Gravitational waves: a foundational review

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Abstract The standard linear approach to the gravitational waves theory is critically reviewed. Contrary to the prevalent understanding, it is pointed out that this theory contains many conceptual and technical obscure issues that require further analysis.

1 Introduction

For many years in the past, the existence of gravitational waves was a controversial issue. The discovery of a binary pulsar whose orbital period changes in accordance with the predicted gravitational wave emission [1] put an end to that controversy. In fact, that discovery provided a compelling evidence for the existence of gravitational waves (for a textbook reference, see Ref. [2]). That evidence, however, did not provide any clue on their form and effects. The only it has done was to confirm the quadrupole radiation formula. Despite this fact, together with the quadrupole radiation formula, the standard linear approach to the gravitational waves theory became widely considered a finished topic, a theory not to be questioned anymore (see, for example, Ref. [3], page 313). In other words, it became a dogma.

However, as a careful analysis of the current theory shows, it is actually plagued by many obscure points [4]. From one hand, owing to the nonlinear nature of gravitation, which makes it difficult to deal with, it is understandable the existence of some obscure, or even controversial points. On the other hand, these difficulties cannot be used as an excuse for our leniency with the established theory. In these notes, by using the potential (or Lagrangian) form of Einstein field equation, even at the risk of committing a heresy, I will critically review the foundations of the standard linear approach to the gravitational waves theory, pointing out precisely where it lacks consistency and why it requires further attention.

2 Gauge versus gravitational waves

It is well-known that, in order to transport its own charge (or source), a gauge field must satisfy a nonlinear field equation. For example, the gauge field equations of chromodynamics must be nonlinear to allow the field to transport color charge. In the language of differential forms (we use here the same notations and conventions of Ref. [5]), the Yang-Mills equation* is written as [6]

$$dH - j = J, \tag{1}$$

*The Yang-Mills theory will be adopted as the paradigm of nonlinear gauge theories.

where $H = -\partial \mathcal{L}/\partial F$ is the excitation 2-form,† with $\mathcal{L}$ the gauge Lagrangian and $F = DA$ the field strength of the gauge potential A. In addition, j stands for the gauge pseudo-current, and J is the source current. Due to the property

$$dd = 0, \tag{2}$$

known as Poincaré lemma [8], the field equation implies the conservation of the total current:

$$d\,(j + J) = 0. \tag{3}$$

Electromagnetism is a particular case of Yang-Mills theories, with the Abelian unitary group $U(1)$ as the gauge group. In this case, the Yang-Mills equation reduces to the *linear* Maxwell equation

$$dH = J, \tag{4}$$

where $H = -\partial \mathcal{L}/\partial F$ is the electromagnetic excitation 2-form, with $\mathcal{L}$ the Lagrangian and $F = dA$ the electromagnetic field strength. The source J in this case is the electric current, which is conserved on account of the Poincaré lemma:

$$dJ = 0. \tag{5}$$

This conservation law says that a source cannot lose electric charge when emitting electromagnetic waves. In fact, remembering that currents are quadratic in the field variable, the linearity of Maxwell equation restricts the gauge self-current j to be also linear, and consequently to vanish

$$j = 0. \tag{6}$$

This is the reason why an electromagnetic wave is unable to transport its own source, that is, electric charge, a result consistent with the conservation law (5). Observe that the source current J is quadratic in the source field variables, but linear in the electromagnetic field. Differently from the self-current j, therefore, the linearity of Maxwell equation does not restrict it to vanish. Then comes the crucial point: *since neither energy nor momentum is source of the electromagnetic field, the energy-momentum current does not appear explicitly in the electromagnetic field equation, and for this reason the linearity of Maxwell equation does not restrict the energy-momentum tensor of the electromagnetic field to be linear. This means that, even though electromagnetic waves are unable to carry electric charge, they do carry energy and momentum, whose intensity is given by the (quadratic) Poynting vector.*

Let us consider now the gravitational case. Denoting by $\mathcal{L}_g$ the gravitational Lagrangian, the potential (or Lagrangian) form of Einstein equation reads [9] ($k = 8\pi G/c^4$)

$$dH_\alpha - k\, t_\alpha = k\, T_\alpha, \tag{7}$$

where

$$H_\alpha = -\frac{k}{h}\frac{\partial \mathcal{L}_g}{\partial dh^\alpha} \tag{8}$$

†In Yang-Mills theory, as well as in electromagnetism, the field excitation 2-form coincides with the field strength. However, there are theories in which they do not coincide. This is the case, for example, of teleparallel gravity, a gauge theory for the translation group [7].

is the gravitational field excitation 2-form (also called superpotential), with h^α the tetrad (or coframe) field and $h = \det(h^\alpha)$. In addition,

$$t_\alpha = -\frac{1}{h}\frac{\partial \mathcal{L}_g}{\partial h^\alpha} \tag{9}$$

stands for the gravitational self-current, which in this case represents the gravitational energy-momentum pseudotensor, and

$$T_\alpha = -\frac{1}{h}\frac{\partial \mathcal{L}_s}{\partial h^\alpha} \tag{10}$$

is the source energy-momentum current, with $\mathcal{L}_s$ the source Lagrangian. Notice that in this form, Einstein equation (7) is similar, in structure, to the Yang-Mills equation (1). *Its main property is to explicitly exhibit the complex defining the energy-momentum pseudo-current of the gravitational field.* From the Poincaré lemma (2), the total energy-momentum density is found to be conserved as a consequence of the field equation:

$$d(t_\alpha + T_\alpha) = 0. \tag{11}$$

Now, owing to the weakness of the gravitational interaction, and considering that the sources of gravitational waves are at enormous distances from Earth, it is sensible to assume that the amplitude of a gravitational wave when reaching a detector on Earth will be very small. These facts allow the use of a perturbative analysis, where the gravitational field variable is expanded in powers of a small parameter. Namely,

$$h^\alpha = \delta^\alpha + h^{(1)\alpha} + h^{(2)\alpha} + \cdots, \tag{12}$$

where δ^α is a trivial tetrad related to Minkowski spacetime. In the linear, or first-order approximation, the gravitational field equation becomes mathematically similar to Maxwell equation. In fact, at this order the field equation (7) reduces to

$$dH^{(1)}_\alpha = k\, T^{(1)}_\alpha, \tag{13}$$

which is mathematically similar to the Maxwell equation (4).

However, in spite of this similarity, there is a fundamental difference between the two cases. As we have already seen, the linearity of Maxwell equation restricts the electromagnetic self-current to vanish. As a consequence, an electromagnetic wave is unable to transport electric charge. Analogously, since the gravitational energy-momentum pseudotensor t_α is at least quadratic in the field variables [10], it vanishes in the linear approximation:

$$t^{(1)}_\alpha = 0. \tag{14}$$

This means that a linear gravitational wave is unable to transport its own source, that is, energy and momentum. This is consistent with the Poincaré lemma, which when applied to the first-order field equation (13) implies that the source energy-momentum tensor is conserved:

$$dT^{(1)}_\alpha = 0. \tag{15}$$

Strictly speaking, this conservation law says that, at this order, a mechanical system cannot lose energy in the form of gravitational waves. Since any wave must have energy to exist, what this conservation law is saying is that linear (or dipole) gravitational radiation does not exist.

3 The standard approach to gravitational waves

3.1 Linear or nonlinear: that is the question

Even though there seems to be a certain agreement that the transport of energy and momentum by gravitational waves is a nonlinear phenomenon (see, for example, Ref. [11]), instead of going to the second order, the standard approach to gravitational waves follows a kind of "mixed procedure", which *consists basically in assuming that gravitational waves carry energy (are nonlinear), but at the same time, because the amount of energy transported is so small, it is also assumed that its dynamics can be approximately described by a linear equation* [12]. More precisely, it can be described by the (sourceless version of the) linear wave equation (13).

Conceptually speaking, however, this is a questionable assumption. The reason is that either a gravitational wave does or does not carry energy. If it carries, it cannot satisfy a linear equation. If applied to a Yang-Mills propagating field, it would correspond to assume that, for a gauge field with small-enough amplitude, its evolution could be accurately described by a linear equation. Of course, this is plainly wrong: a Yang-Mills propagating field must be nonlinear to carry its own source, otherwise it is not a Yang-Mills field. Analogously, a gravitational wave must be nonlinear to transport its own source, otherwise it is not a gravitational wave. *This is not a matter of approximation, but a conceptual question.*[‡]

It is opportune at this point to recall some properties of solitary waves, whose existence depends on a precise compensation between dispersion and nonlinearity [15]. In the specific case of surface waves in shallow water, solitary waves are obtained from the Navier-Stokes equation (for an inviscid fluid) through a perturbation scheme. At the first order, one obtains a linear wave-equation whose solution determines the dispersion relation of the system, *not the physical wave*. At the second order, the first-order solution appears multiplied by itself, giving rise to a nonlinear evolution equation—the so-called Korteweg-de Vries equation. The solitary wave, which is the physical wave observed in nature, is then obtained as a solution to this nonlinear equation. Of course, in order to obtain a more precise solution, one has to go to higher orders in the perturbation scheme. The important point is to observe that, even for very small wave amplitudes, a solitary wave can never be approximately described by a linear equation. This is a general property of nonlinear waves, of which gravitational waves are just an example.

3.2 Messing with the perturbation scheme

The next step of the standard approach is to compute the energy and momentum transported by these linear waves. The common procedure is to make use of the second-order energy-momentum pseudo-current [12].[§] The argument normally used to justify this procedure is that this is similar to the electromagnetic wave, which in spite of being linear, its energy-momentum tensor is quadratic in the field variables. However, this is a misleading argument.

[‡]It is interesting to remark that even the well-known *exact plane gravitational wave* solution of Einstein equations [13] transports neither energy nor momentum [14]. This is in accordance with the nonlinear nature of the transport of energy-momentum by gravitational waves.

[§]The problem of the non-localizability of the energy and momentum of the gravitational field [16] is not relevant for the present discussion, and will not be considered here.

In fact, as discussed in the previous section, the linearity of Maxwell equation does not impose any restriction on the (quadratic) energy-momentum tensor of the electromagnetic field, which can then be used to compute the energy and momentum transported by electromagnetic waves. In the gravitational case, on the other hand, the linearity of the first-order gravitational field equation (13) restricts the gravitational energy-momentum current to be *linear*, and consequently to vanish at this order. A quadratic energy-momentum pseudo-tensor can only appear in orders higher than one. This is the case, for example, of the second-order gravitational field equation (17) below, where $t_\alpha^{(2)}$ represents the second-order gravitational energy-momentum pseudotensor, which is quadratic in the first-order field variable. It must, for this reason, represent the energy and momentum transported by second-order gravitational waves. The simultaneous use of quantities belonging to different orders of the perturbation scheme constitutes a permissive, unacceptable procedure.

4 Problems and obscure points

On account of some unjustifiable assumptions, as well as of the misuse of the perturbation scheme, the standard approach to gravitational waves becomes plagued by many inconsistencies and obscure points. In this section, some of these points are discussed.

4.1 The question of the gravitational wave frequency

Gravitational waves are generated, and act on free particles through tidal effects (see, for example, Ref. [3], page 310). These effects, described by the geodesic deviation equation, are well-known to be produced by inhomogeneities in the gravitational field, and like the ocean tides on Earth, occur twice for each complete cycle of the system Moon-Earth. In fact, according to the quadrupole radiation formula, gravitational radiation comes out from the source with a frequency that is twice the source frequency [17]. However, the plane gravitational wave that emerges from the standard linear approach propagates with the same frequency of the source. To circumvent this problem, one has to artificially adjust by hands the wave frequency, as explained in side-note 8, page 105 of Ref. [2]. This is a drawback of the linear approach, which seems to tell us that the first-order wave does not represent the physical, or quadrupole gravitational wave.

4.2 The question of the effects on free particles

Similarly to the electromagnetic wave, the field components of the first-order gravitational wave (in transverse-traceless coordinates) are orthogonal to the propagation direction. As a consequence, by using the geodesic deviation equation it is concluded that, when passing through two vertically separated particles, the first-order gravitational wave would make them to oscillate orthogonally around the original point. A circumference of free particles would be distorted in such a way that it would become an ellipse, first (let us say) vertically, then horizontally, and so on. The question then arises: how a strictly attractive field like gravitation could give rise to *orthogonal oscillations* around the original position? This orthogonal oscillation can be easily understood in the electromagnetic case, where the Lorentz force is either attractive or repulsive depending on the sign of the field component. However, in the strictly attractive case of gravitation, it is not clear at all how such orthogonal particles oscillation could be possible.

4.3 The question of linear curvature

The existence of a non-vanishing first-order Riemann tensor is perhaps one of the main puzzles of the gravitational wave theory. The usual lore is that, if a linear gravitational wave produces a non-vanishing linear curvature tensor, it must exist physically. However, there are a number of points that should be considered. As discussed in Section 2, due to the fact that the gravitational energy-momentum current is at least quadratic in the field variables, the energy-momentum density associated to any *linear* field configuration must vanish. A non-vanishing energy density can only appear in orders higher than one. This does not mean that the first-order gravitational field is physically meaningless. As a matter of fact, at the second-order the first-order solutions will appear multiplied by itself, giving rise to nonlinear field configurations with non-vanishing energy-momentum density.

Furthermore, recall that the components of the Riemann tensor themselves are not physically meaningful in the sense that they are different in different coordinate systems. For example, starting with the "electric components" R_{i0j0} of the Riemann tensor, through a general coordinate transformation one can get non-vanishing "magnetic components" R_{i0jk}. By inspecting the components of the Riemann tensor, therefore, it is not possible to know whether they represent a true gravitomagnetic field produced by a rotating source, or just effects of coordinates. This can only be done by inspecting the invariants constructed out of the Riemann tensor (see Ref. [18], page 355). Now, as a simple computation shows, *all invariants constructed out of the first-order Riemann tensor of the linear gravitational waves vanish identically* [19–22]. This includes the Kretschmann and the pseudo-scalar invariants. Considering that these invariants are proportional to the mass/energy and angular momentum of the field configuration (see Ref. [18], page 356-357), it can be immediately concluded that first-order gravitational waves are empty of physical meaning as neither mass/energy nor angular momentum can be attributed to them.

5 Second-order gravitational waves

If, instead of the usual standard approach [12], we accept that all first-order equations are fully correct up to that order, we arrive at the inexorable result that linear gravitational waves transport neither energy nor momentum. In this case, it is straightforward to see that they are unable to produce any effect on free particles [23]. The natural way to proceed is then to go to the next order. At the second order, the source energy-momentum tensor is found to be conserved in the covariant sense,

$$DT_\alpha^{(2)} \equiv dT_\alpha^{(2)} + \Gamma^{(1)\beta}_{\alpha} \wedge T_\beta^{(1)} = 0, \tag{16}$$

with $\Gamma^{(1)\beta}_{\alpha}$ the first-order Levi-Civita connection. As is well-known, it is not a true conservation law, but just an identity (called Noether identity) governing the exchange of energy and momentum between the source and gravitation [24]. This means that, differently from what happens at first order, at the second order a mechanical system can lose energy in the form of gravitational waves. The amount of energy and momentum released is correctly predicted by the (nonlinear) quadrupole radiation formula.

At this order, instead of the Maxwell-type field equation (13), the gravitational field equation acquires the Yang-Mills form

$$dH_\alpha^{(2)} - k\, t_\alpha^{(2)} = k\, T_\alpha^{(2)}, \tag{17}$$

with $t_\alpha^{(2)}$ the second-order gravitational energy-momentum pseudotensor, which (we repeat) is quadratic in the first-order field variable. Similarly to a gluon field, which is able to transport color charge, the radiative solution of the sourceless version of the gravitational field equation (17) is able to transport energy and momentum. It must, for this reason, represent the physical gravitational wave.

Nonlinear evolution equations are far more difficult to deal with than linear equations. In spite of this difficulty, it has already been possible to get a glimpse of the properties of the second-order gravitational waves [25]. To begin with, owing to its quadratic nonlinearity, the second-order gravitational wave naturally emerges propagating with a frequency that is twice the source frequency. This is in agreement with the quadrupole radiation formula, as well as with the tidal nature of the gravitational radiation (see the discussion of Section 4.1). This result can be considered an additional evidence that second-order gravitational waves might represent the physical wave.

In general, the amplitude of second-order effects are considered to fall off with $1/r^2$, where r is the distance from the source. Due to the large distances from the sources, second-order effects in gravitational wave theory are usually considered to be neglectful. However, owing to the intricacies of nonlinear equations, the sourceless version of the field equation (17) has a wave solution whose amplitude falls off with $1/r$ [25]. Of course, this is not enough to affirm that such waves are detectable because their intensities depend also on the parameters appearing in the amplitude definition. In particular, like in all nonlinear waves, the amplitude of the second-order gravitational wave depends explicitly on the frequency of the wave. The magnitude of such amplitudes is as yet a question to be explored.

Using the second-order wave solution, it is possible to compute the components of the second-order Riemann tensor components. Substituting these components in the geodesic deviation equation, one can find the effects of the second-order gravitational wave on free particles. Following this procedure, it has been found that when a gravitational wave passes through two particles separated by a small distance *along the direction of propagation of the wave*, both particles begin moving towards the source due to the attraction of gravitation. In addition, owing to the inhomogeneity of the gravitational field of the wave, their distance will oscillate as they move. Observe that the particles will not oscillate around an equilibrium point, but will move towards the source with different velocities in such a way that their distance oscillates along the direction of propagation of the wave. *This means that second-order gravitational waves are longitudinal waves* [25]. Such property is consistent with the tidal origin of gravitational waves, as well as with the strictly attractive character of gravitation. According to the second-order approach, therefore, this longitudinal oscillation is the sign to be looked for in the search of gravitational waves.

6 Final remarks

In spite of its beauty, the traditional form of Einstein equation hides important aspects of general relativity. In addition, it may not be the appropriate form to deal with certain problems of gravitation. On the other hand, owing to its mathematical similarity to the Yang-Mills equation, the potential form of Einstein equation unveils important aspects of general relativity. In particular, because the energy-momentum pseudotensor of the gravitational field appears explicitly in this field equation, the similarities and mainly the differences in relation to Maxwell equation become much more visible. The reason why such form of Einstein equation has seldom been used in gravitation is probably the non-existence of an *invariant* Lagrangian for general relativity that depends on the tetrad and on the first derivative of the tetrad only.

Taking advantage of the potential form of Einstein equation, a critical review of the foundations of the standard approach to the gravitational waves theory has shown the existence of several obscure points and inconsistencies, which are not easily visible in the standard approach. It has also shown that a natural way to circumvent all these problems is to resignedly accept all results emerging from the first-order expansion of Einstein equation, which in turn amounts to accept that *gravitational waves cannot be described by a linear equation, even approximately*. One should then go to the second order to look for the physical gravitational waves. Even though the first-order solution, which does not represent the physical wave, is transverse, *the physically relevant second-order gravitational wave is longitudinal*. This is consistent with both the tidal origin of gravitational waves and the strictly attractive character of gravitation. The second-order wave is furthermore found to propagate with a frequency that is twice the source frequency, in agreement with the quadrupole radiation formula.

It is clear by now that none of the existing antennas has succeeded in detecting any sign of gravitational waves. Of course, it is possible that the detectors did not meet the necessary sensibility to detect them, or that the magnitude of the gravitational waves when reaching a detector on Earth is smaller than originally expected. However, it is also possible that a faulty approach has led all detectors to look for the wrong sign. The analysis presented in these notes, whose purpose was to call the attention for potential problems in the currently accepted theory, suggests that this possibility should not be neglected. After all, one call always ask: where are the waves?

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