Covariant Differential Identities and Conservation Laws in Metric-Torsion Theories of Gravitation. II. Manifestly Generally Covariant Theories[a)]

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The present paper continues the work of the authors [J. Math. Phys. **54**, 062504 (2013)] where manifestly covariant differential identities and conserved quantities in generally covariant metric-torsion theories of gravity of the most general type have been constructed. Here, we study these theories presented more concretely, setting that their Lagrangians $\mathscr{L}$ are *manifestly* generally covariant scalars: algebraic functions of contractions of tensor functions and their covariant derivatives. It is assumed that Lagrangians depend on metric tensor $\mathbf{g}$, curvature tensor $\mathbf{R}$, torsion tensor $\mathbf{T}$ and its first $\boldsymbol{\nabla}\mathbf{T}$ and second $\boldsymbol{\nabla}\boldsymbol{\nabla}\mathbf{T}$ covariant derivatives, besides, on an arbitrary set of other tensor (matter) fields $\boldsymbol{\varphi}$ and their first $\boldsymbol{\nabla}\boldsymbol{\varphi}$ and second $\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi}$ covariant derivatives: $\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R};\ \mathbf{T}, \boldsymbol{\nabla}\mathbf{T}, \boldsymbol{\nabla}\boldsymbol{\nabla}\mathbf{T};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi})$. Thus, both the standard minimal coupling with the Riemann-Cartan geometry and non-minimal coupling with the curvature and torsion tensors are considered.

The studies and results are as follow. (a) A physical interpretation of the Noether and Klein identities is examined. It was found that they are the basis for constructing equations of balance of energy-momentum tensors of various types (canonical, metrical and Belinfante symmetrized). The equations of balance are presented. (b) Using the generalized equations of balance, new (generalized) manifestly generally covariant expressions for canonical energy-momentum and spin tensors of the matter fields are constructed. In the cases, when the matter Lagrangian contains both the higher derivatives and non-minimal coupling with curvature and torsion, such generalizations are non-trivial. (c) The Belinfante procedure is generalized for an arbitrary Riemann-Cartan space. (d) A more convenient in applications generalized expression for the canonical superpotential is obtained. (e) A total system of equations for the gravitational fields and matter sources are presented in the form more naturally generalizing the Einstein-Cartan equations with matter. This result, being a one of more important results itself, is to be also a basis for constructing physically sensible conservation laws and their applications.

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I. INTRODUCTION

The present work is the second one of the series of works related to constructing manifestly covariant differential identities and conserved quantities, and their study in generally covariant metric-torsion theories of gravity. In the first work of the series[1] (we will call it as the Paper I), in an arbitrary Riemann-Cartan space $\mathcal{C}(1, D)$ the next manifestly covariant expressions and relations have been obtained: (a) the generalized Noether current $\mathbf{J}[\delta\boldsymbol{\xi}]$; (b) the system of differential Klein and Noether identities; (c) the generalized superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$, with the use of which the generalized Noether current is presented; (d) the generalized symmetrized Noether current $\overset{sym}{\mathbf{J}}[\delta\boldsymbol{\xi}]$ as a result of an application of the generalized Belinfante procedure to the generalized Noether current.

For the sake of a definiteness, let us repeat the definitions, which we use. A theory is called as *generally covariant* if it is invariant with respect to general diffeomorphisms. At the same time, a form of its presentation can be arbitrary. Because *gauge covariant* theories that are invariant with respect to internal gauge transformations are very similar to the generally covariant ones (have the same structure of currents, etc), for the sake of an universality we call theories of both these types as a *gauge-invariant theories*. On the other hand, the usual gauge theories with an internal gauge group we call separately as the gauge theories of Utiyama-Yang-Mills type.

Thus, in the Paper I, the quantities and relations in the theories of the most general type have been constructed. In the present work, we concretize them. We apply the developed formalism for the study of manifestly generally covariant theories, which are a more interesting and important example of diffeomorphically invariant theories. We call a theory as *manifestly generally covariant* if its Lagrangian $\mathscr{L}$ is a generally covariant scalar constructed as algebraic scalar function of manifestly covariant objects that are transformed following the lin-

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ear *homogeneous* representations of the diffeomorphism group. This means that $\mathscr{L}$ is an algebraic function of scalar contractions of tensor (and/or spinor) field functions and their covariant derivatives; besides manifest dependence on field variables, $\mathscr{L}$ can also depend on curvature and torsion tensors independently. It seems that almost all the physically interesting theories are manifestly generally covariant or can be presented in such a form. Exceptions are, e.g., topological theories of the Chern-Simons type (see reviews[2–10] and references therein), Lagrangians of which are presented by a secondary characteristic class of a topological invariant (Chern-Simons form). Such Lagrangians explicitly contain connections that are transformed following a linear *non-homogeneous* representation. At the same time, under gauge transformations the Lagrangians themselves change to a total divergence. In the Chern-Simons theories, conserved quantities were constructed in the works[11–18]. Notice also that Lagrangians in such theories can be presented in the exactly gauge invariant form by expanding the Chern-Simons form to the transgression form[13,18–27].

Earlier, manifestly generally covariant theories both in Riemannian spacetime (see, for example, Refs.[28–50]), and in the Riemann-Cartan space (see, for example, Refs.[51–57]) were studied already. In particular, in the works by Trautman[51–54] and by Hehl at al[55–57] the simplest theory of gravity with torsion, the Einstein-Cartan theory (ECT), with matter, presented by the Lagrangian $\mathscr{L} = \mathscr{L}^G + \mathscr{L}^M = -\frac{1}{2k}R + \mathscr{L}^M(\mathbf{g};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi})$ is examined. Their main results are: (a) clarification of the role of the *canonical* energy-momentum tensor (EMT) of matter as a source of the metric field; (b) determination of the connection between the variation derivative $\Delta I^M/\Delta\mathbf{T}$ with respect to the torsion field $\mathbf{T}$ and the Belinfante tensor $\mathbf{B}$, induced by the spin tensor (ST) $\mathbf{S}$ in the matter sector of the system related to the action functional I^M; (c) clarification of the role of the *canonical* ST of matter as a source of the torsion field; (d) in the Riemann-Cartan space, construction of the *universal* balance equation for the canonical EMT of matter.

In the present work, we consider significantly more general manifestly generally covariant theories, the total Lagrangians of which $\mathscr{L} = \mathscr{L}^G + \mathscr{L}^M$ have the form: $\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R};\ \mathbf{T}, \boldsymbol{\nabla}\mathbf{T}, \boldsymbol{\nabla}\boldsymbol{\nabla}\mathbf{T};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi})$ that, besides of the second derivatives of matter fields $\boldsymbol{\varphi}$, includes a dependence on non-minimal coupling both with the curvature $\mathbf{R}$ and with the torsion $\mathbf{T}$. By this, we significantly generalize the results of earlier works. At first, following the recommendations of the Paper I, we recalculate the elements necessary for constructing currents and superpotentials of various types. Basing on this, generalized covariant dynamical quantities are constructed. They are total canonical both ST $\mathbf{s}$ and EMT $\mathbf{t}$, so-called modified canonical both ST $\overset{mod}{\mathbf{s}}$ and EMT $\overset{mod}{\mathbf{t}}$, at last, symmetrized EMT $\overset{sym}{\mathbf{t}}$ and metric EMT $\overset{met}{\mathbf{t}}$. Connections between these quantities are clarified, besides, for each of the types of the dynamic characteristics equations of balance are presented. Next, the correspondent currents and superpotentials are constructed. The generalized equations of balance are also the basis for constructing the total system of the equations of the aforementioned theories generalizing the equations of the ECT with matter.

The main original results of the present work related to the manifestly generally covariant metric-torsion theories of gravity are:

- a physical interpretation of the Klein and Noether identities, which are a basis for constructing equations of balance for EMTs of various types;
- a construction of manifestly generally covariant expressions for canonical EMT and ST of matter fields. In the more complicated cases, when $\mathscr{L}^M$ has a generalized dependence as remarked above, canonical EMT and ST of matter fields *cannot be obtained* with the use of the standard procedure, namely, applying the 1-st Noether theorem in the Minkowski space and covariantization of the expressions. In the complicated cases, the generalized equations of balance are crucial for constructing canonical EMT and ST;
- a *nontrivial* generalization of the Belinfante procedure applied to the canonical EMT $\mathbf{t}$ for constructing symmetrized EMT $\overset{sym}{\mathbf{t}}$ *in an arbitrary Riemann-Cartan space*;
- a construction of a more simple and convenient in applications generalized expression for the canonical superpotential;
- a presentation of a total system of equations for the gravitational fields and matter sources in the form, which more naturally generalizes the Einstein-Cartan equations with matter. This result is to be a basis for constructing physically sensible conservation laws and their applications in the Paper III of the series.

The most of the calculations are very cumbersome and intricate. Therefore, to give a possibility to a reader to repeat them, many steps are presented in the main text. Besides, the more important formulae are given in boxes. It is important also that initial identities are analyzed under different assumptions, when either the total set of field equations hold, or a part of field equations (say, the gravitational ones only, or the matter ones only) hold. In future, this will be useful for studying both gravitational theories with sources of the general type and field theories on a given geometrical background. At last, one has to note that, in spite of the present work (Paper II) is the second work of the series, developing the Paper I, it presents a quite independent research.

The paper is organized as follows. In Sec. II, necessary formulae for the current and superpotentials, and the Klein-Noether identities obtained in the Paper I are given. In the present work, namely for all of them concrete expressions in the framework of the manifestly generally covariant theories are constructed. Also problems, which are elaborated in the present work are formulated.

All the next studies are related to the theories with the generalized manifestly covariant Lagrangians $\mathscr{L}$ of the described above type. In Sec. III, corresponding to the formulae (3) - (5), the covariant tensors $\mathbf{K}$ and $\mathbf{L}$ are constructed. They are necessary to construct the tensors $\mathbf{U}$, $\mathbf{M}$ and $\mathbf{N}$ determining the generalized Noether current $\mathbf{J}[\delta\boldsymbol{\xi}]$.

In Sec. IV, the manifestly covariant expressions for the tensors $\mathbf{U}$, $\mathbf{M}$ and $\mathbf{N}$ themselves are carried out. The generalized covariant expressions for the total canonical EMT $\mathbf{t}$ and ST $\mathbf{s}$ are found. A contribution initiated both by the curvature tensor $\mathbf{R}$ and by the non-minimal coupling with the torsion tensor $\mathbf{T}$ is taken into account. It is shown that in the manifestly covariant theories the tensor $\mathbf{N}$ does not vanish only, when the Lagrangian $\mathscr{L}$ contains the curvature tensor $\mathbf{R}$ explicitly. Also additional (with respect to the standard ones) symmetries of $\mathbf{N}$ are clarified.

In Sec. V, a structure of variational derivatives $\Delta I/\Delta\mathbf{T}$ and $\Delta I/\Delta\mathbf{g}$ of the action functional I is analyzed. This gives a basis to clarify a physical sense both of the Noether identity and of all the Klein identities. It is shown that the results by Trautman and by Hehl at al: (27) and (28) are held in a more general case, when the Lagrangian has the form: $\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi})$. However, in a case of the next generalization, when the Lagrangian contains a non-minimal coupling with the torsion, the results become more complicated: *additional* terms appear. Therefore, one needs to modify the basic dynamical characteristics. We introduce such a modification in an explicit form and construct *modified* total canonical ST $\overset{mod}{\mathbf{s}}$ and EMT $\overset{mod}{\mathbf{t}}$. The use of these quantities permits to conserve a connection between $\Delta I^M/\Delta\mathbf{T}$ and $\mathbf{B}$, and the equations of balance in the standard form. Besides, in this section, manifestly covariant equations of balance for both the total symmetrized EMT $\overset{sym}{\mathbf{t}}$ and the total canonical EMT $\mathbf{t}$ are carried out. It is shown also that symmetrized EMT $\overset{sym}{\mathbf{t}}$ and the metrical EMT $\overset{met}{\mathbf{t}}$ are equivalent if the matter equations hold. In the general case we prove that the generalized symmetrized Noether current $\overset{sym}{\mathbf{J}}[\delta\boldsymbol{\xi}]$ is determined by the symmetrized EMT $\overset{sym}{\mathbf{t}}$ only. Then, it turns out that surface terms in the functional action do not influence to constructing both $\overset{sym}{\mathbf{t}}$ and $\overset{sym}{\mathbf{J}}[\delta\boldsymbol{\xi}]$ (see the Paper I, Sec. V).

Sec. VI is devoted to calculating the superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ (21) and clarifying the role of the dynamical characteristics EMT and ST in the structure of the generalized current $\mathbf{J}[\delta\boldsymbol{\xi}]$ (2). The obtained manifestly covariant formula for the superpotential is quite simple, it is expressed only through the Belinfante tensor $\mathbf{b}$ and a tensor $\mathbf{G}$, the last exists only if the Lagrangian $\mathscr{L}$ depends on the curvature tensor $\mathbf{R}$ explicitly.

In Sec. VII, the general structure of the equations of motion of the gravitational fields is examined. The point of view, which is beginning from Lorentz is discussed. Following it, in the background independent field theories, the total EMT and ST are equal to zero identically. The Einstein arguments against are given. We show that the equations of balance for the pure gravitational part hold identically and have a clear geometrical sense: they generalize twice contracted the Bianchi identities onto the case of an arbitrary metric-torsion theory of gravity in the Riemann-Cartan space. Basing on this result, we suggest a more preferable (decomposed) form for the equations of the gravitational fields, where the pure gravitational part is placed on the left hand side of the equations, whereas the other (matter) part is transformed to the right hand side. This generalizes the form of the equations in the ECT with matter as well as the Einstein equations themselves. Their structure is more natural: the modified canonical EMT of matter $\overset{mod}{\mathbf{T}}$ is a source of the metric field $\mathbf{g}$, whereas the modified canonical ST of matter $\overset{mod}{\mathbf{S}}$ is a source of the torsion field $\mathbf{T}$. Such a presentation of the equations is interesting and important itself. However, besides of that, it is the basis for constructing physically sensible conservation laws in the next Paper III of the series. By this, one concludes also that the total dynamical characteristics of the physical system are not equal to zero identically, adding the Einstein arguments.

A calculation of auxiliary quantities is presented in Appendixes. In Appendix A, universal tensors $\{\Delta^{\alpha\beta\gamma}_{\underline{\lambda\mu\nu}}\}$ and $\{\Delta^{\overline{\alpha\beta\gamma}}_{\lambda\mu\nu}\}$, and various related identities are carried out. The use of them permits significantly to simplify a presentation of many formulae. In Appendix B, manifestly covariant expressions for general variations of various quantities, which appear under calculation of the functional variation of the action functional, are found. In Appendix C, the general theory of the Belinfante-Rosenfeld symbols, which permits to present the main relations of the Riemann-Cartan geometry more generally and economically, is developed. At last, in Appendix D, the general identity, which is a central one in constructing modified canonical EMT $\overset{mod}{\mathbf{t}}$ and ST $\overset{mod}{\mathbf{s}}$, is proved.

II. PRELIMINARY FORMULAE AND A STATEMENT OF TASKS

In this Section, we present the main results of the Paper I, which are necessary below. Also, here, we formulate the goals of the present paper. In the Paper I, an arbitrary generally covariant theory of tensor fields $\boldsymbol{\Phi}$, including both gravitational and matter ones, with the action functional

$$I[\boldsymbol{\Phi}; \Sigma_{1,2}] = \int_{\Sigma_1}^{\Sigma_2} dx\, \sqrt{-g}\mathscr{L}, \qquad (1)$$

is studied. In (1), the integration is provided over an arbitrary $(D+1)$-dimensional volume in the Riemann-Cartan space $\mathcal{C}(1,D)$, restricted by two spacelike D-dimensional hypersurfaces Σ_1 and Σ_2; the Lagrangian $\mathscr{L}$ is a local function of the field variables $\mathbf{\Phi} = \{\Phi^A(x); A = \overline{1,N}\}$ and their derivatives up to a second order. One of the main results of the Paper I is a construction of the manifestly covariant expression for the generalized Noether current:

$$J^\mu[\delta\boldsymbol{\xi}] = U_\alpha{}^\mu\delta\xi^\alpha + M_\alpha{}^{\beta\mu}\nabla_\beta\delta\xi^\alpha + N_\alpha{}^{\beta\gamma\mu}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha. \tag{2}$$

The displacement vectors $\delta\boldsymbol{\xi}$ are induced by diffeomorphisms; the tensors $\mathbf{U}$, $\mathbf{M}$ and $\mathbf{N}$ are presented by expressions:

$$\begin{aligned} U_\alpha{}^\mu &\stackrel{def}{=} \mathscr{L}\delta^\mu_\alpha + K^\mu|_A\Phi_\alpha|^A \\ &\quad + L^{\kappa\mu}|_A\left(\nabla_\kappa\Phi_\alpha|^A + \frac{1}{2}R^\varepsilon{}_{\alpha\kappa\lambda}\Phi_\varepsilon{}^\lambda|^A\right); \end{aligned} \tag{3}$$

$$\begin{aligned} M_\alpha{}^{\beta\mu} &\stackrel{def}{=} K^\mu|_A\Phi_\alpha{}^\beta|^A + L^{\beta\mu}|_A\Phi_\alpha|^A \\ &\quad + L^{\kappa\mu}|_A\left(\nabla_\kappa\Phi_\alpha{}^\beta|^A - \frac{1}{2}T^\beta{}_{\kappa\lambda}\Phi_\alpha{}^\lambda|^A\right); \end{aligned} \tag{4}$$

$$N_\alpha{}^{\beta\gamma\mu} \stackrel{def}{=} L^{(\gamma|\mu}|_A\Phi_\alpha{}^{|\beta)}|^A. \tag{5}$$

Important definitions and relations in the Riemann-Cartan geometry are given in the Paper I. Now, recall necessary notations only. The torsion tensor $\mathbf{T}$ and the curvature tensor $\mathbf{R}$ are presented as

$$T^\lambda{}_{\mu\nu} = -2\Gamma^\lambda{}_{[\mu\nu]}; \tag{6}$$

$$R^\kappa{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\kappa{}_{\lambda\nu} - \partial_\nu\Gamma^\kappa{}_{\lambda\mu} + \Gamma^\kappa{}_{\alpha\mu}\Gamma^\alpha{}_{\lambda\nu} - \Gamma^\kappa{}_{\alpha\nu}\Gamma^\alpha{}_{\lambda\mu}. \tag{7}$$

Here, the connection $\mathbf{\Gamma} \stackrel{def}{=} \{\Gamma^\lambda{}_{\mu\nu}\}$ is defined by a metric compatible condition

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\alpha{}_{\mu\lambda}g_{\alpha\nu} - \Gamma^\alpha{}_{\nu\lambda}g_{\mu\alpha} = 0, \tag{8}$$

where the standard covariant derivative $\boldsymbol{\nabla} \stackrel{def}{=} \{\nabla_\lambda\}$ is used. The modified covariant derivative $\stackrel{*}{\boldsymbol{\nabla}}$ is

$$\stackrel{*}{\nabla}_\lambda = \nabla_\lambda + T^\alpha{}_{\lambda\alpha}. \tag{9}$$

Quantities presented by the notations $\{\Phi_\alpha|^A\}$ and $\{\Phi_\alpha{}^\beta|^A\}$ are defined by the transformation properties of the fields $\mathbf{\Phi}$ under diffeomorphisms:

$$\delta_\xi\Phi^A(x) = \Phi_\alpha|^A\delta\xi^\alpha(x) + \Phi_\alpha{}^\beta|^A\nabla_\beta\delta\xi^\alpha(x). \tag{10}$$

The tensors $\mathbf{K}$ and $\mathbf{L}$ are defined as a result of a comparison of the variation of the action functional (1):

$$\begin{aligned} \delta_\Phi I[\mathbf{\Phi};\Sigma_{1,2}] &\stackrel{def}{=} I[\mathbf{\Phi}+\delta\mathbf{\Phi};\Sigma_{1,2}] - I[\mathbf{\Phi};\Sigma_{1,2}] \\ &= \int_{\Sigma_1}^{\Sigma_2} dx\,\delta_\Phi\left(\sqrt{-g}\mathscr{L}\right) \end{aligned} \tag{11}$$

with the formula

$$\begin{aligned} \delta_\Phi I[\mathbf{\Phi};\Sigma_{1,2}] &= \int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\frac{\Delta I}{\Delta\Phi^A}\delta\Phi^A \\ &+ \int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\stackrel{*}{\nabla}_\mu\left\{K^\mu|_A\delta\Phi^A + L^{\beta\mu}|_A\nabla_\beta\delta\Phi^A\right\}. \end{aligned} \tag{12}$$

Hereinafter, instead of the usual variational derivative we use a quantity proportional it

$$\frac{\Delta I}{\Delta\Phi^A} = \frac{1}{\sqrt{-g}}\frac{\delta I}{\delta\Phi^A} \tag{13}$$

— covariant variational derivative.

The tensors $\mathbf{U}$, $\mathbf{M}$ and $\mathbf{N}$ are not independent, they satisfy the system of the Klein-Noether differential identities:

$$\left\{\begin{aligned} &\stackrel{*}{\nabla}_\mu I_\alpha{}^\mu \equiv I_\alpha; && (14)\\ &\left(U_\alpha{}^\beta - \frac{1}{3}N_\lambda{}^{\beta\rho\sigma}R^\lambda{}_{\alpha\rho\sigma}\right) + \stackrel{*}{\nabla}_\mu\left(M_\alpha{}^{[\beta\mu]} - \frac{2}{3}\stackrel{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\beta\mu]} + \frac{1}{3}N_\alpha{}^{[\beta|\rho\sigma}T^{|\mu]}{}_{\rho\sigma}\right) \\ &\quad + \frac{1}{2}\left(M_\alpha{}^{[\rho\sigma]} - \frac{2}{3}\stackrel{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\rho\sigma]} + \frac{1}{3}N_\alpha{}^{[\rho|\kappa\lambda}T^{|\sigma]}{}_{\kappa\lambda}\right)T^\beta{}_{\rho\sigma} \equiv -I_\alpha{}^\beta; && (15)\\ &M_\alpha{}^{(\beta\gamma)} + \stackrel{*}{\nabla}_\mu N_\alpha{}^{\beta\gamma\mu} + N_\alpha{}^{(\beta|\mu\nu}T^{|\gamma)}{}_{\mu\nu} \equiv 0; && (16)\\ &N_\alpha{}^{(\beta\gamma\delta)} \equiv 0, && (17) \end{aligned}\right.$$

where

$$I_\alpha \stackrel{def}{=} \frac{\Delta I}{\Delta\Phi^A}\Phi_\alpha|^A; \tag{18}$$

$$I_\alpha{}^\beta \stackrel{def}{=} \frac{\Delta I}{\Delta\Phi^A}\Phi_\alpha{}^\beta|^A \tag{19}$$

are equal to zero if the equations of motion $\Delta I/\Delta\Phi^A = 0$ hold.

The analysis of the identities (14) – (17) lead to the boundary Klein theorem (the 3-rd Noether theorem), which states that the current (2) can be presented in the form:

$$J^\mu[\delta\boldsymbol{\xi}] = -I_\alpha{}^\mu\delta\xi^\alpha + \left(\overset{*}{\nabla}_\nu\theta^{\mu\nu}[\delta\boldsymbol{\xi}] + \frac{1}{2}\theta^{\rho\sigma}[\delta\boldsymbol{\xi}]T^\mu{}_{\rho\sigma}\right), \tag{20}$$

where the generalized canonical superpotential is

$$\theta^{\mu\nu}[\delta\boldsymbol{\xi}] = \left\{-M_\alpha{}^{[\mu\nu]} + \frac{2}{3}\left(\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\mu\nu]} + \frac{1}{2}T^{[\mu}{}_{\rho\sigma}N_\alpha{}^{\nu]\rho\sigma}\right)\right\}\delta\xi^\alpha + \left\{-\frac{4}{3}N_\alpha{}^{\beta[\mu\nu]}\right\}\nabla_\beta\delta\xi^\alpha. \tag{21}$$

With the use of the generalized Belinfante procedure the generalized symmetrized Noether current

$$\overset{sym}{J}{}^\mu[\delta\boldsymbol{\xi}] \stackrel{def}{=} J^\mu[\delta\boldsymbol{\xi}] - \left(\overset{*}{\nabla}_\nu\mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}] + \tfrac{1}{2}\mathscr{B}^{\rho\sigma}[\delta\boldsymbol{\xi}]T^\mu{}_{\rho\sigma}\right) = \overset{sym}{U}{}_\alpha{}^\mu\delta\xi^\alpha \tag{22}$$

has been constructed. It turns out that the generalized Belinfante tensor $\mathscr{B}[\delta\boldsymbol{\xi}]$, determining the procedure, coincides with the generalized canonical superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ (21). Thus the current $\overset{sym}{\mathbf{J}}[\delta\boldsymbol{\xi}]$ (22), by (20), is proportional to the operators of the equations of motion, that is proportional to the variational derivatives of the action. This means that it does not depend on divergences in the Lagrangian (in the other words, it does not depend on surface terms in the action functional (1)) and is equal to zero on the equations of motion.

In the present paper, we consider more concrete theories presented by the action (1), examining Lagrangians in a manifestly covariant form:

$$\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R};\ \mathbf{T}, \boldsymbol{\nabla}\mathbf{T}, \boldsymbol{\nabla\nabla}\mathbf{T};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla\nabla}\boldsymbol{\varphi}). \tag{23}$$

Here, the total set of the fields $\boldsymbol{\Phi}$ is presented by the metric tensor $\mathbf{g}$, by the torsion tensor $\mathbf{T}$ and by a set of the matter fields $\boldsymbol{\varphi} \stackrel{def}{=} \{\varphi^a(x);\ a = \overline{1,n}\}$, which are considered as tensorial ones also. Lagrangians of the type (23) include, together with the minimal coupling, the non-minimal coupling related both to the curvature and to the torsion. The main task of the present paper is to present relations and conserved quantities (currents and superpotentials) constructed in the Paper I in a maximally concrete form that follows from the concrete structure of the Lagrangian (23).

Recall, see formulae (2) and (21), that for constructing the generalized current $\mathbf{J}[\delta\boldsymbol{\xi}]$ and superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ one needs the tensors $\mathbf{U}$ (3), $\mathbf{M}$ (4) and $\mathbf{N}$ (5). For constructing the last the other tensors, $\mathbf{K}$ and $\mathbf{L}$, defined in (12) have to be calculated. To do this one has to compare (12) with (11), for which one has to know the variation $\delta_\Phi(\sqrt{-g}\mathscr{L})$. Because the fields $\mathbf{T}$ and $\boldsymbol{\varphi}$ are included in the Lagrangian in a similar way, for simplification of the calculations we unite them into the unique set $\boldsymbol{\phi}$:

$$\mathbf{T}, \boldsymbol{\varphi} \quad\to\quad \boldsymbol{\phi} \stackrel{def}{=} \{\phi^a\} \stackrel{def}{=} \{\mathbf{T}, \boldsymbol{\varphi}\}. \tag{24}$$

If necessary one can decompose the set $\boldsymbol{\phi}$ again. Now, the Lagrangian (23) is presented as

$$\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R};\ \boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla\nabla}\boldsymbol{\phi}). \tag{25}$$

One has to keep in mind that the torsion $\mathbf{T}$ is included in the Lagrangian (23) not only *explicitly* as arguments $\mathbf{T}$, $\boldsymbol{\nabla}\mathbf{T}$ and $\boldsymbol{\nabla\nabla}\mathbf{T}$, but *not explicitly* also over the connection $\boldsymbol{\Gamma}$, which is used for constructing the covariant derivative $\boldsymbol{\nabla}$ and the curvature tensor $\mathbf{R}$.

As it was remarked in Introduction, already in the works by Trautman[51–54] and by Hehl at al[55–57] the construction of the conservation laws and conserved quantities in the framework of the manifestly covariant theories has been carried out. Theories with the Lagrangians of the type

$$\mathscr{L} = \mathscr{L}(\mathbf{g};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}). \tag{26}$$

were considered. It was shown that the general relations

$$\frac{\Delta I}{\Delta T^\alpha{}_{\beta\gamma}} = \frac{1}{2}b^{\gamma\beta}{}_\alpha; \tag{27}$$

$$\overset{*}{\nabla}_\mu t^\mu{}_\nu = -t^\mu{}_\lambda T^\lambda{}_{\mu\nu} + \frac{1}{2}s^{\pi,\,\rho\sigma}R_{\rho\sigma\pi\nu} \quad \text{(on the } \boldsymbol{\varphi}\text{-equations)} \tag{28}$$

take a place. The first of them shows that the variational derivative of the action functional I with respect to the torsion $\mathbf{T}$ is equal to the half of the Belinfante tensor $\mathbf{b} \stackrel{def}{=} \{b^{\gamma\beta\alpha}\}$ induced by the canonical ST $\mathbf{s} \stackrel{def}{=} \{s^\pi{}_{\rho\sigma}\}$. The second one is the equation of balance for the canonical EMT $\mathbf{t} \stackrel{def}{=} \{t^\mu{}_\nu\}$. Of course, the study of the theories with the Lagrangians of the type (23), generalizing (27) and (28), has not to lead to contradictions with them.

III. CALCULATION OF THE TENSORS K AND L

Variate the lagrangian (25):

$$\delta\left(\sqrt{-g}\mathscr{L}\right) = \left(\delta\sqrt{-g}\right)\mathscr{L} + \sqrt{-g}\,\delta\mathscr{L}. \tag{29}$$

The variation of the first term is defined by the well known relation $\delta\sqrt{-g} = \sqrt{-g}g^{\beta\gamma}\delta g_{\beta\gamma}/2$. The second one, taking into account the above, can be presented in the form:

$$\delta\mathscr{L} = \frac{\partial^*\mathscr{L}}{\partial g_{\beta\gamma}}\delta g_{\beta\gamma} + \frac{\partial\mathscr{L}}{\partial R^\kappa{}_{\lambda\mu\nu}}\delta R^\kappa{}_{\lambda\mu\nu} + \frac{\partial^*\mathscr{L}}{\partial\phi^a}\delta\phi^a + \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\phi^a)}\delta(\nabla_\mu\phi^a) + \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\phi^a)}\delta(\nabla_\mu\nabla_\nu\phi^a). \tag{30}$$

Hereinafter, $\partial^*\mathscr{L}/\partial g_{\beta\gamma}$ means *explicit* derivative with respect to $g_{\beta\gamma}$, that is the differentiation is provided only with respect to $g_{\beta\gamma}$, which do not included into $\mathbf{R}$ and $\boldsymbol{\nabla}$; analogously, $\partial^*\mathscr{L}/\partial\phi^a$ means differentiation only with respect to ϕ^a, which do not included into $\boldsymbol{\nabla\phi}$ and $\boldsymbol{\nabla\nabla\phi}$. Substituting the expressions for variations $\delta R^\kappa{}_{\lambda\mu\nu}$ (B7), $\delta(\nabla_\mu\phi^a)$ (B10) and $\delta(\nabla_\mu\nabla_\nu\phi^a)$ (B11) into (30), providing the differentiation by parts and grouping similar terms, obtain

$$\begin{aligned}\delta\mathscr{L} &= \left\{\frac{\partial^*\mathscr{L}}{\partial g_{\beta\gamma}}\right\}\delta g_{\beta\gamma} + \left\{\frac{\Delta I}{\Delta\Gamma^\kappa{}_{\lambda\tau}}\right\}\delta\Gamma^\kappa{}_{\lambda\tau} + \left\{\frac{\Delta^* I}{\Delta\phi^a}\right\}\delta\phi^a \\ &+\overset{*}{\nabla}_\mu\left[\left\{G_\kappa{}^{\lambda\mu\nu} + \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\phi^a)}(\Delta^\lambda{}_\kappa)\,{}^a|_b\,\phi^b\right\}\delta\Gamma^\kappa{}_{\lambda\nu} + \left\{\frac{\Delta I}{\Delta(\nabla_\mu\phi^a)}\right\}\delta\phi^a + \left\{\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\alpha\phi^a)}\right\}\nabla_\alpha\delta\phi^a\right],\end{aligned} \tag{31}$$

where the notations

$$\begin{aligned}\frac{\Delta I}{\Delta\Gamma^\kappa{}_{\lambda\tau}} &\overset{def}{=} \left(\overset{*}{\nabla}_\nu G_\kappa{}^{\lambda\tau\nu} + \frac{1}{2}G_\kappa{}^{\lambda\rho\sigma}T^\tau{}_{\rho\sigma}\right) \\ &+\frac{\Delta I}{\Delta(\nabla_\tau\phi^a)}(\Delta^\lambda{}_\kappa)\,{}^a|_b\,\phi^b \\ &+\frac{\partial\mathscr{L}}{\partial(\nabla_\tau\nabla_\nu\phi^a)}\left[(\Delta^\lambda{}_\kappa)\,{}^a|_b\,\nabla_\nu\phi^b - \delta^\lambda_\nu\nabla_\kappa\phi^a\right];\end{aligned} \tag{32}$$

$$G_\kappa{}^{\lambda\mu\nu} \overset{def}{=} 2\frac{\partial\mathscr{L}}{\partial R^\kappa{}_{\lambda\mu\nu}}; \tag{33}$$

$$\frac{\Delta^* I}{\Delta\phi^a} \overset{def}{=} \frac{\partial^*\mathscr{L}}{\partial\phi^a} - \overset{*}{\nabla}_\mu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\phi^a)}\right) + \overset{*}{\nabla}_\nu\overset{*}{\nabla}_\mu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\phi^a)}\right); \tag{34}$$

$$\frac{\Delta I}{\Delta(\nabla_\mu\phi^a)} \overset{def}{=} \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\phi^a)} - \overset{*}{\nabla}_\nu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\nu\nabla_\mu\phi^a)}\right) \tag{35}$$

are used, and $\{(\Delta^\lambda{}_\kappa)\,{}^a|_b\}$ are the Belinfante-Rosenfeld symbols (see Appendix C 1). Next, substituting the expression $\{\delta\Gamma^\kappa{}_{\lambda\tau}\}$ (B6) into (31), providing in the term $\{\Delta I/\Delta\Gamma^\kappa{}_{\lambda\tau}\}\,\delta\Gamma^\kappa{}_{\lambda\tau}$ differentiation by parts and again grouping similar terms, one obtains

$$\begin{aligned}\delta\mathscr{L} &= \left\{\frac{\partial^*\mathscr{L}}{\partial g_{\beta\gamma}} - \overset{*}{\nabla}_\mu\left(\frac{\Delta I}{\Delta\Gamma^\kappa{}_{\lambda\tau}}\right)g^{\kappa\pi}\Delta^{\mu\beta\gamma}_{\underline{\pi\lambda\tau}}\right\}\delta g_{\beta\gamma} + \left\{\frac{\Delta I}{\Delta\Gamma^\kappa{}_{\lambda\tau}}g^{\kappa\pi}\Delta^{\mu\beta\gamma}_{\underline{\pi\lambda\tau}}g_{\mu\varepsilon}\right\}\delta T^\varepsilon{}_{\beta\gamma} + \left\{\frac{\Delta I}{\Delta\phi^a}\right\}\delta\phi^a \\ &+\overset{*}{\nabla}_\mu\left[\left\{\frac{\Delta I}{\Delta\Gamma^\kappa{}_{\lambda\tau}}g^{\kappa\pi}\Delta^{\mu\beta\gamma}_{\underline{\pi\lambda\tau}}\right\}\delta g_{\beta\gamma} + \left\{\left(G_\kappa{}^{\lambda\mu\nu} + \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\phi^a)}(\Delta^\lambda{}_\kappa)\,{}^a|_b\,\phi^b\right)g^{\kappa\pi}\Delta^{\alpha\beta\gamma}_{\underline{\pi\lambda\nu}}\right\}\nabla_\alpha\delta g_{\beta\gamma}\right. \\ &\left.+\left\{\left(G_\kappa{}^{\lambda\mu\nu} + \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\phi^a)}(\Delta^\lambda{}_\kappa)\,{}^a|_b\,\phi^b\right)g^{\kappa\pi}\Delta^{\alpha\beta\gamma}_{\underline{\pi\lambda\nu}}g_{\alpha\varepsilon}\right\}\delta T^\varepsilon{}_{\beta\gamma} + \left\{\frac{\Delta I}{\Delta(\nabla_\mu\phi^a)}\right\}\delta\phi^a + \left\{\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\alpha\phi^a)}\right\}\nabla_\alpha\delta\phi^a\right].\end{aligned} \tag{36}$$

At last, substituting (36) into (29) and recalling that due to the convention (24) the set of fields $\boldsymbol{\phi}$ consists of the torsion field $\mathbf{T}$ and a set of matter fields $\boldsymbol{\varphi}$, one obtains the search expression for the functional variation of the action:

$$
\begin{aligned}
\delta_\Phi I = &\int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\,\Bigg[\left\{\frac{1}{2}\mathscr{L} g^{\beta\gamma} + \frac{\partial^*\mathscr{L}}{\partial g_{\beta\gamma}} - \overset{*}{\nabla}_\mu\left(\frac{\Delta I}{\Delta\Gamma^\kappa{}_{\lambda\tau}}\right) g^{\kappa\pi}\Delta^{\mu\beta\gamma}_{\underline{\pi\lambda\tau}}\right\}\delta g_{\beta\gamma}\\
&+\left\{\frac{\partial^*\mathscr{L}}{\partial T^\varepsilon{}_{\beta\gamma}} - \overset{*}{\nabla}_\mu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\mu T^\varepsilon{}_{\beta\gamma})}\right) + \overset{*}{\nabla}_\nu\overset{*}{\nabla}_\mu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu T^\varepsilon{}_{\beta\gamma})}\right) + \frac{\Delta I}{\Delta\Gamma^\kappa{}_{\lambda\tau}} g^{\kappa\pi}\Delta^{\mu\beta\gamma}_{\underline{\pi\lambda\tau}} g_{\mu\varepsilon}\right\}\delta T^\varepsilon{}_{\beta\gamma}\\
&+\left\{\frac{\partial^*\mathscr{L}}{\partial\varphi^a} - \overset{*}{\nabla}_\mu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\varphi^a)}\right) + \overset{*}{\nabla}_\nu\overset{*}{\nabla}_\mu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\varphi^a)}\right)\right\}\delta\varphi^a\Bigg]\\
&+\int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\,\overset{*}{\nabla}_\mu\Bigg[\left\{\frac{\Delta I}{\Delta\Gamma^\kappa{}_{\lambda\tau}} g^{\kappa\pi}\Delta^{\mu\beta\gamma}_{\underline{\pi\lambda\tau}}\right\}\delta g_{\beta\gamma}\\
&+\left\{\left(G_\kappa{}^{\lambda\mu\nu} + \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu T^\tau{}_{\theta\phi})}(\Delta^\lambda{}_\kappa)\,{}^\tau{}_{\theta\phi}\big|_\omega{}^{\rho\sigma}\,T^\omega{}_{\rho\sigma} + \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\varphi^a)}(\Delta^\lambda{}_\kappa)\,{}^a\big|_b\,\varphi^b\right) g^{\kappa\pi}\Delta^{\alpha\beta\gamma}_{\underline{\pi\lambda\nu}}\right\}\nabla_\alpha\delta g_{\beta\gamma}\\
&+\left\{\left(G_\kappa{}^{\lambda\mu\nu} + \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu T^\tau{}_{\theta\phi})}(\Delta^\lambda{}_\kappa)\,{}^\tau{}_{\theta\phi}\big|_\omega{}^{\rho\sigma}\,T^\omega{}_{\rho\sigma} + \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\varphi^a)}(\Delta^\lambda{}_\kappa)\,{}^a\big|_b\,\varphi^b\right) g^{\kappa\pi}\Delta^{\alpha\beta\gamma}_{\underline{\pi\lambda\nu}} g_{\alpha\varepsilon}\right.\\
&\left.+\frac{\Delta I}{\Delta(\nabla_\mu T^\varepsilon{}_{\beta\gamma})}\right\}\delta T^\varepsilon{}_{\beta\gamma} + \left\{\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\alpha T^\varepsilon{}_{\beta\gamma})}\right\}\nabla_\alpha\delta T^\varepsilon{}_{\beta\gamma} + \left\{\frac{\Delta I}{\Delta(\nabla_\mu\varphi^a)}\right\}\delta\varphi^a + \left\{\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\alpha\varphi^a)}\right\}\nabla_\alpha\delta\varphi^a\Bigg],
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
\frac{\Delta I}{\Delta\Gamma^\kappa{}_{\lambda\tau}} \overset{def}{=} &\left(\overset{*}{\nabla}_\nu G_\kappa{}^{\lambda\tau\nu} + \frac{1}{2}G_\kappa{}^{\lambda\rho\sigma}T^\tau{}_{\rho\sigma}\right)\\
&+\frac{\Delta I}{\Delta(\nabla_\tau T^\varepsilon{}_{\beta\gamma})}(\Delta^\lambda{}_\kappa)\,{}^\varepsilon{}_{\beta\gamma}\big|_\omega{}^{\rho\sigma}\,T^\omega{}_{\rho\sigma} + \frac{\partial\mathscr{L}}{\partial(\nabla_\tau\nabla_\nu T^\varepsilon{}_{\beta\gamma})}\left[(\Delta^\lambda{}_\kappa)\,{}^\varepsilon{}_{\beta\gamma}\big|_\omega{}^{\rho\sigma}\,\nabla_\nu T^\omega{}_{\rho\sigma} - \delta^\lambda_\nu\nabla_\kappa T^\varepsilon{}_{\beta\gamma}\right]\\
&+\frac{\Delta I}{\Delta(\nabla_\tau\varphi^a)}(\Delta^\lambda{}_\kappa)\,{}^a\big|_b\,\varphi^b + \frac{\partial\mathscr{L}}{\partial(\nabla_\tau\nabla_\nu\varphi^a)}\left[(\Delta^\lambda{}_\kappa)\,{}^a\big|_b\,\nabla_\nu\varphi^b - \delta^\lambda_\nu\nabla_\kappa\varphi^a\right];
\end{aligned}
\tag{38}
$$

$$
\frac{\Delta I}{\Delta(\nabla_\mu T^\varepsilon{}_{\beta\gamma})} \overset{def}{=} \frac{\partial\mathscr{L}}{\partial(\nabla_\mu T^\varepsilon{}_{\beta\gamma})} - \overset{*}{\nabla}_\nu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\nu\nabla_\mu T^\varepsilon{}_{\beta\gamma})}\right); \tag{39}
$$

$$
\frac{\Delta I}{\Delta(\nabla_\mu\varphi^a)} \overset{def}{=} \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\varphi^a)} - \overset{*}{\nabla}_\nu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\nu\nabla_\mu\varphi^a)}\right). \tag{40}
$$

On the other hand, for the Lagrangian (23) the formula (12) acquires the form:

$$
\begin{aligned}
\delta_\Phi I = &\int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\left[\frac{\Delta I}{\Delta g_{\beta\gamma}}\delta g_{\beta\gamma} + \frac{\Delta I}{\Delta T^\varepsilon{}_{\beta\gamma}}\delta T^\varepsilon{}_{\beta\gamma} + \frac{\Delta I}{\Delta\varphi^a}\delta\varphi^a\right]\\
&+\int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\,\overset{*}{\nabla}_\mu\left[\left(K^\mu|^{\beta\gamma}\delta g_{\beta\gamma} + L^{\alpha\mu}|^{\beta\gamma}\nabla_\alpha\delta g_{\beta\gamma}\right) + \left(K^\mu|_\varepsilon{}^{\beta\gamma}\delta T^\varepsilon{}_{\beta\gamma} + L^{\alpha\mu}|_\varepsilon{}^{\beta\gamma}\nabla_\alpha\delta T^\varepsilon{}_{\beta\gamma}\right)\right.\\
&\left.+\left(K^\mu|_a\delta\varphi^a + L^{\alpha\mu}|_a\nabla_\alpha\delta\varphi^a\right)\right].
\end{aligned}
\tag{41}
$$

Comparing (37) with the last expression, one can recognize expressions and quantities interesting in our study:

$$
\left\{
\begin{aligned}
&\frac{\Delta I}{\Delta g_{\beta\gamma}} = \frac{1}{2}\mathscr{L} g^{\beta\gamma} + \frac{\partial^*\mathscr{L}}{\partial g_{\beta\gamma}} - \overset{*}{\nabla}_\mu\left(\frac{\Delta I}{\Delta\Gamma^\kappa{}_{\lambda\tau}} g^{\kappa\pi}\right)\Delta^{\mu(\beta\gamma)}_{\underline{\pi\lambda\tau}}; && (42)\\
&\frac{\Delta I}{\Delta T^\varepsilon{}_{\beta\gamma}} = \frac{\Delta^* I}{\Delta T^\varepsilon{}_{\beta\gamma}} + \left(\frac{\Delta I}{\Delta\Gamma^\kappa{}_{\lambda\tau}} g^{\kappa\pi}\right)\Delta^{\mu[\beta\gamma]}_{\underline{\pi\lambda\tau}} g_{\mu\varepsilon}; && (43)\\
&\frac{\Delta I}{\Delta\varphi^a} = \frac{\partial^*\mathscr{L}}{\partial\varphi^a} - \overset{*}{\nabla}_\mu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\varphi^a)}\right) + \overset{*}{\nabla}_\nu\overset{*}{\nabla}_\mu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\varphi^a)}\right); && (44)
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
&K^\mu|^{\beta\gamma} = \frac{\Delta I}{\Delta\Gamma^\kappa{}_{\lambda\tau}} g^{\kappa\pi}\Delta^{\mu(\beta\gamma)}_{\underline{\pi\lambda\tau}}; && (45)\\
&L^{\alpha\mu}|^{\beta\gamma} = \left(G_\kappa{}^{\lambda\mu\nu} + \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu T^\tau{}_{\theta\phi})}(\Delta^\lambda{}_\kappa)\,{}^\tau{}_{\theta\phi}\big|_\omega{}^{\rho\sigma}\,T^\omega{}_{\rho\sigma} + \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\varphi^a)}(\Delta^\lambda{}_\kappa)\,{}^a\big|_b\,\varphi^b\right) g^{\kappa\pi}\Delta^{\alpha(\beta\gamma)}_{\underline{\pi\lambda\nu}}; && (46)
\end{aligned}
\right.
$$

$$\begin{cases} K^{\mu}|_{\varepsilon}{}^{\beta\gamma} = \left(G_{\kappa}{}^{\lambda\mu\nu} + \dfrac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu T^{\tau}{}_{\theta\phi})}(\Delta^{\lambda}{}_{\kappa})\,{}^{\tau}{}_{\theta\phi}\big|_{\omega}{}^{\rho\sigma}\,T^{\omega}{}_{\rho\sigma} + \dfrac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\varphi^{a})}(\Delta^{\lambda}{}_{\kappa})\,{}^{a}\big|_{b}\,\varphi^{b}\right) g^{\kappa\pi}\Delta^{\alpha[\beta\gamma]}_{\underline{\pi\lambda\nu}}g_{\alpha\varepsilon} \\ \qquad + \dfrac{\Delta I}{\Delta(\nabla_\mu T^{\varepsilon}{}_{\beta\gamma})}; & (47) \\ L^{\alpha\mu}|_{\varepsilon}{}^{\beta\gamma} = \dfrac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\alpha T^{\varepsilon}{}_{\beta\gamma})}; & (48) \end{cases}$$

$$\begin{cases} K^{\mu}|_{a} = \dfrac{\Delta I}{\Delta(\nabla_\mu\varphi^{a})}; & (49) \\ L^{\alpha\mu}|_{a} = \dfrac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\alpha\varphi^{a})}. & (50) \end{cases}$$

Here,

$$\frac{\Delta^* I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} \stackrel{def}{=} \frac{\partial^*\mathscr{L}}{\partial T^{\varepsilon}{}_{\beta\gamma}} - \overset{*}{\nabla}_{\mu}\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\mu T^{\varepsilon}{}_{\beta\gamma})}\right) + \overset{*}{\nabla}_{\nu}\overset{*}{\nabla}_{\mu}\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu T^{\varepsilon}{}_{\beta\gamma})}\right), \tag{51}$$

and $\Delta I/\Delta\Gamma^{\kappa}{}_{\lambda\tau}$, $\Delta I/\Delta(\nabla_\mu T^{\varepsilon}{}_{\beta\gamma})$ and $\Delta I/\Delta(\nabla_\mu\varphi^{a})$ are defined by the formulae (38), (39) and (40), respectively. Thus, calculating the tensors **K** and **L** is finalized.

To simplify calculations remark the following. Comparing the formulae (48) and (50) with the formula (46), it is easily to find that there is a connection between the quantities $\{L^{\alpha\mu}|^{\beta\gamma}\}$ and $\{L^{\alpha\mu}|_{\varepsilon}{}^{\beta\gamma}\}$, $\{L^{\alpha\mu}|_{a}\}$

$$\begin{aligned} L^{\alpha\mu}|^{\beta\gamma} &= \left(G_{\kappa}{}^{\lambda\mu\nu} + L^{\nu\mu}|_{\tau}{}^{\theta\varphi}(\Delta^{\lambda}{}_{\kappa})\,{}^{\tau}{}_{\theta\varphi}\big|_{\omega}{}^{\rho\sigma}\,T^{\omega}{}_{\rho\sigma} \right. \\ &\quad \left. + L^{\nu\mu}|_{a}(\Delta^{\lambda}{}_{\kappa})\,{}^{a}\big|_{b}\,\varphi^{b}\right) g^{\kappa\pi}\Delta^{\alpha(\beta\gamma)}_{\underline{\pi\lambda\nu}} \\ &\stackrel{def}{=} \left(G_{\kappa}{}^{\lambda\mu\nu} + L^{\nu\mu}|_{a}(\Delta^{\lambda}{}_{\kappa})\,{}^{a}\big|_{b}\,\phi^{b}\right) g^{\kappa\pi}\Delta^{\alpha(\beta\gamma)}_{\underline{\pi\lambda\nu}}. \end{aligned} \tag{52}$$

Analogously, comparing the formulae (48) and (50) with the formula (47), one finds the connection between the quantities $\{K^{\mu}|_{\varepsilon}{}^{\beta\gamma}\}$ and $\{L^{\alpha\mu}|_{\varepsilon}{}^{\beta\gamma}\}$, $\{L^{\alpha\mu}|_{a}\}$:

$$\begin{aligned} K^{\mu}|_{\varepsilon}{}^{\beta\gamma} &= \frac{\Delta I}{\Delta(\nabla_\mu T^{\varepsilon}{}_{\beta\gamma})} \\ &\quad + \left(G_{\kappa}{}^{\lambda\mu\nu} + L^{\nu\mu}|_{\tau}{}^{\theta\varphi}(\Delta^{\lambda}{}_{\kappa})\,{}^{\tau}{}_{\theta\varphi}\big|_{\omega}{}^{\rho\sigma}\,T^{\omega}{}_{\rho\sigma} \right. \\ &\quad \left. + L^{\nu\mu}|_{a}(\Delta^{\lambda}{}_{\kappa})\,{}^{a}\big|_{b}\,\varphi^{b}\right) g^{\kappa\pi}\Delta^{\alpha[\beta\gamma]}_{\underline{\pi\lambda\nu}}g_{\alpha\varepsilon} \\ &\stackrel{def}{=} {}^{*}K^{\mu}|_{\varepsilon}{}^{\beta\gamma} + \left(G_{\kappa}{}^{\lambda\mu\nu} + L^{\nu\mu}|_{a}(\Delta^{\lambda}{}_{\kappa})\,{}^{a}\big|_{b}\,\phi^{b}\right) g^{\kappa\pi}\Delta^{\alpha[\beta\gamma]}_{\underline{\pi\lambda\nu}}g_{\alpha\varepsilon}, \end{aligned} \tag{53}$$

where

$${}^{*}K^{\mu}|_{\varepsilon}{}^{\beta\gamma} \stackrel{def}{=} \frac{\Delta I}{\Delta(\nabla_\mu T^{\varepsilon}{}_{\beta\gamma})}, \tag{54}$$

compare with (49). Using the formulae (48) – (50), (54) in the expression (38), one can present it in a more compact form:

$$\begin{aligned} \frac{\Delta I}{\Delta\Gamma^{\kappa}{}_{\lambda\tau}} &= \left(\overset{*}{\nabla}_{\nu}G_{\kappa}{}^{\lambda\tau\nu} + \frac{1}{2}G_{\kappa}{}^{\lambda\rho\sigma}T^{\tau}{}_{\rho\sigma}\right) \\ &\quad + {}^{*}K^{\tau}|_{\varepsilon}{}^{\beta\gamma}(\Delta^{\lambda}{}_{\kappa})\,{}^{\varepsilon}{}_{\beta\gamma}\big|_{\omega}{}^{\rho\sigma}\,T^{\omega}{}_{\rho\sigma} + L^{\nu\tau}|_{\varepsilon}{}^{\beta\gamma}\left[(\Delta^{\lambda}{}_{\kappa})\,{}^{\varepsilon}{}_{\beta\gamma}\big|_{\omega}{}^{\rho\sigma}\,\nabla_{\nu}T^{\omega}{}_{\rho\sigma} - \delta^{\lambda}_{\nu}\nabla_{\kappa}T^{\varepsilon}{}_{\beta\gamma}\right] \\ &\quad + K^{\tau}|_{a}(\Delta^{\lambda}{}_{\kappa})\,{}^{a}\big|_{b}\,\varphi^{b} + L^{\nu\tau}|_{a}\left[(\Delta^{\lambda}{}_{\kappa})\,{}^{a}\big|_{b}\,\nabla_{\nu}\varphi^{b} - \delta^{\lambda}_{\nu}\nabla_{\kappa}\varphi^{a}\right] \\ &\stackrel{def}{=} \left(\overset{*}{\nabla}_{\nu}G_{\kappa}{}^{\lambda\tau\nu} + \frac{1}{2}G_{\kappa}{}^{\lambda\rho\sigma}T^{\tau}{}_{\rho\sigma}\right) + {}^{*}K^{\tau}|_{a}(\Delta^{\lambda}{}_{\kappa})\,{}^{a}\big|_{b}\,\phi^{b} + L^{\nu\tau}|_{a}\left[(\Delta^{\lambda}{}_{\kappa})\,{}^{a}\big|_{b}\,\nabla_{\nu}\phi^{b} - \delta^{\lambda}_{\nu}\nabla_{\kappa}\phi^{a}\right]. \end{aligned} \tag{55}$$

IV. THE CALCULATION OF THE TENSORS U, M AND N

Next, we calculate the tensors **U**, **M** and **N** following the formulae (3), (4) and (5) in the manifestly covariant theories with the Lagrangians of the type (23). As we think, we have found the most economical scheme of calculations. Now, we follow it in details. For this we need the formulae

$$g_{\alpha}|_{\beta\gamma} = 2T_{(\beta,\,\gamma)\alpha}; \tag{56}$$

$$\phi_{\alpha}|^{a} = -\left\{\nabla_{\alpha}\phi^{a} + T^{\gamma}{}_{\beta\alpha}(\Delta^{\beta}{}_{\gamma})\,{}^{a}\big|_{b}\,\phi^{b}\right\}; \tag{57}$$

$$\begin{aligned} T_{\alpha}|^{\varepsilon}{}_{\beta\gamma} &= -\nabla_{\alpha}T^{\varepsilon}{}_{\beta\gamma} \\ &\quad -(T^{\varepsilon}{}_{\kappa\alpha}T^{\kappa}{}_{\beta\gamma} + T^{\varepsilon}{}_{\kappa\beta}T^{\kappa}{}_{\gamma\alpha} + T^{\varepsilon}{}_{\kappa\gamma}T^{\kappa}{}_{\alpha\beta}); \end{aligned} \tag{58}$$

$$g_{\alpha}{}^{\beta}|_{\kappa\lambda} = -2g_{\alpha(\kappa}\delta^{\beta}_{\lambda)}; \tag{59}$$

$$\phi_{\alpha}{}^{\beta}|^{a} = (\Delta^{\beta}{}_{\alpha})\,{}^{a}\big|_{b}\,\phi^{b}; \tag{60}$$

$$T_\alpha{}^\lambda|^\varepsilon{}_{\beta\gamma} = \delta^\varepsilon_\alpha T^\lambda{}_{\beta\gamma} + 2T^\varepsilon{}_{\alpha[\beta}\delta^\lambda_{\gamma]}, \tag{61}$$

proved in Appendix C 3.

A. The calculation of the tensor $\mathbf{N}$

For the Lagrangian (23) the formula (5) transforms to

$$\begin{aligned}&N_\alpha{}^{\kappa\lambda\mu} \overset{def}{=} L^{(\lambda|\mu|}{}_A\Phi_\alpha{}^{|\kappa)}|^A\\ &= L^{(\lambda|\mu|}{}^{\beta\gamma}g_\alpha{}^{|\kappa)}|_{\beta\gamma} + L^{(\lambda|\mu|}{}_\varepsilon{}^{\beta\gamma}T_\alpha{}^{|\kappa)}|^\varepsilon{}_{\beta\gamma} + L^{(\lambda|\mu|}{}_a\varphi_\alpha{}^{|\kappa)}|^a.\end{aligned}$$

Let us present the calculation in the next steps.

1. Return to the united field $\boldsymbol{\phi} = \{\mathbf{T}, \boldsymbol{\varphi}\}$ (24). Then

$$N_\alpha{}^{\kappa\lambda\mu} = L^{(\lambda|\mu|}{}^{\beta\gamma}g_\alpha{}^{|\kappa)}|_{\beta\gamma} + L^{(\lambda|\mu|}{}_a\phi_\alpha{}^{|\kappa)}|^a.$$

2. Take into account the connection (52) and obtain

$$\begin{aligned}N_\alpha{}^{\kappa\lambda\mu} &= \left(G_\varepsilon{}^{\eta\mu\nu} + L^{\nu\mu}|_a(\Delta^\eta{}_\varepsilon)\,{}^a|_b\,\phi^b\right) g^{\varepsilon\pi}\Delta^{(\lambda|(\beta\gamma)}_{\underline{\pi\eta\nu}}\, g_\alpha{}^{|\kappa)}|_{\beta\gamma}\\ &+L^{(\lambda|\mu|}{}_a\phi_\alpha{}^{|\kappa)}|^a.\end{aligned}$$

3. Substitute here the expressions (59) and (60). Then, with using the identity (A10), the tensor $\mathbf{N}$ is transformed to the form:

$$\begin{aligned}N_\alpha{}^{\kappa\lambda\mu} &= -\left(G_\varepsilon{}^{\eta\mu\nu} + L^{\nu\mu}|_a(\Delta^\eta{}_\varepsilon)\,{}^a|_b\,\phi^b\right)\delta^\varepsilon_\alpha\delta^{(\kappa}_\eta\delta^{\lambda)}_\nu\\ &+L^{(\lambda|\mu|}{}_a(\Delta^{|\kappa)}{}_\alpha)\,{}^a|_b\,\phi^b = G_\alpha{}^{(\kappa\lambda)\mu}.\end{aligned}$$

Thus, the expression for the tensor $\mathbf{N}$ gets the form:

$$\boxed{N_\alpha{}^{\beta\gamma\mu} = G_\alpha{}^{(\beta\gamma)\mu}.} \tag{62}$$

It is important to note that the *tensor* $\mathbf{N}$ *is not equal to zero only if the Lagrangian contains explicitly the curvature tensor* $\mathbf{R}$ (see the definition (33)).

B. The calculation of the tensor $\mathbf{M}$

For the Lagrangian (23) the formula (4) transforms to

$$\begin{aligned}M_\alpha{}^{\lambda\mu} &\overset{def}{=} K^\mu|_A\Phi_\alpha{}^\lambda|^A + L^{\lambda\mu}|_A\Phi_\alpha|^A + L^{\kappa\mu}|_A\left(\nabla_\kappa\Phi_\alpha{}^\lambda|^A - \frac{1}{2}T^\lambda{}_{\kappa\eta}\Phi_\alpha{}^\eta|^A\right)\\ &= \left\{K^\mu|^{\beta\gamma}g_\alpha{}^\lambda|_{\beta\gamma} + K^\mu|_\varepsilon{}^{\beta\gamma}T_\alpha{}^\lambda|^\varepsilon{}_{\beta\gamma} + K^\mu|_a\varphi_\alpha{}^\lambda|^a\right\}\\ &\quad+\left\{L^{\lambda\mu}|^{\beta\gamma}g_\alpha|_{\beta\gamma} + L^{\lambda\mu}|_\varepsilon{}^{\beta\gamma}T_\alpha|^\varepsilon{}_{\beta\gamma} + L^{\lambda\mu}|_a\varphi_\alpha|^a\right\} + \left\{L^{\rho\mu}|^{\beta\gamma}\left(\nabla_\rho g_\alpha{}^\lambda|_{\beta\gamma} - \frac{1}{2}T^\lambda{}_{\rho\sigma}g_\alpha{}^\sigma|_{\beta\gamma}\right)\right.\\ &\quad\left.+L^{\rho\mu}|_\varepsilon{}^{\beta\gamma}\left(\nabla_\rho T_\alpha{}^\lambda|^\varepsilon{}_{\beta\gamma} - \frac{1}{2}T^\lambda{}_{\rho\sigma}T_\alpha{}^\sigma|^\varepsilon{}_{\beta\gamma}\right) + L^{\rho\mu}|_a\left(\nabla_\rho\varphi_\alpha{}^\lambda|^a - \frac{1}{2}T^\lambda{}_{\rho\sigma}\varphi_\alpha{}^\sigma|^a\right)\right\}.\end{aligned} \tag{63}$$

Provide the calculation of (63) step by step also.

1. Denote the first, second and third braces on the right hand side of (63) as $\{\dots\}_1$, $\{\dots\}_2$ and $\{\dots\}_3$, respectively. Then

$$M_\alpha{}^{\lambda\mu} \overset{def}{=} \{\dots\}_1 + \{\dots\}_2 + \{\dots\}_3.$$

2. In $\{\dots\}_1$, take into account the expression (53) and return to the united field $\boldsymbol{\phi} = \{\mathbf{T}, \boldsymbol{\varphi}\}$. Then

$$\begin{aligned}\{\dots\}_1 &= K^\mu|^{\beta\gamma}g_\alpha{}^\lambda|_{\beta\gamma} + {}^*K^\mu|_a\phi_\alpha{}^\lambda|^a\\ &+\left(G_\kappa{}^{\eta\mu\nu} + L^{\nu\mu}|_a(\Delta^\eta{}_\kappa)\,{}^a|_b\,\phi^b\right) g^{\kappa\pi}\Delta^{\rho[\beta\gamma]}_{\underline{\pi\eta\nu}}\, g_{\rho\varepsilon}\,T_\alpha{}^\lambda|^\varepsilon{}_{\beta\gamma},\end{aligned}$$

where

$${}^*K^\mu|_a\phi_\alpha{}^\lambda|^a \overset{def}{=} {}^*K^\mu|_\varepsilon{}^{\beta\gamma}T_\alpha{}^\lambda|^\varepsilon{}_{\beta\gamma} + K^\mu|_a\varphi_\alpha{}^\lambda|^a. \tag{64}$$

3. In $\{\dots\}_2$, take into account the expression (52) and return to the united field $\boldsymbol{\phi}$ again. Then

$$\begin{aligned}\{\dots\}_2 &= \left(G_\kappa{}^{\eta\mu\nu} + L^{\nu\mu}|_a(\Delta^\eta{}_\kappa)\,{}^a|_b\,\phi^b\right) g^{\kappa\pi}\Delta^{\lambda(\beta\gamma)}_{\underline{\pi\eta\nu}}\, g_\alpha|_{\beta\gamma}\\ &+L^{\lambda\mu}|_a\phi_\alpha|^a.\end{aligned}$$

4. To provide $\{\dots\}_3$ apply the similar steps and obtain

$$\begin{aligned}\{\dots\}_3 &= \left(G_\kappa{}^{\eta\mu\nu} + L^{\nu\mu}|_a(\Delta^\eta{}_\kappa)\,{}^a|_b\,\phi^b\right) g^{\kappa\pi}\\ &\times\Delta^{\rho(\beta\gamma)}_{\underline{\pi\eta\nu}}\left(\nabla_\rho g_\alpha{}^\lambda|_{\beta\gamma} - \frac{1}{2}T^\lambda{}_{\rho\sigma}g_\alpha{}^\sigma|_{\beta\gamma}\right)\\ &+L^{\rho\mu}|_a\left(\nabla_\rho\phi_\alpha{}^\lambda|^a - \frac{1}{2}T^\lambda{}_{\rho\sigma}\phi_\alpha{}^\sigma|^a\right).\end{aligned}$$

5. Combining the results of the points 2 – 4 and collecting the similar terms, find

$$
\begin{aligned}
M_\alpha{}^{\lambda\mu} = {} & K^\mu|^{\beta\gamma} g_\alpha{}^\lambda|_{\beta\gamma} + {}^*K^\mu|_a \phi_\alpha{}^\lambda|^a + L^{\nu\mu}|_a \left[\delta^\lambda_\nu \phi_\alpha|^a + \nabla_\nu \phi_\alpha{}^\lambda|^a - \frac{1}{2} T^\lambda{}_{\nu\beta}\phi_\alpha{}^\beta|^a\right] \\
& + \left(G_\kappa{}^{\eta\mu\nu} + L^{\nu\mu}|_a (\Delta^\eta{}_\kappa)\,{}^a|_b\, \phi^b\right) g^{\kappa\pi} \Delta^{\rho\beta\gamma}_{\underline{\pi\eta\nu}} \left[g_{\rho\varepsilon} T_\alpha{}^\lambda|^\varepsilon{}_{\beta\gamma} + \delta^\lambda_\rho g_\alpha|_{\beta\gamma} + \nabla_\rho g_\alpha{}^\lambda|_{\beta\gamma} - \frac{1}{2} T^\lambda{}_{\rho\sigma} g_\alpha{}^\sigma|_{\beta\gamma}\right].
\end{aligned}
\tag{65}
$$

6. Corresponding to the formulae (45), (55) and (59) one has

$$
\begin{aligned}
K^\mu|^{\beta\gamma} g_\alpha{}^\lambda|_{\beta\gamma} = -2 & \Big[-\left(\overset{*}{\nabla}_\nu G^{\eta\pi\tau\nu} + \frac{1}{2} G^{\eta\pi\rho\sigma} T^\tau{}_{\rho\sigma}\right) \\
& + {}^*K^\tau|_a (\Delta^{\eta\pi})\,{}^a|_b\, \phi^b \\
& + L^{\nu\tau}|_a \left((\Delta^{\eta\pi})\,{}^a|_b\, \nabla_\nu \phi^b - \delta^\eta_\nu \nabla^\pi \phi^a\right)\Big] \Delta^{\mu(\lambda\beta)}_{\underline{\pi\eta\tau}} g_{\beta\alpha}.
\end{aligned}
\tag{66}
$$

7. Taking into account (57) and (60), and adding with subtracting the expression $\left(\overset{*}{\nabla}_\nu G_\alpha{}^{\lambda\mu\nu} + \frac{1}{2} G_\alpha{}^{\lambda\rho\sigma} T^\mu{}_{\rho\sigma}\right)$, one obtains for the sum of ${}^*K^\mu|_a \phi_\alpha{}^\lambda|^a$ and the next item on the right hand side of (65):

$$
\begin{aligned}
& \left[-\left(\overset{*}{\nabla}_\nu G^{\lambda\beta\mu\nu} + \frac{1}{2} G^{\lambda\beta\rho\sigma} T^\mu{}_{\rho\sigma}\right) + {}^*K^\mu|_a (\Delta^{\lambda\beta})\,{}^a|_b\, \phi^b + L^{\nu\mu}|_a \left((\Delta^{\lambda\beta})\,{}^a|_b\, \nabla_\nu \phi^b - \delta^\lambda_\nu \nabla^\beta \phi^a\right)\right] g_{\beta\alpha} \\
& \quad - \left[\overset{*}{\nabla}_\nu G_\alpha{}^{\lambda\mu\nu} + \frac{1}{2} G_\alpha{}^{\lambda\rho\sigma} T^\mu{}_{\rho\sigma}\right] - \frac{1}{2} L^{\nu\mu}|_a \left[2\delta^\lambda_\nu T^\gamma{}_{\beta\alpha} + T^\lambda{}_{\nu\beta}\delta^\gamma_\alpha\right] (\Delta^\beta{}_\gamma)\,{}^a|_b\, \phi^b.
\end{aligned}
$$

8. For the expression inside brackets of the last item in (65), taking into account the formulae (56), (59) and (61), and the identities (A12) and (A13), provide simple identical transformations and obtain

$$
\begin{aligned}
& \Delta^{\rho\beta\gamma}_{\underline{\pi\eta\nu}} \left[g_{\rho\varepsilon} \left(\delta^\varepsilon_\alpha T^\lambda{}_{\beta\gamma} + 2T^\varepsilon{}_{\alpha[\beta}\delta^\lambda_{\gamma]}\right) + 2\delta^\lambda_\rho T_{(\beta,\,\gamma)\alpha} + T^\lambda{}_{\rho\sigma} g_{\alpha(\beta}\delta^\sigma_{\gamma)}\right] \\
& \quad = \Delta^{\rho\beta\gamma}_{\underline{\pi\eta\nu}} \left[\left(g_{\alpha(\rho} T^\lambda{}_{\beta)\gamma} + 2\delta^\lambda_{(\rho} T_{\beta),\,\gamma\alpha}\right) + \left(g_{\alpha[\rho|} T^\lambda{}_{\beta|\gamma]} + 2\delta^\lambda_{[\rho} T_{\gamma],\,\beta\alpha}\right)\right] \\
& \quad = \Delta^{(\rho\beta)\gamma}_{\underline{\pi\eta\nu}} \left(g_{\alpha\rho} T^\lambda{}_{\beta\gamma} + 2\delta^\lambda_\rho T_{\beta,\,\gamma\alpha}\right) + \Delta^{[\rho|\beta|\gamma]}_{\underline{\pi\eta\nu}} \left(g_{\alpha\rho} T^\lambda{}_{\beta\gamma} + 2\delta^\lambda_\rho T_{\gamma,\,\beta\alpha}\right) \\
& \quad = \frac{1}{2}\delta^\rho_{(\pi}\delta^\beta_{\nu)}\delta^\gamma_\eta \left(g_{\alpha\rho} T^\lambda{}_{\beta\gamma} + 2\delta^\lambda_\rho T_{\beta,\,\gamma\alpha}\right) - \frac{1}{2}\delta^\beta_\eta \delta^\rho_{[\pi}\delta^\gamma_{\nu]} \left(g_{\alpha\rho} T^\lambda{}_{\beta\gamma} + 2\delta^\lambda_\rho T_{\gamma,\,\beta\alpha}\right) \\
& \quad = \frac{1}{2}\left[2\left(\delta^\lambda_{(\nu} T_{\pi),\,\eta\alpha} + \delta^\lambda_{[\nu} T_{\pi],\,\eta\alpha}\right) - \left(g_{\alpha(\pi|} T^\lambda{}_{\eta|\nu)} + g_{\alpha[\pi|} T^\lambda{}_{\eta|\nu]}\right)\right] \\
& \quad = \frac{1}{2}\left(2\delta^\lambda_\nu T_{\pi,\,\eta\alpha} - g_{\alpha\pi} T^\lambda{}_{\eta\nu}\right).
\end{aligned}
$$

Thus, the last item in (65) is equal to

$$
\frac{1}{2} G_\alpha{}^{\beta\gamma\mu} T^\lambda{}_{\beta\gamma} - \left(G_\gamma{}^{\beta\lambda\mu} T^\gamma{}_{\beta\alpha}\right) + \frac{1}{2} L^{\nu\mu}|_a \left[2\delta^\lambda_\nu T^\gamma{}_{\beta\alpha} + T^\lambda{}_{\nu\beta}\delta^\gamma_\alpha\right] (\Delta^\beta{}_\gamma)\,{}^a|_b\, \phi^b.
$$

9. Combining the results of the points 6 – 8, one finds

$$
\begin{aligned}
M_\alpha{}^{\lambda\mu} = -2 & \left[-\left(\overset{*}{\nabla}_\nu G^{\eta\pi\tau\nu} + \frac{1}{2} G^{\eta\pi\rho\sigma} T^\tau{}_{\rho\sigma}\right) + {}^*K^\tau|_a (\Delta^{\eta\pi})\,{}^a|_b\, \phi^b + L^{\nu\tau}|_a \left[(\Delta^{\eta\pi})\,{}^a|_b\, \nabla_\nu \phi^b - \delta^\eta_\nu \nabla^\pi \phi^a\right]\right] \\
\times & \left(\Delta^{\mu(\lambda\beta)}_{\underline{\pi\eta\tau}} - \frac{1}{2}\delta^\mu_\tau \delta^\lambda_\eta \delta^\beta_\pi\right) g_{\beta\alpha} - \left[\overset{*}{\nabla}_\nu G_\alpha{}^{\lambda\mu\nu} + \frac{1}{2} G_\alpha{}^{\lambda\rho\sigma} T^\mu{}_{\rho\sigma}\right] + \frac{1}{2} G_\alpha{}^{\beta\gamma\mu} T^\lambda{}_{\beta\gamma} - \left(G_\gamma{}^{\beta\lambda\mu} T^\gamma{}_{\beta\alpha}\right).
\end{aligned}
$$

10. At last, using the identity (A11) and formulae (48)–(50), (54), and denoting

$$
\boxed{
\begin{aligned}
s^\pi{}_{\rho\sigma} \overset{def}{=} & -2\left(\overset{*}{\nabla}_\eta G_{\rho\sigma}{}^{\pi\eta} + \frac{1}{2} G_{\rho\sigma}{}^{\alpha\beta} T^\pi{}_{\alpha\beta}\right) \\
& + 2\left(\frac{\Delta I}{\Delta(\nabla_\pi \phi^a)} (\Delta_{[\rho\sigma]})\,{}^a|_b\, \phi^b + \frac{\partial \mathscr{L}}{\partial(\nabla_\pi \nabla_\alpha \phi^a)} \left[(\Delta_{[\rho\sigma]})\,{}^a|_b\, \nabla_\alpha \phi^b - g_{\alpha[\rho} \nabla_{\sigma]} \phi^a\right]\right),
\end{aligned}
}
\tag{67}
$$

one obtains the finalized expression for the tensor **M**:

$$\boxed{M_{\kappa}{}^{\lambda\mu} = -\left(\Delta^{\overline{\mu\lambda\alpha}}_{\pi\rho\sigma} s^{\pi,\,\rho\sigma}\right) g_{\alpha\kappa} - \left(\overset{*}{\nabla}_{\nu} G_{\kappa}{}^{\lambda\mu\nu} + \frac{1}{2} G_{\kappa}{}^{\lambda\rho\sigma} T^{\mu}{}_{\rho\sigma}\right) + \frac{1}{2} G_{\kappa}{}^{\alpha\beta\mu} T^{\lambda}{}_{\alpha\beta} - \left(G_{\alpha}{}^{\beta\lambda\mu} T^{\alpha}{}_{\beta\kappa}\right).} \tag{68}$$

Remark that the tensor $\mathbf{s} \overset{def}{=} \{s^{\pi}{}_{\rho\sigma}\}$ (67) is just the *generalized canonical spin tensor*, corresponding to the Lagrangian (23). This statement follows from the results of Sec. VII. Namely, basing on the above definition of the ST, one obtains the standard equations of balance for the EMT. Besides, the gravitational field equations acquire the form, naturally generalizing the ECT equations. Remark that the items in the first parentheses on the right hand side of (67) are induced by the non-minimal coupling with the metric field. These items *in principal* cannot be obtained with the use of the 1-st Noether theorem in the Minkowski space and covariantization of the expressions.

C. The calculation of the tensor U

For the Lagrangian (23) the formula (3) has the form:

$$\begin{aligned} U_{\alpha}{}^{\mu} &\overset{def}{=} \mathscr{L}\delta^{\mu}_{\alpha} + K^{\mu}|_{A}\Phi_{\alpha}|^{A} + L^{\kappa\mu}|_{A}\left(\nabla_{\kappa}\Phi_{\alpha}|^{A} + \frac{1}{2}R^{\varepsilon}{}_{\alpha\kappa\lambda}\Phi_{\varepsilon}{}^{\lambda}|^{A}\right) \\ &= \mathscr{L}\delta^{\mu}_{\alpha} + \left\{K^{\mu}|^{\beta\gamma} g_{\alpha}|_{\beta\gamma} + K^{\mu}|_{\varepsilon}{}^{\beta\gamma} T_{\alpha}|^{\varepsilon}{}_{\beta\gamma} + K^{\mu}|_{a}\varphi_{\alpha}|^{a}\right\} + \left\{L^{\kappa\mu}|^{\beta\gamma}\left(\nabla_{\kappa} g_{\alpha}|_{\beta\gamma} + \frac{1}{2}R^{\sigma}{}_{\alpha\kappa\lambda} g_{\sigma}{}^{\lambda}|_{\beta\gamma}\right)\right. \\ &\quad \left. + L^{\kappa\mu}|_{\varepsilon}{}^{\beta\gamma}\left(\nabla_{\kappa} T_{\alpha}|^{\varepsilon}{}_{\beta\gamma} + \frac{1}{2}R^{\sigma}{}_{\alpha\kappa\lambda} T_{\sigma}{}^{\lambda}|^{\varepsilon}{}_{\beta\gamma}\right) + L^{\kappa\mu}|_{a}\left(\nabla_{\kappa}\varphi_{\alpha}|^{a} + \frac{1}{2}R^{\sigma}{}_{\alpha\kappa\lambda}\varphi_{\sigma}{}^{\lambda}|^{a}\right)\right\}. \end{aligned} \tag{69}$$

Transform it. The main steps are as follows.

1. The first and second braces on the right hand side of (69) denote as $\{\dots\}_4$ and $\{\dots\}_5$, respectively. Thus,

$$U_{\alpha}{}^{\mu} \overset{def}{=} \mathscr{L}\delta^{\mu}_{\alpha} + \{\dots\}_4 + \{\dots\}_5.$$

2. In $\{\dots\}_4$, take into account the expression (53) and return to the united field $\boldsymbol{\phi} = \{\mathbf{T}, \boldsymbol{\varphi}\}$. Then

$$\begin{aligned} \{\dots\}_4 &= K^{\mu}|^{\beta\gamma} g_{\alpha}|_{\beta\gamma} + {}^{*}K^{\mu}|_{a}\phi_{\alpha}|^{a} \\ &\quad + \left(G^{\pi\eta\mu\nu} + L^{\nu\mu}|_{a}(\Delta^{\eta\pi})\,{}^{a}|_{b}\,\phi^{b}\right)\Delta^{\rho[\beta\gamma]}_{\underline{\pi\eta\nu}} g_{\rho\varepsilon}\, T_{\alpha}|^{\varepsilon}{}_{\beta\gamma}, \end{aligned}$$

where

$${}^{*}K^{\mu}|_{a}\phi_{\alpha}|^{a} \overset{def}{=} {}^{*}K^{\mu}|_{\varepsilon}{}^{\beta\gamma} T_{\alpha}|^{\varepsilon}{}_{\beta\gamma} + K^{\mu}|_{a}\varphi_{\alpha}|^{a}. \tag{70}$$

3. In $\{\dots\}_5$, take into account the expression (52) and return to the united field $\boldsymbol{\phi} = \{\mathbf{T}, \boldsymbol{\varphi}\}$. Then

$$\begin{aligned} \{\dots\}_5 &= \left(G^{\pi\eta\mu\nu} + L^{\nu\mu}|_{a}(\Delta^{\eta\pi})\,{}^{a}|_{b}\,\phi^{b}\right) \\ &\quad \times \Delta^{\kappa(\beta\gamma)}_{\underline{\pi\eta\nu}}\left(\nabla_{\kappa} g_{\alpha}|_{\beta\gamma} + \frac{1}{2}R^{\varepsilon}{}_{\alpha\kappa\lambda} g_{\varepsilon}{}^{\lambda}|_{\beta\gamma}\right) \\ &\quad + L^{\kappa\mu}|_{a}\left(\nabla_{\kappa}\phi_{\alpha}|^{a} + \frac{1}{2}R^{\varepsilon}{}_{\alpha\kappa\lambda}\phi_{\varepsilon}{}^{\lambda}|^{a}\right). \end{aligned}$$

4. Combining the results of the points 2 – 3 and collecting the similar terms, find

$$\begin{aligned} U_{\alpha}{}^{\mu} &= \mathscr{L}\delta^{\mu}_{\alpha} + K^{\mu}|^{\beta\gamma} g_{\alpha}|_{\beta\gamma} + {}^{*}K^{\mu}|_{a}\phi_{\alpha}|^{a} + \left(G^{\pi\eta\mu\nu} + L^{\nu\mu}|_{a}(\Delta^{\eta\pi})\,{}^{a}|_{b}\,\phi^{b}\right)\Delta^{\kappa\beta\gamma}_{\underline{\pi\eta\nu}} \\ &\quad \times \left[g_{\kappa\varepsilon} T_{\alpha}|^{\varepsilon}{}_{\beta\gamma} + \nabla_{\kappa} g_{\alpha}|_{\beta\gamma} + \frac{1}{2}R^{\varepsilon}{}_{\alpha\kappa\lambda} g_{\varepsilon}{}^{\lambda}|_{\beta\gamma}\right] + L^{\kappa\mu}|_{a}\left(\nabla_{\kappa}\phi_{\alpha}|^{a} + \frac{1}{2}R^{\varepsilon}{}_{\alpha\kappa\lambda}\phi_{\varepsilon}{}^{\lambda}|^{a}\right). \end{aligned} \tag{71}$$

5. Remark that, corresponding to the formulae (45), (55) and (56),

$$\begin{aligned} &K^{\mu}|^{\beta\gamma} g_{\alpha}|_{\beta\gamma} \\ &= 2\left[-\left(\overset{*}{\nabla}_{\nu} G^{\eta\pi\tau\nu} + \frac{1}{2} G^{\eta\pi\rho\sigma} T^{\tau}{}_{\rho\sigma}\right) + {}^{*}K^{\tau}|_{a}(\Delta^{\eta\pi})\,{}^{a}|_{b}\,\phi^{b}\right. \\ &\quad \left. + L^{\nu\tau}|_{a}\left((\Delta^{\eta\pi})\,{}^{a}|_{b}\,\nabla_{\nu}\phi^{b} - \delta^{\eta}_{\nu}\nabla^{\pi}\phi^{a}\right)\right]\Delta^{\mu(\beta\gamma)}_{\underline{\pi\eta\tau}}\; T_{\gamma,\,\beta\alpha}. \end{aligned}$$

6. Turn to the right hand side of (71). Substitute the expression (57) into the third item, and substitute the expressions (57) and (60) into the last one. After that commutate the second covariant derivatives $\nabla_\kappa \nabla_\alpha \phi^a$ by the rule (C32). In the result one obtains

$$
\begin{aligned}
&L^{\nu\mu}|_a \left(\nabla_\nu \phi_\alpha|^a + \frac{1}{2} R^\gamma{}_{\alpha\nu\beta} \phi_\gamma{}^\beta|^a\right) \\
&\quad = -L^{\nu\mu}|_a \nabla_\alpha \nabla_\nu \phi^a \\
&\qquad -L^{\nu\mu}|_a \left[\left(\Delta^{\beta\gamma}\right){}^a|_b \nabla_\nu \phi^b - \delta^\beta_\nu \nabla^\gamma \phi^a\right] T_{\gamma,\,\beta\alpha} \\
&\qquad +L^{\nu\mu}|_a (\Delta^\beta{}_\gamma){}^a|_b\, \phi^b \left(R^\gamma{}_{\beta\alpha\nu} - \nabla_\nu T^\gamma{}_{\beta\alpha} + \frac{1}{2} R^\gamma{}_{\alpha\nu\beta}\right).
\end{aligned}
$$

7. Combining the results of the points 5 and 6, one can see that the sum of the first, second, third and fifth items, after adding and subtracting the combination $\left(\overset{*}{\nabla}_\nu G^{\beta\gamma\mu\nu} + \frac{1}{2} G^{\beta\gamma\rho\sigma} T^\mu{}_{\rho\sigma}\right) T_{\gamma,\,\beta\alpha}$, becomes

$$
\begin{aligned}
&[\mathscr{L}\delta^\mu_\alpha - {}^*K^\mu|_a \nabla_\alpha \phi^a - L^{\nu\mu}|_a \nabla_\alpha \nabla_\nu \phi^a] + 2\left[-\left(\overset{*}{\nabla}_\nu G^{\eta\pi\tau\nu} + \frac{1}{2} G^{\eta\pi\rho\sigma} T^\tau{}_{\rho\sigma}\right)\right. \\
&\quad \left. + {}^*K^\tau|_a (\Delta^{\eta\pi}){}^a|_b\, \phi^b + L^{\nu\tau}|_a \left(\left(\Delta^{\eta\pi}\right){}^a|_b \nabla_\nu \phi^b - \delta^\eta_\nu \nabla^\pi \phi^a\right)\right] \times \left(\Delta^{\mu(\beta\gamma)}_{\underline{\pi\eta\tau}} - \frac{1}{2}\delta^\mu_\tau \delta^\beta_\eta \delta^\gamma_\pi\right) T_{\gamma,\,\beta\alpha} \\
&\quad - \left(\overset{*}{\nabla}_\nu G^{\beta\gamma\mu\nu} + \frac{1}{2} G^{\beta\gamma\rho\sigma} T^\mu{}_{\rho\sigma}\right) T_{\gamma,\,\beta\alpha} + L^{\nu\mu}|_a (\Delta^\beta{}_\gamma){}^a|_b\, \phi^b \left(R^\gamma{}_{\beta\alpha\nu} - \nabla_\nu T^\gamma{}_{\beta\alpha} + \frac{1}{2} R^\gamma{}_{\alpha\nu\beta}\right) \qquad (72) \\
&= [\mathscr{L}\delta^\mu_\alpha - {}^*K^\mu|_a \nabla_\alpha \phi^a - L^{\nu\mu}|_a \nabla_\alpha \nabla_\nu \phi^a] + \left(\Delta^{\overline{\mu\beta\gamma}}_{\tau\eta\pi} s^{\tau,\,\eta\pi}\right) T_{\gamma,\,\beta\alpha} \\
&\quad - \left(\overset{*}{\nabla}_\nu G^{\beta\gamma\mu\nu} + \frac{1}{2} G^{\beta\gamma\rho\sigma} T^\mu{}_{\rho\sigma}\right) T_{\gamma,\,\beta\alpha} + L^{\nu\mu}|_a (\Delta^\beta{}_\gamma){}^a|_b\, \phi^b \left(R^\gamma{}_{\beta\alpha\nu} - \nabla_\nu T^\gamma{}_{\beta\alpha} + \frac{1}{2} R^\gamma{}_{\alpha\nu\beta}\right),
\end{aligned}
$$

where the identity (A11) and the definition (67) have been taken into account.

8. After substituting the equality

$$
-\overset{*}{\nabla}_\nu G^{\beta\gamma\mu\nu} T_{\gamma,\,\beta\alpha} = \overset{*}{\nabla}_\nu \left(G^{\beta\gamma\mu\nu} T_{\beta,\,\gamma\alpha}\right) - G_\gamma{}^{\beta\mu\nu} \nabla_\nu T^\gamma{}_{\beta\alpha},
$$

and adding with subtracting the expression

$$
\begin{aligned}
&G^{\gamma\beta\nu\mu} R_{\gamma\beta\nu\alpha} - \tfrac{1}{2} G^{\gamma\beta\nu\mu} R_{\alpha\gamma\beta\nu} \\
&\quad = G_\gamma{}^{\beta\mu\nu} \left(R^\gamma{}_{\beta\alpha\nu} + \tfrac{1}{2} R^\gamma{}_{\alpha\nu\beta}\right),
\end{aligned}
$$

in the right hand side of (72), it acquires the form:

$$
\begin{aligned}
&\left[\mathscr{L}\delta^\mu_\alpha - {}^*K^\mu|_a \nabla_\alpha \phi^a - L^{\nu\mu}|_a \nabla_\alpha \nabla_\nu \phi^a - G^{\gamma\beta\nu\mu} R_{\gamma\beta\nu\alpha}\right] + \left(\Delta^{\overline{\mu\beta\gamma}}_{\tau\eta\pi} s^{\tau,\,\eta\pi}\right) T_{\gamma,\,\beta\alpha} \\
&\quad + \left[\overset{*}{\nabla}_\nu \left(G^{\beta\gamma\mu\nu} T_{\beta,\,\gamma\alpha}\right) + \frac{1}{2}\left(G^{\beta\gamma\rho\sigma} T_{\beta,\,\gamma\alpha}\right) T^\mu{}_{\rho\sigma}\right] \\
&\quad + \left(G_\gamma{}^{\beta\mu\nu} + L^{\nu\mu}|_a (\Delta^\beta{}_\gamma){}^a|_b\, \phi^b\right) \left[R^\gamma{}_{\beta\alpha\nu} - \nabla_\nu T^\gamma{}_{\beta\alpha} + \frac{1}{2} R^\gamma{}_{\alpha\nu\beta}\right].
\end{aligned}
\qquad (73)
$$

9. At last, turn to the fourth item in the formula (71). Substituting the expressions (56), (59) and (58) into the brackets

$$
[\ldots] \overset{def}{=} \left[g_{\kappa\varepsilon} T_\alpha|^\varepsilon{}_{\beta\gamma} + \nabla_\kappa g_\alpha|_{\beta\gamma} + \frac{1}{2} R^\varepsilon{}_{\alpha\kappa\lambda} g_\varepsilon{}^\lambda|_{\beta\gamma}\right],
$$

one presents it as

$$
\begin{aligned}
[\ldots] &= -g_{\kappa\varepsilon}\left\{\nabla_\alpha T^\varepsilon{}_{\beta\gamma} + \left(T^\varepsilon{}_{\lambda\alpha} T^\lambda{}_{\beta\gamma} + T^\varepsilon{}_{\lambda\beta} T^\lambda{}_{\gamma\alpha} + T^\varepsilon{}_{\lambda\gamma} T^\lambda{}_{\alpha\beta}\right)\right\} + 2\nabla_\kappa T_{(\beta,\,\gamma)\alpha} - R^\varepsilon{}_{\alpha\kappa\lambda} g_{\varepsilon(\beta} \delta^\lambda_{\gamma)} \\
&= -g_{\kappa\varepsilon}\left\{-\nabla_\beta T^\varepsilon{}_{\gamma\alpha} - \nabla_\gamma T^\varepsilon{}_{\alpha\beta} + R^\varepsilon{}_{\alpha\beta\gamma} + R^\varepsilon{}_{\beta\gamma\alpha} + R^\varepsilon{}_{\gamma\alpha\beta}\right\} + \nabla_\kappa T_{\beta,\,\gamma\alpha} + \nabla_\kappa T_{\gamma,\,\beta\alpha} - \frac{1}{2} R_{\beta\alpha\kappa\gamma} - \frac{1}{2} R_{\gamma\alpha\kappa\beta},
\end{aligned}
$$

where the Ricci identity $R^{\varepsilon}{}_{[\alpha\beta\gamma]} \equiv \nabla_{[\alpha}T^{\varepsilon}{}_{\beta\gamma]} + T^{\varepsilon}{}_{\lambda[\alpha}T^{\lambda}{}_{\beta\gamma]}$ has been used. Regrouping terms one obtains

$$\begin{aligned}[\ldots] &= \nabla_\beta T_{\kappa,\,\gamma\alpha} - \nabla_\gamma T_{\kappa,\,\beta\alpha} + R_{\alpha\kappa\beta\gamma} + R_{\beta\kappa\gamma\alpha} + R_{\kappa\gamma\beta\alpha} + \nabla_\kappa T_{\beta,\,\gamma\alpha} + \nabla_\kappa T_{\gamma,\,\beta\alpha} + \frac{1}{2}R_{\alpha\beta\kappa\gamma} + \frac{1}{2}R_{\alpha\gamma\kappa\beta} \\ &= \left(2\nabla_{(\kappa}T_{\beta),\,\gamma\alpha} + R_{\alpha(\kappa\beta)\gamma}\right) + \left(2\nabla_{[\kappa}T_{\gamma],\,\beta\alpha} - R_{\alpha[\kappa\gamma]\beta} + R_{[\kappa\gamma]\beta\alpha}\right) - R_{\kappa\beta\gamma\alpha}.\end{aligned}$$

10. From the last, using the identities (A12) and (A13), and the definition (A1), one gets

$$\begin{aligned}\Delta^{\kappa\beta\gamma}_{\underline{\pi\eta\nu}}[\ldots] &= \Delta^{(\kappa\beta)\gamma}_{\underline{\pi\eta\nu}}\left(2\nabla_\kappa T_{\beta,\,\gamma\alpha} + R_{\alpha\kappa\beta\gamma}\right) + \Delta^{[\kappa|\beta|\gamma]}_{\underline{\pi\eta\nu}}\left(2\nabla_\kappa T_{\gamma,\,\beta\alpha} - R_{\alpha\kappa\gamma\beta} + R_{\kappa\gamma\beta\alpha}\right) - \Delta^{\kappa\beta\gamma}_{\underline{\pi\eta\nu}}R_{\kappa\beta\gamma\alpha} \\ &= \frac{1}{2}\delta^{\kappa}_{(\pi}\delta^{\beta}_{\nu)}\delta^{\gamma}_{\eta}\left(2\nabla_\kappa T_{\beta,\,\gamma\alpha} + R_{\alpha\kappa\beta\gamma}\right) - \frac{1}{2}\delta^{\beta}_{\eta}\delta^{\kappa}_{[\pi}\delta^{\gamma}_{\nu]}\left(2\nabla_\kappa T_{\gamma,\,\beta\alpha} - R_{\alpha\kappa\gamma\beta} + R_{\kappa\gamma\beta\alpha}\right) \\ &\quad -\frac{1}{2}\left(R_{\eta\pi\nu\alpha} + R_{\nu\pi\eta\alpha} - R_{\pi\eta\nu\alpha}\right) = -R_{\pi\eta\alpha\nu} + \nabla_\nu T_{\pi,\,\eta\alpha} - \frac{1}{2}R_{\pi\alpha\nu\eta}.\end{aligned}$$

11. Taking into account the result of the previous point, one concludes that the fourth item in the formula (71) is equal to

$$-\left(G_{\pi}{}^{\eta\mu\nu} + L^{\nu\mu}|_a(\Delta^{\eta}{}_{\pi})\,{}^a|_b\,\phi^b\right) \times \left[R^{\pi}{}_{\eta\alpha\nu} - \nabla_\nu T^{\pi}{}_{\eta\alpha} + \frac{1}{2}R^{\pi}{}_{\alpha\nu\eta}\right]. \tag{74}$$

Notice that this expression exactly equal (up to a sign) to the last term in the formula (73).

12. Summing (73) and (74), keeping in mind (48) – (50), (54) and denoting

$$\boxed{t^{\mu}{}_{\alpha} \stackrel{def}{=} \mathscr{L}\delta^{\mu}_{\alpha} - \frac{\Delta I}{\Delta(\nabla_\mu\phi^a)}\nabla_\alpha\phi^a - \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\phi^a)}\nabla_\alpha\nabla_\nu\phi^a - G^{\beta\gamma\varepsilon\mu}R_{\beta\gamma\varepsilon\alpha},} \tag{75}$$

one obtains the finalized expression for the tensor $\mathbf{U}$:

$$\boxed{U_{\alpha}{}^{\mu} = t^{\mu}{}_{\alpha} + \left(\Delta^{\overline{\mu\beta\gamma}}_{\pi\rho\sigma}s^{\pi,\,\rho\sigma}\right)T_{\gamma,\,\beta\alpha} + \frac{1}{2}G^{\beta\gamma\varepsilon\mu}R_{\alpha\beta\gamma\varepsilon} + \left[\overset{*}{\nabla}_\nu\left(G^{\beta\gamma\mu\nu}T_{\beta,\,\gamma\alpha}\right) + \frac{1}{2}\left(G^{\beta\gamma\rho\sigma}T_{\beta,\,\gamma\alpha}\right)T^{\mu}{}_{\rho\sigma}\right].} \tag{76}$$

Notice also that the tensor $\mathbf{t} \stackrel{def}{=} \{t^{\mu}{}_{\nu}\}$ (75) is just the *generalized canonical energy-momentum tensor*, corresponding to the Lagrangian (23). This statement follows also from the results of Sec. VII. Namely, basing on the above definition of the EMT, one obtains the standard equations of balance for itself. Besides, the gravitational field equations acquire the form, naturally generalizing the ECT equations. It is worse to note that the sequence of the second derivative in the multiplier $\{\nabla_\alpha\nabla_\nu\phi^a\}$ in (75) is *reverse* to the sequence that follows from the construction of the canonical EMT by the direct application of the 1-st Noether theorem. The last term in (75) has appeared due to the non-minimal coupling with the metric field, and also cannot be obtained *in principal* by the application of the 1-st Noether theorem in the Minkowski space and covariantization of the expressions.

V. A PHYSICAL SENSE OF THE KLEIN AND NOETHER IDENTITIES

A. Structure of the variational derivatives

In the following subsections of the present section, we discuss the physical sense of the Klein and Noether identities in the manifestly generally covariant theories. The identities include various combinations of the variational derivatives of the action functional I with respect to fields $\mathbf{g}$, $\mathbf{T}$ and $\boldsymbol{\varphi}$ as ingredients. By this, it is useful to analyze in more details the structure of such derivatives. At first, define the tensors:

$$^{(R)}s^{\pi}{}_{\rho\sigma} \stackrel{def}{=} (-2)\left(\overset{*}{\nabla}_\eta G_{\rho\sigma}{}^{\pi\eta} + \frac{1}{2}G_{\rho\sigma}{}^{\alpha\beta}T^{\pi}{}_{\alpha\beta}\right); \tag{77}$$

$$\begin{aligned}^{(\phi)}f^{\pi}{}_{\rho\sigma} &\stackrel{def}{=} 2\frac{\Delta I}{\Delta(\nabla_\pi\phi^a)}(\Delta_{\rho\sigma})\,{}^a|_b\,\phi^b \\ &+2\frac{\partial\mathscr{L}}{\partial(\nabla_\pi\nabla_\eta\phi^a)}\left[(\Delta_{\rho\sigma})\,{}^a|_b\,\nabla_\eta\phi^b - g_{\eta\rho}\nabla_\sigma\phi^a\right],\end{aligned} \tag{78}$$

which are determined by the dependence of the Lagrangian $\mathscr{L}$ on the curvature tensor $\mathbf{R}$ and on the fields

$\boldsymbol{\phi}$, respectively. It is worse to note that the sum of the tensor (77) and the antisymmetrical part of the tensor (78): ${}^{(\phi)}s^{\pi}{}_{\rho\sigma} \overset{def}{=} {}^{(\phi)}f^{\pi}{}_{[\rho\sigma]}$ presents the canonical ST (67):

$$s^{\pi}{}_{\rho\sigma} = {}^{(R)}s^{\pi}{}_{\rho\sigma} + {}^{(\phi)}s^{\pi}{}_{\rho\sigma}. \tag{79}$$

Now, let us discuss the structure of the variational derivative with respect to the torsion tensor. In the terms of the quantities (77) and (78) the derivative (32) can be rewritten as

$$\frac{\Delta I}{\Delta\Gamma^{\kappa}{}_{\lambda\tau}} = \frac{1}{2}\left({}^{(R)}s^{\tau,\lambda}{}_{\kappa} + {}^{(\phi)}f^{\tau,\,\lambda}{}_{\kappa}\right). \tag{80}$$

Then, due to the identity (A7), the variational derivative (43) is equal to

$$\frac{\Delta I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} = \frac{\Delta^{*} I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} + \frac{1}{2}\left(\Delta^{\overline{\gamma\beta\alpha}}_{\pi\rho\sigma} s^{\pi,\,\rho\sigma}\right) g_{\alpha\varepsilon}, \tag{81}$$

where the first term on the right hand side is defined by the formula (51). Denote the quantity

$$b^{\gamma\beta\alpha} \overset{def}{=} \Delta^{\overline{\gamma\beta\alpha}}_{\pi\rho\sigma} s^{\pi,\,\rho\sigma}; \qquad b^{[\gamma\beta]\alpha} = b^{\gamma\beta\alpha} \tag{82}$$

and call it as the *Belinfante tensor:* $\mathbf{b} \overset{def}{=} \left\{b^{\gamma\beta\alpha}\right\}$, *induced by the ST* $\mathbf{s}$. Then

$$\frac{\Delta I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} = \frac{\Delta^{*} I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} + \frac{1}{2} b^{\gamma\beta}{}_{\varepsilon}. \tag{83}$$

This means *that in the case of only minimal T-coupling (when the Lagrangian $\mathscr{L}$ does not contain the torsion tensor $\mathbf{T}$ explicitly) one has*

$$\frac{\Delta I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} = \frac{1}{2} b^{\gamma\beta}{}_{\varepsilon}. \tag{84}$$

Earlier, the same result (27) has been proved only for the Lagrangians of the type $\mathscr{L} = \mathscr{L}(\mathbf{g};\boldsymbol{\varphi},\boldsymbol{\nabla}\boldsymbol{\varphi})$ (26) with a more simple presentations both of the ST and of the Belinfante tensor (see Refs.[51–57]). We have proved a more general claim: the formula (27) is left valid for the Lagrangians of a more general type $\mathscr{L} = \mathscr{L}(\mathbf{g},\mathbf{R};\boldsymbol{\varphi},\boldsymbol{\nabla}\boldsymbol{\varphi},\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi})$, as this follows from (84) .

The formula (83) *shows that the presence of a non-minimal coupling with torsion changes* (84). The requirement (the desire) to conserve a sense of the variational derivative (84) even at the presence of a non-minimal T-coupling leads to a necessity to modify both the initial Belinfante tensor and the initial ST. Let us demonstrate the modification step by step. Rewrite the formula (83) in form of (84):

$$\boxed{\frac{\Delta I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} = \frac{1}{2} \overset{mod}{b}{}^{\gamma\beta}{}_{\varepsilon},} \tag{85}$$

where the modified Belinfante tensor $\overset{mod}{\mathbf{b}} = \{\overset{mod}{b}{}^{\gamma\beta\alpha}\}$ is defined analogously to the initial one (that is with the use of *any* ST):

$$\overset{mod}{b}{}^{\gamma\beta\alpha} \overset{def}{=} \Delta^{\overline{\gamma\beta\alpha}}_{\pi\rho\sigma} \overset{mod}{s}{}^{\pi,\,\rho\sigma}. \tag{86}$$

The *modified* Belinfante tensor and canonical ST can be represented as initial ones and correspondent additions:

$$\overset{mod}{b}{}^{\gamma\beta\alpha} \overset{def}{=} b^{\gamma\beta\alpha} + \overset{add}{b}{}^{\gamma\beta\alpha} \tag{87}$$

and

$$\overset{mod}{s}{}^{\pi}{}_{\rho\sigma} \overset{def}{=} s^{\pi}{}_{\rho\sigma} + \overset{add}{s}{}^{\pi}{}_{\rho\sigma}. \tag{88}$$

Finally, combining (83) - (86), one obtains the definitions for the additional Belinfante tensor and ST:

$$\overset{add}{b}{}^{\gamma\beta}{}_{\varepsilon} = 2\frac{\Delta^{*} I}{\Delta T^{\varepsilon}{}_{\beta\gamma}}; \tag{89}$$

$$\overset{add}{s}{}^{\pi,\,\rho\sigma} = -4g^{[\sigma|\varepsilon}\frac{\Delta^{*} I}{\Delta T^{\varepsilon}{}_{|\rho]\pi}}. \tag{90}$$

Now, let us turn to the variational derivative with respect to the metric tensor $\mathbf{g}$. By the formulae (42) and (80),

$$\begin{aligned}\frac{\Delta I}{\Delta g_{\beta\gamma}} &= \frac{1}{2}\mathscr{L} g^{\beta\gamma} + \frac{\partial^{*}\mathscr{L}}{\partial g_{\beta\gamma}} \\ &-\frac{1}{2}\overset{*}{\nabla}_{\mu}\left({}^{(R)}s^{\tau,\,\lambda\pi} + {}^{(\phi)}f^{\tau,\,\lambda\pi}\right)\Delta^{\mu(\beta\gamma)}_{\underline{\pi\lambda\tau}}.\end{aligned} \tag{91}$$

Using the identity (A11) and the formula (79), rewrite the last as

$$\begin{aligned}\frac{\Delta I}{\Delta g_{\beta\gamma}} &= \frac{1}{2}\mathscr{L} g^{\beta\gamma} + \frac{\partial^{*}\mathscr{L}}{\partial g_{\beta\gamma}} \\ &-\frac{1}{4}\overset{*}{\nabla}_{\mu}\left({}^{(R)}s^{\mu,\,\beta\gamma} + {}^{(\phi)}f^{\mu,\,\beta\gamma}\right) - \frac{1}{2}\overset{*}{\nabla}_{\mu} b^{\mu\beta\gamma}.\end{aligned} \tag{92}$$

Substituting this expression into the standard definition of the *metric EMT* $\overset{met}{\mathbf{t}} \overset{def}{=} \{\overset{met}{t}{}^{\beta\gamma}\}$:

$$\boxed{\frac{1}{2}\overset{met}{t}{}^{\beta\gamma} \overset{def}{=} \frac{\Delta I}{\Delta g_{\beta\gamma}}}, \tag{93}$$

one obtains

$$\boxed{\overset{met}{t}{}^{\beta\gamma} = \mathscr{L} g^{\beta\gamma} + 2\frac{\partial^* \mathscr{L}}{\partial g_{\beta\gamma}} - \frac{1}{2}\overset{*}{\nabla}_\mu \left({}^{(R)}s^{\mu,\,\beta\gamma} + {}^{(\phi)}f^{\mu,\,\beta\gamma}\right) - \overset{*}{\nabla}_\mu b^{\mu\beta\gamma}.} \tag{94}$$

B. The physical sense of the Noether identity

Turn to the Noether identity (14)

$$\overset{*}{\nabla}_\mu I_\nu{}^\mu \equiv I_\nu. \tag{95}$$

Here, corresponding to the identities (19) and (18),

$$I_\nu{}^\mu \overset{def}{=} \frac{\Delta I}{\Delta \Phi^A}\Phi_\nu{}^\mu|^A = \frac{\Delta I}{\Delta g_{\beta\gamma}} g_\nu{}^\mu|_{\beta\gamma} + \frac{\Delta I}{\Delta T^\varepsilon{}_{\beta\gamma}} T_\nu{}^\mu|^\varepsilon{}_{\beta\gamma} + \frac{\Delta I}{\Delta \varphi^a}\varphi_\nu{}^\mu|^a \overset{def}{=} \frac{\Delta I}{\Delta g_{\beta\gamma}} g_\nu{}^\mu|_{\beta\gamma} + \frac{\Delta I}{\Delta \phi^a}\phi_\nu{}^\mu|^a; \tag{96}$$

$$I_\nu \overset{def}{=} \frac{\Delta I}{\Delta \Phi^A}\Phi_\nu|^A = \frac{\Delta I}{\Delta g_{\beta\gamma}} g_\nu|_{\beta\gamma} + \frac{\Delta I}{\Delta T^\varepsilon{}_{\beta\gamma}} T_\nu|^\varepsilon{}_{\beta\gamma} + \frac{\Delta I}{\Delta \varphi^a}\varphi_\nu|^a \overset{def}{=} \frac{\Delta I}{\Delta g_{\beta\gamma}} g_\nu|_{\beta\gamma} + \frac{\Delta I}{\Delta \phi^a}\phi_\nu|^a. \tag{97}$$

Taking into account the formulae (93), (56), (57), (59) and (60), we rewrite these expressions as

$$I_\nu{}^\mu = -\overset{met}{t}{}^\mu{}_\nu + \frac{\Delta I}{\Delta \phi^a}\left(\Delta^\mu{}_\nu\right){}^a|_b\,\phi^b; \tag{98}$$

$$I_\nu = \overset{met}{t}{}^\mu{}_\lambda T^\lambda{}_{\mu\nu} - \frac{\Delta I}{\Delta \phi^a}\left[\nabla_\nu \phi^a + \left(\Delta^\mu{}_\lambda\right){}^a|_b\,\phi^b\, T^\lambda{}_{\mu\nu}\right]. \tag{99}$$

Using the formulae (85) and (C17), we present the above tensors in the expanded form:

$$I_\nu{}^\mu = -\left[\overset{met}{t}{}^\mu{}_\nu + \left(\frac{1}{2}\overset{mod}{b}{}^{\beta\gamma}{}_\nu T^\mu{}_{\beta\gamma} + \overset{mod}{b}{}^{\mu\beta}{}_\gamma T^\gamma{}_{\beta\nu}\right)\right] + \frac{\Delta I}{\Delta \varphi^a}\left(\Delta^\mu{}_\nu\right){}^a|_b\,\varphi^b; \tag{100}$$

$$\begin{aligned} I_\nu = {} & \left[\overset{met}{t}{}^\mu{}_\lambda + \left(\frac{1}{2}\overset{mod}{b}{}^{\beta\gamma}{}_\lambda T^\mu{}_{\beta\gamma} + \overset{mod}{b}{}^{\mu\beta}{}_\gamma T^\gamma{}_{\beta\lambda}\right)\right] T^\lambda{}_{\mu\nu} - \frac{1}{2}\overset{mod}{b}{}^{\gamma\beta}{}_\varepsilon \nabla_\nu T^\varepsilon{}_{\beta\gamma} \\ & - \frac{\Delta I}{\Delta \varphi^a}\left[\nabla_\nu \varphi^a + \left(\Delta^\mu{}_\lambda\right){}^a|_b\,\varphi^b\, T^\lambda{}_{\mu\nu}\right]. \end{aligned} \tag{101}$$

Substituting the expressions (98) and (99) into (95), we obtain the explicit form of the Noether identity:

$$\boxed{\overset{*}{\nabla}_\mu \overset{met}{t}{}^\mu{}_\nu \equiv -\overset{met}{t}{}^\mu{}_\lambda T^\lambda{}_{\mu\nu} + \left\{\overset{*}{\nabla}_\mu\left[\frac{\Delta I}{\Delta \phi^a}\left(\Delta^\mu{}_\nu\right){}^a|_b\,\phi^b\right] + \frac{\Delta I}{\Delta \phi^a}\left[\nabla_\nu \phi^a + \left(\Delta^\mu{}_\lambda\right){}^a|_b\,\phi^b\, T^\lambda{}_{\mu\nu}\right]\right\}.} \tag{102}$$

After simplifying the Lagrangian $\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R};\ \mathbf{T}, \boldsymbol{\nabla}\mathbf{T}, \boldsymbol{\nabla}\boldsymbol{\nabla}\mathbf{T};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi})$ to the form $\mathscr{L} = \mathscr{L}(\mathbf{g};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi})$ the identity (102) degenerates to the identity obtained in Refs.[51–57] and, thus, generalizes the result of these works.

The formula (102) shows that, when the equations of motion for fields $\{\mathbf{T}, \boldsymbol{\varphi}\} = \boldsymbol{\phi}$ hold (on $\boldsymbol{\phi}$-equations), the Noether identity transforms to the *equations of balance* for the metric EMT $\overset{met}{\mathbf{t}}$:

$$\boxed{\overset{*}{\nabla}_\mu \overset{met}{t}{}^\mu{}_\nu = -\overset{met}{t}{}^\mu{}_\lambda T^\lambda{}_{\mu\nu} \qquad \text{(on the } \boldsymbol{\phi}\text{-equations).}} \tag{103}$$

If one substitutes the expressions (100) and (101) into (95), then, after taking into account the identity (D2), one obtains the Noether identity in the expanded form:

$$\boxed{\begin{aligned} \overset{*}{\nabla}_\mu \left(\overset{met}{t}{}^\mu{}_\nu + \overset{*}{\nabla}_\eta \overset{mod}{b}{}^{\eta\mu}{}_\nu\right) \equiv {} & -\left(\overset{met}{t}{}^\mu{}_\lambda + \overset{*}{\nabla}_\eta \overset{mod}{b}{}^{\eta\mu}{}_\lambda\right) T^\lambda{}_{\mu\nu} + \frac{1}{2}\overset{mod}{s}{}^\pi{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} \\ & + \left\{\overset{*}{\nabla}_\mu\left[\frac{\Delta I}{\Delta \varphi^a}\left(\Delta^\mu{}_\nu\right){}^a|_b\,\varphi^b\right] + \frac{\Delta I}{\Delta \varphi^a}\left[\nabla_\nu \varphi^a + \left(\Delta^\mu{}_\lambda\right){}^a|_b\,\varphi^b\, T^\lambda{}_{\mu\nu}\right]\right\}. \end{aligned}} \tag{104}$$

It is clearly that on the equations of motion of $\{\mathbf{T},\boldsymbol{\varphi}\} = \boldsymbol{\phi}$-fields this equation turns again to (103). However, on the equations of motion of *only $\boldsymbol{\varphi}$-fields* the expanded equations of balance acquire the form:

$$\boxed{\overset{*}{\nabla}_\mu \left(\overset{met}{t}{}^{\mu}{}_{\nu} + \overset{*}{\nabla}_\eta \overset{mod}{b}{}^{\eta\mu}{}_{\nu}\right) = -\left(\overset{met}{t}{}^{\mu}{}_{\lambda} + \overset{*}{\nabla}_\eta \overset{mod}{b}{}^{\eta\mu}{}_{\lambda}\right) T^{\lambda}{}_{\mu\nu} + \frac{1}{2}\overset{mod}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} \qquad \text{(on the } \boldsymbol{\varphi}\text{-equations).}} \tag{105}$$

Thus, the Noether identity is the basis for defining the equations of balance for the metric EMT.

C. The 4-th and 3-rd Klein identities

Notice that, by the definition (33), the tensor $\mathbf{G} \overset{def}{=} \{G_\kappa{}^{\lambda\mu\nu}\}$ has the same symmetries, like the curvature tensor. Using the definition of the tensor $\mathbf{N}$ (62) and the antisymmetry of $\mathbf{G}$ in the second pare of indexes, $G_\alpha{}^{\beta\delta\gamma} = -G_\alpha{}^{\beta\gamma\delta}$, we are convinced that *the 4-th Klein identity* (17)*:*

$$N_\alpha{}^{(\beta\gamma\delta)} = \frac{1}{3}\left(N_\alpha{}^{\beta\gamma\delta} + N_\alpha{}^{\gamma\delta\beta} + N_\alpha{}^{\delta\beta\gamma}\right) \equiv 0 \tag{106}$$

is satisfied automatically. By the antisymmetry of $\mathbf{G}$ in the first pare of indexes, $G^{\beta\alpha\gamma\delta} = -G^{\alpha\beta\gamma\delta}$, *the tensor* $\mathbf{N}$ *satisfies also the new identity*:

$$N^{(\alpha\beta\gamma)}{}_\delta = \frac{1}{3}\left(N^{\alpha\beta\gamma}{}_\delta + N^{\beta\gamma\alpha}{}_\delta + N^{\gamma\alpha\beta}{}_\delta\right) \equiv 0. \tag{107}$$

In the case of a pure metric theory another logic leads also to this conclusion. In the Riemannian geometry $\mathscr{R}(1,D)$ (but not in the Riemann-Cartan geometry $\mathscr{C}(1,D)$!) the tensor $\{G^{\alpha\beta\gamma\delta}\}$ is symmetrical with respect to the permutation of the first and the second pairs of indexes: $\{G^{\gamma\delta\alpha\beta} = G^{\alpha\beta\gamma\delta}\}$, like the curvature tensor $\{R_{\alpha\beta\gamma\delta}\}$. Then, the tensor $\mathbf{N}$ becomes also symmetrical in external indexes, $N^{\alpha\beta\gamma\delta} = N^{\delta\beta\gamma\alpha}$. Namely this property together with (106) gives (107).

In arbitrary generally covariant theories with the Lagrangians $\mathscr{L}$, containing derivatives of the metric up to a second order, see Refs.[44–49,58,59], the quantity $\mathbf{n} \overset{\mathbf{def}}{=} \{\mathbf{n}^{\alpha\mu\nu\beta}\}$ is defined as $n^{\alpha\mu\nu\beta} \overset{def}{=} \partial\mathscr{L}/\partial g_{\mu\nu,\alpha\beta}$. The same as the tensor $\mathbf{N}$, it satisfies the identity of the type (106). Then, because $\mathbf{n}$ is symmetrical both in inner and in external indexes, it satisfies also the identity of the type (107). Thus, our conclusions related to the properties of $\mathbf{N}$ in the manifestly generally covariant theories generalize the results of the aforementioned works.

Now, let us turn to the 3-rd Klein identity (16)

$$M_\kappa{}^{(\lambda\mu)} + \overset{*}{\nabla}_\nu N_\kappa{}^{\lambda\mu\nu} + N_\kappa{}^{(\lambda|\rho\sigma}T^{|\mu)}{}_{\rho\sigma} \equiv 0. \tag{108}$$

Taking into account (82), and (62), calculate the symmetrical in the upper indexes part of the tensor $\mathbf{M}$ (68):

$$M_\kappa{}^{(\lambda\mu)} = -\overset{*}{\nabla}_\nu N_\kappa{}^{\lambda\mu\nu} - N_\kappa{}^{(\lambda|\rho\sigma}T^{|\mu)}{}_{\rho\sigma}. \tag{109}$$

From here it follows that *the 3-rd Klein identity is satisfied automatically also.* Thus, the concrete form of the Lagrangian (23) is enough to be convinced in the identities (16) - (17).

D. The physical sense of the 2-nd Klein identity

It is convenient to represent the 2-nd Klein identity (15) in the form:

$$\overset{sym}{U}{}_{\nu}{}^{\mu} \equiv -I_\nu{}^\mu \tag{110}$$

(see the Paper I, Sec. V, formulae (83) and (84)), where

$$\begin{aligned}\overset{sym}{U}{}_{\nu}{}^{\mu} = &\left(U_\nu{}^\mu - \frac{1}{3}N_\alpha{}^{\mu\beta\gamma}R^\alpha{}_{\nu\beta\gamma}\right)\\ &- \left(\overset{*}{\nabla}_\lambda \theta_\nu{}^{\mu\lambda} + \frac{1}{2}\theta_\nu{}^{\alpha\beta}T^\mu{}_{\alpha\beta}\right);\end{aligned} \tag{111}$$

$$\theta_\nu{}^{\mu\lambda} \overset{def}{=} -M_\nu{}^{[\mu\lambda]} + \frac{2}{3}\left(\overset{*}{\nabla}_\eta N_\nu{}^{\eta[\mu\lambda]} + \frac{1}{2}T^{[\mu}{}_{\rho\sigma}N_\nu{}^{\lambda]\rho\sigma}\right). \tag{112}$$

Let us calculate the expression (111).

1. At first, notice that, by (62), the relation

$$N_\nu{}^{\eta[\mu\lambda]} = \frac{1}{2}\left(G_\nu{}^{\eta\mu\lambda} - G_\nu{}^{[\mu\lambda]\eta}\right) \tag{113}$$

   takes a place. From the last the formulae

$$\begin{aligned}&G_\nu{}^{[\mu\lambda]\eta} + \frac{2}{3}N_\nu{}^{\eta[\mu\lambda]}\\ &= \frac{1}{3}\left(G_\nu{}^{\mu\lambda\eta} + G_\nu{}^{\lambda\eta\mu} + G_\nu{}^{\eta\mu\lambda}\right) = G_\nu{}^{[\mu\lambda\eta]};\end{aligned} \tag{114}$$

$$\frac{1}{2}\left(G_\nu{}^{\mu\rho\sigma} + G_\nu{}^{[\rho\sigma]\mu}\right) - \frac{1}{3}N_\nu{}^{\mu[\rho\sigma]} = G_\nu{}^{[\mu\rho\sigma]} \tag{115}$$

   follow. It is useful also the formula, which follows after contracting (114) with the Riemannian tensor:

$$\frac{1}{2}G^{\alpha\beta\gamma\mu}R_{\nu\alpha\beta\gamma} - \frac{1}{3}N_\alpha{}^{\mu\beta\gamma}R^\alpha{}_{\nu\beta\gamma} = -\frac{1}{2}G_\alpha{}^{[\mu\rho\sigma]}R^\alpha{}_{\nu\rho\sigma}. \tag{116}$$

2. Substituting into the right hand side of (112) the expressions of the tensors $\mathbf{M}$ and $\mathbf{N}$, (68) and (62),

and using the definition (82) and the relations (114) and (115), one finds

$$\theta_\nu{}^{\mu\lambda} = \left[-b^{\mu\lambda}{}_\nu + G_\alpha{}^{\beta\mu\lambda} T^\alpha{}_{\beta\nu}\right] + \left[\overset{*}{\nabla}_\eta G_\nu{}^{[\mu\lambda\eta]} + \tfrac{1}{2}\left(G_\nu{}^{[\mu\rho\sigma]} T^\lambda{}_{\rho\sigma} - G_\nu{}^{[\lambda\rho\sigma]} T^\mu{}_{\rho\sigma}\right)\right]. \tag{117}$$

3. Substituting (117) into the expression in the second parentheses in (111), one gets

$$\begin{aligned}&\overset{*}{\nabla}_\lambda \theta_\nu{}^{\mu\lambda} + \frac{1}{2}\theta_\nu{}^{\alpha\beta} T^\mu{}_{\alpha\beta}\\ &= -\left[\overset{*}{\nabla}_\lambda b^{\mu\lambda}{}_\nu + \frac{1}{2} b^{\rho\sigma}{}_\nu T^\mu{}_{\rho\sigma}\right] + \left[\overset{*}{\nabla}_\lambda \left(G^{\alpha\beta\mu\lambda} T_{\alpha,\,\beta\nu}\right) + \frac{1}{2}\left(G^{\alpha\beta\rho\sigma} T_{\alpha,\,\beta\nu}\right) T^\mu{}_{\rho\sigma}\right]\\ &\quad -\frac{1}{2} G_\alpha{}^{[\mu\rho\sigma]} R^\alpha{}_{\nu\rho\sigma} + \left[G_\nu{}^{\pi\rho\sigma}\left(R^\mu{}_{[\pi\rho\sigma]} - \nabla_{[\pi} T^\mu{}_{\rho\sigma]} - T^\mu{}_{\varepsilon[\pi} T^\varepsilon{}_{\rho\sigma]}\right)\right].\end{aligned} \tag{118}$$

Here, the use of the identity

$$\overset{*}{\nabla}_\lambda \left[\overset{*}{\nabla}_\eta \tilde\theta_\nu{}^{\mu\lambda\eta} + \frac{1}{2}\tilde\theta_\nu{}^{\mu\rho\sigma} T^\lambda{}_{\rho\sigma}\right] \equiv \frac{1}{2}\left(-R^\lambda{}_{\nu\rho\sigma}\tilde\theta_\lambda{}^{\mu\rho\sigma} + R^\mu{}_{\lambda\rho\sigma}\tilde\theta_\nu{}^{\lambda\rho\sigma}\right) \tag{119}$$

has been essential (see the Paper I, Appendix C.1, formula (C3)), with the exchange:

$$\tilde\theta_\nu{}^{\mu\lambda\eta} = G_\nu{}^{[\mu\lambda\eta]}.$$

4. By the Ricci identity: $R^\mu{}_{[\pi\rho\sigma]} \equiv \nabla_{[\pi} T^\mu{}_{\rho\sigma]} - T^\mu{}_{\varepsilon[\pi} T^\varepsilon{}_{\rho\sigma]}$, the last item on the right hand side of the formula (118) disappears and it acquires the form:

$$\begin{aligned}&\overset{*}{\nabla}_\lambda \theta_\nu{}^{\mu\lambda} + \frac{1}{2}\theta_\nu{}^{\alpha\beta} T^\mu{}_{\alpha\beta}\\ &= -\left[\overset{*}{\nabla}_\lambda b^{\mu\lambda}{}_\nu + \frac{1}{2} b^{\rho\sigma}{}_\nu T^\mu{}_{\rho\sigma}\right] + \left[\overset{*}{\nabla}_\lambda \left(G^{\alpha\beta\mu\lambda} T_{\alpha,\,\beta\nu}\right) + \frac{1}{2}\left(G^{\alpha\beta\rho\sigma} T_{\alpha,\,\beta\nu}\right) T^\mu{}_{\rho\sigma}\right] - \frac{1}{2} G_\alpha{}^{[\mu\rho\sigma]} R^\alpha{}_{\nu\rho\sigma}.\end{aligned} \tag{120}$$

5. Substituting into the first parentheses in (111) the expression for the tensor **U** (76) and using (116), one obtains

$$\begin{aligned}&U_\nu{}^\mu - \frac{1}{3} N_\alpha{}^{\mu\beta\gamma} R^\alpha{}_{\nu\beta\gamma}\\ &= t^\mu{}_\nu + b^{\mu\beta\gamma} T_{\gamma,\,\beta\nu} - \frac{1}{2} G_\alpha{}^{[\mu\rho\sigma]} R^\alpha{}_{\nu\rho\sigma}\\ &\quad + \left[\overset{*}{\nabla}_\eta \left(G^{\beta\gamma\mu\eta} T_{\beta,\,\gamma\nu}\right) + \frac{1}{2}\left(G^{\beta\gamma\rho\sigma} T_{\beta,\,\gamma\nu}\right) T^\mu{}_{\rho\sigma}\right],\end{aligned} \tag{121}$$

6. Finally, substituting (120) and (121) into the right hand side of (111), one finds the search expression:

$$\overset{sym}{U}{}_\nu{}^\mu = t^\mu{}_\nu + \left[\overset{*}{\nabla}_\lambda b^{\mu\lambda}{}_\nu + \frac{1}{2} b^{\rho\sigma}{}_\nu T^\mu{}_{\rho\sigma} + b^{\mu\beta\gamma} T_{\gamma,\,\beta\nu}\right]. \tag{122}$$

At the absence of the torsion, the right hand side of this expression presents the known expression for the Belinfante symmetrized EMT[28,29]. Therefore, when the torsion presents it is naturally to define the right hand side of (122) as *symmetrized EMT* $\overset{sym}{\mathbf{t}} \overset{def}{=} \{\overset{sym}{t}{}^\mu{}_\nu\}$:

$$\boxed{\begin{aligned}\overset{sym}{t}{}^\mu{}_\nu &\overset{def}{=} t^\mu{}_\nu + \left[\overset{*}{\nabla}_\lambda b^{\mu\lambda}{}_\nu + \frac{1}{2} b^{\rho\sigma}{}_\nu T^\mu{}_{\rho\sigma} + b^{\mu\beta}{}_\alpha T^\alpha{}_{\beta\nu}\right]\\ &= t^\mu{}_\nu - \left[\overset{*}{\nabla}_\lambda b^{\lambda\mu}{}_\nu + \frac{1}{2} b^{\gamma\beta}{}_\alpha (\Delta^\mu{}_\nu)\,{}^\alpha{}_{\beta\gamma}\big|_\theta{}^{\varphi\xi}\, T^\theta{}_{\varphi\xi}\right],\end{aligned}} \tag{123}$$

and, in the result one obtains

$$\overset{sym}{U}{}_\nu{}^\mu = \overset{sym}{t}{}^\mu{}_\nu. \tag{124}$$

Substituting (124) and (98) into the 2-nd Klein identity (110), one has

$$\boxed{\overset{sym}{t}{}^\mu{}_\nu \equiv \overset{met}{t}{}^\mu{}_\nu - \frac{\Delta I}{\Delta\phi^a}(\Delta^\mu{}_\nu)\,{}^a|_b\,\phi^b.} \tag{125}$$

From the identity (125) it follows that *on the* $\boldsymbol{\phi}$*-equations symmetrized EMT* $\overset{sym}{\mathbf{t}}$ (123) *is equal to the metric EMT*

$\overset{met}{\mathbf{t}}$ (93):

$$\boxed{\overset{sym}{t}{}^{\mu}{}_{\nu} = \overset{met}{t}{}^{\mu}{}_{\nu} \qquad \text{(on the } \boldsymbol{\phi}\text{-equations).}} \tag{126}$$

On the other hand, if on the right hand side of (110) instead of (98) one uses (100) then, keeping in mind (124), (123) and (86), the 2-nd Klein identity can be presented in the other form:

$$\boxed{t^{\mu}{}_{\nu} - \overset{*}{\nabla}_{\lambda} b^{\lambda\mu}{}_{\nu} \equiv \overset{met}{t}{}^{\mu}{}_{\nu} + \left(\frac{1}{2}\overset{add}{b}{}^{\rho\sigma}{}_{\nu} T^{\mu}{}_{\rho\sigma} + \overset{add}{b}{}^{\mu\beta}{}_{\alpha} T^{\alpha}{}_{\beta\nu}\right) - \frac{\Delta I}{\Delta\varphi^{a}}(\Delta^{\mu}{}_{\nu})^{\ a}{}|_{b}\,\varphi^{b}.} \tag{127}$$

In particular, on $\boldsymbol{\varphi}$-equations one has

$$\boxed{t^{\mu}{}_{\nu} - \overset{*}{\nabla}_{\lambda} b^{\lambda\mu}{}_{\nu} = \overset{met}{t}{}^{\mu}{}_{\nu} + \left(\frac{1}{2}\overset{add}{b}{}^{\rho\sigma}{}_{\nu} T^{\mu}{}_{\rho\sigma} + \overset{add}{b}{}^{\mu\beta}{}_{\alpha} T^{\alpha}{}_{\beta\nu}\right) \qquad \text{(on the } \boldsymbol{\varphi}\text{-equations).}} \tag{128}$$

Finalizing, one can conclude that *the 2-nd Klein identity permits to define the Belinfante symmetrized EMT* $\overset{sym}{\mathbf{t}}$ (see (123)), *and to prove an equivalence of the symmetrized* $\overset{sym}{\mathbf{t}}$ *and metric* $\overset{met}{\mathbf{t}}$ *EMTs on* $\boldsymbol{\phi}$*-fields equations* (see (126)).

E. The physical sense of the 1-st Klein identity

We call the identity

$$\overset{*}{\nabla}_{\mu} U_{\nu}{}^{\mu} - \frac{1}{2} M_{\mu}{}^{\rho\sigma} R^{\mu}{}_{\nu\rho\sigma} - \frac{1}{3} N_{\kappa}{}^{\lambda\rho\sigma}\left(\nabla_{\lambda} R^{\kappa}{}_{\nu\rho\sigma} + \frac{1}{2} R^{\kappa}{}_{\nu\lambda\varepsilon} T^{\varepsilon}{}_{\rho\sigma}\right) \equiv -I_{\nu} \tag{129}$$

as the 1-st Klein identity (see the Paper I, Sec. III, formula (38)). Calculate the left hand side of this for the manifestly generally covariant theories.

1. Using the expression for the tensor $\mathbf{U}$ (76), the definition (82), the identity

$$\overset{*}{\nabla}_{\mu}\left[\overset{*}{\nabla}_{\eta}\tilde{\theta}_{\nu}{}^{\mu\eta} + \frac{1}{2}\tilde{\theta}_{\nu}{}^{\rho\sigma} T^{\mu}{}_{\rho\sigma}\right] \equiv -\frac{1}{2} R^{\lambda}{}_{\nu\rho\sigma}\tilde{\theta}_{\lambda}{}^{\rho\sigma}, \tag{130}$$

(see the Paper I, Appendix C.1, formula (C2)) and $\tilde{\theta}_{\nu}{}^{\mu\eta} = G^{\alpha\beta\mu\eta} T_{\alpha,\,\beta\nu}$, one obtains the expression for the first item in (129):

$$\overset{*}{\nabla}_{\mu} U_{\nu}{}^{\mu} = \overset{*}{\nabla}_{\mu} t^{\mu}{}_{\nu} + \overset{*}{\nabla}_{\mu}\left(b^{\mu\beta}{}_{\alpha} T^{\alpha}{}_{\beta\nu}\right) - \frac{1}{2}\left(\overset{*}{\nabla}_{\mu} G_{\alpha}{}^{\rho\sigma\mu}\right) R^{\alpha}{}_{\nu\rho\sigma} - \frac{1}{2} G_{\alpha}{}^{\beta\gamma\mu}\left(\nabla_{\mu} R^{\alpha}{}_{\nu\beta\gamma}\right) - \frac{1}{2}\left(G^{\alpha\beta\rho\sigma} T_{\alpha,\,\beta\lambda}\right) R^{\lambda}{}_{\nu\rho\sigma}.$$

2. The expression for the second item in (129) can be obtained after using the formulae for $\mathbf{M}$ (68) and the definition (82):

$$\begin{aligned} &-\frac{1}{2} M_{\mu}{}^{\rho\sigma} R^{\mu}{}_{\nu\rho\sigma} \\ &= -\frac{1}{2} b^{\sigma\rho\pi} R_{\nu\pi\rho\sigma} + \frac{1}{2}\left(\overset{*}{\nabla}_{\eta} G_{\mu}{}^{\rho\sigma\eta}\right) R^{\mu}{}_{\nu\rho\sigma} \\ &\quad + \frac{1}{4}\left(G_{\mu}{}^{\rho[\alpha\beta]} + G_{\mu}{}^{[\alpha\beta]\rho}\right)\left[R^{\mu}{}_{\nu\rho\sigma} T^{\sigma}{}_{\alpha\beta}\right] \\ &\quad + \frac{1}{2}\left(G^{\alpha\beta\rho\sigma} T_{\alpha,\,\beta\mu}\right) R^{\mu}{}_{\nu\rho\sigma}. \end{aligned}$$

3. Combining the results of the points 1 and 2, taking into account (114) and (115), and using the Bianchi identity $\nabla_{[\pi|}R^{\mu}{}_{\nu|\rho\sigma]} \equiv -R^{\mu}{}_{\nu\varepsilon[\pi}T^{\varepsilon}{}_{\rho\sigma]}$, one gets

$$\begin{aligned}&\overset{*}{\nabla}_{\mu}U_{\nu}{}^{\mu} - \frac{1}{2}M_{\mu}{}^{\rho\sigma}R^{\mu}{}_{\nu\rho\sigma}\\ &-\frac{1}{3}N_{\kappa}{}^{\lambda\rho\sigma}\left(\nabla_{\lambda}R^{\kappa}{}_{\nu\rho\sigma} + \frac{1}{2}R^{\kappa}{}_{\nu\lambda\varepsilon}T^{\varepsilon}{}_{\rho\sigma}\right)\\ &= \overset{*}{\nabla}_{\mu}t^{\mu}{}_{\nu} - \overset{*}{\nabla}_{\mu}\left(b^{\mu\beta}{}_{\alpha}T^{\alpha}{}_{\beta\nu}\right) - \frac{1}{2}b^{\sigma\rho\pi}R_{\nu\pi\rho\sigma}.\end{aligned} \tag{131}$$

Substituting the expressions (131) and (99) into (129), one finds the explicit form of the 1-st Klein identity:

$$\begin{aligned}\overset{*}{\nabla}_{\mu}t^{\mu}{}_{\nu} \equiv &-\left(\overset{met}{t}{}^{\mu}{}_{\lambda} + \overset{*}{\nabla}_{\eta}b^{\eta\mu}{}_{\lambda}\right)T^{\lambda}{}_{\mu\nu}\\ &+\frac{1}{2}b^{\sigma\rho\pi}R_{\nu\pi\rho\sigma} - b^{\eta\mu}{}_{\lambda}\nabla_{\eta}T^{\lambda}{}_{\mu\nu}\\ &+\frac{\Delta I}{\Delta\phi^{a}}\left[\nabla_{\nu}\phi^{a} + (\Delta^{\mu}{}_{\lambda})\,{}^{a}|_{b}\,\phi^{b}\,T^{\lambda}{}_{\mu\nu}\right].\end{aligned} \tag{132}$$

Transform this identity as follows.

1. Using the 2-nd Klein identity (125) and the formula (123), transform the combination $(\overset{met}{\mathbf{t}} + \overset{*}{\boldsymbol{\nabla}}\mathbf{b})$:

$$\begin{aligned}\overset{met}{t}{}^{\mu}{}_{\lambda} + \overset{*}{\nabla}_{\eta}b^{\eta\mu}{}_{\lambda} \equiv\; &t^{\mu}{}_{\lambda} - \tfrac{1}{2}b^{\gamma\beta}{}_{\alpha}(\Delta^{\mu}{}_{\lambda})\,{}^{\alpha}{}_{\beta\gamma}|_{\theta}\,{}^{\varphi\xi}\,T^{\theta}{}_{\varphi\xi}\\ &+\frac{\Delta I}{\Delta\phi^{a}}(\Delta^{\mu}{}_{\lambda})\,{}^{a}|_{b}\,\phi^{b}.\end{aligned}$$

2. Substitute the above expression into (132) and collect the similar terms, then the terms containing the Belinfante tensor $\mathbf{b}$ are presented by the combination

$$\begin{aligned}&\frac{1}{2}b^{\gamma\beta}{}_{\alpha}\left[(\nabla_{\beta}T^{\alpha}{}_{\gamma\nu} + \nabla_{\gamma}T^{\alpha}{}_{\nu\beta}) + (T^{\alpha}{}_{\varepsilon\nu}T^{\varepsilon}{}_{\beta\gamma} + T^{\alpha}{}_{\varepsilon\beta}T^{\varepsilon}{}_{\gamma\nu} + T^{\alpha}{}_{\varepsilon\gamma}T^{\varepsilon}{}_{\nu\beta}) - R^{\alpha}{}_{\nu\beta\gamma}\right]\\ &\quad= \frac{1}{2}b^{\gamma\beta}{}_{\alpha}\left[-\nabla_{\nu}T^{\alpha}{}_{\beta\gamma} + R^{\alpha}{}_{\beta\gamma\nu} + R^{\alpha}{}_{\gamma\nu\beta}\right] = \frac{1}{2}b^{\gamma\beta}{}_{\alpha}\nabla_{\nu}T^{\alpha}{}_{\beta\gamma} + b^{\gamma[\beta\alpha]}R_{\alpha\beta\gamma\nu} = \frac{1}{2}b^{\gamma\beta}{}_{\alpha}\nabla_{\nu}T^{\alpha}{}_{\beta\gamma} + \frac{1}{2}s^{\pi}{}_{\rho\sigma}R^{\rho\sigma}{}_{\pi\nu},\end{aligned}$$

where at the first equality the Ricci identity has been used, whereas at the third equality the definition (82) and identity (A9) have been taken into account.

After all the above steps the 1-st Klein identity (132) acquires the form:

$$\boxed{\overset{*}{\nabla}_{\mu}t^{\mu}{}_{\nu} \equiv -t^{\mu}{}_{\lambda}T^{\lambda}{}_{\mu\nu} + \frac{1}{2}s^{\pi}{}_{\rho\sigma}R^{\rho\sigma}{}_{\pi\nu} - \frac{1}{2}b^{\gamma\beta}{}_{\alpha}\nabla_{\nu}T^{\alpha}{}_{\beta\gamma} + \frac{\Delta I}{\Delta\phi^{a}}\nabla_{\nu}\phi^{a}} \tag{133}$$

or, in the decomposed form:

$$\boxed{\overset{*}{\nabla}_{\mu}t^{\mu}{}_{\nu} \equiv -t^{\mu}{}_{\lambda}T^{\lambda}{}_{\mu\nu} + \frac{1}{2}s^{\pi}{}_{\rho\sigma}R^{\rho\sigma}{}_{\pi\nu} + \frac{1}{2}\overset{add}{b}{}^{\gamma\beta}{}_{\alpha}\nabla_{\nu}T^{\alpha}{}_{\beta\gamma} + \frac{\Delta I}{\Delta\varphi^{a}}\nabla_{\nu}\varphi^{a}.} \tag{134}$$

From (133) and (134) the *equations of balance for the canonical EMT* $\mathbf{t}$ follow

$$\boxed{\overset{*}{\nabla}_{\mu}t^{\mu}{}_{\nu} = -t^{\mu}{}_{\lambda}T^{\lambda}{}_{\mu\nu} + \frac{1}{2}s^{\pi}{}_{\rho\sigma}R^{\rho\sigma}{}_{\pi\nu} - \frac{1}{2}b^{\gamma\beta}{}_{\alpha}\nabla_{\nu}T^{\alpha}{}_{\beta\gamma} \qquad \text{(on the } \boldsymbol{\phi}\text{-equations)};} \tag{135}$$

and

$$\boxed{\overset{*}{\nabla}_{\mu}t^{\mu}{}_{\nu} = -t^{\mu}{}_{\lambda}T^{\lambda}{}_{\mu\nu} + \frac{1}{2}s^{\pi}{}_{\rho\sigma}R^{\rho\sigma}{}_{\pi\nu} + \frac{1}{2}\overset{add}{b}{}^{\gamma\beta}{}_{\alpha}\nabla_{\nu}T^{\alpha}{}_{\beta\gamma} \qquad \text{(on the } \boldsymbol{\varphi}\text{-equations).}} \tag{136}$$

In Refs.[51–57], for the Lagrangian of the type $\mathscr{L} = \mathscr{L}(\mathbf{g};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi})$ the equation of balance for the canonical EMT (28) has been obtained. The result (28) is left valid in a more general case also, when the Lagrangian has a form: $\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi})$ because the last term in (136) does not appear. In the case of non-minimal $\mathbf{T}$-coupling the right hand side of (136) contains *additional* term $\left(\frac{1}{2} \overset{add}{b}{}^{\gamma\beta}{}_{\alpha}\nabla_\nu T^{\alpha}{}_{\beta\gamma}\right)$. However, the *new* equation (136) can be also transformed to the form (28). For this, using the identity (D2) in the last term in (134), one obtains

$$\overset{*}{\nabla}_\mu\left[t^\mu{}_\nu - \left(\overset{*}{\nabla}_\lambda \overset{add}{b}{}^{\mu\lambda}{}_\nu + \frac{1}{2}\overset{add}{b}{}^{\kappa\lambda}{}_\nu T^\mu{}_{\kappa\lambda} + \overset{add}{b}{}^{\mu\lambda}{}_\kappa T^\kappa{}_{\lambda\nu}\right)\right] \equiv -\left[t^\mu{}_\lambda - \left(\overset{*}{\nabla}_\eta \overset{add}{b}{}^{\mu\eta}{}_\lambda + \frac{1}{2}\overset{add}{b}{}^{\kappa\eta}{}_\lambda T^\mu{}_{\kappa\eta} + \overset{add}{b}{}^{\mu\eta}{}_\kappa T^\kappa{}_{\eta\lambda}\right)\right] T^\lambda{}_{\mu\nu} + \frac{1}{2}\overset{mod}{s}{}^{\pi}{}_{\rho\sigma}R^{\rho\sigma}{}_{\pi\nu} + \frac{\Delta I}{\Delta\varphi^a}\nabla_\nu\varphi^a. \tag{137}$$

Here, the expression in the brackets we denote as the *modified canonical EMT* $\overset{mod}{\mathbf{t}} \overset{def}{=} \{\overset{mod}{t}{}^\mu{}_\nu\}$:

$$\overset{mod}{t}{}^\mu{}_\nu \overset{def}{=} t^\mu{}_\nu + \overset{add}{t}{}^\mu{}_\nu, \tag{138}$$

where the *additional EMT* $\overset{add}{\mathbf{t}} \overset{def}{=} \{\overset{add}{t}{}^\mu{}_\nu\}$ is defined as

$$\overset{add}{t}{}^\mu{}_\nu \overset{def}{=} -\left(\overset{*}{\nabla}_\lambda \overset{add}{b}{}^{\mu\lambda}{}_\nu + \frac{1}{2}\overset{add}{b}{}^{\kappa\lambda}{}_\nu T^\mu{}_{\kappa\lambda} + \overset{add}{b}{}^{\mu\lambda}{}_\kappa T^\kappa{}_{\lambda\nu}\right) = \overset{*}{\nabla}_\lambda \overset{add}{b}{}^{\lambda\mu}{}_\nu + \frac{1}{2}\overset{add}{b}{}^{\gamma\beta}{}_\alpha(\Delta^\mu{}_\nu)\,{}^\alpha{}_{\beta\gamma}\big|_\theta{}^{\varphi\xi}\, T^\theta{}_{\varphi\xi}. \tag{139}$$

Note that this modification is analogous to the modification of the canonical ST in (88) and (90). It is evidently that the canonical EMT $\overset{mod}{\mathbf{t}}$ (138) in the case of minimal T-coupling only transforms to (usual) canonical EMT $\mathbf{t}$. By the definition (139), The Belinfante symmetrization of the type (123) applied to $\overset{add}{\mathbf{t}}$ leads to zero identically. Therefore, the *symmetrized EMT* $\overset{sym}{\mathbf{t}}$ constructed by the symmetrization of $\overset{mod}{\mathbf{t}}$ with the use of $\overset{mod}{\mathbf{b}}$ by the rule

$$\begin{aligned}\overset{sym}{t}{}^\mu{}_\nu &= \overset{mod}{t}{}^\mu{}_\nu + \left[\overset{*}{\nabla}_\lambda \overset{mod}{b}{}^{\mu\lambda}{}_\nu + \frac{1}{2}\overset{mod}{b}{}^{\kappa\lambda}{}_\nu T^\mu{}_{\kappa\lambda} + \overset{mod}{b}{}^{\mu\lambda}{}_\kappa T^\kappa{}_{\lambda\nu}\right]\\ &= \overset{mod}{t}{}^\mu{}_\nu - \left[\overset{*}{\nabla}_\lambda \overset{mod}{b}{}^{\lambda\mu}{}_\nu + \frac{1}{2}\overset{mod}{b}{}^{\gamma\beta}{}_\alpha(\Delta^\mu{}_\nu)\,{}^\alpha{}_{\beta\gamma}\big|_\theta{}^{\varphi\xi}\, T^\theta{}_{\varphi\xi}\right]\end{aligned} \tag{140}$$

exactly coincides with the (usual) symmetrized EMT $\overset{sym}{\mathbf{t}}$ (123).

In the terms of the modified canonical EMT $\overset{mod}{\mathbf{t}}$ the identities (133) and (134) can be rewritten as

$$\boxed{\begin{aligned}\overset{*}{\nabla}_\mu \overset{mod}{t}{}^\mu{}_\nu \equiv &-\overset{mod}{t}{}^\mu{}_\lambda T^\lambda{}_{\mu\nu} + \frac{1}{2}\overset{mod}{s}{}^\pi{}_{\rho\sigma}R^{\rho\sigma}{}_{\pi\nu}\\ &-\frac{1}{2}\overset{mod}{b}{}^{\gamma\beta}{}_\alpha\nabla_\nu T^\alpha{}_{\beta\gamma} + \frac{\Delta I}{\Delta\phi^a}\nabla_\nu\phi^a\end{aligned}} \tag{141}$$

and

$$\boxed{\begin{aligned}\overset{*}{\nabla}_\mu \overset{mod}{t}{}^\mu{}_\nu \equiv &-\overset{mod}{t}{}^\mu{}_\lambda T^\lambda{}_{\mu\nu} + \frac{1}{2}\overset{mod}{s}{}^\pi{}_{\rho\sigma}R^{\rho\sigma}{}_{\pi\nu}\\ &+\frac{\Delta I}{\Delta\varphi^a}\nabla_\nu\varphi^a.\end{aligned}} \tag{142}$$

They are the basis for the *equations of balance for the modified canonical EMT* $\overset{mod}{\mathbf{t}}$:

$$\boxed{\begin{aligned}\overset{*}{\nabla}_\mu \overset{mod}{t}{}^\mu{}_\nu = &-\overset{mod}{t}{}^\mu{}_\lambda T^\lambda{}_{\mu\nu} + \frac{1}{2}\overset{mod}{s}{}^\pi{}_{\rho\sigma}R^{\rho\sigma}{}_{\pi\nu}\\ &-\frac{1}{2}\overset{mod}{b}{}^{\gamma\beta}{}_\alpha\nabla_\nu T^\alpha{}_{\beta\gamma} \qquad \text{(on the } \boldsymbol{\phi}\text{-equations)}\end{aligned}} \tag{143}$$

and

$$\boxed{\overset{*}{\nabla}_\mu \overset{mod}{t}{}^{\mu}{}_{\nu} = -\overset{mod}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2}\overset{mod}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} \quad \text{(on the } \boldsymbol{\varphi}\text{-equations).}} \tag{144}$$

Now, the equation (144) has the same structure as the equation (28). Also notice that, when the equations for the torsion field $\Delta I/\Delta \mathbf{T} = 0$ hold, then

$$\overset{mod}{t}{}^{\mu}{}_{\nu} = \overset{sym}{t}{}^{\mu}{}_{\nu} \qquad \text{(on the } \mathbf{T}\text{-equations),} \tag{145}$$

as it follows from (140) and (85).

At last, let us find the identities and the equations of balance for the symmetrized EMT $\overset{sym}{\mathbf{t}}$. Use (123) for rewriting $\mathbf{t}$ as a function of $\overset{sym}{\mathbf{t}}$ and $\mathbf{b}$, substitute the result into (133) and (134) and find, respectively,

$$\boxed{\overset{*}{\nabla}_\mu \overset{sym}{t}{}^{\mu}{}_{\nu} \equiv -\overset{sym}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{\Delta I}{\Delta \phi^a}\nabla_\nu \phi^a;} \tag{146}$$

and

$$\boxed{\begin{aligned}\overset{*}{\nabla}_\mu \overset{sym}{t}{}^{\mu}{}_{\nu} \equiv& -\overset{sym}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu}\\ &+\frac{1}{2}\overset{mod}{b}{}^{\gamma\beta}{}_{\alpha}\nabla_\nu T^{\alpha}{}_{\beta\gamma} + \frac{\Delta I}{\Delta \varphi^a}\nabla_\nu\varphi^a.\end{aligned}} \tag{147}$$

The same identities can be obtained by another way. Namely, express the metric EMT $\overset{met}{\mathbf{t}}$ through the symmetrized EMT $\overset{sym}{\mathbf{t}}$ from the 2-nd Klein identity (125) and substitute the result into the Noether identity (102).

Next, the *equations of balance for the symmetrized EMT* $\overset{sym}{\mathbf{t}}$, which follow from the identities (146) and (147) are

$$\boxed{\overset{*}{\nabla}_\mu \overset{sym}{t}{}^{\mu}{}_{\nu} = -\overset{sym}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} \qquad \text{(on the } \boldsymbol{\phi}\text{-equations)}} \tag{148}$$

and

$$\boxed{\begin{aligned}\overset{*}{\nabla}_\mu \overset{sym}{t}{}^{\mu}{}_{\nu} =& -\overset{sym}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu}\\ &+\frac{1}{2}\overset{mod}{b}{}^{\gamma\beta}{}_{\alpha}\nabla_\nu T^{\alpha}{}_{\beta\gamma} \qquad \text{(on the } \boldsymbol{\varphi}\text{-equations).}\end{aligned}} \tag{149}$$

Finalizing subsection, one concludes that *the 1-st Klein identity is the basis for constructing the equations of balance for the canonical EMT* $\mathbf{t}$. These relations coincide with the known (standard) ones, when a non-minimal coupling with torsion is absent. When a non-minimal coupling with torsion presents the canonical EMT $\mathbf{t}$ is changed by $\overset{mod}{\mathbf{t}}$ with the use of the modified Belinfante tensor, and then the equations of balance for the modified EMT $\overset{mod}{\mathbf{t}}$ acquire the standard form again. Also, the *1-st Klein identity, as well as the 2-nd one, are the basis for constructing the equations of balance for the symmetrized EMT* $\overset{sym}{\mathbf{t}}$.

VI. THE GENERALIZED SUPERPOTENTIAL AND NOETHER CURRENT

A. The calculation of the superpotential

At first, let us calculate the generalized superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}] = \{\theta^{[\mu\nu]}[\delta\boldsymbol{\xi}] = \theta^{\mu\nu}[\delta\boldsymbol{\xi}]\}$ in the explicit form. Corresponding the formula (21), one has

$$\theta^{\mu\nu}[\delta\boldsymbol{\xi}] = \theta_\alpha{}^{\mu\nu}\delta\xi^\alpha + \theta_\alpha{}^{\beta\mu\nu}\nabla_\beta\delta\xi^\alpha, \tag{150}$$

where

$$\theta_\alpha{}^{\mu\nu} = -M_\alpha{}^{[\mu\nu]} + \frac{2}{3}\left(\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\mu\nu]} + \frac{1}{2}T^{[\mu}{}_{\rho\sigma}N_\alpha{}^{\nu]\rho\sigma}\right); \tag{151}$$

$$\theta_\alpha{}^{\beta\mu\nu} = -\frac{4}{3}N_\alpha{}^{\beta[\mu\nu]}. \tag{152}$$

In fact, we have calculated the tensor $\{\theta_\alpha{}^{\mu\nu}\}$ already. It is defined by the expression (117). For the tensor $\{\theta_\alpha{}^{\beta\mu\nu}\}$, using (113), one finds

$$\begin{aligned}\theta_\alpha{}^{\beta\mu\nu} &= -\tfrac{2}{3}G_\alpha{}^{\beta\mu\nu} + \tfrac{2}{3}G_\alpha{}^{[\mu\nu]\beta}\\ &= -G_\alpha{}^{\beta\mu\nu} + \tfrac{1}{3}\left(G_\alpha{}^{\beta\mu\nu} + G_\alpha{}^{\mu\nu\beta} + G_\alpha{}^{\nu\beta\mu}\right)\\ &= -G_\alpha{}^{\beta\mu\nu} + G_\alpha{}^{[\beta\mu\nu]}.\end{aligned} \tag{153}$$

Finally one obtains

$$\boxed{\begin{aligned}\theta^{\mu\nu}[\delta\boldsymbol{\xi}] \ =& \left\{\left[-b^{\mu\nu}{}_\alpha + G_\gamma{}^{\beta\mu\nu}T^\gamma{}_{\beta\alpha}\right] + \left[\overset{*}{\nabla}_\lambda G_\alpha{}^{[\mu\nu\lambda]} + \frac{1}{2}\left(G_\alpha{}^{[\mu\rho\sigma]}T^\nu{}_{\rho\sigma} - G_\alpha{}^{[\nu\rho\sigma]}T^\mu{}_{\rho\sigma}\right)\right]\right\}\delta\xi^\alpha\\ &+\left\{G_\alpha{}^{\beta\mu\nu} + G_\alpha{}^{[\beta\mu\nu]}\right\}\nabla_\beta\delta\xi^\alpha.\end{aligned}} \tag{154}$$

B. Dynamical quantities in the structure of the generalized currents

More useful and interesting, however, to construct the superpotential starting from the generalized canonical Noether current $\mathbf{J}[\delta\boldsymbol{\xi}]$ (2)

$$J^\mu[\delta\boldsymbol{\xi}] = U_\alpha{}^\mu\delta\xi^\alpha + M_\alpha{}^{\beta\mu}\nabla_\beta\delta\xi^\alpha + N_\alpha{}^{\beta\gamma\mu}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha. \tag{155}$$

Such a construction lets us understand better the connections of the generalized currents $\mathbf{J}[\delta\boldsymbol{\xi}]$, $\overset{sym}{\mathbf{J}}[\delta\boldsymbol{\xi}]$, on the one hand, with the dynamical characteristics $\mathbf{t}$, $\overset{sym}{\mathbf{t}}$, $\mathbf{s}$, ..., on the other hand.

Substituting (76), (68) and (62) into the formula (155), one finds the explicit presentation for the current $\mathbf{J}[\delta\boldsymbol{\xi}]$:

$$\begin{aligned} J^\mu[\delta\boldsymbol{\xi}] \; &= \left\{ t^\mu{}_\alpha + b^{\mu\lambda}{}_\kappa T^\kappa{}_{\lambda\alpha} + \frac{1}{2} G^{\pi\rho\sigma\mu} R_{\alpha\pi\rho\sigma} + \left[\overset{*}{\nabla}_\nu \left(G_\kappa{}^{\lambda\mu\nu} T^\kappa{}_{\lambda\alpha} \right) + \frac{1}{2} \left(G_\kappa{}^{\lambda\rho\sigma} T^\kappa{}_{\lambda\alpha} \right) T^\mu{}_{\rho\sigma} \right] \right\} \delta\xi^\alpha \\ &+ \left\{ -b^{\mu\beta}{}_\alpha - \left[\overset{*}{\nabla}_\nu G_\alpha{}^{\beta\mu\nu} + \frac{1}{2} G_\alpha{}^{\beta\rho\sigma} T^\mu{}_{\rho\sigma} \right] + \frac{1}{2} G_\alpha{}^{\rho\sigma\mu} T^\beta{}_{\rho\sigma} - \left(G_\kappa{}^{\lambda\beta\mu} T^\kappa{}_{\lambda\alpha} \right) \right\} \nabla_\beta \delta\xi^\alpha \\ &+ \left\{ G_\alpha{}^{(\beta\gamma)\mu} \right\} \nabla_{(\gamma} \nabla_{\beta)} \delta\xi^\alpha . \end{aligned} \tag{156}$$

As is seen, the *canonical* current $\mathbf{J}[\delta\boldsymbol{\xi}]$ essentially is constructed by the *canonical* dynamic quantities $\mathbf{t}$, $\mathbf{s}$ and the tensor $\mathbf{G}$.

Now, apply the identical transformations to the terms at the right hand side of (156) as follows.

1. For a first part of items in (156), differentiating by parts, adding and subtracting the combination $\left(\frac{1}{2} b^{\kappa\lambda}{}_\alpha T^\mu{}_{\kappa\lambda} \delta\xi^\alpha\right)$, one finds

$$\begin{aligned} &\left(t^\mu{}_\alpha + b^{\mu\lambda}{}_\kappa T^\kappa{}_{\lambda\alpha} \right) \delta\xi^\alpha - b^{\mu\beta}{}_\alpha \nabla_\beta \delta\xi^\alpha = \left\{ t^\mu{}_\alpha + \left(\overset{*}{\nabla}_\lambda b^{\mu\lambda}{}_\alpha + \frac{1}{2} b^{\kappa\lambda}{}_\alpha T^\mu{}_{\kappa\lambda} + b^{\mu\lambda}{}_\kappa T^\kappa{}_{\lambda\alpha} \right) \right\} \delta\xi^\alpha \\ &+ \left\{ \overset{*}{\nabla}_\nu \left(-b^{\mu\nu}{}_\alpha \delta\xi^\alpha \right) + \frac{1}{2} \left(-b^{\kappa\lambda}{}_\alpha \delta\xi^\alpha \right) T^\mu{}_{\kappa\lambda} \right\} = \left\{ \overset{sym}{t}{}^\mu{}_\alpha \right\} \delta\xi^\alpha + \left\{ \overset{*}{\nabla}_\nu \left[-b^{\mu\nu}{}_\alpha \delta\xi^\alpha \right] + \frac{1}{2} \left[-b^{\rho\sigma}{}_\alpha \delta\xi^\alpha \right] T^\mu{}_{\rho\sigma} \right\}, \end{aligned} \tag{157}$$

where, at the second equality, the definition (123) has been taken into account.

2. For a second part of items in (156), differentiating by parts and collecting the similar terms, one obtains

$$\begin{aligned} &\left[\overset{*}{\nabla}_\nu \left(G_\kappa{}^{\lambda\mu\nu} T^\kappa{}_{\lambda\alpha} \right) + \frac{1}{2} \left(G_\kappa{}^{\lambda\rho\sigma} T^\kappa{}_{\lambda\alpha} \right) T^\mu{}_{\rho\sigma} \right] \delta\xi^\alpha \\ &\quad - \left(G_\kappa{}^{\lambda\beta\mu} T^\kappa{}_{\lambda\alpha} \right) \nabla_\beta \delta\xi^\alpha \\ &= \overset{*}{\nabla}_\nu \left[G_\kappa{}^{\lambda\mu\nu} T^\kappa{}_{\lambda\alpha} \delta\xi^\alpha \right] + \frac{1}{2} \left[G_\kappa{}^{\lambda\rho\sigma} T^\kappa{}_{\lambda\alpha} \delta\xi^\alpha \right] T^\mu{}_{\rho\sigma} . \end{aligned} \tag{158}$$

3. For the last part of items in (156), again differentiating by parts and collecting the similar terms, one finds

$$\begin{aligned} &\left(\frac{1}{2} G^{\pi\rho\sigma\mu} R_{\alpha\pi\rho\sigma} \right) \delta\xi^\alpha - \left[\overset{*}{\nabla}_\nu G_\alpha{}^{\beta\mu\nu} + \frac{1}{2} G_\alpha{}^{\beta\rho\sigma} T^\mu{}_{\rho\sigma} \right] \nabla_\beta \delta\xi^\alpha + \frac{1}{2} G_\alpha{}^{\rho\sigma\mu} T^\beta{}_{\rho\sigma} \nabla_\beta \delta\xi^\alpha + G_\alpha{}^{(\beta\gamma)\mu} \nabla_\gamma \nabla_\beta \delta\xi^\alpha \\ &= \frac{1}{2} \left(G^{\lambda[\rho\sigma]\mu} \delta\xi^\alpha \right) R_{\alpha\lambda\rho\sigma} + \left[\overset{*}{\nabla}_\gamma G_\alpha{}^{[\beta\gamma]\mu} + \frac{1}{2} G_\alpha{}^{[\rho\sigma]\mu} T^\beta{}_{\rho\sigma} \right] \nabla_\beta \delta\xi^\alpha + \overset{*}{\nabla}_\nu \left[G_\alpha{}^{(\beta\nu)\mu} \nabla_\beta \delta\xi^\alpha \right] + \frac{1}{2} \left[-G_\alpha{}^{\beta\rho\sigma} \nabla_\beta \delta\xi^\alpha \right] T^\mu{}_{\rho\sigma} . \end{aligned} \tag{159}$$

4. Differentiating by parts the second term on the right hand side of (159), subsequently using the identities (119) and (130) with $\tilde{\theta}_\alpha{}^{\mu\nu\kappa} = G_\alpha{}^{[\nu\kappa\mu]}$ and $\tilde{\theta}^{\nu\kappa\mu} = G_\alpha{}^{[\nu\kappa\mu]} \delta\xi^\alpha$, respectively, and collecting the similar terms, one gets

$$\begin{aligned} &\left[\overset{*}{\nabla}_\gamma G_\alpha{}^{[\beta\gamma]\mu} + \frac{1}{2} G_\alpha{}^{[\rho\sigma]\mu} T^\beta{}_{\rho\sigma} \right] \nabla_\beta \delta\xi^\alpha \\ &= \overset{*}{\nabla}_\nu \left[\overset{*}{\nabla}_\kappa \left(G_\alpha{}^{[\nu\kappa]\mu} \delta\xi^\alpha \right) + \frac{1}{2} \left(G_\alpha{}^{[\rho\sigma]\mu} \delta\xi^\alpha \right) T^\nu{}_{\rho\sigma} \right] \\ &\quad - \overset{*}{\nabla}_\nu \left[G_\alpha{}^{[\nu\beta]\mu} \nabla_\beta \delta\xi^\alpha \right] \\ &\quad - \frac{1}{2} \left(G_\alpha{}^{[\rho\sigma]\lambda} R^\mu{}_{\lambda\rho\sigma} - G_\lambda{}^{[\rho\sigma]\mu} R^\lambda{}_{\alpha\rho\sigma} \right) \delta\xi^\alpha \\ &= \overset{*}{\nabla}_\nu \left[G_\alpha{}^{[\beta\nu]\mu} \nabla_\beta \delta\xi^\alpha \right] - \frac{1}{2} \left(G^{\lambda[\rho\sigma]\mu} \delta\xi^\alpha \right) R_{\alpha\lambda\rho\sigma} . \end{aligned} \tag{160}$$

5. Substituting this result into (159) one obtains

$$\text{R.H.S. of the eq. (159)} = \overset{*}{\nabla}_\nu \left[-G_\alpha{}^{\beta\mu\nu}\nabla_\beta\delta\xi^\alpha\right] + \tfrac{1}{2}\left[-G_\alpha{}^{\beta\rho\sigma}\nabla_\beta\delta\xi^\alpha\right] T^\mu{}_{\rho\sigma}. \tag{161}$$

Combining the points 1 – 5, one finds that the formula (156) is presented equivalently as

$$\boxed{J^\mu[\delta\boldsymbol{\xi}] = \overset{sym}{J}{}^{\mu}[\delta\boldsymbol{\xi}] + \left\{\overset{*}{\nabla}_\nu \theta'^{\mu\nu}[\delta\boldsymbol{\xi}] + \frac{1}{2}\theta'^{\rho\sigma}[\delta\boldsymbol{\xi}]T^\mu{}_{\rho\sigma}\right\},} \tag{162}$$

where

$$\boxed{\overset{sym}{J}{}^{\mu}[\delta\boldsymbol{\xi}] \overset{def}{=} \overset{sym}{t}{}^{\mu}{}_\alpha\delta\xi^\alpha} \tag{163}$$

is the generalized symmetrized Noether current (see the Paper I, Sec. V), and

$$\boxed{\begin{aligned}\theta'^{\mu\nu}[\delta\boldsymbol{\xi}] &\overset{def}{=} \left[-b^{\mu\nu}{}_\alpha + G_\kappa{}^{\lambda\mu\nu}T^\kappa{}_{\lambda\alpha}\right]\delta\xi^\alpha\\ &+ \left[-G_\alpha{}^{\beta\mu\nu}\right]\nabla_\beta\delta\xi^\alpha.\end{aligned}} \tag{164}$$

The formula (163) shows that the *symmetrized current* $\overset{sym}{\mathbf{J}}[\delta\boldsymbol{\xi}]$ is expressed thorough *only the symmetrized EMT* $\overset{sym}{\mathbf{t}}$ even in the case of the Lagrangian of the most general type (23). Analogously, the formula (164) shows that the superpotential $\boldsymbol{\theta}'[\delta\boldsymbol{\xi}]$ is expressed through *only the Belinfante tensor* $\mathbf{b}$ induced by the *canonical ST* $\mathbf{s}$ and the tensor $\mathbf{G}$.

Combining the 2-nd Klein identity (125) and (98), one finds

$$\overset{sym}{t}{}^{\mu}{}_\nu = -I_\mu{}^\nu, \tag{165}$$

that is the symmetrized EMT $\overset{sym}{\mathbf{t}}$ depends on only the Lagrangian derivatives (see definition (19)), and, consequently, *does not depend on divergences in Lagrangian* $\mathscr{L}$. By (125) and (98) also, the formula (163), can be represented as

$$\overset{sym}{J}{}^{\mu}[\delta\boldsymbol{\xi}] = \left[\overset{met}{t}{}^{\mu}{}_\alpha - \frac{\Delta I}{\Delta\phi^a}(\Delta^\mu{}_\alpha)\ {}^a|_b\,\phi^b\right]\delta\xi^\alpha = -I_\alpha{}^\mu\delta\xi^\alpha. \tag{166}$$

Comparing (162) and (166) with the boundary Klein-Noether theorem (20), one concludes that the superpotential $\boldsymbol{\theta}'[\delta\boldsymbol{\xi}]$ (164) has to be equivalent to the canonical superpotential (21). Nevertheless, comparing the right hand sides of (164) and (154) directly, we do not see this! However, the difference is not essential. Recall the remark in the Paper I (Sec. IV, formulae (55)–(69)) that is related to *arbitrary* two superpotentials, $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ and $\boldsymbol{\theta}'[\delta\boldsymbol{\xi}]$, which differ in a term of the type

$$\begin{aligned}\Delta\theta^{\mu\nu}[\delta\boldsymbol{\xi}] &\overset{def}{=} \theta'^{\mu\nu}[\delta\boldsymbol{\xi}] - \theta^{\mu\nu}[\delta\boldsymbol{\xi}]\\ &= \left[\overset{*}{\nabla}_\lambda C_\alpha{}^{\mu\nu\lambda} + C_\alpha{}^{[\mu|\rho\sigma}T^{|\nu]}{}_{\rho\sigma}\right]\delta\xi^\alpha + \left[C_\alpha{}^{\beta\mu\nu}\right]\nabla_\beta\delta\xi^\alpha,\end{aligned} \tag{167}$$

where an *arbitrary* tensor $\{C_\alpha{}^{\lambda\mu\nu}\}$ is totally antisymmetric in contravariant indexes:

$$C_\alpha{}^{[\lambda\mu\nu]} = C_\alpha{}^{\lambda\mu\nu}. \tag{168}$$

Then, such superpotentials, $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ and $\boldsymbol{\theta}'[\delta\boldsymbol{\xi}]$, are related to the *same* Noether current! One can see easily that the difference of $\boldsymbol{\theta}'[\delta\boldsymbol{\xi}]$ (164) and $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ (154) has just the above form with $C_\alpha{}^{\lambda\mu\nu} = -G_\alpha{}^{[\lambda\mu\nu]}$.

Rather, by a simplicity, the superpotential $\boldsymbol{\theta}'[\delta\boldsymbol{\xi}]$ (164) could be more preferable in applications.

VII. STRUCTURE AND INTERPRETATION OF THE EQUATIONS OF GRAVITATIONAL FIELDS

A. The field equations with the total EMT and ST

The system of the equations of motion of all the fields $\mathbf{g}$, $\mathbf{T}$ and $\boldsymbol{\varphi}$, as usual, is obtained by variation of the action functional, thus

$$\begin{cases}\Delta I/\Delta g_{\mu\nu} = 0; & (169)\\ \Delta I/\Delta T^\lambda{}_{\mu\nu} = 0; & (170)\\ \Delta I/\Delta\varphi^a = 0. & (171)\end{cases}$$

Combining (93), (125), (140) and (85), it is not difficult to obtain

$$2\frac{\Delta I}{\Delta g_{\mu\nu}} \equiv \overset{mod}{t}{}^{\mu\nu} - \overset{*}{\nabla}_\lambda \overset{mod}{b}{}^{\lambda\mu\nu} + \frac{\Delta I}{\Delta\varphi^a}(\Delta^{\mu\nu})\ {}^a|_b\,\varphi^b. \tag{172}$$

Then, again turning to (85), one finds that the system (169) - (171) is equivalent to

$$\begin{cases}\overset{mod}{t}{}^{\mu\nu} = 0; & (173)\\ \overset{mod}{b}{}^{\nu\mu}{}_\lambda = 0; & (174)\\ \Delta I/\Delta\varphi^a = 0. & (175)\end{cases}$$

Remark that, by the identity (A9), the equation (174) induces the equation

$$\overset{mod}{s}{}^{\pi}{}_{\rho\sigma} = 0, \tag{176}$$

and conversely. Recall also that the dynamic characteristics of the physical system $\overset{mod}{\mathbf{t}}$ and $\overset{mod}{\mathbf{s}}$ are total because are related to the total action of the system.

A direct interpretation of the equations of the gravitational fields that follows from the visible presentations of (173) and (176) is evident: In an arbitrary metric-torsion theory of gravity without background structures, both the *total* modified EMT $\overset{mod}{\mathbf{t}}$ and the *total* modified canonical $\overset{mod}{\mathbf{s}}$ are equal to nil. The claim that the total dynamic characteristics in a gravitational theory have to be equal to zero is not new. In GR, it has being defended by Lorentz[60,61] and Levi-Civita[62,63], later by Soriau[64].

Comparatively recent, Szabados[39,40] has approved this result, examining the Belinfante procedure. In the works by Logunov and Folomeshkin[35–38] this claim is treated as unavoidable conclusion in a pure metric theory of gravity.

However, under a more detailed consideration such an interpretation meets serious objections, which lead to a necessity to reject it. The first who was against is Einstein[65,66]. Replying the Lorentz work[60,61], he noted that there is no a logic argument against the Lorentz interpretation. But, basing on the equation (173), one cannot to obtain conclusions that usually follow from the conservation laws. Indeed, due to (173), the components of the total energy tensor everywhere during all the time are equal to zero, that is the total energy of the system from the beginning is equal to zero. However, the conservation of "zero" does not require the next existence of the system: one "permits" a disappearance of the physical system at all. Such a conclusion looks at as extremely non-physical. Of course, the Einstein arguments can be applied to discuss the total ST $\overset{mod}{\mathbf{s}}$.

B. Pure gravitational and matter parts of the physical system

Recall the basis of constructing the GR and other metric theories. One of the main requirements is that the dynamic physical picture is postulated as follows. A bend of a curved space-time, in which the matter propagate, is provided by the matter itself. Then, by a natural way it turns out that the physical system is presented as a union of divided the pure gravitational part and the matter part. As it will be shown in the Paper III, in the last one of the series of the works, the problem of defining physically sensible conserved quantities can be solved just in the framework of such a presentation. Below, in subsection VII D, we give the other arguments supporting the split presentation of the equations and against the nil interpretation of the total dynamic characteristics of the system. Now, we give and discuss the main formulae and relations for the split presentation in the framework of the manifestly generally covariant metric-torsion theories given in the Riemann-Cartan space.

Keeping in mind the above, represent the Lagrangian (23) as a sum of the pure gravitational $\mathscr{L}^G$ and matter $\mathscr{L}^M$ parts:

$$\begin{aligned}\mathscr{L} &= \mathscr{L}(\mathbf{g},\mathbf{R};\ \mathbf{T},\boldsymbol{\nabla}\mathbf{T},\boldsymbol{\nabla\nabla}\mathbf{T};\ \boldsymbol{\varphi},\boldsymbol{\nabla}\boldsymbol{\varphi},\boldsymbol{\nabla\nabla}\boldsymbol{\varphi})\\ &\overset{def}{=} \mathscr{L}(\mathbf{g},\mathbf{R};\ \boldsymbol{\phi},\boldsymbol{\nabla}\boldsymbol{\phi},\boldsymbol{\nabla\nabla}\boldsymbol{\phi}) = \mathscr{L}^G+\mathscr{L}^M,\end{aligned} \tag{177}$$

where

$$\mathscr{L}^G = \mathscr{L}^G(\mathbf{g},\mathbf{R}) \overset{def}{=} \mathscr{L}(\mathbf{g},\mathbf{R};0,0,0); \tag{178}$$

$$\mathscr{L}^M = \mathscr{L}^M(\mathbf{g},\mathbf{R};\ \boldsymbol{\phi},\boldsymbol{\nabla}\boldsymbol{\phi},\boldsymbol{\nabla\nabla}\boldsymbol{\phi}) \overset{def}{=} \mathscr{L}-\mathscr{L}^G. \tag{179}$$

Remark that in the definition (178), of course, the connection $\boldsymbol{\Gamma}$ (with the use of that the curvature tensor $\mathbf{R}$ is constructed) continues to depend on the torsion $\mathbf{T}$. Thus, the gravitational Lagrangian $\mathscr{L}^G$ not explicitly (through $\mathbf{R}$) depends on $\mathbf{T}$ also.

However, in all the cases, it is not possible to define $\mathscr{L}^G$ as in (178), for example, in the scalar-tensor Jordan-Brans-Dicke theories[67,68], or, in more general theories gravity with dilaton[69,70], in "sting" presentation (see Refs.[71,72]). At the same time, in the "Einstein" presentation such a splitting can be provided easily. Then it is necessary to define clearly what unusual fields (except of metric and torsion ones) are related to gravitational fields.

It is evidently that the splitting of the Lagrangian (177) leads to a correspondent splitting of the action functional:

$$\begin{aligned}I &= \int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\mathscr{L} = \int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\mathscr{L}^G + \int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\mathscr{L}^M\\ &\overset{def}{=} I^G + I^M.\end{aligned}$$

Of course, the Lagrangian of the vacuum system $\mathscr{L}^G$ has to be *generally covariant scalar*, and then the matter Lagrangian $\mathscr{L}^M$ is, like this, also. Therefore all the above results and conclusions related to the total Lagrangian $\mathscr{L}$ are left valid for each of the Lagrangians $\mathscr{L}^G$ and $\mathscr{L}^M$ themselves.

Define next *matter tensors*.

$$S^\pi{}_{\rho\sigma} \overset{def}{=} s^\pi{}_{\rho\sigma}\big|_{\mathscr{L}=\mathscr{L}^M} \tag{180}$$
(the canonical ST of matter);

$$\overset{add}{S}{}^\pi{}_{\rho\sigma} \overset{def}{=} \overset{add}{s}{}^\pi{}_{\rho\sigma}\Big|_{\mathscr{L}=\mathscr{L}^M} \tag{181}$$
(the additional ST of matter);

$$\overset{mod}{S}{}^\pi{}_{\rho\sigma} \overset{def}{=} \overset{mod}{s}{}^\pi{}_{\rho\sigma}\Big|_{\mathscr{L}=\mathscr{L}^M} \tag{182}$$
(the modified canonical ST of matter).

$$B^{\gamma\beta\alpha} \overset{def}{=} b^{\gamma\beta\alpha}\big|_{\mathscr{L}=\mathscr{L}^M} \tag{183}$$
(the Belinfante tensor for ST $\mathbf{S}$);

$$\overset{add}{B}{}^{\gamma\beta\alpha} \overset{def}{=} \overset{add}{b}{}^{\gamma\beta\alpha}\Big|_{\mathscr{L}=\mathscr{L}^M} \tag{184}$$
(the Belinfante tensor for ST $\overset{add}{\mathbf{S}}$);

$$\overset{mod}{B}{}^{\gamma\beta\alpha} \overset{def}{=} \overset{mod}{b}{}^{\gamma\beta\alpha}\Big|_{\mathscr{L}=\mathscr{L}^M} \tag{185}$$
(the Belinfante tensor for ST $\overset{mod}{\mathbf{S}}$).

$$T^{\mu}{}_{\nu} \overset{def}{=} t^{\mu}{}_{\nu}\big|_{\mathscr{L}=\mathscr{L}^M} \tag{186}$$

(the canonical EMT of matter);

$$\overset{add}{T}{}^{\mu}{}_{\nu} \overset{def}{=} \overset{add}{t}{}^{\mu}{}_{\nu}\Big|_{\mathscr{L}=\mathscr{L}^M} \tag{187}$$

(the additional EMT of matter);

$$\overset{mod}{T}{}^{\mu}{}_{\nu} \overset{def}{=} \overset{mod}{t}{}^{\mu}{}_{\nu}\Big|_{\mathscr{L}=\mathscr{L}^M} \tag{188}$$

(the modified canonical EMT of matter);

$$\overset{sym}{T}{}^{\mu\nu} \overset{def}{=} \overset{sym}{t}{}^{\mu\nu}\Big|_{\mathscr{L}=\mathscr{L}^M} \tag{189}$$

(the symmetrized EMT of matter);

$$\overset{met}{T}{}^{\mu\nu} \overset{def}{=} \overset{met}{t}{}^{\mu\nu}\Big|_{\mathscr{L}=\mathscr{L}^M} \tag{190}$$

(the metrical EMT of matter).

For the above defined matter tensors, relations analogous to those between the total tensors take a place. In particular, analogously to (85) and (172), one has

$$\boxed{\frac{\Delta I^M}{\Delta T^{\alpha}{}_{\beta\gamma}} = \frac{1}{2}\,\overset{mod}{B}{}^{\gamma\beta}{}_{\alpha};} \tag{191}$$

$$\boxed{\begin{aligned} 2\frac{\Delta I^M}{\Delta g_{\mu\nu}} &= \overset{met}{T}{}^{\mu\nu} \\ &\equiv \overset{mod}{T}{}^{\mu\nu} - \overset{*}{\nabla}_{\lambda}\overset{mod}{B}{}^{\lambda\mu\nu} + \frac{\Delta I^M}{\Delta\varphi^a}(\Delta^{\mu\nu})\,{}^{a}|_{b}\,\varphi^b. \end{aligned}} \tag{192}$$

Now, define the *Cartan tensor* $\mathscr{C} \overset{def}{=} \{\mathscr{C}^{[\gamma\beta]}{}_{\alpha} = \mathscr{C}^{\gamma\beta}{}_{\alpha}\}$ and the *(generalized) Einstein tensor* $\mathscr{E} \overset{def}{=} \{\mathscr{E}^{(\mu\nu)} \neq \mathscr{E}^{\mu\nu}\}$:

$$\frac{-1}{2k}\mathscr{C}^{\gamma\beta}{}_{\alpha} \overset{def}{=} \frac{\Delta I^G}{\Delta T^{\alpha}{}_{\beta\gamma}} = \frac{1}{2}\,b^{\gamma\beta}{}_{\alpha}\big|_{\mathscr{L}=\mathscr{L}^G} \tag{193}$$

(the Cartan tensor);

$$\frac{-1}{2k}\left(\mathscr{E}^{(\mu\nu)} - \overset{*}{\nabla}_{\lambda}\mathscr{C}^{\lambda(\mu\nu)}\right) \overset{def}{=} \frac{\Delta I^G}{\Delta g_{\mu\nu}} = \frac{1}{2}\,\overset{met}{t}{}^{\mu\nu}\Big|_{\mathscr{L}=\mathscr{L}^G} \tag{194}$$

(symmetric part of the generalized Einstein tensor);

$$\begin{aligned} \frac{-1}{2k}\mathscr{E}^{\mu}{}_{\nu} &\overset{def}{=} \frac{1}{2}\,t^{\mu}{}_{\nu}\big|_{\mathscr{L}=\mathscr{L}^G} \\ &= \frac{1}{2}\left(\mathscr{L}^G\delta^{\mu}_{\nu} - {}^{(G)}G^{\alpha\beta\gamma\mu}R_{\alpha\beta\gamma\nu}\right) \end{aligned} \tag{195}$$

(the generalized Einstein tensor).

Here,

$${}^{(G)}G_{\alpha}{}^{\beta\gamma\delta} \overset{def}{=} G_{\alpha}{}^{\beta\gamma\delta}\big|_{\mathscr{L}=\mathscr{L}^G} = 2\frac{\partial\mathscr{L}^G}{\partial R^{\alpha}{}_{\beta\gamma\delta}}; \tag{196}$$

$$k \overset{def}{=} (D-1)\Omega_{(D-1)}\,\varkappa, \tag{197}$$

$\Omega_{(D-1)}$ is an area of $(D-1)$-dimensional unit sphere, and $\varkappa$ is the Newtonian gravitational constant in $(D+1)$-dimensional space-time.

A restriction of the 2-nd Klein identity (125) and the definition (123) to the case of the Lagrangian $\mathscr{L} = \mathscr{L}^G$ gives, with taking into account the definitions (193) - (195), the identity

$$\begin{aligned} &-\mathscr{E}^{\mu\nu} + \left\{\overset{*}{\nabla}_{\lambda}\mathscr{C}^{\lambda\mu\nu} + \frac{1}{2}\mathscr{C}^{\gamma\beta}{}_{\alpha}(\Delta^{\mu\nu})\,{}^{\alpha}{}_{\beta\gamma}|_{\theta}{}^{\varphi\xi}\,T^{\theta}{}_{\varphi\xi}\right\} \\ &\equiv -\left(\mathscr{E}^{(\mu\nu)} - \overset{*}{\nabla}_{\lambda}\mathscr{C}^{\lambda(\mu\nu)}\right) + \frac{1}{2}\mathscr{C}^{\gamma\beta}{}_{\alpha}(\Delta^{\mu\nu})\,{}^{\alpha}{}_{\beta\gamma}|_{\theta}{}^{\varphi\xi}\,T^{\theta}{}_{\varphi\xi}, \end{aligned}$$

or

$$\boxed{\mathscr{E}^{[\mu\nu]} \equiv \overset{*}{\nabla}_{\lambda}\mathscr{C}^{\lambda[\mu\nu]}.} \tag{198}$$

Thus, the *antisymmetric part of the generalized Einstein tensor is the divergence of the antisymmetric part of the Cartan tensor*. Using the identity (198), one can represent (194) in the form:

$$\begin{aligned} 2\frac{\Delta I^G}{\Delta g_{\mu\nu}} &= \frac{-1}{k}\left(\mathscr{E}^{(\mu\nu)} - \overset{*}{\nabla}_{\lambda}\mathscr{C}^{\lambda(\mu\nu)}\right) \\ &\equiv \frac{-1}{k}\left(\mathscr{E}^{\mu\nu} - \overset{*}{\nabla}_{\lambda}\mathscr{C}^{\lambda\mu\nu}\right). \end{aligned} \tag{199}$$

C. The gravitational field equations in the split form

By the equations (199) and (190), the equations of motion of the metric field, $\Delta(I^G + I^M)/\Delta g_{\mu\nu} = 0$, can be rewritten as

$$\mathscr{E}^{(\mu\nu)} - \overset{*}{\nabla}_{\lambda}\mathscr{C}^{\lambda(\mu\nu)} = k\,\overset{met}{T}{}^{\mu\nu}, \tag{200}$$

or, using the formulae (191) - (193), (199), one can represent them in the equivalent form:

$$\begin{aligned} \mathscr{E}^{\mu\nu} = {}& k\,\overset{mod}{T}{}^{\mu\nu} + \overset{*}{\nabla}_{\lambda}\left(\mathscr{C}^{\lambda\mu\nu} - k\,\overset{mod}{B}{}^{\lambda\mu\nu}\right) \\ &+ k\frac{\Delta I^M}{\Delta\varphi^a}(\Delta^{\mu\nu})\,{}^{a}|_{b}\,\varphi^b. \end{aligned} \tag{201}$$

If here one takes into account the equations of motion for the torsion field

$$\frac{\Delta(I^G + I^M)}{\Delta T^{\nu}{}_{\mu\lambda}} = 0 \qquad \Leftrightarrow \qquad \mathscr{C}^{\lambda\mu\nu} = k\,\overset{mod}{B}{}^{\lambda\mu\nu} \tag{202}$$

and the equations of motion for the $\boldsymbol{\varphi}$-fields: $\Delta I^M/\Delta\varphi^a = 0$, one obtains the equation for the metric field only:

$$\mathscr{E}^{\mu\nu} = k\,\overset{mod}{T}{}^{\mu\nu} \qquad \text{(on the } \boldsymbol{\phi}\text{-equations).} \tag{203}$$

Now, turn to (202). After antisymmetrization in indexes μ and ν, using the definitions (185), (86) and (182), and the identity (A9), the equation (202) acquires an equivalent form:

$$-2\mathscr{C}^{\lambda[\mu\nu]} = k \overset{mod}{S}{}^{\lambda,\,\mu\nu}. \tag{204}$$

Thus, the total system of the field equations acquires the form:

$$\boxed{\begin{cases} \mathscr{E}^{\mu\nu} = k \overset{mod}{T}{}^{\mu\nu} & \text{(the } \mathbf{g}\text{-equations);} \quad (205)\\ -2\mathscr{C}^{\lambda}{}_{[\mu\nu]} = k \overset{mod}{S}{}^{\lambda}{}_{\mu\nu} & \text{(the } \mathbf{T}\text{-equations);} \quad (206)\\ \Delta I^M/\Delta\varphi^a = 0 & \text{(the } \boldsymbol{\varphi}\text{-equations).} \quad (207)\end{cases}}$$

The interpretation of the gravitational equations of the system is as follows. *The source of the metric field* $\mathbf{g}$ *is the modified canonical EMT of matter* $\overset{mod}{\mathbf{T}}$*, whereas the source of the torsion field* $\mathbf{T}$ *is the modified canonical ST of matter* $\overset{mod}{\mathbf{S}}$.

D. Geometrical identities and the equations of balance for the matter sources

In the present subsection, we show why the total system of the equations is more preferable just in the form (205) - (207). At first, let us discuss the matter part. The identity (142) for the Lagrangian $\mathscr{L}^M$ with taking into account the definitions (188) and (182) leads to the identity

$$\boxed{\begin{aligned}\overset{*}{\nabla}_\mu \overset{mod}{T}{}^{\mu}{}_{\nu} \equiv & - \overset{mod}{T}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} \overset{mod}{S}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} \\ & + \frac{\Delta I^M}{\Delta\varphi^a}\nabla_\nu\varphi^a.\end{aligned}} \tag{208}$$

From here the equations of balance for the matter modified canonical EMT $\overset{mod}{\mathbf{T}}$ follows

$$\boxed{\begin{aligned}\overset{*}{\nabla}_\mu \overset{mod}{T}{}^{\mu}{}_{\nu} = - \overset{mod}{T}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} \overset{mod}{S}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} \\ \text{(on the } \boldsymbol{\varphi}\text{-equations).}\end{aligned}} \tag{209}$$

It is important to note: in order the equation (209) to take a place *it is necessary only* that the $\boldsymbol{\varphi}$-equations hold, it is not necessary to take into account the $\mathbf{g}$- and $\mathbf{T}$-equations. Besides, the equation (209) is related only to $\mathscr{L}^M$, it does not relate to $\mathscr{L}^G$. If, analogously to (188) one defines the pure "gravitational EMT" as $\overset{mod}{\mathbf{t}}\Big|_{\mathscr{L}=\mathscr{L}^G}$, then (in the covariant sense) both the matter and gravitational EMTs, each itself will satisfy *its own* equation of balance. By the Teitelboim terminology[73], they are *dynamically independent.* Moreover, the restriction of the identity (142) to the Lagrangian $\mathscr{L}^G$ leads to the identity

$$\begin{aligned}\overset{*}{\nabla}_\mu \overset{mod}{t}{}^{\mu}{}_{\nu}\Big|_{\mathscr{L}=\mathscr{L}^G} \equiv & - \overset{mod}{t}{}^{\mu}{}_{\lambda}\Big|_{\mathscr{L}=\mathscr{L}^G} T^{\lambda}{}_{\mu\nu} \\ & + \frac{1}{2} \overset{mod}{s}{}^{\pi}{}_{\rho\sigma}\Big|_{\mathscr{L}=\mathscr{L}^G} R^{\rho\sigma}{}_{\pi\nu},\end{aligned} \tag{210}$$

that holds *without any equations of motion.* The identity (210) reflects the fact only that the Lagrangian $\mathscr{L}^G$ is a generally covariant scalar.

In order to fill the real sense of the identity (210), one has to find concrete expressions, to which the quantities $\overset{mod}{\mathbf{t}}\Big|_{\mathscr{L}=\mathscr{L}^G}$ and $\overset{mod}{\mathbf{s}}\Big|_{\mathscr{L}=\mathscr{L}^G}$ correspond. After using the definition (138) for the Lagrangian $\mathscr{L}^G$, the definitions (195) and (193), and the formula

$$2\mathscr{C}^{\lambda[\mu\nu]} = k \overset{mod}{s}{}^{\lambda,\,\mu\nu}\Big|_{\mathscr{L}=\mathscr{L}^G},$$

which is carried out from (193), the identity (210) is rewritten in the form:

$$\boxed{\overset{*}{\nabla}_\mu \mathscr{E}^{\mu}{}_{\nu} \equiv \mathscr{E}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} - \mathscr{C}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu}.} \tag{211}$$

As is seen, it is the *pure geometrical differential identity*, which connects the divergence of the Einstein tensor $\mathscr{E}$ with the Cartan tensor $\mathscr{C}$.

So, the fact that the "equation of balance" for the "gravitational EMT" $\overset{mod}{\mathbf{t}}\Big|_{\mathscr{L}=\mathscr{L}^G}$ is satisfied identically is the direct consequent of the diffeomorphism invariance of the pure geometrical action. Therefore, one has to conclude that in background independent metric-torsion theories of gravity, there are no generally covariant expressions for EMTs and STs defined *classically* for the properly gravitational fields. This claim can be stated not only for the gravitational fields, but it has an universal character. The claim takes a place in an arbitrary gauge invariant (in the sense of the definition in Introduction) theory: *There is no a gauge invariant expression for a current namely of the gauge field because the theory is gauge invariant.*

At the end, let us discuss the role of the identity (211). Namely its existence defines the fact that the form of the gravitational equations (205) - (206) is more preferable. Indeed, substituting the Einstein $\mathscr{E}$ and Cartan $\mathscr{C}$ tensors with the use of the $\mathbf{g}$- and $\mathbf{T}$-equations (205) and (206), respectively, into the identity (211), one obtains the equation of balance for the matter modified EMT $\overset{mod}{\mathbf{T}}$:

$$\begin{aligned}\overset{*}{\nabla}_\mu \overset{mod}{T}{}^{\mu}{}_{\nu} = - \overset{mod}{T}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} \overset{mod}{S}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} \\ \text{(on the } \mathbf{g}\text{- and } \mathbf{T}\text{-equations).}\end{aligned} \tag{212}$$

Recall that the copy of this equation, namely (209), conversely, has been carried out without using the gravitational equations, but only with the use of the $\boldsymbol{\varphi}$-equations. Therefore, one concludes that the role of the identity (211) is to *state the self-consistence of the total system of the field equations* (205) - (207). Just in this sense we generalize the interpretation of the gravitational equations in ECT[51–57,74–77]:

$$\begin{cases} E^{\mu\nu} = k\,T^{\mu\nu}; & (213)\\ \overset{*}{T}{}^{\lambda}{}_{\mu\nu} = k\,S^{\lambda}{}_{\mu\nu};, & (214)\end{cases}$$

where

$$E^{\mu\nu} \overset{def}{=} R^{\mu\nu} - \frac{1}{2}g_{\mu\nu}R; \quad \overset{*}{T}{}^{\lambda}{}_{\mu\nu} \overset{def}{=} T^{\lambda}{}_{\mu\nu} + \delta^{\lambda}_{\mu}T_{\nu} - \delta^{\lambda}_{\nu}T_{\mu}.$$

Here, twice contracted the Bianchi identity

$$\overset{*}{\nabla}_{\mu}E^{\mu}{}_{\nu} \equiv -E^{\mu}{}_{\lambda}T^{\lambda}{}_{\mu\nu} + \frac{1}{2}\overset{*}{T}{}^{\pi}{}_{\rho\sigma}R^{\rho\sigma}{}_{\pi\nu} \tag{215}$$

is treated as a *dynamic conservation of the source*. Recall the Wheeler words[78] related to GR: the "gravitational field watch for the conservation of its sources". We see that the same can be repeated also for the *general metric-torsion theories of gravity*. Moreover, this statement is not related to gravitational theories only, but has an universal character and is related to an arbitrary gauge invariant theory. Namely, the *gauge field watch for the conservation of its matter sources*. Returning to the identity (211), one sees that it is, thus, the generalization of twice contracted the Bianchi identity (215).

At last, notice that in the case of the Lagrangians considered in the works[51–57], the general system of the gravitational equations (205) and (206) exactly is simplified to the equations (213) and (214) obtained in these works.

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Appendix A: The tensors $\{\Delta^{\alpha\beta\gamma}_{\underline{\lambda\mu\nu}}\}$, $\{\Delta^{\overline{\alpha\beta\gamma}}_{\lambda\mu\nu}\}$ and their properties

In the main text, for a significant simplification of expressions we use the tensor:

$$\boxed{\Delta^{\alpha\beta\gamma}_{\underline{\lambda\mu\nu}} \overset{def}{=} \frac{1}{2}\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\lambda}\delta^{\gamma}_{\nu} + \delta^{\alpha}_{\nu}\delta^{\beta}_{\lambda}\delta^{\gamma}_{\mu} - \delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu}\delta^{\gamma}_{\nu}\right).} \tag{A1}$$

It is obtained by differentiating the connection $\Gamma_{\lambda,\,\mu\nu} = g_{\lambda\varepsilon}\Gamma^{\varepsilon}{}_{\mu\nu}$ with respect to derivatives of metric $\partial_{\alpha}g_{\beta\gamma}$. Thus, the use of (A1) leads to the compact presentation:

$$\Gamma_{\lambda,\,\mu\nu} = \Delta^{\alpha\beta\gamma}_{\underline{\lambda\mu\nu}}(\partial_{\alpha}g_{\beta\gamma} + T_{\alpha,\,\beta\gamma}). \tag{A2}$$

The tensor

$$\boxed{\Delta^{\overline{\alpha\beta\gamma}}_{\lambda\mu\nu} \overset{def}{=} \frac{1}{2}\left(\delta^{\beta}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\gamma}_{\nu} + \delta^{\gamma}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu}\delta^{\gamma}_{\nu}\right)} \tag{A3}$$

with the converse symmetry of the co- and contravariant indexes is also useful. It is easily to obtain that

$$\Delta^{\alpha(\beta\gamma)}_{\underline{\nu\mu\lambda}} = \frac{1}{2}\left\{\delta^{\alpha}_{\lambda}\delta^{(\beta}_{\mu}\delta^{\gamma)}_{\nu} - \delta^{(\gamma}_{\lambda}\delta^{\beta)\alpha}_{\mu\nu}\right\}, \tag{A4}$$

where

$$\delta^{\alpha\beta}_{\mu\nu} \overset{def}{=} \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu} = \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu} \tag{A5}$$

is the *generalized Kronecker symbol*. As a consequence of (A4) one has

$$\Delta^{\alpha(\beta\gamma)}_{\underline{[\nu\mu]\lambda}} = -\frac{1}{2}\delta^{(\gamma}_{\lambda}\delta^{\beta)\alpha}_{\mu\nu}. \tag{A6}$$

The next formulae are also valid:

$$\Delta^{\alpha[\beta\gamma]}_{\underline{\nu\mu\lambda}} = \Delta^{\overline{\gamma\beta\alpha}}_{\lambda[\mu\nu]}; \tag{A7}$$

$$\Delta^{\overline{[\gamma\beta]\alpha}}_{\lambda\mu\nu} = -\frac{1}{2}\delta^{\gamma\beta}_{\lambda\mu}\delta^{\alpha}_{\nu} - \frac{1}{4}\delta^{\alpha}_{\lambda}\delta^{\beta\gamma}_{\mu\nu}; \tag{A8}$$

$$\Delta^{\overline{\gamma[\beta\alpha]}}_{\lambda\mu\nu} = -\frac{1}{4}\delta^{\gamma}_{\lambda}\delta^{\beta\alpha}_{\mu\nu}; \tag{A9}$$

$$\Delta^{(\alpha|\beta|\gamma)}_{\underline{\sigma\rho\pi}} + \Delta^{(\alpha\gamma)\beta}_{\underline{\sigma\rho\pi}} = \delta^{\beta}_{\sigma}\delta^{\gamma}_{(\rho}\delta^{\alpha}_{\pi)}; \tag{A10}$$

$$\Delta^{\alpha(\beta\gamma)}_{\underline{\sigma\rho\pi}} - \frac{1}{2}\delta^{\alpha}_{\pi}\delta^{\beta}_{\rho}\delta^{\gamma}_{\sigma} = \Delta^{\overline{\alpha\beta\gamma}}_{\pi[\rho\sigma]}; \tag{A11}$$

$$\Delta^{(\alpha\beta)\gamma}_{\underline{\sigma\rho\pi}} = \frac{1}{2}\delta^{\alpha}_{(\sigma}\delta^{\beta}_{\pi)}\delta^{\gamma}_{\rho}; \tag{A12}$$

$$\Delta^{[\alpha|\beta|\gamma]}_{\underline{\sigma\rho\pi}} = -\frac{1}{2}\delta^{\beta}_{\rho}\delta^{\alpha}_{[\sigma}\delta^{\gamma}_{\pi]}. \tag{A13}$$

Appendix B: The general variations of fields functions

1. The functional and total variations

Let a set of variables, tensor fields, $\mathbf{\Phi}(x) = \{\Phi^{A}(x);\ A = \overline{1,N}\}$, be given in a spacetime. Let a result of an infinitesimal transformations be as

$$\begin{cases} x \;\to\; x'; \\ \mathbf{\Phi}(x) \;\to\; \mathbf{\Phi}'(x'). \end{cases} \tag{B1}$$

The transformation (B1) we will call as the *active* transformation. Then under its action a spacetime point with coordinates x transforms into a *new* point with coordinates x', and a function (physical field) $\mathbf{\Phi}(x)$ transforms to a *new* function $\mathbf{\Phi}'(x')$. At the same time. the coordinate system is *fixed/the same*.

The *total variation* $\bar{\delta}\mathbf{\Phi}(x)$ of field functions $\mathbf{\Phi}(x)$ appears as a result of a comparison of a new function $\mathbf{\Phi}'(x')$, calculated in a new point x', with the initial function $\mathbf{\Phi}(x)$, calculated in the initial point x:

$$\bar{\delta}\mathbf{\Phi}(x) \overset{def}{=} \mathbf{\Phi}'(x') - \mathbf{\Phi}(x). \tag{B2}$$

Unlike this, comparing the new and the old functions calculated in the same (initial) point x, one obtains the perturbation defined as

$$\delta\mathbf{\Phi}(x) \overset{def}{=} \mathbf{\Phi}'(x) - \mathbf{\Phi}(x). \tag{B3}$$

This perturbation, in fact, is the *functional variation* $\delta\mathbf{\Phi}(x)$ of a field function $\mathbf{\Phi}(x)$. Unlike the total variation, it commutes with partial derivatives, and up to a sign coincides with the Lie derivative, which appears under an infinitesimal mapping a spacetime onto itself.

2. The variation of the connection

To obtain an explicitly covariant variation of the connection $\{\delta\Gamma^\lambda{}_{\mu\nu}\}$ by a more economical way one has to use the metric compatible condition (8). One obtains

$$\nabla_\lambda(\delta g_{\mu\nu}) = g_{\mu\alpha}\delta\Gamma^\alpha{}_{\nu\lambda} + g_{\nu\alpha}\delta\Gamma^\alpha{}_{\mu\lambda} = 2g_{(\mu|\alpha}\delta\Gamma^\alpha{}_{|\nu)\lambda}.$$

Now, using (A1) and the formula (6) after variating

$$\delta\Gamma^\lambda{}_{[\mu\nu]} = -\frac{1}{2}\delta T^\lambda{}_{\mu\nu}, \tag{B4}$$

one has

$$\Delta^{\alpha\beta\gamma}_{\underline{\lambda\mu\nu}}\nabla_\alpha(\delta g_{\beta\gamma}) = g_{\lambda\alpha}\delta\Gamma^\alpha{}_{(\mu\nu)} - g_{(\mu|\alpha}\delta T^\alpha{}_{\lambda|\nu)}.$$

This gives

$$\delta\Gamma^\lambda{}_{(\mu\nu)} = g^{\lambda\pi}\Delta^{\alpha\beta\gamma}_{\underline{\pi\mu\nu}}\nabla_\alpha(\delta g_{\beta\gamma}) + g^{\lambda\pi}g_{(\mu|\alpha}\delta T^\alpha{}_{\pi|\nu)}. \tag{B5}$$

Substituting the last formula and (B4) into the evident equality $\delta\Gamma^\lambda{}_{\mu\nu} = \delta\Gamma^\lambda{}_{(\mu\nu)} + \delta\Gamma^\lambda{}_{[\mu\nu]}$, one obtains finally

$$\boxed{\delta\Gamma^\lambda{}_{\mu\nu} = g^{\lambda\pi}\Delta^{\alpha\beta\gamma}_{\underline{\pi\mu\nu}}\left(\nabla_\alpha\delta g_{\beta\gamma} + g_{\alpha\sigma}\delta T^\sigma{}_{\beta\gamma}\right).} \tag{B6}$$

3. The variation of the curvature tensor

Varying the relation (7)

$$R^\kappa{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\kappa{}_{\lambda\nu} - \partial_\nu\Gamma^\kappa{}_{\lambda\mu} + \Gamma^\kappa{}_{\alpha\mu}\Gamma^\alpha{}_{\lambda\nu} - \Gamma^\kappa{}_{\alpha\nu}\Gamma^\alpha{}_{\lambda\mu},$$

keeping in mind that $\{\delta\Gamma^\lambda{}_{\mu\nu}\}$ is a tensor and taking into account (6), one finds

$$\delta R^\kappa{}_{\lambda\mu\nu} = 2\nabla_{[\mu|}\delta\Gamma^\kappa{}_{\lambda|\nu]} + T^\tau{}_{\mu\nu}\delta\Gamma^\kappa{}_{\lambda\tau}$$

or

$$\boxed{\delta R^\kappa{}_{\lambda\mu\nu} = \left(T^\tau{}_{\mu\nu} + \delta^{\pi\tau}_{\mu\nu}\nabla_\pi\right)\delta\Gamma^\kappa{}_{\lambda\tau}.} \tag{B7}$$

4. The variation of the 1-st covariant derivative

Varying the definition of the covariant derivative

$$\nabla_\mu\phi^a = \partial_\mu\phi^a + \Gamma^\kappa{}_{\lambda\mu}(\Delta^\lambda{}_\kappa)\,{}^a|_b\,\phi^b, \tag{B8}$$

where $\{(\Delta^\lambda{}_\kappa)\,{}^a|_b\}$ are the Belinfante-Rosenfeld symbols (see Appendix C 1), one gets

$$\delta(\nabla_\mu\phi^a) = \partial_\mu(\delta\phi^a) + \Gamma^\kappa{}_{\lambda\mu}(\Delta^\lambda{}_\kappa)\,{}^a|_b\,(\delta\phi^b) + (\Delta^\lambda{}_\kappa)\,{}^a|_b\,\phi^b\delta\Gamma^\kappa{}_{\lambda\mu}. \tag{B9}$$

From here one has

$$\boxed{\delta(\nabla_\mu\phi^a) = \nabla_\mu(\delta\phi^a) + (\Delta^\lambda{}_\kappa)\,{}^a|_b\,\phi^b\,\delta\Gamma^\kappa{}_{\lambda\mu}.} \tag{B10}$$

5. The variation of the 2-nd covariant derivative

Let us calculate $\delta(\nabla_\mu\nabla_\nu\phi^a)$. For the sake of simplicity, temporarily denote $\phi^a{}_\nu \overset{def}{=} \nabla_\nu\phi^a$. Then, taking into account the fact that the tensor $\{\phi^a{}_\nu\}$ has for a one index more than the tensor $\{\phi^a\}$ and using (B10), one obtains

$$\begin{aligned}\delta(\nabla_\mu\phi^a{}_\nu) &= \nabla_\mu\delta\phi^a{}_\nu + (\Delta^\lambda{}_\kappa)\,{}^a|_b\,\phi^b{}_\nu\delta\Gamma^\kappa{}_{\lambda\mu} - \phi^a{}_\lambda\delta\Gamma^\lambda{}_{\nu\mu}\\ &= \nabla_\mu\left[\nabla_\nu\delta\phi^a + (\Delta^\lambda{}_\kappa)\,{}^a|_b\,\phi^b\,\delta\Gamma^\kappa{}_{\lambda\nu}\right] + \left[(\Delta^\lambda{}_\kappa)\,{}^a|_b\,\nabla_\nu\phi^b - (\nabla_\kappa\phi^a)\delta^\lambda_\nu\right]\delta\Gamma^\kappa{}_{\lambda\mu}.\end{aligned}$$

Consequently,

$$\boxed{\delta(\nabla_\mu\nabla_\nu\phi^a) = \nabla_\mu\nabla_\nu\delta\phi^a + \left[(\Delta^\lambda{}_\kappa)\,{}^a|_b\,\nabla_\nu\phi^b - (\nabla_\kappa\phi^a)\delta^\lambda_\nu\right]\delta\Gamma^\kappa{}_{\lambda\mu} + \nabla_\nu\left[(\Delta^\lambda{}_\kappa)\,{}^a|_b\,\phi^b\,\delta\Gamma^\kappa{}_{\lambda\nu}\right].} \tag{B11}$$

Appendix C: The Belinfante-Rosenfeld symbols

1. The definition and properties

Let the field functions $\mathbf{\Phi}(x) = \{\Phi^A(x)\} = \{P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}(x)\}$ present generally covariant tensor of the rank $\binom{r}{s}$. Then under a diffeomorphism

$$x^\mu \quad \to \quad x'^\mu = x'^\mu(x) \tag{C1}$$

its components are transformed by the known rule:

$$P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}(x) \;\to\; P'^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}(x'); \tag{C2}$$

$$P'^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\frac{\partial x'^{\mu_2}}{\partial x^{\alpha_2}}\cdot\,\ldots\,\cdot\frac{\partial x'^{\mu_r}}{\partial x^{\alpha_r}}\cdot\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}}\frac{\partial x^{\beta_2}}{\partial x'^{\nu_2}}\cdot\,\ldots\,\cdot\frac{\partial x^{\beta_s}}{\partial x'^{\nu_s}}\,P^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}(x). \tag{C3}$$

For the infinitesimal diffeomorphism

$$x^\mu \quad \to \quad x'^\mu = x^\mu + \delta\xi^\mu(x), \tag{C4}$$

induced by the vector field $\delta\boldsymbol{\xi}(x) = \{\delta\xi^\mu(x)\}$, up to the first order in $\delta\xi^\mu$, the converse diffeomorphism has the form:

$$x'^\beta \quad \to \quad x^\beta \approx x'^\beta - \delta\xi^\beta(x'). \tag{C5}$$

Thus, one has

$$\frac{\partial x'^\mu}{\partial x^\alpha} = \delta^\mu_\alpha + \partial_\alpha\delta\xi^\mu(x); \tag{C6}$$

$$\frac{\partial x^\beta}{\partial x'^\nu} \approx \delta^\beta_\nu - \partial'_\nu\delta\xi^\beta(x') \approx \delta^\beta_\nu - \partial_\nu\delta\xi^\beta(x). \tag{C7}$$

Then

$$\begin{aligned}
&P'^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}(x')\\
&= \left(\delta^{\mu_1}_{\alpha_1} + \partial_{\alpha_1}\delta\xi^{\mu_1}\right)\left(\delta^{\mu_2}_{\alpha_2} + \partial_{\alpha_2}\delta\xi^{\mu_2}\right)\cdot\,\ldots\,\cdot\left(\delta^{\mu_r}_{\alpha_r} + \partial_{\alpha_r}\delta\xi^{\mu_r}\right)\\
&\quad\times\left(\delta^{\beta_1}_{\nu_1} - \partial_{\nu_1}\delta\xi^{\beta_1}\right)\left(\delta^{\beta_2}_{\nu_2} - \partial_{\nu_2}\delta\xi^{\beta_2}\right)\cdot\,\ldots\,\cdot\left(\delta^{\beta_s}_{\nu_s} - \partial_{\nu_s}\delta\xi^{\beta_s}\right)\,P^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}(x)\\[1ex]
&\approx P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}(x)\\
&\quad+\left(\partial_{\alpha_1}\delta\xi^{\mu_1}\right)\left(\delta^{\mu_2}_{\alpha_2}\delta^{\mu_3}_{\alpha_3}\cdot\,\ldots\,\cdot\delta^{\mu_r}_{\alpha_r}\cdot\delta^{\beta_1}_{\nu_1}\delta^{\beta_2}_{\nu_2}\cdot\,\ldots\,\cdot\delta^{\beta_s}_{\nu_s}\right)P^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}(x)\\
&\quad+\left(\partial_{\alpha_2}\delta\xi^{\mu_2}\right)\left(\delta^{\mu_1}_{\alpha_1}\delta^{\mu_3}_{\alpha_3}\cdot\,\ldots\,\cdot\delta^{\mu_r}_{\alpha_r}\cdot\delta^{\beta_1}_{\nu_1}\delta^{\beta_2}_{\nu_2}\cdot\,\ldots\,\cdot\delta^{\beta_s}_{\nu_s}\right)P^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}(x)\\
&\quad+\ldots\\
&\quad+\left(\partial_{\alpha_r}\delta\xi^{\mu_r}\right)\left(\delta^{\mu_1}_{\alpha_1}\delta^{\mu_2}_{\alpha_2}\cdot\,\ldots\,\cdot\delta^{\mu_{r-1}}_{\alpha_{r-1}}\cdot\delta^{\beta_1}_{\nu_1}\delta^{\beta_2}_{\nu_2}\cdot\,\ldots\,\cdot\delta^{\beta_s}_{\nu_s}\right)P^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}(x)\\[1ex]
&\quad-\left(\partial_{\nu_1}\delta\xi^{\beta_1}\right)\left(\delta^{\mu_1}_{\alpha_1}\delta^{\mu_2}_{\alpha_2}\cdot\,\ldots\,\cdot\delta^{\mu_r}_{\alpha_r}\cdot\delta^{\beta_2}_{\nu_2}\delta^{\beta_3}_{\nu_3}\cdot\,\ldots\,\cdot\delta^{\beta_s}_{\nu_s}\right)P^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}(x)\\
&\quad-\left(\partial_{\nu_2}\delta\xi^{\beta_2}\right)\left(\delta^{\mu_1}_{\alpha_1}\delta^{\mu_2}_{\alpha_2}\cdot\,\ldots\,\cdot\delta^{\mu_r}_{\alpha_r}\cdot\delta^{\beta_1}_{\nu_1}\delta^{\beta_3}_{\nu_3}\cdot\,\ldots\,\cdot\delta^{\beta_s}_{\nu_s}\right)P^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}(x)\\
&\quad-\ldots\\
&\quad-\left(\partial_{\nu_s}\delta\xi^{\beta_s}\right)\left(\delta^{\mu_1}_{\alpha_1}\delta^{\mu_2}_{\alpha_2}\cdot\,\ldots\,\cdot\delta^{\mu_r}_{\alpha_r}\cdot\delta^{\beta_1}_{\nu_1}\delta^{\beta_2}_{\nu_2}\cdot\,\ldots\,\cdot\delta^{\beta_{s-1}}_{\nu_{s-1}}\right)P^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}(x)
\end{aligned}$$

$$\begin{aligned}
&= P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}(x)\\
&\quad+\partial_\beta\delta\xi^\alpha\Big[(\delta^{\mu_1}_{\alpha}\delta^{\beta}_{\alpha_1})\delta^{\mu_2}_{\alpha_2}\delta^{\mu_3}_{\alpha_3}\cdot\,\ldots\,\cdot\delta^{\mu_r}_{\alpha_r}\cdot\delta^{\beta_1}_{\nu_1}\delta^{\beta_2}_{\nu_2}\cdot\,\ldots\,\cdot\delta^{\beta_s}_{\nu_s}\\
&\qquad+\delta^{\mu_1}_{\alpha_1}(\delta^{\mu_2}_{\alpha}\delta^{\beta}_{\alpha_2})\delta^{\mu_3}_{\alpha_3}\cdot\,\ldots\,\cdot\delta^{\mu_r}_{\alpha_r}\cdot\delta^{\beta_1}_{\nu_1}\delta^{\beta_2}_{\nu_2}\cdot\,\ldots\,\cdot\delta^{\beta_s}_{\nu_s}\\
&\qquad+\ldots\\
&\qquad+\delta^{\mu_1}_{\alpha_1}\delta^{\mu_2}_{\alpha_2}\cdot\,\ldots\,\cdot\delta^{\mu_{r-1}}_{\alpha_{r-1}}(\delta^{\mu_r}_{\alpha}\delta^{\beta}_{\alpha_r})\cdot\delta^{\beta_1}_{\nu_1}\delta^{\beta_2}_{\nu_2}\cdot\,\ldots\,\cdot\delta^{\beta_s}_{\nu_s}\\[1ex]
&\qquad-\delta^{\mu_1}_{\alpha_1}\delta^{\mu_2}_{\alpha_2}\cdot\,\ldots\,\cdot\delta^{\mu_r}_{\alpha_r}\cdot(\delta^{\beta}_{\nu_1}\delta^{\beta_1}_{\alpha})\delta^{\beta_2}_{\nu_2}\delta^{\beta_3}_{\nu_3}\cdot\,\ldots\,\cdot\delta^{\beta_s}_{\nu_s}\\
&\qquad-\delta^{\mu_1}_{\alpha_1}\delta^{\mu_2}_{\alpha_2}\cdot\,\ldots\,\cdot\delta^{\mu_r}_{\alpha_r}\cdot\delta^{\beta_1}_{\nu_1}(\delta^{\beta}_{\nu_2}\delta^{\beta_2}_{\alpha})\delta^{\beta_3}_{\nu_3}\cdot\,\ldots\,\cdot\delta^{\beta_s}_{\nu_s}\\
&\qquad-\ldots\\
&\qquad-\delta^{\mu_1}_{\alpha_1}\delta^{\mu_2}_{\alpha_2}\cdot\,\ldots\,\cdot\delta^{\mu_r}_{\alpha_r}\cdot\delta^{\beta_1}_{\nu_1}\delta^{\beta_2}_{\nu_2}\cdot\,\ldots\,\cdot\delta^{\beta_{s-1}}_{\nu_{s-1}}(\delta^{\beta}_{\nu_s}\delta^{\beta_s}_{\alpha})\Big]\,P^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}(x).
\end{aligned} \tag{C8}$$

Consequently, the total variation (B2) for the tensor of rank $\binom{r}{s}$ has the form

$$\bar{\delta}P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}(x) = \left(\Delta^\beta{}_\alpha\right)^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}\big|_{\alpha_1\alpha_2\ldots\alpha_r}{}^{\beta_1\beta_2\ldots\beta_s}\;P^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}(x)\;\partial_\beta\delta\xi^\alpha(x), \tag{C9}$$

where

$$\boxed{\begin{aligned}&\left(\Delta^{\beta}{}_{\alpha}\right)\,{}^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}\big|_{\alpha_1\alpha_2\ldots\alpha_r}{}^{\beta_1\beta_2\ldots\beta_s}\\ &\stackrel{def}{=}\left[(\delta^{\mu_1}_{\alpha}\delta^{\beta}_{\alpha_1})\delta^{\mu_2}_{\alpha_2}\delta^{\mu_3}_{\alpha_3}\cdot\ldots\cdot\delta^{\mu_r}_{\alpha_r}+\delta^{\mu_1}_{\alpha_1}(\delta^{\mu_2}_{\alpha}\delta^{\beta}_{\alpha_2})\delta^{\mu_3}_{\alpha_3}\cdot\ldots\cdot\delta^{\mu_r}_{\alpha_r}+\cdots+\delta^{\mu_1}_{\alpha_1}\delta^{\mu_2}_{\alpha_2}\cdot\ldots\cdot\delta^{\mu_{r-1}}_{\alpha_{r-1}}(\delta^{\mu_r}_{\alpha}\delta^{\beta}_{\alpha_r})\right]\delta^{\beta_1}_{\nu_1}\delta^{\beta_2}_{\nu_2}\cdot\ldots\cdot\delta^{\beta_s}_{\nu_s}\\ &\quad-\delta^{\mu_1}_{\alpha_1}\delta^{\mu_2}_{\alpha_2}\cdot\ldots\cdot\delta^{\mu_r}_{\alpha_r}\left[(\delta^{\beta}_{\nu_1}\delta^{\beta_1}_{\alpha})\delta^{\beta_2}_{\nu_2}\delta^{\beta_3}_{\nu_3}\cdot\ldots\cdot\delta^{\beta_s}_{\nu_s}+\delta^{\beta_1}_{\nu_1}(\delta^{\beta}_{\nu_2}\delta^{\beta_2}_{\alpha})\delta^{\beta_3}_{\nu_3}\cdot\ldots\cdot\delta^{\beta_s}_{\nu_s}+\cdots+\delta^{\beta_1}_{\nu_1}\delta^{\beta_2}_{\nu_2}\cdot\ldots\cdot\delta^{\beta_{s-1}}_{\nu_{s-1}}(\delta^{\beta}_{\nu_s}\delta^{\beta_s}_{\alpha})\right].\end{aligned}} \tag{C10}$$

The functional variation $\delta_\xi\mathbf{\Phi}$ of the field function $\mathbf{\Phi}$ induced by the diffeomorphism (C4) is connected with the total variation $\bar\delta\mathbf{\Phi}$ by the evident relation:

$$\bar\delta\Phi^A(x)=\left(\partial_\alpha\Phi^A(x)\right)\delta\xi^\alpha+\delta_\xi\Phi^A(x). \tag{C11}$$

Then, using (C9), one obtains for the tensor of rank $\binom{r}{s}$:

$$\begin{aligned}&\delta_\xi P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}(x)\\ &\quad=-\partial_\alpha P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}(x)\,\delta\xi^\alpha+\left(\Delta^{\beta}{}_{\alpha}\right)\,{}^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}\big|_{\alpha_1\alpha_2\ldots\alpha_r}{}^{\beta_1\beta_2\ldots\beta_s}\;P^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}(x)\,\partial_\beta\delta\xi^\alpha(x),\end{aligned} \tag{C12}$$

Returning to the collective indexes $A={}^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}$ and $B={}^{\alpha_1\alpha_2\ldots\alpha_r}{}_{\beta_1\beta_2\ldots\beta_s}$, the last formula is rewritten in the compact form:

$$\boxed{\delta_\xi\Phi^A=-\partial_\alpha\Phi^A\delta\xi^\alpha+(\Delta^\alpha{}_\beta)\;{}^A\big|_B\,\Phi^B\partial_\beta\delta\xi^\alpha.} \tag{C13}$$

The formulae of the type (C13) can be used also both for the tensor densities of an arbitrary weight and for the spinors of an arbitrary rank $\left(\begin{smallmatrix}k & | & \bar m\\ l & | & \bar n\end{smallmatrix}\right)$. As we know, firstly the symbols $(\Delta^\beta{}_\alpha)\;{}^A\big|_B$ where introduced in the works by Belinfante[28] and by Rosenfeld[30,31]. Therefore we call them as the *Belinfante-Rosenfeld symbols*.

In our consideration, the tensors of the ranks $\binom{0}{2}$ and $\binom{1}{2}$ are more important, for them one has to use

$$(\Delta^\beta{}_\alpha)\;{}_{\mu\nu}\big|^{\varepsilon\kappa}=-(\delta^\varepsilon_\alpha\delta^\beta_\mu)\delta^\kappa_\nu-\delta^\varepsilon_\mu(\delta^\kappa_\alpha\delta^\beta_\nu); \tag{C14}$$

$$(\Delta^\beta{}_\alpha)\;{}^\lambda{}_{\mu\nu}\big|_\pi{}^{\rho\sigma}=\left(\delta^\lambda_\alpha\delta^\beta_\pi\right)\delta^\rho_\mu\delta^\sigma_\nu-\delta^\lambda_\pi\left[(\delta^\rho_\alpha\delta^\beta_\mu)\delta^\sigma_\nu+\delta^\rho_\mu(\delta^\sigma_\alpha\delta^\beta_\nu)\right]. \tag{C15}$$

Thus, for the metric tensor $\mathbf{g}=\{g_{\mu\nu}\}$ and for the torsion tensor $\mathbf{T}=\{T^\lambda{}_{\mu\nu}\}$ one finds, respectively,

$$\boxed{(\Delta^\beta{}_\alpha)\;{}_{\mu\nu}\big|^{\varepsilon\kappa}\,g_{\varepsilon\kappa}=-2g_{\alpha(\mu}\delta^\beta_{\nu)}} \tag{C16}$$

and

$$\boxed{(\Delta^\beta{}_\alpha)\;{}^\lambda{}_{\mu\nu}\big|_\pi{}^{\rho\sigma}\,T^\pi{}_{\rho\sigma}=\delta^\lambda_\alpha T^\beta{}_{\mu\nu}+2T^\lambda{}_{\alpha[\mu}\delta^\beta_{\nu]}.} \tag{C17}$$

2. The covariant derivative

With the use of the Belinfante-Rosenfeld symbols (C10) the covariant derivative of the tensor of the rank $\binom{r}{s}$

$$\begin{aligned}\nabla_\lambda P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}&=\partial_\lambda P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}\\ &+\Gamma^{\mu_1}{}_{\beta\lambda}\,P^{\beta\mu_2\mu_3\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}+\Gamma^{\mu_2}{}_{\beta\lambda}\,P^{\mu_1\beta\mu_3\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}+\cdots+\Gamma^{\mu_r}{}_{\beta\lambda}\,P^{\mu_1\mu_2\ldots\mu_{r-1}\beta}{}_{\nu_1\nu_2\ldots\nu_s}\\ &-\Gamma^\alpha{}_{\nu_1\lambda}\,P^{\mu_1\mu_2\ldots\mu_r}{}_{\alpha\nu_2\nu_3\ldots\nu_s}-\Gamma^\alpha{}_{\nu_2\lambda}\,P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\alpha\nu_3\ldots\nu_s}-\cdots-\Gamma^\alpha{}_{\nu_s\lambda}\,P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_{s-1}\alpha}\end{aligned} \tag{C18}$$

can be presented in the form:

$$\nabla_\lambda P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}=\partial_\lambda P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}+\Gamma^\alpha{}_{\beta\lambda}\left(\Delta^\beta{}_\alpha\right)\,{}^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}\big|_{\gamma_1\gamma_2\ldots\gamma_r}{}^{\delta_1\delta_2\ldots\delta_s}\;P^{\gamma_1\gamma_2\ldots\gamma_r}{}_{\delta_1\delta_2\ldots\delta_s}, \tag{C19}$$

or, returning to the collective indexes A and B, as

$$\boxed{\nabla_\lambda\Phi^A=\partial_\lambda\Phi^A+\Gamma^\alpha{}_{\beta\lambda}(\Delta^\beta{}_\alpha)\;{}^A\big|_B\,\Phi^B.} \tag{C20}$$

3. The covariant form of the variation $\delta_\xi\mathbf{\Phi}$

In the Paper I, Sec. I, for the functional variation $\delta_\xi\mathbf{\Phi}$ the generalized formula (15) has been stated, due to which the formula (C13) has to acquire the form:

$$\delta_\xi \Phi^A = \Phi_\alpha|^A \delta\xi^\alpha + \Phi_\alpha{}^\beta|^A \nabla_\beta \delta\xi^\alpha. \qquad \text{(C21)}$$

Define the explicit expressions for the coefficients in this formula. Using (C20), find the quantity $\{\partial_\lambda \Phi^A\}$, and using

$$\nabla_\beta \delta\xi^\alpha = \partial_\beta \delta\xi^\alpha + \Gamma^\alpha{}_{\gamma\beta}\delta\xi^\gamma \qquad \text{(C22)}$$

find the quantity $\{\partial_\beta \delta\xi^\alpha\}$. Next, substitute the results into (C13) and obtain

$$\begin{aligned}\delta_\xi \Phi^A \ &= \left\{-\nabla_\alpha \Phi^A + 2\Gamma^\gamma{}_{[\beta\alpha]}(\Delta^\beta{}_\gamma)\ {}^A\big|_B\ \Phi^B\right\}\delta\xi^\alpha \\ &+ \left\{(\Delta^\beta{}_\alpha)\ {}^A\big|_B\ \Phi^B\right\}\nabla_\beta \delta\xi^\alpha.\end{aligned}$$

Substituting here $2\Gamma^\gamma{}_{[\beta\alpha]} = -T^\gamma{}_{\beta\alpha}$, obtain finally

$$\boxed{\begin{aligned}\delta_\xi \Phi^A \ &= -\left\{\nabla_\alpha \Phi^A + T^\gamma{}_{\beta\alpha}(\Delta^\beta{}_\gamma)\ {}^A\big|_B\ \Phi^B\right\}\delta\xi^\alpha \\ &+ \left\{(\Delta^\beta{}_\alpha)\ {}^A\big|_B\ \Phi^B\right\}\nabla_\beta \delta\xi^\alpha.\end{aligned}} \qquad \text{(C23)}$$

Comparing this formula with (C21) one finds

$$\boxed{\Phi_\alpha|^A = -\left\{\nabla_\alpha \Phi^A + T^\gamma{}_{\beta\alpha}(\Delta^\beta{}_\gamma)\ {}^A\big|_B\ \Phi^B\right\};} \qquad \text{(C24)}$$

$$\boxed{\Phi_\alpha{}^\beta|^A = (\Delta^\beta{}_\alpha)\ {}^A\big|_B\ \Phi^B.} \qquad \text{(C25)}$$

In particular, for the metric tensor $\mathbf{g}$, using the metric compatible condition $\nabla_\alpha g_{\beta\gamma} = 0$ and the formula (C16), one gets

$$\boxed{g_\alpha|_{\beta\gamma} = 2T_{(\beta,\,\gamma)\alpha};} \qquad \text{(C26)}$$

$$\boxed{g_\alpha{}^\beta|_{\kappa\lambda} = -2g_{\alpha(\kappa}\delta^\beta_{\lambda)}.} \qquad \text{(C27)}$$

Analogously, using the formula (C17), one has for the torsion tensor $\mathbf{T}$:

$$\boxed{\begin{aligned}&T_\alpha|^\varepsilon{}_{\beta\gamma} = -\nabla_\alpha T^\varepsilon{}_{\beta\gamma} \\ &\quad -(T^\varepsilon{}_{\kappa\alpha}T^\kappa{}_{\beta\gamma} + T^\varepsilon{}_{\kappa\beta}T^\kappa{}_{\gamma\alpha} + T^\varepsilon{}_{\kappa\gamma}T^\kappa{}_{\alpha\beta});\end{aligned}} \qquad \text{(C28)}$$

$$\boxed{T_\alpha{}^\beta|^\varepsilon{}_{\kappa\lambda} = \delta^\varepsilon_\alpha T^\beta{}_{\kappa\lambda} + 2T^\varepsilon{}_{\alpha[\kappa}\delta^\beta_{\lambda]}.} \qquad \text{(C29)}$$

4. Commutator of the covariant derivatives

With the use of the Belinfante-Rosenfeld symbols (C10) the commutator of the covariant derivatives of the tensor of the rank $\binom{r}{s}$

$$\begin{aligned}&(\nabla_\rho\nabla_\sigma - \nabla_\sigma\nabla_\rho)P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s} = -T^\lambda{}_{\rho\sigma}\nabla_\lambda P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s} \\ &\quad + R^{\mu_1}{}_{\lambda\rho\sigma}P^{\lambda\mu_2\mu_3\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s} + R^{\mu_2}{}_{\lambda\rho\sigma}P^{\mu_1\lambda\mu_3\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s} + \cdots + R^{\mu_r}{}_{\lambda\rho\sigma}P^{\mu_1\mu_2\ldots\mu_{r-1}\lambda}{}_{\nu_1\nu_2\ldots\nu_s} \\ &\quad - R^\kappa{}_{\nu_1\rho\sigma}P^{\mu_1\mu_2\ldots\mu_r}{}_{\kappa\nu_2\nu_3\ldots\nu_s} - R^\kappa{}_{\nu_2\rho\sigma}P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\kappa\nu_3\ldots\nu_s} - \cdots - R^\kappa{}_{\nu_s\rho\sigma}P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_{s-1}\kappa}\end{aligned} \qquad \text{(C30)}$$

can be presented in the form:

$$\begin{aligned}(\nabla_\rho\nabla_\sigma - \nabla_\sigma\nabla_\rho)P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s} \ &= -T^\lambda{}_{\rho\sigma}\nabla_\lambda P^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s} \\ &+ R^\kappa{}_{\lambda\rho\sigma}\left(\Delta^\lambda{}_\kappa\right)\ {}^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}\big|_{\gamma_1\gamma_2\ldots\gamma_r}{}^{\delta_1\delta_2\ldots\delta_s}\,P^{\gamma_1\gamma_2\ldots\gamma_r}{}_{\delta_1\delta_2\ldots\delta_s},\end{aligned} \qquad \text{(C31)}$$

Returning to the collective indexes A, B, …, it is presented as

$$\boxed{(\nabla_\rho\nabla_\sigma - \nabla_\sigma\nabla_\rho)\Phi^A = -T^\lambda{}_{\rho\sigma}\nabla_\lambda \Phi^A + R^\kappa{}_{\lambda\rho\sigma}(\Delta^\lambda{}_\kappa)\ {}^A\big|_B\ \Phi^B.} \qquad \text{(C32)}$$

Appendix D: The transformation of the expression $\left(\frac{-1}{2}b^{\gamma\beta}{}_\alpha\nabla_\nu T^\alpha{}_{\beta\gamma}\right)$

Let $\{b^{\gamma\beta\alpha}\} \overset{def}{=} \{\Delta^{\overline{\gamma\beta\alpha}}_{\pi\rho\sigma}s^{\pi,\,\rho\sigma}\}$, where $\{s^{\pi,\,[\rho\sigma]} = s^{\pi,\,\rho\sigma}\}$ be an *arbitrary* tensor with such a symmetry. Then $b^{[\gamma\beta]\alpha} = b^{\gamma\beta\alpha}$. Basing on this, transform the expression $\left(\frac{-1}{2}b^{\gamma\beta}{}_\alpha\nabla_\nu T^\alpha{}_{\beta\gamma}\right)$ as follows.

1. Substituting the Ricci identity in the form

$$\begin{aligned}\nabla_\nu T^\alpha{}_{\beta\gamma} \ &\equiv R^\alpha{}_{\nu\beta\gamma} + R^\alpha{}_{\beta\gamma\nu} + R^\alpha{}_{\gamma\nu\beta} - \left(\nabla_\beta T^\alpha{}_{\gamma\nu} + \nabla_\gamma T^\alpha{}_{\nu\beta} + T^\alpha{}_{\lambda\nu}T^\lambda{}_{\beta\gamma} + T^\alpha{}_{\lambda\beta}T^\lambda{}_{\gamma\nu} + T^\alpha{}_{\lambda\gamma}T^\lambda{}_{\nu\beta}\right) \\ &= R^\alpha{}_{\nu\beta\gamma} + 2R^\alpha{}_{[\beta\gamma]\nu} - 2\nabla_{[\beta}T^\alpha{}_{\gamma]\nu} - T^\alpha{}_{\lambda\nu}T^\lambda{}_{\beta\gamma} - 2T^\alpha{}_{\lambda[\beta}T^\lambda{}_{\gamma]\nu},\end{aligned}$$

one obtains

$$\begin{aligned}&-\frac{1}{2}b^{\gamma\beta}{}_\alpha\nabla_\nu T^\alpha{}_{\beta\gamma} \equiv -\frac{1}{2}b^{\gamma\beta}{}_\alpha R^\alpha{}_{\nu\beta\gamma} - b^{\gamma\beta}{}_\alpha R^\alpha{}_{\beta\gamma\nu} \\ &\quad + b^{\gamma\beta}{}_\alpha\nabla_\beta T^\alpha{}_{\gamma\nu} + \frac{1}{2}b^{\gamma\beta}{}_\alpha T^\alpha{}_{\lambda\nu}T^\lambda{}_{\beta\gamma} + b^{\gamma\beta}{}_\alpha T^\alpha{}_{\lambda\beta}T^\lambda{}_{\gamma\nu}.\end{aligned} \qquad \text{(D1)}$$

2. Turn to the first term on the right hand side of

(D1). Then, recall the identity (C2) in the Paper I, Appendix C.1:

$$\overset{*}{\nabla}_\mu \left[\overset{*}{\nabla}_\eta \theta_\nu{}^{\mu\eta} + \frac{1}{2}\theta_\nu{}^{\rho\sigma} T^\mu{}_{\rho\sigma}\right] \equiv -\frac{1}{2} R^\lambda{}_{\nu\rho\sigma}\theta_\lambda{}^{\rho\sigma},$$

change here $\theta_\nu{}^{\mu\eta} = b^{\mu\eta}{}_\nu$ and obtain for this term:

$$-\frac{1}{2} b^{\gamma\beta}{}_\alpha R^\alpha{}_{\nu\beta\gamma} = -\overset{*}{\nabla}_\mu \left[\overset{*}{\nabla}_\eta b^{\mu\eta}{}_\nu + \frac{1}{2} b^{\varepsilon\eta}{}_\nu T^\mu{}_{\varepsilon\eta}\right].$$

3. The second term on the right hand side of (D1) is equal to

$$\begin{aligned} -b^{\gamma\beta}{}_\alpha R^\alpha{}_{\beta\gamma\nu} &= -b^{\gamma\beta\alpha} R_{\alpha\beta\gamma\nu} = -\Delta^{\overline{\gamma\beta\alpha}}_{\pi\rho\sigma} s^{\pi,\,\rho\sigma} R_{\alpha\beta\gamma\nu} \\ &= -\frac{1}{2}\left(s^{\beta,\,\gamma\alpha} + s^{\alpha,\,\gamma\beta} - s^{\gamma,\,\beta\alpha}\right) R_{\alpha\beta\gamma\nu} \\ &= \left(s^{(\alpha,\,\beta)\gamma} - \tfrac{1}{2} s^{\gamma,\,\alpha\beta}\right) R_{\alpha\beta\gamma\nu} = -\frac{1}{2} s^{\pi,\,\rho\sigma} R_{\rho\sigma\pi\nu}; \end{aligned}$$

4. Using the differentiation by part in the third term on the right hand side of (D1), one finds

$$b^{\gamma\beta}{}_\alpha \nabla_\beta T^\alpha{}_{\gamma\nu} = -\overset{*}{\nabla}_\mu \left(b^{\mu\beta}{}_\alpha T^\alpha{}_{\beta\nu}\right) - \left(\overset{*}{\nabla}_\eta b^{\mu\eta}{}_\lambda\right) T^\lambda{}_{\mu\nu};$$

5. At last, one rewrites fourth and fifth terms on the right hand side of (D1), respectively, as

$$\frac{1}{2} b^{\gamma\beta}{}_\alpha T^\alpha{}_{\lambda\nu} T^\lambda{}_{\beta\gamma} = -\frac{1}{2}\left(b^{\varepsilon\eta}{}_\lambda T^\mu{}_{\varepsilon\eta}\right) T^\lambda{}_{\mu\nu}.$$

and

$$b^{\gamma\beta}{}_\alpha T^\alpha{}_{\lambda\beta} T^\lambda{}_{\gamma\nu} = -\left(b^{\mu\beta}{}_\alpha T^\alpha{}_{\beta\lambda}\right) T^\lambda{}_{\mu\nu};$$

Combining the results of the points 2 – 5 in the formula (D1), one obtains the search identity:

$$\boxed{\begin{aligned} -\frac{1}{2} b^{\gamma\beta}{}_\alpha \nabla_\nu T^\alpha{}_{\beta\gamma} \equiv& -\overset{*}{\nabla}_\mu \left[\overset{*}{\nabla}_\eta b^{\mu\eta}{}_\nu + \frac{1}{2} b^{\varepsilon\eta}{}_\nu T^\mu{}_{\varepsilon\eta} + b^{\mu\beta}{}_\alpha T^\alpha{}_{\beta\nu}\right] \\ &- \left[\overset{*}{\nabla}_\eta b^{\mu\eta}{}_\lambda + \frac{1}{2} b^{\varepsilon\eta}{}_\lambda T^\mu{}_{\varepsilon\eta} + b^{\mu\beta}{}_\alpha T^\alpha{}_{\beta\lambda}\right] T^\lambda{}_{\mu\nu} - \frac{1}{2} s^\pi{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu}. \end{aligned}} \tag{D2}$$

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