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General Relativity for Pedestrians - First 6 lectures

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Even after 100 years, general relativity and gravitational radiation continue to create great tidal waves among physicists and undergraduate students. These notes are based on my lectures on various occasions in the University of Delhi, immediately after the exciting direct detection of gravitational waves. In the last section, I address the issue of black hole thermodynamics in connection with the gravitational wave event GW150914, and show that this event is consistent with Hawking's black hole area theorem.

I. INTRODUCTION

Gravity is universal. Everything creates gravity as well as gets affected by gravity since everything has mass (or, equivalently, energy). However, unless the mass of an object is very large, the gravity it generates is very weak.

According to Newton's laws of gravity, acceleration of a test particle due to the gravitational field of a massive object, is proportional to latter's mass, is directed towards the massive body, is inversely proportional to the square of the distance between the two objects, and furthermore, is independent of the test particle's mass.

Newton, with a flash of brilliance, had realized that Moon's orbiting of the Earth is nothing but its continuous fall under Earth's gravity. He estimated from the Moon's orbital period of about 28 days that its acceleration directed towards us due to Earth's gravity is smaller than the acceleration of an apple falling on Newton's head by a factor of square of the ratio of Earth's radius to the Moon-Earth distance. With a leap of generalization, Newton deduced the inverse square law for gravity.

But Newton's theory is inconsistent with special theory of relativity (STR). If the Sun were to disappear at this instant then Newton's theory predicts that at this very instant the Earth will fly off tangentially (in the absence of a centripetal force). But according to STR, no information can travel faster than the speed of light, so the disappearance of Sun cannot instantaneously affect Earth's trajectory.

Einstein corrected the situation by proposing in 1915, about hundred years back, a consistent theory of gravity through his theory of general relativity. General theory of relativity (GTR) is a relativistic theory of gravitation. GTR is based on the observation that the trajectory of a test particle in any arbitrary gravitational field is independent of its inertial mass m (as the acceleration does not dependent on m), and therefore, it must be the geometry of space-time that determines test particle trajectories.

Note that for no other force, acceleration of a test particle is independent of its inertial mass (e.g. in the presence of electromagnetic fields, acceleration of a test charge is proportional to the ratio of its charge to mass).

II. GRAVITY, INERTIAL FRAMES AND EQUIVALENCE PRINCIPLE

In an inertial frame, according to Newtonian laws of gravity, the magnitude of gravitational force between two objects 1 and 2, separated by a distance d, is given by $F = -\frac{GM_1M_2}{d^2}$, where M_1 and M_2 are the gravitational masses. Gravitational mass M plays the role of gravitational charge. This is analogous to the Coulombic case of electric force between electric charges. But the magnitude of acceleration of object 1 due to this gravitational force is $a = \frac{F}{m_1}$ where m_1 is the inertial mass (which appears in $F = ma$ or momentum $p = mv$, etc.).

But from experiments we know that a is independent of m_1 (e.g. Galileo's, and later the torsion balance experiments). In other words, the gravitational acceleration of a test particle is independent of its inertial mass. This is called the weak equivalence principle. Because of this, the gravitational mass M_1 divided by the inertial mass m_1 has to be a constant for all objects. Hence, we can choose units for masses so that this constant has the value 1.

But what is an inertial frame? One operational way of defining an inertial frame is that it is a frame of reference in which if there is no real force acting on an object, then the object either remains at rest or it moves with a uniform velocity. Such a definition rules out an accelerating frame to be an inertial frame.

But the key condition is that there should be no real force acting on the object. One can always shield it from electromagnetic forces, and weak and nuclear forces are anyway short-ranged. But what about gravity? Anything that has energy has mass too, and therefore will be a source of Newtonian gravity. So, how does one create a frame that has no gravity in it?

Einstein, with his brilliant insight, offered an ingenious solution to this predicament. Basically, he employed the Galilean-Newtonian weak equivalence principle which states that in a given region, acceleration of a body due to external gravity is independent of its inertial mass. Imagine that a small bundle of test particles are freely falling in an arbitrary gravitational field. Since their accelerations due to gravity are nearly identical as their relative separations are small, if one were to sit on one such particle and observe the rest, one would find that the other test particles are freely floating as though gravity has simply disappeared!

This is Einstein's principle of equivalence according to which no matter how strong or how time varying the gravity is, one can always choose a small enough frame of reference, for a sufficiently small time interval such that gravity vanishes in this frame. So, one has obtained a truly inertial frame, albeit of a limited size one! In other words, from Einstein's argument, no matter where, one can always construct a local inertial frame, the size of the frame depending on the scale on which the gravity varies. However, we have a queer situation here: according to a freely falling observer A, there is no gravitational force in her/his neighbourhood, while on the other hand, according to an outside observer B who is at rest on the surface of the earth, there is gravity acting on the falling observer. In the Newtonian paradigm, the existence of a genuine force cannot depend on the choice of frames of reference.

To highlight the above point further, let us look at Einstein's equivalence principle from another angle. Consider a frame of reference that is far removed from sources of gravitation so that there is no external gravity felt by an observer C anywhere in this frame. But if this frame C is accelerating with respect to an inertial frame (i.e. C is a non-inertial frame), then the observer C will experience a pseudo-gravitational force. No measurement in C can distinguish between real gravity and the pseudo-gravity if weak equivalence principle is correct. Is gravity then a 'real force'? We will see shortly how this perplexing issue is resolved in Einstein's GTR.

III. GRAVITY AND SPECIAL THEORY OF RELATIVITY

Newton's theory of gravitation also demands that the gravitational force be instantaneously transmitted by the source to the test particle, since it is inversely proportional to the square of the instantaneous separation between the two. Instant transmission is unsatisfactory, as Einstein's special theory of relativity demands that no physical effect can propagate faster than $c = 3 \times 10^8 m\ s^{-1}$, the speed of light in vacuum or, for that matter, speed of any particle with zero rest mass. This ensues from the relativistic expression for energy E of a free particle with rest mass m given by,

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1}$$

From the above, it is evident that if $v > c$, the energy becomes pure imaginary, ruling out faster than light motions. Clearly, gravitational theory needs to incorporate relativity. But, how? The clue comes from Einstein's version of equivalence principle. We have seen in the previous section that weak equivalence principle guarantees that in the presence of gravitation it is always possible to choose a limited size frame of reference for a short enough time in which gravity disappears (e.g. a freely falling frame). This limited region constitutes a local inertial frame of reference, so that a Cartesian coordinate system can be set up for specifying spatial coordinates here, and clocks can be arranged to measure proper time. Such a coordinate system is referred to as the Minkowskian coordinate system. Therefore, in this local inertial frame (LIF), the laws of physics (other than the gravitational phenomena) must take the same form as they do in special theory of relativity. The proper distance ds between two nearby events in the LIF with space-time coordinates $x^\mu = (ct, x, y, z)$ and $x^\mu + dx^\mu = (ct + cdt, x + dx, y + dy, z + dz)$ is evaluated, as in STR, using,

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \equiv \eta_{\mu\nu} dx^\mu dx^\nu \ . \tag{2}$$

Note that in eq.(2), x^i, i=1,2,3 are the Cartesian coordinates of the event, and $\eta_{\mu\nu}$ is the Minkowski metric with $\eta_{00} = 1 = -\eta_{ii}$, rest of the off-diagonal components of the metric being zero. Einstein summation convention has been used in eq.(2), so that repetition of Greek indices imply summation over 0,1,2 and 3.

(From now on we will adopt the Einstein summation convention wherein whenever a Greek index repeats in an expression it means that the expression is being summed over with the index running from 0 to 3.)

STR not only proclaimed that time is the fourth dimension but also necessitated a departure from the Euclidean notion of distances. For events that are not causally connected ds^2 is negative, which was unthinkable in Euclidean paradigm.

In other words, by going over to a small freely falling frame and choosing a locally inertial or Minkowskian coordinate system, one manages not only to make gravity vanish locally but also express the non-gravitational laws of physics exactly as one does in special relativity. But, what about the laws pertaining to gravitation itself? And, what if one requires to express laws of physics over larger regions of space-time?

Let us first deal with the simplest and hypothetical 'gravitational' set up - the case of uniform and static gravity, so that the acceleration vector due to gravity is same everywhere, and at all times. But this situation, according to the equivalence principle is identical to zero gravity case, for, one has to just consider a freely falling reference frame as large as and for as long as one wants, and in this frame gravity simply vanishes. Hence, one can choose Minkowski coordinates globally and eq.(2) describes the space-time geometry everywhere in such a frame. Thus, uniform gravity everywhere is equivalent to zero gravity.

Our next case is: gravity around a massive, spherically symmetric body of radius R and mass M. From the Newtonian point of view, the acceleration due to gravity caused by it at any external point P, is inversely proportional to the distance between P and the centre of the massive object, and is directed towards the centre. Now, if one considers a large freely falling frame (LFFF) at an initial distance $d \gg R$, then does gravity totally vanish in this frame?

Clearly the answer is no. For, if one takes two test particles 1 and 2 separated by a vector $\vec{L}$, that are freely falling along with this LFFF, then if $\vec{L}$ is perpendicular to the radial direction of fall, an observer in LFFF will notice 1 and 2 to be accelerating towards each other with a magnitude,

$$a_{12} \approx \frac{G\ M\ L}{d^3} \tag{3}$$

because each of the particles will be accelerating radially towards the centre of the massive body.

On the other hand, if $\vec{L}$ was along the radial direction of the free fall, the observer in LFFF would measure 1 and 2 to be accelerating away from each other with a magnitude given by eq.(3), as the particle nearer to the massive object would be falling with a greater acceleration than the one farther away. These are nothing but instances of tidal acceleration, ubiquitous whenever gravity is non-uniform.

Although gravitational acceleration vanishes in a local inertial frame (LIF), tidal acceleration does not. It is just that in a LIF, the magnitude of $\vec{L}$ is small as the frame itself is of limited size, so that according to eq.(3), the value of the tidal acceleration is negligibly small here. But when the frame is large, its different parts encounter varying degree of tidal stretching or tidal compression. For instance, we do not experience Sun's gravity as Earth is freely falling towards the Sun.

Nevertheless oceans exhibit high and low tides, since our planet is large enough for Sun's tidal forces to be non-negligible. The above example shows that, in general, one cannot eliminate the effects of gravitation entirely. The LIFs, however, are very useful through the use of STR, for the extraction of physical meanings of various mathematical expressions.

Since one cannot have in general global inertial (i.e. Minkowski) coordinates in the presence of gravity, it is necessary to develop a formalism that employs arbitrary coordinates like curvilinear coordinates in the analysis. One can motivate the necessity of using coordinate systems other than the Minkowskian ones, from a physical standpoint.

Consider the case of a sufficiently large reference frame that is made up of 3-dimensional Cartesian grid of standard rods with clocks arranged at their intersections. Such a framework of Minkowskian coordinate system cannot be maintained as a LFFF, when there is non-uniform gravitation present, because of the following reason.

From STR, the condition that nothing can travel faster than c implies that no object can be absolutely rigid. Otherwise, one could simply transfer energy (and therefore, signals) from one spatial point to another with infinite speed, just by tapping one end of a long 'rigid' rod causing the other end to move instantaneously. Now, non-uniform gravity would mean that different portions of the LFFF would fall with different accelerations, leading to stretching and compression of the (initially) cubical grid of rods and clocks (forces other than gravity will enter the analysis), so that it is no longer possible to maintain a global Minkowskian coordinate system in any LFFF. Curvilinear coordinates, therefore, become indispensable in relativistic gravitational physics.

In STR, square of the proper (i.e. Lorentz invariant) distance between any two infinitesimally events is given by eq.(2) when Minkowskian coordinates are chosen. Preceding arguments make it clear that when gravitation is included, one would need to modify eq.(2) and, instead of the Minkowski metric, one would require a general metric tensor. Similarly, the concept of tidal acceleration has to be made precise from the point of view of arbitrary frames of reference that use general coordinate systems.

IV. CURVILINEAR COORDINATES, SCALAR, VECTOR AND TENSOR FIELDS

Let an **event** occur at some space-time point P, which is assigned a coordinate $x^\mu(\mathrm{P})$ by an observer O, with $\mu = 0, 1, 2, 3$, which are four real numbers corresponding to one time and three space coordinates. The same point P, in general, will have a different coordinate $x'^{\,\mu}(\mathrm{P})$ according to another observer O'. Note that the observers O and O' may not be inertial observers so that the coordinates x^μ and $x'^{\,\mu}$ are, in general, curvilinear coordinates. A **space-time manifold** is defined to be the set of all events. In GTR, the mathematical forms of physical laws remain the same even when one makes an arbitrary coordinate transformation.

In general, any arbitrary event belonging to a space-time manifold can be assigned coordinates x^μ and x'^μ by O and O', respectively. Since labeling of an event with coordinates by an observer involves a mapping from the space-time manifold to R^4, it follows that there is a mapping between x^μ and x'^μ, i.e. there exists a function that relates coordinate system employed by O to that of O'. Therefore, one may either treat x'^μ to be a continuous and differentiable function of x^α or vice-versa (i.e. x^μ as a smooth function of x'^α), with $\mu\ \ , \alpha = 0, 1, 2, 3$.

The demand for a smooth function can be justified from a classical physics standpoint that events can be arbitrary close to each other with no holes or discreteness in the space-time manifold. Going from one set of coordinates x^α to another set x'^α is called a **general coordinate transformation**. In a given coordinate system, each coordinate component of an event is functionally independent of the other coordinate so that,

$$\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu \tag{4}$$

Consider some physical variable (e.g. comoving energy density or pressure of a fluid) that can be described by observer O as a real valued function $\phi(x^\alpha)$ of space-time coordinates. Observer O', however, will find the same physical variable to be represented by a different function $\phi'(x'^\alpha)$. The function ϕ is said to be a **scalar field** if everywhere on the space-time manifold,

$$\phi'(x'^\alpha) = \phi(x^\alpha)$$

given that x^α and x'^α are the space-time coordinates assigned by observers O and O', respectively, to the same event.

Physically, what a scalar field signifies is that, at every event P, the value of the physical variable $\phi(x^\alpha(P))$ as measured by the observer O is identical to the value $\phi'(x'^\alpha(P))$ as measured by O', although the functional form of the physical variable depends on the observer.

How does one define vector components when one is using general curvilinear coordinates? Intuitively, a vector has magnitude as well as direction, and hence it resembles an arrow. Suppose, we have two events P and P' which are temporally as well as spatially near each other, so that they have coordinates x^μ and $x^\mu + dx^\mu$, respectively, for observer O. Clearly, the directed line PP' from P to P' looks like an infinitesimally short arrow and thereby qualifies to be called a vector with dx^μ as the vector components.

According to O', however, the directed line PP' has components dx'^μ since P and P' have coordinates x'^μ and $x'^\mu + dx'^\mu$, respectively, in her/his frame. The relation between the components is given by the usual rules of partial derivatives.

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \tag{5}$$

The above equation suggests that a contravariant vector field $V^\mu(x^\alpha)$ ought to be defined as an entity that transforms under a general coordinate transformation $x^\gamma \to x'^\gamma$ in the following manner,

$$V^\mu(x^\alpha) \to V'^\mu(x'^\alpha) = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x^\alpha) \tag{6}$$

The next question that arises is: What about objects like $\frac{\partial \phi(x)}{\partial x^\mu}$, where $\phi(x^\alpha)$ is a scalar field? Let us see how this entity transforms under general coordinate transformation. When $x^\gamma \to x'^\gamma$, we find that,

$$\frac{\partial \phi(x)}{\partial x^\mu} \to \frac{\partial \phi'(x')}{\partial x'^\mu}$$

$$= \frac{\partial \phi(x)}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \phi(x)}{\partial x^\nu} \tag{7}$$

Clearly, the transformation given by eq.(7) is different from the one in eq.(6). This motivates one to introduce another kind of vector field called covariant vectors.

A covariant vector field $V_\mu(x)$ is defined to be an object such that under $x^\gamma \to x'^\gamma$,

$$V_\mu(x) \to V'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu(x) \tag{8}$$

To summarize, while the transformation property and the directional nature of infinitesimal vector dx^μ leads to the notion of contravariant vectors, similar considerations concerning the partial derivative $\frac{\partial}{\partial x^\mu}$ entails the concept of covariant vectors. In 3+1 dimensional space-time, vector fields have 4 components corresponding to $\mu = 0, 1, 2, 3$. In pictorial terms, contravariant vectors are like arrows while the covariant vectors are like normal vectors to surfaces.

We can now wrap up the above considerations to arrive at a generalization - tensor fields of arbitrary ranks. A tensor field $V^{\mu_1\mu_2...\mu_{n-1}\mu_n}_{\nu_1\nu_2...\nu_{m-1}\nu_m}(x)$ of rank $n+m$ is an entity such that under the coordinate transformation $x^\alpha \to x'^\alpha$,

$$V^{\mu_1\mu_2...\mu_{n-1}\mu_n}_{\nu_1\nu_2...\nu_{m-1}\nu_m}(x) \to V'^{\mu_1\mu_2...\mu_{n-1}\mu_n}_{\nu_1\nu_2...\nu_{m-1}\nu_m}(x')$$

where,

$$V'^{\mu_1\mu_2...\mu_{n-1}\mu_n}_{\nu_1\nu_2...\nu_{m-1}\nu_m}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\frac{\partial x'^{\mu_2}}{\partial x^{\alpha_2}}...\frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}}\frac{\partial x^{\beta_2}}{\partial x'^{\nu_2}}...\frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}V^{\alpha_1\alpha_2...\alpha_{n-1}\alpha_n}_{\beta_1\beta_2...\beta_{m-1}\beta_m}(x) \quad (9)$$

We should note that in the above equation the arguments x^α and x'^α of V and V', respectively, are the coordinates of the same event, as emphasized in the first paragraph of this section. In other words, transformation of tensors are completely local because of which tensors of identical ranks can be added and subtracted.

An important result that follows from eq.(9) is that if a tensor vanishes at an event in one coordinate system, it is identically zero at that event in all coordinate systems.

A fundamental entity in GTR that describes space-time geometry is the space-time dependent metric tensor $g_{\mu\nu}(x^\alpha)$, which determines the invariant proper distance ds between any two nearby events with coordinates x^μ and $x^\mu + dx^\mu$,

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \quad (10)$$

Here, x^μ, μ, ν=0,1,2,3, now represents a general curvilinear coordinate, specifying the location of an event. The metric $g_{\mu\nu}(x)$ is a generalization of $\eta_{\mu\nu}$, the Minkowski metric tensor.

Is $g_{\mu\nu}(x)$ a tensor field? If ds^2 is invariant under general coordinate transformation, then we can readily prove that $g_{\mu\nu}$ is a covariant tensor of rank 2. This is because, under $x^\alpha \to x'^\alpha$, we have,

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \to ds^2 = g'_{\alpha\beta}(x')dx'^\alpha dx'^\beta$$

$$= g'_{\alpha\beta}(x')\left(\frac{\partial x'^\alpha}{\partial x^\mu}dx^\mu\right)\left(\frac{\partial x'^\beta}{\partial x^\nu}dx^\nu\right)$$

$$= \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}g'_{\alpha\beta}(x')dx^\mu dx^\nu \quad (11)$$

Comparing eqs.(10) and (11) as well as using the fact that dx^μ is an arbitrary infinitesimal separation, we get the result,

$$g_{\mu\nu}(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}g'_{\alpha\beta}(x') \quad (12)$$

implying that $g_{\mu\nu}$ is a covariant tensor of second rank. This result readily connects with equivalence principle in the following manner.

If the space-time geometry was not curved, one could choose a coordinate system such that everywhere the metric tensor is just the Minkowski metric tensor. But GTR states that energy and momentum associated with matter warp the space-time geometry, entailing that in general it is not possible to choose inertial coordinates everywhere so that the metric is globally Minkowskian.

However, according to the principle of equivalence, by choosing an appropriate coordinate system, even in an arbitrarily curved space-time, the metric tensor can be made to take the form of $\eta_{\mu\nu}$ in a sufficiently small space-time region (physically, this corresponds to choosing a sufficiently small freely falling frame). This is precisely what eq.(12) entails. One can choose a new set of coordinates ξ^α such that in a small region,

$$\eta_{\mu\nu} = \frac{\partial x'^\alpha}{\partial \xi^\mu}\frac{\partial x'^\beta}{\partial \xi^\nu}g'_{\alpha\beta}(x') \quad (13)$$

For the dynamics of bodies moving in pure gravity, the notion of gravitational mass becomes superfluous in GTR since particle trajectories are geodesics of space-time geometry determined from the line-element,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

Hence, it is not surprising that the world lines of freely falling test particles are independent of their inertial masses.

One can also define a contravariant metric tensor $g^{\mu\nu}(x)$ by demanding that,

$$g^{\mu\alpha}(x)g_{\alpha\nu}(x) = \delta^{\mu}_{\nu} \tag{14}$$

If one considers $g_{\mu\nu}$ to be a 4×4 matrix, then eq.(14) implies that the contravariant metric tensor $g^{\mu\nu}$ can be viewed as the inverse of the corresponding matrix.

Both $g_{\mu\nu}(x)$ and $g^{\mu\nu}(x)$ are symmetric tensor fields, i.e. $g_{\mu\nu} = g_{\nu\mu}$ and $g^{\mu\nu} = g^{\nu\mu}$ everywhere in the space-time manifold.

Employing $g^{\mu\nu}(x)$, one can raise indices of a covariant tensor, just like lowering the indices of a contravariant tensor field can be achieved by using $g_{\mu\nu}$. Therefore,

$$V^{\mu}(x) = g^{\mu\nu}(x)V_{\nu}(x) \tag{15}$$

is a contravariant vector field corresponding to the covariant vector field $V_{\mu}(x)$. While,

$$W_{\mu}(x) = g_{\mu\nu}(x)W^{\nu}(x) \tag{16}$$

is a covariant vector field corresponding to the contravariant vector field $W^{\mu}(x)$. Hence, raising and lowering of indices can be done freely for any tensor field of any rank by making suitable use of the metric tensors.

So far we have done some amount of tensor algebra. Let us now take up some tensor calculus. Suppose $A^{\mu}(x)$ is a contravariant vector field. Is $\frac{\partial A^{\mu}(x)}{\partial x^{\nu}}$ a second rank tensor? Under a general coordinate transformation $x^{\gamma} \to x'^{\gamma}$,

$$\frac{\partial A^{\mu}(x)}{\partial x^{\nu}} \to \frac{\partial A'^{\mu}(x')}{\partial x'^{\nu}} = \frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial x'^{\mu}}{\partial x^{\beta}}A^{\beta}(x)\right)$$

$$= \frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial x'^{\mu}}{\partial x^{\beta}}\frac{\partial A^{\beta}(x)}{\partial x^{\alpha}} + \frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial^2 x'^{\mu}}{\partial x^{\alpha}\,\partial x^{\beta}}\;A^{\beta}(x) \tag{17}$$

It is obvious from eq.(17) that because of the second term in its right hand side, $\frac{\partial A^{\mu}(x)}{\partial x^{\nu}}$ does not transform like a tensor. What is the remedy?

This is where the concept of covariant derivative comes in. We introduce a new mathematical object $\Gamma^{\mu}_{\alpha\beta}(x^{\lambda})$ referred to as Christoffel symbol (and also as affine connection and Levi Civita connection), and define the covariant derivative of $A^{\mu}(x)$ as follows,

$$A^{\mu};\nu = A^{\mu},\nu + \Gamma^{\mu}_{\nu\alpha}A^{\alpha} \tag{18}$$

where,

$$A^{\mu},\nu \equiv \frac{\partial A^{\mu}(x)}{\partial x^{\nu}} \tag{19}$$

(From now on partial derivative w.r.t. x^{α} will be denoted by $,\alpha$)

Using eq.(17), it can be easily shown that $A^{\mu};\nu$, defined by eq.(18), transforms as a tensor of rank 1+1 provided the Christoffel symbol $\Gamma^{\mu}_{\alpha\beta}(x^{\lambda})$ transforms under a general coordinate transformation as,

$$\Gamma^{\mu}_{\alpha\beta}(x^{\lambda}) \;\to\; \Gamma'^{\,\mu}_{\alpha\beta}(x'^{\lambda}) = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{\partial x^{\sigma}}{\partial x'^{\alpha}}\frac{\partial x^{\gamma}}{\partial x'^{\beta}}\;\Gamma^{\nu}_{\sigma\gamma}(x^{\lambda}) + \;\frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{\partial^2 x^{\nu}}{\partial x'^{\alpha}\partial x'^{\beta}} \tag{20}$$

Clearly, because of the second term in the right hand side of eq.(20), Christoffel symbol is not a tensor. One can use precisely this feature to choose local Minkowski coordinates ξ^{μ} to make $\Gamma^{\mu}_{\alpha\beta}$ vanish at a space-time point. This dovetails nicely with equivalence principle, since we know that in a freely falling frame, the metric is $\eta_{\mu\nu}$ in a small neighborhood so that it has vanishing derivatives at a point.

Now, since for any arbitrary scalar field $\phi(x)$, $\frac{\partial\phi(x)}{\partial x^{\mu}}$ is already a covariant vector field (see eq.(7)), the covariant derivative of any scalar field is its usual partial derivative,

$$\phi;\nu = \phi,\nu \tag{21}$$

Now, if $V^{\mu}(x)$ and $U_{\mu}(x)$ are any two contravariant and covariant vector fields, respectively, $V^{\mu}(x)U_{\mu}(x)$ is a scalar field (see Prob.1(a)). Hence, its covariant derivative according to eqs. (18) and (21) is given by,

$$(V^{\mu}(x)U_{\mu}(x));\nu = V^{\mu};\nu U_{\mu} + V^{\mu}U_{\mu};\nu = (V^{\mu}(x)U_{\mu}(x)),\nu$$

$$= V^{\mu}, \nu U_{\mu} + V^{\mu} U_{\mu}, \nu$$

$$= (V^{\mu}, \nu + \Gamma^{\mu}_{\nu\alpha} V^{\alpha}) U_{\mu} + V^{\mu} U_{\mu}; \nu \tag{22}$$

From eq.(22) it ensues,

$$U_{\mu}; \nu = U_{\mu}, \nu - \Gamma^{\alpha}_{\nu\mu} U_{\alpha} \tag{23}$$

as $V^{\mu}(x)$ is an arbitrary contravariant vector field. This procedure can be deployed to obtain covariant derivatives of any tensor field of arbitrary rank.

Therefore, as particular examples,

$$T_{\mu\nu}; \alpha = T_{\mu\nu}, \alpha - \Gamma^{\beta}_{\alpha\mu} T_{\beta\nu} - \Gamma^{\beta}_{\alpha\nu} T_{\mu\beta} \tag{24}$$

and,

$$A^{\mu\nu}; \alpha = A^{\mu\nu}, \alpha + \Gamma^{\mu}_{\alpha\beta} A^{\beta\nu} + \Gamma^{\nu}_{\alpha\beta} A^{\mu\beta} \tag{25}$$

In 3-dimensional Euclidean geometry, the line-element $dl^2 = dx^2 + dy^2 + dz^2$ has the same form whether you shift the Cartesian coordinate system by any constant vector or rotate the coordinate system about any axis by any constant angle. The line-element given by eq.(2) is similarly invariant under Lorentz transformations as well as constant space-time translations. According to the equivalence principle, whatever is the gravity around, in a locally inertial frame (i.e. freely falling frame), the line-element is given by eq.(2) and non-gravitational laws of physics take the same form as in special relativity. But, what is the connection between this feature of gravitation and geometry?

Consider a generally curved two-dimensional surface (e.g. the surface of, say, a pear). No matter how greatly the surface is curved, one can always choose a tiny enough patch on it, such that it is sufficiently flat for Euclidean geometry to hold good over it. As one increases the size of the patch, the curvature of the pear's surface becomes apparent. This is so similar to the main characteristic of gravity that we discussed in the preceding paragraph. The small patch on the pear over which the line-element is Euclidean ($dl^2 = dx^2 + dy^2$) is analogous to the local inertial frame in the case of 4-dimensional space-time where the line-element is described by eq.(2).

V. CHRISTOFFEL SYMBOL, CURVATURE TENSOR AND THE EINSTEIN EQUATIONS

Now, from eqs.(15) and (16), in order that,

$$V^{\mu}; \alpha = g^{\mu\nu}; \alpha V_{\nu} + g^{\mu\nu} V_{\nu}; \alpha = g^{\mu\nu} V_{\nu}; \alpha$$

and,

$$W_{\mu}; \alpha = g_{\mu\nu}; \alpha W^{\nu} + g_{\mu\nu} W^{\nu}; \alpha = g_{\mu\nu} W^{\nu}; \alpha$$

we require,

$$g^{\mu\nu}; \alpha = 0 = g_{\mu\nu}; \alpha \tag{26}$$

Making use of eq.(26), we have,

$$g_{\mu\alpha}; \nu + g\alpha\nu; \mu - g_{\mu\nu}; \alpha = 0 \tag{27}$$

From the eqs.(24) and (27), it can be easily proved that,

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda}, \beta + g_{\beta\lambda}, \alpha - g_{\alpha\beta}, \lambda) \tag{28}$$

displaying an important fact that Christoffel symbol is related to metric tensor and its derivatives.

Because of eq.(28), one can show that the trajectory (i.e. the worldline) $x^{\mu}(\lambda)$, where λ is an affine parameter characterizing the worldline, that extremizes the proper length (invariant under general coordinate transformations),

$$S \equiv \int ds = \int \sqrt{g_{\mu\nu} dx^{\mu} dx^{\nu}} = \int \sqrt{g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda}} d\lambda \tag{29}$$

satisfies the geodesic equation,

$$\frac{d^2x^\alpha}{d\lambda^2}+\Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}=0\ . \tag{30}$$

In weak fields (small departure from Minkowski space-time), one can choose quasi-Minkowskian coordinates so that,

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} \tag{31}$$

with,

$$|h_{\mu\nu}|\ll 1 \tag{32}$$

for μ,ν=0,1,...,3.

For static and weak gravitational fields where test particles move with speeds much less than c, one must have eq.(30) reduce to Newtonian gravitational dynamics, where a particle with spatial coordinate $x^i,\ i=1,2,3$ satisfies,

$$\frac{d^2x^i}{dt^2}=-\frac{\partial\phi_N(\vec{r})}{\partial x^i} \tag{33}$$

where $\phi_N(\vec{r})$ is the Newtonian gravitational potential at $\vec{r}$.

Eq.(30) indeed leads to eq.(33) in the weak and static field approximation provided,

$$g_{00}=1+h_{00}\approx 1+\frac{2\phi_N(\vec{r})}{c^2} \tag{34}$$

$$g_{0i}\approx 0\ ,\quad g_{ij}\approx-\delta_{ij} \tag{35}$$

so that,

$$\Gamma^i_{00}\approx\frac{1}{c^2}\frac{\partial\phi_N}{\partial x^i} \tag{36}$$

We know from STR that the time elapsed in a clock (comoving with an observer O') cruising with uniform velocity with respect to an inertial observer O is given by,

$$\tau=\frac{1}{c}\int\sqrt{\eta_{\mu\nu}dx^\mu dx^\nu}=\sqrt{1-v^2/c^2}\ \ t\ , \tag{37}$$

where x^μ, v and t are the space-time coordinates of O', speed of O' and time as measured by the inertial observer O, respectively. This is called the proper time that elapses in the frame of O', and is invariant under Lorentz transformations. The time dilation result ensues from eq.(37).

What can we say about the proper time elapsed for a test particle as it moves along an arbitrary worldline in a curved space-time? Let the worldline in a space-time manifold whose geometry is described by metric $g_{\mu\nu}(x^\alpha)$ be described by $x^\mu(\lambda)$ from λ_1 to λ_2, λ being an affine parameter characterizing the worldline.

Since one understands good clocks and good measuring rods in the framework of STR, the way to measure proper time τ elapsed in an arbitrarily accelerating clock in arbitrary gravity is clearly by adding the infinitesmal time intervals elapsed in local inertial frames that lie along the trajectory of the clock at different instants of time and that co-move with the clock at those instants of time (for comoving clocks $v=0$ so that the proper time is just the time elapsed in these clocks as seen from eq.(37)). But by virtue of eqs.(11) and (13), this sum is just,

$$\tau=\frac{1}{c}\int_1^2\sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda \tag{38}$$

Another way of stating the above argument is that the infinitesimal proper time interval between two time-like separated close by events is ds/c, and since time is additive, the total proper time elapsed is simply the integral given by eq.(38).

We can apply the above result to determine the proper time elapsed in a clock at rest in a weak and static gravity. Using eqs.(34) and (35) in eq.(38) for a clock at rest ($dx^i=0$) at a point A, one obtains the proper time elapsed to be given by,

$$\tau_A\approx(1+\frac{\phi_N(A)}{c^2})\ \ t\ , \tag{39}$$

t being the proper time elapsed in the frame of a static inertial observer at infinity where $h_{00} = 0$. From this one concludes that not only time runs slow in attractive gravitational fields but also radiation emitted from regions with stronger and attractive gravitational potentials get redshifted as they move out to weaker gravity regions.

To summarize, in order to connect LIFs at different space-time points, and to express physical laws in terms of arbitrary coordinates in reference frames of size as large as one wishes, one needs the language of tensor calculus so that one acquires an affine connection $\Gamma^{\mu}_{\alpha\beta}$ derivable from the metric tensor $g_{\mu\nu}$ and its derivatives. Although, this affine connection (or, Christoffel symbol) vanishes at a point in a LIF, its derivative does not.

This brings us to the Riemann curvature tensor $R^{\mu}{}_{\nu\alpha\beta}$ which represents how curved is the space-time geometry, and is constructed out of Christoffel symbol and its derivatives in the following way,

$$R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta}, \alpha - \Gamma^{\mu}_{\nu\alpha}, \beta + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha} \tag{40}$$

From eq.(40), it is obvious that,

$$R^{\mu}{}_{\nu\alpha\beta} = -R^{\mu}{}_{\nu\beta\alpha}$$

One can obtain a symmetric second rank tensor called the Ricci tensor from the Riemann tensor,

$$R_{\nu\beta} = R^{\mu}{}_{\nu\mu\beta} = R_{\beta\nu} \tag{41}$$

The Ricci scalar is simply,

$$R = g^{\mu\nu}R_{\mu\nu} \tag{42}$$

From eq.(40), one can easily prove that if,

$$R_{\mu\nu\alpha\beta} = g_{\mu\lambda}R^{\lambda}{}_{\nu\alpha\beta}$$

then,

$$R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha} = -R_{\nu\mu\alpha\beta} = R_{\alpha\beta\mu\nu} \tag{43}$$

To summarize, Christoffel symbol is like gravitational field since it involves derivatives of the metric. However, because it is not a tensor (see eq.(20)), it can also represent pseudo-gravity. For instance, even when there is no true gravity so that one can choose global Minkowski coordinates everywhere (i.e. global inertial frame), if one goes to another coordinate system (i.e. non-inertial frame), the Christoffel symbol will be non-zero describing fictitious force. The true gravitation is represented by the Riemann tensor. Although in a local inertial frame, gravitational force disappears, tidal force does not. The Christoffel symbol is not a tensor. In LIF it is zero at a point, while the Riemann tensor in general is not. This reminds us of the acceleration due to gravity vanishing while the tidal force does not, in a LIF.

For instance, earth is freely falling towards the sun because of latter's pull. But we do not feel sun's gravity since the freely falling earth constitutes a local inertial frame. However, as sun's gravity is non-uniform, portions of earth closer to the sun feel a greater tug than those located farther. This differential pull is the source of tidal force which causes the commonly observed ocean tides. In GTR, the tidal acceleration is due to the fourth rank Riemann tensor that is constructed out of the metric and its first as well as second derivatives. Therefore, the ocean tides owe their existence to the non-zero Riemann tensor describing the curvature of space-time geometry around the sun (as well as the moon).

Therefore, true gravity represented by the tidal gravitational field is related to the Riemann curvature tensor $R^{\mu}_{\nu\alpha\beta}$, a fourth rank tensor constructed out of the connection $\Gamma^{\mu}_{\alpha\beta}$ and its derivatives. In mathematics, $R^{\mu}_{\nu\alpha\beta}$ determines whether the geometry is flat or curved. This, in a sense, completes the identification of gravity with geometry. While in the gauge theory framework, $\Gamma^{\mu}_{\alpha\beta}$ is analogous to gauge potential with $R^{\mu}_{\nu\alpha\beta}$ as the corresponding gauge covariant field strength.

In later lectures, we will see that the dynamics of space-time geometry is determined by the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{44}$$

where the Ricci tensor and Ricci scalar are $R_{\mu\nu} \equiv R^{\alpha}_{\mu\alpha\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$, respectively. $T_{\mu\nu}$ is the matter energy-momentum tensor whose various components represent the flux of energy and momentum carried by matter in appropriate directions. When the gravity is weak and static, eq.(33) reduces to Newton's gravity,

$$\nabla^2\phi = 4\pi G\rho \tag{45}$$

for a non-relativistic source with mass density ρ and negligible pressure. The Newtonian gravitational potential ϕ is identified with the geometrical entity $(g_{00} - 1)c^2/2$.

GTR tells us that matter distorts the space-time from an Minkowskian geometry to a non-Minkowskian one, and test bodies just move along straightest possible paths in such a curved space-time. As to, how the matter warps the space-time geometry, is given by the so called Einstein equations which relate tensors created out of the metric and the Riemann tensor to the matter energy-momentum tensor multiplied by a combination of Newton's constant G and light speed c.

Einstein equations possess a pristine beauty, with space-time geometry on one side, and the energy and momentum of matter on the other. When the geometrical curvature of space-time is small and the motion within the source is slow enough, GTR leads automatically to Newton's laws of gravitation.

VI. GRAVITATIONAL RADIATION

GTR as a theory of gravitation gained immediate acceptance among the physics community as soon as its prediction of bending of light was actually seen during the solar eclipse of 1919. Of course, GTR had already correctly explained the anomalous precession of the perihelion of Mercury. Since GTR is based on special relativity, gravitational perturbations too propagate as gravitational waves (or, undulations in space-time geometry) with finite speed c. Later, indirect evidence for gravitational waves predicted by Einstein was corroborated with the discovery of slowly inspiralling Hulse-Taylor binary pulsar (PSR 1913 + 16).

Very far away from a source whose energy and momentum distributions are changing asymmetrically, if the ensuing perturbation in the space-time geometry, represented by $h_{\mu\nu}$, is sufficiently weak, one can choose a quasi-Minkowskian coordinate system and express the metric tensor as,

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}(\vec{r}, t) \ , \tag{46}$$

with the perturbation or the gravitational wave amplitude satisfying,

$$|h_{\mu\nu}(\vec{r}, t)| \ll 1 \tag{47}$$

at large distances from the source.

The gravitational wave (GW) amplitude $h_{\mu\nu}(\vec{r}, t)$ is determined by the second time derivative of the mass quadrupole moment of the source that is undergoing changes in its matter distribution. Since the metric tensor governs the proper distance (or equivalently, proper time) by virtue of eq.(29), the proper distance between two test particles will undulate when a GW is incident on them. By measuring the relative separation between the test particles as a function of time, one can gain information about $h_{\mu\nu}$.

Do GWs transport energy? Now, only entities that have energy can possibly be perceived or measured, since exchange of energy between an object and the sensors is crucial for its detection. Even in quantum theory, two subsystems can influence each other only via an interaction Hamiltonian. Feynman and Hermann Bondi had used the following thought experiment to demonstrate that GWs carry energy [1]: Consider two loose metal rings around a rod that is held in a horizontal position. If a GW passes by, the rings will move and oscillate with respect to the rod (elasticity of the rod will prevent appreciable change in its length because of the incident GW). Hence, the rings and the rod will get heated up because of friction. This energy certainly has to be at the expense of the energy carried by the GW.

Physical effects of GWs on test particles are best understood in the transverse, traceless gauge. So, if a linearly polarized GW is propagating in the z-direction, it can have only $h_{11} = -h_{22}$ and $h_{12} = h_{21}$ as the non-zero components that are orthogonal to the direction of propagation. Consider now, for simplicity, the case of a near monochromatic GW for which $h_{11} = -h_{22} = h$ and $h_{12} = h_{21} = 0$. Suppose one has two test particles scattered on the xy-plane. In order to monitor the change in the proper distance between the test particles because of the metric perturbation, one has to have the initial separation L between these particles to be much less than the radius of curvature of space-time geometry associated with the incident GW, so that one can use either eq.(29) or the geodesic deviation equation to determine the change in the separation.

For a GW, the radius of curvature of space-time geometry is of the order of its wavelength. Hence, the above condition then is,

$$L \ll c(\omega/2\pi)^{-1} \tag{48}$$

where ω is the angular frequency of the GW. If the two test particles lie on the x-axis then according to eq.(29), the proper distance $l(t)$ is given by,

$$l(t) = \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = \int_0^L \sqrt{1 + h_{11}} \ dx \cong \left(1 + \frac{1}{2} \ h\right) L \ , \tag{49}$$

where one has used the condition given by eq.(47). Therefore, the strain is simply related to the GW amplitude in the following manner,

$$\frac{\Delta L}{L} \equiv \frac{l(t) - L}{L} = \frac{1}{2}\ h\ . \tag{50}$$

Eq.(49) tells us that as the GW passes by, the proper distance between the test particles changes with time. Therefore, if light is emitted from one test particle at time t_1 towards the other, the arrival time t_2 of light at the other test particle is given by,

$$t_2 \approx t_1 + l(t_1)/c = t_1 + \left(1 + \frac{1}{2}\ h\right) L/c\ . \tag{51}$$

It is essentially this feature, represented by eq.(51), that is used in a laser interferometer to detect GW, since changes in the arrival times from the two arms of the interferometer correspond directly to the changes in the phase difference that lead to GW amplitude measurements from the fringe shifts in the interference pattern (see [2] for a pedagogical introduction to this subject).

In a time interval $T = 2\pi/\omega$ (where ω is the angular frequency of the near monochromatic GW), the variation in the proper distance ΔL between two test particles intercepting the GW can change by an amount of the order of $hL/2$ (eq.(50)), where h and L are the magnitude of GW amplitude and initial separation between the particles (provided L is much much less than the wavelength of the GW, which is $\sim 2\pi c/\omega$). An interesting question to ask is: Can the change ΔL happen so rapidly that $\Delta L/T > c$? Now, $\Delta L/T = hL/2T = hL\omega/4\pi$, which is much less than hc because of eq.(48). Since, h is much much less than unity (eq.(47)), $\Delta L/T$ is much much less than c. Therefore, the rate of change of separation between the test particles can never exceed the speed of light.

The two LIGOs, laser interferometric GW detectors of USA, independently achieved direct detection of GWs from a black hole binary merger event on September 14, 2015, with a time delay of about 7 ms [3]. This gravitational wave source GW150914 is at a redshift of about 0.09 corresponding to a luminosity distance of about 410 Mpc. The binary system consisted of two coalescing black holes of mass 29 $M_\odot$ and 36 $M_\odot$ that eventually collided with each other to settle down into a bigger rotating black hole of mass 62 $M_\odot$. The mass deficit of 3 $M_\odot$ was carried away by GWs. Significantly, this event not only corroborates GW results that ensue from GTR but is also consistent with the prediction of quasi-normal mode emission of GWs from a perturbed black hole [4].

Therefore, it is pertinent to ask whether GW150914 is consistent with black hole thermodynamics (BHT) too. From laws of BHT, it follows that the event horizon area of the final black hole must be larger than the sum of the event horizon areas of the binary components. The radius of the event horizon of a black hole of mass M and spin angular momentum L is given by (see, for example,[5]),

$$R_{EH} = \frac{G}{c^2}\left(M + \sqrt{M^2 - (L\ c/G\ M)^2}\right) \tag{52a}$$

while the area of the event horizon is given by (see, for example,[6]),

$$A = 8\pi\left(\frac{G}{c^2}\right)^2 M\left(M + \sqrt{M^2 - (L\ c/G\ M)^2}\right) \tag{52b}$$

There is considerable uncertainty about the spin angular momenta of the two initial black holes of GW150914. In our analysis, we may take them to be Schwarzschild black holes (i.e. $L = 0$) so that we may start with the maximum event horizon area (eq.(52b)). The initial total area of the event horizons is then given by,

$$A_i = 16\pi(G/c^2)^2[M_1^2 + M_2^2] \tag{53}$$

where $M_1 = 29\ M_\odot$ and $M_2 = 36\ M_\odot$. The black hole formed after the merger has a mass $M_f = 62\ M_\odot$ and spin angular momentum $L_f = 0.67\ \frac{GM_f^2}{c}$. Hence, the final area of the event horizon according to eq.(52b) is given by,

$$A_f = 8\pi(G/c^2)^2 M_f\left(M_f + \sqrt{M_f^2 - (L_f\ c/G\ M_f)^2}\right) \tag{54}$$

Therefore, from eqs.(53) and (54), the ratio of final area to the initial is,

$$\frac{A_f}{A_i} = \frac{M_f^2\left(1 + \sqrt{1 - (L_f\ c/G\ M_f^2)^2}\right)}{2\left(M_1^2 + M_2^2\right)} = 1.57 \tag{55}$$

where one has used,

$$L_f \ c/G \ M_f^2 = 0.67 \ .$$

Thus, even after overestimating the initial area by assuming that the initial black holes were Schwarzschild black holes, eq.(55) demonstrates that the parameters deduced from the event GW150914 are consistent with Hawking's area theorem [6].

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Practice Problems

1 If $V^\mu(x)$ and $W_\mu(x)$ are two vector fields, then show that:
(a) $V^\mu W_\mu$ is a scalar field.
(b) $V^\mu W_\nu$ is a tensor field of rank 1+1
(c)$g^{\nu\alpha} V^\mu W_\alpha$ is a tensor field of rank 2.

2 (a) A tensor $A^{\mu\nu}$ is found to be anti-symmetric in a particular coordinate system $\{x^\mu\}$. Prove that $A^{\mu\nu} = - \ A^{\nu\mu}$ in all coordinate systems.

(b) It is given that a zero-rest mass particle is moving along the world line $x^\mu(\lambda)$ in 3+1-dimensions, where λ is an affine parameter. Show that the tangent vector $u^\mu = \frac{dx^\mu}{d\lambda}$ satisfies the equation,

$$g_{\mu\nu} u^\mu u^\nu = 0 \ .$$

(c) (i) If $A^\mu_\nu(x)$ is a second rank tensor field of (1+1) type, then show that,

$$A^\mu_\nu;\alpha = A^\mu_\nu,\alpha + \Gamma^\mu_{\alpha\beta} A^\beta_\nu - \Gamma^\beta_{\alpha\nu} A^\mu_\beta$$

(ii) Show that $\delta^\mu_\nu;\alpha = 0$

3 (a) Consider the 2-dimensional manifold constituted by the surface of a sphere of radius $r = 1$.
(i) Choosing a convenient coordinate system, obtain $g_{\mu\nu}$, $g^{\mu\nu}$ and $\Gamma^\mu_{\alpha\beta}$.
(ii) A vector v^μ starting out with components $v^1 = A$ and $v^2 = B$ from the point P: $(\theta, \phi) = (\pi/2, 0)$ is parallel transported from P to $(\pi/2, \pi/2)$ first, then to $(\alpha, \pi/2)$, and thereafter to $(\alpha, 0)$, and finally back to P. Here, $0 \leq \alpha < \pi/2$. Find the components of v^μ when it returns to P. Does your result change when $\alpha \to 0$?

(b) For a vector field $V^\mu(x^\gamma)$, prove that,

$$V^\mu;\alpha;\beta \ - \ V^\mu;\beta;\alpha = - \ R^\mu{}_{\nu\alpha\beta} V^\nu$$

(c) If $\psi(x^\lambda)$ is a scalar field then prove that,

$$g^{\mu\nu}\psi;\mu;\nu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu}\left(\sqrt{-g} \ g^{\mu\nu} \frac{\partial \psi}{\partial x^\mu}\right)$$

(d) For weak and static gravity, show that clocks tick at a slower rate in regions of stronger gravity.

4 (a) The Einstein-Hilbert action is given by,

$$A_G = -\frac{c^3}{16\pi G}\int R\sqrt{-g}\ d^4x$$

Prove that under an infinitesimal variation $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$, the variation in A_G is given by,

$$\delta A_G = -\frac{c^3}{16\pi G}\int (R_{\mu\nu} - 1/2\ g_{\mu\nu}R)\delta g^{\mu\nu}\sqrt{-g}\ d^4x$$

provided $\delta g^{\mu\nu}(x^\alpha)$ vanishes at infinity.

(b) If $\delta g_{\mu\nu}$ is an infinitesimal variation of the metric tensor $g_{\mu\nu}$ and $T^{\mu\nu}$ is the energy-momentum tensor, prove that

$$T^{\mu\nu}\delta g_{\mu\nu} = -\ T_{\mu\nu}\delta g^{\mu\nu}$$

(c) If $\bar{G}^{\mu\nu} \equiv R^{\mu\nu} - 1/2\ g^{\mu\nu}R + \Lambda g^{\mu\nu}$ then prove that $\bar{G}^{\mu\nu};\nu = 0$. Assume that Λ is a constant.

5 (a) Show that a light ray emitted radially outward from $r < R_s = 2\ G\ M/c^2$ can never cross the event horizon of a Schwarzschild blackhole of mass M.

(b) For a Schwarzschild blackhole of mass M, show that a test particle of non-zero rest mass cannot have constant r trajectories when it is inside the event horizon (i.e. $r < R_s \equiv 2\ G\ M/c^2$)).

5 (a) For a particular space-time, $\xi^\mu(x^\alpha)$ is given to be a Killing vector field. Consider a test particle falling freely in this space-time along a geodesic $x^\mu(\lambda)$, where λ is an affine parameter. Show that $u^\mu\xi_\mu$ is a constant of motion, given that $u^\mu \equiv \frac{dx^\mu}{d\lambda}$.

(b) Suppose the metric tensor $g_{\mu\nu}$ is independent of the particular coordinate x^σ for a fixed value of σ so that $\frac{\partial g_{\mu\nu}}{\partial x^\sigma} = 0$. Then, show that $\xi^\mu = \delta^\mu{}_\sigma$ satisfies the Killing equation,

$$\xi_{\mu\ ;\nu}\ +\ \xi_{\nu\ ;\mu} = 0$$

(b) Given a spherically symmetric white dwarf of mass 2×10^{33} gm and radius 6000 km, find the maximum energy that can be extracted by lowering a test particle of mass 10^6 gm very very slowly towards the white dwarf, by first obtaining an expression for the conserved energy of the test particle at rest in this space-time.

6 (a) Show that a test particle of rest mass m falling freely due to gravity of a spherically symmetric body of mass $M \gg m$ moves along a geodesic $r(\phi)$ that satisfies the differential equation,

$$\text{(i)} \qquad \frac{d^2u}{d\phi^2}\ +\ u = \frac{R_s}{2l^2}\ +\ \frac{3R_s u^2}{2}$$

where $l \equiv \frac{L_z}{mc}$, $u(\phi) \equiv 1/r(\phi)$, $R_s \equiv 2\ G\ M/c^2$ and L_z is the angular momentum of the test particle.

(ii) Under what approximation is,

$$r(\phi) \cong \frac{a(1-e^2)}{1 + e\cos\phi}$$

a solution of the differential equation in (i)? [a and e are positive constants]

(b) In the case of static and weak gravity, $g_{\mu\nu} \approx \eta_{\mu\nu}\ +\ h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ and $\partial h_{\mu\nu}/\partial t = 0$. If $h_{00} = 2\phi_N/c^2$ show that Newton's gravity,

$$\nabla^2\phi_N = 4\pi G\rho$$

follows from the Einstein equation,

$$R_{\mu\nu} = \frac{8\pi G}{c^4}[T_{\mu\nu} - 1/2\ g_{\mu\nu}T]$$

where ϕ_N and ρ are the Newtonian gravitational potential and mass density, respectively.

(c) Consider a blackhole of mass $M = 2 \times 10^{34}$ gm and a test body of mass $m = 10^{-4}\ M$ orbiting around the blackhole in the $\theta = \pi/2$ plane, with a closest radial approach $r_{min} = 10^{12}$ cm when it has the maximum speed $v_{max} = 10^{-3}\ c$. The test body follows, to a good approximation, the geodesic

$$r(\phi) \cong \frac{2l^2/R_s}{1 + e\ \psi(\phi)}$$

where

$$\psi(\phi) \equiv \cos\phi + \frac{3R_s^2}{4l^2}\ \phi\ \sin\phi\ ,$$

$l \equiv \frac{L_z}{mc}$, $u(\phi) \equiv 1/r(\phi)$, $R_s \equiv 2\ G\ M/c^2$, e is the eccentricity and L_z is the angular momentum of the test body. The orbital period P is given to be 10^6 seconds. Estimate the rate of precession of the point of closest approach. Does the result depend on m?

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