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# Construction of energy-momentum tensor of gravitation

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## Abstract

We argue the possibility that the gravitational energy-momentum tensor is constructed in general relativity through the Noether theorem. In particular, we explicitly demonstrate that the constructed quantity can vary as a tensor under the general coordinate transformation. Furthermore, we verify that the energy-momentum conservation is satisfied because one of the two indices of the energy-momentum tensor should be in the local Lorentz frame. It is also shown that the gravitational energy and the matter one cancel out in certain space-times.

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## I. INTRODUCTION

The way of defining the gravitational energy-momentum tensor is a long standing issue in general relativity, and it has not completely been solved yet. There have been proposed several definitions of the gravitational energy momentum tensor. However, those forms are different with each other, and it is considered that they are not appropriate tensors [1].

To derive the expression of the gravitational energy-momentum tensor is a significant fundamental problem in general relativity. Hence, it is strongly expected that such an explicit representation on the gravitational energy-momentum tensor in general relativity can be useful and helpful to further consider the so-called modified gravity theories including $F(R)$ gravity, which have widely been studied to explain the late-time cosmic acceleration, i.e., the dark energy problem (for recent reviews on the dark energy problem and modified gravity theories, see, for example, Refs. [2–4]).

In this paper, by using the Noether theorem, we attempt to construct a proper form of a gravitational energy-momentum tensor in general relativity. What is the most crucial point in our approach is that the general coordinate transformation is described with the tetrad. The energy and momentum of gravitation are induced by the tetrad. It has an index of the local Lorentz frame, so that the energy-momentum conservation law can be satisfied. The gravitational energy-momentum tensor has also been examined in Ref. [5], and its further investigations including the energy-momentum conservation law have been executed in Ref. [6]. Indeed, however, its explicit form was not been presented. In this work, therefore, we attempt to explicitly construct the expression of the gravitational energy-momentum tensor.

Particularly, as concrete examples, we investigate the gravitational energy of the homogeneous and isotropic Friedman-Lemaître-Robertson-Walker (FLRW) universe with its different curvatures. Moreover, we analyze an energy in an outer region of the Schwarzschild black hole with its mass $M$, whose metric describes the assymptotically flat, vacuum, and spherically symmetric space-time. As a consequence, we find that the energy becomes $-M/2$. It is known that the gravitational energy takes the negative signature of the matter energy [7]. We use units of $k_{\mathrm{B}} = c = \hbar = 1$, and the Planck mass is given by $M_{\mathrm{Pl}} = G_{\mathrm{N}}^{-1/2} = 1.2 \times 10^{19}$ GeV, where $G_{\mathrm{N}}$ is the gravitational constant.

The organization of the paper is the following. In Sec. II, we analyze the gravitational energy-momentum tensor in general relativity. Especially, we present the Noether theorem, and with it the gravitational energy-momentum tensor is constructed. In Sec. III, we derive the gravitational energy in the flat, closed, and open FLRW universes and the Schwarzschild black hole. We find that in these space-times, the sum of the gravitational energy and the matter one becomes zero. Conclusions are presented in Sec. IV.

## II. GRAVITATIONAL ENERGY-MOMENTUM TENSOR IN GENERAL RELATIVITY

In this section, we first formulate the Noether theorem. By using it, we next construct the gravitational energy-momentum tensor.

### A. Noether theorem

The action describing general relativity is represented as

$$I = \frac{1}{16\pi}\int d^4xeR \tag{II.1}$$

$$= \frac{1}{16\pi}\int d^4xe\left(-\frac{1}{4}K_{\mu\nu\lambda}K^{\mu\nu\lambda} - \frac{1}{2}K_{\mu\nu\lambda}K^{\lambda\nu\mu} + K^{\mu\nu}{}_{\mu}K^{\rho}{}_{\nu\rho} + 2\nabla_{\mu}K^{\nu\mu}{}_{\nu}\right) \tag{II.2}$$

$$= \int d^4x\left(F + \partial_\lambda D^\lambda\right)\,, \tag{II.3}$$

where $e$ is a determinant of a tetrad, and the gravitational constant $G_{\mathrm{N}}$ has been set to be unity. Here, we have defined the torsion tensor $K_{\mu\nu\lambda} \equiv e_{a\lambda}(e^a_{\mu},_{\nu} - e^a_{\nu},_{\mu})$ and $K^{\mu\nu\lambda} \equiv g^{\mu\rho}g^{\nu\sigma}e^{\lambda}_a(e^a_{\rho},_{\sigma} - e^a_{\sigma},_{\rho})$, where the comma denotes the partial derivative of $e^a_{\mu},_{\nu} \equiv \partial_\mu e^a_\mu$. The Latin index means the local Lorentz coordinate, whereas the Greece index shows the world coordinate. Both the Latin and Greece indices run over $0, 1, 2, 3$. Moreover, $F$ is the volume term, and $\partial_\lambda D^\lambda$ is the surface one.

It is seen that under the following general coordinate transformation

$$\begin{aligned} x'^{\mu} &= x^\mu + \delta x^\mu \\ &= x^\mu + e^\mu_a \xi^a\,, \end{aligned} \tag{II.4}$$

the action in Eqs. (II.1)–(II.3) is invariant. Here, $\xi^a$ is an arbitrary function, and it becomes zero on a boundary. Accordingly, the Noether theorem is satisfied as follows.

$$\partial_\mu\left[\delta x^\mu(F + \partial_\lambda D^\lambda)\right] + \delta^L F + \partial_\lambda \delta^L D^\lambda = 0\,, \tag{II.5}$$

where $\delta^L F$ and $\partial_\lambda \delta^L D^\lambda$ are the Lie derivatives of $F$ and $\partial_\lambda D^\lambda$, respectively. Here, we have used the fact that the Lie derivative and the usual derivative can be commutative with each other. The Lie derivatives of $F$ and $D^\lambda$ are given by

$$\delta^L F = \frac{\partial F}{\partial e^a_\nu}\delta^L e^a_\nu + \frac{\partial F}{\partial e^a_{\nu,\mu}}\delta^L e^a_{\nu,\mu} \,, \tag{II.6}$$

$$\delta^L D^\lambda = \frac{\partial D^\lambda}{\partial e^a_\nu}\delta^L e^a_\nu + \frac{\partial D^\lambda}{\partial e^a_{\nu,\mu}}\delta^L e^a_{\nu,\mu} \,. \tag{II.7}$$

In addition, the Lie derivative of the tetrad reads

$$\delta^L e^a_\nu = -e^a_\sigma \partial_\nu (e^\sigma_b \xi^b) - e^\sigma_b \xi^b \partial_\sigma e^a_\nu \tag{II.8}$$

$$= -\xi^b (e^\sigma_b \partial_\sigma e^a_\nu - e^\sigma_b \partial_\nu e^a_\sigma) - \partial_\nu \xi^a \tag{II.9}$$

$$= -\xi^b K_{\nu b}{}^a - \partial_\nu \xi^a \,. \tag{II.10}$$

By substituting these expressions of the Lie derivatives (II.6), (II.7), and (II.10) into the Noether theorem described in Eq. (II.5), we acquire

$$\partial_\mu (e t^\mu_a \xi^a - V^{\mu\nu}_a \partial_\nu \xi^a - W^{\mu\nu\lambda}_a \partial_\lambda \partial_\nu \xi^a) + e T^\nu_a (\xi^b K_{\nu b}{}^a + \partial_\nu \xi^a) = 0 \,, \tag{II.11}$$

where we have used the gravitational field equation, and $T^\nu_a$ is the energy-momentum tensor of matter. Furthermore, $t^\mu_a$, $V^{\mu\nu}_a$, and $W^{\mu\nu\lambda}_a$ are defined as

$$e t^\mu_a \equiv e^\mu_a (F + \partial_\nu D^\nu) + \frac{\partial D^\mu}{\partial e^b_\nu} K_{a\nu}{}^b + \frac{\partial D^\mu}{\partial e^b_{\nu,\lambda}} \partial_\lambda K_{a\nu}{}^b - \frac{\partial F}{\partial e^b_{\nu,\mu}} K_{\nu a}{}^b \,, \tag{II.12}$$

$$V^{\mu\nu}_a \equiv \frac{\partial F}{\partial e^a_{\nu,\mu}} + \frac{\partial D^\mu}{\partial e^a_\nu} + \frac{\partial D^\mu}{\partial e^b_{\rho,\nu}} K_{\rho a}{}^b \,, \tag{II.13}$$

$$W^{\mu\nu\lambda}_a \equiv \frac{\partial D^\mu}{\partial e^a_{\nu,\lambda}} \,. \tag{II.14}$$

Since $\xi^a$ is an arbitrary function, Eq. (II.11) can be reduced to the four equations, each of which are proportional to $\xi^a$, ${\xi^a}_{,\nu}$, ${\xi^a}_{,\nu\mu}$, and ${\xi^a}_{,\mu\nu\lambda}$. These equations are

$$\partial_\mu (e t^\mu_a) + e T^\nu_b K_{\nu a}{}^b = 0 \,, \tag{II.15}$$

$$e t^\mu_a - \partial_\nu V^{\nu\mu}_a + e T^\mu_a = 0 \,, \tag{II.16}$$

$$(V^{\mu\nu}_a + \partial_\rho W^{\rho\mu\nu}_a) \partial_\mu \partial_\nu \xi^a = 0 \,, \tag{II.17}$$

$$W^{\mu\nu\rho}_a \partial_\mu \partial_\nu \partial_\rho \xi^a = 0 \,. \tag{II.18}$$

It can easily be shown that $V^{\mu\nu}_a$ is antisymmetric with respect to $(\mu, \nu)$. Thus, the derivative of Eq. (II.16) is written as

$$\partial_\mu e (t^\mu_a + T^\mu_a) = 0 \,. \tag{II.19}$$

## B. Energy-momentum tensor of gravitation

It is natural to consider that $t_a^\mu$ will behave as a gravitational energy momentum tensor, because $T_a^\mu$ is the energy momentum tensor of matter. Hence, Eq. (II.19) implies the conservation law of the total energy momentum. This equation is invariant under the general coordinate transformation, because the energy-momentum tensor has only one world coordinate index, Eq. (II.19) is invariant under the general coordinate transformation in Eq. (II.4). It follows from Eq. (II.12) that the gravitational energy momentum tensor is expressed as

$$\begin{aligned}16\pi t_a^\mu &= e_a^\mu R + 2\nabla_\nu K_a{}^{\mu\nu} - 2\nabla^\mu K_{a\nu}{}^\nu + (K_a{}^{\mu\nu} - K^\nu{}_a{}^\mu + K^{\mu\nu}{}_a)K_{\nu\lambda}{}^\lambda \\ &\quad + K_{a\nu\lambda}k^{\mu\nu\lambda} + K_{a\lambda\nu}K^{\mu\nu\lambda} - K_{\nu\lambda a}K^{\mu\nu\lambda}\,. \end{aligned} \tag{II.20}$$

It is remarked that with the following several identities

$$R = -\frac{1}{4}K_{\mu\nu\lambda}K^{\mu\nu\lambda} - \frac{1}{2}K_{\mu\nu\lambda}K^{\lambda\nu\mu} + K^{\mu\nu}{}_\mu K^\rho{}_{\nu\rho} + 2\nabla_\mu K^{\nu\mu}{}_\nu\,, \tag{II.21}$$

$$\begin{aligned}R_a^\mu &= -\frac{1}{2}\nabla_\nu K^{\mu\nu}{}_a - \frac{1}{2}\nabla_\nu K_a{}^{\nu\mu} - \frac{1}{2}\nabla_\nu K^\mu{}_a{}^\nu - \nabla_a K^{\nu\mu}{}_\nu \\ &\quad + \frac{1}{2}K^{\lambda\nu\mu}K_{a\lambda\nu} - \frac{1}{4}K^{\lambda\nu\mu}K_{\lambda\nu a} + \frac{1}{2}\left(K_a{}^{\mu\nu} - K^{\mu\nu}{}_a + K^\nu{}_a{}^\mu\right)K_{\nu\rho}{}^\rho\,,\end{aligned} \tag{II.22}$$

and

$$\nabla_\nu K_a{}^{\mu\nu} + \nabla_a K^{\mu\nu}{}_\nu + \nabla^\mu K_{\nu a}{}^\nu + K_a{}^{\mu\nu}K_{\nu\rho}{}^\rho + \frac{1}{2}K_{a\rho\nu}K^{\rho\nu\mu} - \frac{1}{2}K_{\rho\nu a}K^{\mu\rho\nu} = 0\,, \tag{II.23}$$

it is possible to rewrite the gravitational energy-momentum tensor in several different forms.

From Eqs. (II.15) and (II.19), we find

$$\partial_\mu(eT_a^\mu) = eT_b^\mu K_{\mu a}{}^b\,. \tag{II.24}$$

The energy-momentum tensor of matter $T_a^\mu$ can be replaced with the Einstein tensor $G_a^\mu$ through the gravitational field equation $G_a^\mu = 8\pi T_a^\mu$. Therefore, Eq. (II.24) becomes

$$e(\partial_\mu G_a^\mu + \Gamma^\nu_{\nu\mu}G_a^\mu - K_{\nu a}{}^b G_b^\mu) = 0\,. \tag{II.25}$$

Since the Ricci's rotation coefficient is given by

$$\Omega_{ab\mu} = \frac{1}{2}(K_{ab\mu} - K_{b\mu a} - K_{\mu ab})\,, \tag{II.26}$$

Eq. (II.25) reads

$$\nabla_\mu G_a^\mu = 0\,. \tag{II.27}$$

Hence, the Bianchi identity can be derived.

It is the novel point to present the explicit representation of the gravitational energy-momentum tensor, in comparison with the preceding work [5], in which the expression of the gravitational energy-momentum tensor was not given.

## III. GRAVITATIONAL ENERGY

In this section, we demonstrate that the gravitational energy and the matter one cancel out, particularly, the FLRW universe with different curvatures and the Schwarzschild spacetime.

### A. FLRW universe

We study the gravitational energy in the FLRW universe with the metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right] \,, \tag{III.1}$$

where $a(t)$ is the scale factor, $K$ is the cosmic curvature, and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ is the line element on the two-dimensional sphere. If $K = 0$, the universe is flat, while if $K > 0 (< 0)$, it is closed (open). In the FLRW background, the $(t, t)$ component of the gravitational energy-momentum tensor is represented as

$$t_{tt} = t^t_0 e^0_t g_{tt} \tag{III.2}$$

$$= \frac{3}{8\pi}\left(-\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) \,, \tag{III.3}$$

where the dot denotes the time derivative, and the Hubble parameter $H$ is defined as $H = \dot{a}/a$. The total gravitational energy is given by

$$E_{\rm G} = \int \sqrt{\gamma} t_{tt} dr d\theta d\phi \tag{III.4}$$

$$= -\frac{3}{2} a(\dot{a}^2 + K) \int \frac{r^2}{\sqrt{1-Kr^2}} dr \,, \tag{III.5}$$

where $\gamma$ is the determinant of the three-dimensional space metric.

*1. $K = 0$ (Flat universe)*

When $K = 0$, the total gravitational energy is written as

$$E_{\mathrm{G}} = -\frac{3}{2} a\dot{a}^2 \int_0^{\infty} r^2 dr \,. \tag{III.6}$$

By combining this equation with the Friedmann equation

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}\rho \,, \tag{III.7}$$

where $\rho$ is the energy density of the dust matter, we obtain

$$E_{\mathrm{G}} = -4\pi\rho_0 \int_0^{\infty} r^2 dr \,. \tag{III.8}$$

Here, we express $\rho = \rho_0/a^3$ with $\rho_0$ the value of $\rho$ at the present time, when the scale factor is taken as $a = a_0 = 1$. On the other hand, the $(t, t)$ component of energy-momentum tensor of matter reads $\rho = \rho_0/a^3$. Consequently, the total energy of matter is described as

$$\begin{aligned} E_{\mathrm{m}} &= \int_0^{\infty} \sqrt{\gamma} T_{tt} dr d\theta \, d\phi \\ &= 4\pi\rho_0 \int_0^{\infty} r^2 dr \,. \end{aligned} \tag{III.9}$$

By comparing Eq. (III.8) with Eq. (III.9), it is clearly seen that the gravitational energy and the matter energy cancel out, because the difference between them is only the signature and their absolute values are the same with each other. This is consistent with the investigations in Ref. [7].

*2. $K > 0$ (Closed universe)*

For $K > 0$, the gravitational energy is

$$\begin{aligned} E_{\mathrm{G}} &= -\frac{3}{2} a \left( \dot{a}^2 + \frac{K}{a^2} \right) \int_0^{r_{\mathrm{c}}} \frac{r^2}{\sqrt{1 - Kr^2}} dr \\ &= -\frac{3\pi}{8} \frac{a(\dot{a}^2 + K)}{K^{3/2}} \,, \end{aligned} \tag{III.10}$$

where $r_{\mathrm{c}}$ is the radius of the curvature, and we have $K = 1/r_{\mathrm{c}}^2$. With the Friedmann equation

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}\rho - \frac{K}{a^2} \,, \tag{III.11}$$

we get

$$E_{\mathrm{G}} = -\pi^2 r_{\mathrm{c}}^3 \rho_0 \,. \tag{III.12}$$

Hence, the value of the matter energy is equal to the positive signature of $E_{\mathrm{G}}$ in Eq. (III.12), i.e., $|E_{\mathrm{G}}|$.

### *3. $K < 0$ (Open universe)*

In the case of $K < 0$, the gravitational energy is given by

$$E_{\mathrm{G}} = -4\pi\rho_0 \int_0^\infty \frac{r^2}{\sqrt{1-Kr^2}} dr \,. \tag{III.13}$$

The energy of matter is also equivalent to the positive signature of value of $E_{\mathrm{G}}$ in Eq. (III.13).

As a result, in all the cases of the FLRW universe, the gravitational energy and the energy of matter cancel out. The energy conservation law means the total energy is constant. However, the examples shown above indicate the fact that the total energy vanishes.

## B. Schwarzschild black hole

As the last example, we explore the energy in the outer region of the Schwarzschild black hole. The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1-2M/r} dr^2 + r^2 d\Omega^2 \,. \tag{III.14}$$

The $(t, t)$ component of the energy-momentum tensor for the Schwarzschild black hole is represented as

$$t_{tt} = -\frac{M^2}{8\pi r^4} \,. \tag{III.15}$$

Therefore, the total gravitational energy in the outer region of the Schwarzschild black hole becomes

$$\begin{aligned} E_{\mathrm{G}} &= \int_{2M}^\infty \sqrt{\gamma} t_{tt} d^3x \\ &= -\frac{M^2}{2} \int_{2M}^\infty \frac{1}{\sqrt{r-2M} r^{3/2}} dr \\ &= -\frac{M}{2} \,. \end{aligned} \tag{III.16}$$

This is the energy seen by the observer who is in the rest frame in the infinite distance.

## IV. CONCLUSIONS

In the present paper, with the Noether theorem, we have attempted to construct a gravitational energy-momentum tensor in general relativity. Indeed, we have examined that under the general coordinate transformation, the quantity constructed in our approach can behave as a tensor. We have also confirmed that the energy-momentum conservation law can be met. It originates from the fact that in the expression of the energy-momentum tensor, one of the two indices should be the one in terms of the local Lorentz frame. In addition, we find that the gravitational energy and the matter ones cancel out in the homogeneous and isotropic FLRW universe and the Schwarzschild space-time.

It should be emphasized that the most significant idea of our construction method is to describe the general coordinate transformation by using the tetrad. Furthermore, the tetrad induces the gravitational energy and momentum and has an index of the local Lorentz frame. Thus, the energy-momentum conservation law can be met.

If the gravitational energy-momentum tensor is multiplied by a tetrad, both of its two indices will be in the world coordinate. As a consequence, the gravitational energy-momentum tensor could not be symmetric. In this sense, it might be difficult to consider that general relativity is a complete theory of gravitation.

Regarding the tetrad, it is meaningful to mention that as an alternative description of gravity to general relativity, the so-called teleparallelism has attracted much attention in the literature. In the teleparallelism, the gravity theory is written with the torsion scalar $T$ constructed with the Weitzenböck connection, and not the scalar curvature $R$ constructed with the Levi-Civita connection [8] (for a recent review, see, e.g., [9]). The torsion scalar is derived from the torsion tensor described by using the tetrad, as shown in Sec. II A. It has been indicated that in an extended theory of teleparallelism, so-called $F(T)$ gravity, the inflationary universe [10] and the dark energy dominated stage [11] can be realized (for more detailed explanations and references, see [3, 4, 12]). In fact, the energy and momentum have been explored in the Poincare gauge theory [13], and the energy-momentum conservation law has recently been discussed in teleparallelism [14]. In the light of such a recent study on teleparallelism, it is considered that there exist the cases in which the tetrad is much more useful to describe the theory of gravitation.

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