REVIEW ARTICLE

The Kerr Metric

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Abstract. This review describes the events leading up to the discovery of the Kerr metric in 1963 and the enormous impact the discovery has had in the subsequent 50 years. The review discusses the Penrose process, the four laws of black hole mechanics, uniqueness of the solution, and the no-hair theorems. It also includes Kerr perturbation theory and its application to black hole stability and quasi-normal modes. The Kerr metric's importance in the astrophysics of quasars and accreting stellar-mass black hole systems is detailed. A theme of the review is the "miraculous" nature of the solution, both in describing in a simple analytic formula the most general rotating black hole, and in having unexpected mathematical properties that make many calculations tractable. Also included is a pedagogical derivation of the solution suitable for a first course in general relativity.

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1. Introduction

The Schwarzschild solution was found within only a few months of the publication of Einstein's field equations [1]. It is hard to imagine how different the development of general relativity would have been without this exact solution in hand. Instead of dealing only with small weak-field corrections to Newtonian gravity, as Einstein had initially imagined would be the case, fully nonlinear features of the theory could be studied, most notably gravitational collapse and singularity formation.

The existence of the Schwarzschild solution set in motion a search for other exact solutions. None was more eagerly sought than the metric for a rotating axisymmetric source. Already in 1918, Lense and Thirring [2] had found the exterior field of a rotating sphere to first order in the angular momentum, but was there a simple exact solution that was physically relevant? It took almost 50 years to find such a solution: the Kerr metric [3]. Now that another 50 years have elapsed, we can see what an enormous impact this discovery has had. Practically every subfield of general relativity has been influenced. And in astrophysics the discovery of rotating black holes together with a simple way to treat their properties has revolutionized the subject.

In this review, I will first give some forms of the metric for reference. Next I will describe how Kerr found the solution, before moving on to detailing its properties and applications.

2. Forms of the Kerr metric

Recall that a metric is *stationary* if it has a Killing vector field that is timelike at infinity. A metric is *static* if it is stationary and invariant under time reversal, or equivalently, if the time Killing vector is hypersurface orthogonal. A rotating metric is not invariant under $t \to -t$, and so must be stationary without being static.

In Kerr's original paper, he presented the metric in the following form‡

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right)\left(\mathrm{d}v - a\sin^2\theta\,\mathrm{d}\tilde{\phi}\right)^2 \\ &+ 2\left(\mathrm{d}v - a\sin^2\theta\,\mathrm{d}\tilde{\phi}\right)\left(\mathrm{d}r - a\sin^2\theta\,\mathrm{d}\tilde{\phi}\right) \\ &+ \left(r^2 + a^2\cos^2\theta\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\tilde{\phi}^2\right). \end{aligned} \tag{2.1}$$

Nowadays, it is easy to check with computer algebra that this metric satisfies the vacuum Einstein equations. Deriving it from some reasonable assumptions is still not easy, however. I will return to the derivation in §3 and §7.

When $a = 0$, the above metric reduces to Schwarzschild in ingoing Eddington-Finkelstein coordinates, so it is usually called the ingoing Eddington-Finkelstein form of the Kerr metric. The coordinate $\tilde{\phi}$ has a tilde to distinguish it from the Boyer-Lindquist coordinate ϕ below. The ingoing principal null vector (see §3) is particularly simple in these coordinates—it is simply $-\partial/\partial r$. This form of the metric has three off-diagonal terms and so is quite cumbersome for calculations.

The Schwarzschild curvature singularity at $r = 0$ is replaced in the Kerr metric by $r^2 + a^2\cos^2\theta = 0$, that is, $r = 0$ and $\theta = \pi/2$. It is not exactly clear what the geometry of this singularity is if we interpret r and θ as being like ordinary spherical

‡ Kerr's coordinate u is here denoted v to be consistent with the convention that u denotes a retarded time ($u = t - r$ in flat space) whereas v denotes an advanced time ($v = t + r$ in flat space). The sign of a has also been corrected from the original paper, as Kerr himself quickly noted later.

polar coordinates. The situation becomes clearer in the so-called Kerr-Schild form, described next.

The Kerr-Schild form is very useful for finding exact solutions to the field equations, although this is not how Kerr originally derived the solution. A Kerr-Schild metric has the form

$$\mathrm{d}s^2 = (\eta_{\alpha\beta} + 2H\ell_\alpha\,\ell_\beta)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta \tag{2.2}$$

where ℓ_α is a null vector with respect to both $g_{\alpha\beta}$ and $\eta_{\alpha\beta}$. Kerr also gave the metric in this form in his original paper, with

$$H = \frac{mr^3}{r^4 + a^2 z^2}, \qquad \ell_\alpha = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right). \tag{2.3}$$

Here r is not a coordinate but is implicitly defined by

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1. \tag{2.4}$$

We now see that $r = 0$ corresponds to the ring $x^2 + y^2 = a^2$, $z = 0$.

A very convenient coordinate system for the Kerr metric was introduced by Boyer and Lindquist [4] in 1967. The transformation from ingoing Eddington-Finkelstein coordinates is defined by

$$\mathrm{d}v = \mathrm{d}t + (r^2 + a^2)\,\mathrm{d}r/\Delta \tag{2.5}$$

$$\mathrm{d}\tilde{\phi} = \mathrm{d}\phi + a\,\mathrm{d}r/\Delta \tag{2.6}$$

where $\Delta \equiv r^2 - 2Mr + a^2$. The metric in these coordinates has only one off-diagonal term:

$$\begin{aligned}\mathrm{d}s^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\phi + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 \\ &+ \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta\,\mathrm{d}\phi^2\end{aligned} \tag{2.7}$$

where $\Sigma \equiv r^2 + a^2\cos^2\theta$. In these coordinates, the metric is manifestly asymptotically flat, and M and $J = aM$ are easily identified as the mass and angular momentum by letting $r \to \infty$. When $a = 0$, the metric reduces to Schwarzschild in standard curvature coordinates. Boyer-Lindquist coordinates will be used exclusively in the rest of this paper.

Note that the Boyer-Lindquist form of the metric is singular also at $\Delta = 0$. This is a coordinate singularity, since the ingoing Eddington-Finkelstein form is regular there. This behavior is reminiscent of the situation in Schwarzschild, where $\Delta = 0$ at $r = 2M$, the event horizon. In fact, it is easy to check that the normal vector to surfaces $r =$ constant satisfies

$$n_\alpha n_\beta g^{\alpha\beta} = g^{rr} = \frac{\Delta}{\Sigma}. \tag{2.8}$$

Accordingly, the normal vector is null when $\Delta = 0$; the $\Delta = 0$ surfaces are null hypersurfaces. Examination of the null geodesics of Kerr near these surfaces shows that they are in fact horizons. The roots of $\Delta = 0$ are

$$r_\pm = M \pm \sqrt{M^2 - a^2} \tag{2.9}$$

which defines the outer and inner horizons. As we will see later, the region near r_- is very likely not important physically, and so I will refer to r_+ simply as the event horizon of the rotating black hole.

The charged generalization of the Kerr solution was found in 1965 [5] and is called the Kerr-Newman metric. In Boyer-Lindquist coordinates, one simply replaces Δ by $r^2 - 2Mr + a^2 + Q^2$ to get the solution.

3. History of the discovery

Why was its so difficult to find a rotating generalization of the Schwarzschild metric? The straightforward approach was to follow Schwarzschild: Write down the most general line element that reflected the symmetries of the problem (stationarity and axisymmetry), then get the field equations and try to solve them. But here Einstein's instinct was correct: the equations were so complicated that nobody succeeded. Lewis [6] carried out the first notable attempt in 1932. Weyl in 1917 [7] had simplified the static (nonrotating) case by introducing "canonical coordinates," and Lewis showed that similar coordinates could be found in the rotating case. The equations were simpler in these coordinates, but still intractable in general. Lewis found some solutions but, setting a pattern that persisted until Kerr's work, none that was asymptotically flat and nonsingular. A combination historical/personal account of this work is given by Dautcourt [8]. He singles out several attempts to solve the Lewis or related equations [9–11], and describes his own futile attempts. As Ehlers and Kundt wrote in their 1962 review [11], *"the old problem of constructing rigorously the field of a finite rotating body is as yet unsolved, even as to its exterior part."* It was not until Ernst's reformulation of the equations in 1968 [12] that the Kerr solution might possibly have been discovered using this approach, had Kerr not already done so five years earlier by a completely different method.

3.1. Kerr's method of attack

This section is based on an account given by Kerr himself [13] of the discovery of the Kerr metric.

Kerr's success had its origin in a paper by Petrov in 1954 [14]. Petrov classified the algebraic properties of the Weyl tensor at any point into three types plus some subcases. The classification describes the properties of four null "eigenvectors" determined by the Weyl tensor, now called principal null vectors. If two or more of the eigenvectors coincide, the metric is called *algebraically special* in modern terminology. Pirani [15] made researchers in the West aware of Petrov's work, rechristening Petrov's types and subtypes with distinct labels (Type I, Type D, ...) that have become standard. Kerr heard about this work in a seminar by Pirani in 1957.

The Petrov classification had a huge impact on general relativity theory, leading for example to the famous peeling theorem for gravitational waves. An important result for the story of the Kerr metric was the Goldberg-Sachs theorem [16]: *A vacuum metric is algebraically special if and only if it admits a geodesic and shearfree null congruence.* Robinson and Trautman [17] used this result and the assumption that the congruence was hypersurface orthogonal to reduce the complete solution of the field equations to a single nonlinear PDE.

In 1962 Kerr met Alfred Schild at a meeting in Santa Barbara. Schild invited Kerr to come to the newly founded Center for Relativity at the University of Texas as a postdoc for the next academic year.

At this time, Kerr was playing around with Einstein's equations using complex null tetrads, a formalism mathematically equivalent to the Newman-Penrose spinor formalism, which was introduced in that same year. Kerr was trying to extend the Robinson and Trautman work to the case where the null congruence of the algebraically special spacetime was "twisting," that is, not hypersurface orthogonal. The solutions found by Robinson and Trautman were all static, and Kerr hoped that generalizing the Robinson-Trautman approach to twisting congruences might allow him to find the metric of a rotating source.

During this period, an important question being studied was the fate of a star undergoing gravitational collapse. It was generally accepted that a perfectly spherical star would collapse to a black hole described by the Schwarzschild metric.§ But was this merely an artifact of perfect symmetry? Maybe the slightest angular momentum would halt the collapse before the formation of an event horizon, or at least before the formation of a singularity. Finding a metric for a rotating star would be very helpful in answering these questions.

Kerr was not the only one pursing such a solution. Robinson and Trautman presented some results on metrics with twisting null congruences at the GR3 conference in Warsaw in 1962 that Kerr attended, but still had not found any useful rotating solutions [19]. Then a preprint appeared by Newman, Tabourino and Unti where they seemed to have proved that the only algebraically special spacetime with a twisting principal null congruence is the so-called NUT space (after the authors' initials). This metric is a one-parameter generalization of Schwarzschild that is not asymptotically flat except for the pure Schwarzschild case. If true, this would mean that the rotating generalization of Schwarzschild would not be found among the algebraically special solutions. Kerr and his colleague Alan Thompson checked the Newman *et al* paper and found an error, which was corrected in the published version [20]. So Newman *et al* had not ruled out the possibility of algebraically special rotating solutions; Kerr continued his search.

The earlier attempts to find a rotating axisymmetric solution by Lewis, Papapetrou, and others, were based on using the symmetries to simplify the metric and then trying to solve the resulting field equations. By contrast, Kerr's method was to assume first that the metric would be algebraically special like Schwarzschild, simplify the metric accordingly (e.g., with the Goldberg-Sachs theorem), and only then impose the t and ϕ symmetries. Although the simplification of the metric and field equations for the algebraically special case now appears in standard monographs (e.g., [21]), it was not trivial at the time. The next step, imposing the symmetries, was done very cleverly. Instead of trying to solve Killing's equation directly, which probably would have been impossible, Kerr used the action of the symmetry group explicitly to deduce the allowed form of the Killing vectors. Kerr imposed the t and ϕ symmetries each in turn, and found a solution that was asymptotically flat with two parameters, M and a. When $a = 0$, the solution reduced to Schwarzschild. Did a nonzero a turn it into the long-sought rotating solution?

Kerr describes how Schild sat excitedly in Kerr's office while Kerr calculated the angular momentum of the solution. Schild puffed away at his pipe while Kerr chain smoked cigarettes. Kerr transformed the metric to coordinates in the standard form for asymptotically flat metrics, (7.1) below, read off the Lense-Thirring term, and then announced to Schild "It's rotating!" The discovery paper [3] was sent to Physical Review Letters in July 1963 and appeared in September. It was only 1 1/2 pages long and gave little hint of how the solution was found. To those not expert in the long

§ The term "black hole" was actually only introduced by Wheeler in 1968 [18].

history of failure to find such a solution, the physical importance of the paper was almost certainly not evident. And anyone trying to check the solution in those pre-computer-algebra days would have been mystified. In fact, the details only appeared seven years later [22]. An alternative derivation, starting from the Kerr-Schild ansatz, appeared earlier [23]. Some of the details of the rather terse Kerr-Schild paper are explained in [24].

In December of that year (1963), Kerr went to the first Texas Symposium on Relativistic Astrophysics in Dallas. The conference was prompted by the discovery of quasars earlier in the year. Their high redshift suggested that relativity might have something to do with their underlying mechanism, so the conference was an attempt to bring together relativists and astrophysicists. Kerr's conference paper [25] was the first major announcement of the new metric. In the paper, Kerr pointed out that gravitational collapse to a Schwarzschild black hole had difficulty in explaining the prodigious energy output of quasars because of the "frozen star" behavior for distant observers. However, the properties of the event horizon were different with rotation taken into account, as shown with the newly-discovered Kerr solution. (In modern language, a naked singularity was visible for $a > M$.)

Kip Thorne [26] has given an amusing description of how Kerr's presentation had absolutely zero impact on the meeting's audience. Papapetrou was so incensed that he stood up to lecture the participants on the importance of Kerr's discovery, but without effect. Ironically, the Kerr solution is generally accepted today as underlying the explanation of quasars (see §10.2), although not because it is the external metric of a rotating star nor because it describes a naked singularity.

4. Properties of the Kerr metric

Many important properties of the Kerr metric were figured out remarkably quickly, as I now describe.

4.1. Geodesics: a surprise

In a stationary axisymmetric metric, the existence of the two Killing vectors ∂_t and ∂_ϕ implies that the corresponding momenta of a test particle, p_t and p_ϕ, are constants of the motion. The Hamiltonian itself always gives a constant of the motion, but for a complete solution of the geodesic equations in four dimensions we require a fourth constant of the motion. Given the lack of any obvious symmetry in r and θ in the Kerr metric, there was no reason to expect the geodesics to be completely integrable. So it came as a complete surprise when in 1968 Carter [27] showed that a fourth constant could be found because the Hamilton-Jacobi equation for the geodesics was separable in r and θ. The treatment of Kerr geodesics is now a standard feature of essentially all relativity textbooks, so I will not discuss it further here. Important for later developments, as we will see, is that also in 1968 Carter noted the separability of the scalar wave equation in the Kerr metric [28].

4.2. Maximal extensions, matter sources, and Birkhoff's Theorem

In 1965 Penrose [29] had published the first black hole singularity theorem, showing that singularity formation was an inevitable result of gravitational collapse, and not some special feature of spherical symmetry where all the collapsing particles were aimed at the same point. Penrose also introduced the idea of using conformally compactified diagrams to describe the causal relations in a spacetime, and Penrose diagrams were applied to black hole spacetimes by Carter [30]. Analogous to the Kruskal diagram for Schwarzschild, the maximal analytic extension of Kerr was worked out [4, 27, 30], revealing a complicated structure of "universes" patched together.

The modern viewpoint is that all the complications of the Kerr solution beyond the inner horizon are irrelevant to real astrophysics. There are two reasons for this. First, the inner horizon is generically singular in the sense of being an infinite blue-shift surface that magnifies perturbations (see, e.g., [31–34]). Second, there is no Birkhoff Theorem for the Kerr metric. Outside a rotating star, the metric is *not* described by Kerr. As will be discussed below, a generic rotating star can have gravitational multipoles that are not the same as Kerr. The monopole and magnetic dipole moments of Kerr are described by M and a. Kerr does have higher multipole moments, but they are all expressible in terms of M and a. What we do expect is that if a rotating star collapses to a black hole, then the exterior metric will asymptotically approach Kerr. By contrast, during spherical gravitational collapse the exterior metric is Schwarzschild at all times. This means that if we draw the worldline of the surface of the collapsing star on the Kruskal diagram, the region inside the horizon but outside the stellar surface is physical. It does not make sense, however, to draw a similar diagram for Kerr. The bottom line is that talk of analytically extending metrics to expand out into other universes has come to be realized to be of mathematical interest at best.

A related issue is that of a possible interior matter solution for the Kerr metric. I will take the viewpoint that, while there may be some mathematical interest in searching for such a solution, there is no physical interest. The reason is again no Birkhoff's Theorem: The Kerr metric does not represent the exterior metric of a physically likely source, nor the metric during any realistic gravitational collapse. Rather, it gives the asymptotic metric at late times as whatever dynamical process produced the black hole settles down.

4.3. The Penrose process

In the Kerr metric, the time Killing vector ∂_t changes from being timelike to being spacelike in a region outside the event horizon. The boundary of this region occurs where the Killing vector becomes null:

$$\partial_t \cdot \partial_t = g_{tt} = 0 \quad \Rightarrow \quad r = M + \sqrt{M^2 - a^2 \cos^2\theta}. \tag{4.1}$$

This surface is called the *stationary limit* or the *ergosurface*. The region between the event horizon and the ergosurface is the *ergosphere*. The existence of the ergosphere allows various kinds of energy extraction mechanisms for a rotating black hole. For example, inside the ergosphere, a particle with 4-momentum $\mathbf{p}$ can have negative conserved energy $E = -\mathbf{p} \cdot \partial_t$. In the Penrose process [35] a particle falls into the ergosphere where it splits into two. One of the particles is created with negative energy and falls into the hole. The other comes out to infinity with more energy than the original particle, with the black hole losing some rotational energy.

The original Penrose process is not likely to be astrophysically important [36]. Its wave analogue, superradiant scattering, is much more interesting (see §8.2). And its manifestation in the Blandford-Znajek process for electromagnetic energy extraction from black holes is at the foundation of current explanations of prodigious astrophysical energy sources such as quasars (see §10.2).

5. The four laws of black hole mechanics

5.1. The classical era

The development of black hole thermodynamics began with analogies to the second law. Penrose and Floyd [35] had noted that the surface area of a Kerr black hole increased during the Penrose process, and suggested this might be a general feature of black hole interactions. Independently, Christodoulou [37] had shown that a quantity he called the irreducible mass could never decrease when particles were captured by Kerr black holes, and investigated "reversible transformations," where it remained unchanged. The decisive step came with Hawking's Area Theorem, that the surface area of a generic black hole could never decrease [38]. Bekenstein [39], who was also a graduate student at Princeton and so very familiar with Christodoulou's thesis work, proposed that black hole area was not just analogous to entropy, but actually was proportional to its entropy. He based his argument on information theory: The black hole entropy was a measure of the information about the interior that was inaccessible to an external observer. He argued that the constant of proportionality should be the square of the Planck length, up to a numerical factor. Bekenstein also proposed a generalization of the second law of thermodynamics: The sum of the black hole entropy plus the ordinary entropy in the exterior never decreases.

At about the same time, the first law of black hole mechanics was developed. For the Kerr metric, the area of the horizon is

$$A = 8\pi M r_+. \tag{5.1}$$

Replacing a in this expression by $J = aM$, the angular momentum of the hole, and differentiating gives

$$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega\delta J \tag{5.2}$$

where

$$\kappa = \frac{r_+ - M}{2Mr_+}, \qquad \Omega = \frac{a}{2Mr_+}. \tag{5.3}$$

Here Ω is the "angular velocity of the horizon" and κ is the surface gravity, related to the acceleration of a particle corotating with the black hole at the horizon. Bekenstein [39] and Smarr [40] had noted the similarity between (5.2) and the first law of thermodynamics, where if A was proportional to the black hole entropy then κ would have to be proportional to the black hole temperature.

These ideas were synthesized and extended by Bardeen, Carter, and Hawking [41]. They proved the validity of the first law (5.2) for general stationary black hole configurations, including external matter distributions. They also gave a proof of the zeroth law, that the surface gravity κ is constant on the black hole horizon. This gives further support to the idea of κ as related to temperature, which is constant for a body in thermal equilibrium. They finally conjectured that the third law would hold in the form that the temperature of a black hole, i.e., κ, cannot be reduced to zero by a finite sequence of operations. It follows from the third law that nonextremal black holes cannot be made extremal in a finite number of steps. The third law was later proved by Israel [42]. Note that Planck's formulation of the third law of thermodynamics does not hold in black hole mechanics: The entropy (area) of an extremal black hole is finite even though it has zero temperature (surface gravity).

By now, there exist many different proofs of these laws, making slightly different assumptions (see [43] for a review and references).

5.2. The quantum era

In the classical treatment, the laws of black hole mechanics were considered to be only analogies to the real laws of thermodynamics. For example, since a black hole could not radiate, it always had to have zero temperature. This meant that κ could not be the actual temperature of the hole. All this changed with Hawking's discovery that when quantum effects were considered, a black hole *can* radiate [44, 45]. Hawking showed that the radiation was thermal with a temperature

$$T = \frac{\hbar\kappa}{2\pi} \tag{5.4}$$

where Boltzmann's constant is set to unity in the denominator. By the first law (5.2), this fixes the constant of proportionality between area and entropy:

$$S = \frac{1}{4}\frac{A}{\hbar}. \tag{5.5}$$

As Bekenstein had argued, the entropy was indeed proportional to the area in Planck units. (Hawking had actually not been persuaded by Bekenstein's original proposal, and had a different motivation for his calculation. See [46] for some historical remarks and references.)

The discovery of a rational basis for assigning entropy to black holes and the formulation of a generalized second law that includes black hole interactions have spawned a huge amount of research that would take us too far afield to discuss here. See the companion article by Jacobson [47] and also [48] for reviews and references. Perhaps the most interesting question that has come out of this is what the entropy of a black hole represents microscopically. If entropy is fundamentally the log of the number of states accessible to the system, then what are these accessible degrees of freedom for a black hole? Some of the proposed ideas are discussed in [46], for example. Perhaps the most intriguing comes from string theory, where microstates counted for D-brane configurations allow the entropy of certain black holes to be calculated. Who could have imagined that, in such a short time, Kerr's playing around with principal null vectors of the Weyl tensor would lead to direct explorations of quantum gravity in string theory?

6. Black Hole Uniqueness

In 1967 Werner Israel [49] published a remarkable theorem: the only static vacuum black holes are the Schwarzschild black holes (and hence are spherically symmetric).‖ Unlike equilibrium matter objects, it was not possible to find a static black hole with a quadrupole moment, for example. Israel quickly extended his result to a uniqueness theorem for static black holes with charge: only Reissner-Nordstrom black holes are allowed [50].

Israel's paper set off a burst of activity to see if the uniqueness result holds also for the stationary case: does the Kerr metric describe all possible rotating black holes? This question is related to the conjecture that a black hole formed by gravitational collapse will asymptotically settle down to a member of the Kerr family. (We usually assume that the charge of a real black hole is negligible because

‖ Here and in most of this review, I will gloss over the exact assumptions that go into proving the various theorems that I quote. Among other things, it is here assumed that one is dealing with asymptotically flat black holes with no cosmological constant.

macroscopic astrophysical charged objects will rapidly be neutralized by surrounding plasma. However, in theoretical general relativity the uniqueness of the Kerr-Newman family has also been an important question.)

More precisely, the conjecture was that the only stationary, asymptotically flat solution of the vacuum Einstein equations that is nonsingular from infinity to a regular event horizon is the Kerr metric.

Carter [51] showed that axisymmetric black holes could depend on only two parameters, the mass and angular momentum. He also showed that the Kerr metric was the only one that included a zero angular momentum black hole. (More precisely, he showed that there were no linear perturbations near Kerr that were stationary and axisymmetric other than changes in the mass and angular momentum.) These results strongly suggested that Kerr black holes were the only ones. Next, Hawking [52] proved that all stationary black holes must be either static or axisymmetric, making one of Carter's assumptions unnecessary. He also showed that the topology of the event horizon had to be spherical, another assumption that was used in the Israel/Carter work. Finally, in a spectacular feat of ingenuity, Robinson [53] gave a definitive proof of Carter's results without comparing Kerr only to nearby solutions.

For most physicists and astronomers, this was the end of the subject. Kerr black holes were all they needed to understand. However, among the more mathematically inclined the fun had only begun. Kerr-Newman uniqueness was finally nailed down in the 1980's by Mazur [54] and Bunting [55]. Their work could also be used to simplify and improve the earlier results for uncharged black holes. Interestingly, in the case of extremal charged black holes, with $Q^2 = M^2$, the uniqueness theorem also allows Majumdar-Papapetrou solutions [56]. These solutions describe an arbitrary number of black holes in equilibrium with the gravitational attraction exactly balanced by Coulomb repulsion.

Much of the work since the 1970's has focused on removing the "technical" assumptions made in the earlier proofs and on improving the mathematical rigor, but there is still the sense that some of the assumptions are not necessary. Excellent reviews are given in [57–60]. The main fly in the ointment today is still the cosmic censorship hypothesis: if singularities can occur outside an event horizon, then the proofs fail. The hypothesis was introduced by Penrose [61], who postulated the existence of a "cosmic censor" who forbids the appearance of "naked" singularities not clothed by an event horizon. Despite the failure to find any convincing example of a naked singularity forming from well-behaved generic initial data, there is still no proof that this is impossible.

The uniqueness theorems have led to two strands of ongoing research: generalizations of the no-hair theorem, and questions of uniqueness for black holes in higher dimensions.

6.1. No-hair theorems

In parallel with the work on black hole uniqueness in the 1960's, the community's understanding of gravitational collapse began to shift away from the frozen-star viewpoint. With a comoving vantage point instead, the picture that developed was that the gravitational field decouples from its matter source in the late stages of collapse and radiates away all the multipoles it can, leaving only the "charges" associated with a conserved flux integral at infinity.

This idea was catalyzed by Doroshkevich, Zel'dovich and Novikov [62], who showed that the higher multipole moments of a system died out during gravitational collapse—the beginnings of the no-hair theorem. Exactly how this occurred only became clear later, with Price's Theorem [63, 64]: "Whatever can be radiated is radiated." Price gave quantitative estimates of the decay of multipole moments measured at infinity.

By 1969 Penrose could write [61]:

> Doubts have frequently been expressed concerning [the no-hair conjecture], since it is felt that a body would be unlikely to throw off all its excess multipole moments just as it crosses the Schwarzschild radius. ... On the other hand, the gravitational field *itself* has a lot of settling-down to do after the body has fallen into the "hole". The asymptotic measurement of the multipole moments need have very little to do with the detailed structure of the body itself; the *field* can contribute very significantly. In the process of settling down, the field radiates gravitationally—and electromagnetically too, if electromagnetic field is present. Only the mass, angular momentum and charge need survive as ultimate independent parameters.

Wheeler introduced the aphorism "black holes have no hair" in 1971 [65].

More recent work has made it clear that the matter fields play a crucial role in the applicability of the no-hair theorem. A trivial example is that one can construct stationary black holes surrounded by equilibrium rotating disks of perfect fluid or collisionless matter. However, we now know that even matter fields analogous to electromagnetic fields, with generalized conserved charges, can violate the no-hair theorem. In particular, static solutions for Yang-Mills fields coupled to gravity can not be classified simply by mass, angular momentum, and conserved charges. In fact, a wide variety of scalar field models violate the no-hair theorem. Since these scalar fields are precisely of the type that are expected to occur in the low-energy limit of string theory, they are not just of mathematical interest. The situation is reviewed in [57]. These violations of the no-hair theorem are static and spherically symmetric. There is also numerical evidence that with suitable matter fields, static black holes are not necessarily spherically symmetric or even axisymmetric, and nonrotating black holes need not be static. Whether these results are astrophysically important or simply points of principle is not clear.

Interestingly, experimental tests of the no-hair theorem have been proposed. The idea is to measure the spin and quadrupole moment of a black hole and see if they satisfy the Kerr relationship. See, for example, [66, 67].

6.2. Black holes in higher dimensions

Interest in black holes in higher dimensions is motivated by string theory, which describes gravity and requires more than four dimensions. But this is not the only motivation: We learn about black holes and general relativity in four dimensions by studying what happens in other dimensions. In four dimensions, black hole properties include

- uniqueness
- no-hair theorem, i.e., characterization by conserved charges
- spherical topology
- dynamical stability

None of these properties holds in higher dimensions. Excellent reviews of these and other properties are given in [68–70].

An example of a regular, stationary, asymptotically-flat, vacuum solution in five dimensions is the Myers-Perry black hole [71]. Such solutions in fact exist in all higher dimensions and are generalizations of the Kerr metric. They have spherical horizon topology and can be rotating in several independent rotation planes. But in five dimensions there are also the Emparan-Reall black rings with $S^2 \times S^1$ horizon topology [72]; the Pomeransky-Senkov black rings, which have a second angular momentum parameter [73]; and the "Black Saturn" solutions, which are multi-component black holes where a spherical horizon is surrounded by a black ring [74].

Looking at these solutions, we see that there are black rings and Myers-Perry black holes with the same mass and angular momentum. In fact, black rings are not fully characterized by their conserved charges. The Black Saturns are regular vacuum multi-black-hole solutions, which do not to exist in four dimensions. Clearly the situation in higher dimensions is much more complicated than in four dimensions.

6.3. Numerical relativity and black hole uniqueness

In the past ten years, fully 3-d numerical computations with black holes have become possible. A natural question is to ask whether the final state of, for example, the inspiral and merger of a binary black hole system is a Kerr black hole. The question has been studied most thoroughly by Owen [75, 76], who gives references to earlier work.

Owen [75] used numerical data from a high-precision simulation of an equal-mass, nonspinning black hole-black hole binary carried out with the `SpEC` code [77]. He devised a (spatially) gauge invariant definition of multipoles for nonaxisymmetric dynamical metrics, and confirmed that the multipoles settle down to the expected Kerr values. The agreement could be measured to at least a part in 10^5 because of the high accuracy of the numerical code. Owen also showed that that the exponential falloff of the dominant quasi-normal mode at late times agreed with the expected Kerr value. (Quasi-normal modes are discussed in §8.5.) Higher quasi-normal modes are more difficult to analyze because the assignment of l mode numbers to the modes is not straightforward. (See §8 for a discussion of angular modes of Kerr.) Accordingly, Owen also analyzed the quasi-normal modes of a head-on collision between two black holes, one with spin up and one with spin down [78]. Here the final state is expected to be Schwarzschild, which makes the assignment of l mode numbers to the modes unambiguous. Again he found agreement limited only by the numerical accuracy of the simulations.

In [76], Owen extended work by Campanelli *et al* [79] to confirm that the spacetime of a black hole merger approaches Petrov Type D at late times.

7. Pedagogical Derivation of the Kerr Metric

As we have seen, the original derivation of the Kerr metric was a tour de force of complicated calculations. Even the alternative derivation in the Kerr-Schild paper [23] admitted, "It is well worth pointing out that the calculations giving these results are by no means simple." This causes a problem for teaching a course in the subject. One can simply present the metric to students without derivation, or have them verify with computer algebra that it satisfies the vacuum Einstein equations. However, it would be nice if there were a pedagogical derivation on a par with the derivation of the Schwarzschild solution that is usually done in a course. By "pedagogical" I mean that computer algebra is allowed for computing the Einstein or Ricci tensor from a given metric (e.g., using the Mathematica program at the website [80]), but that the assumptions and techniques have to be suitable for beginning students. So, for example, the derivation in Chandrasekhar's book [81] is "straightforward" in the sense that it does not make use of concepts like algebraic specialness. But even with evaluation of the field equations via computer algebra, the subsequent solution is far from trivial.

There have been several attempts in the literature to formulate a pedagogical derivation. Enderlein [82] makes use of the Lorentz-transformed basis of 1-forms for a flat spacetime in oblate spheroidal coordinates. However, the field equations still require complicated manipulations to solve. Deser and Franklin [83] start with a rather obscure (for beginning students) motivation of the metric. They then use "Weyl's trick" to simplify the variational principle for the field equations. (This trick is to impose the symmetries before carrying out the variation, which under certain conditions gives the correct final equations.) It is necessary to manipulate the scalar curvature R in the integrand of the variational principle by suitable integrations by parts to make the problem tractable. While suitable for experts, this does not meet my definition of pedagogical.

Dadhich [84] does not actually use the field equations. Instead, he makes some arbitrary-seeming assumptions about geodesics. However, he starts with a good form for the metric.

It seems that the only way to construct a derivation transparent enough for a first course is to make some heuristic assumptions that allow one to start with a metric in suitable form. The field equations that follow from the assumed metric must be simple enough to be solved and yield the desired metric. The question of what heuristic assumptions are plausible is, of course, subjective and also tainted by knowing the desired answer. Here I will assemble what I consider to be the best elements of earlier attempts into what I would use to present a derivation to beginning students.

The metric outside a weak-field, slowly rotating source is approximately

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2+\left(1+\frac{2M}{r}\right)\mathrm{d}r^2+r^2\mathrm{d}\theta^2+r^2\sin^2\theta\,\mathrm{d}\phi^2-\frac{4aM}{r}\sin^2\theta\,\mathrm{d}\phi\,\mathrm{d}t \tag{7.1}$$

to first order in a. Here M is the mass and $a = J/M$ is the angular momentum per unit mass. The last term describes the Lense-Thirring "dragging of inertial frames." Grouping the mass terms in g_{tt} and $g_{t\phi}$ together suggests rewriting this as

$$\mathrm{d}s^2 = -\mathrm{d}t^2+\left(1+\frac{2M}{r}\right)\mathrm{d}r^2+r^2\mathrm{d}\theta^2+r^2\sin^2\theta\,\mathrm{d}\phi^2+\frac{2M}{r}(\mathrm{d}t-a\sin^2\theta\,\mathrm{d}\phi)^2, \tag{7.2}$$

but still only really valid to first order in a.

Now consider the effect of rotation on the spatial geometry. With no rotation, the Schwarzschild geometry is spherically symmetric. We expect rotation to "flatten" the geometry and so we want to choose coordinates in which this effect will appear simple. Following [85], the simplest choice is to use ellipsoids to define the coordinate system. In axisymmetry, a family of confocal ellipsoids $r =$ constant is defined by

$$\begin{aligned} x &= \sqrt{r^2+a^2}\sin\theta\cos\phi \\ y &= \sqrt{r^2+a^2}\sin\theta\sin\phi \\ z &= r\cos\theta. \end{aligned} \tag{7.3}$$

The flat space metric in these oblate spheroidal coordinates is

$$-\,\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2=-\mathrm{d}t^2+\frac{\Sigma}{r^2+a^2}\mathrm{d}r^2+\Sigma\,\mathrm{d}\theta^2+(r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2 \tag{7.4}$$

$$=-\frac{r^2+a^2}{\Sigma}(\mathrm{d}t-a\sin^2\theta\,\mathrm{d}\phi)^2+\frac{\Sigma}{r^2+a^2}\mathrm{d}r^2+\Sigma\,\mathrm{d}\theta^2+\frac{\sin^2\theta}{\Sigma}[(r^2+a^2)\mathrm{d}\phi-a\,\mathrm{d}t]^2 \tag{7.5}$$

where $\Sigma=r^2+a^2\cos^2\theta$. The form (7.5) is a simple rewriting of (7.4).

Although the $\mathrm{d}t\,\mathrm{d}\phi$ terms actually cancel out of (7.5), it provides a good starting point for finding a rotating metric, where these terms do not cancel. Comparing (7.2) and (7.5) suggests making the ansatz of replacing r^2+a^2 in the first and second terms of (7.5) with arbitrary functions of r. We could make the more general ansatz of arbitrary functions of r and θ, but that turns out to be too complicated. So we try the simpler assumption first. In particular, let the coefficient of the first term in (7.5) become

$$\frac{r^2+a^2}{\Sigma}\to\frac{Y(r)}{\Sigma}\equiv\frac{r^2+a^2-Z(r)}{\Sigma} \tag{7.6}$$

while in the second term let $r^2+a^2\to F(r)$. Then undoing the operation¶ that led from (7.4) to (7.5) gives

$$\mathrm{d}s^2=-\mathrm{d}t^2+\frac{\Sigma}{F(r)}\mathrm{d}r^2+\Sigma\,\mathrm{d}\theta^2+(r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2+\frac{Z(r)}{\Sigma}(\mathrm{d}t-a\sin^2\theta\,\mathrm{d}\phi)^2. \tag{7.7}$$

Note that in getting from (7.2) to (7.7) I have made the unjustified assumption that the quantity a in the Lense-Thirring term is the same as the parameter a describing the oblateness of the spheroidal coordinates.

Now compute the vacuum field equations $G_{\mu\nu}=0$ from the metric (7.7) using a computer algebra program (e.g., the Mathematica program [80]). To avoid complications with trigonometric functions, let $q=\cos^2\theta$ so that $\mathrm{d}\theta^2=\mathrm{d}q^2/[q(1-q)]$. All the equations are quite complicated, except for the equation $G_{rr}=0$. This equation has a coefficient of q and a constant term, which must separately vanish. This gives two equations:

$$-\,r^2(r^2+a^2-Z)+[Z+r(r-Z')]F=0 \tag{7.8}$$

$$(r^2-Z)(r^2+a^2-Z)+r(Z'-r)F=0. \tag{7.9}$$

Solving (7.9) for F gives

$$F=\frac{(r^2-Z)(r^2+a^2-Z)}{r(r-Z')}. \tag{7.10}$$

Substituting for F in (7.8) and ignoring the spurious solution $Z=r^2+a^2$ (which would give a singular metric with $F=0$), we find

$$Z'=Z/r \qquad \Rightarrow \qquad Z=2Mr, \qquad F=r^2-2Mr+a^2. \tag{7.11}$$

The constant of integration in solving for Z is set by comparing with the weak-field limit (7.2) when $r\to\infty$. One can easily check that the remaining components of the Einstein tensor are zero for the solution (7.11). The final form of the solution is the Boyer-Lindquist metric (2.7).

¶ It turns out that using the form (7.4) directly with the replacements $Y(r)$ and $F(r)$ is also tractable and only slightly more complicated than using $Z(r)$ and $F(r)$.

8. Kerr Perturbations

The study of black hole perturbations was initiated in a groundbreaking paper by Regge and Wheeler in 1957 [86]. Their goal was to prove that the Schwarzschild metric described a solution that was stable to linear perturbations. The spherical symmetry and time independence of the metric allowed the perturbations to be decomposed into spherical harmonics and Fourier modes, and they derived a radial equation describing the odd-parity perturbations. This work was brought to fruition by Vishveshwara [87] and Zerilli [88, 89] in the early 1970's. In particular, Zerilli derived a radial equation for the even-parity modes that allowed the stability proof to be completed. The theory of Schwarzschild perturbations has turned out to be rich in unexpected applied mathematics as well as in practical applications. Much of this work in summarized in Chandrasekhar's book [81].

With the success of Vishveshwara and Zerilli for the Schwarzschild case, attention naturally turned to Kerr. While it was clear that one would still be able to separate out the t and ϕ dependence in the perturbation equations, there was no obvious reason why the r and θ dependence would separate. Nevertheless, Carter's discovery [28] of the unexpected separability of the scalar wave equation on the Kerr background gave some hope.

Rather than tackle the full gravitational perturbation problem head on, Fackerell and Ipser [90] decided to look first at Maxwell's equations on a Kerr background. They used the Newman-Penrose (NP) formalism, inspired by Price's work on Schwarzschild perturbations [64]. Price had used the NP formalism to recover the Regge-Wheeler equation in a relatively simple way and to prove important results about the late-time decay of perturbations. Since the explicit form of the equations in NP language can be chosen to encode the principal null vectors, it is reasonable to expect the formalism to be well-suited to Type D metrics like Schwarzschild and Kerr. In the NP formalism, the gravitational field is described by an overdetermined set of equations that includes the Bianchi identities for the Weyl tensor explicitly. The ten independent components of this tensor appear as five complex quantities, Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4. In Price's work, the Regge-Wheeler equation appeared as the equation governing the imaginary part of the "middle" quantity, Ψ_2.

For Maxwell's equations, the NP formalism uses three complex quantities Φ_0, Φ_1, and Φ_2, which encode the six components of **E** and **B**. Fackerell and Ipser combined these equations on the Kerr background and found a decoupled equation for the middle quantity Φ_1. Unfortunately, this equation did not separate in r and θ and so was of limited use. The gravitational problem still appeared out of reach.

Meanwhile, Bardeen and Press [91] had been collaborating on using Price's NP approach to study radiation fields on the Schwarzschild background. At this time, I was a beginning second-year graduate student at Caltech and not making much progress on the problem that my advisor, Kip Thorne, had given me. One day, Press, who was a year ahead of me, told me that Bardeen had found decoupled equations in Schwarzschild for the "extreme" quantities Ψ_0 and Ψ_4 (and Φ_0 and Φ_2 in the electromagnetic case). Since Price, Fackerell and Ipser had all been at Caltech, I was very familiar with their work and I wondered if there might be decoupled equations for the extreme quantities in Kerr as well. In the NP formalism there are certain commutation relations that are used to effect the decoupling, and I suspected that the decoupling might go through in any Type D metric if the choice of null vectors was all that mattered. The calculation took only a few hours, and indeed there were decoupled equations. I was quite excited at first, but when I wrote out the equation for Ψ_4 in Boyer-Lindquist coordinates it did not separate, just like the Fackerell-Ipser equation for Φ_1.

For the next few weeks I tried all sorts of algebraic contortions to see if I could make the equation separate. There didn't seem to be any useful information in the mathematical literature on separability of PDEs beyond the usual Killing vector-based separability and the classification of separable coordinate systems for Poisson's equation. Eventually Kip Thorne decided to send me to Maryland for a month to consult with Charles Misner, an expert in PDEs (among many other things). I still made no progress, and returned feeling that I had somehow let Kip down. I put the work aside and took up other problems in an effort to finish my thesis. Every few weeks I would take out my notes and try a few more manipulations. One evening, about six months after finding the decoupling, something made me try a new set of substitutions. A key variable in the NP formalism is ρ, whose real and imaginary parts encode the divergence and curl of the outgoing principal null congruence. In the Kerr metric in Boyer-Lindquist coordinates,

$$\rho = -\frac{1}{r - ia\cos\theta}. \tag{8.1}$$

I found that if I considered the equation satisfied by $\rho^{-4}\Psi_4$ instead of Ψ_4, the equation suddenly separated. Similarly, the electromagnetic equation for $\rho^{-2}\Phi_2$ was separable. Interestingly, the Ψ_0 and Φ_0 equations were separable without modification. However, I had never looked at them before because Ψ_4 and Φ_2 describe outgoing as opposed to incoming radiation at large r and so were considered to be physically more relevant. Had I looked at Ψ_0 or Φ_0 earlier, I could have saved myself a lot of anguish.

The final result could be written as a single master equation describing all the perturbations [92, 93]:

$$\begin{aligned}
&\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\phi^2} \\
&\quad - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\phi} \\
&\quad - 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 4\pi\Sigma T.
\end{aligned} \tag{8.2}$$

Here the spin weight parameter s takes on the values 0, $\frac{1}{2}$, 1, and 2 for scalar, neutrino, electromagnetic, and gravitational perturbations. The quantity ψ is the field (scalar, $\rho^{-4}\Psi_4$, etc.), and the explicit form of the source terms T is given in [93]. (The neutrino equation was given independently by Unruh [94].)

Consider first the vacuum case ($T = 0$). Inspection of (8.2) shows that it will separate in the form

$$\psi = e^{-i\omega t}e^{im\phi}S(\theta)R(r). \tag{8.3}$$

The equations for S and R are

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right) + \left(a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta - \frac{(m+s\cos\theta)^2}{\sin^2\theta} + s + A\right)S = 0 \tag{8.4}$$

$$\Delta^{-s}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{s+1}\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right)R = 0 \tag{8.5}$$

where $K \equiv (r^2+a^2)\omega - am$ and $\lambda \equiv A + a^2\omega^2 - 2am\omega$. The angular equation (8.4) is an eigenvalue equation for the separation constant A. The eigenfunctions ${}_sS_{lm}(\theta)$

are called spin-weighted spheroidal harmonics, since they generalize spheroidal wave functions (when $s = 0$) and spin-weighted spherical harmonics (when $a\omega = 0$). The radial equation (8.5) is in effective potential form, except the potential is complex and not short-range, unlike the Regge-Wheeler or Zerilli equations.

When sources are present ($T \neq 0$), one can use the angular eigenfunctions to expand the right-hand side of (8.2). This gives a radial equation identical to (8.5) but with a source term on the right-hand side. Boundary conditions and the computation of energy fluxes for the radial equation are discussed in [93, 95]. In particular, for outgoing waves at infinity, Ψ_4 is related to the transverse traceless metric perturbation by

$$\Psi_4 = -\tfrac{1}{2}\omega^2(h_{\hat{\theta}\hat{\theta}} - i h_{\hat{\theta}\hat{\phi}}) \tag{8.6}$$

and encodes the two polarization states of a gravitational wave.

The Dirac equation turned out to be tricky. However, Chandrasekhar [96] showed how to handle it by first separating the variables and then decoupling the field components.

In the remainder of this section I will describe some applications of the Kerr perturbation equations and some later developments of the theory.

8.1. The miraculous identities

Suppose that one has solved the equation for a gravitational perturbation Ψ_4 (or an electromagnetic perturbation Φ_2). Then the corresponding quantity Ψ_0 (or Φ_0) should be completely determined without having to re-solve the master perturbation equation, and vice versa. It is in fact possible to relate the quantities directly by means of some remarkable identities that follow from Einstein's or Maxwell's equations [95].

Let us first display the identities for the electromagnetic case. Following the normalization conventions of Chandrasekhar [81], write the separated quantities in the form

$$\Phi_0 = R_1(r)S_1(\theta), \qquad \tfrac{1}{2}\rho^{-2}\Phi_2 = R_{-1}(r)S_{-1}(\theta) \tag{8.7}$$

where S and R satisfy (8.4) and (8.5) and S is normalized to unity over the sphere. The subscripts ± 1 denote the spin weight s of the quantities. Now define the radial and angular operators

$$\mathcal{D}_n = \partial_r - \frac{iK}{\Delta} + 2n\frac{r-M}{\Delta}, \qquad \mathcal{L}_n = \partial_\theta + Q + n\cot\theta \tag{8.8}$$

where $Q = -a\omega\sin\theta + m/\sin\theta$ and K defined after (8.5) is the negative of the variable defined by Chandrasekhar. The "adjoints" of these operators are defined by changing the signs of K and Q and are denoted by $\mathcal{D}^\dagger$ and $\mathcal{L}^\dagger$. Then Maxwell's equations imply

$$\Delta\mathcal{D}_0\mathcal{D}_0 R_{-1} = \mathcal{C}\Delta R_1 \qquad \Delta\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger\Delta R_1 = \mathcal{C}R_{-1} \tag{8.9}$$

$$\mathcal{L}_0\mathcal{L}_1 S_1 = \mathcal{C}S_{-1} \qquad \mathcal{L}_0^\dagger\mathcal{L}_1^\dagger S_{-1} = \mathcal{C}S_1 \tag{8.10}$$

where the constant $\mathcal{C}$ is given by

$$\mathcal{C} = \sqrt{\lambda_{\mathrm{Ch}}^2 + 4ma\omega - 4a^2\omega^2}. \tag{8.11}$$

Here $\lambda_{\mathrm{Ch}} = \lambda + s^2 + s$ is the separation constant used by Chandrasekhar, which has the advantage that it is independent of s. We see that the operators $\mathcal{D}_n$ and $\mathcal{L}_n$ and their adjoints act like raising and lowering operators for the radial and angular eigenfunctions.

Similar relations follow from the perturbed Einstein equations for the $s = \pm 2$ functions [81, 95], but now requiring four operators to change $s = 2$ to $s = -2$, for example. Chandrasekhar [81] has called these various identities the Teukolsky-Starobinsky identities. For $s = \pm 2$, a new constant analogous to (8.11) appears, which turns out to be complex. Chandrasekhar has called constants like (8.11) Starobinsky constants.[+]

The identities described in this section are quite unexpected and "miraculous." In spherical symmetry, we understand the existence of raising and lowering operators on spin-weighted spherical harmonics from the algebra of the rotation group. Here equations following from certain laws of physics lead to the discovery of identities that must be satisfied by a new class of special functions of mathematical physics, the spin-weighted spheroidal harmonics and the related radial functions. For a discussion of the identities from the special-function point of view, see [99].

8.2. Superradiant scattering

Section 4.3 described the Penrose process for particles. There is also its wave analogue, *superradiance* [100]. For wave modes proportional to $\exp -i(\omega t - m\phi)$, a scattered wave can come off with more energy than the incident wave if $m > 0$ and $0 < \omega < m\Omega$. (The angular velocity of the horizon Ω was defined in (5.3).) Essentially some negative energy flux is absorbed by the hole.

The existence of superradiance made it seem more likely that Kerr black holes might be dynamically unstable. Numerical calculations [95, 101] showed that the maximum superradiant amplification of low-m modes is bounded, reaching 138% for the $l = m = 2$ gravitational mode. The stability question is taken up in the next subsection.

Superradiance was invoked to design the "black-hole bomb" [101], where a mirror around the black hole reflects radiation back toward the hole multiple times. The amplitude grows exponentially until the mirror explodes. Leaving aside the possibility that an advanced civilization might be able to control the power flow through suitable ports in the mirror, this scenario suggests that a scalar field with mass would be unstable near a Kerr black hole. Negative energy could radiate down the hole, but no energy would leave at infinity because the solution itself decays exponentially fast. Such instability has in fact been rigorously demonstrated [102].

8.3. Stability of Kerr black holes

8.3.1. Mode stability As already mentioned, the primary motivation of the early work on Schwarzschild perturbations was to prove that spherical black holes were stable objects, and hence might actually exist in the physical universe. After separating out the angular dependence, for modes of the form

$$\psi = R(r)e^{-i\omega t} \tag{8.12}$$

[+] These constants first appeared, without derivation, in a paper by Starobinsky and Churilov [97]. A reader consulting this paper may be confused to see that only the electromagnetic constant is given. The much more complicated gravitational constant is mentioned, but not explicitly given. The resolution of this puzzle is that in the pre-Internet era, preprints of papers were mailed around, and the preprint contained the gravitational expression. However, the expression was inexplicably omitted in the published version. The constants appear in expressions for energy fluxes of waves and so I gave a derivation in [98] by looking at the asymptotic behavior of the fields at large r, presumably the same derivation used by Starobinsky. A little later I realized that the raising and lowering properties hold in general, and not just at large r [95].

both the Regge-Wheeler and Zerilli equations can be written in the form of Schrodinger-like equations:

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^{*2}} - V(r)R = -\omega^2 R. \tag{8.13}$$

Here $r^*(r)$ is the so-called tortoise coordinate that maps the interval (r_+, ∞) the interval $(-\infty, \infty)$. The potential V goes to zero at both ends of the interval and is real, positive, and independent of ω. There are two standard arguments to show the stability of modes satisfying (8.13).

First, (8.13) is a linear, self-adjoint eigenvalue problem for ω^2. By self-adjointness, ω^2 must be real, so any instability must lie on the positive imaginary axis, $\omega = i\alpha$ say. Thus the equation takes the form

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^{*2}} = (\text{positive-definite function})R \tag{8.14}$$

which manifestly has no solution that is regular at $r^* \to \pm\infty$.

A second way of proving stability is to note that the modes (8.12) satisfy the wave equation

$$\frac{\partial^2\psi}{\partial r^{*2}} - \frac{\partial^2\psi}{\partial t^2} - V(r)\psi = 0. \tag{8.15}$$

Associated with this equation is the energy

$$E = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}r^* \left(\left|\frac{\partial\psi}{\partial t}\right|^2 + \left|\frac{\partial\psi}{\partial r^*}\right|^2 + V|\psi|^2 \right). \tag{8.16}$$

Since the potential is nonnegative, so is the energy. The boundary conditions for well-behaved solutions of (8.15) are

$$\psi \to \begin{cases} Ae^{-i\omega(t-r^*)}, & r^* \to \infty \\ Be^{-i\omega(t+r^*)}, & r^* \to -\infty. \end{cases} \tag{8.17}$$

Physically, these correspond to waves leaving the domain at both infinity and the event horizon. Evaluating $\mathrm{d}E/\mathrm{d}t$, simplifying using the wave equation (8.15), and integrating by parts gives

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -|\omega|^2(|A|^2 + |B|^2). \tag{8.18}$$

Thus the energy cannot grow without bound, so the terms in the integrand of (8.16) are bounded, proving stability.

For the Kerr metric, the effective potential associated with the radial equation (8.5) is not real nor independent of ω. Thus there does not seem to be a stability proof analogous to the self-adjointness argument for Schwarzschild. Turning to the second method of proving stability, one can write down a conserved energy based on (8.2). However, the energy density is not positive in the ergosphere. Thus the total energy can be finite while the field still grows exponentially in parts of spacetime. Press and Teukolsky [98] noted that boundary conditions analogous to (8.17) imply that an instability corresponds to the incoming wave amplitude from infinity having a zero for ω in the upper half-plane. Since Schwarzschild is stable, for $a = 0$ all such zeros must be in the lower half-plane. Assuming that an instability could occur only if a zero migrated smoothly from the lower half plane as a increases from zero, one needs to search only the real axis for various values of a, which we did numerically for the lowest lying angular modes. Paper [95] found a conserved energy for the radial equation (8.5) from the Wronskian of two linearly independent solutions and hence showed that an instability could occur only by a zero migrating in the finite superradiant frequency range between 0 and $m\Omega$. Hartle and Wilkins [103] proved that the assumption of smooth migration was valid, but stability still rested on the numerical search results. Finally, in 1989, Whiting [104] gave a rigorous proof that there could be no exponentially growing modes in Kerr. In the proof, Whiting constructed a "miraculous" conserved quantity with a positive definite integrand.

For most physicists, this was the end of the story. Black holes were unequivocally established as stable objects firmly predicted by general relativity and likely to be found in various astrophysical settings. For the more mathematically inclined, the story was only just beginning.

8.3.2. Linear and nonlinear stability While an unstable mode implies that the system is unstable, showing that all the modes are stable does not necessarily imply that the system is stable. One needs to be sure that any perturbation can be expressed as a superposition of modes. This completeness is guaranteed for self-adjoint problems like Schwarzschild, but requires further work in Kerr. More important, superposing an infinite number of stable Fourier modes does not guarantee that the result is stable. The standard method of proving stability in this case relies on a conserved energy like (8.16). But instead of relying on modes like (8.17) to infer that the energy does not grow as is done in (8.18), one has to find quantitative estimates of the decay of the field in time to show boundedness of the energy.

Showing boundedness of perturbations is called proving linear stability, which is all that is necessary for linear equations. Boundedness of all solutions of a nonlinear system gives nonlinear stability. Often, proving linear stability first makes it easy to prove nonlinear stability.

The first full linear stability proof for Schwarzschild was carried out by Kay and Wald [105], but the Kerr case is still not completely done. See [106–108] for status reports and references.

8.4. Metric reconstruction

In Schwarzschild, the Regge-Wheeler and Zerilli equations directly give the odd- and even-parity perturbations for certain combinations of the metric. Given solutions of these equations, the full metric can be reconstructed from the remaining perturbation equations. In Kerr, by contrast, the perturbation equation (8.2) describes either the Ψ_4 or Ψ_0 component of the Weyl tensor. Since this tensor consists of second derivatives of the metric, reconstructing the corresponding metric perturbation thus requires two integrations. Moreover, finding the complete metric involves solving the remaining Newman-Penrose equations. And finally, Ψ_4 and Ψ_0 have spin weight 2 and so give no information about $l = 0$ and $l = 1$ metric perturbations corresponding to shifts in mass and angular momentum; they must be separately reconstructed.

Chandrasekhar [81] has carried out this reconstruction in an amazing feat of analysis. The procedure is so complicated that it does not seem to have been used, at least in its entirety, in any application. Attention has instead focused on an alternative strategy that uses the analogue of Hertz potentials. This idea was introduced by Chrzanowski [109] and then by a somewhat different route by Cohen and Kegeles [110, 111], following their earlier work on the electromagnetic case [112]. Later work by Wald [113] and Stewart [114] further elucidated the method. The Hertz potential Ψ satisfies the homogeneous master equation (8.2), but it is not the Ψ_0 or Ψ_4 arising from the metric perturbation it generates. Instead, the actual reconstruction procedure is as follows:

(i) Solve the master equation for ψ with the physical boundary conditions and source, where ψ is one of the quantities Ψ_0 or Ψ_4.

(ii) Solve an equation of the form $\mathcal{D}\Psi^* = \psi$, where $\mathcal{D}$ is a certain fourth-order partial differential operator and $*$ denotes complex conjugation. This equation can be solved by separation of variables using spin-weighted spheroidal harmonics. The Hertz potential Ψ must also satisfy the homogeneous master equation (8.2). This procedure was first explicitly carried out for Kerr by Ori [115].

(iii) Compute the metric by applying a certain second-order partial differential operator to Ψ. (This is the formula originally given by Chrzanowski, Cohen, and Kegeles [109–111].)

(iv) Complete the solution by finding and adding the $l = 0$ and $l = 1$ pieces to the metric. This part of the procedure does not seem to have a general formulation, and so far is carried out case-by-case.

The resulting metric is given in the so-called "radiation gauge." When sources are present, the gauge leads to singularities even in vacuum regions, and a great deal of effort has been expended in dealing with these complications.

A nice pedagogical example of metric reconstruction is given in [116]. The principal application of reconstruction has been to the self-force problem for small bodies interacting with Kerr black holes. See [117] for a review and references.

8.5. Black hole quasi-normal modes

If a black hole is perturbed in some way, the radiation produced generally has three components. At early times, there is prompt emission that reflects the details of the source and whose Fourier components basically propagate out along null rays. At very late times, there is "tail" emission produced by backscatter of the prompt radiation off the curvature of spacetime. The amplitude of tail emission decays as a power law in time, as shown originally by Price [63, 64] for the Schwarzschild case. In between these two components the radiation is dominated by the black hole *ringdown*. This can be thought of as radiation produced by the dynamical excitation of the black hole spacetime, which then emits radiation much like a struck bell rings down by emitting certain characteristic tones. Like a real bell, a black hole system is dissipative since the waves carry off energy. Accordingly, the modes have an imaginary component in the frequency that represents the damping rate.

Ringdown was noticed very early in studies of perturbations of Schwarzschild black holes [118, 119]. In 1971 Press [120] introduced the viewpoint that ringdown waves should be thought of as the free oscillations of the black hole and called them quasi-normal modes. Since then many applications of these modes have been found and a huge literature has developed. A comprehensive review is provided in [121], which references a number of other more specialized reviews. Here I will describe just a few important applications.

First, quasi-normal modes are a ubiquitous feature of gravitational radiation from astrophysical black holes, and searches by detectors such as LIGO and VIRGO try to include their effects. In this case we expect that only the low-lying least damped modes are important. We may be lucky enough to measure the frequencies and decay times for two modes from, for example, the inspiral and merger of two black holes. Since the frequency and decay time depend only on the mass and spin of the final black hole, with four measured quantities we would have a very clean test of general relativity, namely that the final state is indeed described by the Kerr metric.

A prescient theoretical application of quasi-normal modes was the paper by York in 1983 [122], who showed how the statistical mechanics of the modes was related to the black hole entropy. Further interest was sparked by Hod [123] in 1998, who used asymptotic values for the highly damped modes to conjecture the spacing of the area spectrum for quantized black holes. Soon after this, Horowitz and Hubeny [124] showed how quasi-normal modes could be used in the AdS/CFT correspondence: modes for black holes in AdS space are related to relaxation times in the dual conformal field theory. Since then there has been an explosion of work exploiting quasi-normal modes to elucidate the AdS/CFT correspondence and quantum gravity (see [121, 125] for reviews and references).

8.6. Analytic solutions

Analytic treatment of the perturbation equations (8.4) and (8.5) was initiated in an important paper by Leaver [126] on Kerr quasi-normal modes. Leaver noted that the equations were similar to those solved by Jaffé in 1934 [127] to determine the electronic spectrum of the hydrogen molecule ion. In particular, the radial equation (8.5) has two regular singular points, at the horizon radii r_+ and r_-, the roots of $\Delta = 0$. It also has an irregular singular point at infinity. Leaver constructed solutions as expansions about the horizon,

$$R(r) \sim \sum_{n=0}^{\infty} a_n z^n, \qquad z \equiv \frac{r - r_+}{r - r_-} \tag{8.19}$$

where the $\sim$ symbol means I have left out a known prefactor function of r for simplicity. The prefactor includes the boundary conditions of ingoing waves at $r = r_+$ and outgoing waves at infinity. Equation (8.19) will be a valid solution all the way to $r = \infty$ only if the series is convergent for any $z \in [0, 1]$. Substituting the series into the radial equation leads to a three-term recurrence relation for the coefficients:

$$\alpha_0 a_1 + \beta_0 a_0 = 0 \tag{8.20}$$

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0, \qquad n = 1, 2, \ldots \tag{8.21}$$

Here the coefficients α_n, β_n, and γ_n depend only on the parameters in the differential equation. Investigating the behavior of a_n in (8.21) for large n shows that of the two linearly independent solutions, only the one for which a_n decreases with n will allow the series to converge. The standard theory of three-term recurrence relations (cf., e.g., [128] or [129]) says that the corresponding solution of (8.21) is the so-called minimal solution. This solution can be found from the continued fraction that results from rewriting (8.21) for the ratio a_{n+1}/a_n and iterating:

$$\frac{a_{n+1}}{a_n} = \frac{-\gamma_{n+1}}{\beta_{n+1} -} \, \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} -} \, \frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3} -} \cdots \tag{8.22}$$

Evaluating (8.22) for $n = 0$ and equating this to a_1/a_0 from (8.20) gives an implicit equation for the characteristic frequency of the normal mode that can be solved numerically. For other values of ω, the equation will not be satisfied.

In the Schwarzschild case, the eigenvalue of the angular equation is known analytically, $A(a\omega = 0) = l(l+1) - s(s+1)$, and is independent of ω. In the Kerr case, the angular equation has to be solved simultaneously with the radial equation to determine A. As a function of $u = \cos\theta$, the angular equation also has two regular singular points and one irregular singular point. Thus it can be handled in exactly the same way as the radial equation, this time as a series in $z = 1 + \cos\theta$. The three-term recurrence relation leads to a continued fraction for the minimal solution and hence an implicit equation for A. So Kerr modes can be found by solving the two implicit equations simultaneously for A and ω.

Leaver [130] also found several other representations of solutions of the perturbation equations. The most useful is an expansion as a series of Coulomb wave functions. The coefficients in the expansion once again lead to a three-term recurrence relation, but this time the expansion index n ranges from $-\infty$ to ∞. These solutions are convergent for $r_+ < r \leq \infty$.

The power series expansions (8.19) valid near the horizon converge quite slowly. Mano, Suzuki, and Takasugi [131] instead found expansions in hypergeometric functions that are valid for $r_+ \leq r < \infty$. The expansion coefficients have the *same* three-term recurrence relation as the Coulomb wave function expansion, which is valid from infinity inward. They made some improvements in [132], and a summary can be found in §4 of [133]. A key element of the analysis was a set of connection formulas that allow one to match the expansion valid near the horizon to the expansion valid at infinity, and thus impose appropriate boundary conditions. A number of applications are given in [133]. A recent application I cannot help mentioning is that of Fujita [134], who computed the analytic post-Newtonian expansion of the gravitational waveform for a particle in circular orbit about a Schwarzschild black hole to 22nd order! This kind of analysis is useful not just for waveforms of extreme mass-ratio inspiral binaries, but also for studying the range of validity of PN expansions, which is not well understood at all.

Instead of solving the angular equation as a power series in $1+\cos\theta$, one can make an expansion in Jacobi polynomials, again leading to a three-term recurrence relation. This method is due to Fackerell and Crossman [135], with some improvements in [136].

Ordinary differential equations like the radial and angular perturbation equations are examples of the confluent Heun equation. There has been growing interest in analyzing the equations from this point of view (see e.g., [137]). A result that emerges from this analysis is that the radial and angular equations are essentially "the same," related by (possibly complex) coordinate transformations. This is the reason that the identities of §8.1 exist for both the angular and radial functions: If they exist for one variable, they must exist for the other.

9. "Explanation" of the miracles of Kerr: the Killing-Yano tensor

Everyone who spends time working with the Kerr metric comes away convinced that that there is something magical about it. There is first the very fact of its existence. We may not be surprised that the special features of Type D metrics allow us to find analytic solutions, but there is no a priori reason why the most general rotating black hole solution should be a member of this class. There are also small surprises, like the unexpectedly simple formula for the angular velocity of a particle in an equatorial circular orbit,

$$\Omega_{\text{circular}} = \pm \frac{M^{1/2}}{r^{3/2} \pm a M^{1/2}}. \tag{9.1}$$

Here the upper and lower signs refer to prograde and retrograde orbits. Only a simple modification of Kepler's Third Law to incorporate rotation is required.

But the real feeling that something is special begins with encountering the unexpected separability of the Hamilton-Jacobi and scalar wave equations. The additional integral of motion provided by Carter's constant makes the geodesic equations completely integrable. Then an even bigger surprise, the decoupling and separability of the equations for electromagnetic, neutrino, and gravitational perturbations. And among the various functions encountered in the solution of these perturbation equations, a panoply of identities and relationships required by the Maxwell or Einstein equations and in fact satisfied by the solutions as new "special functions" of mathematical physics. No wonder Chandrasekhar [81] was led to refer to the "...many properties which have endowed the Kerr metric with an aura of the miraculous."

The existence of t and ϕ constants of the motion or separability follows directly from the stationarity and axisymmetry of the metric, or equivalently from the existence of the two Killing vectors ∂_t and ∂_ϕ. But there is no group theory to account for the separability in r and θ. The separation is characterized by Carter's constant, which is quadratic in the particle momenta and generalizes the total angular momentum of Schwarzschild. In 1970, Walker and Penrose [138] showed that its existence follows from the existence of a symmetric Killing tensor for Kerr that satisfies a generalization of Killing's equation

$$K_{(ab;c)} = 0. \tag{9.2}$$

The Killing tensor, in turn, is the "square" of the antisymmetric Killing-Yano tensor [139, 140], which satisfies

$$K_{ab} = f_{ac} f_b{}^c, \qquad f_{a(b;c)} = 0. \tag{9.3}$$

A metric that admits a Killing-Yano tensor must be Type D [141], but not all Type D metrics actually have one [142].

Carter [143] showed that in a vacuum geometry, a Killing vector or Killing tensor can be used to construct operators that commute with the scalar Laplacian and hence imply its separability. Carter and McLenaghan [144] next constructed an operator from the Killing-Yano tensor that commutes with the operator in the Dirac equation, explaining its separability. However, the separability of the higher spin equations is still mysterious. The separation constant itself has been characterized as the eigenvalue of an operator constructed from the Killing-Yano tensor [145, 146], but commutation relations or similar criteria that would imply separability have not been found. The situation is summarized in [147].

Even if one could show that the Killing-Yano tensor implies all the separability properties of Kerr, it is not clear in what sense this would be an explanation rather than a restatement. Symmetry or group theory is an explanation of separability because they are general and fundamental properties, but the existence of the Killing-Yano tensor seems to be a miracle on the same level as separability.

Another way of seeing why the Killing-Yano tensor explanation is inadequate is to consider the remarkable identities discussed in §8.1. In spherical symmetry, the angular identities are related to the raising and lowering operators for spin-weighted spherical harmonics that follow from group theory. The radial identities then follow by mapping the angular equation to the radial equation, as mentioned at the end of §8.6. In Kerr, however, these identities emerge as unexpected delights, with no known deep reason for their existence.

Many of the above results on "hidden symmetries" for the Kerr metric can be extended to more general spacetimes or to higher dimensions (see [148] for a discussion and references).

10. The solution enters astrophysics

The first Texas symposium in 1963 was spurred by the discovery of quasars and the belief that relativity might have something to do with explaining them, but it was not until a few years later that the discipline of black hole astrophysics was really born. The discovery of pulsars in 1967 [149] and their quick identification as rotating neutron stars [150] introduced the first highly relativistic objects into astrophysics. The launch of X-ray satellites dedicated to astronomy starting with Uhuru in late 1970 led to a spate of discoveries of compact objects accreting matter from companion stars. Among these X-ray sources was Cyg X-1, the first reliable stellar-mass black hole candidate. Soon after this, strong observational evidence began to emerge that quasars and other powerful radio sources at centers of galaxies are powered by supermassive black holes. By now relativistic astrophysics is a vast subject, and here I will focus only on the central role of the Kerr metric and on attempts to verify that Kerr black holes actually exist with all the properties predicted by general relativity.

10.1. Observations of black holes

10.1.1. Masses of stellar-mass black holes For a stellar-mass compact object, the main argument used to claim that it is a black hole is to determine its mass reliably. If this mass is greater than the maximum mass of a neutron star, then the object must be a black hole.

The maximum mass of a neutron star is uncertain because of our lack of understanding of the properties of nuclear matter at high densities. However, it is generally believed to be less than $2.5\,M_\odot$, with an absolute upper bound around $3\,M_\odot$ (for a recent review and references, see [151]). The most massive reliable observed neutron star mass is $2.01 \pm 0.04\,M_\odot$ [152].

All the accurate black hole mass determinations come from X-ray binary systems. Gas flows from the companion star onto a compact object through an accretion disk. (Here compact object means a white dwarf, neutron star, or black hole. A normal star is ruled out by the small size of the binary orbit.) The hot disk emits X-rays. The mass of the compact object is measured by a venerable technique of classical astronomy, measuring the radial velocity curve of the companion by the Doppler shift of its spectral lines. From the velocity curve and Kepler's Third Law one gets the mass function

$$f(M) = \frac{M \sin^3 i}{(1+q)^2}. \tag{10.1}$$

Here i is the orbital inclination of the binary to the line of sight and q is the ratio of companion mass to black hole mass M. Determining M requires further observational and theoretical inputs to pin down i and q. This has been done for about two dozen sources where M is accurately measured to be greater than $3M_\odot$ [153, 154], the maximum neutron star mass.

Note that one can rewrite (10.1) as

$$M = \frac{(1+q)^2}{\sin^3 i} f. \tag{10.2}$$

Since $q \geq 0$ and $\sin i \leq 1$, the *minimum* mass is given by the value of the mass function f, an extremely robust observable. About ten of the known compact X-ray sources have values of f in the range $3 - 8\,M_\odot$. There is no wiggle room to argue about uncertainties in astrophysical modeling; these sources must be black holes.

10.1.2. Spins of stellar-mass black holes Black hole spin measurements have nowhere near the reliability of the precise mass measurements possible using orbital dynamics. Determining the spin requires difficult observations from matter very close to the event horizon, where the metric is most affected by the value of a.

For stellar-mass black holes, the continuum fitting method relies on models of the accretion disk. The gravity produced by the disk is negligible compared with that of the black hole, so the model can be constructed in a fixed Kerr background metric. In the thin disk model, the inner edge of the disk occurs at the innermost stable circular orbit $r_{\rm ISCO}$, a simple function of a [36]. Fitting the continuum spectrum of the disk to the thin-disk model thus gives an estimate of a. About ten black hole spins have been estimated in this way, with values ranging up to near extremal. For more details, see for example [155].

A second method for determining the black hole spin is relativistic X-ray reflection spectroscopy, also called broad iron line spectroscopy. A number of compact X-ray sources show a hard component in the spectrum that comes from a "corona" just above and below the accretion disk. The irradiated disk produces a reflection signature in the X-ray spectrum, most prominently an iron-Kα emission line. The line is Doppler and gravitationally redshifted by different amounts at different radii in the disk, and fitting a model to the radiation gives an estimate of a. More than a dozen spins have been measured in this way. For more details, see [156]. In a few cases measurements have also been made on the same sources by the continuum fitting method, and the results are roughly consistent. However, both methods are still subject to quite large uncertainties.

10.1.3. Masses of supermassive black hole Estimating the mass of a supermassive black hole can be done in several ways. For example, the radiation is produced by accretion, with the luminosity limited by the Eddington limit at which the outward radiation pressure is balanced by the input pull of gravity,

$$L_{\rm E} = \frac{4\pi c G M m_p}{\sigma_{\rm T}} \sim 10^{38} \frac{M}{M_\odot}\,{\rm erg\,s^{-1}}. \tag{10.3}$$

Here m_p is the proton mass and $\sigma_{\rm T}$ the Thomson cross section for electron scattering. Bright quasars have luminosities $L \sim 10^{46}\,{\rm erg\,s^{-1}}$, so $M \gtrsim 10^8 M_\odot$.

For such an object to be a black hole, its size has to be sufficiently small. Quasars show variability on a timescale of days. Setting this timescale to the light crossing time implies that the object is smaller than a few light days, or $\sim 100M$. Even tighter limits for some sources come from Doppler measurements of X-ray lines, showing that matter is orbiting at a substantial fraction of c [157].

The black hole at the center of our Galaxy, Sgr A*, has a very well-determined mass. (For a review of the arguments that this radio source is the black hole, see, e.g., [158].) The most precise value to date is $(4.26 \pm 0.14) \times 10^6\, M_\odot$ [159], determined from a combined analysis of individual stellar orbits around the black hole and the dynamics of the nuclear star cluster.

10.1.4. Spins of supermassive black holes Spins of supermassive black holes are typically determined using the X-ray reflection spectroscopy method, which requires the presence of a bright accretion disk. (The continuum fitting method is difficult to apply in the supermassive case.) About two dozen spins have been estimated [156, 160], many suggesting that rapidly spinning massive Kerr black holes do actually exist.

Since Sgr A* does not have a bright X-ray emitting accretion disk, the prospects of distinguishing between Schwarzschild and Kerr for its metric are not good at present.

10.2. Energy extraction from astrophysical black holes

What exactly is the mechanism by which supermassive black holes at the centers of galaxies power such energetic phenomena as quasars, AGNs, and relativistic jets? And what about gamma-ray bursts and relativistic outflows in galactic X-ray binaries? It is widely believed that the Blandford-Znajek process [161] is one of the most promising mechanisms in many of these cases.

The Blandford-Znajek process is related to the Penrose process, in that rotational energy is extracted from the black hole because negative energy can flow down the hole if there is an ergosphere. However, it is a purely electromagnetic process—plasma is present to support the fields, but its inertia and energy are negligible. Moreover, the presence of plasma allows the process to work even with stationary fields.

In the 1990's, there was some controversy about whether the Blandford-Znajek process was actually theoretically viable, but that has been resolved and the process itself is now well understood (see, e.g., [162–166]). However, complete and convincing models of the accretion, jet formation, and energy emission are still beyond the reach of current large-scale numerical simulations.

10.3. Are astrophysical black hole candidates really black holes?

The astronomical observations described above are all of the form "there are regions of the Universe with a lot of mass in a small volume." *If* we assume general relativity is correct, then these must be black holes. But there are still no convincing observations that tell us directly that these objects have the bizarre properties predicted by relativity, such as event horizons or an exterior geometry described by the Kerr metric. As long as we depend only on electromagnetic observations, we will be at the mercy of the uncertainties in the theoretical models of the radiation.

This situation is about to change. Within a few years, gravitational wave detectors such as LIGO and VIRGO will reach a sensitivity at which they are expected to detect waves from the inspiral and merger of binary systems containing black holes and neutron stars. In particular, waves from a black hole-black hole merger will probe for the first time the strong-field regime as the black holes merge. Comparison with numerical solutions of the full Einstein equations will provide powerful tests of general relativity in this regime. And measurement of the ringdown waves will show directly the settling down of the final state to a Kerr black hole. Or maybe to something unexpected . . . .

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