Mass and Angular Momentum in General Relativity

José Luis Jaramillo[1,2] and Eric Gourgoulhon[2]

[1] Instituto de Astrofísica de Andalucía, CSIC, Apartado Postal 3004, Granada 18080, Spain jarama@iaa.es
[2] Laboratoire Univers et Théories, Observatoire de Paris, CNRS, Université Paris Diderot, 5 place Jules Janssen, F-92190 Meudon, France
eric.gourgoulhon@obspm.fr

We present an introduction to mass and angular momentum in General Relativity. After briefly reviewing energy-momentum for matter fields, first in the flat Minkowski case (Special Relativity) and then in curved spacetimes with or without symmetries, we focus on the discussion of energy-momentum for the gravitational field. We illustrate the difficulties rooted in the Equivalence Principle for defining a local energy-momentum density for the gravitational field. This leads to the understanding of gravitational energy-momentum and angular momentum as non-local observables that make sense, at best, for extended domains of spacetime. After introducing Komar quantities associated with spacetime symmetries, it is shown how total energy-momentum can be unambiguously defined for isolated systems, providing fundamental tests for the internal consistency of General Relativity as well as setting the conceptual basis for the understanding of energy loss by gravitational radiation. Finally, several attempts to formulate quasi-local notions of mass and angular momentum associated with extended but finite spacetime domains are presented, together with some illustrations of the relations between total and quasi-local quantities in the particular context of black hole spacetimes. This article is not intended to be a rigorous and exhaustive review of the subject, but rather an invitation to the topic for non-experts. In this sense we follow essentially the expositions in [1, 2, 3, 4] and refer the reader interested in further developments to the existing literature, in particular to the excellent and comprehensive review by Szabados [1].

1 Issues around the notion of gravitational energy in General Relativity

1.1 Energy-momentum density for matter fields

Let us first consider mass and angular momentum associated with matter in the absence of gravity, in a flat Minkowski spacetime. The density of energy and linear momentum associated with a distribution of matter are encoded in the energy-momentum tensor $T_{\mu\nu}$, corresponding to the Noether current conserved under infinitesimal spacetime translations in a Lagrangian framework. This general conservation property, namely $\partial_\mu T^{\mu\nu} = 0$ in inertial Minkowski coordinates, plays a key role in our discussion. Indeed, together with the presence of symmetries, it permits the introduction of conserved quantities or *charges*. Given a space-like hypersurface Σ and considering the unit time-like vector n^μ normal to it, we can define the conserved quantity associated with the symmetry k^μ and the domain D ($\subset \Sigma$) as

$$Q_D[k^\mu] = \int_D k^\rho T_{\nu\rho} n^\nu \sqrt{\gamma}\, d^3x \quad , \tag{1}$$

where $\sqrt{\gamma}\, d^3x$ denotes the induced volume element in D. The conservation of $T_{\mu\nu}$ and the characterisation of k^μ as a symmetry imply the conservation of the vector $T^\mu{}_\nu k^\nu$, i.e. $\partial_\mu \left(T^\mu{}_\nu k^\nu\right) = 0$. Applying then the Stokes theorem, it follows the equality between the change in time of $Q_D[k^\mu]$ and the flux of $\gamma^\mu{}_\rho T^{\rho\nu} k_\nu$ through the boundary of D (where $\gamma^\mu{}_\nu$ is the projector on D). Minkowski spacetime symmetries are given by Poincaré transformations. Therefore, we can associate conserved quantities with the infinitesimal generators corresponding to translations T^ν_a, rotations J^μ_i, and boosts K^μ_i (here the label a for translation generators runs in $\{0,1,2,3\}$, whereas i is a space-like index in $\{1,2,3\}$). In this manner, a 4-momentum $P_a[D]$ and an angular momentum $J_i[D]$ associated with the distribution of matter in $D \subset \Sigma$ can be defined as

$$P_a[D] = \int_D T_{\mu\nu} \mathrm{T}^\nu_a n^\mu \sqrt{\gamma}\, d^3x \quad , \quad J_i[D] = \int_D T_{\mu\nu} \mathrm{J}^\nu_i n^\mu \sqrt{\gamma}\, d^3x \quad . \tag{2}$$

More generally, we can combine together the rotation and boost generators J^μ_i and K^μ_i into a vector-field-valued antisymmetric matrix $\mathrm{M}^\mu_{[ab]}$ (where $\mathrm{J}^\mu_i = {}^3\epsilon_i{}^{jk} \mathrm{M}^\mu_{[jk]}$ and $\mathrm{K}^\mu_i = \mathrm{M}^\mu_{[0i]}$) and write the conserved quantities

$$J_{[ab]}[D] = \int_D T_{\mu\nu} \mathrm{M}^\nu_{[ab]} n^\mu \sqrt{\gamma}\, d^3x \quad . \tag{3}$$

The mass and (Pauli-Lubanski) spin are constructed as

$$m^2[D] := -\eta^{ab} P_a[D] P_b[D] \quad , \quad S^a[D] := \frac{1}{2} {}^4\epsilon^{abcd} P_b[D] J_{[cd]}[D] \quad , \tag{4}$$

in terms of which Poincaré Casimirs (invariant under Poincaré transformations) can be expressed.

In the non-flat case, (matter) energy-momentum tensor acts as the source of gravity through the Einstein equation and, consistently with Bianchi identities, satisfies the divergence-free condition analogous to the flat conservation law:

$$G_{\mu\nu} := {}^4R_{\mu\nu} - \frac{1}{2} {}^4R\, g_{\mu\nu} = 8\pi T_{\mu\nu} \quad , \quad \nabla_\nu T^{\nu\mu} = 0 \; . \tag{5}$$

The same strategy employed in the flat case for defining physical quantities associated with matter, i.e. from conserved currents corresponding to some symmetry, can be followed in non-flat spacetimes $(\mathcal{M}, g_{\mu\nu})$ presenting Killing vectors k^μ. The vector $T^\mu{}_\nu k^\nu$ is conserved, i.e. $\nabla_\mu \left(T^\mu{}_\nu k^\nu\right) = 0$, and provides a current-density for the conserved quantity $Q_D[k^\mu]$ defined by expression (1). The physical interpretation of $Q_D[k^\mu]$ depends of course on the nature of the Killing vector k^μ. Actually, $Q_D[k^\mu]$ does not actually depend on the slice Σ in the sense that its value is the same in the domain of dependence of D (this precisely corresponds to the conserved nature of this charge).

In a general spacetime with no symmetries the previous strategy ceases to work, and ambiguities in the definition of mass and angular momentum enter into scene. One can still calculate the flux of $T^\mu{}_\nu \xi^\nu$ for a given vector ξ^ν, and define the associated quantity $Q_D[\xi^\mu]$. However, the latter will now depend on the slice Σ and, in addition, its explicit dependence on ξ^μ introduces some degree of arbitrariness in the discussion. In this context, given a space-like 3+1 foliation $\{\Sigma_t\}$ of the spacetime with time-like normal vector n^μ, the current $P^\mu := -T^{\mu\nu} n_\nu$ can be interpreted as the energy-momentum density associated with (Eulerian) observers *at rest* with respect to Σ_t. That is, $E := T^{\mu\nu} n_\mu n_\nu$ stands as the matter energy density and $p^\mu := -\gamma^\mu{}_\rho T^{\rho\nu} n_\nu$ as the momentum density, where $\gamma_{\mu\nu}$ is the induced metric on Σ_t (see Eq. (12) below for the complete 3+1 decomposition of $T_{\mu\nu}$). In particular, we can calculate the matter energy associated with observers n^μ over the spatial region D by direct integration

$$E[D] = \int_D E\sqrt{\gamma}\, d^3x = \int_D T^{\mu\nu} n_\mu n_\nu \sqrt{\gamma}\, d^3x \ . \tag{6}$$

By imposing the dominant energy condition on the matter energy-momentum tensor (see section 3.3), the vector $-T^{\mu\nu} n_\nu$ is future directed and non-space-like. Its Lorentzian norm is therefore non-positive and an associated matter mass density m can be given as $m^2 := -P^\mu P_\mu = -(-T^{\mu\rho} n_\rho)(-T^{\nu\sigma} n_\sigma) g_{\mu\nu} = E^2 - p^i p_i \geq 0$. The corresponding mass $\mathrm{M}[D]$ in the extended region D would be

$$\mathrm{M}[D] := \int_D \sqrt{E^2 - p^i p_i}\, \sqrt{\gamma}\, d^3x \ . \tag{7}$$

Note the difference between the construction of $\mathrm{M}[D]$ and that of $m[D]$ in the Minkowskian case: for the latter one first integrates to obtain the charges and then calculates a Minkowskian norm, whereas for constructing $\mathrm{M}[D]$ that order is reversed; in addition, different metrics are employed in each case (cf. section 2.2. in [1]).

1.2 Problems when defining a gravitational energy-momentum

In the characterisation of the physical properties of the gravitational field, in particular its energy-momentum and angular momentum, we could try to follow a similar strategy to that employed for the matter fields. This would amount to identify appropriate local densities that would then be integrated over finite spacetime regions. However such an approach rapidly meets important conceptual difficulties.

A local (point-like) density of energy associated with the gravitational field cannot be defined in General Relativity. Reasons for this can be tracked to the Equivalence Principle. Illustrated in a heuristic manner, this principle can be used to get rid of the gravitational field on a given point of spacetime. Namely, a free falling point-like particle does not *feel* any gravitational field so that, in particular, no gravitational energy density can be identified at spacetime points.

In a Lagrangian setting, these basic conceptual difficulties are reflected in the attempts to construct a gravitational energy-momentum tensor, when mimicking the methodological steps followed in the matter field case. We can write generically the gravitational-matter action as

$$S = S_{\rm EH} + S_{\rm m} = \frac{1}{16\pi}\int_{\mathcal{M}} {}^4\!R\sqrt{-g}\; d^4x + \int_{\mathcal{M}} L_{\rm m}(g_{\mu\nu}, \Phi_i, \nabla_\mu \Phi_i, ...)\sqrt{-g}\; d^4x \quad . \tag{8}$$

where $S_{\rm EH}$ denotes the Einstein-Hilbert action and Φ_i in the matter Lagrangian $L_{\rm m}$ account for the matter fields. The symmetric energy-momentum for matter is obtained from the variation of the matter action $S_{\rm m}$ with respect to the metric

$$T_{\mu\nu} := \frac{-2}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta g^{\mu\nu}} \quad , \tag{9}$$

whereas the field equations for the matter fields follow from the variation with respect to the matter fields Φ_i. On the contrary, the gravitational action $S_{\rm EH}$ only depends on the gravitational field, since any further background structure would be precluded by diffeomorphism invariance (a feature closely tied to the physical Equivalence Principle). Einstein equation for the gravitational field follows from the variation of the total action with respect to the metric field $g_{\mu\nu}$, with no gravitational analogue of the symmetric matter energy-momentum tensor $T_{\mu\nu}$. Attempts to construct a symmetric energy-momentum tensor for the gravitational field either recover the Einstein tensor $G_{\mu\nu}$ or can only be related to higher-order gravitational energy-momentum objects, such as the Bel-Robinson tensor (see e.g. [5]). Again, the absence of a tensorial (i.e. point-like geometric) quantity representing energy-momentum for the gravitational field is consistent with, and actually a consequence of, the Equivalence Principle.

The natural interpretation of the symmetric matter energy-momentum tensor $T_{\mu\nu}$ as introduced in Eq. (9) is that of the *current-source* for the gravitational field, obtained as a conserved current associated with spacetime translations. Alternative, in terms of the Noether theorem [6] it is natural to introduce a (non-symmetric) *canonical* energy-momentum tensor for matter from which a symmetric one can be constructed through the Belinfante-

Rosenfeld procedure [7, 8, 9]. The application of this construction to the gravitational field naturally leads to the discussion of gravitational energy-momentum *pseudo-tensors* [1]. The underlying idea consists in decomposing the Einstein tensor $G_{\mu\nu}$ into a part that can be identified with the energy-momentum and a second piece that can be expressed in terms of a pseudo-potential. That is [10]

$$G_\mu{}^\nu := -8\pi \, t_\mu{}^\nu + \frac{1}{2\sqrt{-g}}\partial_\lambda(H_\mu{}^{\nu\lambda}) \quad , \tag{10}$$

where $t_{\mu\nu}$ is the gravitational energy-momentum pseudo-tensor and $H_{\mu\nu}{}^\lambda$ is the superpotential. Einstein equation is then written as

$$\partial_\lambda(H_\mu{}^{\nu\lambda}) = 16\pi\sqrt{-g}\,(t_\mu{}^\nu + T_\mu{}^\nu) =: 16\pi\mathcal{T}_\mu{}^\nu \quad . \tag{11}$$

Objects $t_\mu{}^\nu$ and $H_\mu{}^{\nu\lambda}$ are not tensorial quantities. This means that their value at a given spacetime point is not a well-defined notion. Moreover, their very definition needs the introduction of some additional background structure and some choice of preferred coordinates is naturally involved. Different pseudo-tensors exist in the literature, e.g. those introduced by Einstein, Papapetrou, Bergmann, Landau and Lifshitz, Moller or Weinberg (e.g. see references in [10]).

As an alternative to the pseudo-tensor approach, there also exist attempts in the literature aiming at constructing truly tensorial energy-momentum quantities. However they also involve the introduction of some additional structure, either in the form of a background object or by fixing a gauge in some given formulation of General Relativity (cf. comments on the tetrad formalism approach in [1]).

Non-local character of gravitational energy

As illustrated above, crucial conceptual and practical caveats are involved in the association of energy and angular momentum with the gravitational field. For these reasons, one might legitimately consider gravitational energy and angular momentum in General Relativity as intrinsically meaningless notions in *generic* situations, in such a way that the effort to derive explicit general local expressions actually represents an ill-defined problem (cf. remarks in [11] referring to the quest for a local expression of energy in General Relativity). Having said this and after accepting the non-existence of a *local* (point-like) notion of energy density for the gravitational field, one may also consider gravitational energy-momentum and angular momentum as notions intrinsically associated with extended domains of the spacetime and then look for restricted settings or appropriate limits where they can be properly defined.

In fact, making a sense of the energy and angular momentum for the gravitational field in given regions of spacetime is extremely important in different contexts of gravitational physics, as it can be illustrated with examples coming from mathematical relativity, black hole physics, lines of research in Quantum Gravity, or relativistic astrophysics. From a structural point of view, having a well-defined *mass positivity* result is crucial for the internal consistency of the theory, as well as for the discussion of the solutions stability. Moreover, the possibility of introducing appropriate positive-definite (*energy*) quantities is often a key step in different developments in mathematical relativity, in particular when using variational principles. In the study of the physical picture of black holes, appropriate notions of mass and angular momentum are employed. In particular, they play a key role in the formulation of black hole thermodynamics (e.g. [12]), a cornerstone in different approaches to Quantum Gravity. In the context of relativistic astrophysics and numerical relativity, the study of relativistic binary mergers, gravitational collapse and the associated generation/propagation of gravitational radiation also requires appropriate notions of energy and angular momentum (see e.g. [13] for a further discussion on the intersection between numerical and mathematical relativity).

Once the non-local nature of the gravitational energy-momentum and angular momentum is realised, the conceptual challenge is translated into the manner of determining the appropriate physical parameters associated with the gravitational field in an extended region of spacetime. An unambiguous answer has been given in the case of the total mass of an isolated system. However, the situation is much less clear in the case of extended but finite spacetime domains. In a broad sense, existing attempts either enforce some additional structure that restricts the study to an appropriate subset of the solution space of General Relativity, or alternatively they look for a genuinely geometric characterisation aiming at fulfilling some expected *physical* requirements. In this article we present an overview of some of the relevant existing attempts and illustrate the kind of additional structures they involve.

1.3 Notation

Before proceeding further, we set the notation, some of whose elements have already been anticipated above. The signature of spacetime $(\mathcal{M}, g_{\mu\nu})$ is chosen to be diag$[-1,1,1,1]$ and Greek letters are used for spacetime indices in $\{0,1,2,3\}$. We denote the Levi-Civita connection by ∇_μ and the volume element by ${}^4\epsilon = \sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. We make $G = c = 1$ throughout.

3+1 decompositions

In our presentation of the subject, 3+1 foliations of spacetime $(\mathcal{M}, g_{\mu\nu})$ by space-like 3-slices $\{\Sigma_t\}$ will play an important role. Given a height-function t, the time-like unit normal to Σ_t will be denoted by n^μ and the 3+1 decomposition of the evolution vector field by $t^\mu = Nn^\mu + \beta^\mu$, where N is the lapse function and β^μ is the shift vector. The induced metric on the space-like 3-slice Σ_t is expressed as $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, with D_μ the associated

Levi-Civita connection and volume element ${}^3\epsilon = \sqrt{\gamma}dx^1 \wedge dx^2 \wedge dx^3$, so that ${}^3\epsilon_{\mu\nu\rho} = n^\sigma {}^4\epsilon_{\sigma\mu\nu\rho}$. The extrinsic curvature of $(\Sigma_t, \gamma_{\mu\nu})$ in $(\mathcal{M}, g_{\mu\nu})$ is defined as $K_{\mu\nu} := -\frac{1}{2}\mathcal{L}_n\gamma_{\mu\nu} = -\gamma_\mu{}^\rho\nabla_\rho n_\nu$. The 3+1 decomposition of the (matter) stress-energy tensor, in terms of an Eulerian observer n^μ in rest with respect to the foliation $\{\Sigma_t\}$, is

$$T_{\mu\nu} = E\ n_\mu n_\nu + p_{(\mu}n_{\nu)} + S_{\mu\nu} \quad , \tag{12}$$

where the matter energy and momentum densities are given by $E := T_{\mu\nu}n^\mu n^\nu$ and $p_\mu := -T_{\nu\rho}n^\nu\gamma^\rho{}_\mu$, respectively, whereas the matter stress tensor is $S_{\mu\nu} := T_{\rho\sigma}\gamma^\rho{}_\mu\gamma^\sigma{}_\nu$. Latin indices running in $\{1, 2, 3\}$ will be employed in expressions only involving objects intrinsic to space-like Σ_t slices.

Closed 2-surfaces

Closed 2-surfaces $\mathcal{S}$, namely topological spheres in our discussion, will also be relevant in the following. The normal bundle $T^\perp\mathcal{S}$ can be spanned by a time-like unit vector field n^μ and a space-like unit vector field s^μ, that we choose to satisfy the orthogonality condition $n^\mu s_\mu = 0$. When considering $\mathcal{S}$ as embedded in a space-like 3-surface Σ, n^μ can be identified with the time-like normal to Σ and s^μ with the normal to $\mathcal{S}$ tangent to Σ. In the generic case, n^μ and s^μ can be defined up to a boost transformation: $n'^\mu = \cosh(\eta)n^\mu + \sinh(\eta)s^\mu$ and $s'^\mu = \sinh(\eta)n^\mu + \cosh(\eta)s^\mu$, with η a real parameter. Alternatively, one can span $T^\perp_p\mathcal{S}$ at $p \in \mathcal{S}$ in terms of the null normals defined by the intersection between the normal plane to $\mathcal{S}$ and the light-cone at the spacetime point p. The directions defined by the *outgoing* ℓ^μ and the *ingoing* k^μ null normals (satisfying $k^\mu\ell_\mu = -1$) are uniquely determined, though it remains a boost-normalization freedom: $\ell'^\mu = f \cdot \ell^\mu$, $k'^\mu = \frac{1}{f} \cdot k^\mu$. The induced metric on $\mathcal{S}$ is given by: $q_{\mu\nu} = g_{\mu\nu} + k_\mu\ell_\nu + \ell_\mu k_\nu = g_{\mu\nu} + n_\mu n_\nu - s_\mu s_\nu = \gamma_{\mu\nu} - s_\mu s_\nu$, the latter expression applying when $\mathcal{S}$ is embedded in $(\Sigma, \gamma_{\mu\nu})$. The Levi-Civita connection associated with $q_{\mu\nu}$ will be denoted by ${}^2D_\mu$ and the volume element by ${}^2\epsilon = \sqrt{q}dx^1 \wedge dx^2$, i.e. ${}^2\epsilon_{\mu\nu} = n^\rho s^\sigma {}^4\epsilon_{\rho\sigma\mu\nu}$. When integrating tensors on $\mathcal{S}$ with components normal to the sphere, it is convenient to express the volume element as $dS_{\mu\nu} = (s_\mu n_\nu - n_\mu s_\nu)\sqrt{q}d^2x$ (this is just a convenient manner of re-expressing ${}^4\epsilon_{\mu\nu\rho\sigma}$ for integrating over $\mathcal{S}$ after a contraction with the appropriate tensor; cf. for example Eq.(13)).

The *second fundamental tensor* of $(\mathcal{S}, q_{\mu\nu})$ in $(\mathcal{M}, g_{\mu\nu})$ is defined as $\mathcal{K}^\alpha_{\mu\nu} := q^\rho{}_\mu q^\sigma{}_\nu\nabla_\rho q^\alpha{}_\sigma$, that can be expressed as $\mathcal{K}^\alpha_{\mu\nu} = n^\alpha\Theta^{(n)}_{\mu\nu} + s^\alpha\Theta^{(s)}_{\mu\nu} = k^\alpha\Theta^{(\ell)}_{\mu\nu} + \ell^\alpha\Theta^{(k)}_{\mu\nu}$, where the *deformation tensor* $\Theta^{(v)}_{\mu\nu}$ associated with a vector v^μ normal to $\mathcal{S}$ is defined as $\Theta^{(v)}_{\mu\nu} = q^\rho{}_\mu q^\sigma{}_\nu\nabla_\rho v_\sigma$. We set a specific notation for the cases corresponding to s^μ and n^μ, namely $H_{\mu\nu} := \Theta^{(s)}_{\mu\nu}$, the extrinsic curvature of $(\mathcal{S}, q_{\mu\nu})$ inside a 3-slice $(\Sigma, \gamma_{\mu\nu})$, and $L_{\mu\nu} := -\Theta^{(n)}_{\mu\nu}$.

Information about the extrinsic curvature of $(\mathcal{S}, q_{\mu\nu})$ in $(\mathcal{M}, g_{\mu\nu})$ is completed by the *normal fundamental forms* associated with normal vectors v^μ. In particular, we define the 1-form $\Omega^{(\ell)}_\mu := k^\rho q^\sigma{}_\mu \nabla_\sigma \ell_\rho$. This form is not invariant under a boost transformation, and transforms as $\Omega^{(\ell')}_\mu = \Omega^{(\ell)}_\mu + {}^2D_\mu \mathrm{ln} f$ in the notation above. Other normal fundamental forms can be defined in terms of normals k^μ, n^μ and s^μ, but they are all related up to total derivatives.

2 Spacetimes with Killing vectors: Komar quantities

As commented above, some additional structure is needed to introduce meaningful notions of gravitational energy and angular momentum. Let us first consider spacetimes admitting isometries. This represents the most straightforward generalization of the definition of physical parameters as conserved quantities under existing symmetries. Requiring the presence of Killing vectors represents our first example of the enforcement of an additional structure on the considered spacetime.

Given a Killing vector field k^μ in the spacetime $(\mathcal{M}, g_{\mu\nu})$ and $\mathcal{S}$ a space-like closed 2-surface, let us define the Komar quantity [14] k_{K} as

$$k_{\mathrm{K}} := -\frac{1}{8\pi} \oint_{\mathcal{S}} \nabla^\mu k^\nu \, dS_{\mu\nu} \quad , \tag{13}$$

(see previous section for the notation $dS_{\mu\nu}$ for the volume element on $\mathcal{S}$). Let us consider $\mathcal{S}$ as embedded in a space-like 3-slice Σ and let us take a second closed 2-surface $\mathcal{S}'$ such that either $\mathcal{S}'$ is completely contained in $\mathcal{S}$ or vice-versa, and let us denote by V the region in Σ contained between $\mathcal{S}$ and $\mathcal{S}'$. The previously defined Komar quantity k_{K} is then *conserved* in the sense that its value does not depend on the chosen 2-surface as long as no matter is present in the intermediate region V

$$k_{\mathrm{K}}^{\mathcal{S}} = 2 \int_V \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) n^\mu k^\nu \sqrt{\gamma} \, d^3x + k_{\mathrm{K}}^{\mathcal{S}'} \quad , \tag{14}$$

where $T = T_{\mu\nu} g^{\mu\nu}$.

Remark 1. Two important points must be stressed: a) the definition of k_{K} is geometric and therefore coordinate independent, and b) k_{K} is associated with a closed 2-surface with no need to refer to any particular embedding in a 3-slice Σ (in the discussion above the latter has been only introduced for pedagogical reasons).

2.1 Komar mass

Stationary spacetimes admit a time-like Killing vector field k^μ. The associated conserved Komar quantity is known as the Komar mass

$$M_{\mathrm{K}} := -\frac{1}{8\pi} \oint_{\mathcal{S}} \nabla^\mu k^\nu \, dS_{\mu\nu} \quad . \tag{15}$$

This represents our first notion of mass in General Relativity. It is instructive to write the Komar mass in terms of 3+1 quantities. Given a 3-slicing $\{\Sigma_t\}$ and choosing the evolution vector $t^\mu = Nn^\mu + \beta^\mu$ to coincide with the time-like Killing symmetry, we find

$$M_{\mathrm{K}} = \frac{1}{4\pi}\oint_{\mathcal{S}} \left(s^i D_i N - K_{ij} s^i \beta^j\right) \sqrt{q}\, d^2x \quad . \tag{16}$$

2.2 Komar angular momentum

Let us consider now an axisymmetric spacetime, where the axial Killing vector is denoted by ϕ^μ. That is, ϕ^μ is a space-like Killing vector whose action on $\mathcal{M}$ has compact orbits, two stationary points (the poles), and is normalized so that its natural affine parameter takes values in $[0, 2\pi)$. The Komar angular momentum is defined as

$$J_{\mathrm{K}} := \frac{1}{16\pi}\oint_{\mathcal{S}_t} \nabla^\mu \phi^\nu \, dS_{\mu\nu} \quad . \tag{17}$$

Note (apart from the sign choice) the factor 1/2 with respect to the Komar quantity ϕ_{K}, known as the *Komar anomalous factor* (it can be explained in the context of a bimetric formalism by writing the conserved quantities in terms of an *Einstein energy-momentum flux* density that can be expressed as the sum of half the Komar contribution plus a second term: in the angular momentum case this second piece vanishes, whereas for the mass case it equals half the Komar term; cf. [15]). Adopting a 3-slicing adapted to axisymmetry, i.e. $n^\mu \phi_\mu = 0$, we have:

$$J_{\mathrm{K}} = \frac{1}{8\pi}\oint_{\mathcal{S}} K_{ij} s^i \phi^j \sqrt{q}\, d^2x = \frac{1}{8\pi}\oint_{\mathcal{S}} \Omega^{(\ell)}_\mu \phi^\mu \sqrt{q}\, d^2x \quad . \tag{18}$$

3 Total mass of Isolated Systems in General Relativity

3.1 Asymptotic Flatness characterisation of Isolated Systems

The characterisation of an isolated system in General Relativity aims at capturing the idea that spacetime becomes flat when we move *sufficiently far* from the system, so that spacetime approaches that of Minkowski. However, the very notion of *far away* becomes problematic due to the absence of an a priori background spacetime. In addition, we must consider different *kinds of infinities*, since we can move away from the system in space-like and also in null directions. Different strategies exist in the literature for the formalization of this asymptotic flatness idea, and not all of them are mathematically equivalent. Traditional approaches attempt to specify the adequate fall-off conditions of the curvature in appropriate coordinate systems at *infinity*.

These approaches have the advantage of embodying the weakest versions of asymptotic flatness. We will illustrate their use in the discussion of spatial infinity in section 3.2. However, the use of coordinate expressions in this strategy also introduces the need of verifying the intrinsic nature of the obtained results, something that it is not always straightforward. For this reason, a geometric manner of describing asymptotic flatness is also desirable, without relying on specific coordinates. This has led to the conformal compactification picture, where infinity is *brought* to a *finite distance* by an appropriate spacetime conformal transformation. More concretely, one works with an unphysical spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ with boundary, such that the physical spacetime $(\mathcal{M}, g_{\mu\nu})$ is conformally equivalent to the interior of $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$, i.e. $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. *Infinity* is captured by the boundary $\partial\tilde{\mathcal{M}}$ and is characterised by the vanishing of the conformal factor, $\Omega = 0$. The whole picture is inspired in the structure of the conformal compactification of Minkowski spacetime. The conformal boundary is the union of different pieces, which are classified according to the metric-type of the geodesics reaching their points. This defines (past and future) null infinity $\mathscr{I}^{\pm}$, spatial infinity i^0 and (past and future) time-like infinity $i^{\pm}$, i.e. $\partial\tilde{\mathcal{M}} = \mathscr{I}^{\pm} \cup i^0 \cup i^{\pm}$. The conformal spacetime is represented in the so-called Carter-Penrose diagram. Fall-off conditions for the characterisation of asymptotic flatness are substituted by differentiability conditions on the fields at null and spatial infinity (isolated systems do not require flatness conditions on time-like infinity). Null infinity was introduced in the conformal picture by Penrose [16, 17], the discussion of asymptotic flatness at spatial infinity was developed by Geroch [18] and a unified treatment was presented in [19, 20]. We will briefly illustrate the different approaches to asymptotic flatness in the following sections, but we refer the reader to the existing bibliography (e.g. [21, 4]) for further details.

3.2 Asymptotic Euclidean slices

The following two sections are devoted to the discussion of conserved quantities at spatial infinity, but they also illustrate the coordinate-based approach to asymptotic flatness. A slice Σ endowed with a space-like 3-metric γ_{ij} is *asymptotically Euclidean* (flat), if there exists a Riemannian background metric f_{ij} such that:

i) f_{ij} is flat, except possibly on a compact domain D of Σ.
ii) There exists a coordinate system $(x^i) = (x, y, z)$ such that outside D, $f_{ij} = \mathrm{diag}(1, 1, 1)$ (*Cartesian-type coordinates*) and the variable $r := \sqrt{x^2 + y^2 + z^2}$ can take arbitrarily large values on Σ.
iii) When $r \to +\infty$

$$\begin{aligned} \gamma_{ij} = f_{ij} + O(r^{-1}) \quad &, \quad \frac{\partial \gamma_{ij}}{\partial x^k} = O(r^{-2}) \quad ; \\ K_{ij} = O(r^{-2}) \quad &, \quad \frac{\partial K_{ij}}{\partial x^k} = O(r^{-3}) \quad . \end{aligned} \tag{19}$$

Given an asymptotically flat spacetime foliated by asymptotically Euclidean slices $\{\Sigma_t\}$, *spatial infinity* is defined by $r \to +\infty$ and denoted as i^0.

Asymptotic symmetries at spatial infinity

As commented in the discussion of the Komar quantities, the existence of symmetries provides a natural manner of defining physical parameters as conserved quantities. In the context of spatial infinity, the spacetime diffeomorphisms preserving the asymptotic Euclidean structure (19) are referred to as asymptotic symmetries. Asymptotic symmetries close a Lie group. Since the spacetime is asymptotically flat, one would expect this group to be isomorphic to the Poincaré group. However, the set of diffeomorphisms $(x^\mu) = (t, x^i) \to (x'^\mu) = (t', x'^i)$ preserving conditions (19) is given by

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + c^\mu(\theta, \varphi) + O(r^{-1}) \ \ , \tag{20}$$

where $\Lambda^\mu{}_\nu$ is a Lorentz matrix and the c^μ's are four functions of the angles (θ, φ) related to coordinates $(x^i) = (x, y, z)$ by the standard spherical formulæ: $x = r \sin\theta \cos\varphi, y = r \sin\theta \sin\varphi, z = r \cos\theta$. This group indeed contains the Poincaré symmetry, but it is actually much larger due to the presence of *angle-dependent* translations. The latter are known as *supertranslations* and are defined by $c^\mu(\theta, \varphi) \neq \mathrm{const}$ and $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$ in the group representation (20). The corresponding abstract infinite-dimensional symmetry preserving the structure of spatial infinity (*Spi*) is referred to as the Spi group [19, 20]. The existence of this (infinite-dimensional) Lie structure of asymptotic symmetries has implications in the definition of a global physical mass, linear and angular momentum at spatial infinity (see below).

3.3 ADM quantities

Hamiltonian techniques are particularly powerful for the systematic study of physical parameters, considered as conserved quantities under symmetries acting as canonical transformations in the solution (phase) space of a theory. In this sense, the Hamiltonian formulation of General Relativity provides a natural framework for the discussion of global quantities at spatial infinity. This was the original approach adopted by Arnowitt, Deser and Misner in [22] and we outline here the basic steps.

First, a variational problem for the class of spacetimes we are considering must be set. For a correct formulation we need to specify: a) the dynamical fields we are varying, b) the domain $\mathcal{V}$ over which these fields are varied together with the prescribed value of their variations at the boundary $\partial\mathcal{V}$, and c) the action functional S compatible with the field equations. As integration domain $\mathcal{V}$ we consider the region bounded by two space-like 3-slices Σ_{t_1} and Σ_{t_2} and an outer time-like tube $\mathcal{B}$. Σ_{t_1} and Σ_{t_2} can be seen as part of a 3-slicing $\{\Sigma_t\}$ with metric and extrinsic curvature given by (γ_{ij}, K^{ij}), whereas $\mathcal{B}$ has $(\chi_{\mu\nu}, P^{\mu\nu})$ as induced metric and extrinsic curvature. That is, $\chi_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ and $P_{\mu\nu} = -\gamma_\mu{}^\rho \nabla_\rho u_\nu$, where u^μ is the unit space-like normal to $\mathcal{B}$. The dynamical field whose variation we consider is the spacetime metric $g_{\mu\nu}$, under boundary conditions $\delta g_{\mu\nu}|_{\partial\mathcal{V}} = 0$ (note that we impose nothing on variations of the derivatives of $g_{\mu\nu}$). The appropriate gravitational Einstein-Hilbert action then reads (cf. for example [3]; the discussion has a straightforward extension to incorporate matter)

$$S = \frac{1}{16\pi}\int_{\mathcal{V}} {}^4R\sqrt{-g}\, d^4x + \frac{1}{8\pi}\left\{ -\int_{\Sigma_{t_2}} (K - K_0)\sqrt{\gamma}\, d^3x \right. \tag{21}$$
$$\left. + \int_{\Sigma_{t_1}} (K - K_0)\sqrt{\gamma}\, d^3x + \int_{\mathcal{B}} (P - P_0)\sqrt{-\chi}\, d^3x \right\} \quad ,$$

where K and P are the traces of the extrinsic curvatures of the hypersurfaces Σ_{t_i} and $\mathcal{B}$, respectively, as embedded in $(\mathcal{M}, g_{\mu\nu})$. The subindex 0 corresponds to their extrinsic curvatures as embedded in $(\mathcal{M}, \eta_{\mu\nu})$. The boundary term guarantees the well-posedness of the variational principle, i.e. the functional differentiability of the action and the recovery of the correct Einstein field equation, under the assumed boundary conditions for the dynamical fields.

Making use of the 3+1 fields decompositions, and considering the intersections $\mathcal{S}_t := \mathcal{B} \cap \Sigma_t$ between space-like 3-slices Σ_t and the time-like hypersurface $\mathcal{B}$, we can express the action (21) as

$$S = \frac{1}{16\pi}\int_{t_1}^{t_2} \left\{ \int_{\Sigma_t} N\left({}^3R + K_{ij}K^{ij} - K^2\right)\sqrt{\gamma}\, d^3x + 2\oint_{\mathcal{S}_t} (H - H_0)\sqrt{q}\; d^2x \right\} dt \tag{22}$$

where H and H_0 denote the trace of the extrinsic curvature of the 2-surface $\mathcal{S}_t$ as embedded in (Σ_t, γ_{ij}) and (Σ_t, f_{ij}), respectively. The Lagrangian density L can be read from the form of the action (22). The 3-metric γ_{ij} plays the role of the dynamical variable and the dependence of L on $\dot{\gamma}_{ij}$ follows from the explicit expression of the extrinsic curvature K_{ij} in terms of the lapse and the shift, that is

$$K_{ij} = \frac{1}{2N}\left(\gamma_{ik}D_j\beta^k + \gamma_{jk}D_i\beta^k - \dot{\gamma}_{ij}\right) \quad . \tag{23}$$

In particular no derivatives of N and β^i appear in (22), indicating that the lapse function and the shift vector are not dynamical variables. The Hamiltonian description is obtained by performing a Legendre transformation from variables $(\gamma_{ij}, \dot{\gamma}_{ij})$ to canonical ones (γ_{ij}, π^{ij}), where

$$\pi^{ij} := \frac{\delta L}{\delta \dot{\gamma}_{ij}} = \frac{1}{16\pi}\sqrt{\gamma}\left(K\gamma^{ij} - K^{ij}\right) . \tag{24}$$

The Hamiltonian density $\mathcal{H}$ is then given by

$$\mathcal{H} = \pi^{ij}\dot{\gamma}_{ij} - L \quad , \tag{25}$$

and the Hamiltonian follows from an integration over a 3-slice, resulting in (cf. [2, 3] for details)

$$\begin{aligned} H = \frac{1}{16\pi}\Big\{ &- \int_{\Sigma_t} \left(NC_0 + 2\beta^i C_i\right)\sqrt{\gamma}\, d^3x \\ &-2\oint_{\mathcal{S}_t} \left[N(H - H_0) - \beta^i(K_{ij} - K\gamma_{ij})s^j\right]\sqrt{q}\, d^2x \Big\} \quad , \end{aligned} \tag{26}$$

where

$$\begin{aligned} C_0 &:= {}^3R + K^2 - K_{ij}K^{ij} \quad , \\ C_i &:= D_j K^j{}_i - D_i K \quad . \end{aligned} \tag{27}$$

Functionals C_0 and C_i vanish on solutions of the Einstein equation (in vacuum). More specifically, equations $C_0 = 0$ and $C_i = 0$ respectively represent the Hamiltonian and momentum constraints of General Relativity, corresponding to the contraction of the Einstein equation (5) with n^μ. From a geometric point of view, they are referred to as the Gauss-Codazzi relations and represent conditions for the embedding of (Σ_t, γ_{ij}) as a submanifold of a spacetime $(\mathcal{M}, g_{\mu\nu})$ with vanishing $n^\mu G_{\mu\nu}$. The evaluation of the gravitational Hamiltonian (26) on solutions to the Einstein equation yields

$$H_{\text{solution}} = -\frac{1}{8\pi}\oint_{\mathcal{S}_t} \left[N(H - H_0) - \beta^i(K_{ij} - K\gamma_{ij})s^j\right]\sqrt{q}\, d^2x \quad . \tag{28}$$

Remark 2. Note that in the absence of boundaries the gravitational Hamiltonian vanishes on physical solutions. This is a feature of diffeomorphism invariant theories [23] and reflects the fact that the Hamiltonian, considered as the generator of a canonical transformation, does not move points in the solution space of the theory. In other words, it is a generator of gauge transformations, something consistent with the interpretation of the Hamiltonian as the generator of diffeomorphisms. Note also that the situation changes in the presence of boundaries, where diffeomorphisms not preserving boundary conditions do not correspond to gauge transformations, indicating the presence of residual degrees of freedom (this is of relevance, for instance, in certain aspects of the quantum theory).

ADM energy

We focus on solutions corresponding to isolated systems and consider 3-slices Σ_t that are asymptotically Euclidean in the sense of conditions (19). We choose the lapse and the shift so that the evolution vector t^μ is associated with some asymptotically inertial observer for which $N = 1$ and $\beta^i = 0$

at spatial infinity. In particular, this flow vector t^μ generates asymptotic time translations that, in this asymptotically flat context, constitute actual (asymptotic) symmetries. Conserved quantities under time translations have the physical meaning of an energy. In the present case, the conserved quantity is referred to as the ADM energy. The latter is obtained from expression (28) by making $N = 1$ and $\beta^i = 0$ and taking the limit to spatial infinity, namely $r \to \infty$ in the well-defined sense of section 3.2. That is

$$E_{\mathrm{ADM}} := -\frac{1}{8\pi} \lim_{\mathcal{S}_{(t,r\to\infty)}} \oint_{\mathcal{S}_t} (H - H_0)\sqrt{q}\, d^2x \quad . \tag{29}$$

This ADM energy represents the total energy contained in the slice Σ_t. Using the explicit expression of the extrinsic curvature in terms of metric components, the ADM energy can be written as

$$E_{\mathrm{ADM}} = \frac{1}{16\pi} \lim_{\mathcal{S}_{(t,r\to\infty)}} \oint_{\mathcal{S}_t} \left[\mathcal{D}^j \gamma_{ij} - \mathcal{D}_i(f^{kl}\gamma_{kl})\right] s^i \sqrt{q}\, d^2x \quad , \tag{30}$$

where $\mathcal{D}_i$ stands for the connection associated with the metric f_{ij} and, consistently with notation in section 1.3, s^i corresponds to the unit normal to $\mathcal{S}_t$ tangent to Σ_t and oriented towards the exterior of $\mathcal{S}_t$ (note that when $r \to \infty$ the normalization with respect to γ_{ij} and f_{ij} are equivalent). In particular, if we use the Cartesian-like coordinates employed in (19) we recover the standard form (see e.g, [4])

$$E_{\mathrm{ADM}} = \frac{1}{16\pi} \lim_{\mathcal{S}_{(t,r\to\infty)}} \oint_{\mathcal{S}_t} \left(\frac{\partial \gamma_{ij}}{\partial x^j} - \frac{\partial \gamma_{jj}}{\partial x^i}\right) s^i \sqrt{q}\, d^2x \quad . \tag{31}$$

Remark 3. We note that asymptotic flatness conditions (19) guarantee the finite value of the integral since the $O(r^2)$ part of the measure $\sqrt{q}\, d^2x$ is compensated by the $O(r^{-2})$ parts of $\partial\gamma_{ij}/\partial x^j$ and $\partial\gamma_{jj}/\partial x^i$. It is very important to point out that finiteness of the ADM energy relies on the subtraction of the *reference* value H_0 in Eq. (29).

Conformal decomposition expression of the ADM energy.

A useful expression for the ADM energy in certain formulations of the Einstein equation is given in terms of a conformal decomposition of the 3-metric

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij} \quad . \tag{32}$$

Choosing the representative $\tilde{\gamma}_{ij}$ of the conformal class by the unimodular condition $\det(\tilde{\gamma}_{ij}) = \det(f_{ij}) = 1$, conditions (19) translate into

$$\begin{aligned} \Psi &= 1 + O(r^{-1}) \qquad , \qquad \frac{\partial \Psi}{\partial x^k} = O(r^{-2}) \quad ; \\ \tilde{\gamma}_{ij} &= f_{ij} + O(r^{-1}) \qquad , \qquad \frac{\partial \tilde{\gamma}_{ij}}{\partial x^k} = O(r^{-2}) \quad , \end{aligned} \tag{33}$$

for the conformal factor and the conformal metric. Then it follows [2]

$$E_{\mathrm{ADM}} = -\frac{1}{2\pi} \lim_{\mathcal{S}_{(t,r\to\infty)}} \oint_{\mathcal{S}_t} s^i \left(\mathcal{D}_i \Psi - \frac{1}{8} \mathcal{D}^j \tilde{\gamma}_{ij} \right) \sqrt{q}\, d^2x \quad . \tag{34}$$

Note that, whereas in the time-symmetric ($K_{\mu\nu} = 0$) conformally flat case the Komar mass is given in terms of the monopolar term in the asymptotic expansion of the (adapted) lapse, the ADM energy is given by the monopolar term in ψ (the latter holds more generally under a vanishing *Dirac-like* gauge condition on $\mathcal{D}^j \tilde{\gamma}_{ij}$).

Example 1 (Newtonian limit). As an application of expression (34) we check that the ADM energy recovers the standard result in the Newtonian limit. For this we assume that the gravitational field is weak and static. In this setting it is always possible to find a coordinate system $(x^\mu) = (x^0 = ct, x^i)$ such that the metric components take the form

$$-d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = -(1 + 2\Phi)\, dt^2 + (1 - 2\Phi)\, f_{ij}\, dx^i dx^j \quad , \tag{35}$$

where again f_{ij} is the flat Euclidean metric in the 3-dimensional slice and Φ is the Newtonian gravitational potential, solution of the Poisson equation $\Delta\Phi = 4\pi\rho$ where ρ is the mass density (we recall that we use units in which the Newton's gravitational constant G and the light velocity c are unity). Then, using $\Psi = (1 - 2\Phi)^{1/4} \approx 1 - \frac{1}{2}\Phi$, Eq. (34) translates into

$$E_{\mathrm{ADM}} = \frac{1}{4\pi} \lim_{\mathcal{S}_{(t,r\to\infty)}} \oint_{\mathcal{S}_t} s^i \mathcal{D}_i \Phi \sqrt{q}\, d^2x = \frac{1}{4\pi} \int_{\Sigma_t} \Delta\Phi \sqrt{f}\, d^3x \quad . \tag{36}$$

where in the second step we have assumed that Σ_t has the topology of $\mathbb{R}^3$ and have applied the Gauss-Ostrogradsky theorem (with $\Delta = \mathcal{D}_i \mathcal{D}^i$). Using now that Φ is a solution of the Poisson equation, we can write

$$E_{\mathrm{ADM}} = \int_{\Sigma_t} \rho \sqrt{f}\, d^3x \quad , \tag{37}$$

and we recover the standard expression for the total mass of the system at the Newtonian limit (as it will be seen in next section, in a non-boosted slice like this, mass is directly given by the energy expression).

ADM 4-momentum. ADM mass

ADM linear momentum

Linear momentum corresponds to the conserved quantity associated with an invariance under spatial translations. In the asymptotically flat case, the ADM momentum is associated with space translations preserving the fall-off conditions (19) expressed in terms of the Cartesian-type coordinates (x^i).

Given one of such coordinate systems, the three vectors $(\partial_i)_{i\in\{1,2,3\}}$ represent asymptotic symmetries generating asymptotic spatial translations that correspond to a choice $N = 0$ and $\beta^i_{(\partial_j)} = \delta^i_j$ in the evolution vector t^μ. Substituting these values for the lapse and shift in the Hamiltonian expression evaluated on solutions (28), we obtain the conserved quantity under the infinitesimal translation ∂_i:

$$P_i := \frac{1}{8\pi} \lim_{\mathcal{S}_{(t,r\to\infty)}} \oint_{\mathcal{S}_t} \left(K_{ik} - K\gamma_{ik}\right) s^k \sqrt{q}\, d^2x \quad . \tag{38}$$

Remark 4. Asymptotic fall-off conditions (19) guarantee the finiteness of expression (38) for P_i.

The *ADM momentum* associated with the hypersurface Σ_t is defined as the linear form $(P_i) = (P_1, P_2, P_3)$. Its components actually transform as those of a linear form under changes of Cartesian coordinates $(x^i) \to (x'^i)$ which asymptotically correspond to a rotation and/or a translation. For discussing transformations under the full Poincaré group, we must introduce the ADM 4-momentum defined as

$$(P^{\mathrm{ADM}}_\mu) := (-E_{\mathrm{ADM}}, P_1, P_2, P_3) \quad . \tag{39}$$

Under a coordinate change $(x^\mu) = (t, x^i) \to (x'^\mu) = (t', x'^i)$ which preserves the asymptotic conditions (19), i.e. any coordinate change of the form (20), components P^{ADM}_μ transform under the vector linear representation of the Lorentz group

$$P'^{\mathrm{ADM}}_\mu = (\Lambda^{-1})^\nu{}_\mu \, P^{\mathrm{ADM}}_\nu \quad , \tag{40}$$

as first shown by Arnowitt, Deser and Misner in [22]. Therefore (P^{ADM}_μ) can be seen as a linear form acting on vectors at spatial infinity i^0 and is called the *ADM 4-momentum*.

ADM mass

Having introduced the ADM 4-momentum, its Minkowskian length provides a notion of mass. The ADM mass is therefore defined as:

$$M^2_{\mathrm{ADM}} := -P^{\mathrm{ADM}}_\mu P^\mu_{\mathrm{ADM}} \quad , \quad M_{\mathrm{ADM}} = \sqrt{E^2_{\mathrm{ADM}} - P_i P^i} \quad . \tag{41}$$

Remark 5. In the literature, references are found where the term *ADM mass* actually refers to this length of the ADM 4-momentum and other references where it refers to its time component, that we have named here as the ADM energy. These differences somehow reflect traditional usages in Special Relativity where the term *mass* is sometimes reserved to refer to the Poincaré invariant (rest-mass) quantity, and in other occasions is used to denote the boost-dependent time component of the energy-momentum.

The ADM mass is a time independent quantity. Time evolution is generated by the Hamiltonian in expression (26). The time variation of a given quantity F defined on the phase space is expressed as the sum of its Poisson bracket with the Hamiltonian (accounting for the implicit time dependence through the time variation of the phase space variables) and the partial derivative of F with respect to time. Since in expression (26) there is no explicit time dependence, constancy of the ADM mass follows:

$$\frac{d}{dt} M_{\mathrm{ADM}} = 0 \quad . \tag{42}$$

As a consequence of this, the ADM mass is a property of the whole (asymptotically flat) spacetime.

Remark 6 (Relation between ADM and Komar masses). Komar mass is defined only in the presence of a time-like Killing vector k^μ. However, in the asymptotically flat case we can discuss the relation between the ADM energy and the Komar mass associated with an asymptotic inertial observer. Though the relation is not straightforward from explicit expressions (18) and (30), it can be shown [24, 25] that, for any foliation $\{\Sigma_t\}$ such that the associated unit normal n^μ coincides with the time-like Killing vector k^μ at infinity (i.e. $N \to 1$ and $\beta^i \to 0$) we have

$$M_{\mathrm{K}} = M_{\mathrm{ADM}} \quad . \tag{43}$$

As a practical application, this relation has been used as a quasi-equilibrium condition in the construction of initial data for compact objects in quasi-circular orbits (e.g. [26]).

Positivity of the ADM mass.

One of the most important results in General Relativity is the proof of the positivity of the ADM mass under appropriate energy conditions for the matter energy-momentum tensor. This is important first on conceptual grounds, since it represents a crucial test of the internal consistency of the theory. A violation of this result would evidence an essential instability of the solutions of the theory. It is also relevant on a practical level, since this theorem (and/or related results) pervades the everyday practice of (mathematical) relativists.

The theorem states that, under the *dominant energy condition*, the ADM mass cannot be negative, i.e. $M_{\mathrm{ADM}} \geq 0$. Moreover, $M_{\mathrm{ADM}} = 0$ if and only if the spacetime is Minkowski. This result was first obtained by Schoen and Yau [27, 28] and then recovered using spinorial techniques by Witten [29].

The dominant energy condition essentially states that the local energy measured by a causal observer is always positive, and that the flow of energy associated with this observer cannot travel faster than light. More precisely, given a future-directed time-like vector v^μ, this conditions states that the vector $-T^\mu{}_\nu v^\nu$ is a future-oriented causal vector. Vector $-T^\mu{}_\nu v^\nu$ represents the energy-momentum 4-current density as seen by the observer associated with v^μ, in an analogous decomposition to that in (12). From the dominant energy condition it follows $E := T_{\mu\nu} v^\mu v^\nu \geq 0$, i.e. the local density cannot be negative (*weak energy condition*) and, more generally, $E \geq \sqrt{P^i P_i}$.

ADM angular momentum

Pushing forward the strategy followed for defining the ADM mass and linear momentum, one would attempt to introduce total angular momentum as the conserved quantity associated with rotations at spatial infinity. More specifically, in the Cartesian-type coordinates used for characterising asymptotically Euclidean slices (19), infinitesimal generators $(\phi_i)_{i\in\{1,2,3\}}$ for rotations around the three spatial axes are

$$\phi_x = -z\partial_y + y\partial_z \quad , \quad \phi_y = -x\partial_z + z\partial_x \quad , \quad \phi_z = -y\partial_x + x\partial_y \quad , \tag{44}$$

which constitute Killing symmetries of the asymptotically flat metric. When using the associated *lapse* functions and *shift* vectors in the Hamiltonian expression (28), namely $N = 0$ and $\beta^i_{(\phi_j)} = (\phi_j)^i$, the following three quantities result

$$J_i := \frac{1}{8\pi} \lim_{\mathcal{S}_{(t, r\to\infty)}} \oint_{\mathcal{S}_t} (K_{jk} - K\gamma_{jk})\, (\phi_i)^j\, s^k \sqrt{q}\, d^2x, \qquad i \in \{1,2,3\} \quad . \tag{45}$$

However, the interpretation of J_i as the components of an angular momentum faces two problems:

1. First, asymptotic fall-off conditions (19) are not sufficient to guarantee the finiteness of expressions (45).
2. Second, in contrast with the linear momentum case, the quantity $(J_i) = (J_1, J_2, J_3)$ does not transform appropriately under transformations (20) preserving (19). This can be tracked to the existence of supertranslations. In particular, the so-defined angular-momentum vector (J_i) depends non-covariantly on the particular coordinates we have chosen.

For this reason, it is not appropriate to refer to an ADM angular momentum in the same sense that we use the ADM term for mass and linear momentum quantities. A manner of removing the above-commented ambiguities consists in identifying an appropriate subclass of Cartesian-type coordinates where, first, the J_i components are finite and, second, they transform as the components of a linear form. Among the different strategies proposed in the literature, we comment here on the one proposed by York [30] in terms of further conditions on the conformal metric $\tilde{\gamma}_{ij}$ introduced in (32) and the trace of the extrinsic curvature K. Namely

$$\frac{\partial \tilde{\gamma}_{ij}}{\partial x^j} = O(r^{-3}) \quad , \quad K = O(r^{-3}) \quad , \tag{46}$$

representing *asymptotic gauge conditions*. That is, they actually impose restrictions on the choice of coordinates but not on the geometric properties of spacetime at spatial infinity. First condition in (46) is known as the *quasi-isotropic gauge*, whereas the second one is referred to as the *asymptotic maximal gauge*.

Remark 7. Note that, in contrast with the total angular momentum defined at spatial infinity, no ambiguity shows up in the definition of the Komar angular momentum in Eq. (17).

3.4 Bondi energy and linear momentum

We could introduce Bondi (or Trautman-Bondi-Sachs) energy at null infinity following the same approach we have employed for the ADM energy, i.e. by taking the appropriate limit of (28) with $N = 1$ and $\beta^i = 0$. In the present case, instead of keeping t constant and making $r \to \infty$ as we did in (29), we should introduce retarded and advanced time coordinates (respectively, $u = t - r$ and $v = t + r$) and consider the limit

$$E_{\mathrm{BS}} := -\frac{1}{8\pi} \lim_{\mathcal{S}_{(u,v\to\infty)}} \oint_{\mathcal{S}_u} (H - H_0)\sqrt{q}\, d^2x \quad . \tag{47}$$

The full discussion of this limit would require the introduction of the appropriate fall-off conditions for the metric components in a special class of coordinate system adapted to null infinity (Bondi coordinates). This is in the spirit of the original discussion on the energy flux of gravitational radiation from an isolated system by Bondi, Van der Burg and Metzner [31], and Sachs [32]. However, aiming at providing some flavour of the geometric approach to asymptotic flatness, we rather outline here a discussion in the setting of the conformal compactification approach.

Null infinity

A smooth spacetime $(\mathcal{M}, g)$ is *asymptotically simple* [33] (see e.g. also [21]) if there exists another (unphysical) smooth Lorentz manifold $(\tilde{\mathcal{M}}, \tilde{g})$ such that:

i) $\mathcal{M}$ is an open submanifold of $\tilde{\mathcal{M}}$ with (smooth) boundary $\partial\tilde{\mathcal{M}}$.
ii) There is a smooth scalar field Ω on $\tilde{\mathcal{M}}$, such that: $\Omega > 0$, $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ on $\mathcal{M}$, and $\Omega = 0$, $\partial_\mu \Omega \neq 0$ on $\partial\tilde{\mathcal{M}}$.
iii) Every null geodesic in $\mathcal{M}$ begins and ends on $\partial\tilde{\mathcal{M}}$.

An asymptotically simple spacetime is *asymptotically flat* (at null infinity) if, in addition, Einstein vacuum equation is satisfied in a neighbourhood of $\partial\tilde{\mathcal{M}}$ (or the energy-momentum decreases sufficiently fast in the matter case). In this case the boundary $\partial\tilde{\mathcal{M}}$ consists, at least, of a null hypersurface with two connected components $\mathscr{I} = \mathscr{I}^- \cup \mathscr{I}^+$, each one with topology $S^2 \times \mathbb{R}$ (note that in Minkowski $\partial\tilde{\mathcal{M}}$ also contains the points $i^0, i^\pm$). Boundaries $\mathscr{I}^-$ and $\mathscr{I}^+$ represent past and future null infinity, respectively.

Symmetries at null infinity

In order to characterise a vector ξ^μ in $\mathcal{M}$ as an infinitesimal asymptotic symmetry at (future) null infinity $\mathscr{I}^+$, we must assess the vanishing of $\mathcal{L}_\xi g_{\mu\nu}$ as one gets to $\mathscr{I}^+$. For this, we require first that ξ^μ, considered as a vector field in the unphysical spacetime (i.e. under the immersion of $\mathcal{M}$ into $\tilde{\mathcal{M}}$), can be smoothly extended to $\mathscr{I}^+$. Then ξ^μ is characterised as an asymptotic symmetry by demanding that $\Omega^2\mathcal{L}_\xi g_{\mu\nu}$ can also be smoothly extended to $\mathscr{I}^+$ and vanishes there, that is

$$\left(\tilde{\nabla}_\mu\xi_\nu + \tilde{\nabla}_\nu\xi_\mu - 2\Omega^{-1}\xi^\rho\tilde{\nabla}_\rho\Omega\,\tilde{g}_{\mu\nu}\right)\Big|_{\mathscr{I}^+} = 0 \quad . \tag{48}$$

Two vector fields ξ^μ and ξ'^μ are considered to generate the same infinitesimal asymptotic symmetry if their extensions to $\mathscr{I}^+$ coincide. The equivalence class of such vector fields, that we will still denote by ξ^μ, generates the asymptotic symmetry group at $\mathscr{I}^+$. This is known as the Bondi-Metzner-Sachs (BMS) group and is universal in the sense that it is same for every asymptotically flat spacetime. The BMS group is infinite-dimensional, as it was the case of the Spi group at spatial infinity. It does not only contain the Poincaré group, but actually is a semi-direct product of the Lorentz group and the infinite-dimensional group of *angle dependent* supertranslations (see details in e.g. [4]). The key point for the present discussion is that it possesses a unique *canonical* set of asymptotic 4-translations characterised as the only 4-parameter subgroup of the supertranslations that is a *normal* subgroup of the BMS group. This leads us to the Bondi-Sachs 4-momentum.

Bondi-Sachs 4-momentum

As mentioned above, the original introduction of the Bondi energy was based in the identification of certain expansion coefficients in the line element of radiative spacetimes in adapted (Bondi) coordinates [31]. A Hamiltonian analysis, counterpart of the approach adopted in section 3.3 for introducing the ADM mass, can be found in [34]. Here we rather follow a construction based on the Komar mass expression. Though Eq. (13) only defines a conserved quantity for a Killing vector k^μ, the vector fields ξ^μ_a ($a \in \{0,1,2,3\}$) corresponding to the 4-translations at $\mathscr{I}^+$ get closer to an infinitesimal symmetry as one approaches $\mathscr{I}^+$. Therefore, one can expect that a Komar-like expression makes sense for a given cross-section S_u of $\mathscr{I}^+$. This is indeed the case and the evaluation of the integral does not depend on how we get to S_u. However, the integral does depend on the representative ξ^μ in the class of vectors corresponding to the asymptotic symmetry. This is cured by imposing a divergence-free condition on ξ^μ [35]. Bondi-Sachs 4-momentum at $S_u \subset \mathscr{I}^+$ is then defined as

$$P_a^{\mathrm{BS}} := -\frac{1}{8\pi}\lim_{(\mathcal{S}\to\mathcal{S}_u)}\oint_{\mathcal{S}}\nabla^\mu\xi^\nu_a dS_{\mu\nu} \quad , \quad \nabla_\mu\xi^\mu_a = 0 \quad . \tag{49}$$

Alternatively, ambiguities in the Komar integral can be solved by dropping the condition on the divergence and adding a term $\alpha \nabla_\mu \xi^\mu_a$ to the surface integral. When $\alpha = 1$ the resulting integral is called the *linkage* [36]. The discussion of Bondi-Sachs angular momentum is more delicate. We refer the reader to the discussion in section 3.2.4 of [1].

Bondi energy and positivity of gravitational radiation energy

Bondi energy E_{BS} (the zero component of the Bondi-Sachs 4-momentum) is a decreasing function of the retarded time. More concretely, Bondi energy satisfies a *loss equation*

$$\frac{dE_{\mathrm{BS}}}{du} = -\int_{\mathcal{S}_u} F\sqrt{q}\, d^2x \quad , \tag{50}$$

where $F \geq 0$ can be expressed in terms of the squares of the so-called *news* functions. In [25] it is shown that, if the *news* tensor satisfies the appropriate conditions, then Bondi mass coincides initially with the ADM mass. Bondi energy is interpreted as the remaining of the ADM energy in the process of energy extraction by gravitational radiation. As for the ADM mass, a positivity result holds for the Bondi mass [37, 38]. These properties constitute the underlying conceptual/structural justification of our understanding of energy radiation by gravitational waves: gravitational radiation carries positive energy away from isolated radiating systems, and the total radiated energy cannot be bigger than the original total ADM energy.

4 Notions of mass for bounded regions: quasi-local masses

As commented in section 1.2, the convenience of associating energy-momentum with the gravitational field in given regions of the spacetime is manifest in very different contexts of gravity physics. More specifically, mathematical and numerical General Relativity or approaches to Quantum Gravity provide examples where we need to associate such an energy-momentum with a *finite* region of spacetime. This can be either motivated by the need to define appropriate physical/astrophysical quantities, or by the convenience of finding quasi-local quantities with certain desirable mathematical properties (e.g. positivity, monotonicity...) in the study of a specific problem.

There exist many different approaches for introducing quasi-local prescriptions for the mass and angular momentum. Some of them can be seen as *quasi-localizations* of successful notions for the physical parameters of the total system, such as the ADM mass, whereas other attempts constitute genuine *ab initio* methodological constructions, mainly based on Lagrangian or Hamiltonian approaches. An important drawback of most of them in the context of the present article is that, typically, they involve *constructions* that are difficult to capture in short mathematical definitions without losing the underlying physical/geometrical insights. An excellent and comprehensive review is reference [1] by Szabados.

Ingredients in the quasi-local constructions.

First, the relevant bounded spacetime domain must be identified. Typically, these are compact space-like domains D with a boundary given by a closed 2-surface $\mathcal{S}$. Explicit expressions, such as relevant associated integrals, are formulated in terms of either the (3-dimensional) domain D itself or on its boundary $\mathcal{S}$. In particular, conserved-current strategies permit to pass from the 3-volume integral to a conserved-charge-like 2-surface integral. In other cases, 2-surface integrals are a consequence of the need of including boundary terms for having a correct variational formulation (as it was the case in the Hamiltonian formulation of section 3.3).

We have already presented an example of quasi-local quantity in section 2, namely the Komar quantities. Since symmetries will be absent in the generic case, an important ingredient in most quasi-local constructions is the prescription of some vector field that plays the role that infinitesimal symmetries had played in case of being present. In connection with this, one usually needs to introduce some background structure that can be interpreted as a kind of gauge choice.

Finally, different *plausibility* criteria for the assessment of the proposed quasi-local expressions (e.g. positivity, monotonicity, recovery of known limits...) need to be considered (see [1]).

4.1 Some relevant quasi-local masses

Round spheres. Misner-Sharp energy

In some special situations, as it is the case of isolated systems above and some exact solutions, there is agreement on the form of the gravitational field energy-momentum. Another interesting case is that of spherically symmetric spacetimes, where the rotation group $SO(3)$ acts transitively as an isometry. Orbits under this rotation group are *round* spheres $\mathcal{S}$. Then, using the areal radius r_A as a coordinate ($4\pi r_A^2 = A$), an appropriate notion of mass/energy was given by Misner and Sharp [39]

$$E(\mathcal{S}) := \frac{1}{8} r_A^3 R_{\mu\nu\rho\sigma} {}^2\epsilon^{\mu\nu} \, {}^2\epsilon^{\rho\sigma} \quad , \tag{51}$$

where ${}^2\epsilon_{\mu\nu} = n^\rho s^\sigma {}^4\epsilon_{\rho\sigma\mu\nu}$ (cf. section 1.3) is the volume element on $\mathcal{S}$. This expression is related to the so-called Kodama vector K^μ, that can be defined in spherically symmetric spacetimes and such that $\nabla_\mu(G^{\mu\nu}K_\nu) = 0$. The current $S^\mu = G^{\mu\nu}K_\nu$ is thus conserved and, taking D as a solid ball of radius r_A, the flux of S^μ through the round boundary ∂D actually equals the change in time of the mass expresion (51). Misner-Sharp proposal is considered as the *standard form* of quasi-local mass for round spheres.

Brown-York energy

The rationale of the approach in Ref. [40] to quasi-local energy strongly relies on the well-posedness of a variational problem for the gravitational action. The adopted variational formulation is essentially the one outlined in section 3.3 (where the discussion was in fact based in the treatment in [3] adapted from [40]). However, if the main interest is placed in the expressions of quasi-local parameters and not in the details of the symplectic geometry of the system phase space, a full Hamiltonian analysis does not need to be undertaken and one can rather follow a Hamilton-Jacobi one. The latter starts from action (21) defined on the spacetime domain $\mathcal{V}$. We recall that the boundary $\partial\mathcal{V}$ is given by two space-like hypersurfaces Σ_1 and Σ_2 and a time-like tube $\mathcal{B}$, such that the 2-spheres $\mathcal{S}_i$ are the intersections between Σ_i and $\mathcal{B}$. The metric and extrinsic curvatures on Σ_i are given by $\gamma_{\mu\nu}$ and $K^{\mu\nu}$, whereas those on $\mathcal{B}$ are denoted by $\chi_{\mu\nu}$ and $P^{\mu\nu}$. A Hamilton-Jacobi principal function can then be introduced by evaluating the action S on classical trajectories. An arbitrary function S^0 of the data on the boundaries can be added to S [it is the responsible of the *reference* terms with subindex 0 in expression (21)]. The principal function is given by $S_{\mathrm{Cl}} := \left(S - S^0\right)|_{\mathrm{Cl}}$ and Hamilton-Jacobi equations are obtained from its variation with respect to the data at the final slice Σ_2. One of the Hamilton-Jacobi equations leads to the definition of a *surface stress-energy-momentum* tensor as

$$\tau^{\mu\nu} := \frac{-2}{\sqrt{-\chi}}\frac{\delta S_{\mathrm{Cl}}}{\delta\chi_{\mu\nu}} = \frac{1}{8\pi}\left\{(P\chi^{\mu\nu} - P^{\mu\nu}) - (P_0\chi^{\mu\nu} - P_0^{\mu\nu})\right\} \quad . \tag{52}$$

This tensor satisfies a conservation-like equation with a source given in terms of the matter energy-momentum tensor $T^{\mu\nu}$. This motivates the definition of the *charge* $Q_{\mathcal{S}}(\xi^\mu)$ associated with a vector ξ^μ as

$$Q_{\mathcal{S}}(\xi^\mu) := \oint_{\mathcal{S}} \xi_\rho \tau^{\rho\nu} n_\nu \sqrt{q}\, d^2x \quad , \tag{53}$$

whose change along the tube $\mathcal{B}$ is given by a matter flux. This expression is analogous to (1) in the matter case (here $\mathcal{S} \subset \mathcal{B}$ and n^μ is the time-like unit normal to $\mathcal{S}$ and tangent to $\mathcal{B}$).

Using the 2+1 decomposition induced by a 3+1 space-like slicing $\{\Sigma_t\}$, we can decompose the tensor $\tau^{\mu\nu}$ as we did for the matter energy-momentum tensor $T^{\mu\nu}$ in Eq. (12). Writing explicitly the time-like components, it results

$$\begin{aligned} \varepsilon &:= n_\mu n_\nu \tau^{\mu\nu} = -\frac{1}{8\pi}(H - H^0) \quad , \\ j_\mu &:= -q_{\mu\nu} n_\rho \tau^{\nu\rho} = \frac{1}{8\pi} q_{\mu\nu} s_\rho \left(K\gamma^{\nu\rho} - K^{\nu\rho}\right)\Big|_0^{\mathrm{Cl}} \quad . \end{aligned} \tag{54}$$

Expressing the vector ξ^μ in the 3+1 decomposition $\xi^\mu = \xi n^\mu + \xi^\mu_\perp$ and considering a 2-surface $\mathcal{S}$ lying in a slice of $\{\Sigma_t\}$, we have

$$Q_{\mathcal{S}}(\xi^\mu) = \oint_{\mathcal{S}} \xi_\rho \tau^{\rho\nu} n_\nu \sqrt{q}\, d^2x = \oint_{\mathcal{S}} \left(\xi\varepsilon - \xi_\perp^\rho j_\rho\right) \sqrt{q}\, d^2x \quad . \tag{55}$$

The Brown-York energy is then [cf. with the ADM mass expression (41)]

$$E_{\rm BY}(\mathcal{S}, n^\mu) := Q_{\mathcal{S}}(n^\mu) = -\frac{1}{8\pi}\oint_{\mathcal{S}} (H - H_0)\sqrt{q}\, d^2x \quad . \tag{56}$$

Note that this expression explicitly depends on the manner in which $\mathcal{S}$ is inserted in some space-like 3-slice. In this sense, it corresponds to an energy (depending on a boost) rather than a mass.

Kijowski, Epp, Liu-Yau and Kijowski-Liu-Yau expressions

We briefly comment on some expressions that can be related to the Brown-York energy. Studying more general boundary conditions than the ones in [40], Kijowski proposed the following quasi-local expression for the mass [41]

$$E_{\rm Kij} := \frac{1}{16\pi}\oint_{\mathcal{S}} \frac{(H_0)^2 - (H^2 - L^2)}{H_0}\sqrt{q}\, d^2x \quad , \tag{57}$$

where $H = H_{\mu\nu}q^{\mu\nu}$ and $L = L_{\mu\nu}q^{\mu\nu}$ are the traces of the extrinsic curvatures of $\mathcal{S}$ with respect to unit orthogonal space-like s^μ and time-like n^μ vectors, i.e. $n^\mu s_\mu = 0$ (cf. notation in section 1.3). Apart from the choice of the background terms H_0, this expression only depends on $\mathcal{S}$, and not in the manner of embedding it into some space-like hypersurface. Using a different set of boundary conditions, another quasi-local quantity was introduced by Kijowski (referred to as a *free energy*). The same quantity was later derived by Liu and Yau, using a different approach [42]. We will refer to the resulting quasi-local energy as the Kijowski-Liu-Yau energy, having the form

$$E_{\rm KLY} := \frac{1}{8\pi}\oint_{\mathcal{S}} \left(H^0 - \sqrt{H^2 - L^2}\right) \quad . \tag{58}$$

On the other hand, aiming at removing the dependence of Brown-York energy on the space-like hypersurface, Epp [43] proposed the following boost-invariant expression

$$E_{\rm E} := \frac{1}{8\pi}\oint_{\mathcal{S}} \left(\sqrt{(H^0)^2 - (L^0)^2} - \sqrt{H^2 - L^2}\right) \quad . \tag{59}$$

Note that Brown-York energy can be seen as a gravitational field version of the quasi-local matter energy (6), whereas Epp's expression rather corresponds to the matter mass (7). For further recent work along this approach to quasi-local mass, see [44, 45].

Hawking, Geroch and Hayward energies

Hawking energy.

Given a topological sphere $\mathcal{S}$, its Hawking energy is defined as [46]

$$E_{\mathrm{H}}(\mathcal{S}) = \sqrt{\frac{A(\mathcal{S})}{16\pi}} \left(1 + \frac{1}{8\pi}\oint_{\mathcal{S}} \theta_+\theta_-\right) \sqrt{q}\, d^2x \quad , \tag{60}$$

where $\theta_+ = q^{\mu\nu}\Theta^{(\ell)}_{\mu\nu}$ and $\theta_- = q^{\mu\nu}\Theta^{(k)}_{\mu\nu}$ are the expansions associated with outgoing and ingoing null normals (cf. notation in section 1.3). It can be motivated by understanding the mass surrounded by the 2-sphere $\mathcal{S}$ as an estimate of the bending of ingoing at outgoing light rays from $\mathcal{S}$. An average, boost-independent measure of this convergence-divergence behaviour of light rays is given by $\oint_{\mathcal{S}} \theta_+\theta_- {}^2\epsilon$. Then, from the Ansatz $A + B\oint_{\mathcal{S}} \theta_+\theta_- {}^2\epsilon$, the constants A and B are fixed from round spheres in Minkowski and from the horizon sections in Schwarzschild spacetime.

Hawking energy depends only on the surface $\mathcal{S}$ and not on any particular embedding of it in a space-like hypersurface. In the spherically symmetric case it recovers the standard Misner-Sharp energy (51). For apparent horizons, or more generally for marginally trapped surfaces, it reduces to the *irreducible mass* accounting for the energy that cannot be extracted from a black hole by a Penrose process and that is given entirely in terms of the area. Hawking energy does not satisfy a positivity criterion, since it can be negative even in Minkowski spacetime. However, for large spheres approaching null infinity, $E_{\mathrm{H}}(\mathcal{S})$ recovers Bondi-Sachs energy, whereas for spheres approaching spatial infinity it tends to the ADM energy. Though it is not monotonic in the generic case, monotonicity can be proved for sequences of spheres obtained from appropriate geometric flows. This has a direct interest for the extension of Huisken & Ilmanen proof [47] of the Riemaniann Penrose inequality to the general case.

Geroch energy.

For a surface $\mathcal{S}$ embedded in a space-like hypersurface Σ, Geroch energy [48] is defined as

$$E_{\mathrm{G}}(\mathcal{S}) := \frac{1}{16\pi}\sqrt{\frac{A(\mathcal{S})}{16\pi}} \oint_{\mathcal{S}} \left(2\,{}^2R - H^2\right) \sqrt{q}\, d^2x \quad , \tag{61}$$

where H is again the trace of the extrinsic curvature of $\mathcal{S}$ inside Σ. Geroch energy is never larger than Hawking energy, but it can be proved that it also tends to the ADM mass for spheres approaching spatial infinity.

The relevance of Geroch energy lies on its key role in the first proof of the Riemaniann Penrose, by Huisken & Ilmanen [47] (see also section 5.1). In particular, use is made of the monotonicity properties of E_{G} under an *inverse mean curvature* flow in Σ.

Hayward energy.

Some generalizations of Hawking energy exist. A vanishing expression for flat spacetimes can be obtained by considering the modified expression

$$E'_{\mathrm{H}}(\mathcal{S}) = \sqrt{\frac{A(\mathcal{S})}{16\pi}} \left(1 + \frac{1}{8\pi}\oint_{\mathcal{S}} \theta_+\theta_- - \frac{1}{2}\sigma^+_{\mu\nu}\sigma^{\mu\nu}_-\right)\sqrt{q}\, d^2x \quad , \tag{62}$$

where the shears $\sigma^+_{\mu\nu}$ and $\sigma^-_{\mu\nu}$ are the traceless parts of $\Theta^{(\ell)}_{\mu\nu}$ and $\Theta^{(k)}_{\mu\nu}$, respectively. E'_{H} still asymptotes to the ADM energy at spatial infinity, but does not recover Bondi-Sachs energy at null infinity (but rather Newman-Unti one; cf. references in [1]). Related to this modified Hawking energy, Hayward has proposed [49] another quasi-local energy expression by taking into account the anholonomicity form Ω_μ, one of the normal fundamental 1-forms introduced in section 1.3

$$E_{\mathrm{Hay}}(\mathcal{S}) = \sqrt{\frac{A(\mathcal{S})}{16\pi}} \left(1 + \frac{1}{8\pi}\oint_{\mathcal{S}} \theta_+\theta_- - \frac{1}{2}\sigma^+_{\mu\nu}\sigma^{\mu\nu}_- - 2\Omega_\mu\Omega^\mu\right)\sqrt{q}\, d^2x \quad . \tag{63}$$

Though the divergence-free part of Ω_μ can be related to angular momentum (see below), this 1-form is a gauge dependent object changing by a total differential under a boost transformation. Therefore, some natural gauge for fixing the boost freedom is needed.

Bartnik mass

Bartnik quasi-local mass is an example of *quasi-localization* of a global quantity, in particular the ADM mass. In very rough terms, the idea in Bartnik's construction consists in defining the mass of a compact space-like 3-domain D as the ADM mass of that asymptotically Euclidean slice Σ that contains D without any other *source* of energy. The strategy to address this absence of further energy is to consider all plausible extensions of D into Euclidean slices, calculate the ADM mass for all them, and then consider the infimum of this set of ADM masses. In more precise terms, let us consider a compact, connected 3-hypersurface D in spacetime, with boundary $\mathcal{S}$ and induced metric γ_{ij}. Bartnik's construction actually focuses on time-symmetric $K_{ij} = 0$ domains D. Let us also assume that a dominant energy condition (though the original formulation in [50] makes use of a weak-energy-constraint condition) is satisfied. In a time-symmetric context this amounts to the positivity of the Ricci scalar, ${}^3R \geq 0$. One can then define $\mathcal{P}(D)$ as the set of Euclidean time-symmetric initial data sets (Σ, γ_{ij}) satisfying the dominant energy condition, with a single asymptotic end, finite ADM mass $M_{\mathrm{ADM}}(\Sigma)$, not containing horizons (minimal surfaces in this context) and extending D through its boundary $\mathcal{S}$. Then, Bartnik's mass [50] is defined as

$$M_{\mathrm{B}}(D) := \inf\left\{M_{\mathrm{ADM}}(\Sigma),\ \text{such that}\ \Sigma \in \mathcal{P}(D)\right\} \quad . \tag{64}$$

The *no-horizon* condition is needed to avoid extensions (Σ, γ_{ij}) with arbitrarily small ADM mass. There is also a spacetime version of Bartnik's construction, not relying on an initial data set on D but only on the geometry of 2-surfaces $\mathcal{S}$. Let us define $\mathcal{P}(\mathcal{S})$ as the set of globally hyperbolic spacetimes $(\mathcal{M}, g_{\mu\nu})$ satisfying the dominant energy condition, admitting an asymptotically Euclidean Cauchy hypersurface Σ with finite ADM mass, not presenting an event horizon and such that $\mathcal{S}$ is embedded (i.e. both its intrinsic and extrinsic geometry) in $(\mathcal{M}, g_{\mu\nu})$. Then, one defines

$$M_{\mathrm{B}}(\mathcal{S}) := \inf \{M_{\mathrm{ADM}}(\mathcal{M}), \text{ such that } \mathcal{M} \in \mathcal{P}(\mathcal{S})\} \quad . \tag{65}$$

The comparison between $M_{\mathrm{B}}(D)$ and $M_{\mathrm{B}}(\partial D)$ is not straightforward, due to issues regarding the horizon characterisation. From the positivity of the ADM mass it follows the non-negativity of the Bartnik mass $M_{\mathrm{B}}(D)$. In fact, $M_{\mathrm{B}}(D) = 0$ characterises D as locally flat. From the definition (64) it also follows the monotonicity of $M_{\mathrm{B}}(D)$, i.e. if $D_1 \subset D_2$ then $M_{\mathrm{B}}(D_1) \leq M_{\mathrm{B}}(D_2)$. Bartnik mass tends to the ADM mass, as domains D tend to Euclidean slices (the proof makes use of the Hawking energy introduced above). Another interesting feature, consequence of the proof of the Riemannian Penrose conjecture [47], is that Bartnik mass reduces to the *standard form* $E(\mathcal{S})$ in Eq. (51) for round spheres. However, the explicit calculation of the Bartnik mass is problematic. An approach to its practical computability is provided by Bartnik's conjecture stating that the infimum in (64) is actually a minimum realised by an element in $\mathcal{P}(D)$ characterised by its stationarity outside D. Further developments of these ideas have been proposed by Bray (cf. [51]).

4.2 Some remarks on quasi-local angular momentum

Spinorial techniques provide a natural setting for the discussion of angular momentum. This does not only apply to angular momentum, since spinorial and also twistor techniques define a framework where further quasi-local mass notions can be introduced (e.g. Penrose mass), and known results can be reformulated in particularly powerful formulations (e.g. the discussion of positive mass theorems using the Nester-Witten form). However, in this article we will not discuss these approaches and we refer the reader to the relevant sections in Ref. [1]. We will focus on certain aspects of quasi-local expressions for angular momentum of Komar-like type. As it was shown in section 2.2, choosing a two-sphere $\mathcal{S}$ in a 3-slice adapted to the axial symmetry ϕ^μ, a 1-form L_μ can be found such that the Komar angular momentum is expressed as

$$J(\phi^\mu) = \frac{1}{8\pi} \oint_{\mathcal{S}} L_\nu \phi^\nu \sqrt{q}\, d^2x \quad . \tag{66}$$

In particular, in the Komar expression (18) we have $L_\mu = q_\mu{}^\nu K_{\nu\rho} s^\rho$, whereas in the spatial infinity expression (45) this is modified by a term proportional to the trace K of the extrinsic curvature. The same applies for an angular momentum defined from the Brown-York charge (53) when plugging the expression for j_μ in (54) into (55), where ϕ^μ does not need to be a symmetry. The normal fundamental 1-forms Ω_μ on $\mathcal{S}$ (cf. section 1.3) provide another avenue to L_μ. In this section we assume the form (66) for the angular momentum and comment on some approaches to the determination of the (*quasi-symmetry*) axial vector ϕ^μ.

Divergence-free and quasi-Killing axial vectors

No ambiguity for ϕ^μ is present when an axial symmetry exists on $\mathcal{S}$: ϕ^μ is taken as the corresponding Killing vector. In the absence of such a symmetry, we must address two issues. First, expression (66) depends on the space-like 3-slice in which $\mathcal{S}$ is embedded. This follows from the modification of the 1-form L_μ by a total differential under a boost transformation: $L_\mu \to L_\mu + {}^2D_\mu f$ (cf. boost/normalization transformation of $\Omega^{(\ell)}_\mu$ in section 1.3). Angular momentum can be associated with $\mathcal{S}$, independently of any hypersurface Σ, by demanding the axial vector to be divergence-free: ${}^2D_\mu\phi^\mu = 0$. Then, the boost-induced modification vanishes under integration. Second, the physical meaning of $J(\phi^\mu)$ is unclear if references to a symmetry notion are completely dropped. In this sense, different approaches exist aiming at defining appropriate *quasi-Killing* notions. We simply mention here some recent works along these lines. In the context of isolated horizons (see next subsection) a prescription for the determination of a quasi-Killing axial vector on black hole horizons has been proposed in [52], though the divergence-free character is not guaranteed. Ref. [53] presents an approach for finding an approximate Killing vector by means of a minimization variational prescription that respects the divergence-free character of ϕ^μ. In the context of dynamical or trapping horizons [54, 55, 56], a unique divergence-free vector ϕ^μ can be chosen such that it is preserved by the unique slicing of the (space-like) horizon worldtube by marginally outer trapped surfaces [57]. Also in the context of dynamical horizons, a proposal for ϕ^μ has been made in [58] relying on a conformal decomposition of the metric $q_{\mu\nu}$ on $\mathcal{S}$. See Ref. [59] for a discussion of the divergence-free character of vector fields associated with quasi-local observables on $\mathcal{S}$.

Equation (66) only provides the expression for the component of the angular momentum vector that is associated with the vector ϕ^μ. If we are interested in determining the total angular momentum vector, a sensible prescription for the other two components is needed. This is an important practical issue in numerical simulations (see e.g. [60]).

4.3 A study case: quasi-local mass of black hole isolated horizons

The need of introducing some additional structure has been discussed above in different settings (e.g. symmetries for Komar quantities and asymptotic flatness for ADM and Bondi masses). We illustrate now this issue in a quasi-local context related to equilibrium black hole horizons.

A brief review of isolated horizons

The isolated horizon framework introduced by Ashtekar and collaborators [56] provides a quasi-local setting for characterising black hole horizons in quasi-equilibrium inside an otherwise dynamical spacetime. It presents a hierarchical structure with different quasi-equilibrium levels. The minimal notion of quasi-equilibrium is provided by the so-called *non-expanding horizons* (NEH). Given a Lorentzian manifold, a NEH is a hypersurface $\mathcal{H}$ such that:

i) $\mathcal{H}$ is a null hypersurface of topology $S^2 \times \mathbb{R}$ that is sliced by marginally (outer) trapped surfaces, i.e. the expansion of the null congruence associated with the null generator ℓ^μ vanishes on $\mathcal{H}$: $\theta^{(\ell)} = q^{\mu\nu}\Theta^{(\ell)}_{\mu\nu} = 0$.
ii) Einstein equation is satisfied on $\mathcal{H}$.
iii) The vector $-T^\mu{}_\nu \ell^\nu$ is future directed.

The geometry of a NEH is characterised by the pair $(q_{\mu\nu}, \hat{\nabla}_\mu)$, where $q_{\mu\nu}$ is the induced null metric on $\mathcal{H}$ and $\hat{\nabla}_\mu$ is the unique connection (not a Levi-Civita one) induced from the ambient spacetime connection. $\hat{\nabla}_\mu$ characterises the *extrinsic geometry* of the NEH. A certain combination of components in $\hat{\nabla}_\mu$ can be put together to define an intrinsic object on $\mathcal{H}$, namely the 1-form ω_μ characterised by

$$\hat{\nabla}_\mu \ell^\nu = \omega_\mu \ell^\nu \quad . \tag{67}$$

Defining a *surface gravity* as $\kappa_{(\ell)} := \ell^\mu \omega_\mu$, the acceleration expression for ℓ^μ is given by: $\hat{\nabla}_\ell \ell^\mu = \kappa_{(\ell)} \ell^\mu$. On the other hand, the projection of ω_μ on $\mathcal{S}$ recovers the fundamental normal 1-form: $\Omega^{(\ell)}_\mu = q_\mu{}^\rho \omega_\rho$. The quasi-equilibrium hierarchy is introduced by demanding the invariance of the null hypersurface geometry under the ℓ^μ (evolution) flow in a progressive manner:

1. A NEH is characterised by the *time-invariance* of the intrinsic geometry $q_{\mu\nu}$: $\mathcal{L}^{\mathcal{H}}_\ell q_{\mu\nu} = 0$.
2. A weakly isolated horizon (WIH) is a NEH, together with an equivalence class of null normals $[\ell^\mu]$, for which the 1-form ω_μ is time-invariant: $\mathcal{L}^{\mathcal{H}}_\ell \omega_\mu = 0$. This is equivalent to the (time and angular) constancy of the surface gravity: $\hat{\nabla}_\mu \kappa_{(\ell)} = 0$.
3. An isolated horizon (IH) is a WIH on which the whole extrinsic geometry is time-invariant: $[\mathcal{L}^{\mathcal{H}}_\ell, \hat{\nabla}_\mu] = 0$.

The NEH and IH quasi-equilibrium levels represent genuine restrictions on the geometry of $\mathcal{H}$ as a hypersurface in the ambient spacetime. On the contrary, a WIH structure can always be implemented on a NEH by an appropriate choice of the null normal ℓ^μ normalization. In this sense, a WIH does not represent a higher level of quasi-equilibrium than a NEH. However, from the point of view of the Hamiltonian analysis of spacetimes with a black hole in quasi-equilibrium as an inner boundary, the WIH notion proves to be crucial for the correct definition of the phase space symplectic structure and, more concretely, for the sound formulation of the quasi-local mass and angular momentum of the horizon.

An overview of the Hamiltonian analysis of isolated horizons

Conserved quantities under horizon symmetries

As in the presentation of ADM quantities in section 3.3, mass and angular momentum of isolated horizons are introduced as conserved quantities under appropriate symmetries (see [61, 62] and the outline in Appendix C of [63] for further details on the following discussion). One starts from a symmetry of the horizon structure in the Lorentzian spacetime manifold and then constructs an associated canonical transformation in the phase or solution space of the system. The conserved quantity under this canonical transformation provides the relevant physical quantity. In view of the variational problem (see below), a WIH is the relevant horizon structure to be considered in this context. A vector field W^μ preserves the WIH structure (W^μ is a WIH-symmetry) if

$$\mathcal{L}^{\mathcal{H}}_W \ell^\mu = \mathrm{const} \cdot \ell^\mu \quad , \quad \mathcal{L}^{\mathcal{H}}_W q_{\mu\nu} = 0 \quad , \quad \mathcal{L}^{\mathcal{H}}_W \omega_\mu = 0 \quad . \tag{68}$$

WIH-symmetries are of the form $W^\mu = c_W \ell^\mu + b_W S^\mu$, where c_W and b_W are constants on $\mathcal{H}$ and S^μ is a Killing vector of any spatial section $\mathcal{S}$ of $\mathcal{H}$.

Variational problem for spacetimes containing WIHs

In order to set up the Hamiltonian treatment, we need first to define a well-posed variational problem. Here we are interested in the variational problem for asymptotically flat spacetimes containing a WIH. We will furthermore demand this WIH to contain an axial Killing vector ϕ^μ. The variational problem is then set in the region contained between two asymptotically Euclidean slices Σ_- and Σ_+, spatial infinity i^0 and the part of $\mathcal{H}$ between an initial horizon slice $S_- = \mathcal{H} \cap \Sigma_-$ and a final one $S_+ = \mathcal{H} \cap \Sigma_+$. The action, as in Eq. (21), can be written [61, 62] as the sum of a bulk and a boundary term at spatial infinity, and the variation of the dynamical fields is set to vanish on the slices Σ_- and Σ_+. No boundary term associated with the inner boundary $\mathcal{H}$ is introduced. The variational problem is well-posed, in particular the Einstein equation is recovered, as long as the condition

$$\int_{\mathcal{H}} \delta\omega \wedge {}^2\epsilon = 0 \quad , \tag{69}$$

holds, where ω_μ has been introduced in (67) and ${}^2\epsilon_{\mu\nu}$ is the volume 2-form on sections $\mathcal{S}$ of $\mathcal{H}$. The crucial ingredient in the well-posedness of the problem is precisely the WIH structure. This is the additional structure needed in order to guarantee the vanishing of (69), so that the variational problem is correctly posed and quasi-local quantities can be defined.

Phase space, canonical transformations and physical quantities

The phase space is defined by the couple $(\Gamma, \mathbf{J})$ where Γ is an infinite-dimensional manifold where each point represents a solution to the Einstein equation containing a WIH, and $\mathbf{J}$ is a symplectic form (a closed 2-form) on Γ in terms of which the Poisson bracket is defined. In particular, a vector field X on Γ generates a canonical transformation if it leaves the symplectic form invariant: $\mathcal{L}^{\Gamma}_X \mathbf{J} = 0$. Using the closedness of $\mathbf{J}$ this is equivalent to the exactness of the 1-form $i_X J$, i.e. to the (local) existence of a function H_X such that $i_X \mathbf{J} = \delta H_X$ (where δ denotes the differential in Γ). In particular, the quantity H_X defined on the phase space is preserved along the flow of X. In this context, first, the symplectic form can be obtained from the action by using the *conserved symplectic current* method [64] and, second, a vector field X_W on Γ can be constructed from a WIH-symmetry W^μ on $\mathcal{H}$ (cf. [61, 62] for details). For the correct definition of a physical parameter associated with a given WIH-symmetry W^μ, we must assess if the corresponding X_W preserves the canonical form, i.e. if $i_{X_W}\mathbf{J}$ is locally exact. If this is the case, the conserved quantity is simply read from the associated explicit expression of the Hamiltonian H_{X_W}. When this scheme is applied to the axial symmetry ϕ^μ on $\mathcal{H}$, the corresponding X_ϕ turns out to be automatically an infinitesimal canonical transformation and the conserved quantity has the form

$$J_{\mathcal{H}} := X_\phi = \frac{1}{8\pi} \oint_{\mathcal{S}_t} \omega_\mu \phi^\mu \sqrt{q}\, d^2x = \frac{1}{8\pi} \oint_{\mathcal{S}_t} \Omega^{(\ell)}_\mu \phi^\mu \sqrt{q}\, d^2x \quad , \qquad (70)$$

where $\mathcal{S}_t$ is any spatial section of $\mathcal{H}$. This prescription for $J_{\mathcal{H}}$ exactly coincides with the Komar expression (18). The mass discussion is more subtle. In this case the WIH-symmetry t^μ associated with *time evolution* is chosen as an appropriate linear combination of the null normal ℓ^μ and the axial vector ϕ^μ. It is then found

$$i_{X_t}\mathbf{J} = \delta E_{\mathrm{ADM}} - \left(\frac{\kappa_{(t)}}{8\pi}\, \delta A_{\mathcal{H}} + \Omega_{(t)}\, \delta J_{\mathcal{H}} \right) \quad , \qquad (71)$$

where $\kappa_{(t)}$ and $\Omega_{(t)}$ are functions on Γ and $A_{\mathcal{H}}$ and $J_{\mathcal{H}}$ correspond, respectively, to the area of any section of $\mathcal{H}$ and to the horizon angular momentum in (70). The right-hand-side expression is (locally) exact if functions $\kappa_{(t)}$ and $\Omega_{(t)}$ depend only on $A_{\mathcal{H}}$ and $J_{\mathcal{H}}$, and satisfy: $\frac{\partial \kappa_{(t)}}{\partial J_{\mathcal{H}}} = 8\pi \frac{\partial \Omega_{(t)}}{\partial A_{\mathcal{H}}}$. A function $E^t_{\mathcal{H}}$ *only* depending on $A_{\mathcal{H}}$ and $J_{\mathcal{H}}$ then exists, such that we can write

$$\delta E^t_{\mathcal{H}} = \frac{\kappa_{(t)}(A_{\mathcal{H}}, J_{\mathcal{H}})}{8\pi}\, \delta A_{\mathcal{H}} + \Omega_{(t)}(A_{\mathcal{H}}, J_{\mathcal{H}})\, \delta J_{\mathcal{H}} \quad . \qquad (72)$$

To finally determine the quasi-local mass $M_{\mathcal{H}}$, the functional form of $E^t_{\mathcal{H}}(A_{\mathcal{H}}, J_{\mathcal{H}})$ is normalized to the one in the stationary Kerr family in Γ. Note that this is only justified once $E^t_{\mathcal{H}}$ has been shown to depend *only* on $A_{\mathcal{H}}$ and $J_{\mathcal{H}}$, a non-trivial result. In sum, for isolated horizons $M_{\mathcal{H}}(A_{\mathcal{H}}, J_{\mathcal{H}}) := M_{\mathrm{Kerr}}(A_{\mathcal{H}}, J_{\mathcal{H}})$, given by the Christodoulou mass expression [65].

Remark 8 (Quasi-local first law of black hole dynamics). Expression (72) extends the first law of black hole dynamics (see section 5.2) from the stationary setting to dynamical spacetimes where only the black hole horizon is in equilibrium.

5 Global and quasi-local quantities in black hole physics

As an application, we briefly comment on some relevant issues concerning mass and angular momentum in the particular case of black hole spacetimes.

5.1 Penrose inequality: a *claim* for an improved mass positivity result for black holes

In the context of the established gravitational collapse picture, Penrose [16] proposed an inequality providing an upper bound for the area of the spatial sections of black hole event horizons in terms of the square of the ADM mass. This conjecture followed from a heuristic chain of arguments including rigorous results (singularity and black hole uniqueness theorems), together with conjectures such as weak cosmic censorship and the stationarity of the final state of the evolution of a black hole spacetime. A local-in-time version of the Penrose inequality can be formulated in terms of data on a Euclidean slice. In this version Penrose conjecture states that, given an asymptotically Euclidean slice Σ containing a black hole under the dominant energy condition, the following inequality should be satisfied

$$A_{\mathrm{min}} \leq 16\pi M^2_{\mathrm{ADM}} \quad , \tag{73}$$

where A_{min} is the minimal area enclosing the apparent horizon. In addition, equality is only attained by a slice of Schwarzschild spacetime. Though this was originally proposed in an attempt to construct counter-examples to the weak cosmic censorship conjecture, growing evidence has accumulated supporting its generic validity. Beyond spherical symmetry [66], a formal proof only exists in the Riemannian case, $K_{\mu\nu} = 0$, where the original derivation [47] (see also [67]) makes use of some of the quasi-local expressions presented in section 4 (cf. discussion about Geroch and Hawking energies, that coincide in this time-symmetric case $K_{\mu\nu} = 0$). The intrinsic geometric relevance of the Penrose inequality is reflected in its alternative name as the *isoperimetric inequality for black holes* [68].

Penrose inequality can also be seen as strengthening the positive ADM mass theorem in section 3.3, for the case of black hole spacetimes: the ADM mass is not only positive but must be larger than a certain positive-definite quantity. Though it is tempting to identify this positive quantity with some quasi-local mass associated with the black hole, e.g. with its irreducible mass $A =: 16\pi M^2_{\mathrm{irr}}$ related to the Hawking mass (60), a caveat follows from the fact that the relevant minimal surface of area $A_{\rm min}$ does not necessarily coincide with the apparent horizon, as examples in [69] show. In any case, this geometric inequality represents a bridge between global and quasi-local properties in black hole spacetimes and has become one of the current geometric and physical/conceptual main challenges in General Relativity.

5.2 Black hole (thermo-)dynamics

A set of four laws was established in [70] for stationary black holes. These black hole laws are analogous in form to the standard thermodynamical laws. Though this analogy is compelling, the fundamental nature of such relation was only acknowledged under the light of Hawking's discovery [71] of the (semiclassical) thermal emission of particles from the event horizon (Hawking radiation). Given a stationary black hole spacetime with stationary Killing vector t^μ, black hole rigidity theorems [72] imply the existence of a second Killing vector k^μ that coincides with the null generators ℓ^μ on the horizon. We can write $k^\mu = t^\mu + \Omega_H \phi^\mu$, where ϕ^μ is an axial Killing vector and Ω_H is a constant referred to as the angular velocity of the horizon (see also [4]). We can write $k^\nu \nabla_\nu k^\mu = \kappa k^\mu$ on the horizon, which defines the surface gravity function κ. The zeroth law of black hole mechanics then states the constancy of the surface gravity on the event horizon. The second law, namely Hawking's area theorem [73, 74], guarantees that the area of the event horizon never decreases, whereas the third law states that the surface gravity κ cannot be reduced to zero in a finite (advanced) time (see [75] for a precise statement). In the present context, we are particularly interested in the first law, since it relates the variations of some of the quasi-local and global quantities we have introduced in the text, in the particular black hole context. First law provides an expression for the change of the total mass M of the black hole (a well-defined notion since we deal with asymptotically flat spacetimes) under a small stationary and axisymmetric change in the solution space

$$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega_H J_H \quad , \tag{74}$$

where A is the area of a spatial section of the horizon, and J_H is the Komar angular momentum associated with the axial Killing ϕ^μ. Equation (74) relates the variation of a global quantity $M = M_{\rm ADM}$ at spatial infinity on the left-hand-side, to the variation of quantities locally defined at the horizon, on the right-hand-side. In particular, we could express the variation of the horizon area in terms of the variation of the irreducible local mass $M_{\rm irr}$, as $\delta A = 32\pi M_{\rm irr}\delta M_{\rm irr}$. Such a formulation actually plays a role in the criterion for constructing sequences of binary black hole initial data corresponding to quasi-circular adiabatic inspirals (cf. [26] and the first law of binary black holes in [76]). Derivation of (74) involves the notions of ADM mass, as well as the generalization to stationarity of the Smarr formula for Kerr mass [stating $M = 2\Omega_H J_H + \kappa A/(4\pi)$] by using the Komar mass expression. Result

(72) in section 4.3 provides an *extension* of this law to black hole spacetimes non-necessarily stationary, but containing an isolated horizon for which an unambiguous notion of black hole mass can be introduced. Quasi-local attempts to extend the first law to the fully dynamical case have been explored in the dynamical and trapping horizon framework [54, 55, 56, 77]. However, the lack of a general unambiguous notion of quasi-local mass prevents the derivation of a *result* analogous to (74) or (72), i.e. the equality between the variation of two *independent* well-defined quantities. In the quasi-local dynamical context, an unambiguous law for the area evolution can be determined (see e.g. [78, 79] and references therein). The latter can then be used to *define* a flux of energy through the horizon by comparison with (74).

5.3 Black hole extremality: a mass-angular momentum inequality

Subextremal Kerr black holes are characterised by presenting angular momenta bounded by their total masses. Keeping axisymmetry, it has been recently shown [80, 81, 82] (see also [83]) that the inequality

$$|J_{\mathrm{K}}| \leq M^2_{\mathrm{ADM}} \quad , \tag{75}$$

holds also for vacuum, maximal (K=0), axisymmetric Euclidean data. Moreover, equality only holds for slices of extremal Kerr. Inequality (75) provides a non-trivial relation for black hole spacetimes between precisely the two physical quantities we are focusing on in this review. It is natural to explore if some analogous inequality holds when moving away from axisymmetry and when considering only the local region around the black hole. Attempts have been done in this sense, but they all must face the ambiguities resulting from the absence of canonical expressions for quasi-local masses and angular momenta. In order to illustrate the caveats to keep in mind when undertaking this kind of discussion, one can consider the case in which Komar quantities are used for constructing a truly quasi-local analogue of expression (75) for axisymmetric stationary data: initial data have been constructed [84] where the quotient $|J_{\mathrm{K}}|/M^2_{\mathrm{K}}$ on the black hole horizon can become arbitrarily large. Interestingly, these studies have led to the formulation [85] in axisymmetry of the related conjecture $8\pi|J_{\mathrm{K}}| \leq A$, only involving intrinsic quantities on the horizon. This inequality has been proved to hold in the stationary axisymmetric case [86], as well as in a generalization including the electromagnetic field and the associated electric charge on the horizon [87].

6 Conclusions

The problem of characterising the energy-momentum and angular momentum associated with the gravitational field in General Relativity has been present since the birth of the theory and controversies have plagued its already long-standing history. Understanding that no local density of energy-momentum can be identified for the gravitational field has challenged the validity of the mass and angular momentum *cherished* notions from non-gravitational physics, when trying to perform a straightforward extension of these concepts to the gravitational field in a general relativistic setting.

The study of specific problems suggests concrete and/or partial solutions. In this spirit, at low velocities and weak self-gravities post-Newtonian approaches handle consistent notions of mass and angular momentum and the same holds in perturbative approaches around exact solutions, for which physical quantities can be identified unambiguously. In the same line, a (quasi-local) notion of the energy carried by a gravitational wave can be introduced as an *average* along the wavelength, proving to provide a useful notion in practical applications. A particular setting of singular conceptual importance is that of isolated systems in General Relativity, specifically through their characterisation as asymptotically flat spacetimes. The notions of *total* ADM and Bondi-Sachs energy-momentum provide well-defined quantities that, on the one hand, have clarified important conceptual issues such as the capability of gravitational waves to actually carry energy away from a system and, on the other hand, they also represent inestimable tools in practical applications due to their intrinsic/geometric character. Positivity theorems for the total mass represent without any doubt some of the most important and profound results in General Relativity. The combination of the success in isolated systems, together with the absence of a gravitational local energy-momentum density, has led to the consideration that the whole effort for the search of a *local* expression for the gravitational energy represents an ill-posed or pseudo-problem (see e.g. [11]). But at the same time, and motivated by practical needs and/or fundamental physical reasons (cf. in this sense [88] for a related discussion on *quasi-local* issues regarding observables in Quantum Field Theory), important efforts have been devoted to the introduction of quasi-local notions of gravitational energy-momentum associated with extended but finite regions of the spacetime. In this respect, significant insights into the structure of the gravitational field have been achieved, with applications in diverse conceptual and practical contexts. But it must be acknowledged, as it is referred in [1], that the status of the quasi-local mass studies is in a kind of *post-modern* situation in which the devoted intensive efforts have resulted in a plethora of proposals with no obvious definitive and entirely satisfying candidate.

A moderate (intermediate) position that avoids radical skepticism against the quasi-local approach would consist in assuming that mass and (in a more restricted sense) angular momentum can be unambiguously defined only as global quantities for isolated systems. But accepting, at the same time, that quasi-local expressions provide meaningful and insightful quantities that are inextricably subject to the need of making systematically explicit the specific setting in which they are defined (one can make the analogy with the notion of *effective mass* in solid state physics, where different masses can be *simultaneously* employed for the same particle as long as their specific purposes are clearly stated[3]). The moral of the whole discussion in this article is that the formulation of meaningful global or quasi-local mass and angular momentum notions in General Relativity *always* needs the introduction of some additional structure in the form of symmetries, quasi-symmetries or some other background structure. This point must be explicitly kept in mind whenever employing the so-defined physical quantities, specially when extrapolating or performing compared analysis.

Acknowledgements. The authors wish to thank the organisers of the Orleans School on Mass for their kind invitation and encouragement.

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[3] We thank B. Carter for his many comments and insights in this discussion, and in particular for bringing us to the *solid state analogy* for gravitational masses.

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