Fundamental forces and their dynamics

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In this essay, we wish to propose a general principle: *the equation of motion or dynamics of a fundamental force should not be prescribed but instead be entirely driven by geometry of the appropriate spacetime manifold.* The motivation for this pronouncement comes from the fact that the equation of motion of general relativity follows from the geometry of Riemannian spacetime manifold without appeal to anything else from outside. The driving differential geometric property is the Bianchi identity satisfied by the Riemann curvature tensor. Similarly it is geometry of the principal tangent bundle of fibre spacetime manifold that may account for dynamics of the gauge vector fields. It is the classical electric force for the Abelian gauge symmetry group while the non-Abelian symmetry leads to the non-Abelian forces, the weak and the strong. We shall also reflect on a unified picture of the basic forces, and the duality correspondences it may inspire.

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I. INTRODUCTION

Newton's law of gravity and Coulomb's law of electric force were motivated by the conservation of flux of force across 2-sphere ensuring conservation of mass and electric charge enclosed inside. By synthesizing Coulomb, Ampere and Faraday, and adding his ingenious displacement current, Maxwell wrote the famous equations of motion of electromagnetic field. This synthesis of electric and magnetic fields gave rise to existence of electromagnetic wave that propagated in vacuum with the universally constant invariant velocity, the velocity of light. The realization that electric or magnetic aspect is manifestation of the same force when charge is at rest or moving. The both should therefore be brought on the same footing; i.e., having the same dimension. For that to happen universally for all observers, again a universally invariant velocity would be required. Such a velocity is given by the (electromagnetic wave) velocity of light.

Like the synthesis of electric and magnetic fields, the invariant velocity would also bind space and time into a four dimensional spacetime manifold. That would then accord to a new mechanics, special relativity. We can also envision that special relativity is driven by universalization of the Newtonian mechanics, which described motion of all massive particles, to include zero mass particles. The invariant velocity of zero mass particle is synthesized in spacetime structure. It serves as an excellent spacetime background for the Maxwell's electrodynamics as well as for the rest of physics except, of course, gravity.

Like Newtonian mechanics, how about universalizing the Newtonian gravity by including massless particles as well. That means gravity should also link to zero mass particles; i.e., it is universal with linkage to both massive as well as massless particles. It links to everything that physically exist. Note that motion of zero mass particle is a property of spacetime and hence its interaction with any force has to be negotiated through spacetime itself. For inclusion of zero mass particles in gravitational interaction, space(spacetime) has to be curved [1–3]. For universalization of gravitational force, the framework has to be enlarged from flat to curved spacetime. Gravity is thus described by spacetime curvature and hence it becomes the property of spacetime geometry. That is, the gravitational equation of motion cannot be prescribed instead it has to be determined by the geometry of spacetime. That is what happens in the Einstein gravity, general relativity (GR). In particular Newton's inverse square law is mandated by the Riemannian geometry of spacetime.

It then motivates us to ask, should this not be the case for the other fundamental forces as well ? Of course a particular force would accord to a geometry of appropriate spacetime manifold. For a universal force of gravity, spacetime manifold should be universally accessible, and hence it is spacetime itself which is accessible and shared by all equally without any qualification. The gravitational dynamics follows from the differential geometric property – the Bianchi identity satisfied by the Riemann curvature tensor. In GR gravitational dynamics resides in the spacetime curvature.

What are the other possible choices for spacetime manifold. One obvious candidate is the fibre bundle manifold of principal tangent bundle. With the qualification of fibre bundle, this manifold is accessible only to particles having a particular charge. Again it is the curvature of the principal tangent bundle that leads through the Bianchi identity to an equation of motion of a force – the Maxwell equation of electrodynamics. The dynamics of electric force is thus

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governed by the geometry of principal tangent bundle spacetime. More generally dynamics of a gauge vector field could be described by the tangent bundle spacetime.

A vector field could as well be decorated by internal space indices indicating $SU(N)$ symmetries, where $N = 1$, is the Abelian symmetry of Maxwell field while $N > 1$ describes non-Abelian symmetry of the Yang-Mills fields. In particular $SU(2)$ and $SU(3)$ respectively correspond to the weak and the strong forces.

It then turns out that equation of motion of all the four fundamental forces follow from the geometry of the corresponding spacetime manifold without appeal to anything else from outside. We propose that this should be the defining property of a fundamental force.

The paper is organised as follows: In the next section we would derive the equations of motion of the four fundamental forces from the geometry of relevant spacetime. We shall begin with the derivation of the Einstein's equation of general relativity and then take up derivation of the equations of motion for Abelian and non-Abelian vector gauge forces. We conclude with a discussion.

II. EQUATION OF MOTION OF FUNDAMENTAL FORCES

We shall begin by geometric characterization of homogeneous and inhomogeneous spacetime, and then show that the Einstein's gravitational equation follows naturally when spacetime is inhomogeneous.

A. Universal force – Einstein's gravity

Since we wish to derive the equation of motion of a universal force, the corresponding spacetime should also be accessible to all particles without any exception. It is therefore the four dimensional spacetime manifold in which space and time are bound together by the universally invariant, the constant velocity of light. As a matter of fact, homogeneity of space and time mandates existence of the universally invariant velocity [1–3].

In the absence of all forces, spacetime is naturally homogeneous, and hence its geometry should also be homogeneous; i.e., Riemann curvature tensor should be homogeneous. It should be covariantly constant which means it should be written in terms of the metric, g_{ab} with vanishing covariant derivative. We thus write

$$R_{abcd} = \Lambda(g_{ac}g_{bd} - g_{ad}g_{bc})\,. \qquad (1)$$

This is a spacetime of constant curvature Λ which could, as determined by the experiments and observations, be positive, dS or negative, AdS or zero, flat Minkowski. It is the most general characterization of "free" homogeneous spacetime. It imbibes no dynamics of any force. Dynamics of universal force emerges when spacetime becomes inhomogeneous.

Let us appeal to the famous theorem if differential geometry, $D^2 = 0$ identically where D is a properly defined covariant derivative. It is known as the Bianchi identity. John Wheeler elegantly paraphrased it as *boundary of boundary is zero – a conservation law*. Its familiar well known manifestations are curl of gradient and divergence of curl being zero. These are for scalar and vector fields, and their higher order analogue demands that the Bianchi derivative of the Riemann curvature should vanish identically; i.e.,

$$\nabla_{[e}R_{ab]cd} = 0\,. \qquad (2)$$

This is the Bianchi differential identity satisfied by the Riemann tensor.

Let us now take its trace, multiplying by $g^{ac}g^{bd}$, we get

$$\nabla_b G_a{}^b = 0\,, \qquad (3)$$

where

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}\,,\; R_{ac} = g^{bd}R_{abcd}\,,\; R = g^{ab}R_{ab}\,. \qquad (4)$$

We have derived here a divergence free second rank symmetric tensor, known as the Einstein tensor, from the Riemann curvature which involves second derivative of the metric tensor.

On integrating the divergence equation, we obtain

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}\;,\; \nabla_b T_a{}^b = 0\,. \qquad (5)$$

The Einstein tensor is the creature of the Riemann curvature and hence it contains second derivative of the metric, a second order differential operator like $\nabla^2\Phi$, while Λg_{ab} is a constant of integration relative to covariant derivative. Thus far we have come purely driven by spacetime geometry with no reference or appeal to anything from outside.

The above equation would become the equation of motion of a force if we identify the source with the second rank symmetric tensor T_{ab} with vanishing divergence. Since this source makes spacetime inhomogeneous for all particles without any qualification or exception, it should represent a universal physical property which is shared by everything that physically exists. The obvious answer is energy-momentum. With this, the above equation is the Einstein's equation of gravitation

– general relativity [1–3].

It is remarkable that the Einstein gravity springs up naturally without any prescription from outside when spacetime is inhomogeneous. As a matter of fact we were simply exploring what is it that makes spacetime inhomogeneous without reference to gravity at all. The answer turns out that it is the presence of energy-momentum distribution that makes spacetime universally inhomogeneous and the universal force it gives rise to is uniquely the Einstein gravity.

We have thus derived the dynamics of Einstein's gravitational force from the geometry of spacetime alone. In particular it should be emphasized that the inverse square law is now dictated by the spacetime geometry.

This sets the setting for similar exploration of the other fundamental forces.

B. Abelian Maxwell force

Now let us explore the fibre bundle geometry of the principal tangent bundle. Its curvature is a 2-form F_{ab} which satisfies the Bianchi identity,

$$\nabla^*_b F^{ab} = 0\,, \tag{6}$$

where ${}^*F^{ab}$ is the Hodge dual of F^{ab}. This implies that F_{ab} is a curl of a vector; i.e.,

$$F_{ab} = \nabla_{[a} A_{b]} = -F_{ba}\,. \tag{7}$$

Here A_a is the connection on the tangent bundle spacetime.

On account of the antisymmetry of F_{ab}, we write

$$\nabla_b \nabla_a F^{ab} = 0\,, \tag{8}$$

which on integration yields

$$\nabla_a F^{ab} = -J^b\,,\ ,\nabla_a J^a = 0\,. \tag{9}$$

This is the Maxwell's equation, the equation of motion of the electric force where A_a is the 4-vector potential and J^a is the 4-conserved current.

Like the curvature of spacetime leads to the Einstein's gravitational equation, similarly curvature of the principal tangent bundle spacetime to the Maxwell's equation of electrodynamics. Electric force has the Abelian $U(1)$ symmetry group and is therefore long range, and has massless free propagation.

The dynamics of the two classical long range forces, gravity and electromagnetic, follows respectively from the corresponding geometry of Riemannian spacetime and the principal tangent bundle spacetime. There is no reference or appeal to anything else from outside, and it all flow naturally from purely the geometric properties.

C. Non-Abelian Yang-Mills forces

Non-Abelian character refers to the internal space symmetry group, $SU(N)$ for $N > 1$. In $N = 1$, it includes the Abelian case, $SU(1) = U(1)$. Let ψ_a denote generically the fields which transform under certain representation of the above symmetry group with generators t_A, represented by matrices $(t_A)_\mathsf{a}{}^\mathsf{b}$ (the sans-serif font is used here to distinguish these from spacetime indices) [4]. The fields transform as $\delta\psi_\mathsf{a} = i\epsilon^A (t_A)_\mathsf{a}{}^\mathsf{b}\psi_\mathsf{b}$. The gauge covariant derivative in this case is introduced via the gauge potential G^A_a which transforms as $\delta G^A_i = \partial_i \epsilon^A + i\epsilon^C (\tilde{t}_C)^A{}_B G^B_i$ where $\tilde{t}_A$ indicates the adjoint representation. The gauge covariant derivative is then given by $(D_i\psi)_\mathsf{b} = \partial_i\psi_\mathsf{b} - iG^B_i (t_B)_\mathsf{b}{}^\mathsf{c}\psi_\mathsf{c}$. For such a theory, one may show that $([D_i, D_j]\psi)_\mathsf{a} = -i(t_A)_\mathsf{a}{}^\mathsf{b} F^A_{ij}\psi_\mathsf{b}$ with

$$F^A_{ij} = \partial_i G^A_j - \partial_j G^A_i + i(\tilde{t}_C)^A{}_B G^B_i G^C_j \tag{10}$$

The key departure from the Abelian case is the last term, which makes it subtle to define a conserved charge in terms of a gauge covariant current density. Nevertheless, if one considers the gauge invariant lagrangian of the form:

$$L = \frac{1}{4} F^A_{ij} F^{B\,ij} \delta_{AB} + L_{\text{matter}}(\psi_\mathsf{a}, (D_i\psi)_\mathsf{a}, \ldots) \tag{11}$$

and define the matter current density by

$$J^i_A = -i\frac{\partial L_{\text{matter}}}{\partial (D_i\psi)_\mathsf{a}} (t_A)_\mathsf{a}{}^\mathsf{b}\psi_\mathsf{b}\ , \tag{12}$$

then the equation of motion can be written as

$$D_k F^{ki}_A = -J^i_A\,. \tag{13}$$

Using the transformation law for F^{ki}_A, it can be shown that

$$D_i J^i_A = 0\,. \tag{14}$$

Note that, while the above current is defined in a gauge covariant manner and satisfies a gauge covariant conservation law, it does not yield a local conserved charge.

The critical step in getting at the equation of motion is the vanishing of double divergence of the curvature 2-form. We verify that continues to hold good for the gauge invariant derivative as well.

Then as before we can write the equation of motion for the non-Abelian Yang-Mills force. It is the weak force for $SU(2)$ and the strong for $SU(3)$. The geometry of principal tangent bundle spacetime wraps in all gauge vector forces Abelian as well as non-Abelian.

III. UNIFIED VIEW OF THE FUNDAMENTAL FORCES

We shall now attempt to envisage a unified picture [5] of the four basic forces. A force is characterized by the two properties: (a) its linkage, to what does it link to, and (b) its range, how far is its reach ?

Let us begin with a universal force which links to everything that physically exists, massive as well as massless, and it reaches out everywhere, and hence is long range. Since it links to massless particles, which can feel it only if force curves space, rather spacetime as space and time are already bound together. It has therefore to be described by curvature of spacetime and its dynamics following from the Bianchi identity as we have seen above.

The universal force is then uniquely the Einstein's gravity, GR.

It is envisaged that the other three forces emerge as these two properties are peeled off one by one. If we relax the property (a) that linkage not universal but to a particular charge and retain the long range property (b). Charge has to be bipolar and hence the force would be a vector field. Since it is long range, propagator would be massless. It would then be a vector gauge field with the dynamics given by the Maxwell's equations.

The long range force linking to a charge is therefore uniquely the Maxwell's electric force. Einstein gravity and Maxwell's electric force are the only two long range forces.

Now if we relax the long range property (b). The short range would require either propagator is massive which would then link only to massive particles or coupling is running such that it tends to zero as $r \to 0$, asymptotic freedom. The short range is achieved through the internal space structure $SU(N)$ superposed on the vector field. Then $N = 2, 3$ correspond respectively to the weak and the strong force.

The long range forces are the classical forces and they are uniquely gravity and electromagnetic. In this envision, there can exist no other long range force. On the other hand short range ones are quantum forces where the weak and the strong force are identified with $N = 2, 3$ for $SU(N)$ internal symmetry group. Here however the question remains open for higher N, keeping the possibility open for a new force.

It is remarkably insightful that all the four fundamental forces accord to such a simple unified picture based on the two characteristic properties, linkage and range. Further it also makes some very interesting suggestions for the duality relations. Electric and weak forces are complimentary to each other, the former is long range but not universal while the latter is short range but universal in the sense of linking to all massive particles. This is a pointer to the electro-weak unification. On the other hand, gravity is universal having universal linkage as well as long range while the strong force is neither, which points to their complementarity. The most remarkable result in this line is the famous AdS/CFT correspondence [6] which has been one of the drivers of high energy research for over two decades now. Also note that in the string theoretic understanding of the strong force, there is involvement of spin-2 massless rather than spin-1 particle.

These are the strong pointers for seeking duality relations between electric and weak, and gravity and strong in appropriate framework.

IV. DISCUSSION

We had set out to derive the equations of motion of all the four fundamental forces from geometry of the relevant spacetime. The force which is universal has to have universal linkage as well as it should reach everywhere, and hence long range. It has therefore to be governed by the geometry of spacetime manifold which is accessible to all particles; i.e., the spacetime itself. That universal force is the Einstein's gravity – GR having its dynamics entirely governed by Riemann curvature through the Bianchi identity. The Einstein's gravitational equation follows without reference to anything else.

There is a very satisfying general feature that accord to classical mechanics, that when spacetime is free of all forces including gravity, it is homogeneous having constant curvature. When it is inhomogeneous, Einstein's gravity emerges naturally. The constant curvature of spacetime characterizes force free state. The constant curvature spacetime is also characterized by maximal symmetry that it admits all ten Killing vectors in four dimensions. This means that the Lie derivative of the metric tensor along all ten Killing directions vanishes. The point to be noted is that maximally symmetry does not imply vanishing of Riemann curvature instead it makes it homogeneous with vanishing covariant derivative.

When force links to a particular charge, it has to be bipolar so as to give total charge zero [8] when summed over all charges in the Universe. It is then a vector field and its dynamics would be described by the curvature of principal tangent bundle spacetime.

The absence of force would then require the principal tangent bundle spacetime to be maximally symmetric.

That would mean Lie derivative of the connection on the bundle – vector potential should vanish relative to all ten Killing vectors. It could be easily verified that that would require the curvature 2-form $F_{ab} = 0$. That is, tangent bundle spacetime is homogeneous only when the curvature vanishes. This is in contrast from the gravity case where it required spacetime to be of constant (and not necessarily of zero) curvature. Here again

When force is long range, it is Abelian gauge vector field with $U(1)$ symmetry. A vector field could also have internal symmetry group $SU(N)$ which essentially amounts to putting some "hooks" to make it short range, either by making propagator massive (the weak force) or coupling running tending to zero as $r \to 0$, asymptotic freedom (the strong force). The dynamics of vector field whether Abelian or non-Abelian follows from curvature of the principal tangent bundle spacetime.

In this unified picture there could exist only the two classical long range forces, Einstein's gravity and Maxwell's electromagnetic field, and the both have been uniquely identified. Thus there is no room for any other long range force. Any new force, if it exists, has to be short range. On the other the weak and strong forces are identified but there is no uniqueness about it. Any $SU(N)$ force would be very well accommodated in this overall perception. The question of new force therefore remains open in the short range regime.

This visualization strongly suggests duality between electric (linkage to specific charge and long range) and weak (universal linkage all massive particles and short range) as they are complementary, and electro-weak unification is indicative of that. Similarly there is complementarity between gravity, which has universal linkage as well as long range, and strong, which is neither. The celebrated AdS/CFT correspondence [6] is perhaps indicative of that.

Finally it is a pleasure to dedicate this short discourse to the fond memory of my long time friend and esteemed colleague, Paddy, with much warmth and fondness. He was always interested in the fundamental questions and never hesitated in looking at things in a different and unconventional ways, usually exposing new insight and revealing vista. This is a modest offering in that vein.

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[8] One may ask how does total charge become zero for gravity where its charge, energy-momentum is unipolar and positive? The only way it could be neutralized is that gravitational field it produces has negative energy, which when summed over all space exactly cancels out positive charge [7]. This is why gravity is always attractive.