, p, p, p)$ and the stress tensor $\sigma_{\alpha\beta} = -p\delta_{\alpha\beta}$, giving

$$F^\alpha = -p \oint df^\alpha = -p \oint n^\alpha df, \tag{2.4.78}$$

which states that the force per unit surface df acting on a surface is parallel, with the opposite sign, to the outward normal vector of this surface n^α, $\frac{dF^\alpha}{df} = -pn^\alpha$, and which is referred to as *Pascal's law*. Matter described by the tensor (2.4.77) represents an *ideal fluid*. The relation between ϵ and p is referred to as the *equation of state*. In the Galilean frame of reference, combining (1.6.115), (2.4.71) and (2.4.77) gives

$$W = \frac{\epsilon + pv^2/c^2}{1 - v^2/c^2}, \tag{2.4.79}$$

$$\mathbf{S} = \frac{(\epsilon + p)\mathbf{v}}{1 - v^2/c^2}, \tag{2.4.80}$$

$$\sigma_{\alpha\beta} = -\frac{(\epsilon + p)v_\alpha v_\beta}{c^2 - v^2} - p\delta_{\alpha\beta}. \tag{2.4.81}$$

The relation (2.4.77) gives

$$T = T^i_i = \epsilon - 3p. \tag{2.4.82}$$

The component $T_{00} = \epsilon u_0^2 + p(u_0^2 - g_{00})$ is, using $u_0 = \frac{g_{00}dx^0 + g_{0\alpha}dx^\alpha}{ds}$, (1.4.96) and (1.4.97), equal to

$$T_{00} = \epsilon u_0^2 + pg_{00}\left(\frac{dl}{ds}\right)^2, \tag{2.4.83}$$

so it is positive under physical conditions $\epsilon > 0$, $p > 0$ and $g_{00} > 0$. If $\mathcal{T}_{ik}$ depends also on derivatives of u^i then matter described by the tensor (2.4.77) with the corresponding additional terms represents a *viscous fluid*.

2.4.7 Mass and Papapetrou equations of motion

Consider matter which is distributed over a small region in space and consists of points with the coordinates x^i, forming an extended body whose motion is represented by a world tube in spacetime. The motion of the body as a whole is represented by an arbitrary timelike world line γ inside the world tube, which consists of points with the coordinates $X^i(\tau)$, where τ is the proper time on γ. Define

$$\delta x^i = x^i - X^i, \;\; \delta x^0 = 0, \;\; u^i = \frac{dX^i}{ds}. \tag{2.4.84}$$

Also define the following integrals:

$$M^{ik} = u^0 \int \mathfrak{T}^{ik} dV, \tag{2.4.85}$$

$$M^{ijk} = -u^0 \int \delta x^i \mathfrak{T}^{jk} dV, \tag{2.4.86}$$

$$N^{ijk} = u^0 \int \mathfrak{S}^{ijk} dV, \tag{2.4.87}$$

$$J^{ik} = \int (\delta x^i \mathfrak{T}^{k0} - \delta x^k \mathfrak{T}^{i0} + \mathfrak{S}^{ik0}) dV = \frac{1}{u^0}(-M^{ik0} + M^{ki0} + N^{ik0}). \tag{2.4.88}$$

The quantity J^{ik} is equal to $\int(\delta x^i \mathfrak{T}^{kl} - \delta x^k \mathfrak{T}^{il} + \mathfrak{S}^{ikl})dS_l$ taken for the volume hypersurface, so it is a tensor, which we call the *total spin tensor*. The quantity N^{ijk} is also a tensor. The relation $\delta x^0 = 0$ gives

$$M^{0jk} = 0. \tag{2.4.89}$$

Assume that the dimensions of the body are small, so integrals with two or more factors δx^i multiplying $\mathfrak{T}^{jk}$ and integrals with one or more factors δx^i multiplying $\mathfrak{S}^{jkl}$ can be neglected.

The conservation law for the tetrad energy-momentum density (2.4.32) is

$$\mathfrak{T}^{ji}_{\ \ ,i} + \{^{\,j}_{i\,k}\} \mathfrak{T}^{ik} - C_{ik}^{\ \ j} \mathfrak{T}^{ik} - \frac{1}{2} R_{ikl}^{\ \ \ j} \mathfrak{S}^{ikl} = 0. \tag{2.4.90}$$

Integrating (2.4.90) over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals gives

$$\int \mathfrak{T}^{j0}_{\ \ ,0} dV + \int \{^{\,j}_{i\,k}\} \mathfrak{T}^{ik} dV - \int C_{ik}^{\ \ j} \mathfrak{T}^{ik} dV - \frac{1}{2} \int R_{ikl}^{\ \ \ j} \mathfrak{S}^{ikl} dV = 0. \tag{2.4.91}$$

Expanding

$$\Gamma^{\,i}_{j\,k} = \Gamma^{\,i(0)}_{j\,k} + \Gamma^{\,i(0)}_{j\,k,l} \delta x^l, \tag{2.4.92}$$

where the superscripts 0 denote the values at X^i, and substituting these expressions into (2.4.91) gives (omitting the superscripts)

$$\Big(\int \mathfrak{T}^{j0} dV\Big)_{,0} + \{^{\,j}_{i\,k}\} \int \mathfrak{T}^{ik} dV + \{^{\,j}_{i\,k}\}_{,l} \int \delta x^l \mathfrak{T}^{ik} dV - C_{ik}^{\ \ j} \int \mathfrak{T}^{ik} dV$$
$$-C_{ik\ ,l}^{\ \ j} \int \delta x^l \mathfrak{T}^{ik} dV - \frac{1}{2} \int R_{ikl}^{\ \ \ j} \mathfrak{S}^{ikl} dV = 0 \tag{2.4.93}$$

or, using the definitions (2.4.85), (2.4.86) and (2.4.87),

$$\frac{d}{ds}\Big(\frac{M^{j0}}{u^0}\Big) + \{^{\,j}_{i\,k}\} M^{(ik)} - \{^{\,j}_{i\,k}\}_{,l} M^{l(ik)} - C_{ik}^{\ \ j} M^{[ik]} + C_{ik\ ,l}^{\ \ j} M^{l[ik]} - \frac{1}{2} R_{ikl}^{\ \ \ j} N^{ikl} = 0. \tag{2.4.94}$$

The conservation law (2.4.90) gives

$$(x^l \mathfrak{T}^{ji})_{,i} = \mathfrak{T}^{jl} - x^l \{^{\,j}_{i\,k}\} \mathfrak{T}^{ik} + x^l C_{ik}^{\ \ j} \mathfrak{T}^{ik} + \frac{1}{2} x^l R_{ikm}^{\ \ \ \ j} \mathfrak{S}^{ikm}, \tag{2.4.95}$$

$$(x^l x^m \mathfrak{T}^{ji})_{,i} = x^m \mathfrak{T}^{jl} + x^l \mathfrak{T}^{jm} - x^l x^m \{^{\,j}_{i\,k}\} \mathfrak{T}^{ik} + x^l x^m C_{ik}^{\ \ j} \mathfrak{T}^{ik}$$
$$+\frac{1}{2} x^l x^m R_{ikn}^{\ \ \ \ j} \mathfrak{S}^{ikn}. \tag{2.4.96}$$

Integrating (2.4.95) over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals gives

$$\int (x^l\mathfrak{T}^{j0})_{,0}dV = \int \mathfrak{T}^{jl}dV - \int x^l\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV + \int x^l C_{ik}{}^j\mathfrak{T}^{ik}dV + \frac{1}{2}\int x^l R_{ikm}{}^j\mathfrak{S}^{ikm}dV. \tag{2.4.97}$$

Substituting (2.4.84) into (2.4.97) gives

$$\begin{aligned}&\frac{u^l}{u^0}\int \mathfrak{T}^{j0}dV + X^l\int \mathfrak{T}^{j0}{}_{,0}dV + \int (\delta x^l\mathfrak{T}^{j0})_{,0}dV = \int \mathfrak{T}^{jl}dV - X^l\int \{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV \\ &-\int \delta x^l\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV + X^l\int C_{ik}{}^j\mathfrak{T}^{ik}dV + \int \delta x^l C_{ik}{}^j\mathfrak{T}^{ik}dV \\ &+\frac{1}{2}X^l\int R_{ikm}{}^j\mathfrak{S}^{ikm}dV,\end{aligned} \tag{2.4.98}$$

which reduces, due to (2.4.91), to

$$\frac{u^l}{u^0}\int \mathfrak{T}^{j0}dV + \Big(\int (\delta x^l\mathfrak{T}^{j0}dV\Big)_{,0} = \int \mathfrak{T}^{jl}dV - \int \delta x^l\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV + \int \delta x^l C_{ik}{}^j\mathfrak{T}^{ik}dV. \tag{2.4.99}$$

Substituting (2.4.92) into (2.4.99), omitting the superscripts and using the definitions (2.4.85), (2.4.86) and (2.4.87), turns (2.4.99) into

$$\frac{u^l}{u^0}M^{j0} - \frac{d}{ds}\Big(\frac{M^{lj0}}{u^0}\Big) = M^{jl} + \{{}^{\,j}_{i\,k}\}M^{lik} - C_{ik}{}^j M^{lik}. \tag{2.4.100}$$

Putting $l = 0$ in (2.4.100) gives the identity because of (2.4.89).

Integrating (2.4.96) over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals gives

$$\begin{aligned}&\int (x^l x^m\mathfrak{T}^{j0})_{,0}dV = \int x^m\mathfrak{T}^{jl}dV + \int x^l\mathfrak{T}^{jm}dV - \int x^l x^m\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV \\ &+\int x^l x^m C_{ik}{}^j\mathfrak{T}^{ik}dV + \frac{1}{2}\int x^l x^m R_{ikn}{}^j\mathfrak{S}^{ikn}dV.\end{aligned} \tag{2.4.101}$$

Substituting (2.4.84) into (2.4.97) gives

$$\begin{aligned}&X^l X^m\int \mathfrak{T}^{j0}{}_{,0}dV + \frac{u^l}{u^0}X^m\int \mathfrak{T}^{j0}dV + \frac{u^l}{u^0}\int \delta x^m\mathfrak{T}^{j0}dV + \frac{u^m}{u^0}X^l\int \mathfrak{T}^{j0}dV \\ &+\frac{u^m}{u^0}\int \delta x^l\mathfrak{T}^{j0}dV + X^l\int \delta x^m\mathfrak{T}^{j0}{}_{,0}dV + X^m\int \delta x^l\mathfrak{T}^{j0}{}_{,0}dV \\ &= -X^l X^m\Big(\int \{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV - \int C_{ik}{}^j\mathfrak{T}^{ik}dV - \frac{1}{2}\int R_{ikl}{}^j\mathfrak{S}^{ikl}dV\Big) \\ &+X^l\Big(\int \mathfrak{T}^{jm}dV - \int \delta x^m\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV + \int \delta x^m C_{ik}{}^j\mathfrak{T}^{ik}dV\Big) \\ &+X^m\Big(\int \mathfrak{T}^{jl}dV - \int \delta x^l\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV + \int \delta x^l C_{ik}{}^j\mathfrak{T}^{ik}dV\Big) \\ &+\int \delta x^m\mathfrak{T}^{jl}dV + \int \delta x^l\mathfrak{T}^{jm}dV,\end{aligned} \tag{2.4.102}$$

which reduces, due to (2.4.91) and (2.4.99), to

$$\frac{u^l}{u^0}\int\delta x^m\mathfrak{T}^{j0}dV+\frac{u^m}{u^0}\int\delta x^l\mathfrak{T}^{j0}dV=\int\delta x^m\mathfrak{T}^{jl}dV+\int\delta x^l\mathfrak{T}^{jm}dV \tag{2.4.103}$$

or

$$\frac{u^l}{u^0}M^{mi0}+\frac{u^m}{u^0}M^{li0}=M^{mil}+M^{lim}. \tag{2.4.104}$$

The expressions analogous to (2.4.95) and (2.4.96) with higher multiples of x^i do not introduce new relations.

The conservation law for the angular momentum density (2.4.15) is

$$\mathfrak{S}^{ijk}{}_{,k}-\Gamma^{\,i}_{l\,k}\mathfrak{S}^{jlk}+\Gamma^{\,j}_{l\,k}\mathfrak{S}^{ilk}-2\mathfrak{T}^{[ij]}=0. \tag{2.4.105}$$

Integrating (2.4.105) over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals gives

$$\int\mathfrak{S}^{ij0}{}_{,0}dV-\int\Gamma^{\,i}_{l\,k}\mathfrak{S}^{jlk}dV+\int\Gamma^{\,j}_{l\,k}\mathfrak{S}^{ilk}dV-2\int\mathfrak{T}^{[ij]}dV=0. \tag{2.4.106}$$

Substituting (2.4.92) into (2.4.106), omitting the superscripts and using the definitions (2.4.85) and (2.4.87), turns (2.4.106) into

$$\frac{d}{ds}\Big(\frac{N^{ij0}}{u^0}\Big)-\Gamma^{\,i}_{l\,k}N^{jlk}+\Gamma^{\,j}_{l\,k}N^{ilk}-2M^{[ij]}=0. \tag{2.4.107}$$

The conservation law (2.4.105) gives

$$(x^l\mathfrak{S}^{ijk})_{,k}=\mathfrak{S}^{ijl}+x^l\Gamma^{\,i}_{l\,k}\mathfrak{S}^{jlk}-x^l\Gamma^{\,j}_{l\,k}\mathfrak{S}^{ilk}+2x^l\mathfrak{T}^{[ij]}. \tag{2.4.108}$$

Integrating (2.4.108) over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals gives

$$\int(x^l\mathfrak{S}^{ij0})_{,0}dV=\int\mathfrak{S}^{ijl}dV+\int x^l\Gamma^{\,i}_{m\,k}\mathfrak{S}^{jmk}dV-\int x^l\Gamma^{\,j}_{m\,k}\mathfrak{S}^{imk}dV+2\int x^l\mathfrak{T}^{[ij]}dV. \tag{2.4.109}$$

Substituting (2.4.84) into (2.4.109) gives

$$\frac{u^l}{u^0}\int\mathfrak{S}^{ij0}dV+X^l\int\mathfrak{S}^{ij0}{}_{,0}dV=\int\mathfrak{S}^{ijl}dV+X^l\Big(\int\Gamma^{\,i}_{m\,k}\mathfrak{S}^{jmk}dV-\int\Gamma^{\,j}_{m\,k}\mathfrak{S}^{imk}dV$$
$$+2\int\mathfrak{T}^{[ij]}dV\Big)+2\int\delta x^l\mathfrak{T}^{[ij]}dV, \tag{2.4.110}$$

which reduces, due to (2.4.106), to

$$\frac{u^l}{u^0}\int\mathfrak{S}^{ij0}dV=\int\mathfrak{S}^{ijl}dV+2\int\delta x^l\mathfrak{T}^{[ij]}dV \tag{2.4.111}$$

or

$$M^{l[ij]}=-\frac{1}{2}\Big(\frac{u^l}{u^0}N^{ij0}-N^{ijl}\Big). \tag{2.4.112}$$

Putting $l=0$ in (2.4.112) gives the identity because of (2.4.89). The expressions analogous to (2.4.108) with higher multiples of x^i do not introduce new relations.

Taking the cyclic permutations of the indices i,l,m in (2.4.104), adding the first and second of these relations, and subtracting the third, gives

$$\frac{u^l}{u^0}M^{[mi]0}+\frac{u^i}{u^0}M^{[ml]0}+\frac{u^m}{u^0}M^{(li)0}=M^{l[im]}+M^{i[lm]}+M^{m(il)}. \tag{2.4.113}$$

Substituting (2.4.88) and (2.4.112) into (2.4.113) gives

$$M^{m(il)}=u^{(i}J^{l)m}+\frac{u^m}{u^0}M^{(il)0}+N^{m(il)}. \tag{2.4.114}$$

Putting $m=0$ in (2.4.114), substituting it into (2.4.114) and using (2.4.89) gives

$$M^{m(il)}=u^{(i}J^{l)m}-\frac{u^m}{u^0}(u^{(i}J^{l)0}+N^{0(il)})+N^{m(il)}. \tag{2.4.115}$$

Combining (2.4.112) and (2.4.115) gives

$$M^{mil}=u^{(i}J^{l)m}-\frac{u^m}{u^0}(u^{(i}J^{l)0}+N^{0(il)})+N^{m(il)}-\frac{1}{2}\Big(\frac{u^m}{u^0}N^{il0}-N^{ilm}\Big). \tag{2.4.116}$$

Therefore

$$M^{(ik)0}=-(u^{(i}J^{k)0}+N^{0(ik)}). \tag{2.4.117}$$

Combining the antisymmetric part of (2.4.100) and (2.4.107) gives

$$M^{jl}-M^{lj}=\frac{u^l}{u^0}M^{j0}-\frac{u^j}{u^0}M^{l0}-(\{{}^{\,j}_{i\,k}\}-C_{ik}{}^{j})M^{lik}+(\{{}^{\,l}_{i\,k}\}-C_{ik}{}^{l})M^{jik}$$
$$-\frac{d}{ds}\Big(\frac{M^{lj0}-M^{jl0}}{u^0}\Big). \tag{2.4.118}$$

Using (2.4.88), (2.4.112) and (2.4.115) brings (2.4.118) to

$$-\Gamma^{\,j}_{i\,k}N^{lik}+\Gamma^{\,l}_{i\,k}N^{jik}=\frac{u^l}{u^0}M^{j0}-\frac{u^j}{u^0}M^{l0}-\{{}^{\,j}_{i\,k}\}\Big(u^iJ^{kl}+N^{lik}-\frac{u^l}{u^0}(u^iJ^{k0}+N^{0ik})\Big)$$
$$-\frac{1}{2}C_{ik}{}^{j}\Big(\frac{u^l}{u^0}N^{ik0}-N^{ikl}\Big)+\{{}^{\,l}_{i\,k}\}\Big(u^iJ^{kj}+N^{jik}-\frac{u^j}{u^0}(u^iJ^{k0}+N^{0ik})\Big)$$
$$+\frac{1}{2}C_{ik}{}^{l}\Big(\frac{u^j}{u^0}N^{ik0}-N^{ikj}\Big)+\frac{d}{ds}J^{lj}, \tag{2.4.119}$$

which, using $\frac{D^{\{\}}}{ds}J^{lj}=\frac{d}{ds}J^{lj}+u^k\{{}^{\,l}_{i\,k}\}J^{ij}+u^k\{{}^{\,j}_{i\,k}\}J^{li}$, turns into

$$\frac{D^{\{\}}}{ds}J^{lj}=\Big[\frac{u^j}{u^0}M^{l0}+\frac{u^j}{u^0}\{{}^{\,l}_{i\,k}\}(u^iJ^{k0}+N^{0ik})-\frac{1}{2}C_{ik}{}^{l}\Big(\frac{u^j}{u^0}N^{ik0}-N^{ikj}\Big)+C^{l}_{\ ik}N^{jik}\Big]-\Big[l\leftrightarrow j\Big] \tag{2.4.120}$$

or, using the four-momentum

$$P^l=\frac{1}{c}\int \mathfrak{T}^{l0}dV, \tag{2.4.121}$$

into

$$\frac{D^{\{\}}}{ds}J^{lj} = \Big[cu^j P^l + \frac{u^j}{u^0}\{{}^{\,l}_{i\,k}\}(u^i J^{k0} + N^{0ik}) - \frac{1}{2}C_{ik}{}^l\Big(\frac{u^j}{u^0}N^{ik0} - N^{ikj}\Big) + C^l_{\ ik}N^{jik}\Big] - \Big[l \leftrightarrow j\Big]. \tag{2.4.122}$$

Therefore

$$\begin{aligned}
\frac{D^{\{\}}J^{li}}{ds}u_i &= cP^l + \frac{1}{u^0}\{{}^{\,l}_{i\,k}\}(u^i J^{k0} + N^{0ik}) - \frac{1}{2}C_{ik}{}^l\Big(\frac{1}{u^0}N^{ik0} - N^{ikj}u_j\Big) + C^l_{\ ik}N^{jik}u_j \\
&-cu^l u_j P^j - \frac{u^l u_j}{u^0}\{{}^{\,j}_{i\,k}\}(u^i J^{k0} + N^{0ik}) + \frac{1}{2}C_{ik}{}^j\Big(\frac{u^l}{u^0}N^{ik0} - N^{ikl}\Big)u_j \\
&-C^j_{\ ik}N^{lik}u_j,
\end{aligned} \tag{2.4.123}$$

which gives with (2.4.122)

$$\begin{aligned}
&\frac{D^{\{\}}J^{lj}}{ds} + u^l u_i\frac{D^{\{\}}J^{ji}}{ds} - u^j u_i\frac{D^{\{\}}J^{li}}{ds} = C^l_{\ ik}N^{jik} + \frac{1}{2}C_{ik}{}^l N^{ikj} - C^j_{\ ik}N^{lik} \\
&-\frac{1}{2}C_{ik}{}^j N^{ikl} - u^j u_m\Big(C^l_{\ ik}N^{mik} + \frac{1}{2}C_{ik}{}^l N^{ikm} - C^m_{\ ik}N^{lik} - \frac{1}{2}C_{ik}{}^m N^{ikl}\Big) \\
&+u^l u_m\Big(C^j_{\ ik}N^{mik} + \frac{1}{2}C_{ik}{}^j N^{ikm} - C^m_{\ ik}N^{jik} - \frac{1}{2}C_{ik}{}^m N^{ikj}\Big) \\
&= 2(\delta^l_{[n}\delta^j_{m]} - u^j u_{[m}\delta^l_{n]} + u^l u_{[m}\delta^j_{n]})\Big(C^n_{\ ik}N^{mik} + \frac{1}{2}C_{ik}{}^n N^{ikm}\Big).
\end{aligned} \tag{2.4.124}$$

Multiplying (2.4.124) by u_j gives the identity, so only 3 equations in (2.4.124) are independent. Thus 3 components of J^{ik} are arbitrary and we can impose 3 constraints on J^{ik}. A simple choice is

$$J^{ik}u_k = 0, \tag{2.4.125}$$

which means that in the local rest frame $J^{\alpha 0} = 0$, so the three independent components of J^{ik} are the spatial $J^{\alpha\beta}$. Analogously to the Pauli-Lubański pseudovector (1.6.70), define the *four-spin pseudovector*

$$J^i = \frac{1}{2}e^{ijkl}u_j J_{kl}, \tag{2.4.126}$$

which is orthogonal to u^i,

$$J^i u_i = 0. \tag{2.4.127}$$

The condition (2.4.125) gives the relation inverse to (2.4.126):

$$J^{ik} = -e^{ikjl}u_j J_l. \tag{2.4.128}$$

Differentiating (2.4.126) covariantly with respect to $\{{}^{\,i}_{j\,k}\}$ and using (2.4.124) gives

$$\begin{aligned}
\frac{D^{\{\}}J^i}{ds} &= -\frac{1}{2}e^{ijkl}\frac{D^{\{\}}u_j}{ds}e_{klmn}u^m J^n + e^{ijkl}u_j\delta^k_{[n}\delta^l_{m]}\Big(C^n_{\ pr}N^{mpr} + \frac{1}{2}C_{pr}{}^n N^{prm}\Big) \\
&= -u^i\frac{D^{\{\}}u^k}{ds}J_k + e^{ij}{}_{nm}u_j\Big(C^n_{\ pr}N^{mpr} + \frac{1}{2}C_{pr}{}^n N^{prm}\Big) \\
&= -u^i\frac{D^{\{\}}u^k}{ds}J_k + \frac{D^{\{\}}u^i}{ds}u^k J_k + e^{ij}{}_{nm}u_j\Big(C^n_{\ pr}N^{mpr} + \frac{1}{2}C_{pr}{}^n N^{prm}\Big).
\end{aligned} \tag{2.4.129}$$

Thus the covariant (with respect to the Levi-Civita connection) change of the spin pseudovector along the world line γ is the sum of the corresponding Fermi-Walker transport (with respect to the Levi-Civita connection) and a term which depends on the torsion and spin density.

In (2.4.123), the four-momentum P^l depends on terms proportional to the four-velocity u^l and terms in which the index l appears in other quantities. Define the *mass* of the system described by the energy-momentum density $\mathfrak{T}^{ik}$ as the coefficient m of u^l in the expansion for $\frac{P^l}{c}$,

$$\frac{P^l}{c} = mu^l + \dots, \tag{2.4.130}$$

so

$$m = \frac{u_j}{c}P^j + \frac{u_j}{c^2u^0}\{^{\,j}_{i\,k}\}(u^iJ^{k0} + N^{0ik}) - \frac{u_j}{2c^2u^0}C_{ik}{}^jN^{ik0} = \frac{u_j}{c}\Pi^j, \tag{2.4.131}$$

where

$$\Pi^j = P^j + \frac{1}{cu^0}\{^{\,j}_{i\,k}\}(u^iJ^{k0} + N^{0ik}) - \frac{1}{2cu^0}C_{ik}{}^jN^{ik0} \tag{2.4.132}$$

is the *modified four-momentum*. Substituting (2.4.131) and (2.4.132) into (2.4.123) gives

$$\frac{D^{\{\}}}{ds}J^{li}u_i = c\Pi^l - mc^2u^l + u_j\Big(C^l_{\ ik}N^{jik} + \frac{1}{2}C_{ik}{}^lN^{ikj}\Big) - u_j\Big(C^j_{\ ik}N^{lik} + \frac{1}{2}C_{ik}{}^jN^{ikl}\Big), \tag{2.4.133}$$

so $\Pi^l - mcu^l = \Pi^j(\delta^l_j - u^lu_j)$ is a vector. Thus the modified four-momentum Π^i is a vector and the mass m is a scalar. Substituting (2.4.132) into (2.4.122) gives the *Papapetrou equation of motion for the spin*:

$$\frac{D^{\{\}}}{ds}J^{lj} = cu^j\Pi^l - cu^l\Pi^j + C^l_{\ ik}N^{jik} + \frac{1}{2}C_{ik}{}^lN^{ikj} - C^j_{\ ik}N^{lik} - \frac{1}{2}C_{ik}{}^jN^{ikl}. \tag{2.4.134}$$

Putting (2.4.100), (2.4.107), (2.4.112), (2.4.115), (2.4.117) and (2.4.132) into (2.4.94) gives

$$\begin{aligned}
&\frac{d}{ds}\Big(c\Pi^j - \frac{1}{u^0}\{^{\,j}_{i\,k}\}(u^iJ^{k0} + N^{0ik}) + \frac{1}{2u^0}C_{ik}{}^jN^{ik0}\Big) + \{^{\,j}_{i\,k}\}u^k\Big(c\Pi^i - \frac{1}{u^0}\{^{\,j}_{l\,m}\}(u^lJ^{m0} \\
&+N^{0lm}) + \frac{1}{2u^0}C_{lm}{}^jN^{lm0}\Big) - \{^{\,j}_{i\,k}\}(\{^{\,i}_{l\,m}\} - C_{lm}{}^i)M^{klm} - \{^{\,j}_{i\,k}\}\frac{d}{ds}\Big(\frac{M^{(ik)0}}{u^0}\Big) \\
&-\{^{\,j}_{i\,k}\}_{,l}\Big(u^{(i}J^{k)l} - \frac{u^l}{u^0}(u^{(i}J^{k)0} + N^{0(ik)}) + N^{l(ik)}\Big) - \frac{1}{2}C_{ik}{}^j\frac{d}{ds}\Big(\frac{N^{ik0}}{u^0}\Big) \\
&-\frac{1}{2}C_{ik}{}^j(-\Gamma^{\,i}_{l\,m}N^{klm} + \Gamma^{\,k}_{l\,m}N^{ilm}) - \frac{1}{2}C_{ik}{}^j{}_{,l}\Big(\frac{u^l}{u^0}N^{ik0} - N^{ikl}\Big) - \frac{1}{2}R^{iklj}N_{ikl} \\
&= c\frac{D^{\{\}}\Pi^j}{ds} - \{^{\,j}_{i\,k}\}\{^{\,i}_{l\,m}\}(u^lJ^{mk} + N^{klm}) - \{^{\,j}_{i\,k}\}_{,m}(u^iJ^{km} + N^{mik}) \\
&+\frac{1}{2}\{^{\,j}_{i\,k}\}C_{lm}{}^iN^{lmk} + C_{ik}{}^j\Gamma^{\,i}_{l\,m}N^{klm} + \frac{1}{2}C_{ik}{}^j{}_{,m}N^{ikm} - \frac{1}{2}R^{iklj}N_{ikl} = 0.
\end{aligned} \tag{2.4.135}$$

Using (1.4.51) turns (2.4.135) into the *Papapetrou equation of motion for the momentum*:

$$\frac{D^{\{\}}\Pi^j}{ds} = -\frac{1}{2c}P^j_{\ imk}u^i J^{mk} - \frac{1}{2c}N_{ikl}C^{ikl:j}. \tag{2.4.136}$$

If the spin density vanishes then the Einstein-Cartan gravitational field equations reduce to the Einstein-Hilbert gravitational field equations. The conservation law for the spin density (2.4.15) with the condition $\mathfrak{S}^{ijk} = 0$ gives the symmetry of the energy-momentum density, $\mathfrak{T}^{ik} = \mathfrak{T}^{ki}$. The relations (2.4.85), (2.4.86), (2.4.87) and (2.4.88) give then

$$M^{ik} = M^{ki}, \tag{2.4.137}$$
$$M^{ijk} = M^{ikj}, \tag{2.4.138}$$
$$N^{ijk} = 0, \tag{2.4.139}$$
$$J^{ik} = cL^{ik} = \int(\delta x^i \mathfrak{T}^{k0} - \delta x^k \mathfrak{T}^{i0})dV = \frac{1}{u^0}(-M^{ik0} + M^{ki0}), \tag{2.4.140}$$

where L^{ik} is the angular momentum tensor, analogous to (2.4.58). The modified four-momentum (2.4.132) reduces to

$$\Pi^j = P^j + \frac{1}{u^0}\{^{\,j}_{i\,k}\}u^i L^{k0} \tag{2.4.141}$$

and (2.4.133) gives

$$\Pi^l = mcu^l + \frac{D^{\{\}}L^{li}}{ds}u_i. \tag{2.4.142}$$

The relation (2.4.129) reduces to

$$\frac{D^{\{\}}J^i}{ds} = -u^i\frac{D^{\{\}}u^k}{ds}J_k + \frac{D^{\{\}}u^i}{ds}u^k J_k, \tag{2.4.143}$$

so the covariant (with respect to the Levi-Civita connection) change of the spin pseudovector along the world line γ is equal to the corresponding Fermi-Walker transport. Multiplying (2.4.143) by J_i and using (2.4.127) gives

$$J^i J_i = \mathrm{const}, \tag{2.4.144}$$

so the change of the spin pseudovector along a world line is a rotation, called *spin precession*. The Papapetrou equation of motion for the spin (2.4.134) reduces to

$$\frac{D^{\{\}}L^{lj}}{ds} = u^j\Pi^l - u^l\Pi^j, \tag{2.4.145}$$

while the Papapetrou equation of motion for the momentum (2.4.136) reduces to

$$\frac{D^{\{\}}\Pi^j}{ds} = -\frac{1}{2}P^j_{\ imk}u^i L^{mk}. \tag{2.4.146}$$

The change of the mass m along the world line γ is, using (1.4.48), (2.4.125), (2.4.131), (2.4.142) and (2.4.146),

$$\frac{dm}{ds} = \frac{D^{\{\}}m}{ds} = \frac{1}{c}u_j\frac{D^{\{\}}\Pi^j}{ds} + \frac{1}{c}\Pi^j\frac{D^{\{\}}u_j}{ds} = \frac{1}{c}\Pi^j\frac{D^{\{\}}u_j}{ds} = \frac{1}{c}\frac{D^{\{\}}L^{ji}}{ds}u_i\frac{D^{\{\}}u_j}{ds}$$
$$= -\frac{1}{c}L^{ji}\frac{D^{\{\}}u_i}{ds}\frac{D^{\{\}}u_j}{ds} = 0, \tag{2.4.147}$$

so

$$m = \text{const.} \tag{2.4.148}$$

In the absence of the external gravitational field and neglecting the gravitational field of the body, the relation (2.4.141) gives

$$\Pi^j = P^j, \tag{2.4.149}$$

so (2.4.146) reduces to

$$\frac{dP^j}{ds} = 0, \tag{2.4.150}$$

whose integration gives the conservation of the four-momentum along a world line:

$$P^i = \text{const.} \tag{2.4.151}$$

The equation of motion for the spin (2.4.145) becomes

$$\frac{dL^{lj}}{ds} = u^j P^l - u^l P^j, \tag{2.4.152}$$

whose integration gives the conservation of the angular momentum along a world line:

$$L^{ik} + X^i P^k - X^k P^i = \text{const.} \tag{2.4.153}$$

The tensor L^{ik} is the intrinsic angular momentum of the body, while the tensor (in the absence of the gravitational field) $X^i P^k - X^k P^i$ is the orbital angular momentum associated with the motion of the body as a whole. If $L^{ik} = 0$ then (2.4.153) gives $P^i \propto u^i$, so (2.4.151) is equivalent to $u^i = \text{const}$ and thus X^i is a linear function of the proper time τ. If $L^{ik} \neq 0$ then X^i can be given by 3 arbitrary functions of τ (since $u^i u_i = 1$). In the *momentum rest frame*, in which $P^\alpha = 0$, $u^\alpha \neq 0$, so the body has an arbitrary internal motion. The 3 constraints (2.4.125) eliminate this arbitrariness, so the equations of motion entirely determine the motion of the body.

2.4.8 Energy-momentum tensor for particles

If the body is not spatially extended then it is referred to as a *particle*. The corresponding condition $\delta x^\alpha = 0$ gives

$$M^{ijk} = 0, \;\; L^{ik} = 0. \tag{2.4.154}$$

Therefore (2.4.112) reduces to $\frac{u^l}{u^0}N^{ij0} - N^{ijl} = 0$, which with (2.4.88) gives

$$N^{ijk} = u^i J^{jk}, \tag{2.4.155}$$

so

$$J^{ij} = S^{ij} = N^{ijk}u_k, \tag{2.4.156}$$

where S^{ij} is the *intrinsic spin tensor*. If the body is spatially extended then the difference

$$R^{ik} = J^{ik} - S^{ik} \tag{2.4.157}$$

is the *rotational spin tensor*. The difference between the rotational spin tensor and angular momentum tensor is, due to (2.4.88) and (2.4.112),

$$R^{ik} - cL^{ik} = \frac{N^{ik0}}{u^0} - N^{ikl}u_l = -2M^{l[ik]}u_l. \tag{2.4.158}$$

This expression vanishes, because of (2.4.89), in the *velocity rest frame*, in which $u^\alpha = 0$, which is also locally Galilean, so $u_\alpha = 0$.

If a particle is spinless then its four-momentum is proportional to its four-velocity due to (2.4.141) and (2.4.142):

$$P^l = mcu^l, \tag{2.4.159}$$

which gives

$$P^2 = m^2c^2, \tag{2.4.160}$$

in agreement with (1.6.80). Equations (2.4.100), (2.4.121), (2.4.137), (2.4.139) and (2.4.159) give

$$M^{ik} = \frac{u^i}{u^0}M^{k0} = \frac{u^i u^k}{(u^0)^2}M^{00} = mc^2 u^i u^k, \tag{2.4.161}$$

so

$$\int \mathfrak{T}^{ik}dV = mc^2\frac{u^i u^k}{u^0} \tag{2.4.162}$$

or

$$\mathfrak{T}^{ik}(\mathbf{x}) = mc^2\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_0)\frac{u^i u^k}{u^0}, \tag{2.4.163}$$

where $\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_0)$ is the spatial Dirac delta representing a point mass located at $\mathbf{x}_0$. Define the *mass density* μ such that

$$\mu\sqrt{\mathfrak{f}}dV = dm, \tag{2.4.164}$$

where $\mathfrak{f}$ is given by (1.4.105). The mass density for a particle located at $\mathbf{x}_a$ is

$$\mu(\mathbf{x}) = \frac{m}{\sqrt{\mathfrak{f}}}\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_a), \tag{2.4.165}$$

so (2.4.163) turns into

$$\mathfrak{T}^{ik} = \mu c^2\frac{u^i u^k}{\sqrt{\mathfrak{f}}u^0}. \tag{2.4.166}$$

Thus the energy-momentum tensor for a spinless particle is given by

$$T^{ik} = \mu c^2 \frac{u^i u^k}{\sqrt{g_{00}} u^0} = \frac{\mu c}{\sqrt{g_{00}}} \frac{dx^i}{ds} \frac{dx^k}{dt} \tag{2.4.167}$$

or, for a system of particles,

$$T^{ik}(\mathbf{x}) = \sum_a m_a c^2 \boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_a) \frac{u^i u^k}{\sqrt{-\mathfrak{g}} u^0}. \tag{2.4.168}$$

The Papapetrou equation of motion (2.4.146) for a spinless particle reduces to the metric geodesic equation (1.4.80),

$$\frac{D^{\{\}} u^i}{ds} = 0. \tag{2.4.169}$$

In the absence of torsion and in the locally Galilean frame of reference, the conservation law for the energy-momentum tensor is given by (2.4.34), so

$$T_{\alpha\ ,i}^{\ i} = 0. \tag{2.4.170}$$

Consider a closed system of particles which carry out a finite motion, in which all quantities vary over finite ranges. Define the average over a certain time interval τ of a function f of these quantities as $\bar{f} = \frac{1}{\tau} \int_0^\tau f dt$. The average of the derivative of a bounded quantity $\bar{\dot{f}} = \frac{1}{\tau}(f(\tau) - f(0)) \to 0$ as $\tau \to \infty$. Thus averaging (2.4.170) over the time gives

$$\bar{T}_{\alpha\ ,\beta}^{\ \beta} = 0. \tag{2.4.171}$$

Multiplying (2.4.171) by x^α and integrating over the volume gives, omitting surface integrals,

$$\int x^\alpha \bar{T}_{\alpha\ ,\beta}^{\ \beta} dV = -\int \bar{T}_\alpha^{\ \alpha} dV = 0. \tag{2.4.172}$$

The average energy of the system (2.4.52) is thus

$$\bar{E} = \int \bar{T}_0^{\ 0} dV = \int \bar{T}_i^{\ i} dV. \tag{2.4.173}$$

Substituting (1.6.115) into (2.4.168) gives

$$T_i^{\ i}(\mathbf{x}) = \sum_a m_a c^2 \boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_a) \Big(1 - \frac{v^2}{c^2}\Big)^{1/2}, \tag{2.4.174}$$

so $T_i^{\ i} \geq 0$. Putting (2.4.174) into (2.4.173) gives

$$\bar{E} = \sum_a m_a c^2 \overline{\Big(1 - \frac{v^2}{c^2}\Big)^{1/2}}, \tag{2.4.175}$$

which is referred to as the *virial theorem*.

Comparing (2.4.82) with (2.4.174) gives

$$\epsilon - 3p = \sum_a m_a c^2 \Big(1 - \frac{v^2}{c^2}\Big)^{1/2}, \tag{2.4.176}$$

where the summation extends over all particles in unit volume, so $p \leq \epsilon/3$. In the nonrelativistic limit $p \approx 0$, while in the ultrarelativistic limit ($v \sim c$) $p \sim \epsilon/3$. Consider a system of noninteracting identical particles of mass m, which we call an *ideal gas*, with the number of particles in unit volume (*concentration*) n, so

$$\mu = nm. \tag{2.4.177}$$

Comparing (2.4.77) in the locally Galilean rest frame with (2.4.167) gives the kinetic formulae for ideal gases:

$$\epsilon = nmc^2\bar{\gamma}, \tag{2.4.178}$$

$$p = \frac{nm}{3}\overline{\gamma v^2}. \tag{2.4.179}$$

In a locally inertial frame of reference, (1.6.115) and (2.4.159) give

$$P^i = mc\gamma\Big(1, \frac{\mathbf{v}}{c}\Big), \tag{2.4.180}$$

so the energy and momentum of the particle are

$$E = mc^2\gamma, \tag{2.4.181}$$

$$\mathbf{P} = m\gamma\mathbf{v}. \tag{2.4.182}$$

Thus (2.4.160) gives

$$E^2 = (Pc)^2 + (mc^2)^2. \tag{2.4.183}$$

We also obtain

$$\mathbf{v}\cdot d\mathbf{P} = \frac{cu^\alpha}{\gamma}d(mcu_\alpha) = \frac{cu_\beta}{\gamma}d(mcu_\alpha)\delta^{\alpha\beta} = -\frac{cu_\beta}{\gamma}d(mcu_\alpha)g^{\alpha\beta}$$
$$= \frac{cu_0}{\gamma}d(mcu_0)g^{00} = cdP_0 = dE. \tag{2.4.184}$$

In the rest frame of the particle, $\mathbf{P} = 0$, (2.4.183) reduces to Einstein's formula for the *rest energy*,

$$E = mc^2. \tag{2.4.185}$$

The formulae (2.4.181) and (2.4.182) give

$$\mathbf{v} = \frac{c^2}{E}\mathbf{p}. \tag{2.4.186}$$

If a particle is massless, $m = 0$, then (2.4.183) gives

$$E = Pc, \tag{2.4.187}$$

which is consistent with (2.4.186) only if

$$v = c. \tag{2.4.188}$$

References: [2, 3, 5, 6, 7, 9].

2.5 Gravitational field equations

2.5.1 Einstein-Hilbert action and Einstein equations

The *Einstein-Hilbert action* for the gravitational field and matter is, due to (2.2.2),

$$S=-\frac{1}{2\kappa c}\int P\sqrt{-\mathfrak{g}}d\Omega+S_m, \tag{2.5.1}$$

where the metric tensor is regarded as a variational variable and the affine connection is the Levi-Civita connection. Varying (2.5.1) with respect to the metric tensor gives, using (2.3.1) and the identity $\delta\sqrt{-\mathfrak{g}}=-\frac{1}{2}\sqrt{-\mathfrak{g}}g_{ik}\delta g^{ik}$ (which results from $\delta\mathfrak{g}=\mathfrak{g}g^{ik}\delta g_{ik}=-\mathfrak{g}g_{ik}\delta g^{ik}$),

$$\delta S=-\frac{1}{2\kappa c}\int\Big(\delta P_{ik}g^{ik}\sqrt{-\mathfrak{g}}+P_{ik}\delta g^{ik}\sqrt{-\mathfrak{g}}-\frac{1}{2}P\sqrt{-\mathfrak{g}}g_{ik}\delta g^{ik}\Big)d\Omega+\frac{1}{2c}\int T_{ik}\sqrt{-\mathfrak{g}}\delta g^{ik}d\Omega. \tag{2.5.2}$$

Partial integration of the first term on the right-hand side of (2.5.2), using (1.4.57), brings this term to zero:

$$\int\delta P_{ik}\mathsf{g}^{ik}d\Omega=\int\Big((\delta\{{}^{\,l}_{i\,k}\})_{:l}-(\delta\{{}^{\,l}_{i\,l}\})_{:k}\Big)\mathsf{g}^{ik}d\Omega=-\int(\mathsf{g}^{ik}{}_{:l}\delta\{{}^{\,l}_{i\,k}\}-\mathsf{g}^{ik}{}_{:k}\delta\{{}^{\,l}_{i\,l}\})d\Omega=0, \tag{2.5.3}$$

where

$$\mathsf{g}^{ik}=\sqrt{-\mathfrak{g}}g^{ik} \tag{2.5.4}$$

is the contravariant *metric density*, whose covariant derivative with respect to the Christoffel symbols vanishes, $\mathsf{g}^{ik}{}_{:l}=0$. Equaling $\delta S=0$ in (2.5.2) gives the *Einstein equations* of the general theory of relativity:

$$G_{ik}=\kappa T_{ik} \tag{2.5.5}$$

or

$$P_{ik}=\kappa\Big(T_{ik}-\frac{1}{2}Tg_{ik}\Big). \tag{2.5.6}$$

Because $\delta\int P\sqrt{-\mathfrak{g}}d\Omega=\delta\int G\sqrt{-\mathfrak{g}}d\Omega$, where the noncovariant quantity G is given by (2.2.5), the left-hand side of the Einstein equations is

$$G_{ik}=\frac{1}{\sqrt{-\mathfrak{g}}}\frac{\partial(\sqrt{-\mathfrak{g}}G)}{\partial g^{ik}}. \tag{2.5.7}$$

The covariant conservation of the Einstein tensor (1.4.67) imposes the conservation of the metric dynamical energy-momentum tensor (2.4.23). Therefore the gravitational field equations contain the equations of motion of matter. In *vacuum*, where $T_{ik}=0$, the Ricci tensor in (2.5.6) vanishes:

$$P_{ik}=0. \tag{2.5.8}$$

Thus vanishing of P_{ik} at a given point in spacetime is a covariant criterion of whether matter is present or absent at this point.

The Einstein equations (2.5.5) are 10 second-order partial differential equations for: $10-4=6$ independent components of the metric tensor g_{ik} (the factor 4 is the number of the coordinates which can be chosen arbitrarily), 3 independent components of the four-velocity u^i, and either ϵ or p (which are related to each other by the equation of state). The contracted Bianchi identity (1.4.67) gives the equations of motion of matter. In vacuum, the Einstein equations are $10-4=6$ independent equations (the factor 4 is the number of constraints from the contracted Bianchi identity) for 6 independent components of the metric tensor g_{ik}.

In the Einstein equations, the only second time-derivatives of g_{ik} are the derivatives of the spatial components of the metric tensor, $\ddot{g}_{\alpha\beta}$, and they appear only in the $\alpha\beta$ components of the field equations (2.5.5). Therefore the initial values (at $t=0$) for $g_{\alpha\beta}$ and $\dot{g}_{\alpha\beta}$ can be chosen arbitrarily. The first time-derivatives $\dot{g}_{0\alpha}$ and $\dot{g}_{00}$ appear only in the $\alpha\beta$ components of the field equations (2.5.5). The 0α and 00 components of the field equations (2.5.5) give the initial values for $g_{0\alpha}$ and g_{00}. The undetermined initial values for $\dot{g}_{0\alpha}$ and $\dot{g}_{00}$ correspond to 4 degrees of freedom for a free gravitational field. A general gravitational field has 8 degrees of freedom: 4 degrees of freedom for a free gravitational field, 3 related to the four-velocity, and 1 related to ϵ (or p).

2.5.2 Einstein pseudotensor and principle of equivalence

Define

$$\mathsf{G}=\sqrt{-g}G, \tag{2.5.9}$$

where G is the noncovariant quantity (2.2.5). The action for the gravitational field and matter,

$$S=-\frac{1}{2\kappa c}\int \mathsf{G}d\Omega+S_m=\frac{1}{c}\int\Big(-\frac{1}{2\kappa}\mathsf{G}+\mathfrak{L}_m\Big)d\Omega, \tag{2.5.10}$$

produces the Einstein field equations by varying the metric tensor, because G differs from $\sqrt{-g}P$ by a total divergence:

$$\frac{\delta}{\delta g^{ik}}\Big(-\frac{1}{2\kappa}\mathsf{G}+\mathfrak{L}_m\Big)=0. \tag{2.5.11}$$

Construct a canonical energy-momentum density (2.4.35) corresponding to the gravitational field, treating $-\frac{1}{2\kappa}\mathsf{G}$ (which depends only on g^{ij} and its first derivatives $g^{ij}{}_{,k}$) like $\mathfrak{L}_m$ and g^{ik} like a matter field ϕ:

$$\mathfrak{t}_k{}^i=-\frac{1}{2\kappa}\Big(\frac{\partial\mathfrak{G}}{\partial g^{jl}{}_{,i}}g^{jl}{}_{,k}-\delta^i_k\mathfrak{G}\Big). \tag{2.5.12}$$

This quantity is not a tensor density since $\mathfrak{G}$ is not a scalar density and its division by $\sqrt{-\mathfrak{g}}$, $\frac{\mathfrak{t}_k{}^i}{\sqrt{-\mathfrak{g}}}$, is referred to as the *Einstein energy-momentum pseudotensor* for the gravitational field. The four-momentum corresponding to the total energy-momentum density for the gravitational field and matter (which is not a vector) is then

$$P_i=\frac{1}{c}\int(\mathfrak{t}_i{}^k+\mathcal{T}_i{}^k)dS_k, \tag{2.5.13}$$

where the sum $\mathfrak{t}_i^{\ k} + \mathcal{T}_i^{\ k}$ is called the *Einstein energy-momentum complex*. The definition (2.5.12) gives

$$2\kappa\mathfrak{t}_{k\ ,i}^{\ i} = \partial_i\frac{\partial\mathfrak{G}}{\partial g^{jl}_{\ \ ,i}}g^{jl}_{\ \ ,k} - \frac{\partial\mathfrak{G}}{\partial g^{jl}_{\ \ ,i}}g^{jl}_{\ \ ,ki} + \mathfrak{G}_{,k} = \partial_i\frac{\partial\mathfrak{G}}{\partial g^{jl}_{\ \ ,i}}g^{jl}_{\ \ ,k} - \frac{\partial\mathfrak{G}}{\partial g^{jl}_{\ \ ,i}}g^{jl}_{\ \ ,ki} + \frac{\partial\mathfrak{G}}{\partial g^{jl}}g^{jl}_{\ \ ,k}$$
$$+\frac{\partial\mathfrak{G}}{\partial g^{jl}_{\ \ ,i}}g^{jl}_{\ \ ,ik} = \Big(\frac{\partial\mathfrak{G}}{\partial g^{jl}} - \partial_i\frac{\partial\mathfrak{G}}{\partial g^{jl}_{\ \ ,i}}\Big)g^{jl}_{\ \ ,k} = \frac{\delta\mathfrak{G}}{\delta g^{jl}}g^{jl}_{\ \ ,k}, \tag{2.5.14}$$

which, using (2.5.11), gives

$$\mathfrak{t}_{k\ ,i}^{\ i} = \frac{\delta\mathfrak{L}_m}{\delta g^{jl}}g^{jl}_{\ \ ,k} = \frac{1}{2}\mathcal{T}_{jl}g^{jl}_{\ \ ,k}. \tag{2.5.15}$$

The covariant conservation (2.4.23) gives

$$\mathcal{T}_{k\ ,i}^{\ i} = \{^{\,l}_{k\,i}\}\mathcal{T}_l^{\ i} = \frac{1}{2}g^{lm}g_{im,k}\mathcal{T}_l^{\ i} = -\frac{1}{2}g^{lm}_{\ \ \ ,k}\mathcal{T}_{lm}, \tag{2.5.16}$$

so the total energy-momentum density for the gravitational field and matter is ordinarily conserved:

$$(\mathfrak{t}_k^{\ i} + \mathcal{T}_k^{\ i})_{,i} = 0. \tag{2.5.17}$$

Integrating (2.5.17) over the four-dimensional volume and using the Gauß-Stokes theorem gives

$$\oint(\mathfrak{t}_k^{\ i} + \mathcal{T}_k^{\ i})dS_i = 0, \tag{2.5.18}$$

so the four-momentum (2.5.13) is conserved, $P_i =$const. Because the quantity $\mathfrak{t}_{ik}$ is not symmetric in the indices i, k, the total angular momentum constructed from P^i as in (2.4.58),

$$M^{ik} = \int(x^i dP^k - x^k dP^i) = \frac{1}{c}\int(x^i(\mathfrak{t}^{kl} + \mathcal{T}^{kl}) - x^k(\mathfrak{t}^{il} + \mathcal{T}^{il}))dS_l, \tag{2.5.19}$$

is not conserved. The conservation law (2.5.17) gives $\mathfrak{t}_k^{\ l} + \mathcal{T}_k^{\ l} = \eta_{k\ \ ,i}^{\ li}$, where $\eta^{kli} = -\eta^{kil}$, so $\mathfrak{t}_k^{\ l} - \mathfrak{t}^l_{\ k} = (\eta_k^{\ li} - \eta^l_{\ k}{}^i)_{,i}$. Analogously to (2.4.44) and (2.4.45), we could bring $\mathfrak{t}_k^{\ l} + \mathcal{T}_k^{\ l}$ to a symmetric form. However, using $(\eta_k^{\ li} - \eta^l_{\ k}{}^i)$ instead of $\Sigma_k^{\ li}$ in (2.4.45), where $\theta_i^{\ k}$ is replaced by $\mathfrak{t}_i^{\ k} + \mathcal{T}_i^{\ k}$, gives $\tau_i^{\ k} = 0$, so this symmetrization procedure does not work for the Einstein pseudotensor.

The Einstein pseudotensor (2.5.12) can be explicitly written as

$$\mathfrak{t}_k^{\ i} = \frac{1}{2\kappa}(\{^{\ i}_{l\,m}\}\mathfrak{g}^{lm}_{\ \ \ ,k} - \{^{\ l}_{m\,l}\}\mathfrak{g}^{mi}_{\ \ \ ,k} + \delta^i_k\mathfrak{G}), \tag{2.5.20}$$

so it is a homogeneous quadratic function of the Christoffel symbols. Thus it vanishes in the local Galilean frame of reference. It can also differ from zero in the Minkowski spacetime (in the absence of the gravitational field) if we choose the coordinates such that the Christoffel symbols do not vanish. Therefore the energy of the gravitational field is not absolutely localized in spacetime; it depends on the choice of the coordinates. The gravitational field can be always eliminated locally by transforming the coordinate system to the local Galilean frame of reference in which the Einstein pseudotensor vanishes. This property of the gravitational field is referred to as the *principle of equivalence.*

The construction of a conserved four-momentum for the gravitational field and matter is possible because the Lagrangian density for the gravitational field $\mathfrak{L}_g$ is linear in the second derivatives of the metric tensor. The Lagrangian density (2.2.2) can be generalized to

$$\mathfrak{L}_g = -\frac{1}{2\kappa}\sqrt{-\mathfrak{g}}(P + 2\Lambda), \tag{2.5.21}$$

where Λ is referred to as the *cosmological constant*, without altering the Einstein energy-momentum pseudotensor (2.5.12). Another scalar density which is linear in curvature is $\epsilon^{ijkl}P_{ijkl}$, but this parity-violating expression vanishes due to the cyclic identity (1.4.64). Therefore the simplest choice for a gravitational Lagrangian density, linear in P, is the only one that admits ordinary conservation laws for the gravitational field and matter, and thus physical.

2.5.3 Landau-Lifshitz energy-momentum pseudotensor

The covariant conservation (2.4.23) in the local Galilean frame of reference is

$$\mathcal{T}_k{}^i{}_{,i} = 0, \tag{2.5.22}$$

so $\mathcal{T}^{ik}$ can be expressed as $\mathcal{T}^{ik} = \eta^{ikl}{}_{,l}$, where $\eta^{ikl} = -\eta^{ilk}$. The Einstein equations (2.5.5) in the Galilean frame are

$$(-\mathfrak{g})T^{ik} = h^{ikl}{}_{,l}, \tag{2.5.23}$$

where

$$h^{ikl} = \lambda^{iklm}{}_{,m} = -h^{ilk}, \tag{2.5.24}$$

$$\lambda^{iklm} = \frac{1}{2\kappa}(-\mathfrak{g})(g^{ik}g^{lm} - g^{il}g^{km}). \tag{2.5.25}$$

In an arbitrary frame of reference, (2.5.23) is not valid. Define t^{ik} such that

$$(-\mathfrak{g})(t^{ik} + T^{ik}) = h^{ikl}{}_{,l}. \tag{2.5.26}$$

Therefore

$$((-\mathfrak{g})(t^{ik} + T^{ik}))_{,k} = 0, \tag{2.5.27}$$

so there is a conservation of the four-momentum of the gravitational field and matter,

$$P^i = \frac{1}{c}\int(-\mathfrak{g})(t^{ik} + T^{ik})dS_k. \tag{2.5.28}$$

The quantity t^{ik} is not a tensor density, so the conserved four-momentum P^i (2.5.28) is not a vector. The four-momentum P^i is not a vector even for Lorentz transformations, because of the factor $-\mathfrak{g}$ instead of the correct (weight 1) density $\sqrt{-\mathfrak{g}}$ in

(2.5.28). Dividing P^i by $\sqrt{-\mathfrak{g}}$ at some fixed point (a natural choice is infinity) turns it into a vector under Lorentz transformations. Using (2.5.26) turns (2.5.28), for the hypersurface $dS_0 = dV$, into

$$P^i = \frac{1}{c}\int h^{ikl}{}_{,l}dS_k = \frac{1}{2c}\oint h^{ikl}df^*_{kl} = \frac{1}{c}\oint h^{i0\alpha}df_\alpha. \tag{2.5.29}$$

The quantity t^{ik} is referred to as the *Landau-Lifshitz energy-momentum pseudotensor* for the gravitational field, and the sum $(-\mathfrak{g})(t^{ik}+T^{ik})$ is called the *Landau-Lifshitz complex.*

The explicit expression for the Landau-Lifshitz pseudotensor is

$$\begin{aligned}t^{ik} &= \frac{1}{2\kappa}\Big((g^{il}g^{km}-g^{ik}g^{lm})(2\{{}^{\,n}_{lm}\}\{{}^{\,p}_{np}\}-\{{}^{\,n}_{lp}\}\{{}^{\,p}_{mn}\}-\{{}^{\,n}_{ln}\}\{{}^{\,p}_{mp}\})\\ &+g^{il}g^{mn}(\{{}^{\,k}_{lp}\}\{{}^{\,p}_{mn}\}+\{{}^{\,k}_{mn}\}\{{}^{\,p}_{lp}\}-\{{}^{\,k}_{np}\}\{{}^{\,p}_{lm}\}-\{{}^{\,k}_{lm}\}\{{}^{\,p}_{np}\})\\ &+g^{kl}g^{mn}(\{{}^{\,i}_{lp}\}\{{}^{\,p}_{mn}\}+\{{}^{\,i}_{mn}\}\{{}^{\,p}_{lp}\}-\{{}^{\,i}_{np}\}\{{}^{\,p}_{lm}\}-\{{}^{\,i}_{lm}\}\{{}^{\,p}_{np}\})\\ &+g^{lm}g^{np}(\{{}^{\,i}_{ln}\}\{{}^{\,k}_{mp}\}-\{{}^{\,i}_{lm}\}\{{}^{\,k}_{np}\})\Big)\end{aligned} \tag{2.5.30}$$

or

$$\begin{aligned}(-\mathfrak{g})t^{ik} &= \frac{1}{2\kappa}\Big(\mathbf{g}^{ik}{}_{,l}\mathbf{g}^{lm}{}_{,m}-\mathbf{g}^{il}{}_{,l}\mathbf{g}^{km}{}_{,m}+\frac{1}{2}g^{ik}g_{lm}\mathbf{g}^{ln}{}_{,p}\mathbf{g}^{pm}{}_{,n}\\ &-(g^{il}g_{mn}\mathbf{g}^{kn}{}_{,p}\mathbf{g}^{mp}{}_{,l}+g^{kl}g_{mn}\mathbf{g}^{in}{}_{,p}\mathbf{g}^{mp}{}_{,l})+g_{lm}g^{np}\mathbf{g}^{il}{}_{,n}\mathbf{g}^{km}{}_{,p}\\ &+\frac{1}{8}(2g^{il}g^{km}-g^{ik}g^{lm})(2g_{np}g_{qr}-g_{pq}g_{nr})\mathbf{g}^{nr}{}_{,l}\mathbf{g}^{pq}{}_{,m}\Big).\end{aligned} \tag{2.5.31}$$

This pseudotensor is symmetric in the indices i,k, so there is a conservation of the total angular momentum constructed from P^i as in (2.4.58),

$$M^{ik} = \int(x^i dP^k - x^k dP^i) = \frac{1}{c}\int(x^i(t^{kl}+\mathcal{T}^{kl})-x^k(t^{il}+\mathcal{T}^{il}))(-\mathfrak{g})dS_l. \tag{2.5.32}$$

Dividing M^{ik} by $\sqrt{-\mathfrak{g}}$ at infinity turns it into an antisymmetric tensor under Lorentz transformations. Using (2.5.24) and (2.5.26) turns (2.5.28), for the hypersurface $dS_0 = dV$, into

$$\begin{aligned}M^{ik} &= \frac{1}{c}\int(x^i\lambda^{klmn}{}_{,nm}-x^k\lambda^{ilmn}{}_{,nm})dS_l = \frac{1}{2c}\oint(x^i\lambda^{klmn}{}_{,n}-x^k\lambda^{ilmn}{}_{,n})df^*_{lm}\\ &-\frac{1}{c}\oint(\lambda^{klin}-\lambda^{ilkn})_{,n}dS_l = \frac{1}{c}\oint(x^ih^{k0\alpha}-x^kh^{i0\alpha}+\lambda^{i0\alpha k})df_\alpha.\end{aligned} \tag{2.5.33}$$

Choosing the volume hypersurface $dV = dS_0$ gives

$$M^{ik} = \frac{1}{c}\int(x^i(t^{k0}+\mathcal{T}^{k0})-x^k(t^{i0}+\mathcal{T}^{i0}))(-\mathfrak{g})dV. \tag{2.5.34}$$

The conservation of $M^{0\alpha}$ in (2.5.34) divided by the conservation of P^0 in (2.5.28) gives a uniform motion (2.4.61) with velocity (2.4.62) of the center of inertia for the gravitational field and matter, with the coordinates

$$X^\alpha = \frac{\int x^\alpha(t^{00}+\mathcal{T}^{00})(-\mathfrak{g})dV}{\int(t^{00}+\mathcal{T}^{00})(-\mathfrak{g})dV}. \tag{2.5.35}$$

The coordinates of the center of inertia (2.5.35), like (2.4.63), are not the spatial components of a four-dimensional vector.

2.5.4 Utiyama action

The *Utiyama action* for the gravitational field and matter is equal to (2.5.1), where the tetrad is regarded as a variational variable and the spin connection is the Levi-Civita spin connection (1.5.31). Thus (2.3.5) gives

$$\delta S = -\frac{1}{2\kappa c}\int \delta(\mathfrak{e}P)d\Omega + \frac{1}{c}\int \mathfrak{T}_i^{\ a}\delta e^i_a d\Omega. \tag{2.5.36}$$

The Lagrangian density for the gravitational field is given by (2.2.2), with the Riemann scalar P given by (1.5.39) and (1.5.41):

$$\mathfrak{e}P = \mathfrak{e}e^i_a e^{jb}(\varpi^a_{\ bj,i} - \varpi^a_{\ bi,j} + \varpi^a_{\ ci}\varpi^c_{\ bj} - \varpi^a_{\ cj}\varpi^c_{\ bi}) = 2\mathfrak{e}^{ij}_{ab}(\varpi^{ab}_{\ \ j,i} + \varpi^a_{\ ci}\varpi^{cb}_{\ \ j}), \tag{2.5.37}$$

where

$$\mathfrak{e}^{ij}_{ab} = \mathfrak{e}e^{[i}_a e^{j]}_b. \tag{2.5.38}$$

Varying $\mathfrak{e}P$ and omitting total derivatives gives in the absence of torsion, using $\delta\mathfrak{e} = \mathfrak{e}e^i_a\delta e^a_i$ and $\mathfrak{e}^{ij}_{ab|j} = \mathfrak{e}^{ij}_{ab,j} - \varpi^c_{\ aj}\mathfrak{e}^{ij}_{cb} - \varpi^c_{\ bj}\mathfrak{e}^{ij}_{ac} = 0$ (which results from (1.5.22)),

$$\delta(\mathfrak{e}P) = (2P^a_{\ i} - Pe^a_i)\mathfrak{e}\delta e^i_a + 2\mathfrak{e}^{ij}_{ab}\delta(\varpi^{ab}_{\ \ j,i} + \varpi^a_{\ ci}\varpi^{cb}_{\ \ j}) = (2P^a_{\ i} - Pe^a_i)\mathfrak{e}\delta e^i_a$$
$$+2(\mathfrak{e}^{ij}_{ab,j} - \varpi^c_{\ aj}\mathfrak{e}^{ij}_{cb} - \varpi^c_{\ bj}\mathfrak{e}^{ij}_{ac})\delta\varpi^{ab}_{\ \ i} = (2P^a_{\ i} - Pe^a_i)\mathfrak{e}\delta e^i_a. \tag{2.5.39}$$

Equaling $\delta S = 0$ gives the tetrad Einstein equations:

$$P^a_{\ i} - \frac{1}{2}Pe^a_i = \frac{\kappa}{\mathfrak{e}}\mathfrak{T}_i^{\ a}, \tag{2.5.40}$$

equivalent to the metric Einstein equations (2.5.5) because of (2.3.4) and (2.3.32) (in the absence of torsion).

2.5.5 Møller pseudotensor

The Riemann scalar P is linear in derivatives of $\varpi^a_{\ bi}$:

$$\mathfrak{e}P = (\mathfrak{e}e^i_a e^j_b\varpi^{ab}_{\ \ j})_{,i} - (\mathfrak{e}e^i_a e^j_b)_{,i}\varpi^{ab}_{\ \ j} - (\mathfrak{e}e^i_a e^j_b\varpi^{ab}_{\ \ i})_{,j} + (\mathfrak{e}e^i_a e^j_b)_{,j}\varpi^{ab}_{\ \ i} + \mathfrak{e}e^i_a e^j_b\varpi^{ac}_{\ \ i}\varpi_c^{\ b}{}_j$$
$$-\mathfrak{e}e^i_a e^j_b\varpi^{ac}_{\ \ j}\varpi_c^{\ b}{}_i = 2(\mathfrak{e}e^i_a e^j_b\varpi^{ab}_{\ \ j})_{,i} - 2(\mathfrak{e}e^i_a e^j_b)_{,i}\varpi^{ab}_{\ \ j} + \mathfrak{e}e^i_a e^j_b\varpi^{ac}_{\ \ i}\varpi_c^{\ b}{}_j$$
$$-\mathfrak{e}e^i_a e^j_b\varpi^{ac}_{\ \ j}\varpi_c^{\ b}{}_i. \tag{2.5.41}$$

Thus we can subtract from $\mathfrak{e}P$ total derivatives without altering the field equations, replacing P by a noncovariant quantity M:

$$\mathfrak{e}M = -2(\mathfrak{e}e^i_a e^j_b)_{,i}\varpi^{ab}_{\ \ j} + \mathfrak{e}e^i_a e^j_b\varpi^{ac}_{\ \ i}\varpi_c^{\ b}{}_j - \mathfrak{e}e^i_a e^j_b\varpi^{ac}_{\ \ j}\varpi_c^{\ b}{}_i$$
$$= -2\mathfrak{e}(\{{}^{\,k}_{k\,i}\}\varpi^{ij}_{\ \ j} + \varpi^i_{\ ai}\varpi^{aj}_{\ \ j} - \{{}^{\,i}_{k\,i}\}\varpi^{kj}_{\ \ j} + \varpi^j_{\ bi}\varpi^{ib}_{\ \ j} - \{{}^{\,j}_{k\,i}\}\varpi^{ik}_{\ \ j})$$
$$+\mathfrak{e}(\omega^{ic}_{\ \ i}\omega_c^{\ j}{}_j - \omega^{ic}_{\ \ j}\omega_c^{\ j}{}_i) = \mathfrak{e}(\omega^{ia}_{\ \ i}\omega^j_{\ aj} - \omega^{ia}_{\ \ j}\omega^j_{\ ai}), \tag{2.5.42}$$

using (1.4.35) and (1.5.31). Therefore

$$M = \omega^{ia}_{\ \ i}\omega^j_{\ aj} - \omega^{ia}_{\ \ j}\omega^j_{\ ai}. \tag{2.5.43}$$

Because of (1.5.15), the quantity (2.5.43) depends on the tetrad e^i_a and its first derivatives $e^i_{a,j}$. Therefore, analogously to (2.5.12), we can construct a canonical energy-momentum density corresponding to the gravitational field, treating $-\frac{1}{2\kappa}\mathfrak{e}M$:

$$\mathfrak{m}_k{}^i = -\frac{1}{2\kappa}\Big(\frac{\partial(\mathfrak{e}M)}{\partial e^j_{a,i}}e^j_{a,k} - \delta^i_k\mathfrak{e}M\Big). \tag{2.5.44}$$

This quantity is not a tensor density since $\mathfrak{e}M$ is not a scalar density and its division by $\mathfrak{e}$, $\frac{\mathfrak{m}_k{}^i}{\mathfrak{e}}$, is referred to as the *Møller energy-momentum pseudotensor* for the gravitational field. One can show, analogously to the steps leading to (2.5.17, that the total energy-momentum density for the gravitational field and matter is ordinarily conserved:

$$(\mathfrak{m}_k{}^i + \mathcal{T}_k{}^i)_{,i} = 0. \tag{2.5.45}$$

Thus the corresponding total four-momentum is conserved:

$$P_i = \frac{1}{c}\int(\mathfrak{m}_i{}^k + \mathcal{T}_i{}^k)dS_k = \text{const}, \tag{2.5.46}$$

where the sum $\mathfrak{m}_i{}^k + \mathcal{T}_i{}^k$ is called the *Møller energy-momentum complex*. The Møller pseudotensor depends on the choice of both the coordinates and the tetrad. To fix the tetrad, one can impose on it 6 constraints which are covariant under constant Lorentz transformations but not under general Lorentz transformations (otherwise these constraints would not fix the tetrad since Lorentz transformations are tetrad rotations). A natural choice is to constrain the 6 components of the spin connection ω_{ijk} in which the last index is contracted with a covariant derivative or the trace of the spin connection.

2.5.6 Einstein-Cartan action

If we regard the torsion tensor as a variational variable (in addition to the metric tensor) then the action for the gravitational field and matter is, due to (2.2.1),

$$S = -\frac{1}{2\kappa c}\int R\sqrt{-\mathfrak{g}}d\Omega + S_m, \tag{2.5.47}$$

and it is referred to as the *Einstein-Cartan action*. Using (1.4.54) gives

$$S = -\frac{1}{2\kappa c}\int\Big(P - g^{ik}(2C^l{}_{il:k} + C^j{}_{ij}C^l{}_{kl} - C^l{}_{im}C^m{}_{kl})\Big)\sqrt{-\mathfrak{g}}d\Omega + S_m. \tag{2.5.48}$$

Partial integration of the terms with covariant derivatives : and omitting total derivatives (which do not contribute to the field equations) reduces (2.5.48) to

$$S = -\frac{1}{2\kappa c}\int\Big(P - g^{ik}(C^j{}_{ij}C^l{}_{kl} - C^l{}_{im}C^m{}_{kl})\Big)\sqrt{-\mathfrak{g}}d\Omega + S_m. \tag{2.5.49}$$

Varying (2.5.49) with respect to the metric tensor and contortion tensor (which is equivalent to varying with respect to the torsion tensor) gives, using (2.3.19), (2.3.26) and (2.5.3),

$$\delta S = -\frac{1}{2\kappa c}\int\Big(P_{ik} - \frac{1}{2}Pg_{ik} - C^j_{\ ij}C^l_{\ kl} + C^l_{\ im}C^m_{\ kl} + \frac{1}{2}g_{ik}(C^{jm}_{\ \ \ j}C^l_{\ ml} - C^{lj}_{\ \ \ m}C^m_{\ jl})\Big)$$
$$\times\sqrt{-\mathfrak{g}}\delta g^{ik}d\Omega - \frac{1}{\kappa c}\int(C^{kj}_{\ \ \ i} - C^{lj}_{\ \ \ l}\delta^k_i)\sqrt{-\mathfrak{g}}\delta C^i_{\ jk}d\Omega + \frac{1}{2c}\int T_{ik}\sqrt{-\mathfrak{g}}\delta g^{ik}d\Omega$$
$$+\frac{1}{2c}\int s_j^{\ ik}\sqrt{-\mathfrak{g}}\delta C^j_{\ ik}d\Omega. \tag{2.5.50}$$

For variations δg^{ik}, $\delta S = 0$ gives the *first Einstein-Cartan equation*

$$G_{ik} = \kappa(T_{ik} + U_{ik}), \tag{2.5.51}$$

where

$$U_{ik} = \frac{1}{\kappa}\Big(C^j_{\ ij}C^l_{\ kl} - C^l_{\ ij}C^j_{\ kl} - \frac{1}{2}g_{ik}(C^{jm}_{\ \ \ j}C^l_{\ ml} - C^{mjl}C_{ljm})\Big) \tag{2.5.52}$$

or

$$U_{ik} = \frac{1}{\kappa}\Big(-(S^l_{\ ij} + 2S_{(ij)}^{\ \ \ \ l})(S^j_{\ kl} + 2S_{(kl)}^{\ \ \ \ j}) + 4S_iS_k + \frac{1}{2}g_{ik}(S^{mjl} + 2S^{(jl)m})$$
$$\times(S_{ljm} + 2S_{(jm)l}) - 2g_{ik}S^jS_j\Big). \tag{2.5.53}$$

For variations $\delta C^j_{\ ik}$, $\delta S = 0$ gives the *second Einstein-Cartan equation*

$$C^k_{\ [ji]} - \delta^k_{[i}C^l_{\ j]l} = \frac{\kappa}{2}s_{ij}^{\ \ k} \tag{2.5.54}$$

or

$$T^j_{\ ik} = -\frac{\kappa}{2}s_{ik}^{\ \ j}, \tag{2.5.55}$$

where

$$T^j_{\ ik} = S^j_{\ ik} - S_i\delta^j_k + S_k\delta^j_i \tag{2.5.56}$$

is the *modified torsion tensor*. The relation (2.5.55) is equivalent to

$$S^k_{\ ij} = -\frac{\kappa}{2}(s_{ij}^{\ \ k} + \delta^k_{[i}s_{j]l}^{\ \ \ l}). \tag{2.5.57}$$

This relation between the torsion and spin tensors is algebraic: torsion at a given point in spacetime does not vanish only if there is matter at this point, represented in the Lagrangian density by a function which depends on torsion. Unlike the metric, which is related to matter through a differential field equation, torsion does not propagate. Combining (2.5.52) and (2.5.57) gives

$$U_{ik} = \kappa\Big(-s^{ij}_{\ \ [l}s^{kl}_{\ \ \ j]} - \frac{1}{2}s^{ijl}s^k_{\ jl} + \frac{1}{4}s^{jli}s_{jl}^{\ \ k} + \frac{1}{8}g_{ik}(-4s^l_{\ j[m}s^{jm}_{\ \ \ l]} + s^{jlm}s_{jlm})\Big). \tag{2.5.58}$$

The tensor (2.5.58) represents a correction to the dynamical energy-momentum tensor from the spin contributions to the geometry of spacetime, quadratic in the spin density (so the sign of the spin density does not affect this correction) and corresponding to a spin-spin contact interaction. If matter fields do not depend on torsion then $U_{ik} = 0$ and the first Einstein-Cartan equation (2.5.51) reduces to the Einstein equations (2.5.5).

2.5.7 Kibble-Sciama action

The *Kibble-Sciama action* for the gravitational field and matter is equal to (2.5.47), where both the tetrad and spin connection are regarded as variational variables. Thus (2.3.27) gives

$$\delta S = -\frac{1}{2\kappa c}\int \delta(\mathfrak{e}R)d\Omega + \frac{1}{c}\int \mathfrak{T}_i^{\ a}\delta e^i_a d\Omega + \frac{1}{2c}\int \mathfrak{S}_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i}d\Omega. \tag{2.5.59}$$

The Lagrangian density for the gravitational field is given by (2.2.1), with the curvature scalar R given by (1.5.36) and (1.5.38):

$$\mathfrak{e}R = \mathfrak{e}e^i_a e^{jb}(\omega^a_{\ bj,i} - \omega^a_{\ bi,j} + \omega^a_{\ ci}\omega^c_{\ bj} - \omega^a_{\ cj}\omega^c_{\ bi}) = 2\mathfrak{e}^{ij}_{ab}(\omega^{ab}_{\ \ j,i} + \omega^a_{\ ci}\omega^{cb}_{\ \ j}). \tag{2.5.60}$$

Varying $\mathfrak{e}R$ and omitting total derivatives gives, using $\mathfrak{e}^{ij}_{ab|j} = \mathfrak{e}^{ij}_{ab,j} - \omega^c_{\ aj}\mathfrak{e}^{ij}_{cb} - \omega^c_{\ bj}\mathfrak{e}^{ij}_{ac} + \Gamma^{\,i}_{k\,j}\mathfrak{e}^{kj}_{ab} + \Gamma^{\,j}_{k\,j}\mathfrak{e}^{ik}_{ab} - \Gamma^{\,k}_{k\,j}\mathfrak{e}^{ij}_{ab} = 0$ (which results from (1.5.22)),

$$\begin{aligned}\delta(\mathfrak{e}R) &= (2R^a_{\ i} - Re^a_i)\mathfrak{e}\delta e^i_a + 2\mathfrak{e}^{ij}_{ab}\delta(\omega^{ab}_{\ \ j,i} + \omega^a_{\ ci}\omega^{cb}_{\ \ j}) \\ &= (2R^a_{\ i} - Re^a_i)\mathfrak{e}\delta e^i_a + 2(\mathfrak{e}^{ij}_{ab,j} - \omega^c_{\ aj}\mathfrak{e}^{ij}_{cb} - \omega^c_{\ bj}\mathfrak{e}^{ij}_{ac})\delta\omega^{ab}_{\ \ i} \\ &= (2R^a_{\ i} - Pe^a_i)\mathfrak{e}\delta e^i_a - 2(S^i_{\ kj}\mathfrak{e}^{kj}_{ab} + 2S_j\mathfrak{e}^{ij}_{ab})\delta\omega^{ab}_{\ \ i}.\end{aligned} \tag{2.5.61}$$

For variations $\delta\omega^{ab}_{\ \ i}$, $\delta S = 0$ gives

$$S^i_{\ ab} - S_a e^i_b + S_b e^i_a = -\frac{\kappa}{2\mathfrak{e}}\mathfrak{S}_{ab}^{\ \ i}, \tag{2.5.62}$$

equivalent to the second Einstein-Cartan equation (2.5.55). For variations δe^i_a, $\delta S = 0$ gives

$$R^a_{\ i} - \frac{1}{2}Re^a_i = \frac{\kappa}{\mathfrak{e}}\mathfrak{T}_i^{\ a} \tag{2.5.63}$$

or

$$R_{ki} - \frac{1}{2}Rg_{ik} = \frac{\kappa}{\sqrt{-\mathfrak{g}}}\mathfrak{T}_{ik}. \tag{2.5.64}$$

Substituting (2.5.55) and (2.5.64) into the conservation law for the spin density (2.4.16) gives

$$-2(S^k_{\ ij;k} - S_{i;j} + S_{j;i}) = R_{ji} - R_{ij} - 4S_k(S^k_{\ ij} - S_i\delta^k_j + S_j\delta^k_i), \tag{2.5.65}$$

which is equivalent to the contracted cyclic identity (1.4.61). Thus the contracted cyclic identity imposes the conservation law for the spin density in the Einstein-Cartan gravity. Substituting (2.5.55) and (2.5.64) into the conservation law for the energy-momentum density (2.4.31) gives

$$R^j_{\ i;j} - \frac{1}{2}R_{;i} = 2S_j\Big(R^j_{\ i} - \frac{1}{2}R\delta^j_i\Big) + 2S^j_{\ ki}\Big(R^k_{\ j} - \frac{1}{2}R\delta^k_j\Big) - (S^j_{\ kl} - S_k\delta^j_l + S_l\delta^j_k)R^{kl}_{\ \ ji}, \tag{2.5.66}$$

which is equivalent to the contracted Bianchi identity (1.4.62). Thus the contracted Bianchi identity imposes the conservation law for the energy-momentum density in the Einstein-Cartan gravity. Substituting (2.5.55) and (2.5.64) into the Belinfante-Rosenfeld relation (2.3.33) gives

$$\kappa T_{ik} = R_{ki} - \frac{1}{2}Rg_{ik} + \nabla^*_j(S^j_{\ ik} + 2S_k\delta^j_i - 2S_{(ik)}^{\ \ \ j} - 2S^j g_{ik}) = R_{ki} - \frac{1}{2}Rg_{ik}$$
$$+\nabla^*_j(-C^j_{\ ki} + C^l_{\ kl}\delta^j_i - C^{lj}_{\ \ l}g_{ik}). \tag{2.5.67}$$

Combining (1.4.52), (1.4.54) and (2.5.67) gives

$$\kappa T_{ik} = P_{ik} - \frac{1}{2}Pg_{ik} + C^l_{\ ki:l} - C^l_{\ kl:i} + C^j_{\ ki}C^l_{\ jl} - C^j_{\ kl}C^l_{\ ji} - \frac{1}{2}g_{ik}(-2C^{lj}_{\ \ l:j}$$
$$-C^{lj}_{\ \ l}C^m_{\ jm} + C^{mjl}C_{ljm}) - C^j_{\ ki:j} - C^j_{\ lj}C^l_{\ ki} + C^l_{\ kj}C^j_{\ li} + C^l_{\ ij}C^j_{\ kl} + C^j_{\ kj:i}$$
$$-C^l_{\ ki}C^j_{\ lj} - g_{ik}(C^{lj}_{\ \ l:j} + C^j_{\ lj}C^{ml}_{\ \ \ m}) - C_j(-C^j_{\ ki} + C^l_{\ kl}\delta^j_i - C^{lj}_{\ \ l}g_{ik}), \tag{2.5.68}$$

which is equivalent to the first Einstein-Cartan equation (2.5.51). Thus the relation between the Ricci tensor and the Riemannian Ricci tensor is equivalent to the Belinfante-Rosenfeld relation in the Einstein-Cartan gravity, and (2.5.64) is another form of the first Einstein-Cartan equation.

2.5.8 Einstein-Cartan pseudotensor

Replacing the action for the gravitational field and matter (2.5.49) by

$$S = -\frac{1}{2\kappa c}\int\Big(G - g^{ik}(C^j_{\ ij}C^l_{\ kl} - C^l_{\ im}C^m_{\ kl})\Big)\sqrt{-\mathfrak{g}}d\Omega + S_m \tag{2.5.69}$$

produces the first Einstein-Cartan equation by varying the metric tensor, because $\sqrt{-g}G$ differs from $\sqrt{-g}P$ by a total divergence:

$$\frac{\delta}{\delta g^{ik}}\Big(-\frac{1}{2\kappa}\Big(\mathsf{G} - \mathsf{g}^{np}(C^j_{\ nj}C^l_{\ pl} - C^l_{\ nm}C^m_{\ pl})\Big) + \mathfrak{L}_m\Big) = 0. \tag{2.5.70}$$

The canonical energy-momentum density for the gravitational field is also given by (2.5.12). The relations (2.5.14) and (2.5.70) give

$$\mathsf{t}_k^{\ i}{}_{,i} = \frac{\delta(\mathfrak{L}_m + 2\kappa\mathsf{g}^{np}(C^j_{\ nj}C^l_{\ pl} - C^l_{\ nm}C^m_{\ pl}))}{\delta g^{jl}}g^{jl}_{\ \ ,k} = \frac{1}{2}(\mathcal{T}_{jl} + \sqrt{-\mathfrak{g}}U_{jl})g^{jl}_{\ \ ,k}. \tag{2.5.71}$$

The covariant conservation (2.4.23) gives

$$(\mathcal{T}_k^{\ i} + \sqrt{-\mathfrak{g}}U_k^{\ i})_{,i} = \{^{\,l}_{k\,i}\}(\mathcal{T}_l^{\ i} + \sqrt{-\mathfrak{g}}U_l^{\ i}) = \frac{1}{2}g^{lm}g_{im,k}(\mathcal{T}_l^{\ i} + \sqrt{-\mathfrak{g}}U_l^{\ i})$$
$$= -\frac{1}{2}g^{lm}_{\ \ \ ,k}(\mathcal{T}_{lm} + \sqrt{-\mathfrak{g}}U_{lm}), \tag{2.5.72}$$

so the total energy-momentum density for the gravitational field and matter is ordinarily conserved:

$$(\mathsf{t}_k^{\ i} + \mathcal{T}_k^{\ i} + \sqrt{-\mathfrak{g}}U_k^{\ i})_{,i} = 0. \tag{2.5.73}$$

Thus the corresponding four-momentum is conserved:

$$P_i = \frac{1}{c}\int(\mathfrak{t}_i^{\ k} + \mathcal{T}_i^{\ k} + \sqrt{-\mathfrak{g}}U_i^{\ k})dS_k = \text{const.} \tag{2.5.74}$$

The quantity $\frac{\mathfrak{t}_i^{\ k}}{\sqrt{-\mathfrak{g}}} + U_i^{\ k}$ is referred to as the *Einstein-Cartan energy-momentum pseudotensor* for the gravitational field, and the sum $\mathfrak{t}_i^{\ k} + \mathcal{T}_i^{\ k} + \sqrt{-\mathfrak{g}}U_i^{\ k}$ is called the *Einstein-Cartan energy-momentum complex.*

2.5.9 Palatini variation

If the matter action S_m does not depend on the affine connection, its variation with respect to the metric and connection ($\delta\Gamma^{\,j}_{i\,k}$ is a tensor) is referred to as the *Palatini variation.* Varying (2.5.47) with respect to $\Gamma^{\,k}_{i\,j}$ gives, due to (1.3.39),

$$\delta S = -\frac{1}{2\kappa c}\int \delta R_{ik}\mathfrak{g}^{ik}d\Omega = -\frac{1}{2\kappa c}\int((\delta\Gamma^{\,l}_{i\,k})_{;l} - (\delta\Gamma^{\,l}_{i\,l})_{;k} - 2S^j_{\ lk}\delta\Gamma^{\,l}_{i\,j})g^{ik}\sqrt{-\mathfrak{g}}d\Omega. \tag{2.5.75}$$

Partial integration and omitting total derivatives in (2.5.75) gives, using (1.2.33),

$$\delta S = \frac{1}{2\kappa c}\int(\delta\Gamma^{\,l}_{i\,k}\mathfrak{g}^{ik}_{\ \ ;l} - 2S_l\delta\Gamma^{\,l}_{i\,k}\mathfrak{g}^{ik} - \delta\Gamma^{\,l}_{i\,l}\mathfrak{g}^{ik}_{\ \ ;k} + 2S_k\delta\Gamma^{\,l}_{i\,l}\mathfrak{g}^{ik} + 2S^j_{\ lk}\delta\Gamma^{\,l}_{i\,j}\mathfrak{g}^{ik})d\Omega. \tag{2.5.76}$$

Since the affine connection is metric-compatible, $g_{ij;k} = 0$, $\delta S = 0$ turns the torsion tensor into zero, so the connection is formed by the Christoffel symbols and the field equations are the Einstein equations (2.5.5). Thus varying the action for matter fields, which do not depend on the affine connection, with respect to the connection is equivalent to varying it with respect to the torsion tensor. However, if the matter action S_m depends on the affine connection then (2.5.76) becomes

$$\begin{aligned}\delta S = &\frac{1}{2\kappa c}\int(\delta\Gamma^{\,l}_{i\,k}\mathfrak{g}^{ik}_{\ \ ;l} - 2S_l\delta\Gamma^{\,l}_{i\,k}\mathfrak{g}^{ik} - \delta\Gamma^{\,l}_{i\,l}\mathfrak{g}^{ik}_{\ \ ;k} + 2S_k\delta\Gamma^{\,l}_{i\,l}\mathfrak{g}^{ik} + 2S^j_{\ lk}\delta\Gamma^{\,l}_{i\,j}\mathfrak{g}^{ik})d\Omega \\ &+\frac{1}{2c}\int \Pi^{i\ k}_{\ j}\delta\Gamma^{\,j}_{i\,k}d\Omega,\end{aligned} \tag{2.5.77}$$

where the *hypermomentum density* is defined as

$$\Pi^{i\ k}_{\ j} = 2\frac{\delta\mathfrak{L}_m}{\delta\Gamma^{\,j}_{i\,k}}. \tag{2.5.78}$$

Since the connection is metric-compatible, $\delta S = 0$ gives

$$g^{ik}S_j - \delta^k_j S^i - S^{ki}_{\ \ j} = \frac{\kappa}{2\sqrt{-\mathfrak{g}}}\Pi^{i\ k}_{\ j}. \tag{2.5.79}$$

Contracting the indices i, j gives

$$\Pi^{i\ k}_{\ i} = 0, \tag{2.5.80}$$

which also results from the invariance of the Lagrangian density under a projective transformation (1.2.49) (the symmetric part of the Ricci tensor is invariant under this transformation):

$$\delta\mathfrak{L} = \delta\mathfrak{L}_m = \frac{1}{2}\Pi^{i\ \,k}_{\ j}\delta\Gamma^{\ j}_{i\ k} = \frac{1}{2}\Pi^{i\ \,k}_{\ j}\delta^j_i\delta A_k = 0. \tag{2.5.81}$$

The relation (2.5.80) constrains possible forms of matter Lagrangians algebraically, so it is not a conservation law. Therefore varying the action with respect to the affine connection, unlike that with respect to the torsion (or spin connection), does not constitute a physical variational principle. Only the antisymmetric part of the connection (torsion) can be regarded as a dynamical variable; its symmetric part can always be brought locally to zero by a suitable transformation of the coordinates.

2.5.10 Gravitational potential

If the metric tensor g_{ij} is approximately equal to the Minkowski metric tensor η_{ij} then the corresponding gravitational field is *weak*. We can write

$$g_{00} \approx 1 + \frac{2\phi}{c^2}, \tag{2.5.82}$$

where ϕ is referred to as the *gravitational potential*. Thus nonrelativistic gravitational fields, corresponding to the limit $c \to \infty$, are weak. Also $u^0 \approx 1$ and $u^\alpha \approx 0$. In this limit, the leading component of the Levi-Civita connection is

$$\{^{\,\alpha}_{0\,0}\} \approx -\frac{1}{2}g^{\alpha\beta}\frac{\partial g_{00}}{\partial x^\beta} = \frac{1}{c^2}\frac{\partial\phi}{\partial x^\alpha}, \tag{2.5.83}$$

so the metric geodesic equation (1.4.80) reduces to

$$\frac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla}\phi. \tag{2.5.84}$$

The quantity G in (2.2.5) reduces to

$$G = \frac{2}{c^4}(\nabla\phi)^2. \tag{2.5.85}$$

The leading component of the Riemannian Ricci tensor is

$$P_{00} \approx \frac{\partial\{^{\,\alpha}_{0\,0}\}}{\partial x^\alpha} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial x^{\alpha 2}} = \frac{1}{c^2}\triangle\phi. \tag{2.5.86}$$

The leading component of the energy-momentum tensor (2.4.167) is

$$T_{00} = \mu c^2. \tag{2.5.87}$$

Therefore the Einstein equations in the nonrelativistic limit reduce to the *Poisson equation*:

$$\triangle\phi = 4\pi G\mu, \tag{2.5.88}$$

where

$$G = \frac{c^4\kappa}{8\pi} \tag{2.5.89}$$

is *Newton's gravitational constant*. In vacuum, where $\mu = 0$, the Poisson equation reduces to the Laplace equation:

$$\triangle\phi = 0. \tag{2.5.90}$$

2.5.11 Hydrodynamics

The covariant conservation (2.4.24) of the metric energy-momentum tensor (2.4.77) gives

$$((\epsilon+p)u^k)_{:k}u^i+(\epsilon+p)u^k u^i_{\ :k}=p_{,k}g^{ik}. \tag{2.5.91}$$

Multiplying (2.5.91) by u_i gives

$$((\epsilon+p)u^k)_{:k}=p_{,k}u^k, \tag{2.5.92}$$

which, upon substituting into (2.5.91) yields the *Euler equation*:

$$(\epsilon+p)\frac{D^{\{\}}u^i}{ds}=p_{,k}h^{ik}. \tag{2.5.93}$$

If $p_{,i}\propto u_i$ (which includes the case $p=$ const) then (2.5.93) reduces to the metric geodesic equation (1.4.80). Defining a quantity w such that

$$\frac{dw}{w}=\frac{d\epsilon}{\epsilon+p} \tag{2.5.94}$$

brings (2.5.92) to

$$(wu^i)_{:i}=0. \tag{2.5.95}$$

In the nonrelativistic limit, $c\rightarrow\infty$, $u^0\sim 1$, $u^\alpha\approx\frac{v^\alpha}{c}$, $\epsilon\approx\mu c^2$ and $p\ll\epsilon$, so (2.5.92) reduces to the *equation of continuity*:

$$\frac{\partial\mu}{\partial t}+\operatorname{div}\mathbf{s}=0, \tag{2.5.96}$$

where

$$\mathbf{s}=\mu\mathbf{v} \tag{2.5.97}$$

is referred to as the *mass current*. Integrating (2.5.96) over the volume gives

$$\frac{\partial}{\partial t}\int\mu dV+\oint\mathbf{s}\cdot d\mathbf{f}=0, \tag{2.5.98}$$

which means that the change in time of the total mass inside a volume, $m=\int\mu dV$, is balanced by the *mass flux* through the surface bounding this volume, representing the conservation of the total mass of a fluid. The Euler equation (2.5.93) reduces in this limit to

$$\mu\Big(\frac{\partial v^\alpha}{\partial t}+v^\alpha_{\ ,\beta}v^\beta\Big)=\mu\phi_{,\beta}\eta^{\alpha\beta}+p_{,\beta}\eta^{\alpha\beta} \tag{2.5.99}$$

or

$$\mu\frac{d\mathbf{v}}{dt}=\mu\Big(\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big)=-\mu\boldsymbol{\nabla}\phi-\boldsymbol{\nabla}p. \tag{2.5.100}$$

Integrating (2.5.100) over the volume gives, using $\mathbf{P}=\int\mu\mathbf{v}dV$, the change in time of the total momentum of a fluid:

$$\frac{d\mathbf{P}}{dt}=-\int\boldsymbol{\nabla}\phi dm-\oint pd\mathbf{f}. \tag{2.5.101}$$

Without pressure gradients, (2.5.100) reduces to (2.5.84).
References: [1, 2, 3, 4, 5, 6, 7]

2.6 Spinor fields

2.6.1 Dirac matrices

The Dirac matrices defined by (1.7.1) are complex. A particular solution of (1.7.1) is given by the *Dirac representation*:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix}, \tag{2.6.1}$$

where I is the unit 2×2 matrix and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.6.2}$$

are the *Pauli matrices* (all indices are coordinate invariant). The Pauli matrices are traceless $\mathrm{tr}(\sigma^\alpha) = 0$ and Hermitian $\sigma^{\alpha\dagger} = \sigma^\alpha$ (the Hermitian conjugation of a matrix A is the combination of the complex conjugation and transposition, $A^\dagger = A^{*T}$), satisfy

$$\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\epsilon_{\alpha\beta\gamma}\sigma_\gamma, \tag{2.6.3}$$

and their square is I. The identity (2.6.3) gives the anticommutation relation

$$\Big[\frac{\sigma_\alpha}{2}, \frac{\sigma_\beta}{2}\Big] = i\epsilon_{\alpha\beta\gamma}\frac{\sigma_\gamma}{2}, \tag{2.6.4}$$

so $\frac{\gamma_\alpha}{2}$ form the lowest, two-dimensional representation of the angular momentum operator M_α (1.6.76). The properties of σ^α imply that the Dirac matrices are traceless $\mathrm{tr}(\gamma^i) = 0$ and satisfy

$$\gamma^{0\dagger} = \gamma^0, \;\; \gamma^{\alpha\dagger} = -\gamma^\alpha, \;\; \gamma^{i\dagger} = \gamma^0\gamma^i\gamma^0. \tag{2.6.5}$$

Define

$$\gamma^5 = -\frac{i}{24}e_{ijkl}\gamma^i\gamma^j\gamma^k\gamma^l = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.6.6}$$

which is traceless $\mathrm{tr}(\gamma^5) = 0$ and Hermitian $\gamma^{5\dagger} = \gamma^5$, and satisfies

$$\{\gamma^i, \gamma^5\} = 0, \;\; (\gamma^5)^2 = 1. \tag{2.6.7}$$

In the Dirac representation

$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{2.6.8}$$

The anticommutation relation (1.7.1) gives

$$\gamma^i\gamma_i = 4, \tag{2.6.9}$$
$$\gamma^i\gamma^j\gamma_i = -2\gamma^j, \tag{2.6.10}$$
$$\gamma^i\gamma^j\gamma^k\gamma_i = 4\eta^{jk}I, \tag{2.6.11}$$
$$\gamma^i\gamma^j\gamma^k\gamma^l\gamma_i = -2\gamma^l\gamma^k\gamma^j, \tag{2.6.12}$$
$$\gamma^i\gamma^j\gamma^k = \eta^{ij}\gamma^k + \eta^{jk}\gamma^i - \eta^{ik}\gamma^j + i\epsilon^{ijkl}\gamma_l\gamma^5. \tag{2.6.13}$$

The Dirac representation is not unique; the relation (1.7.1) is invariant under a similarity transformation $\gamma^i \to S\gamma^i S^{-1}$, where S is a nondegenerate ($\det S \neq 0$) matrix. Accordingly, $\psi \to S\psi$ and $\bar{\psi} \to \bar{\psi}S^{-1}$. Taking $S = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}$ turns the Dirac repreentation into the *Weyl representation*, in which

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2.6.14}$$

For an infinitesimal Lorentz transformation (1.6.7), the relations (1.7.5) and (1.7.6) give $L = I + \frac{1}{8}\epsilon_{ab}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, so

$$L^\dagger = I + \frac{1}{8}\epsilon_{ab}(\gamma^{b\dagger}\gamma^{a\dagger} - \gamma^{a\dagger}\gamma^{b\dagger}) \tag{2.6.15}$$

is equal to L^{-1} (so L is unitary) for rotations and equal to L for boosts. The relation (2.6.5) gives then

$$L^\dagger\gamma^0 = \gamma^0 + \frac{1}{8}\epsilon_{ab}(\gamma^{b\dagger}\gamma^{a\dagger} - \gamma^{a\dagger}\gamma^{b\dagger})\gamma^0 = \gamma^0 - \frac{1}{8}\epsilon_{ab}\gamma^0(\gamma^a\gamma^b - \gamma^b\gamma^a) = \gamma^0 L^{-1}. \tag{2.6.16}$$

Thus the quantity $\psi^\dagger\gamma^0$ transforms under (1.7.7) like an adjoint spinor:

$$\psi^\dagger\gamma^0 \to \psi^\dagger L^\dagger\gamma^0 = \psi^\dagger\gamma^0 L^{-1}. \tag{2.6.17}$$

The spinors ψ and $\psi^\dagger\gamma^0$ can be used to construct tensors, as in (1.7.11): $\psi^\dagger\gamma^0\psi$ transforms like a scalar, $\psi^\dagger\gamma^0\gamma^i\psi$ is a vector, $\psi^\dagger\gamma^0\gamma^5\psi$ is a pseudoscalar, $\psi^\dagger\gamma^0\gamma^i\gamma^5\psi$ is a pseudovector, and $\psi^\dagger\gamma^0\gamma^{[i}\gamma^{j]}\psi$ is an antisymmetric tensor. Higher-rank tensors constructed from ψ and $\psi^\dagger\gamma^0$ reduce to the above 5 kinds of tensors because of (2.6.13). Hereinafter, we will use $\bar{\psi}$ to denote $\psi^\dagger\gamma^0$.

Define the *chirality projection operators*

$$P_\pm = \frac{I \pm \gamma^5}{2}, \quad P_+ + P_- = I, \quad P_\pm^2 = I, \quad P_+P_- = P_-P_+ = 0. \tag{2.6.18}$$

They project a spinor ψ into the *right-handed* spinor ψ_R and *left-handed* spinor ψ_L,

$$\psi_R = P_+\psi, \quad \psi_L = P_-\psi, \quad \psi = \psi_R + \psi_L. \tag{2.6.19}$$

2.6.2 Dirac equation

A Lagrangian density for dynamical spinor fields must contain first derivatives of spinors. The simplest scalar containing derivatives of spinors is quadratic in ψ, $\bar{\psi}\gamma^i\psi_{;i}$, where $\psi_{;i}$ is the covariant derivative of ψ (1.7.14). This quantity is complex. In the locally inertial frame of reference, its complex conjugate is

$$(\bar{\psi}\gamma^i\psi_{,i})^* = (\bar{\psi}\gamma^i\psi_{,i})^\dagger = \psi^\dagger_{,i}\gamma^{i\dagger}\bar{\psi}^\dagger = \bar{\psi}_{,i}\gamma^0\gamma^{i\dagger}\gamma^0\psi = \bar{\psi}_{,i}\gamma^i\psi, \tag{2.6.20}$$

so both $\bar{\psi}\gamma^i\psi_{,i} + \bar{\psi}_{,i}\gamma^i\psi$ and $i(\bar{\psi}\gamma^i\psi_{,i} - \bar{\psi}_{,i}\gamma^i\psi)$ are real. The former is, however, equal to a total divergence $(\bar{\psi}\gamma^i\psi)_{,i}$, so a Lagrangian density proportional to such term does not contribute to field equations. Thus the simplest dynamical part of a spinor Lagrangian density is proportional to $i(\bar{\psi}\gamma^i\psi_{,i} - \bar{\psi}_{,i}\gamma^i\psi)$. Another scalar that can be used in a spinor Lagrangian is proportional to $\bar{\psi}\psi$. Therefore the simplest Lagrangian density for spinor fields, in the locally Galilean frame of reference, has the form

$$\mathfrak{L}_\psi = \frac{i}{2}(\bar{\psi}\gamma^i\psi_{,i} - \bar{\psi}_{,i}\gamma^i\psi) - m\bar{\psi}\psi, \tag{2.6.21}$$

where m is a real scalar constant called the *spinor mass*, and it is referred to as the *Dirac Lagrangian density*. For any frame of reference,

$$\mathfrak{L}_\psi = \frac{i\mathfrak{e}}{2}(\bar{\psi}\gamma^i\psi_{;i} - \bar{\psi}_{;i}\gamma^i\psi) - m\mathfrak{e}\bar{\psi}\psi = \frac{i\mathfrak{e}}{2}e^i_a(\bar{\psi}\gamma^a\psi_{;i} - \bar{\psi}_{;i}\gamma^a\psi) - m\mathfrak{e}\bar{\psi}\psi. \tag{2.6.22}$$

Consider the metric formulation of gravity with the Einstein-Hilbert action (2.5.1). Therefore spacetime has the Riemannian geometry, so $\psi_{;i} = \psi_{:i}$. Varying (2.6.22) with respect to $\bar{\psi}$ and omitting total derivatives gives

$$\delta\mathfrak{L}_\psi = \delta\bar{\psi}(i\gamma^i\psi_{:i} - m\psi), \tag{2.6.23}$$

so the stationarity of the action $\delta S = 0$ under $\delta\bar{\psi}$ gives the *Dirac equation*:

$$i\gamma^i\psi_{:i} = m\psi. \tag{2.6.24}$$

Varying (2.6.22) with respect to ψ and omitting total derivatives gives the adjoint conjugate of (2.6.24):

$$-i\bar{\psi}_{:i}\gamma^i = m\bar{\psi}. \tag{2.6.25}$$

The Dirac equation is linear in ψ, so ψ can be multiplied by an arbitrary constant without altering (2.6.24). Varying (2.6.22) with respect to e^i_a gives the tetrad energy-momentum density for the spinor field,

$$\mathfrak{T}_i^{\ a} = \frac{i\mathfrak{e}}{2}(\bar{\psi}\gamma^a\psi_{:i} - \bar{\psi}_{:i}\gamma^a\psi - e^a_i\bar{\psi}\gamma^j\psi_{:j} + e^a_i\bar{\psi}_{:j}\gamma^j\psi) + m\mathfrak{e}e^a_i\bar{\psi}\psi, \tag{2.6.26}$$

so

$$T_{ik} = \frac{i}{2}(\bar{\psi}\gamma_{(k}\psi_{:i)} - \bar{\psi}_{:(i}\gamma_{k)}\psi - g_{ik}\bar{\psi}\gamma^j\psi_{:j} + g_{ik}\bar{\psi}_{:j}\gamma^j\psi) + mg_{ik}\bar{\psi}\psi. \tag{2.6.27}$$

The conservation law (2.4.24) applied to the energy-momentum tensor (2.6.27) gives the Dirac equations (2.6.24) and (2.6.25).

Subtracting (2.6.25) multiplied by ψ from (2.6.24) multiplied by $\bar{\psi}$ gives, using (1.7.33) and $\psi_{|i} = \psi_{:i}$,

$$(\bar{\psi}\gamma^i\psi)_{|i} = (\bar{\psi}\gamma^i\psi)_{:i} = 0, \tag{2.6.28}$$

so the vector density

$$\mathsf{j}^i_V = \mathfrak{e}\bar{\psi}\gamma^i\psi, \tag{2.6.29}$$

called the *vector Dirac current*, is conserved: $\mathsf{j}^i_{V,i} = 0$ or

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathsf{j} = 0, \tag{2.6.30}$$

where

$$c\rho = \mathfrak{e}\psi^\dagger\psi, \ \ \mathsf{j} = \mathfrak{e}\bar{\psi}\boldsymbol{\gamma}\psi. \tag{2.6.31}$$

The spinor density ρ is real and positive. The conservation law (2.6.30) is referred to as the equation of continuity, like (2.5.96).

The Dirac equation (2.6.24) gives

$$-\gamma^j(\gamma^i\psi_{:i})_{|j} = im\gamma^j\psi_{:j} \tag{2.6.32}$$

or, due to (1.7.33),

$$-\gamma^j\gamma^i\psi_{|ij} = m^2\psi. \tag{2.6.33}$$

Using (1.7.30) and (1.7.39) turns (2.6.33) into the *Klein-Gordon-Fock equation*:

$$\psi_{|i}{}^{i} + m^2\psi = \frac{1}{8}R_{klij}\gamma^k\gamma^l\gamma^i\gamma^j\psi. \tag{2.6.34}$$

If a spinor is equal to its either left- or right-handed projection, $\psi = \psi_L$ or $\psi = \psi_R$, then it is called a *Weyl spinor*. Multiplying (2.6.24) by $P_\pm$ gives

$$iP_\pm\gamma^i\psi_{:i} = i\gamma^i P_\mp\psi_{:i} = mP_\pm\psi \tag{2.6.35}$$

or

$$i\gamma^i\psi^{L(R)}_{:i} = m\psi^{R(L)}. \tag{2.6.36}$$

Thus if ψ is a Weyl spinor then $m = 0$.

2.6.3 Spinors in Einstein-Cartan gravity

Consider the metric-affine formulation of gravity with the Einstein-Cartan action (2.5.47), in which spacetime has the Riemann-Cartan geometry. Varying (2.6.22) with respect to e^i_a gives the tetrad energy-momentum density for the spinor field,

$$\mathfrak{T}_i{}^a = \frac{i\mathfrak{e}}{2}(\bar{\psi}\gamma^a\psi_{;i} - \bar{\psi}_{;i}\gamma^a\psi - e^a_i\bar{\psi}\gamma^j\psi_{;j} + e^a_i\bar{\psi}_{;j}\gamma^j\psi) + m\mathfrak{e}e^a_i\bar{\psi}\psi. \tag{2.6.37}$$

Putting the definition of the covariant derivative of a spinor (1.7.14) into (2.6.22) gives

$$\mathfrak{L}_\psi = \frac{i\mathfrak{e}}{2}(\bar{\psi}\gamma^i\psi_{,i} - \bar{\psi}_{,i}\gamma^i\psi) - \frac{i\mathfrak{e}}{2}\bar{\psi}\{\gamma^i, \Gamma_i\}\psi - m\mathfrak{e}\bar{\psi}\psi. \tag{2.6.38}$$

Using the Fock-Ivanenko coefficients (1.7.28) as the spinor connection Γ_i turns (2.6.38) into

$$\mathfrak{L}_\psi = \frac{i\mathfrak{e}}{2}(\bar{\psi}\gamma^i\psi_{,i} - \bar{\psi}_{,i}\gamma^i\psi) + \frac{i\mathfrak{e}}{8}\omega_{abi}\bar{\psi}\{\gamma^i, \gamma^a\gamma^b\}\psi - m\mathfrak{e}\bar{\psi}\psi. \tag{2.6.39}$$

The spin density (2.3.17) corresponding to the Lagrangian density (2.6.39) is, due to the identity $\{\gamma^i, \gamma^j\gamma^k\} = 2\gamma^{[i}\gamma^j\gamma^{k]}$,

$$\mathfrak{S}^{ijk} = \frac{i\mathfrak{e}}{2}\bar{\psi}\gamma^{[i}\gamma^j\gamma^{k]}\psi \tag{2.6.40}$$

or

$$s^{ijk} = \frac{i}{2}\bar{\psi}\gamma^{[i}\gamma^{j}\gamma^{k]}\psi. \tag{2.6.41}$$

The spin density (2.6.40) is independent of m and totally antisymmetric,

$$\mathfrak{s}^{ijk} = \mathfrak{s}^{[ijk]}. \tag{2.6.42}$$

The second Einstein-Cartan equation (2.5.57) for the spin tensor (2.6.41) gives a totally antisymmetric torsion tensor,

$$S_{ijk} = -\frac{i\kappa}{4}\bar{\psi}\gamma_{[i}\gamma_{j}\gamma_{k]}\psi, \tag{2.6.43}$$

so $S_i = 0$. Thus the contortion tensor is, using (2.6.13),

$$C_{ijk} = \frac{\kappa}{4}\epsilon_{ijkl}\bar{\psi}\gamma^l\gamma^5\psi. \tag{2.6.44}$$

The pseudovector density

$$\mathfrak{j}^i_A = \mathfrak{e}\bar{\psi}\gamma^i\gamma^5\psi \tag{2.6.45}$$

is called the *axial Dirac current*.

Varying (2.6.39) with respect to $\bar{\psi}$ gives, after omitting total divergences,

$$\frac{i}{2}(\mathfrak{e}\gamma^k\psi_{,k} + (\mathfrak{e}\gamma^k\psi)_{,k} - \mathfrak{e}\{\Gamma_k, \gamma^k\}\psi) - \mathfrak{e}m\psi = 0. \tag{2.6.46}$$

Substituting

$$(\mathfrak{e}\gamma^k\psi)_{,k} = \mathfrak{e}\gamma^k\psi_{,k} + \mathfrak{e}\gamma^k{}_{;k}\psi - 2\mathfrak{e}S_k\gamma^k\psi = \mathfrak{e}\gamma^k\psi_{,k} + \mathfrak{e}[\Gamma_k, \gamma^k]\psi \tag{2.6.47}$$

into (2.6.46) gives

$$i\gamma^k\psi_{,k} - i\gamma^k\Gamma_k\psi - m\psi = i\gamma^k\psi_{;k} - m\psi = 0. \tag{2.6.48}$$

The relation (1.5.33) gives

$$\psi_{;k} = \psi_{:k} + \frac{1}{4}C_{ijk}\gamma^i\gamma^j\psi, \tag{2.6.49}$$

from which we obtain, upon substituting (2.6.44),

$$\begin{aligned}\gamma^k\psi_{;k} &= \gamma^k\psi_{:k} + \frac{\kappa}{16}\epsilon_{ijkl}(\bar{\psi}\gamma^l\gamma^5\psi)\gamma^i\gamma^j\gamma^k\psi = \gamma^k\psi_{:k} + \frac{i\kappa}{16}\epsilon_{ijkl}(\bar{\psi}\gamma^l\gamma^5\psi)\epsilon^{ijkm}\gamma_m\gamma^5\psi \\ &= \gamma^k\psi_{:k} + \frac{3i\kappa}{8}(\bar{\psi}\gamma^l\gamma^5\psi)\gamma_l\gamma^5\psi. \end{aligned} \tag{2.6.50}$$

Therefore (2.6.48) becomes the *Heisenberg-Ivanenko* equation:

$$i\gamma^k\psi_{:k} - \frac{3\kappa}{8}(\bar{\psi}\gamma_k\gamma^5\psi)\gamma^k\gamma^5\psi = m\psi. \tag{2.6.51}$$

Varying (2.6.39) with respect to ψ gives the adjoint conjugate of (2.6.51),

$$-i\bar{\psi}_{:k}\gamma^k - \frac{3\kappa}{8}(\bar{\psi}\gamma_k\gamma^5\psi)\bar{\psi}\gamma^k\gamma^5 = m\bar{\psi}. \tag{2.6.52}$$

The Heisenberg-Ivanenko equation (2.6.51) differs from the Dirac equation (2.6.24) by a nonlinear term, cubic in the spinor field and representing a spinor self-interaction, corresponding to a spin-spin interaction in the tensor (2.5.58). The conservation law (2.4.32) applied to the energy-momentum density (2.6.37) gives the Heisenberg-Ivanenko equations (2.6.51) and (2.6.52). Subtracting (2.6.52) multiplied by ψ from (2.6.51) multiplied by $\bar{\psi}$ gives the conservation of the vector Dirac current (2.6.29).

The total antisymmetry of the spin density implies

$$N^{ijk} = N^{[ijk]}, \tag{2.6.53}$$

where N^{ijk} is given by (2.4.87). Also

$$N^{ijk} = 3S^{[ij}u^{k]}, \tag{2.6.54}$$

where S^{ij} is the intrinsic spin tensor (2.4.156). The covariant (with respect to the Levi-Civita connection) change (2.4.129) of the spin pseudovector along a world line becomes

$$\frac{D^{\{\}}J^i}{ds} = -u^i\frac{D^{\{\}}u^k}{ds}J_k + \frac{D^{\{\}}u^i}{ds}u^kJ_k + \frac{3}{2}e^{ij}{}_{nm}u_jS^n{}_{ik}N^{ikm} = 3S^i{}_{jk}u^jN^k, \tag{2.6.55}$$

where

$$N^i = \frac{1}{6}e^{ijkl}N_{jkl}. \tag{2.6.56}$$

If $N^i \propto J^i$ then (2.6.55) gives $J^iJ_i = \text{const}$. For a point particle, M^{ijk} given by (2.4.86) vanishes. Thus (2.4.112) reduces to

$$N^{ijl} = \frac{u^l}{u^0}N^{ij0}, \tag{2.6.57}$$

which for a spinor particle gives $N^{il0} = -\frac{u^l}{u^0}N^{i00}$ and thus $N^{ijk} = 0$ or

$$\psi = 0. \tag{2.6.58}$$

Therefore a spinor field in the Einstein-Cartan gravity cannot be approximated as a point particle.
References: [3, 4, 5, 6, 7].

2.7 Electromagnetic field

2.7.1 Gauge invariance and electromagnetic potential

The Lagrangian density (2.6.22) is a real combination of the complex Dirac matrices γ^i and spinors ψ, $\bar{\psi}$. It is invariant under a *gauge transformation of the first type* of the spinor fields,

$$\psi \to \psi' = e^{ie\alpha}\psi, \ \ \bar{\psi} \to \bar{\psi}' = e^{-ie\alpha}\bar{\psi}, \tag{2.7.1}$$

if $e\alpha$ is a real scalar constant, but it is not invariant for $e\alpha(x^i)$, because

$$\psi'_{;\mu} = e^{ie\alpha}(\psi_{;\mu} + ie\alpha_{,\mu}\psi). \tag{2.7.2}$$

Introduce a compensating vector field A_μ, called the *electromagnetic potential*, such that the Weyl or *electromagnetic covariant derivative*

$$D_\mu = \nabla_\mu - ieA_\mu \tag{2.7.3}$$

of a spinor ψ,

$$D_\mu\psi = \psi_{;\mu} - ieA_\mu, \tag{2.7.4}$$

transforms under (2.7.1) like ψ:

$$D_\mu\psi' = e^{ie\alpha}D_\mu\psi. \tag{2.7.5}$$

This requirement gives

$$\psi'_{;\mu} - ieA'_\mu\psi' = e^{ie\alpha}(\psi_{;\mu} - ieA_\mu\psi), \tag{2.7.6}$$

which, with (2.7.1) and (2.7.2), yields the transformation law for the electromagnetic potential,

$$A'_\mu = A_\mu + \alpha_{,\mu}, \tag{2.7.7}$$

called a *gauge transformation of the second type*. The real scalar constant e is called the *spinor electric charge*. The adjoint conjugation of (2.7.4) is

$$D_\mu\bar{\psi} = \bar{\psi}_{;\mu} + ieA^*_\mu. \tag{2.7.8}$$

The scalar $\bar{\psi}\psi$ is invariant under (2.7.1), so

$$D_\mu(\bar{\psi}\psi) = \partial_\mu(\bar{\psi}\psi), \tag{2.7.9}$$

which constraints the electromagnetic potential to be real:

$$A^*_\mu = A_\mu. \tag{2.7.10}$$

The time component of A^μ, $\phi = A^0$, is called the *electric potential* and the spatial components A^α form the *magnetic potential* $\mathbf{A}$:

$$A^\mu = (\phi, \mathbf{A}). \tag{2.7.11}$$

The gauge transformation (2.7.7) reads

$$\phi' = \phi + \frac{\partial\alpha}{c\partial t}, \quad \mathbf{A}' = \mathbf{A} - \boldsymbol{\nabla}\alpha. \tag{2.7.12}$$

In the local Minkowski spacetime, A^μ transforms according to (1.6.98),

$$\begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta} \\ \gamma\boldsymbol{\beta} & 1 + \frac{(\gamma-1)\boldsymbol{\beta}}{\beta^2}\boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \phi' \\ \mathbf{A}' \end{pmatrix}. \tag{2.7.13}$$

The gauge-invariant modification of the Dirac Lagrangian density (2.6.22) is

$$\mathfrak{L}_\psi = \frac{i\mathfrak{e}}{2}e^i_a(\bar{\psi}\gamma^a D_i\psi - D_i\bar{\psi}\gamma^a\psi) - m\mathfrak{e}\bar{\psi}\psi. \tag{2.7.14}$$

The electromagnetic potential corresponds, up to the multiplication by an arbitrary constant, to the vector multiple of I in the formula for the spinor connection (1.7.26). The electromagnetic potential is analogous to the affine connection: it modifies a derivative of a spinor so such derivative transforms like a spinor under unitary gauge transformations of the first type, while the connection modifies a derivative of a tensor so such derivative transforms like a tensor under coordinate transformations.

2.7.2 Electromagnetic field tensor

The commutator of total covariant derivatives of a spinor is given by (1.7.30) with the curvature spinor K_{ij} given by (1.7.36), where the tensor B_{ij} is related to the vector A_i in (1.7.26) by (1.7.38). Therefore the commutator of the electromagnetic covariant derivatives of a spinor, $[D_i, D_j]\psi$, is given by (1.7.30) with the curvature spinor

$$K_{ij} = \frac{1}{4}R_{klij}\gamma^k\gamma^l + ieF_{ij}I, \tag{2.7.15}$$

where the antisymmetric tensor

$$F_{ij} = A_{j,i} - A_{i,j} = A_{j:i} - A_{i:j} \tag{2.7.16}$$

is referred to as the *electromagnetic field tensor*. The electromagnetic field tensor is analogous to the curvature tensor: it appears in the expression for the commutator of electromagnetic covariant derivatives of a spinor, while the curvature tensor appears in the expression for the commutator of coordinate-covariant derivatives of a tensor. Substituting (2.7.7) into (2.7.16) gives

$$F'_{ij} = F_{ij}, \tag{2.7.17}$$

so the electromagnetic field tensor is gauge invariant. The definition (2.7.16) is equivalent to the *first Maxwell-Minkowski equation*

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = F_{ij:k} + F_{jk:i} + F_{ki:j} = 0 \tag{2.7.18}$$

or

$$\epsilon^{ijkl}F_{jk,l} = \epsilon^{ijkl}F_{jk:l} = 0. \tag{2.7.19}$$

Define

$$E_\alpha = F_{0\alpha}, \tag{2.7.20}$$

$$B_{\alpha\beta} = F_{\alpha\beta},\ \ B^\alpha = -\frac{1}{2\sqrt{\mathfrak{f}}}\epsilon^{\alpha\beta\gamma}B_{\beta\gamma},\ \ B_{\alpha\beta} = -\sqrt{\mathfrak{f}}\epsilon_{\alpha\beta\gamma}B^\gamma, \tag{2.7.21}$$

where $\mathfrak{f}$ is given by (1.4.105). The component of (2.7.18) with all spatial indices, $B_{\alpha\beta,\gamma} + B_{\beta\gamma,\alpha} + B_{\gamma\alpha,\beta} = 0$, gives, using (1.4.116),

$$\mathrm{div}\,\mathbf{B} = 0. \tag{2.7.22}$$

The components of (2.7.18) with one temporal index, $B_{\alpha\beta,0} + E_{alpha,\beta} - E_{\beta,\alpha} = 0$, gives, using (1.4.117),

$$\text{curl}\,\mathbf{E} = -\frac{1}{c\sqrt{\mathfrak{f}}}\frac{\partial(\sqrt{\mathfrak{f}}\mathbf{B})}{\partial t}. \tag{2.7.23}$$

The spatial vector $\mathbf{E}$ is called the *electric field* and the spatial pseudovector $\mathbf{B}$ is the *magnetic field*.

In the locally geodesic and Galilean frame of reference, these fields depend on the components of the electromagnetic potential (2.7.11) according to (2.7.16):

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{c\partial t} - \boldsymbol{\nabla}\phi, \tag{2.7.24}$$

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \tag{2.7.25}$$

and they are invariant under (2.7.12). The tensor F_{ij} is given by

$$F_{ij} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{2.7.26}$$

and transforms according to (1.6.99). Thus the electric and magnetic fields transform according to

$$\mathbf{E}' = \gamma(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}) + \frac{1-\gamma}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{E})\boldsymbol{\beta}, \tag{2.7.27}$$

$$\mathbf{B}' = \gamma(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}) + \frac{1-\gamma}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta}. \tag{2.7.28}$$

In this frame, (2.7.22) and (2.7.23) become the *first pair of the Maxwell equations*:

$$\text{div}\,\mathbf{B} = 0, \tag{2.7.29}$$

$$\text{curl}\,\mathbf{E} = -\frac{\partial\mathbf{B}}{c\partial t}. \tag{2.7.30}$$

Applying the div operator to (2.7.25) gives (2.7.29) and applying the curl operator to (2.7.24) gives (2.7.30). Applying the div operator to (2.7.30) gives (2.7.29). Integrating the first pair of the Maxwell equations over the volume and surface area, respectively, gives

$$\oint\mathbf{B}\cdot d\mathbf{f} = 0, \tag{2.7.31}$$

$$\oint\mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{c\partial t}\Big(\int\mathbf{B}\cdot d\mathbf{f}\Big). \tag{2.7.32}$$

The integral $\oint\mathbf{A}\cdot d\mathbf{f}$ is the flux of a vector $\mathbf{A}$ through the surface $\mathbf{f}$ and the integral $\oint\mathbf{A}\cdot d\mathbf{l}$ is called the *circulation* of $\mathbf{A}$ along the contour $\mathbf{l}$. Thus the flux of the magnetic field through a closed surface vanishes and the circulation of the electric field along a contour, which is called the *electromotive force*, is equal to the minus time derivative of the flux of the magnetic field through the surface enclosed by this contour (*Faraday's law*).

2.7.3 Lagrangian density for electromagnetic field

The simplest gauge-invariant Lagrangian density representing the electromagnetic field is a linear combination of terms quadratic in F_{ij}: $\sqrt{-\mathfrak{g}}F_{ij}F^{ij}$ and $\epsilon^{ijkl}F_{ij}F_{kl}$. The second term is a total divergence because of (2.7.19):

$$\epsilon^{ijkl}F_{ij}F_{kl} = 2(\epsilon^{ijkl}F_{ij}A_l)_{,k}, \tag{2.7.33}$$

so it does not contribute to the field equations. Thus the Lagrangian density for the electromagnetic field is given by

$$\mathfrak{L}_{EM} = -\frac{1}{16\pi}\sqrt{-\mathfrak{g}}F_{ij}F^{ij}, \tag{2.7.34}$$

where the Gaußian factor $\frac{1}{16\pi}$ sets the units of A_i. In the locally geodesic and Galilean frame of reference, (2.7.34) becomes

$$\mathfrak{L}_{EM} = \frac{1}{8\pi}(E^2 - B^2). \tag{2.7.35}$$

Therefore in order for the action S to have a minimum, there must be the minus sign in front of the right-hand side of (2.7.34). Otherwise an arbitrarily rapid change of $\mathbf{A}$ in time would result in an arbitrarily large value of $\mathbf{E}$, according to (2.7.24), and thus an arbitrarily low value of S, so the action would have no minimum. A generalization of the tensor (2.7.16) to a covariant derivative with respect to the affine connection $\Gamma^{\,k}_{i\,j}$, $A_{j;i} - A_{i;j} = F_{ij} + 2S^k_{\ ij}A_k$, is not gauge invariant, so the torsion tensor cannot appear in a gauge-invariant Lagrangian density which is quadratic in F_{ij}. Thus the electromagnetic field, unlike spinor fields, does not couple to torsion.

2.7.4 Electromagnetic current

Define the *electromagnetic current density*

$$\mathfrak{j}^i = -\frac{c\delta\mathfrak{L}_m}{\delta A_i}, \tag{2.7.36}$$

and the *electromagnetic current vector*

$$j^i = \frac{\mathfrak{j}^i}{\sqrt{-\mathfrak{g}}}. \tag{2.7.37}$$

The invariance of the action under an arbitrary infinitesimal gauge transformation $\delta A_i = A'_i - A_i = \phi_{,i}$ gives, upon partial integration and omitting a total divergence,

$$\delta S = -\frac{1}{c^2}\int \mathfrak{j}^i\delta A_j d\Omega = -\frac{1}{c^2}\int \mathfrak{j}^i\phi_{,i}d\Omega = \frac{1}{c^2}\int \mathfrak{j}^i_{\ ,i}\phi d\Omega = 0, \tag{2.7.38}$$

so the electromagnetic current is conserved,

$$\mathfrak{j}^i_{\ ,i} = 0, \ \ j^i_{\ :i} = 0. \tag{2.7.39}$$

The gauge-invariant Lagrangian density (2.6.22) for spinor matter is

$$\mathfrak{L}_\psi = \frac{i\mathfrak{e}}{2}e^i_a(\bar{\psi}\gamma^a\nabla_i\psi - \nabla_i\bar{\psi}\gamma^a\psi) - m\mathfrak{e}\bar{\psi}\psi = \frac{i\mathfrak{e}}{2}e^i_a(\bar{\psi}\gamma^a\psi_{;i} - \bar{\psi}_{;i}\gamma^a\psi) - m\mathfrak{e}\bar{\psi}\psi - eA_i\mathfrak{e}\bar{\psi}\gamma^i\psi, \tag{2.7.40}$$

so the electromagnetic current for the spinor field is

$$\mathsf{j}^i = ec\mathfrak{e}\bar{\psi}\gamma^i\psi, \tag{2.7.41}$$

which is proportional to the conserved vector Dirac current (2.6.29). The electromagnetic current density (2.7.36) corresponds to the current (2.4.7) with ξ^i=0 and $\bar{\delta}\phi = ie\alpha\phi$ (which is equal to the infinitesimal $\phi' - \phi$ due to (2.7.1)).

Consider matter which is distributed over a small region in space, as in section (2.4.7). Integrating (2.7.39) over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals gives

$$\int \mathsf{j}^0{}_{,0}dV = 0. \tag{2.7.42}$$

The conservation law (2.7.39) also gives

$$(x^k\mathsf{j}^i)_{,i} = x^k{}_{,i}\mathsf{j}^i + x^k\mathsf{j}^i{}_{,i} = \delta^k_i\mathsf{j}^i = \mathsf{j}^k, \tag{2.7.43}$$

which, upon integrating over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals, gives

$$\left(\int x^k\mathsf{j}^0dV\right)_{,0} = \int \mathsf{j}^k dV. \tag{2.7.44}$$

Using (2.4.84) turns (2.7.44) into

$$\frac{u^k}{u^0}\int \mathsf{j}^0dV + \left(\int \delta x^k\mathsf{j}^0dV\right)_{,0} = \int \mathsf{j}^k dV. \tag{2.7.45}$$

For a particle located at $\mathbf{x}_a$, $\int \delta x^k\mathsf{j}^0dV = 0$ and $\mathsf{j}^i(\mathbf{x})$ is thus proportional to $\boldsymbol{\delta}(\mathbf{x}-\mathbf{x}_a)$, so

$$\mathsf{j}^k = \frac{u^k}{u^0}\mathsf{j}^0. \tag{2.7.46}$$

Define the *electric charge density* ρ such that

$$j^0 = \frac{c\rho}{\sqrt{g_{00}}}. \tag{2.7.47}$$

The electric charge density is not a tensor density. Define the *electric charge* e such that

$$\rho\sqrt{\mathfrak{f}}dV = de. \tag{2.7.48}$$

The electric charge density for particles with charges e_a located at $\mathbf{x}_a$ is

$$\rho(\mathbf{x}) = \sum_a \frac{e_a}{\sqrt{\mathfrak{f}}}\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_a), \tag{2.7.49}$$

and $\int \mathrm{j}^0 dV$ (which is equal to $\int \mathrm{j}^i dS_i$ for a volume hypersurface, so it is a scalar) is

$$\int \mathrm{j}^0 dV = \sum_a \int \sqrt{-\mathfrak{g}} \frac{ce_a}{\sqrt{g_{00}}\sqrt{\mathfrak{l}}} \boldsymbol{\delta}(\mathbf{x}-\mathbf{x}_a) dV = c\sum_a e_a, \tag{2.7.50}$$

so the electric charge is a scalar. Thus the electromagnetic current vector for a system of charged particles is

$$j^k(\mathbf{x}) = \sum_a \frac{cu^k}{u^0} \frac{e_a}{\sqrt{-\mathfrak{g}}} \boldsymbol{\delta}(\mathbf{x}-\mathbf{x}_a), \tag{2.7.51}$$

analogously to (2.4.168). The relation (2.7.42) represents the conservation of the total electric charge of a physical system.

In the locally geodesic and Galilean frame of reference, $\frac{u^i}{u^0} = (1, \mathbf{v}/c)$, so

$$j^i = (c\rho, \mathbf{j}), \tag{2.7.52}$$

where $\mathbf{j}$ is the *spatial current vector*,

$$\mathbf{j} = \rho \mathbf{v}. \tag{2.7.53}$$

The conservation law (2.7.39) in this frame, $j^i_{\ ,i} = 0$, has the form of the equation of continuity (2.6.30). For one particle located at $\mathbf{x}_0(t)$, $\rho(\mathbf{x}) = e\boldsymbol{\delta}(\mathbf{x}-\mathbf{x}_0)$, (2.6.30) is explicitly satisfied since

$$\begin{aligned} \frac{\partial\rho}{\partial t} &= e\frac{\partial}{\partial t}\boldsymbol{\delta}(\mathbf{x}-\mathbf{x}_0) = e\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}_0}\boldsymbol{\delta}(\mathbf{x}-\mathbf{x}_0) = -e\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}\boldsymbol{\delta}(\mathbf{x}-\mathbf{x}_0) \\ &= -\frac{\partial}{\partial\mathbf{x}}\cdot\Big(e\mathbf{v}\boldsymbol{\delta}(\mathbf{x}-\mathbf{x}_0)\Big) = -\boldsymbol{\nabla}\cdot\mathrm{j}, \end{aligned} \tag{2.7.54}$$

where $\mathbf{v} = \frac{d\mathbf{x}_0}{dt}$. For a system of charged particles, we also have

$$\int \mathbf{j} dV = \sum_a e_a \mathbf{v}_a. \tag{2.7.55}$$

The equation of continuity (2.6.30) represents, upon integrating over the volume, the conservation of the total electric charge:

$$\frac{\partial}{\partial t}\left(\int \rho dV\right) + \oint \mathbf{j}\cdot d\mathbf{f} = 0. \tag{2.7.56}$$

2.7.5 Maxwell equations

The total Lagrangian density for the electromagnetic field and matter is the sum of (2.7.34) and the term $-\sqrt{-\mathfrak{g}}A_i j^i$ due to (2.7.36):

$$\mathfrak{L}_{EM} = -\frac{1}{16\pi}\sqrt{-\mathfrak{g}}F_{ik}F^{ik} - \frac{1}{c}\sqrt{-\mathfrak{g}}A_k j^k, \tag{2.7.57}$$

where we omit the terms corresponding to the gravitational field and matter which does not depend on A_k. Varying (2.7.57) with respect to A_k, integrating partially and omitting total divergences gives

$$\delta\mathfrak{L}_{EM} = -\frac{1}{8\pi}\sqrt{-\mathfrak{g}}F^{ik}\delta F_{ik} - \frac{\mathsf{j}^k}{c}\delta A_k = -\frac{1}{8\pi}\sqrt{-\mathfrak{g}}F^{ik}(\delta A_{k,i} - \delta A_{i,k}) - \frac{\mathsf{j}^k}{c}\delta A_k$$
$$= \frac{1}{4\pi}\sqrt{-\mathfrak{g}}F^{ik}\delta A_{k,i} - \frac{\mathsf{j}^k}{c}\delta A_k = \frac{1}{4\pi}(\sqrt{-\mathfrak{g}}F^{ik})_{,i}\delta A_k - \frac{1}{c}\sqrt{-\mathfrak{g}}j^k\delta A_k, \qquad (2.7.58)$$

so the principle of least action $\delta S = 0$ for arbitrary variations δA_k yields the *second Maxwell-Minkowski equation*

$$(\sqrt{-\mathfrak{g}}F^{ik})_{,i} = \frac{4\pi}{c}\mathsf{j}^k \qquad (2.7.59)$$

or

$$F^{ik}{}_{:i} = \frac{4\pi}{c}j^k. \qquad (2.7.60)$$

The electromagnetic field equation (2.7.59) implies that j^i is conserved, $\mathsf{j}^i{}_{,i} = 0$, which corresponds to the conservation of the total electric charge, but does not constrain the motion of particles. Therefore a configuration of charged particles producing the electromagnetic field can be arbitrary, subject only to the condition that the total charge be conserved, unlike a configuration of particles producing the gravitational field which is not arbitrary but constrained by the gravitational field equations.

Define

$$D^\alpha = -\sqrt{g_{00}}F^{0\alpha}, \qquad (2.7.61)$$
$$H^{\alpha\beta} = \sqrt{g_{00}}F^{\alpha\beta},\ \ H_\alpha = -\frac{1}{2}\sqrt{\mathfrak{f}}\epsilon_{\alpha\beta\gamma}H^{\beta\gamma},\ \ H^{\alpha\beta} = -\frac{1}{\sqrt{\mathfrak{f}}}\epsilon^{\alpha\beta\gamma}H_\gamma. \qquad (2.7.62)$$

The relations $F_{0\alpha} = g_{0i}g_{\alpha j}F^{ij}$ and $F^{\alpha\beta} = g^{\alpha i}g^{\beta j}F_{ij}$ give then

$$D_\alpha = \frac{E_\alpha}{\sqrt{g_{00}}} + g^\beta H_{\alpha\beta}, \qquad (2.7.63)$$
$$B^{\alpha\beta} = \frac{H^{\alpha\beta}}{\sqrt{g_{00}}} - g^\alpha E^\beta + g^\beta E^\alpha, \qquad (2.7.64)$$

or, in the spatial-vector notation,

$$\mathbf{D} = \frac{\mathbf{E}}{\sqrt{g_{00}}} - \mathbf{g}\times\mathbf{H}, \qquad (2.7.65)$$
$$\mathbf{B} = \frac{\mathbf{H}}{\sqrt{g_{00}}} + \mathbf{g}\times\mathbf{E}. \qquad (2.7.66)$$

Using (1.4.102) brings the temporal component of (2.7.59) to

$$\frac{1}{\sqrt{\mathfrak{f}}}(\sqrt{\mathfrak{f}}D^\alpha)_{,\alpha} = 4\pi\rho \qquad (2.7.67)$$

or

$$\operatorname{div}\mathbf{D} = 4\pi\rho. \tag{2.7.68}$$

The spatial components of (2.7.59) read

$$\frac{1}{\sqrt{\gamma}}(\sqrt{\gamma}H^{\alpha\beta})_{,\beta} + \frac{1}{\sqrt{\gamma}}(\sqrt{\gamma}D^{\alpha})_{,0} = -4\pi\rho\frac{dx^{\alpha}}{dx^{0}} \tag{2.7.69}$$

or

$$\operatorname{curl}\mathbf{H} = \frac{1}{c\sqrt{\gamma}}\frac{\partial(\sqrt{\gamma}\mathbf{D})}{\partial t} + \frac{4\pi}{c}\mathbf{j}. \tag{2.7.70}$$

The conservation law (2.7.39) reads

$$\frac{1}{\sqrt{\gamma}}\frac{\partial(\sqrt{\gamma}\rho)}{\partial t} + \operatorname{div}\mathbf{j} = 0. \tag{2.7.71}$$

In the locally geodesic and Galilean frame of reference, (2.7.65) and (2.7.66) reduce to

$$\mathbf{D} = \mathbf{E}, \tag{2.7.72}$$
$$\mathbf{B} = \mathbf{H}. \tag{2.7.73}$$

In this frame, (2.7.68) and (2.7.70) become the *second pair of the Maxwell equations*:

$$\operatorname{div}\mathbf{E} = 4\pi\rho,) \tag{2.7.74}$$
$$\operatorname{curl}\mathbf{B} = \frac{\partial\mathbf{E}}{c\partial t} + \frac{4\pi}{c}\mathbf{j}. \tag{2.7.75}$$

Applying the div operator to (2.7.75) and using (2.7.74) gives (2.6.30). Integrating the second pair of the Maxwell equations over the volume and surface area, respectively, gives

$$\oint \mathbf{E}\cdot d\mathbf{f} = 4\pi q, \tag{2.7.76}$$
$$\oint \mathbf{B}\cdot d\mathbf{l} = \frac{\partial}{c\partial t}\Big(\int \mathbf{E}\cdot d\mathbf{f}\Big) + \frac{4\pi}{c}\int \mathbf{j}\cdot d\mathbf{f}. \tag{2.7.77}$$

Thus the flux of the electric field through a closed surface is proportional to the total charge inside the volume enclosed by the surface $\mathbf{f}$ (*Gauß' law*) and the circulation of the magnetic field along a contour is equal to the time derivative of the flux of the electric field through the surface enclosed by this contour, called the displacement current, plus the surface integral of the current vector (*Ampère's law*).

The two pairs of the Maxwell equations are linear in the fields $\mathbf{E}$ and $\mathbf{B}$. The sum of any two solutions of the Maxwell equations is also a solution of these equations. Thus the electromagnetic field of a system of sources (particles) is the sum of the fields from each source. The additivity of the electromagnetic field is referred to as the *principle of superposition*.

2.7.6 Energy-momentum tensor for electromagnetic field

The metric energy-momentum tensor (2.3.3) for the electromagnetic field T_{ik}^{EM} is given by the Lagrangian density (2.7.34):

$$\delta\mathfrak{L}_{EM} = \frac{1}{32\pi}\sqrt{-\mathfrak{g}}g_{lm}F_{ik}F^{ik}\delta g^{lm} - \frac{1}{8\pi}\sqrt{-\mathfrak{g}}F_{ik}F_{lm}g^{il}\delta g^{km}$$
$$= \frac{1}{8\pi}\sqrt{-\mathfrak{g}}\Big(\frac{1}{4}g_{ik}F_{lm}F^{lm} - F_i^{\ j}F_{kj}\Big)\delta g^{ik}, \tag{2.7.78}$$

so

$$T_{ik}^{EM} = \frac{1}{4\pi}\Big(\frac{1}{4}g_{ik}F_{lm}F^{lm} - F_i^{\ j}F_{kj}\Big). \tag{2.7.79}$$

The corresponding energy density W, energy current $\mathbf{S}$ called the *Poynting vector*, and stress tensor $\sigma_{\alpha\beta}$ called the *Maxwell stress tensor*, are given in the locally geodesic and Galilean frame of reference, due to (2.4.71), by

$$W = \frac{1}{8\pi}(E^2 + B^2), \tag{2.7.80}$$
$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}, \tag{2.7.81}$$
$$\sigma_{\alpha\beta} = \frac{1}{4\pi}\Big(E_\alpha E_\beta + B_\alpha B_\beta - \frac{1}{2}\delta_{\alpha\beta}(E^2 + B^2)\Big). \tag{2.7.82}$$

Multiplying (2.7.30) by $\mathbf{B}$ and (2.7.75) by $\mathbf{E}$ and adding these scalar products gives

$$\frac{1}{c}\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{c}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = -\frac{4\pi}{c}\mathbf{j}\cdot\mathbf{E} - (\mathbf{B}\cdot\mathrm{curl}\,\mathbf{E} - \mathbf{E}\cdot\mathrm{curl}\,\mathbf{B}), \tag{2.7.83}$$

from which we obtain

$$\frac{1}{2c}\frac{\partial}{\partial t}(E^2 + B^2) = -\frac{4\pi}{c}\mathbf{j}\cdot\mathbf{E} - \mathrm{div}(\mathbf{E}\times\mathbf{B}) \tag{2.7.84}$$

or

$$\frac{\partial W}{\partial t} + \mathbf{j}\cdot\mathbf{E} + \mathrm{div}\,\mathbf{S} = 0. \tag{2.7.85}$$

The energy-momentum tensor for the electromagnetic field is traceless,

$$T_{ik}^{EM}g^{ik} = 0, \tag{2.7.86}$$

so (2.4.174) and the virial theorem (2.4.175) remain unchanged if the particles interact electromagnetically. The condition (2.7.86) also gives, using (2.4.82),

$$\epsilon_{EM} = 3p_{EM}, \tag{2.7.87}$$

so (2.4.176) implies that the free electromagnetic field is ultrarelativistic.

2.7.7 Lorentz force

Consider a charge particle interacting with the electromagnetic field. The total energy-momentum tensor for the particle and electromagnetic field is covariantly conserved, which gives the motion of the particle. The electromagnetic part yields, using (2.7.18) and (2.7.60),

$$T_{i\ :k}^{\ k} = \frac{1}{4\pi}\Big(\frac{1}{2}F_{lm:i}F^{lm} - F_{il:k}F^{kl} - F_{il}F^{kl}_{\ \ :k}\Big) = \frac{1}{4\pi}\Big(-\frac{1}{2}F_{mi:l}F^{lm} - \frac{1}{2}F_{il:m}F^{lm}$$
$$-F_{il:k}F^{kl} - F_{il}F^{kl}_{\ \ :k}\Big) = \frac{1}{4\pi}F_{il}F^{kl}_{\ \ :k} = -\frac{1}{c}F_{il}j^l. \tag{2.7.88}$$

The particle part gives, using (2.4.167),

$$T_{i\ :k}^{\ k} = \Big(\mu c^2\frac{u_i u^k}{\sqrt{g_{00}}u^0}\Big)_{:k}, \tag{2.7.89}$$

so we obtain

$$\Big(\mu c^2\frac{u_i u^k}{\sqrt{g_{00}}u^0}\Big)_{:k} - \frac{1}{c}F_{il}j^l = 0. \tag{2.7.90}$$

Multiplying (2.7.90) by u^i and using (2.7.46) gives

$$\Big(\mu c^2\frac{u^k}{\sqrt{g_{00}}u^0}\Big)_{:k}, \tag{2.7.91}$$

which turns (2.7.90) into

$$\mu c^2\frac{u^k}{\sqrt{g_{00}}u^0}u_{i:k} = \frac{1}{c}F_{il}\rho\frac{u^l}{\sqrt{g_{00}}u^0} \tag{2.7.92}$$

or

$$mc\frac{D^{\{\}}u^i}{ds} = \frac{e}{c}F^{ij}u^j, \tag{2.7.93}$$

which is the equation of motion of a particle of mass m and charge e in the electromagnetic field F_{ij}. Multiplying (2.7.93) by u_i gives the identity, so (2.7.93) has 3 independent components. The right-hand side of (2.7.93) is referred to as the *Lorentz force*.

In the locally geodesic and Galilean frame of reference, $\frac{D^{\{\}}}{ds} = \frac{d}{ds} = \frac{u^0}{c}\frac{d}{dt}$ and $u^i = (\gamma, \gamma\mathbf{v}/c)$, so (2.7.93) reads (we choose the spatial components as the 3 independent ones)

$$mc\frac{du^\alpha}{dt} = eF^{\alpha 0} + \frac{e}{c}F^{\alpha\beta}v_\beta \tag{2.7.94}$$

or, using (2.4.159),

$$\frac{d\mathbf{P}}{dt} = e\mathbf{E} + \frac{e}{c}\mathbf{v}\times\mathbf{B}. \tag{2.7.95}$$

The temporal component of (2.7.93) is

$$mc\frac{du_0}{dt} = \frac{e}{c}F_{0\alpha}v^\alpha \tag{2.7.96}$$

or

$$\frac{dE}{dt} = e\mathbf{v} \cdot \mathbf{E}, \tag{2.7.97}$$

which also results from multiplying (2.7.95) by $\mathbf{v}$ and using (2.4.184). Integrating (2.7.85) over the volume gives

$$\frac{\partial}{\partial t} \int W dV + \int \mathbf{j} \cdot \mathbf{E} dV + \oint \mathbf{S} \cdot d\mathbf{f} = 0, \tag{2.7.98}$$

which, with (2.7.55) and (2.7.97), yields the conservation of the total energy (2.4.66) of the electromagnetic field and particles:

$$\frac{\partial}{\partial t} \left(\int W dV + \sum_a E_a \right) + \oint \mathbf{S} \cdot d\mathbf{f} = 0. \tag{2.7.99}$$

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