Conserved currents in the Cartan formulation of general relativity

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ABSTRACT. We derive the expressions for the local, on-shell closed co-dimension 2 forms in the Cartan formulation of general relativity and explicitly show their equivalence to those of the metric formulation.

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1 Introduction

Surface charges in general relativity and gauge theories have a long history that goes back to the founding papers on the Hamiltonian formulation, see [1] for a review and [2] for further developments. Covariant approaches based on the linearized theory are discussed in [3], chapter 20, and also in [4, 5]. A non-exhaustive list of subsequent references includes [6–10]. More recently, there has been interest in first order formulations, see e.g., [11–16].

Our approach here is based on actions, or more precisely, equivalence classes of Lagrangians up to total divergences. It originates in applications of the Batalin-Vilkovisky formalism to the perturbative renormalization of gauge theories [17], [18], but can also be formulated entirely independently of this machinery, see [19–25] for details.

The aim of this note is to provide explicit expressions for the local, on-shell closed co-dimension 2 forms in the Cartan formulation of general relativity and prove their equivalence with those of the metric formulation. The present note is extracted from a more complete investigation that covers other first order formulations of general relativity [26].

2 Generalities

2.1 Local BRST cohomology and generalized auxiliary fields

One of the virtues of the approach is that non-trivial, local, co-dimension 2 forms that are closed for all solutions of the equations of motion can be shown to be isomorphic to local BRST cohomology classes in ghost number -2. In turn, the latter are naturally covariant under field redefinitions as well as suitably invariant under the introduction and elimination of auxiliary and generalized auxiliary fields [17]. Auxiliary fields are a set of fields whose Euler-Lagrange equations of motion can be solved algebraically to determine them in terms of the remaining fields of the variational principle. Generalized auxiliary fields extend this concept to the master action [27, 28]. They are present whenever the vanishing of the gauge transformations of the fields can be solved algebraically for some of the gauge parameters. The associated generalized auxiliary fields are sub-sets of fields which are algebraically pure gauge, in the sense that they can be shifted arbitrarily by gauge transformations that do not involve derivatives.

This is relevant for our purpose since the components of the Lorentz connection in the Cartan formulation are auxiliary fields, while going from the vielbein to the metric formulation involves elimination of generalized auxiliary fields. Indeed, in the linearized formulation the skew-symmetric part of the vielbein fluctuations are algebraically pure gauge since they can be shifted arbitrarily by Lorentz rotations. The argument can then be extended to the non-linear theory as well, for instance by a perturbative analysis.

More details will be provided in [26].

2.2 General case

Let ϕ^i denote the fields of the variational principle, n the spacetime dimension and $\mathcal{L} = L\, d^n x$ the Lagrangian times the volume form. Here and below, we use the notation

$$(d^{n-p}x)_{\mu_1\dots\mu_p} = \frac{1}{p!(n-p)!}\epsilon_{\mu_1\dots\mu_p\mu_{p+1}\dots\mu_n} dx^{\mu_{p+1}}\dots dx^{\mu_n}, \tag{2.1}$$

where the wedge product is omitted, $\epsilon_{\mu_1\dots\mu_n}$ is completely antisymmetric and $\epsilon_{01\dots n-1} = 1$. Let $\delta_\epsilon \phi^i = R^i_\alpha(\epsilon^\alpha)$ denote a generating set of non trivial gauge transformations. Under standard regularity assumptions, one can then show that there is an isomorphism between equivalence classes of local, on-shell closed co-dimension 2 forms, with two such forms being equivalent if they differ on-shell by an exact local form, and equivalence classes of reducibility parameters $\bar f^\alpha[x,\phi]$ satisfying $R^i_\alpha(\bar f^\alpha) \approx 0$, with two sets of reducibility parameters being equivalent if they agree on-shell. In other words, the classification of local, on-shell closed co-dimension 2 forms is done through the classification of reducibility parameters, which is a tractable problem.

The construction of the $n-2$ forms from the reducibility parameters can be summarized as follows. For any f^α, standard integrations by parts allow one to write

$$R^i_\alpha(f^\alpha)\frac{\delta\mathcal{L}}{\delta\phi^i} = f^\alpha R^{+i}_\alpha(\frac{\delta\mathcal{L}}{\delta\phi^i}) + d_H S_f, \tag{2.2}$$

for some weakly vanishing $n-1$ form

$$S_f = S^{i\mu}_\alpha(\frac{\partial}{\partial dx^\mu}\frac{\delta\mathcal{L}}{\delta\phi^i}, f^\alpha). \tag{2.3}$$

The $n-2$ form is then obtained by applying the contracting homotopy ρ_H for the horizontal differential of the variational bi-complex [29, 30]

$$\{d_H, \rho_H\}\omega^p = \omega^p \text{ for } p < n. \tag{2.4}$$

to S_f,

$$k_f = \rho_H S_f. \tag{2.5}$$

Indeed, the Noether identities associated to the generating set of non-trivial gauge transformations are

$$R^{+i}_\alpha(\frac{\delta\mathcal{L}}{\delta\phi^i}) = 0. \tag{2.6}$$

For particular reducibility parameters that satisfy $R^i_\alpha(\bar f^\alpha) = 0$, (2.2) reduces to $d_H S_{\bar f} = 0$ so that (2.4) reduces to

$$d_H k_{\bar f} = S_{\bar f} \approx 0. \tag{2.7}$$

One can then proceed to show that $k_{\bar{f}}$ satisfies (2.7) also for general reducibility parameters (see [19] for details).

In this discussion, we have neglected non-trivial, identically conserved currents, which are related to the topology of the bundle of fields. We have thus neglected "magnetic" charges and concentrated on the "electric" ones. The former can easily be incorporated when taking into account the cohomology of the horizontal differential of the variational bi-complex in lower form degrees, and more specifically, in degree $n-2$ for the present case.

2.3 Linearized theories

For definiteness, let us take the example of the Einstein-Hilbert action in metric formulation, where a generating set of gauge transformations is given by the Lie derivative of the metric, $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$. In spacetime dimension $n \geqslant 3$, one can then show that $\xi^\rho[x,g]$ can be assumed not to depend on the fields, so that reducibility parameters correspond to Killing vectors. Since a generic metric does not admit Killing vectors, there are no non-trivial conserved $n-2$ forms in general relativity. In linearized gravity however, a generating set of gauge transformations is given by $\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the background solution around which one linearizes the theory. There are then as many conserved $n-2$ forms as there are Killing vectors of the background solution. Explicit expressions are obtained by applying the construction described previously, but now in the framework of the linearized theory. For Einstein gravity, this has been done explicitly in [19].

More generally, for gauge theories linearized around a solution $\bar{\phi}^i$ with gauge transformations $\delta_\epsilon \varphi^i = R^i_\alpha[x,\bar{\phi}](\epsilon^\alpha)$, one can show [20] that one may obtain the $n-2$ forms of the linearized theory from the weakly vanishing Noether current S_f of the full theory through

$$k_f[\delta\phi,\phi] = k_f^{\mu\nu}(d^{n-2}x)_{\mu\nu} = \frac{|\lambda|+1}{|\lambda|+2}\partial_{(\lambda)}[\delta\phi^i \frac{\delta}{\delta\phi^i_{((\lambda)\nu)}}\frac{\partial}{\partial dx^\nu} S_f], \tag{2.8}$$

by replacing f by reducibility parameters of the linearized theory, ϕ^i by the background solution $\bar{\phi}^i$ and $\delta\phi^i$ by any solution $\bar{\varphi}^i$ of the theory linearized around $\bar{\phi}^i$. Explicit expressions for the higher order Euler-Lagrange derivatives can be found in [29] and [30]; our conventions and notations for multi-indices are summarized in the appendix of [19].

This construction is applicable in the case of Lagrangians that are of finite, arbitrarily high order in derivatives. In case S_f is of second order in derivatives, which usually requires the Euler-Lagrange equations of motion to be of second order as well, one needs the higher order Euler-Lagrange operators up to order 2,

$$k_f[\delta\phi,\phi] = \frac{1}{2}\delta\phi^i \frac{\delta}{\delta\phi^i_\nu}\frac{\partial}{\partial dx^\nu} S_f + \frac{2}{3}\partial_\sigma[\delta\phi^i \frac{\delta}{\delta\phi^i_{\nu\sigma}}\frac{\partial}{\partial dx^\nu} S_f]. \tag{2.9}$$

For theories for which S_f is of first order in derivatives, only the first higher order Euler-Lagrange operator is involved and reduces to the partial derivative, so that the formula simplifies to

$$k_f[\delta\phi, \phi] = \frac{1}{2}\delta\phi^i \frac{\partial}{\partial\phi^i_\nu} \frac{\partial}{\partial dx^\nu} S_f. \tag{2.10}$$

A first order formulation can always be achieved by introducing suitable auxiliary and generalized auxiliary fields.

For notational simplicity, we take units where the gravitational constant is $G = (16\pi)^{-1}$. More standard choices correspond to multiplying the action and forms below by $(16\pi G)^{-1}$.

2.4 Asymptotics

The strategy to use the linearized theory at infinity with prescribed asymptotics in order to define conservation laws in general relativity is discussed in detail in [3].

Rather than trying to develop a theory for the asymptotic case, as done for instance in [19] for the "asymptotically linear" case, one can take a more pragmatic point view that consists in using the formula for the $n-2$ forms above, while substituting asymptotic reducibility parameters and asymptotic solutions determined by the fall-off conditions instead of exact ones determined by the linearized theory. The approach is reminiscent of the one for current algebras associated to broken global symmetries described in [31]. As a result, the currents are in general neither integrable nor conserved. This is precisely what happens for general relativity with asymptotically flat boundary conditions at null infinity [9, 24, 25].

3 Application to the Cartan formulation of GR

3.1 Cartan formulation

Consider an n dimensional spacetime with a moving, (pseudo-)orthonormal frame,

$$e_a = e_a{}^\mu \frac{\partial}{\partial x^\mu}, \quad e^a = e^a{}_\mu dx^\mu, \tag{3.1}$$

where $e_a{}^\mu e^a{}_\nu = \delta^\mu_\nu$, $e_a{}^\mu e^b{}_\mu = \delta^b_a$, and $\partial_a f = e_a(f)$. The structure functions are defined by

$$[e_a, e_b] = D^c{}_{ab} e_c \iff de^a = -\frac{1}{2} D^a{}_{bc} e^b e^c. \tag{3.2}$$

For further use, note that if $\mathbf{e} = \det e^a{}_\mu$ then

$$\partial_\mu(\mathbf{e}\, e^\mu{}_a) = \mathbf{e}\, D^b{}_{ba}, \tag{3.3}$$

We thus assume that there is a pseudo-Riemannian metric,

$$g_{\mu\nu} = e^a{}_\mu \eta_{ab} e^b{}_\nu, \tag{3.4}$$

with a flat (Lorentz) metric in tangent space, $\eta_{ab} = \mathrm{diag}((-)1, 1, \dots, 1)$. As usual, tangent space indices $a, b, \dots$ and world indices $\mu, \nu, \dots$ are lowered and raised with g_{ab}, $g_{\mu\nu}$, and their inverses, and converted into each other using the vielbeins $e_a{}^\mu$ and their inverse.

Local (Lorentz) rotations are denoted by $\Lambda_a{}^b(x)$ with $\Lambda_a{}^b \eta_{bc} \Lambda_d{}^c = \eta_{ad}$, or equivalently, $\Lambda^d{}_b \Lambda_a{}^b = \delta^d_a$. Under a combined frame rotation and coordinate transformation, we have

$$e'^{\,\mu}_a(x') = \Lambda_a{}^b(x) e_b{}^\nu(x) \Lambda^\mu{}_\nu(x), \tag{3.5}$$

with $\Lambda^\mu{}_\nu = \dfrac{\partial x'^\mu}{\partial x^\nu}$.

In addition, assume that there is an affine connection defined by

$$D_c e_a = \Gamma^b{}_{ac} e_b, \tag{3.6}$$

and that metricity holds,

$$D_a \eta_{bc} = 0. \tag{3.7}$$

This implies in particular that

$$\Gamma_{abc} = -\Gamma_{bac}, \tag{3.8}$$

In terms of the Poincaré algebra,

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \quad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \tag{3.9}$$

one defines the Lorentz connection $\Gamma = \frac{1}{2}\Gamma^{ab} J_{ab}$, with $\Gamma^{ab} = \Gamma^{ab}{}_\mu dx^\mu = \Gamma^{ab}{}_c e^c$, and $e = e^a P_a$.

The torsion and curvature tensors are defined by

$$T = T^a P_a = de + [\Gamma, e], \quad R = \frac{1}{2} R^{ab} J_{ab} = d\Gamma + \frac{1}{2}[\Gamma, \Gamma], \tag{3.10}$$

where the wedge product is omitted, and the bracket is the graded commutator.

More explicitly, $T^a = \frac{1}{2} T^a{}_{bc} e^b e^c = de^a + \Gamma^a{}_b e^b$, so that

$$T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \Gamma^a{}_{b\mu} e^b{}_\nu - \Gamma^a{}_{b\nu} e^b{}_\mu, \tag{3.11}$$

$$T^c{}_{ab} = 2\Gamma^c{}_{[ba]} + D^c{}_{ba}, \tag{3.12}$$

where round (square) brackets denote (anti) symmetrization of enclosed indices divided by the factorial of the number of indices involved. In this case,

$$\partial_\mu(\mathbf{e}\, v^\mu) = \mathbf{e}\,(D_\mu + e_b{}^\nu \partial_\mu e^b{}_\nu) v^\mu = D_\mu(\mathbf{e}\, v^\mu), \tag{3.13}$$

with $D_\mu v^\mu = \partial_\mu v^\mu$ for the Lorentz connection and the definition

$$D_\mu \mathbf{e} = \mathbf{e}\,(e_b{}^\nu \partial_\mu e^b{}_\nu). \tag{3.14}$$

In particular, this implies that

$$D_\mu(\mathbf{e}\, e^\mu{}_a) = \mathbf{e}\, T^b{}_{ab}. \tag{3.15}$$

For the curvature components, $R^a{}_b = \frac{1}{2} R^a{}_{bcd} e^c e^d = d\Gamma^a{}_b + \Gamma^a{}_c \Gamma^c{}_b$, we have

$$R^f{}_{c\mu\nu} = \partial_\mu \Gamma^f{}_{c\nu} - \partial_\nu \Gamma^f{}_{c\mu} + \Gamma^f{}_{d\mu}\Gamma^d{}_{c\nu} - \Gamma^f{}_{d\nu}\Gamma^d{}_{c\mu}, \tag{3.16}$$

$$R^f{}_{cab} = \partial_a \Gamma^f{}_{cb} - \partial_b \Gamma^f{}_{ca} + \Gamma^f{}_{da}\Gamma^d{}_{cb} - \Gamma^f{}_{db}\Gamma^d{}_{ca} - D^d{}_{ab}\Gamma^f{}_{cd}. \tag{3.17}$$

Furthermore,

$$[D_a, D_b] v_c = -R^d{}_{cab} v_d - T^d{}_{ab} D_d v_c. \tag{3.18}$$

Under a local frame rotation, we have

$$e' = \Lambda e \Lambda^{-1}, \quad \Gamma' = \Lambda \Gamma \Lambda^{-1} + \Lambda d \Lambda^{-1}, \tag{3.19}$$

so that

$$T' = \Lambda T \Lambda^{-1}, \quad R' = \Lambda R \Lambda^{-1}. \tag{3.20}$$

Defining $\Lambda = \mathbf{1} + \omega + O(\omega^2)$, with $\omega = \frac{1}{2}\omega^{ab} J_{ab}$, $\omega^{ab} = -\omega^{ba}$, we have

$$\delta_\omega \Gamma = -(d\omega + [\Gamma, \omega]) \iff \delta_\omega \Gamma^{ab} = -(d\omega^{ab} + \Gamma^a{}_c \omega^{cb} + \Gamma^b{}_c \omega^{ac}), \tag{3.21}$$

and also

$$\delta_\omega e = [\omega, e] \iff \delta_\omega e^a = \omega^a{}_b e^b. \tag{3.22}$$

Under a coordinate transformation, we have

$$e'^a{}_\mu = \Lambda_\mu{}^\nu e^a{}_\nu, \quad \Gamma'^a{}_{b\mu} = \Lambda_\mu{}^\nu \Gamma^a{}_{b\nu}, \tag{3.23}$$

and for $x'^\mu = x^\mu - \xi^\mu + O(\xi^2)$, $\Lambda^\mu{}_\nu = \delta^\mu_\nu - \partial_\nu \xi^\mu + O(\xi^2)$, so that $\omega_\nu{}^\mu = \partial_\nu \xi^\mu$ and

$$\delta_\xi e^a{}_\mu = \mathcal{L}_\xi e^a_\mu, \quad \delta_\xi \Gamma^a{}_{b\mu} = \mathcal{L}_\xi \Gamma^a{}_{b\mu}, \tag{3.24}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative.

The Bianchi identities are

$$dT + [\Gamma, T] = [R, e], \quad dR + [\Gamma, R] = 0. \tag{3.25}$$

Explicitly,

$$R^a{}_{[bcd]} = D_{[b} T^a{}_{cd]} + T^a{}_{f[b} T^f{}_{cd]}, \quad D_{[f} R^a{}_{|b|cd]} = -R^a{}_{bg[f} T^g{}_{cd]}, \tag{3.26}$$

where a bar encloses indices that are not involved in the (anti) symmetrization. The Ricci tensor is defined by $\mathbf{R}_{ab} = R^c{}_{acb}$, while $S_{ab} = R^c{}_{cab} = 0$. Contracting the Bianchi identities gives

$$\mathbf{R}_{ab} - \mathbf{R}_{ba} = -D_c T^c{}_{ab} - 2D_{[a} T^c{}_{b]c} - T^c{}_{fc} T^f{}_{ab}, \tag{3.27}$$

$$2D_{[f}\mathbf{R}_{|b|d]} + D_c R^c{}_{bdf} = \mathbf{R}_{bg} T^g{}_{df} - 2R^c{}_{b[f|g|} T^g{}_{d]c}. \tag{3.28}$$

The curvature scalar is defined by $\mathbf{R} = g^{ab}\mathbf{R}_{ab}$, the Einstein tensor by

$$G_{ab} = \mathbf{R}_{ab} - \frac{1}{2} g_{ab}\mathbf{R}. \tag{3.29}$$

Contracting (3.28) with η^{bf} gives the contracted Bianchi identities,

$$D_b G^b{}_a = \frac{1}{2} R^{bc}{}_{da} T^d{}_{bc} + \mathbf{R}^b{}_c T^c{}_{ab}. \tag{3.30}$$

For any affine connection, metricity $D_a g_{bc} = 0$, implies that the connection is given by

$$\Gamma_{abc} = \{_{abc}\} + K_{abc} + r_{abc}, \tag{3.31}$$

where the Christoffel symbols are given by

$$\{_{abc}\} = \frac{1}{2}(\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc}) = \{_{acb}\}, \tag{3.32}$$

K_{abc} are the components of the contorsion tensor,

$$K_{abc} = \frac{1}{2}(T_{bac} + T_{cab} - T_{abc}) = -K_{bac}, \tag{3.33}$$

and

$$r_{abc} = \frac{1}{2}(D_{bac} + D_{cab} - D_{abc}) = -r_{bac}. \tag{3.34}$$

Furthermore, one can directly show that

$$\Gamma^a{}_{b\mu} = e^a{}_\nu(\partial_\mu e_b{}^\nu + \Gamma^\nu{}_{\rho\mu} e^\rho{}_b) \iff \Gamma_{abc} = e_{a\nu}\partial_c e_b{}^\nu + e_a{}^\mu e_b{}^\nu e_c{}^\rho \Gamma_{\mu\nu\rho}. \tag{3.35}$$

with

$$\Gamma_{\mu\nu\rho} = \{_{\mu\nu\rho}\} + K_{\mu\nu\rho}. \tag{3.36}$$

Note also that for a Lorentz connection, (3.31) reduces to

$$\Gamma_{abc} = K_{abc} + r_{abc}. \tag{3.37}$$

3.2 Variational principle

In the standard Cartan formulation, the variables of the variational principle are the components of the vielbein $e_a{}^\mu$ and a Lorentz connection 1-form in the coordinate basis, $\Gamma^a{}_{b\mu}$ in terms of which the action is

$$S^C[e_a{}^\mu,\Gamma^b{}_{c\nu}] = \int d^nx\, L^C = \int d^nx\, \mathbf{e}\,(R^{ab}{}_{\mu\nu}e_a{}^\mu e_b{}^\nu - 2\Lambda). \tag{3.38}$$

Using

$$\delta R^a{}_{b\mu\nu} = D_\mu\delta\Gamma^a{}_{b\nu} - D_\nu\delta\Gamma^a{}_{b\mu}, \tag{3.39}$$

the variation of the action is given by

$$\delta S^C = \int d^nx\, \mathbf{e}\big[2(G^a{}_\mu + \Lambda e^a{}_\mu)\delta e_a{}^\mu + e_a{}^\mu e_b{}^\nu(D_\mu\delta\Gamma^{ab}{}_\nu - D_\nu\delta\Gamma^{ab}_\mu)\big]. \tag{3.40}$$

Using now (3.13) and neglecting boundary terms, this gives

$$\delta S^C = \int d^nx\, \big[2\mathbf{e}\,(G^a{}_\mu + \Lambda e^a{}_\mu)\delta e_a{}^\mu + 2D_\nu(\mathbf{e}\,e_a{}^\mu e_b{}^\nu)\delta\Gamma^{ab}{}_\mu\big], \tag{3.41}$$

so that

$$\frac{\delta L^C}{\delta e_a{}^\mu} = 2\mathbf{e}\,(G^a{}_\mu + \Lambda e^a{}_\mu), \tag{3.42}$$

$$\frac{\delta L^C}{\delta\Gamma^{ab}{}_\mu} = 2D_\nu(\mathbf{e}\,e_{[a}{}^\mu e_{b]}{}^\nu) = \mathbf{e}\,(T^\mu{}_{ab} + 2e^\mu_{[a}T^c{}_{b]c}). \tag{3.43}$$

Contracting the equations of motions associated to (3.43) with $e_\mu{}^b$ gives $T^b{}_{ab} = 0$. When re-injecting, this implies $T^a{}_{bc} = 0$. It follows that when the equations of motion for $\Gamma^{ab}{}_\mu$ hold, the connection is torsionless and thus given by $\Gamma_{abc} = r_{abc}$. The fields $\Gamma^{ab}{}_\mu$ are thus entirely determined by $e_a{}^\mu$ so that $\Gamma^{ab}{}_\mu$ are auxiliary fields.

Using (3.40) for an infinitesimal gauge transformation as in (3.21), (3.22), (3.24) under the form

$$\delta_{\xi,\omega}S^C = \int d^nx\, \big[\frac{\delta L^C}{\delta e_a{}^\mu}\delta_{\xi,\omega}e_a{}^\mu + \frac{\delta L^C}{\delta\Gamma^{ab}{}_\mu}\delta_{\xi,\omega}\Gamma^{ab}{}_\mu\big], \tag{3.44}$$

and integrating by parts in order to isolate undifferentiated gauge parameters as in (2.6) gives the Noether identities

$$\frac{\delta L^C}{\delta e^{[a|\mu|}}e_{b]}{}^\mu + D_\mu\frac{\delta L^C}{\delta\Gamma^{ab}{}_\mu} = 0, \tag{3.45}$$

$$\frac{\delta L^C}{\delta e_a{}^\mu}\partial_\rho e_a{}^\mu + \frac{\delta L^C}{\delta\Gamma^{ab}{}_\mu}\partial_\rho\Gamma^{ab}_\mu + \partial_\mu\Big(\frac{\delta L^C}{\delta e_a{}^\rho}e_a{}^\mu - \frac{\delta L^C}{\delta\Gamma^{ab}{}_\mu}\Gamma^{ab}{}_\rho\Big) = 0. \tag{3.46}$$

Equation (3.45) can be shown to be equivalent to (3.27). Using (3.45), equation (3.46) can be written as

$$\partial_\mu\Big(\frac{\delta L^C}{\delta e_a{}^\rho}e_a{}^\mu\Big) + \frac{\delta L^C}{\delta e_a{}^\mu}D_\rho e_a{}^\mu + \frac{\delta L^C}{\delta\Gamma^{ab}{}_\mu}R^{ab}{}_{\rho\mu} = 0, \tag{3.47}$$

and then be shown to be equivalent to (3.30).

3.3 Construction of the co-dimension 2 forms

When keeping the boundary term, one finds the weakly vanishing Noether current associated to the gauge symmetries as

$$S^\mu_{\xi,\omega} = \frac{\delta L^C}{\delta \Gamma^{ab}{}_\mu}(-\omega^{ab} + \Gamma^{ab}{}_\rho \xi^\rho) - \frac{\delta L^C}{\delta e_a{}^\rho} e_a{}^\mu \xi^\rho. \tag{3.48}$$

The associated co-dimension 2 form $k_{\xi,\omega} = k^{\mu\nu}_{\xi,\omega}(d^{n-2}x)_{\mu\nu}$ computed through (2.10) is given by

$$\begin{aligned} k^{\mu\nu}_{\xi,\omega} = \mathbf{e}\big[(2\delta e_a{}^\mu e_b{}^\nu + e^c{}_\lambda \delta e_c{}^\lambda e_a{}^\nu e_b{}^\mu)(-\omega^{ab} + \Gamma^{ab}{}_\rho \xi^\rho) \\ + \delta\Gamma^{ab}{}_\rho(\xi^\rho e_a{}^\mu e_b{}^\nu + 2\xi^\mu e_a{}^\nu e_b{}^\rho) - (\mu \longleftrightarrow \nu)\big]. \end{aligned} \tag{3.49}$$

This can also be written as

$$k_{\xi,\omega} = -\delta K^K_{\xi,\omega} + K^K_{\delta\xi,\delta\omega} - \xi^\nu \frac{\partial}{\partial dx^\nu}\Theta_\xi, \tag{3.50}$$

where

$$K^K_{\xi,\omega} = 2\mathbf{e}\, e_a{}^\nu e_b{}^\mu(-\omega^{ab} + \Gamma^{ab}{}_\rho \xi^\rho)(d^{n-2}x)_{\mu\nu}, \quad \Theta_\xi = 2\mathbf{e}\,\delta\Gamma^{ab}{}_\rho e_a{}^\mu e_b{}^\rho (d^{n-1}x)_\mu. \tag{3.51}$$

According to the general results reviewed in section 2, the co-dimension 2 form is closed, $d_H k_{\xi,\omega} = 0$, or equivalently, $\partial_\nu k^{\mu\nu}_{\xi,\omega} = 0$, if $e_a{}^\mu, \Gamma^{ab}{}_\mu$ are solutions to the Euler-Lagrange equations of motion, and thus to the Einstein equations, $\delta e_a{}^\mu, \delta\Gamma^{ab}{}_\mu$ solutions to the linearized equations and ω^{ab}, ξ^ρ satisfy

$$\mathcal{L}_\xi e_a{}^\mu + \omega_a{}^b e_b{}^\mu \approx 0, \quad \mathcal{L}_\xi \Gamma^{ab}{}_\mu \approx D_\mu \omega^{ab}, \tag{3.52}$$

where $\approx$ now denotes on-shell for the background solution and is relevant in case the parameters ω^{ab}, ξ^ρ explicitly depend on the background solution $e_a{}^\mu, \Gamma^{ab}{}_\mu$ around which one linearizes. Note that the first equation also implies in particular that ξ^ρ is a possibly field dependent Killing vector of the background solution $g_{\mu\nu}$,

$$\mathcal{L}_\xi g_{\mu\nu} \approx 0, \tag{3.53}$$

and that

$$\omega^{ab} \approx -e^b{}_\mu \mathcal{L}_\xi e^{a\mu} \approx -e^{[b}{}_\mu \mathcal{L}_\xi e^{a]\mu}. \tag{3.54}$$

3.4 Reduction to the metric formulation

In order to compare with the results in the metric formulation, let us go on-shell for the auxiliary fields $\Gamma^{ab}{}_\mu$ and eliminate ω^{ab} using (3.54). The former implies that we are in the torsionless case with the Lorentz connection simplified to $\Gamma^{ab}{}_{\mu} = r^{ab}{}_{\mu}$, while (3.35) reduces to

$$\Gamma^{ab}{}_{\mu} = e^{a}{}_{\nu}\nabla_{\mu}e^{b\nu} = e^{[a}{}_{\nu}\nabla_{\mu}e^{b]\nu}, \tag{3.55}$$

with $\nabla_{\mu}v^{\nu} = \partial_{\mu}v^{\nu} + \{^{\nu}{}_{\rho\mu}\}v^{\rho}$. Note also that the Killing equation can be written as $\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} \approx 0$. Together with (3.55), we have

$$-\,\omega^{ab} + \Gamma^{ab}{}_{\rho}\xi^{\rho} \approx -e^{[a}{}_{\rho}e^{b]}{}_{\sigma}\nabla^{\rho}\xi^{\sigma}, \tag{3.56}$$

$$\delta\Gamma^{ab}{}_{\rho} = \delta e^{[a}{}_{\sigma}\nabla_{\rho}e^{b]\sigma} + e^{[a}{}_{\sigma}\delta\{^{\sigma}{}_{\tau\rho}\}e^{b]\tau} + e^{[a}{}_{\sigma}\nabla_{\rho}\delta e^{b]\sigma}, \tag{3.57}$$

with

$$\delta\{^{\sigma}{}_{\tau\rho}\} = \frac{1}{2}g^{\sigma\delta}(\nabla_{\rho}\delta g_{\delta\tau} + \nabla_{\tau}\delta g_{\delta\rho} - \nabla_{\delta}\delta g_{\tau\rho}). \tag{3.58}$$

Using that

$$\delta e^{a}{}_{\mu}e_{a\nu} = \frac{1}{2}h_{\mu\nu} + \delta e^{a}{}_{[\mu}e_{|a|\nu]}, \tag{3.59}$$

with $h_{\mu\nu} = \delta g_{\mu\nu}$, indices being lowered and raised with $g_{\mu\nu}$ and its inverse, and $h = h^{\mu}_{\mu}$, substitution into (3.49) gives

$$6\sqrt{|g|}\nabla_{\rho}(\delta e_{a}{}^{[\mu}e^{|a|\nu}\xi^{\rho]}) + k^{\mu\nu}_{\xi}, \tag{3.60}$$

where the first term can be dropped since it is trivial in the sense that it corresponds to the exterior derivative of an $n-3$ form, while

$$k^{\mu\nu}_{\xi} = \sqrt{|g|}\,\big[\xi^{\nu}\nabla^{\mu}h + \xi^{\mu}\nabla_{\sigma}h^{\sigma\nu} + \xi_{\sigma}\nabla^{\nu}h^{\sigma\mu} + \frac{1}{2}h\nabla^{\nu}\xi^{\mu} + \frac{1}{2}h^{\mu\sigma}\nabla_{\sigma}\xi^{\nu} + \frac{1}{2}h^{\nu\sigma}\nabla^{\mu}\xi_{\sigma} - (\mu \longleftrightarrow \nu)\big]. \tag{3.61}$$

We have thus recovered the results of the metric formulation since the last expression agrees with the one given in [20][1], which in turn is equivalent to those derived directly in the metric formulation in [19].

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[1] up to a typo in the second term of equation (35) in that reference, where $\tilde{\xi}^{\mu}D_{\sigma}h^{\sigma\mu}$ should read $\tilde{\xi}^{\mu}D_{\sigma}h^{\sigma\nu}$.

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