

Notes on Gravitational Physics

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These notes are self-contained, with the first six chapters used for a one-semester course with recommended texts by Wald[95], Misner, Thorne, & Wheeler (MTW)[60], and, particularly for gravitational waves, by Schutz[74] and Thorne & Blandford [92]. In its treatment of topics covered in these standard texts, the presentation here typically includes steps between equations that are skipped in Wald or MTW. Treatments of gravitational waves, particle orbits in black-hole backgrounds, the Teukolsky equation, and the initial value equations are motivated in part by the dramatic discoveries of gravitational waves from the inspiral and coalescence of binary black holes and neutron stars, advances in numerical relativity, and the expected launch of the LISA space-based observatory. Students are assumed to have encountered special relativity, but these notes give a detailed presentation with a geometrical orientation, starting with time dilation and length contraction and including relativistic particles, fluids, electromagnetism, and curvilinear coordinates. Chapters 2-5 cover curvature, the Einstein equation, relativistic stars, and black holes. Chapter 6, on gravitational waves, includes a discussion of detection and of noise in interferometric detectors.

Chapter 7, on the initial value problem, has a section on the form of the equations used in numerical relativity. Its notation is that used, for example, in Baumgarte & Shapiro[15] and Shibata[78]; the presentation here is taken in part from the text by Friedman and Stergioulas [39]. The notes also have a chapter on the Newman-Penrose formalism and the Teukolsky equation. Following that is a chapter on black-hole thermodynamics and a final chapter on the gravitational action and on conserved quantities for asymptotically flat spacetimes, using Noether's theorem. A detailed appendix on integration is largely taken from Ref. [39]. Cosmology is given as a separate course at UWM and is not covered here.

Corrections and suggestions have been made by several students, particularly by Charalampos Markakis to an earlier version of these notes and recently by Nikolaus Prusinski. A number of the black-and-white illustrations were drawn by Michael Hero.

Notation and conventions

Signature	$- + ++$
Indices $\alpha, \beta, \dots$	spacetime abstract
Indices μ, ν	spacetime concrete
Indices $a, b, \dots$	space and n -dimensional abstract
Indices $i, j, \dots$	space and n -dimensional concrete
Spacetime	M
Metric on M	$g_{\alpha\beta}$
Spacelike hypersurface	Σ
Unit normal to Σ	n^α
Projection $\perp n^\alpha$	γ_α^β
Riemann tensor	$[\nabla_a, \nabla_b]v_c = R_{abc}{}^d v_d$
Ricci tensor	$R_{ab} = R_{acb}{}^c$

In these notes, hyperref links to articles and books available online are in blue, as in this link to a translation of Einstein's 1916 paper and the following links to other notes on general relativity and gravitational physics, listed alphabetically by author:

Steven Balabus's notes

C. P. Burgess notes, short, with a good section on lensing.

Sean Carroll's Lecture Notes on GR (subsequently expanded to Carroll's widely used textbook).

Mike Guidry's Modern General Relativity Lecture Notes

Hollands and Sanders, Lecture Notes on General Relativity

Gerard 't Hooft's Intro to GR

Harvey Reall's General Relativity from a Cambridge course.

For bibliographies of GR books, look [here](#) and [here](#); more recent texts include some listed in the abstract as well as the longer version *Numerical Relativity*[14] of the Baumgarte-Shapiro book, Eric Poisson's *A Relativists Toolkit*[64], *Gravity* by Poisson and Cliff Will[65], and Ericourgoulhon's *3+1 Formalism in General Relativity*[45]. Two gravitational-wave texts are *Gravitational-Wave Physics and Astronomy*, by Jolien Creighton & Warren Anderson, [31] and *Gravitational Waves*, by Michele Maggiore, [56].

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Chapter 1. Introduction:

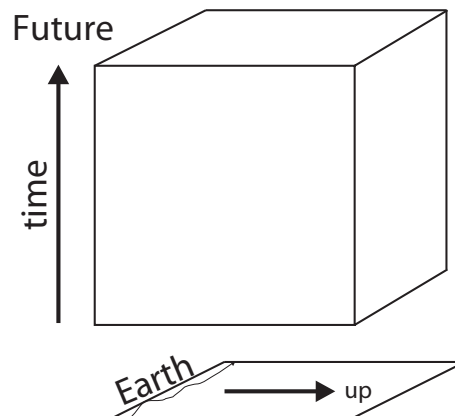
Physics in a Flat Spacetime

1.1 Concepts of Space and Time

The history of the way Western cultures have viewed space and time can – in retrospect – be described by a sequence in which most of what seemed to be the obvious features were gradually discarded. The following description is a retrospective caricature of the history, inaccurate in that, for example, Galileo is said to think of spacetime as flat when he had no precise notion that could properly be called “spacetime.”

Pre Greek

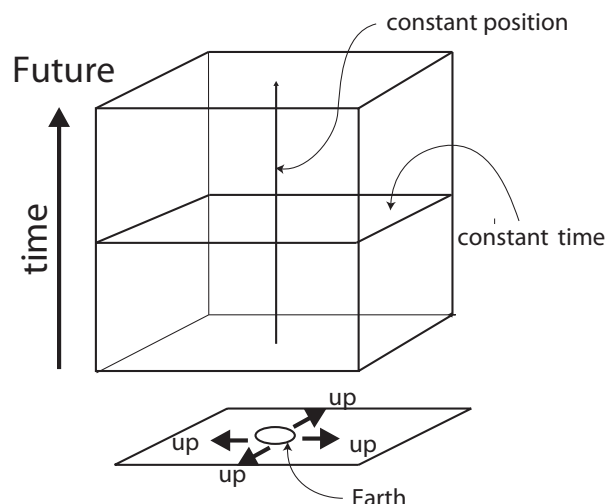
- Earth flat
- Preferred spatial direction (up)
- Space absolute
- Time absolute
- Preferred time direction
(toward future)
- Space flat



Space is *absolute* if it makes sense to say that events at two different times occur at the same place. All observers agree on what it means to be at rest – to stay at the same point of space. Time is *absolute* if it makes sense to say that events at two different places occurred at the same time. All observers agree on what events are simultaneous, and synchronized watches at different places read the same elapsed time between pairs of simultaneous events.

Greek

- Earth curved* (Italics mark change)
- Preferred set of spatial directions (up)*
- Space absolute
- Time absolute
- Preferred time direction
- Space flat



The Copernican Revolution and Galilean relativity

The loss of absolute space begins with Copernicus. His model (and presumably that of Aristarchus, 1700 years earlier) rested on a spectacular coincidence: The retrograde motion of all five planets is simultaneously explained if one simply places the Sun, instead of the Earth, at the center of the universe, regards the Earth as a planet, and places what were then six planets in nearly circular orbits around the Sun. For each outer planet — Mars, say — the explanation is simply this: Because the period of the Earth is shorter than the period of Mars, as the Earth orbits the Sun, it periodically passes Mars. When it does so, Mars appears to go backward relative to the background stars.

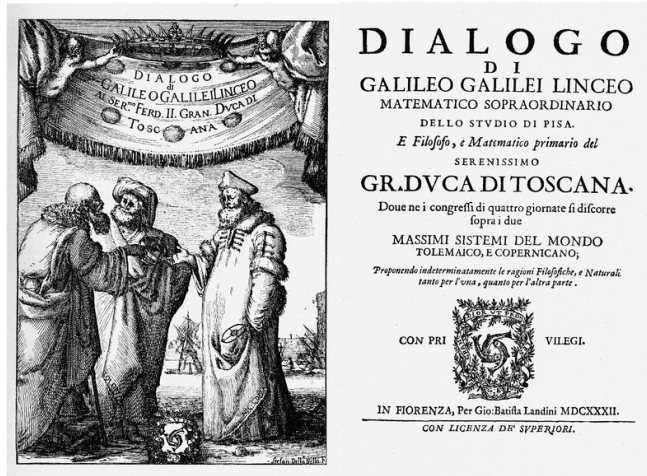
The model explains an additional coincidence: When the outer planets – e.g., Jupiter and Saturn – are in the same place in the sky, their retrograde motion occurs at nearly the same time. (A time lapse video that illustrates this dance, showing Jupiter and Saturn moving by the Pleiades and Hyades, is here: [pas de deux](#))

Finally, the Copernican model gives a relation between the time a planet spends in its retrograde loop and the angular size of the loop in the sky: First, the known periods of the planet and the Earth, together with the time the planet spends moving retrograde, determine the radius of the planet's orbit in terms of the radius of the Earth's orbit. Knowing this radius fixes the angular size of the retrograde loop. So one gets five numerical retrodictions, one number for each known planet, that have to coincide with observation for the model to hold.

That Copernicus found the distance to each of the planets in terms of the Earth's distance from the Sun would have been one final, breathtaking coincidence – and in this case a prediction, except that it was hundreds of years before the distance from the Sun to the planets could finally be measured. Textbooks emphasize the fact that Copernicus' model of retrograde motion was not significantly more accurate than Ptolemy's Earth-centered model. Because he had to make corrections to circular orbits by hand to account for what turned out to be elliptical orbits, the overall number of parameters Copernicus used for the same accuracy in orbital prediction was no smaller than Ptolemy's.

Once one understands that the Earth is not at rest and that it is one of the planets, our picture of the universe changes dramatically. The other planets, no longer heavenly, are ordinary objects like the Earth. They are presumed to be made of the same kind of matter that the Earth is made of and to obey the same laws. There is no longer a natural rest frame, and Galileo's conclusion is that we cannot tell what is at rest and what is moving.

Here are two quotes, one from Galileo and one from Sir James Jeans that may be useful to your students for historical perspective if you are teaching physics or astronomy:



Excerpt from Galileo's Dialogue Concerning the Two Chief World Systems, Ptolemaic and Copernican, 1632 Dialogo.pdf

[SALVATI] Shut yourself up with some friend in the largest room below decks of some large ship and there procure gnats, flies, and other such small winged creatures. Also get a great tub full of water and within it put certain fishes; let also a certain bottle be hung up, which drop by drop lets forth its water into another narrow-necked bottle placed underneath. Then, the ship lying still, observe how those small winged animals fly with like velocity towards all parts of the room; how the fish swim indifferently towards all sides; and how the distilling drops all fall into the bottle placed underneath. And casting anything toward your friend, you need not throw it with more force one way than another, provided the distances be equal; and leaping with your legs together, you will reach as far one way as another. Having observed all these particulars, though no man doubts that, so long as the vessel stands still, they ought to take place in this manner, make the ship move with what velocity you please, so long as the motion is uniform and not fluctuating this way and that. You will not be able to discern the least alteration in all the forenamed effects, nor can you gather by any of them whether the ship moves or stands still. ...in throwing something to your friend you do not need to throw harder if he is towards the front of the ship from you... the drops from the upper bottle still fall into the lower bottle even though the ship may have moved many feet while the drop is in the air ... Of this correspondence of effects the cause is that the ship's motion is common to all the things contained in it and to the air also; I mean if those things be shut up in the room; but in case those things were above the deck in the open air, and not obliged to follow the course of the ship, differences would be observed, ... smoke would stay behind... .

[SAGREDO] Though it did not occur to me to try any of this out when I was at sea, I am sure you are right. I remember being in my cabin wondering a hundred times whether the ship was moving or not, and sometimes I imagined it to be moving one way when in fact it was moving the other way. I am therefore satisfied that no experiment that can be done in a closed cabin can determine the speed or direction of motion of a ship in steady motion.

A description of Galileo's work, from Sir James Jeans, *The Growth of Physical Science*:

He set up a gently sloping plank, some 12 yards in length, and made polished steel balls roll down a narrow groove cut into it. With this simple apparatus, he was able to verify his conjecture that the speed of fall increased uniformly with time – the law of uniform acceleration. It was one of the great moments in the history of science.

For now it became clear that the effect of force was not to *produce* motion, but to *change* motion - to produce acceleration; a body on which no force acts moves at uniform speed.

Galileo's rolling steel balls, moving in a horizontal plane, continued their motion with undiminished speed until they were checked by friction and the resistance of the air. This was not entirely new. Plutarch, in 100 AD, had written, "Everything is carried along by the motion natural to it, if it is not deflected by something else." But Galileo was the first to establish the principle experimentally. Where others had conjectured, Galileo proved.

The idea that force is needed to produce motion is overturned by the realization that we cannot tell what is at rest and what is moving. After the Copernican revolution (after Galileo), it is clear that a force is needed to change the velocity of an object, not to make it move. The understanding is made precise in Newton's first two laws, the first due to Galileo.

1. With no force acting on it, an object moves at constant speed in a straight line.
2. $\mathbf{F} = m\mathbf{a}$.

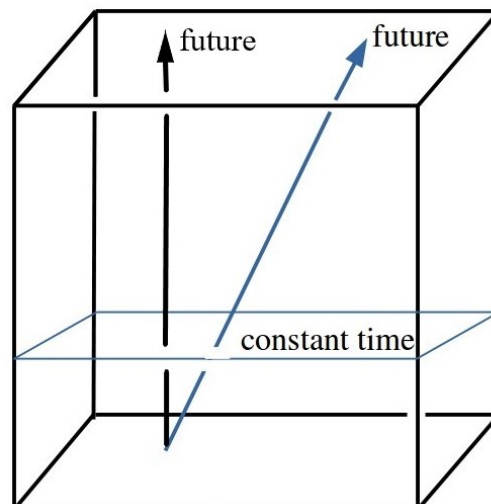
A Galilean boost, changing each particle trajectory $\mathbf{x}(t)$ by the transformation

$$\mathbf{x}(t) \rightarrow \mathbf{x}(t) + \mathbf{v}t \quad (1.1)$$

leaves Newton's laws invariant: Because $\frac{d^2}{dt^2}[\mathbf{x}(t)] = \frac{d^2}{dt^2}[\mathbf{x}(t) + \mathbf{v}t]$, the history of a system of particles for which each particle satisfies $\mathbf{F} = m\mathbf{a}$ is mapped to a boosted history satisfying $\mathbf{F} = m\mathbf{a}$. The trajectory of an observer who says her frame is at rest, with trajectory $(t, \mathbf{x}(t)) = (t, \mathbf{0})$ is mapped to the trajectory $(t, \mathbf{x}(t)) = (t, \mathbf{v}t)$ of an observer who says that *his* frame is at rest: $(t', \mathbf{x}'(t')) = (t, \mathbf{0})$. Time is still absolute, the same t for both observers.

Galilean

Earth curved
 No preferred spatial direction
 No absolute space
 Time absolute
 Preferred set of time directions
 Spacetime flat



Here's the abstract of Einstein's 1905 paper, introducing special relativity.

(Full text: <https://www.fourmilab.ch/etexts/einstein/specrel/www/>)

ON THE ELECTRODYNAMICS OF MOVING BODIES

It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. . .

Examples of this sort, together with the unsuccessful attempts to discover any motion of the Earth relatively to the “light medium,” suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the “Principle of Relativity”) to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies

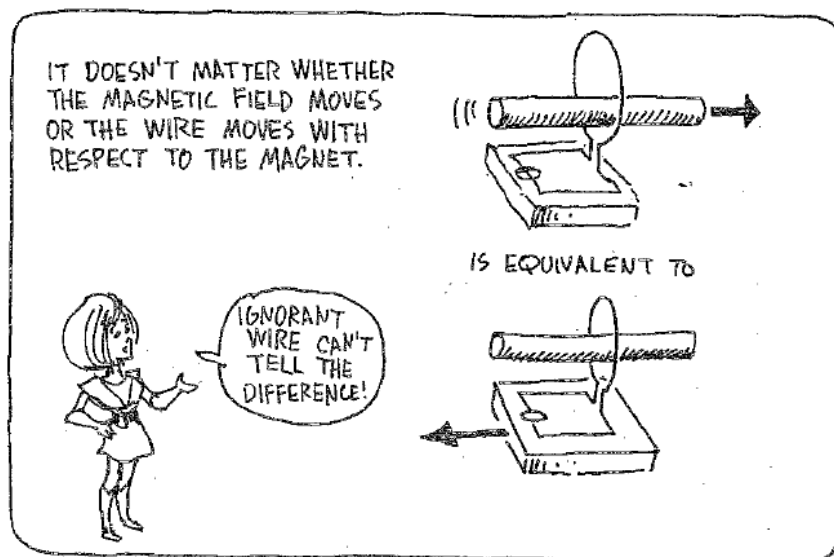
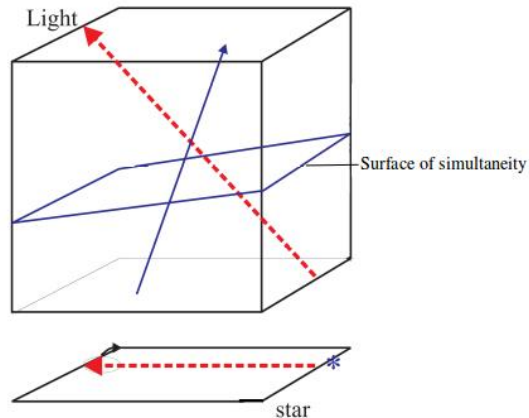


Figure 1.1: From Gonick & Huffman's Cartoon Guide to Physics[44]

Again a remarkable coincidence from the lack of absolute space underlies special relativity. Hidden in Maxwell's equations in their original form – and still in the form you first encounter – is a structure that guarantees their invariance and that of the Lorentz force law: In a frame for which the wire is moving, the $q\mathbf{v} \times \mathbf{B}$ force on the moving electrons produces the current. In the frame in which the magnet moves, the equations conspire to give an electric field whose force $q\mathbf{E}$ produces the *same* current, as measured by the ammeter in Fig. 1.1.

Special Relativity

Earth curved
 No preferred spatial direction
 No absolute space
No absolute time
 Preferred set of time directions
 Spacetime flat



With the understanding that light is a wave, it appeared possible to regain absolute space: You are at rest relative to space (or to the ether that fills it) if light travels at the same speed c in all directions. But Maxwell's equations and hence the speed of light are invariant under a boost of the dynamical system to an identical system moving with a velocity v . In order that the speed of light be unchanged by boosting the emitter or observer, however, one must discard not only absolute space but also absolute time. Observers moving relative to one another must disagree on what events are simultaneous.

If you have taken special relativity, you encountered this historical logic, leading to time dilation, length contraction, and, ultimately, the law that governs the geometry of flat space, the distance between two events:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2.$$

The special case of this Minkowski metric is the Pythagorean metric $dx^2 + dy^2 + dz^2$ for the distance between simultaneous events as defined by a particular observer. If you have not followed the historical path to the Minkowski metric you should read, for example, Chapter 4 of Hartle's *Gravity*. The argument, inferring the Minkowski metric from the observer independence of the speed of light, is also given as an appendix 1.A at the end of this section. (You will first need to read Sect. 1.2.2.) Beginning with the next section, 1.2, we will follow the logic of the law itself, inferring the physics from the geometry of a flat spacetime.

Discarding the final assumption—that space is flat—begins with mathematicians in the 18th century, in particular with Gauss and his student Riemann:

Therefore, either the reality on which our space is based must form a discrete manifold or else the reason for the metric relationships must be sought for, externally, in the binding forces acting on it. On the Hypotheses which lie at the Bases of Geometry., Bernhard Riemann, 1854

<https://www.emis.de/classics/Riemann/WKCGeom.pdf>

Riemann sought but failed to find a relation between the geometry of space and the forces of gravity and electromagnetism. He tried this before Maxwell's equations had shown that light was an electromagnetic wave. Identifying space with an ether comprised of particles, he sought to relate forces to changes in geometry, speaking of

1) the resistance with which a particle opposes a change of its volume, and 2) the resistance with which a physical line element opposes a change of length. Gravity and electric attraction and repulsion are founded on the first part, light, heat propagation, electrodynamic and magnetic attraction and repulsion on the second part. [68] (Translation here from Bottazini-Tazzioli)

Why should gravity in particular be related to geometry? Here's a version from Robert Geroch, appropriate for general audiences and parties: To decide what a straight line is, you could stretch a rope to find the shortest distance between two points. But gravity bends the rope. Stretching the rope more tightly decreases the bending, but there is a fundamental limit to the stretching: As the tension increases, the speed of waves in the string increases, and the speed of the waves cannot be greater than the speed of light. You could extend a solid rod between the two points and measure the length of the rod with a ruler, but gravity bends the rod and the ruler. Choosing increasingly rigid and incompressible rods decreases the bending, but the more rigid the rod, the greater the speed of sound, and the speed of sound must be less than the speed of light. Finally, you could use light itself, shining a laser between the two points, its path defining a straight line in spacetime. But gravity bends the light. Anything you use to decide what a straight line is— or, more generally, to determine the geometry of spacetime— is curved by gravity. That fact suggests gravity is tied in a fundamental way to the geometry of spacetime. Einstein's motivation, based on the equivalence principle and perhaps already familiar to you, comes in the next chapter.

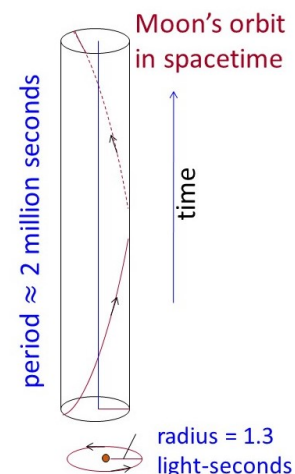
Two obvious obstacles blocked Riemann and later mathematicians (Sylvester, Clifford) from relating curvature to the curved trajectories of particles in a gravitational field:

The geometers thought of the curvature of space, not spacetime.

Space (and spacetime) is very nearly flat in the solar system.

Euclidean geometry is accurate to better than one part in a billion at the Earth's surface, but the spatial path of a ball or planet is nothing like a straight line.

So why isn't it nonsense to say that particles move on geodesics, on paths of shortest length? **The reason is that the spacetime paths are very nearly straight:** If time is measured in seconds, distance is measured in light-seconds. The Earth-Moon distance is about 1.3 light-seconds, and the period of the Moon's orbit is 2.3×10^6 s, so the spacetime trajectory of the Moon is a nearly straight spiral (see diagram on right). Similarly, a ball that is tossed up and comes back down in 1 s reaches a height of 4×10^{-9} light-seconds (1.2 m), so it travels along a nearly straight path in spacetime, deviating from straight by a distance only 4×10^{-9} times the total distance it moves in spacetime. The spacetime curvature of the path is comparable to the departure from Euclidean of the spacetime (and spatial) geometry.



General Relativity

Earth curved

No preferred spatial direction

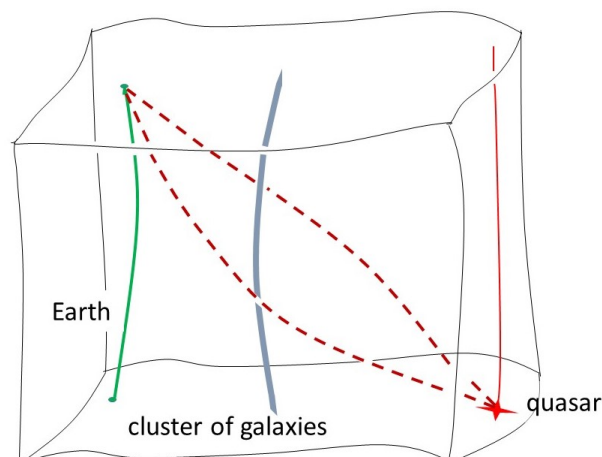
No absolute space

No absolute time

Preferred set of time directions

(if spacetime is time orientable)

Spacetime curved



Light traveling from a quasar to Earth along two geodesics. Gravitational lensing produces two images of the object.

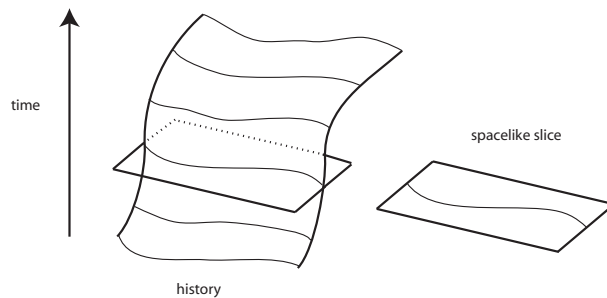
Exercise 1. In units with $c = 1$, the ratio

(radius of the Moon's orbit)/(radius of curvature of the spacetime trajectory of the Moon) is dimensionless and should be small. The radius of curvature of the trajectory is of order $1/a$, with a the acceleration. Find the dimensionless ratio $r/(1/a)$, with r the radius of the Moon's orbit. What is the ratio for a falling object on Earth?

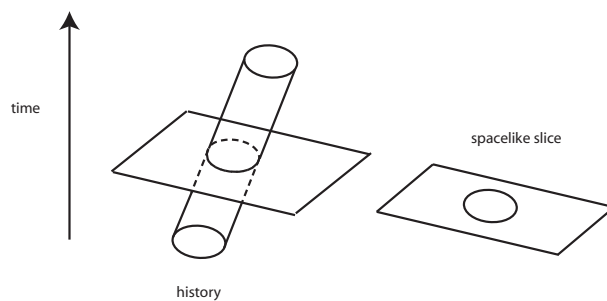
Spacetime diagrams

Once there is no absolute time, histories can no longer be uniquely described by a set of snapshots. Instead, the usual visualization is a spacetime diagram. Here are some examples.

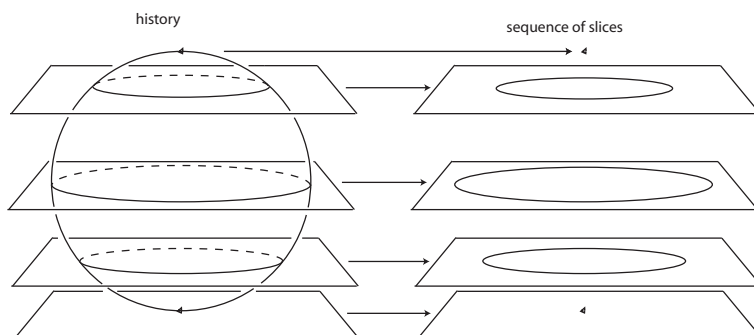
a. The history of a string is a sheet.



b. The history of a circle is a cylinder.

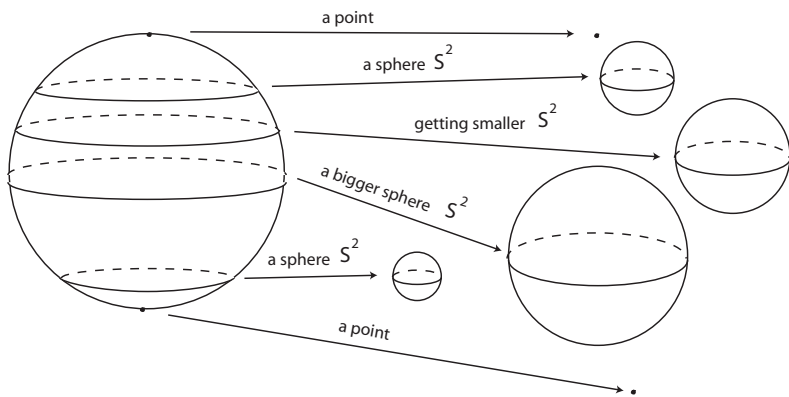


c. The history of a circle that starts as a point, expands, then ends as a point is a sphere.



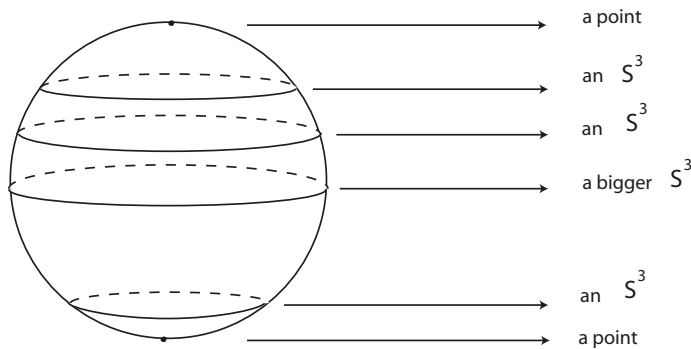
d. The history of a balloon that starts from a point, expands, contracts, and ends as a point is a 3-dimensional sphere S^3 . (This is the set of points in $\mathbb{R}^4$ satisfying $t^2 + x^2 + y^2 + z^2 = 1$.)

history:1 dimension suppressed sequence of slices



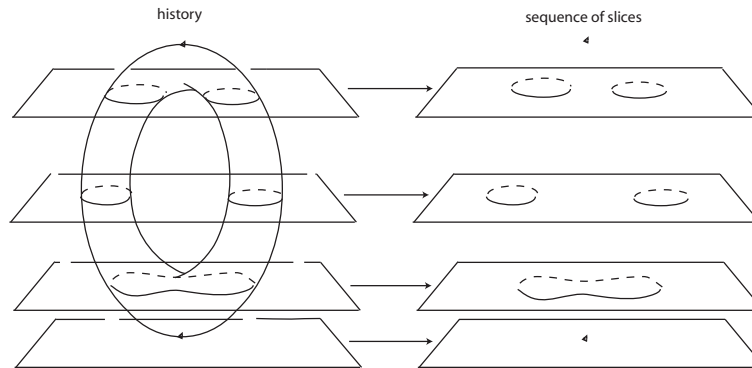
e. Closed universe: The history is a sequence of S^3 's starting from a point, expanding, then contracting to a point.

history :2 dimensions suppressed sequence of slices



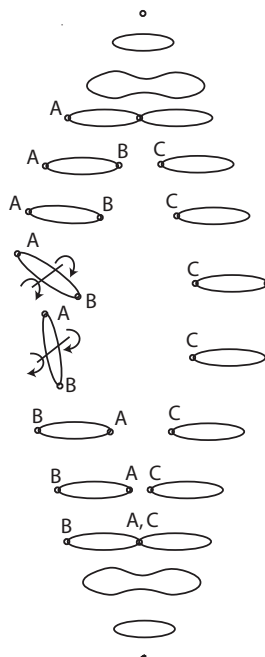
Spacetime diagrams will soon be used to explicate some typical paradoxes of special relativity. In the meantime, it's sometimes useful in understanding objects that can't be embedded in three dimensions - like Klein bottles - to construct them as histories, with time the extra dimension.

A torus as a spacetime history



Klein bottle. A Klein bottle can't be embedded in three dimensions without self-intersection. It can, however, be embedded in four dimensions and so is a smooth non-intersecting history in spacetime: Click on [torus-Klein bottle movie](#). If your browser doesn't play it, download the mp4 file to view it. As shown in this movie, to create a Klein bottle as a smooth submanifold of spacetime, take two circles (modeled as two wire loops) that coincide at $t = 0$, separate them, rotate one by π , and then bring them together at $t = T$.

Here is a somewhat different sequence of slices, similar now to the slices of the torus in the previous figure



1.2 Minkowski space

The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality. – Hermann Minkowski, 1908, translation above by Perrett and Jeffery.

1.2.1 Minkowski metric

The Euclidean geometry of flat 3-dimensional space is completely specified by its flat metric, given in Cartesian coordinates by the Pythagorean relation

$$ds^2 = dx^2 + dy^2 + dz^2. \quad (1.2)$$

Because the space is flat, the distance between finitely separated events is similarly given by

$$\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2, \quad (1.3)$$

and this is the distance Δs between two events that are simultaneous for a given observer in terms of that observer's Cartesian coordinates.

Underlying the unification of space and time is the universe's speed limit, the fact the speed of information c is observer-independent. The Pythagorean relation is the 3-dimensional special case of the distance between any two events in flat 4-dimensional space, given in terms of the Cartesian coordinates of any inertial observer by

$$\Delta s^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2, \quad (1.4)$$

with Δs^2 negative when the separation between events is timelike. The proper time $\Delta\tau$ on a clock moving along a straight line between the events is then given by

$$\Delta\tau^2 = (\Delta t)^2 - [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]/c^2. \quad (1.5)$$

To complete the experimentally implied unification of space and time, we choose units with $c = 1$ and write

$$\Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2. \quad (1.6)$$

Units with $c = 1$. If time and distance are measured in meters, 1 m in time is $1 \text{ m}/c = 3.3336 \text{ ns}$;¹ if both are measured in seconds, 1 second of spatial distance is one light-second; and if both are measured in years, 1 year of spatial distance is one light-year (ly).

The Minkowski metric of spacetime is now

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad (1.7)$$

where

$$\|\eta_{\mu\nu}\| = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \quad (1.8)$$

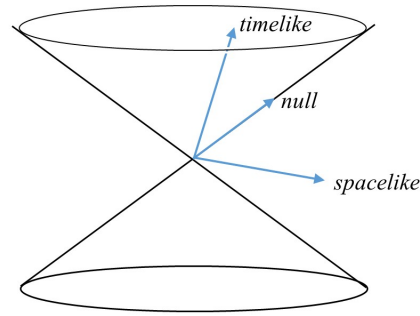
The squared length of a vector v^α is the number $\eta_{\alpha\beta} v^\alpha v^\beta$.

¹Mermin [59], noting that $c \approx 1 \text{ foot/nanosecond}$, defines a phoot to make this exact: $1 \text{ phoot} := c \times 1 \text{ ns} \approx .984 \text{ ft}$. Then $c = 1 \text{ phoot/ns}$.

If $v^\alpha v_\alpha < 0$, v^α is timelike,

if $v^\alpha v_\alpha = 0$, v^α is null, and

if $v^\alpha v_\alpha > 0$, v^α is spacelike.



The null vectors lie on a 3-dimensional double cone, the light cone. Timelike vectors fill the interior of the cone, and spacelike vectors lie outside it. A choice of future and past picks out future and past cones generated by future and past pointing null vectors, respectively.

The distance between two timelike separated events P and Q is the time read on a clock that moves in a straight line (no acceleration) from one event to the other. This is called the proper time of the clock (or, for an observer with a watch, the observer's proper time. With t and x in seconds, the proper time

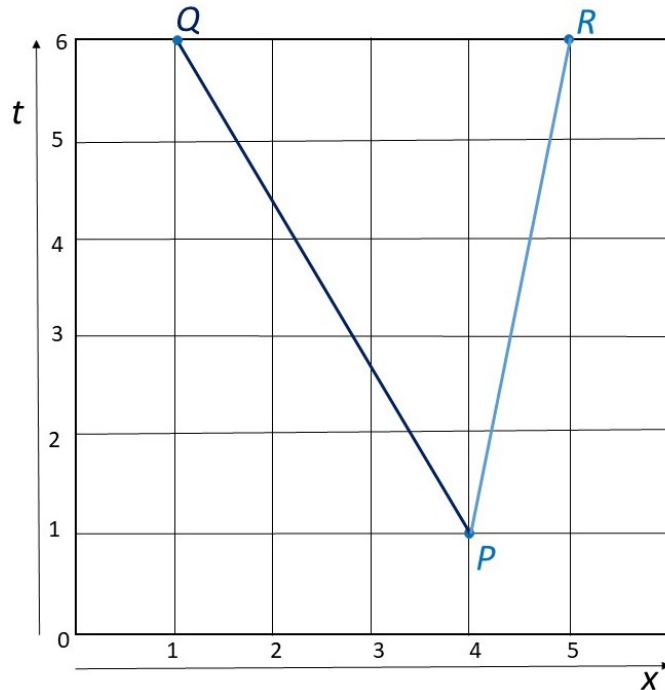


Figure 1.2: The proper time τ read by clocks moving along the dark and light blue trajectories from P to Q and P to R is the spacetime distance between the respective events.

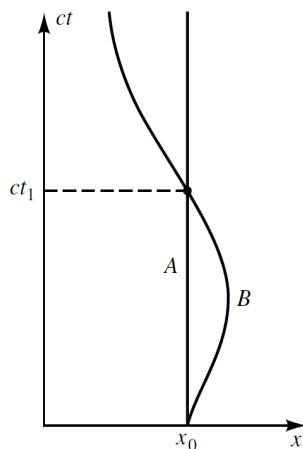
between events P and Q is $\Delta\tau = \sqrt{5^2 - 3^2} = 4$ s; between P and R it is $\sqrt{5^2 - 1} = \sqrt{24} \approx 4.9$ s. Notice that, although the path PQ looks longer in the spatial Euclidean geometry of your monitor (or paper), the length of the path in the actual Lorentzian geometry of spacetime is shorter.

If spacetime diagrams are not familiar to you, read Chapter 4 of Hartle's *Gravity*. Here's an excerpt (continuing until the exercises before the *Surfaces of Simultaneity* section below):

To describe four-dimensional spacetime we first introduce a tool, which is so simple it appears trivial, but so powerful it is indispensable. This is the idea of a spacetime diagram. A spacetime diagram is a

plot of two of the coordinate axes of an inertial frame—two coordinate axes of spacetime. Since there are four axes and only two dimensions on a piece of paper, two or at most three of these axes can be drawn. Spacetime diagrams are slices or sections of spacetime in much the same way as an x - y plot is a two-dimensional slice of three-dimensional space. A typical example is shown in the figure above. It is convenient to use ct rather than t as an axis, because then both have the same dimension. [Hartle will set $c = 1$ later.]

A point P in spacetime can be called an *event* because an event occurs at a particular place at a particular time, that is, at a point in spacetime. For example, a supernova explosion happened at the event in spacetime that occurred in A. D. 1054 at the location of the Crab nebula. An event P can be located in spacetime by giving its coordinates (t_P, x_P, y_P, z_P) in an inertial frame. A particle describes a curve in spacetime called a world line [In these notes, it will be called the particle's *trajectory*]. It is the curve of positions of the particle at different instants, i.e., $x(t)$. The figure below shows a spacetime diagram with two sample world lines.



Hartle Fig. 4.5. World lines in spacetime. A is the world line of a particle that sits at rest at x_0 for all time in the inertial frame (ct, x) . World line B represents an observer who accelerates away from x_0 at time $t = 0$, decelerates, reverses direction, crosses x_0 at $t = t_1$, and heads off toward negative x .

The slope of the world line gives the ratio c/v^x since $d(ct)/dx = cdt/dx = c/v^x$. Zero velocity corresponds to infinite slope. A velocity of c corresponds to a slope of unity. Light rays therefore move along the 45° lines in a spacetime diagram.

[After introducing the Minkowski metric, Hartle continues with:]

Test your understanding of this by answering the following questions about the lengths between points in the figures in the spacetime diagram in Figure 4.8 [copied below].

- Which of the sides of triangle ABC is the longest? Which is the shortest? What are the lengths in the units of the grid?
- Which is the shorter path between points A and C —the straight-line path between A and C or the path through the other sides of ABC ?

Then for (c) and (d), answer the same questions for triangle $A'B'C'$.

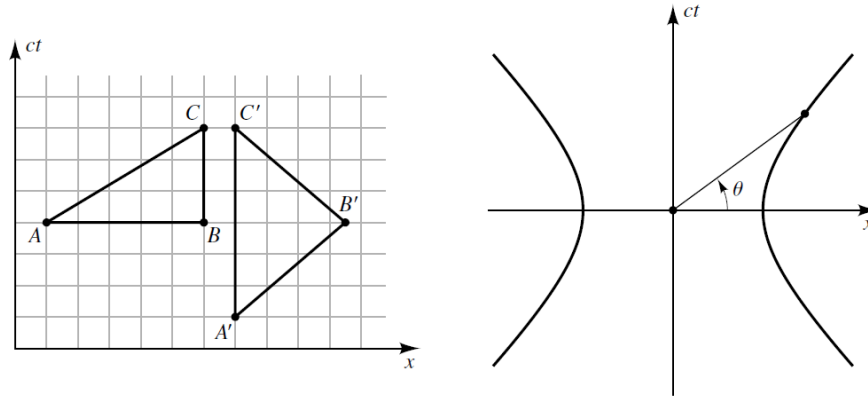


FIGURE 4.8 A little spacetime geometry. The left-hand figure shows a spacetime diagram with two triangles whose properties are discussed in the text. The right-hand figure shows a spacetime analog of a circle—a hyperbola that is a constant spacetime distance from the origin. A hyperbolic angle θ is a ratio of a distance along these hyperbolas to the distance from the origin. The hyperbolic angle θ shown is the ratio of the spacetime distance along the hyperbola from the x -axis to the spacetime distance of the hyperbola from the origin.

Example: Spacetime Diagrams as Maps of Spacetime.

No one would think of confusing the relationships between lengths on a Mercator map of the world with the relationships between true distances on the surface of the Earth. A Mercator map is a projection of the geometry of the globe on a sheet of paper. There are analogies between the elements of plane geometry and the geometry of a spacetime diagram. One is illustrated in Figure 4.8.

The analog of a circle of radius R centered on the origin is the locus of points a constant spacetime distance from the origin. This consists of the hyperbolas $x^2 - (ct)^2 = R^2$. The ratios of arcs along a hyperbola to R define hyperbolic angles, as shown in the figure, with the relation

$$ct = R \sinh \theta, \quad x = R \cosh \theta. \quad (1.9)$$

It's useful to be able to understand these analogies, but it does not prove useful in relativity to pursue them too far. [We'll see later that a hyperbola of this kind is the trajectory of a uniformly accelerating particle.]

The following exercises have one- or two-line solutions.

Exercise 2. τ -Ceti, 12 ly away, hosts the closest potentially habitable planet. Draw a spacetime diagram for a spacecraft traveling to τ -Ceti and back at speed $\frac{12}{13}c$, ignoring time spent accelerating. Use the lengths of the diagram in its Minkowski geometry to find the proper time elapsed on a spacecraft clock. (If you are tempted to convert to m, cm, or s, stop and think.)

Exercise 3. What is the proper time τ read by a clock that moves with trajectory $(t, x(t), y(t))$ from $t = t_1$ to $t = t_2$. Use the Minkowski metric (1.7).

Exercise 4. What is $\tau(t)$ for a circular orbit with speed v ?

1.2.2 Surfaces of simultaneity

The Cartesian coordinates of the diagram in Fig. 1.2 above are those of an inertial observer who moves along a vertical line; the horizontal $t = \text{constant}$ lines are her surfaces of simultaneous events. We can define surfaces of simultaneity in a democratic way, for any inertial observer as follows. To construct a three dimensional space of events she calls simultaneous, she bounces light rays off events and calls the time t of each event the average of the proper times measured on her clock when she sent and got back the signal. Two events will be simultaneous if the times t she computes in this way are the same.

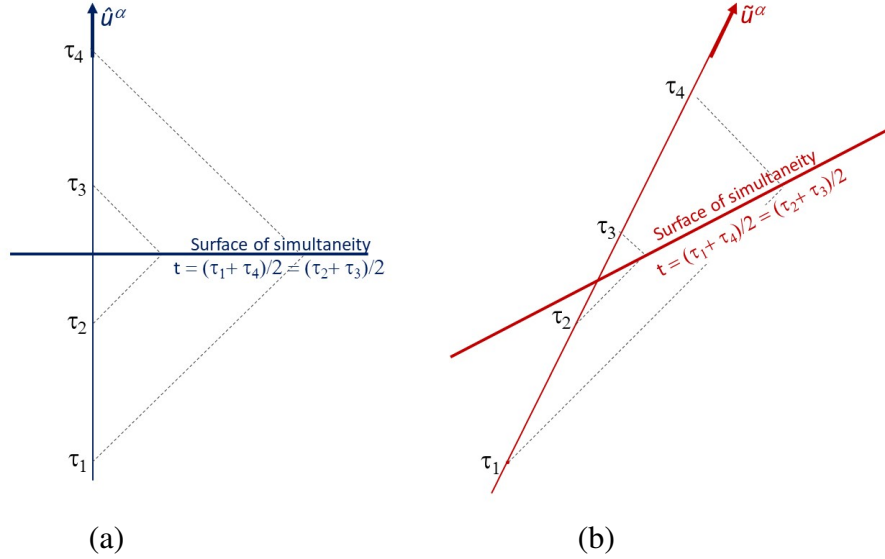


Figure 1.3: (a) (b)
Light rays are shown as dotted lines in each diagram.

The set of all simultaneous events is called the surface of simultaneity of the observer, and two examples are shown in (a) and (b) above. Call observer (a) Alice and observer (b) Bob. In a natural coordinate system of an inertial observer, the surfaces of simultaneity are the $t = \text{constant}$ hyperplanes.

In Fig. 1.3 (a), Alice moves along the vertical blue line through the origin. It is clear from the figure that her trajectory is orthogonal to her surfaces of simultaneity with respect to the Lorentz metric. That is, a unit vector $\hat{u}$ tangent to Alice's trajectory has components $(1, 0)$, while a vector $\hat{x}$ tangent to her surface of simultaneity has components $(0, 1)$, and their dot product is $\eta_{\mu\nu} \hat{u}^\mu \hat{x}^\nu = -(1)(0) + (0)(1) = 0$. Because we don't know who is moving and who is at rest, it must be that Bob's trajectory is orthogonal to his surface of simultaneity.

His trajectory is (t, vt) in Alice's coordinates. Its tangent $(1, v)$ is orthogonal to the line with coordinates $(t, t/v)$ and tangent $(1, 1/v)$:

$$(1, v) \cdot (1, 1/v) = -1 + 1 = 0.$$

In more familiar language, Bob's 4-velocity is the unit tangent vector $\tilde{u}$ with components $(\tilde{u}^\mu) = \gamma(1, v)$ in Alice's frame, and the orthogonal unit vector proportional to $(1, 1/v)$ has components $(\tilde{x}^\mu) = \gamma(v, 1)$.

Exercise 5. Check directly that Bob's surfaces of simultaneity, constructed by bouncing light-rays off events, as in Fig. 1.3(b), have the slope $dx/dt = 1/v$ we have just found. It's easiest to choose as the surface of simultaneity the line from the origin at $\tau = 0$ to the point where the light bounces, taking the times of departure and arrival of the bouncing light (according to Bob's clock) to be $\tau = \mp T$.

1.2.3 Time dilation and length contraction from the Minkowski metric

Time dilation

Consider a slicing of spacetime by the surfaces of simultaneity of an inertial observer, together with a clock that moves with constant speed relative to that observer. *Time dilation* gives the proper time read by the clock as it moves from one of these $t = \text{constant}$ surfaces another, in terms of the difference Δt . For example, time dilation looks at the proper time of the clock carried by Bob as it passes a set of *synchronized clocks* at rest relative to another inertial observer, Alice. Along Alice's surface of simultaneity, the synchronized clocks all read the same time t .

In figure 1.4, Alice and Bob, carrying clocks, move relative to one another at speed v . The same spacetime with the spacetime trajectories of Alice and Bob is shown in (a) and (b). In (a) the spacetime is sliced by Alice's surfaces of simultaneity, in (b) by Bob's.

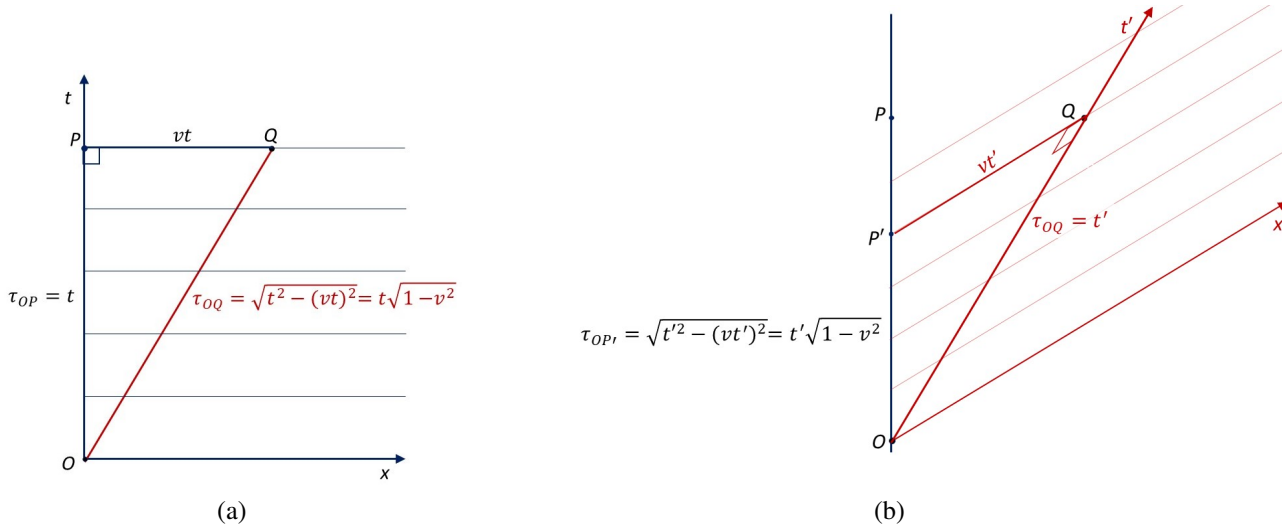


Figure 1.4: Bob moves from O to Q in proper time τ_{OQ} . Alice moves from O to P in proper time τ_{OP} , where, as shown in (a), P lies on Alice's surface of simultaneity through Q , with time coordinate $t = \tau_{OP}$, larger than τ_{OQ} . From triangle OPQ , $\tau_{OQ} = t\sqrt{1-v^2}$.

In (b), Bob's surfaces of simultaneity are shown. When Alice reaches point P' in her proper time $\tau_{OP'}$, she is already crossing Bob's surface of simultaneity through Q , with $\tau_{OP'} = t'\sqrt{1-v^2}$.

Diagram (a) uses the Minkowski metric to compare Bob's proper clock time τ_{OQ} to the larger change in the coordinate $t = \tau_{OP}$ labeling Alice's surface of simultaneity through P and Q : From $\triangle OPQ$,

$$\tau_{OQ} = \tau_{OP}\sqrt{1-v^2}.$$

Diagram (b) shows the same clock trajectories, but with Bob's surfaces of simultaneity. When Alice reaches P' she is already crossing Bob's surface of simultaneity through Q . Diagram (b) uses the Minkowski metric to compare Alice's proper clock time $\tau_{OP'}$ to the larger change in the coordinate $t' = \tau_{OQ}$ labeling Bob's surface of simultaneity through Q and P' : From triangle OQP' (notice $OQ \perp QP'$)

$$\tau_{OP'} = \tau_{OQ}\sqrt{1-v^2}.$$

The result, of course, is that you can't tell whose clock is at rest and whose is moving. In each case the proper time elapsed on a clock is smaller than the difference in coordinate time between the first and last surface of simultaneity of an observer moving with speed v relative to the clock. And it is smaller by the factor $\sqrt{1-v^2}$.

Length contraction

What is the length of a stick? In its own rest frame, it is the spacetime distance between two events, the simultaneous positions of the two ends of the stick as defined by an observer, Alice, moving with the stick.

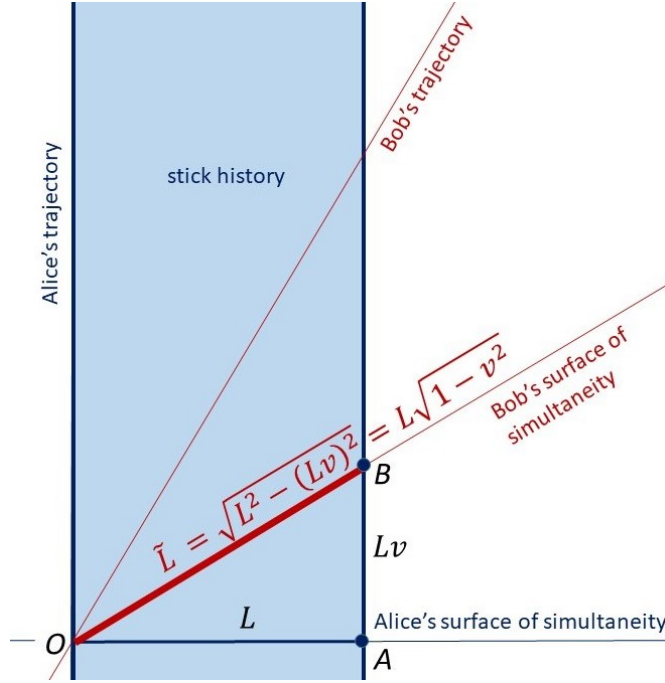


Figure 1.5: Alice's surface of simultaneity through O intersects the stick's history along OA , with length L , the proper length of the stick. Bob's surface of simultaneity intersects the history along OB , with length $\tilde{L}$.

In Fig. 1.5, the blue shaded region is the spacetime stick, the history of the stick in spacetime. Its length L in its own frame is the length of the stick's intersection with Alice's surface of simultaneity. The stick's length in Bob's frame is the length of the stick's intersection with Bob's surface of simultaneity, namely the line $(t, t/v)$ in Alice's (and the stick's) coordinates. It intersects the left end of the stick at the origin and it intersects the right end of the stick when $(t, t/v) = (t, L)$, implying $t = Lv$. The length of the stick in Bob's frame is then the length $\tilde{L} = OB$, the distance between events O and B . From triangle OAB ,

$$\tilde{L} = \sqrt{-(Lv)^2 + L^2} = L\sqrt{1 - v^2}, \quad (1.10)$$

shorter than the proper length by $\sqrt{1 - v^2}$.

Because a boost tilts the surface of simultaneity only in the plane of the boost, lengths orthogonal to that plane are unchanged. For an object whose proper volume V is fixed, a slice boosted by a velocity v then intersects the history of the object in a volume

$$\tilde{V} = V\sqrt{1 - v^2}. \quad (1.11)$$

To make this obvious for an object of arbitrary shape, consider a boost in the t - x plane of the object (t and x the coordinates of an observer at rest relative to the object). Divide the volume into small cubes aligned with the coordinate axes. Then only the length of a cube in the x direction changes with the change of slice, and the volume of each cube is smaller by $\sqrt{1 - v^2}$ in the boosted slice of its history.

1.2.4 Paradoxes

The usual paradoxes of special relativity arise because the notion of simultaneity is observer dependent in reality but observer independent in ordinary language and experience. We have already considered the paradox of two relatively moving observers, each saying that the clock carried by the other is running slow. One paradox will be treated here; another will be a problem. *They can all be similarly resolved, with the system described as a single consistent spacetime, sliced in different ways by the surfaces of simultaneity of different observers.*

Consider a box having length L in its rest frame and a rod of length $\ell > L$ in its (rod's) rest frame. When the rod moves (rapidly) toward the box, the box sees the rod's length contracted to such an extent that when it reaches the box it can fit comfortably inside. The rod, on the other hand, sees the box contracted and thinks it can't come close to making it inside. But the box can actually catch the rod by lowering a flap when the right end of the rod arrives and raising it when the left end of the rod has passed. Everyone must then agree that the closed box has a rod inside. How can this be consistent with what the rod sees? (The flap of the box can be as small as desired, and the time needed to raise and lower it negligible compared to any other times in the problem, so the details of the flap can be ignored).

We will draw a spacetime diagram to describe the history of the rod and box, and then look at the slicings of that history by the rod's and the box's surfaces of simultaneity, showing that both rod and box are correct *and* consistent.

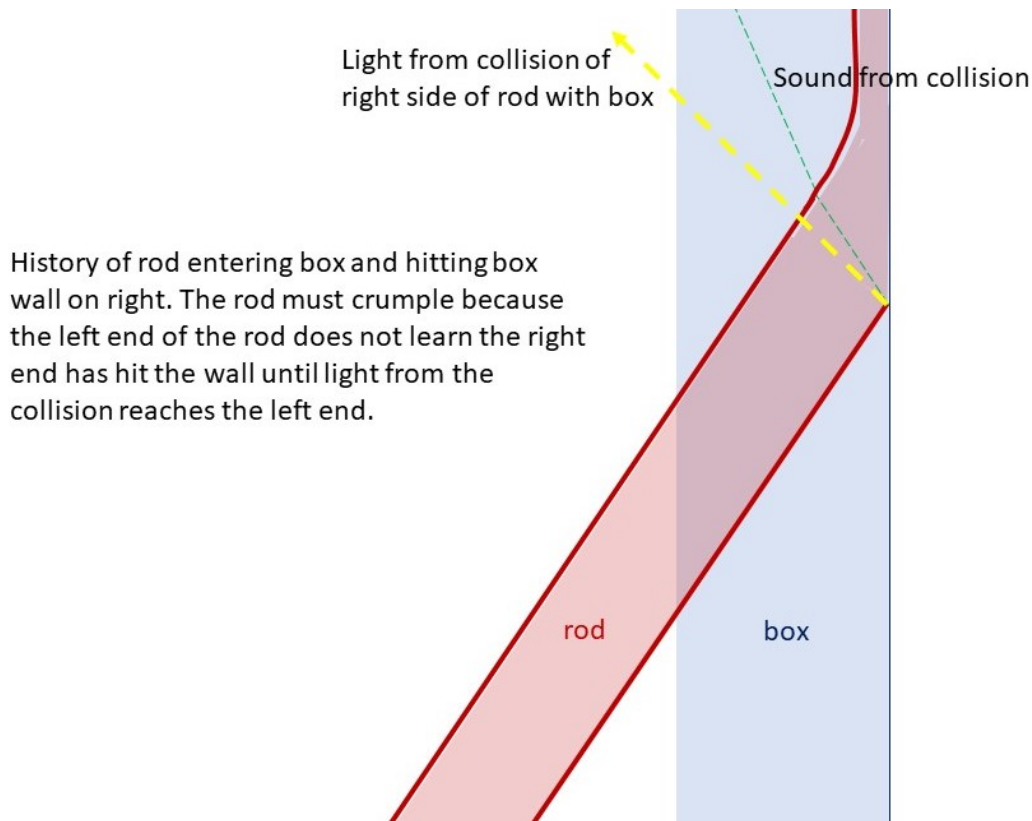
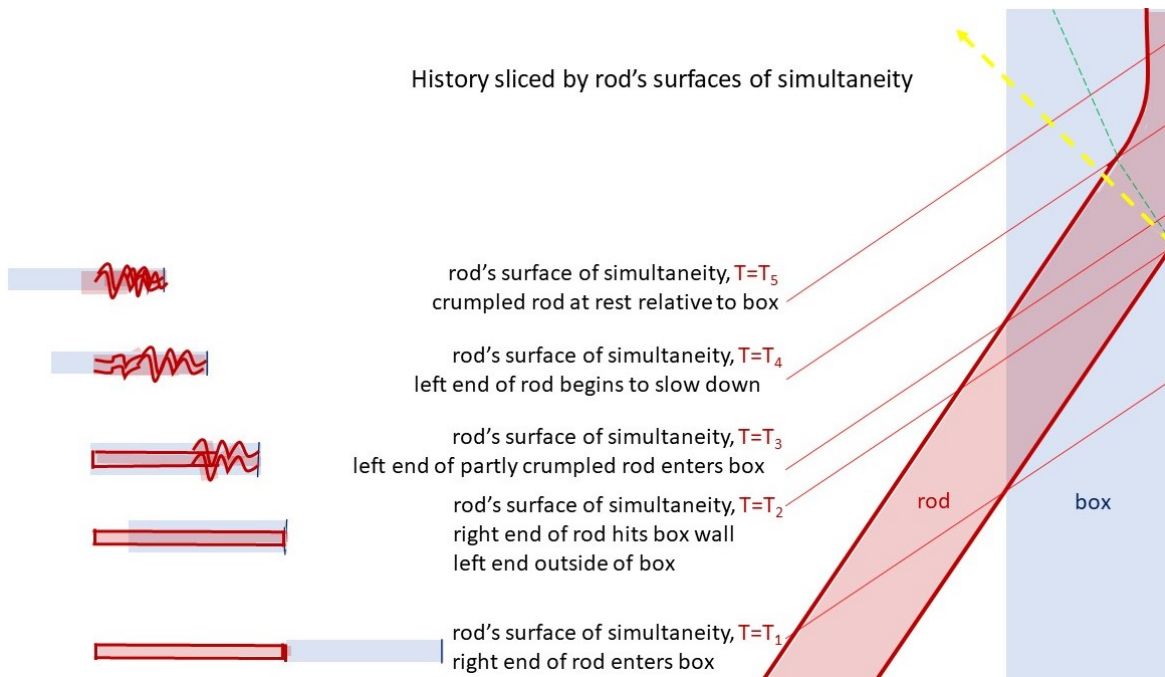
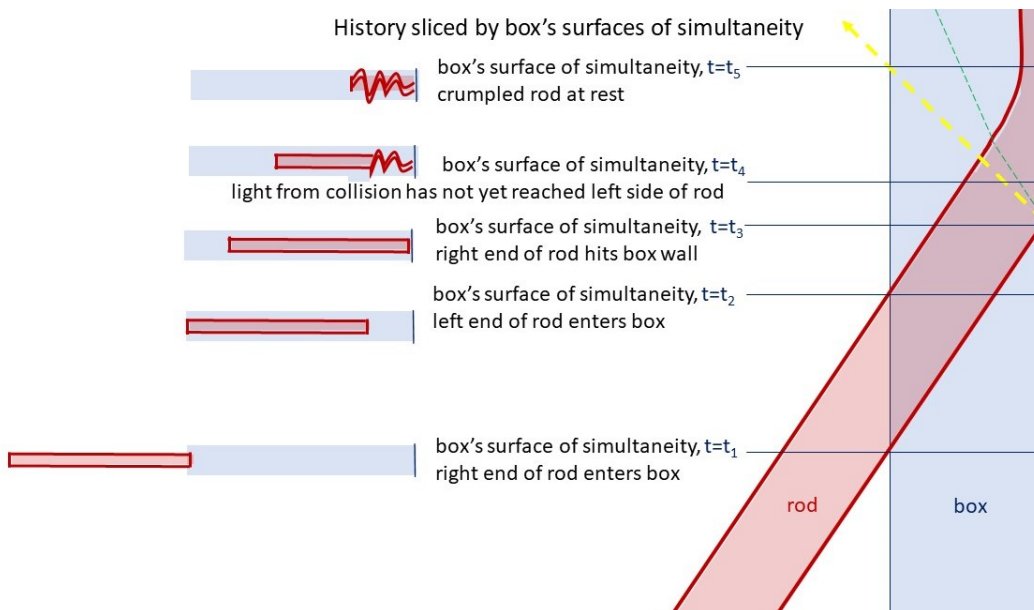


Figure 1.6: Histories, like the history shown here, are observer-independent. What changes with different observers is the way the history is sliced by the observers' surfaces of simultaneity.

In the slicing of history by the surfaces of simultaneity of an observer moving with the velocity of the initial rod, the left side of the rod is still outside the box when its right end hits the wall of the box.



In the slicing of the history by the box's surfaces of simultaneity, the entire rod is inside the box before its right end hits the wall of the box.



The lack of rigidity of the rod is required to resolve the paradox, and the lack of rigidity is implied by causality: The left end of the rod does not change its speed until information from the collision has traveled from right to left across the rod.

Exercise: (Lightman, Press, Price, Teukolsky # 1.24 [55])

Two giant frogs are captured, imprisoned in a large metal cylinder, and placed on an airplane. While in flight, the storage door accidentally opens and the cylinder containing the frogs falls out. Sensing something amiss, the frogs decide to try to break out. Centering themselves in the cylinder, they push off from each other and slam simultaneously into the ends of the cylinder. They instantly push off from the ends and shoot across the cylinder past each other into the opposite ends. This continues until the cylinder hits the ground. Consider how this looks from some other interial frame falling at another speed. In this frame the frogs do *not* hit the ends of the cylinder simultaneously, so the cylinder jerks back and forth about its mean speed v . The cylinder, however, was at rest in one inertial frame. Does that mean that one inertial frame can jerk back and forth with respect to another?

We have derived the usual results of special relativity, time dilation and length contraction, from the Minkowski metric. This is the reverse of the usual historical presentation, in which time dilation and length contraction are inferred from observer independence of the speed of light and are then used to obtain the metric. The next section, starting on the next page, is a supplement for readers who are not familiar with these standard arguments. If the standard presentation is familiar to you, skip to Sect. 1.3 on tensors and vector spaces. Later results will not refer to this supplementary section.

1.A Supplement: Inferring the Minkowski metric from observer independence of the speed of light.

In this section, we'll go over the standard arguments giving time dilation and length contraction and then show how they lead to the Minkowski metric,

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2. \quad (1.12)$$

We'll need Sect. 1.2.2, on surfaces of simultaneity, including *Exercise 5*. Notice that the construction of surfaces of simultaneity, including the exercise, does not use the metric itself; it is a construction based only on light rays and the proper time measured on clocks.

When the right side of Eq. 1.12 is positive, it gives the squared distance between two spacelike-related events, P and Q . That is, PQ is the distance read on a ruler carried by an observer for whom the two events are simultaneous. When the right side is negative, we can write

$$\Delta \tau^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2, \quad (1.13)$$

with $\Delta \tau$ the proper time elapsed on a clock that moves along an unaccelerated path from event P to event Q (where P is the earlier event).

The argument will show that the spacetime version of the Pythagorean relation,

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2,$$

is implied by its special case when $\Delta t = 0$,

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2,$$

together with the fact that $c = 1$ for all observers. For those of you who have not seen the derivations before, they may be clearer with c not set to 1, so c will be kept until we go on to the derivation of the Minkowski metric. First, a quick review of the observer-dependence of simultaneity.

Loss of absolute time.

The fact that the speed of light is the same to all observers—implied by the observer-independence of Maxwell's equations and shown by the Michelson-Morley experiment—requires us to abandon absolute time: The argument that different observers must disagree on what events are simultaneous is simple. It just uses the fact that Alice, on a moving train, and Bob, on the ground, watching the train pass him, each see light moving at speed c in all directions regardless of the motion of the source: Alice sets off a firecracker at the center of her train car and observes the light hit the two sides simultaneously. Bob on the ground watches the train move to the right, so he sees the light hit the right side after it hits the left side. (See, for example, minute 8 onward in this Mechanical Universe video for a clear and entertaining presentation. It goes on to show tilted surfaces of simultaneity in spacetime.)

1.A.1 Time dilation and length contraction

Time dilation

The experiment compares a time interval Δt_B measured by Bob's synchronized clocks to proper time $\Delta\tau = \Delta t_A$ measured by Alice's single clock that moves past them at speed v . Bob's clocks are at rest relative to the ground. The experiment shows that the time measured on Bob's clocks is longer than the time measured on Alice's clock by the factor $\gamma = 1/\sqrt{1 - v^2/c^2}$.

In the experiment, Alice's clock is on the floor of a train of height ℓ that moves relative to Bob at speed v . A light signal from Alice's clock bounces off the ceiling and returns to her clock, traveling vertically as seen by Alice. Alice's clock thus reads a roundtrip time $2\Delta\tau = 2\ell/c$.²

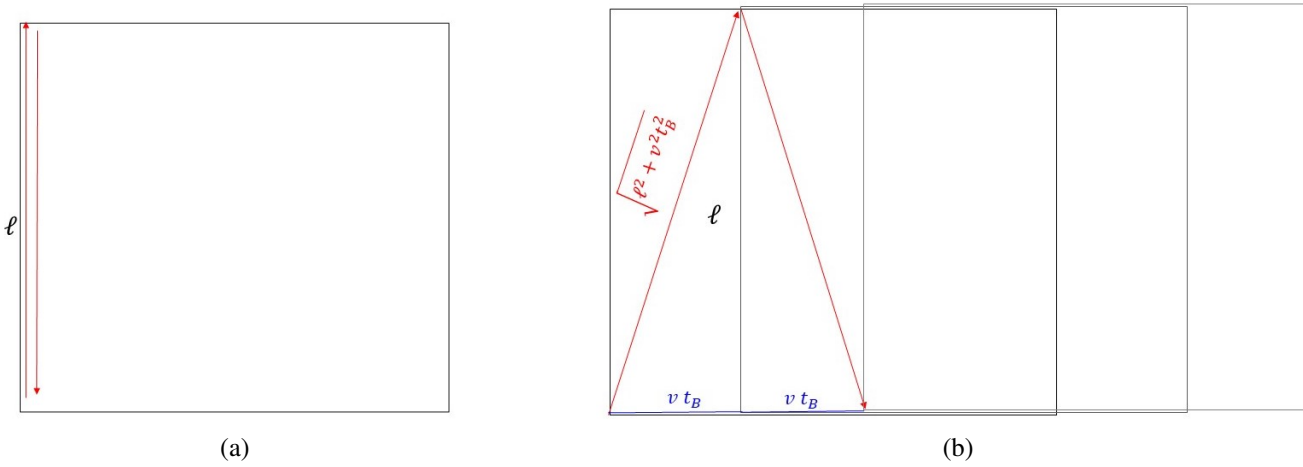


Figure 1.7: (a) In Alice's frame, a photon traverses a vertical path from floor to ceiling and back. (b) In Bob's frame the photon moves a horizontal distance $2v t_B$ while it goes from floor to ceiling and back. Three successive positions of the train are shown, one in black, the others in gray.

Bob sees the light travel along diagonal paths, starting from his ground clock at t_B , bouncing off the ceiling, and meeting a synchronized clock when that clock reads time $t_B + 2\Delta t_B$. By the Pythagorean theorem, the total distance traveled is $2\sqrt{\ell^2 + v^2 \Delta t_B^2} = 2\sqrt{c^2 \Delta\tau^2 + v^2 \Delta t_B^2}$. Because Bob must measure a speed c , that distance is $2c\Delta t_B$:

$$2c\Delta t_B = 2\sqrt{c^2 \Delta\tau^2 + v^2 \Delta t_B^2} \implies \Delta t_B = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}}. \quad (1.14)$$

Length contraction

We next use the time-dilation result of the last section to relate the length of a ruler held by Bob to its length measured by Alice, who sees the ruler move past her at speed v . The ruler's proper length ℓ is the length measured in its rest frame, the length measured by Bob. Because Bob sees Alice moving at speed v , his synchronized clocks measure a time $\Delta t_B = \ell/v$ for Alice to move from one side

²We are looking at the roundtrip path of the photon instead of just the path from floor to ceiling because the comparison is between a single clock that moves relative to a set of synchronized clocks. To directly measure the time for a photon to travel from floor to ceiling, Alice would need to synchronize a clock on the ceiling with her ground clock.

of the ruler to the other: Δt_B is the time interval measured on Bob's synchronized clocks at each end of the ruler, between event P when Alice is at the left end of the ruler and event Q when she is at the right end.

But we have just seen that the time elapsed on Alice's clock between two events on Bob's clocks as Alice's clock passes them is $\Delta t_A = \Delta t_B \sqrt{1 - v^2/c^2}$. Now Alice, watches the ruler move past her with speed v in time t_A , and she thereby measures a length vt_A of the ruler:

$$\ell_A = vt_A = vt_B \sqrt{1 - v^2/c^2} = \ell \sqrt{1 - v^2/c^2}.$$

That is, if Alice moves relative to a ruler at speed v along the line of the ruler, she will measure a contracted length

$$\ell_A = \ell \sqrt{1 - v^2/c^2}. \quad (1.15)$$

If two stars are at rest relative to one another, their proper distance ℓ is the distance measured by an observer at rest relative to them. A spacecraft traveling from one star to another measures the smaller distance $\ell \sqrt{1 - v^2/c^2}$ corresponding to a time elapsed on its clock that is smaller than ℓ/v by the factor $\sqrt{1 - v^2/c^2}$.

1.A.2 Minkowski metric

We now relate time dilation and length contraction, Eqs. (1.14) and (1.15), to the geometry of flat space. This relation between physics and geometry comes from a set of identifications:

1. The set of events is spacetime; the path of an unaccelerated particle is a straight line in spacetime.
2. The time $\Delta\tau$ elapsed on an unaccelerated clock is the spacetime distance between two events: That is, the distance PQ between event P , the spacetime point at which a clock reads τ , and event Q at which the same clock reads $\tau + \Delta\tau$ is $PQ = \Delta\tau$.

This is a physical interpretation of the distance between timelike separated events.

3. The distance between spacelike related events P and Q is the distance measured by an observer for whom the events are simultaneous. It is the distance measured by a ruler for which event P is the left end of the ruler at a time t and Q is the right end at the same time t , according to an observer at rest relative to the ruler. That is, a ruler measures the distance between events that are simultaneous in the rest frame of the ruler.

The usual Pythagorean relation, together with assumption (3) means that on any surface of simultaneity S , the distance between points is given by

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2,$$

where the coordinates measure distances in orthogonal directions along the surface. We take, as the fourth coordinate t , proper time measured by the inertial (unaccelerated) observers for whom S is a surface of simultaneity. Then t, x, y, z will be coordinates for spacetime, with x, y, z constant and t proper time along the trajectory of each of these observers.

We first derive the spacetime Pythagorean relation for timelike related events from the time dilation equation. We take Bob to be an inertial observer, whose coordinates are t, x, y, z as described. As in the time dilation experiment above, Alice's clock will move at speed v past two of Bob's synchronized

clocks. In the spacetime diagram below, the trajectory of Bob's clocks is vertical, and the horizontal line is one of Bob's surfaces of simultaneity, a set of events for which Bob's clocks all read the same time t .

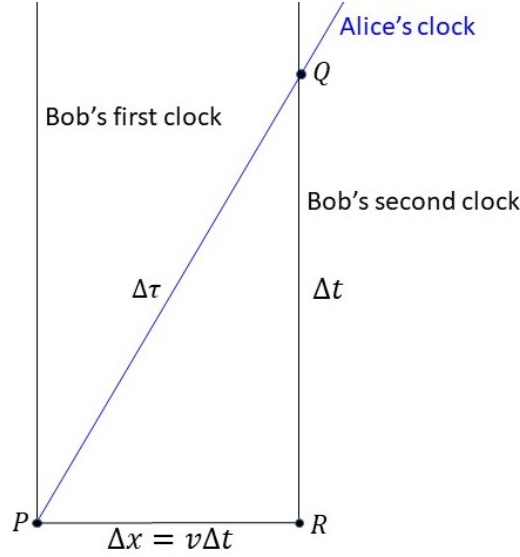


Figure 1.8: Bob's clocks move along vertical trajectories, and Alice's clock moves along the diagonal (blue) trajectory. Alice's clock meeting Bob's first clock is event P ; Alice's clock meeting Bob's second clock is event Q .

Bob's first clock at P is synchronized with his second clock at R , each reading time t , with the two clocks separated in Bob's frame by a distance Δx : Then $PR = \Delta x$, because P and R lie in a surface of simultaneity. Alice's clock, which passes Bob's clock at P , reaches Bob's second clock at a point Q , when that clock reads a time $t + \Delta t$. The statement "Bob measures a speed v for Alice" means $\Delta x = v\Delta t$. That is, the proper time QR elapsed on Bob's second clock is $QR = \Delta t = \Delta x/v$. Finally, the third side PQ of triangle PQR is the proper time elapsed on Alice's clock between events P and Q . From the time dilation equation (1.14), the length PQ is $\Delta\tau = \Delta t\sqrt{1 - v^2}$.

The lengths of the three sides of the triangle therefore satisfy the spacetime Pythagorean relation,

$$(PQ)^2 = (QR)^2 - (PR)^2. \quad (1.16)$$

Check:

$$(PQ)^2 = \Delta\tau^2 = (1 - v^2)\Delta t^2 = \Delta t^2 - (v\Delta t)^2 = (QR)^2 - (PR)^2,$$

or

$$\Delta\tau^2 = \Delta t^2 - \Delta x^2. \quad (1.17)$$

With a generic orientation of Bob's x -axis in his surface of simultaneity, $PR = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$, and we have

$$\Delta\tau^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2, \quad (1.18)$$

as claimed.

Finally, we show that the spacetime Pythagorean relation holds for two spacelike related events. We'll use length-contraction, Eq. (1.15), to relate the proper length Δx of a ruler carried by Bob to its length measured by Alice. Let points P and Q be the two ends of the ruler in one of Alice's surfaces of simultaneity - the left and right ends of the ruler at the same time according to Alice. In the diagram, the history of the ruler is the blue rectangle. Choose Bob's coordinates so that P and Q are in his t - x plane with Δt and Δx positive. If P and Q are spacelike related, then $\Delta t < \Delta x$ in Bob's coordinates. From *Exercise 5*, the surface of simultaneity of an observer (Alice) moving with speed v along Bob's x direction has slope $\Delta t/\Delta x = v$, with Δt and Δx labeled in the diagram below. (With c not set to 1, $v = c^2 \Delta t/\Delta x$.)

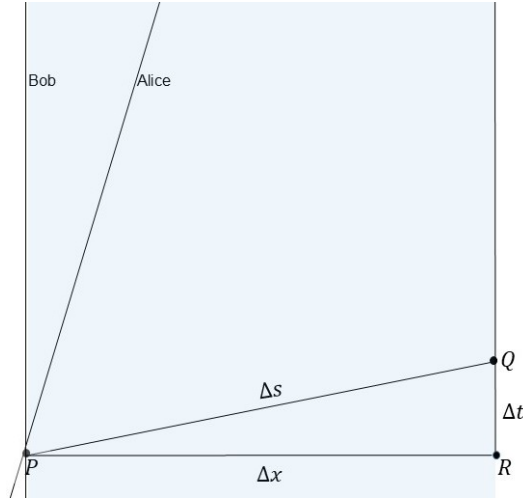


Figure 1.9: The light blue region is the history of a ruler at rest in Bob's frame. PQ and PR are, respectively, intersections of the ruler with Alice's and Bob's surfaces of simultaneity. Because events P and R are simultaneous in the rest frame of the ruler, the length $PR = \Delta x$ is the proper length of the ruler and is its length as measured by Bob. The length $PQ = \Delta s$ is the ruler's length as measured by Alice.

The ruler at Bob's time t is the line PR , the intersection of Bob's surface of simultaneity with the ruler history. The length of the ruler according to Alice is the line PQ , the intersection of her surface of simultaneity with the ruler history. Because Alice sees the ruler move past her with speed v , she measures the contracted length $PQ = \Delta s = \Delta x \sqrt{1 - v^2}$, implying the spacetime Pythagorean relation for the triangle PQR :

$$(PQ)^2 = -(QR)^2 + (PR)^2.$$

Check:

$$\begin{aligned} (PQ)^2 &= (\Delta x \sqrt{1 - v^2})^2 = \Delta x^2 [1 - (\Delta t/\Delta x)^2] = \Delta x^2 - \Delta t^2 \\ &= -(QR)^2 + (PR)^2, \quad \text{or} \\ \Delta s^2 &= -\Delta t^2 + \Delta x^2. \end{aligned}$$

With a general direction of Bob's spatial axes, we then have

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2, \quad (1.19)$$

as claimed.

If PQ is the trajectory of a photon, $\Delta t = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. Requiring the spacetime distance between P and Q to be continuous as a function of P and Q then implies the spacetime distance Δs between lightlike related events is zero.

1.1.3 Lorentz transformations

Symmetries of flat space

The equations of Newtonian physics are invariant under rotations and translations. The active form of the invariance is the statement that a constant rotation or translation takes an isolated system to a physically indistinguishable isolated system. In its passive form, this is the invariance of the equations under a rotation or translation of the coordinates. The invariance is tied to the invariance of flat space under the Euclidean group, the group of transformations

$$\mathbf{x} \mapsto R\mathbf{x} + \mathbf{a}$$

with R a rotation or rotation-reflection.

The rotation group is the group of all linear transformations of flat space that preserve lengths and angles, or, equivalently, dot products of two vectors

$$\mathbf{u} \cdot \mathbf{v} = \delta_{ij} u^i v^j = u^1 v^1 + u^2 v^2 + u^3 v^3.$$

Let $\bar{\mathbf{u}} = R\mathbf{u}$, $\bar{\mathbf{v}} = R\mathbf{v}$. Preserving the dot product means $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = \mathbf{u} \cdot \mathbf{v}$, or

$$\delta_{ij} R^i_k R^j_l u^k v^l = \delta_{ij} u^i v^j, \text{ all } \mathbf{u}, \mathbf{v} \implies \delta_{ij} R^i_k R^j_l = \delta_{kl}. \quad (1.20)$$

That is, a rotation R preserves the metric δ_{ij} of flat space. The Euclidean group is then the group of length and angle preserving maps: These are *isometries*, maps that preserve the metric.

Newton's laws are also invariant, as noted earlier, under Galilean transformations, boosts by an arbitrary velocity $\mathbf{v}$:

$$t \mapsto t, \quad \mathbf{x} \mapsto \mathbf{x} + \mathbf{v}t,$$

mixing space and time, but preserving $t = \text{constant}$ surfaces. These are not isometries of a spacetime metric, but we will see that they are the Newtonian limit of Lorentz boosts, isometries of the Lorentz metric.

Symmetries of flat spacetime

The 4-dimensional dot product of two vectors u, v is given by

$$u \cdot v = \eta_{\mu\nu} u^\mu v^\nu = u_\mu v^\mu = -u^0 v^0 + u^1 v^1 + u^2 v^2 + u^3 v^3,$$

and the Lorentz group is the group of linear transformations preserving dot products. Again, with $\bar{u} = \Lambda u$, $\bar{v} = \Lambda v$, preserving dot products means $\bar{u} \cdot \bar{v} = u \cdot v$ or

$$\eta_{\mu\nu} \Lambda^\mu_\sigma \Lambda^\nu_\tau u^\sigma v^\tau = \eta_{\mu\nu} u^\mu v^\nu, \text{ all } u, v \implies \eta_{\mu\nu} \Lambda^\mu_\sigma \Lambda^\nu_\tau = g_{\sigma\tau}. \quad (1.21)$$

That is, Λ preserves the metric: A Lorentz transformation is a linear map of Minkowski space to itself that preserves the Lorentz metric.

One typically describes the group by picking a timelike direction $\hat{u}$ and slicing the spacetime by spacelike hypersurfaces orthogonal to $\hat{u}$. These are the surfaces of simultaneity of an observer whose velocity is $\hat{u}$. The Lorentz group is generated by the rotations of these hypersurfaces and by boosts.

In an active description, a boost Λ maps a point P of Minkowski space to the point $\tilde{P} = \Lambda(P)$. If P has coordinates (t, x, y, z) , the coordinates of $\tilde{P}$, for a boost in the t - x plane are given by

$$\begin{aligned}\tilde{t} &= \frac{t + vx}{\sqrt{1 - v^2}} = \gamma(t + vx) \\ \tilde{x} &= \frac{x + vt}{\sqrt{1 - v^2}} = \gamma(x + vt), \quad \tilde{y} = y, \quad \tilde{z} = z\end{aligned}$$

or

$$\Lambda = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1.22)$$

where $v < 1$.

A boost maps a 4-velocity $\hat{u}$ with components $\hat{u}^\mu = (1, 0, 0, 0)$ to a velocity $\tilde{u}$ with components $\tilde{u}^\mu = \Lambda^\mu_\nu u^\nu$, or $(\tilde{u}^\mu) = (\gamma, \gamma v, 0, 0)$. The surface of simultaneity of Alice with velocity $\hat{u}$ is mapped to the surface of simultaneity of Bob with velocity $\tilde{u}$. In particular, the unit vector with components $(\hat{x}^\mu) = (0, 1, 0, 0)$ is mapped to

$$(\tilde{x}^\mu) = (\Lambda^\mu_\nu \hat{x}^\nu) = (\gamma v, \gamma, 0, 0),$$

orthogonal to $\tilde{u}^\mu$. These unit vectors are, of course, the vectors tangent to the trajectories of Alice and Bob and to their surfaces of simultaneity in Fig. 1.4.

The Poincaré group is the group of Lorentz transformations and translations,

$$P \mapsto \Lambda(P) + a, \quad (1.23)$$

the group of length and angle preserving maps, isometries of the Lorentz metric.

We can write a boost in a form resembling a rotation, by setting $\cosh \theta = \frac{1}{\sqrt{1-v^2}}$, $\sinh \theta = \frac{v}{\sqrt{1-v^2}}$:

$$\Lambda = \begin{pmatrix} \cosh \theta & \sinh \theta & 0 & 0 \\ \sinh \theta & \cosh \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.24)$$

The relation comes from the replacement $t \rightarrow \tau = it$ that takes the Minkowski metric to the Euclidean metric $d\tau^2 + dx^2 + dy^2 + dz^2$, analytically extended to real τ . The boost of the t - x plane becomes a rotation by ϑ in the τ - x plane,

$$\begin{pmatrix} \cos \vartheta & \sin \vartheta & 0 & 0 \\ \sin \vartheta & \cos \vartheta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1.25)$$

with $\theta = i\vartheta$.

You are more likely to have encountered the passive version of a boost in which the points of spacetime are fixed and the coordinates changed: Again Bob moves with speed v relative to Alice. Alice and

Bob assign to each point P of spacetime coordinates $t(P), x(P), y(P), z(P)$ and $t'(P), x'(P), y'(P), z'(P)$, related by

$$\begin{aligned} t' &= \frac{t - vx}{\sqrt{1 - v^2}} = \gamma(t - vx) \\ x' &= \frac{x - vt}{\sqrt{1 - v^2}} = \gamma(x - vt) \\ y' &= y \\ z' &= z. \end{aligned}$$

Alice's and Bob's surfaces of simultaneity are of course the $t = \text{constant}$ and $t' = \text{constant}$ surfaces. The origin is chosen with Alice's trajectory given by the line $x = y = z = 0$, and Bob's trajectory is $x' = y' = z' = 0$.

1.3 Tensors and Vector Spaces

1.3.1 Mathematical preliminaries

We start with some notation:

$\mathbb{R}$ is the real line, $\mathbb{C}$ the complex plane.

$\mathbb{R}^n$ is the set of n -tuples (ordered sets of n real numbers).

Then $\mathbf{x} = (x^1, \dots, x^n)$ is in $\mathbb{R}^n$ if $x^1, \dots, x^n$ are in $\mathbb{R}$. We'll of course use $(x^0, \dots, x^3)$ for spacetime.

Summation is as usual implied over a repeated index. Our conventions will be such that repeated indices will occur in pairs with one raised and one lowered.

A *group* is a set G together with a binary operation $\cdot$, satisfying

- (1) $a, b \text{ in } G \Rightarrow a \cdot b \text{ in } G$
- (2) $a, b, c \text{ in } G \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c.$
- (3) There is an identity element e in G such that $a \cdot e = e \cdot a = a$, all a in G .
- (4) For every a in G there is an element a^{-1} in G with $a \cdot a^{-1} = a^{-1} \cdot a = e$.

An *abelian* group is a group in which the product $\cdot$ is commutative: $a \cdot b = b \cdot a$, all a, b in G .

Example 1: The integers are an abelian group under addition (the “product” is $+$ and the identity is 0).

Example 2: The reals are an abelian group under addition.

Example 3: The complex numbers are an abelian group under addition.

Example 4: $\mathbb{R}^n$ is an abelian group under addition defined by

$$(x^1, \dots, x^n) + (y^1, \dots, y^n) = (x^1 + y^1, \dots, x^n + y^n).$$

Example 5: The group GL_4 is the group of all linear maps $a : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ for which $a(\mathbf{x}) \neq 0$ unless $\mathbf{x} = 0$. The element a can be represented as an $n \times n$ matrix by writing $x \rightarrow x'$ with $x'^i = a^i_j x^j$. Then the condition that $a(\mathbf{x}) \neq 0$ for $\mathbf{x} \neq 0$ is equivalent to requiring that the matrix a^i_j be nonsingular (that $\det \|a^i_j\| \neq 0$), and also equivalent to the existence of an inverse map a^{-1} in GL_4 .

Example 6: The smooth invertible maps of $\mathbb{R}^4$ onto $\mathbb{R}^4$ form a group with composition the product. This is the group of diffeomorphisms of $\mathbb{R}^4$. (Invertible here means having a smooth inverse.)

Exercise 6. Prove that GL_n , defined as the set of all $n \times n$ matrices with nonzero determinant is a group. In particular, show that each element $\|a_j^i\|$ has an inverse.

1.3.2 Vector Space

Writing laws of Newtonian physics in terms of vectors and tensors makes them manifestly invariant under rotations. And numbers obtained by measurements can be written as dot products of tensors and vectors and are thus manifestly independent of coordinates or a choice of basis. In relativity, the laws of physics are again written in terms of vectors and tensors, now with spacetime indices, and are thereby manifestly Lorentz invariant. Vectors $\mathbf{v}$ and $\mathbf{w}$ can be added, with $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$, and you can multiply them by real numbers r , so a vector space is an abelian group under addition, together with an operation of scalar multiplication:

Definition: A set V is a real vector space if V is an abelian group under an operation (addition) that we denote by $+$ and if there is a second operation (scalar multiplication) that associates with every r in $\mathbb{R}$ and $\mathbf{v}$ in V an element $r\mathbf{v}$ in V and that satisfies

$$\begin{aligned} (1) \quad r(\mathbf{u} + \mathbf{v}) &= r\mathbf{u} + r\mathbf{v} & (2) \quad (r + s)\mathbf{v} &= r\mathbf{v} + s\mathbf{v} \\ (3) \quad r(s\mathbf{v}) &= (rs)\mathbf{v} & (4) \quad 1\mathbf{v} &= \mathbf{v}, \quad \text{all } r, s \text{ in } \mathbb{R}, \mathbf{u}, \mathbf{v} \text{ in } V \end{aligned}$$

A complex vector space is defined in exactly the same way, except that r and s are in $\mathbb{C}$ instead of $\mathbb{R}$.

Because V is an abelian group, there is a zero vector $\mathbf{0}$ for which $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$, all $\mathbf{v}$ in V .

Example 1: $\mathbb{R}^n$ is a vector space with scalar multiplication defined by

$$r(x^0, \dots, x^{n-1}) = (rx^0, \dots, rx^{n-1})$$

Exercise 7. Show that $0 \cdot \mathbf{v} = \mathbf{0}$, all $\mathbf{v}$ in V .

Exercise 8. Show that $\mathbb{R}^n$ is a vector space with $\mathbf{0} = (0, \dots, 0)$, checking that it is an abelian group under addition and that conditions (1)-(4) are satisfied

Example 2: The set of square integrable functions ($\int f^2 dx < \infty$) on the real line is a vector space: Addition is addition and scalar multiplication is scalar multiplication.

Example 3: A Hilbert space is a complete complex vector space together with an inner product.

Definition: $L : V \rightarrow W$ is a *linear* map between two vector spaces V and W if

$$L(r\mathbf{u} + s\mathbf{v}) = rL(\mathbf{u}) + sL(\mathbf{v}), \quad \text{all } r, s \in \mathbb{R}, \mathbf{u}, \mathbf{v} \in V.$$

Two vector spaces V and W are *isomorphic* if there is a one to one and onto linear map $L : V \rightarrow W$.

Example: The group of isomorphisms from $\mathbb{R}^4$ to itself is GL_4 .

A vector space V is finite dimensional if there is a finite set of vectors $\{e_0, \dots, e_{n-1}\}$ that span V , i.e., such that every vector $\mathbf{v}$ in V can be written in the form

$$\mathbf{v} = v^i \mathbf{e}_i, \quad v^i \in \mathbb{R}.$$

The smallest number n of vectors that span V is called the dimension of V and a set of n vectors that span V is a *basis* or frame of V . The numbers v^i are the *components* of $\mathbf{v}$ in the frame $\{\mathbf{e}_0, \dots, \mathbf{e}_{n-1}\}$. Note that the n basis vectors are linearly independent. That is, no one of them – say $\mathbf{e}_0$ – is a linear combination of the rest. If it were, if $\mathbf{e}_0$ could be written in terms of the others,

$$\mathbf{e}_0 = r^1 \mathbf{e}_1 + \dots + r^{n-1} \mathbf{e}_{n-1},$$

then *any* vector $\mathbf{v}$ would be a linear combination of the remaining basis vectors, $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$,

$$\mathbf{v} = v^0 \mathbf{e}_0 + v^1 \mathbf{e}_1 + \dots + v^{n-1} \mathbf{e}_{n-1} = (v^1 + r^1 v^0) \mathbf{e}_1 + \dots + (v^{n-1} + r^{n-1} v^0) \mathbf{e}_{n-1}.$$

The $n - 1$ vectors $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$ would therefore span V , contradicting the assumption that the smallest set spanning V has n vectors.

Example: $\mathbb{R}^n$ has the natural basis

$$\mathbf{E}_0 = (1, 0, \dots, 0)$$

$$\vdots$$

$$\mathbf{E}_{n-1} = (0, \dots, 0, 1),$$

and the i^{th} component of $\mathbf{x} = (x^0, \dots, x^{n-1})$ is the number x^i .

Every n -dimensional vector space is, in fact, isomorphic to $\mathbb{R}^n$, and an isomorphism L is easy to find: Let $\{\mathbf{e}_0, \dots, \mathbf{e}_{n-1}\}$ be a basis of V . Then any $\mathbf{v}$ in V is $\mathbf{v} = v^i \mathbf{e}_i$ and the isomorphism is simply

$$L(\mathbf{v}) = (v^0, \dots, v^{n-1}) = v^i \mathbf{E}_i. \quad (1.26)$$

Exercise 9. Show that L is an isomorphism, that it's linear, onto, and one to one. Show first that a linear map L is one-to-one if and only if

$$L(\mathbf{v}) = 0 \implies \mathbf{v} = 0.$$

To have an isomorphism $V \rightarrow \mathbb{R}^n$ means that if you give me a vector in V , I'll give you n real numbers. And a frame of V provides just that: For each vector $\mathbf{v}$, it gives you the n numbers $(v^0, \dots, v^{n-1})$, the components of $\mathbf{v}$ along the frame.

Change of basis:

We'll use MTW notation, in which one writes the components of a vector $\mathbf{v}$ in two different frames as v^i and $v^{i'}$, with the prime associated with the index. This reflects the fact that the vector itself does not change under a change of basis.

If $(\mathbf{e}_0, \dots, \mathbf{e}_{n-1})$ and $(\mathbf{e}_{0'}, \dots, \mathbf{e}_{n-1'})$ are any two bases of V , then each vector $\mathbf{e}_i$ is a linear combination of the primed basis vectors $\mathbf{e}_{i'}$:

$$\mathbf{e}_i = \mathbf{e}_{j'} a^{j'}_i. \quad (1.27)$$

Analogously,

$$\mathbf{e}_{i'} = \mathbf{e}_j a^j_{i'}. \quad (1.28)$$

Because $\|a^{j'}_i\|$ takes the unprimed frame to the primed frame, and $\|a^j_{i'}\|$ takes the primed frame back to the unprimed frame, they must be inverses of one another.

Claim: The matrices $\|a^{j'}_i\|$ and $\|a^j_{i'}\|$ are inverses:

$$a^i_{k'} a^{k'}_j = \delta^i_j \quad \text{and} \quad a^{i'}_k a^k_{j'} = \delta^{i'}_{j'}. \quad (1.29)$$

Proof: $e_i a^i_{k'} a^{k'}_j = e_{k'} a^{k'}_j = e_j = \delta^i_j e_i$. Equating coefficients of each e_i gives $a^i_{k'} a^{k'}_j = \delta^i_j$. To show the second equation, exchange primed and unprimed indices in the proof.

If a vector $\mathbf{v}$ has components v^i with respect to e_i , its components with respect to e'_i are

$$v^{i'} = a^{i'}_j v^j. \quad (1.30)$$

Similarly,

$$v^i = a^i_{j'} v^{j'}. \quad (1.31)$$

Proof: We have $\mathbf{v} = v^i e_i = v^{i'} e_{i'}$ and $e_{i'} = e_j a^j_{i'}$. Then $e_i v^i = e_j a^j_{i'} v^{i'} = e_i a^i_{j'} v^{j'}$.

Equating coefficients of e_i gives (1.31).

Exercise 10. In the natural basis $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ for $\mathbb{R}^3$, a vector has components $v^1 = 1$, $v^2 = 2$, $v^3 = 3$.

(a) What are the components of $\mathbf{v}$ relative to the bases

$$\mathbf{e}_{1'} = (1, 1, 0), \quad \mathbf{e}_{2'} = (0, 2, 0), \quad \mathbf{e}_{3'} = (1, 1, 1);$$

and

$$\mathbf{e}_{1''} = (2, 0, 0), \quad \mathbf{e}_{2''} = (0, 2, 0), \quad \mathbf{e}_{3''} = (0, 0, 2)?$$

(b) Find the matrices $a^{i'}_j$, $a^i_{j'}$, $a^{i''}_j$, $a^i_{j''}$.

1.3.3 Dual Space

In quantum mechanics, the wave function ψ of a single particle, say, can be regarded as a “ket” vector $|\psi\rangle$ in the Hilbert space H of square integrable functions on space ($\mathbb{R}^3$). It can also be regarded as a “bra” vector $\langle\psi|$, a linear map from H to $\mathbb{C}$ by

$$\langle\psi| : |\tilde{\psi}\rangle \rightarrow \langle\psi|\tilde{\psi}\rangle = \int \psi^* \tilde{\psi} d^3x,$$

where $\tilde{\psi}$ is any other square integrable function ($|\tilde{\psi}\rangle$ a vector in H).

The vector $\mathbf{x} = (x^0, \dots, x^3)$ in $\mathbb{R}^4$ can similarly be regarded as a linear map from $\mathbb{R}^4$ to $\mathbb{R}$ by $\mathbf{x} : \mathbf{y} \mapsto -x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3$, where $\mathbf{y}$ is any vector in $\mathbb{R}^4$. If

$$x_\mu = \eta_{\mu\nu} x^\nu,$$

we can write $\mathbf{x} : \mathbf{y} \longrightarrow x_\mu y^\mu$.

The set of all linear maps from V to $\mathbb{R}$ also is called the dual space, V^* . It is a vector space (Exercise below), and, if V is finite-dimensional of dimension n , then V^* also has dimension n . If σ and τ are two linear maps from V to $\mathbb{R}$, their sum is

$$(\sigma + \tau)(\mathbf{v}) = \sigma(\mathbf{v}) + \tau(\mathbf{v})$$

$$\text{real number} = \text{real number} + \text{real number}$$

Scalar multiplication is given by

$$(r\sigma)(\mathbf{v}) = r(\sigma(\mathbf{v}))$$

$$\text{real number} = \text{real number } r \times \text{real number } \sigma(\mathbf{v})$$

The vectors σ in V^* are called covectors or 1-forms. Wald calls them dual vectors.

Exercise 11. Show that V^* is a vector space.

To see that V^* is n dimensional when V is n dimensional, we introduce a basis $\{e_i\}$ for V . Let σ be any linear map, and $v = v^i e_i$ a vector in V . Because σ is linear, $\sigma(v) = v^i \sigma(e_i)$. So if we define the n real numbers σ_i by

$$\sigma_i = \sigma(e_i), \quad (1.32)$$

we have $\sigma(v) = \sigma_i v^i$. Now comes the key part: we are going to define a basis for V^* in which σ_i will be the i^{th} component of σ . If we let ω^i be the linear map

$$\omega^i(v) = v^i$$

Then σ is a linear combination of the ω^i :

$$\sigma = \sigma_i \omega^i.$$

Check: $\sigma(v) = \sigma_i v^i = \sigma_i \omega^i(v)$.

Thus $\{\omega^i\}$ is a basis for V^* , and the numbers $\sigma_1, \dots, \sigma_n$ are the components of σ in the basis $\{e_i\}$.³ In particular, V^* is n -dimensional. A basis $\{\omega^i\}$ for V^* is called the basis *dual to* $\{e_i\}$ if

$$\omega^i(e_j) = \delta_j^i,$$

and the basis we have just introduced satisfies this relation.

Change of basis

If $e_{i'} = e_j a^j_{i'}$, and σ is a covector, then

$$\sigma_{i'} = a^j_{i'} \sigma_j. \quad (1.33)$$

Proof. $\sigma_{i'} = \sigma(e_{i'}) = \sigma(e_j a^j_{i'}) = a^j_{i'} \sigma(e_j) = a^j_{i'} \sigma_j$.

The basis dual to $\{e_{i'}\}$ is $\{\omega^{i'}\}$, with

$$\omega^{i'} = a^{i'}_j \omega^j.$$

Check:

$$\omega^{i'}(v) = a^{i'}_j \omega^j(v) = a^{i'}_j v^j = v^{i'}. \quad (1.34)$$

Finally, note that $\sigma(v)$ is a number, and it never heard of a basis, so $\sigma_i v^i$ must be independent of the choice of basis. Check:

$$\sigma_{i'} v^{i'} = (a^j_{i'} \sigma_j) (a^{i'}_k v^k) = (a^j_{i'} a^{i'}_k) \sigma_j v^k = \delta^j_k \sigma_j v^k = \sigma_j v^j.$$

Abstract index notation (invented by Roger Penrose):

Instead of using arrows or boldface, we will often write a vector in the manner v^a , and a covector will be written σ_a . The symbol v^a will *not* mean the a^{th} component of the vector – it will be the same as writing v .

$$v^a \equiv v. \quad (1.35)$$

Then $\sigma_a v^a$ and $v^a \sigma_a$ both mean $\sigma(v)$. There is no reason to introduce a basis every time we want to talk about a vector, but the boldface (or arrow) notation is clumsy for tensor algebra.

³MTW and Schutz use this notation for the dual basis, following Cartan. Wald writes e^i instead of ω^i .

Index conventions: Latin indices will be used for any number of dimensions, Greek for spacetime only. Latin and Greek indices at the beginning of the alphabet, $a, b, c, \dots$ and $\alpha, \beta, \gamma, \dots$ will be abstract (a for “abstract”); Latin and Greek indices from i (ι) onward, $i, j, k, l, m, n, \dots$ and $\iota, \kappa, \lambda, \mu, \nu, \dots$, will be concrete.

1.3.4 Tensors

The definition of a connecting vector, as a directed line joining two points does not need a coordinate system or a choice of basis. And the definition specifying magnitude and direction again does not involve a basis. Tensors similarly arise in physics in a way that does not involve the introduction of a basis or the components of the tensor. In particular, the tensor you are likely to have encountered first, the moment of inertial tensor I_{ab} , relates the angular velocity vector Ω^a of a rigid body to the body’s rotational kinetic energy and to its angular momentum.

$$E = \frac{1}{2} I_{ab} \Omega^a \Omega^b, \quad L_a = I_{ab} \Omega^b.$$

That is I_{ab} can be regarded as the map $\Omega^b \rightarrow L_a$ from angular velocity to angular momentum; or as the map $(\Omega^a, \Omega^b) \rightarrow 2E$. The map $\Omega^b \rightarrow L_a$ is linear; and the map $(\Omega^a, \Omega^b) \rightarrow 2E$ is bilinear (separately linear in each argument). Wald uses abstract indices, while MTW use only concrete indices and write $I(u, v)$ instead of $I_{ab} u^a u^b$.

We already introduced a covector as a linear map from vectors v^a to numbers; $\sigma_a : v^a \rightarrow \sigma_a v^a$. A vector can similarly be regarded as a linear map from covectors σ_a to numbers;

$$v_a : \sigma_a \rightarrow \sigma_a v^a.$$

We define a covariant tensor T_{ab} as a map from pairs of vectors to numbers $(u^a, v^b) \rightarrow T_{ab} u^a v^b$, with T_{ab} linear in u^a and v^b separately:

$$T_{ab}(ru^a + sw^a)v^b = rT_{ab}u^a v^b + sT_{ab}w^a v^b \quad (1.36)$$

$$T_{ab}u^a(rv^b + sw^b) = rT_{ab}u^a v^b + sT_{ab}u^a w^b \quad (1.37)$$

The components of T_{ab} in a basis $(e_0^a, e_1^a, e_2^a, e_3^a)$ are, for example,

$$T_{12} = T_{ab} e_1^a e_2^b$$

Exercise 12. a) Show that in a different basis $e_{i'} = e_i a_{i'}^i$

$$T_{i'j'} = a_{i'}^i a_{j'}^j T_{ij}. \quad (1.38)$$

T_{ab} is called a second rank covariant tensor. A (third rank, mixed) tensor S^{ab}_c is analogously a map that takes two covectors and one vector to a number, $(\sigma_a, \tau_b, v^c) \rightarrow S^{ab}_c \sigma_a \tau_b v^c$, with the map linear in each argument.

It is worth reiterating that each component of a tensor is a number. Each component of a tensor field is then a scalar field.

Exercise 8 b). Prove that the components of S^{ab}_c transform under a change of basis in the manner

$$S^{i'j'}_{k'} = a_{i'}^i a_{j'}^j a_{k'}^k S^{ij}_k. \quad (1.39)$$

In general then, a tensor $T^{a\cdots b}_{c\cdots d}$ with r up indices and s down indices is a multilinear map from s vectors and r covectors to $\mathbb{R}$. Its components transform under a change of basis in the manner

$$T^{i'\cdots j'}_{k'\cdots l'} = a^{i'}_i \cdots a^{j'}_j a^k_{k'} \cdots a^l_{l'} T^{i\cdots j}_{k\cdots l}. \quad (1.40)$$

The tensors with r up indices and s down indices form a vector space: Each linear combination of tensors of the same type is again a linear map:

$$pS^{a\cdots b}_{c\cdots d} + qT^{a\cdots b}_{c\cdots d} \text{ maps } (\sigma_a, \dots, \tau_b, u^c, \dots, v^d) \text{ to}$$

$$(pS^{a\cdots b}_{c\cdots d} + qT^{a\cdots b}_{c\cdots d})\sigma_a \cdots \tau_b u^c \cdots v^d = pS^{a\cdots b}_{c\cdots d}\sigma_a \cdots \tau_b u^c \cdots v^d + qT^{a\cdots b}_{c\cdots d}\sigma_a \cdots \tau_b u^c \cdots v^d.$$

In the older mathematics texts, tensors are often defined by the transformation law, Eq. (1.40) – and, in physics texts, by the equivalent law with the $a^{j'}_i$'s restricted to Lorentz transformations. The transformation law (1.38) is equivalent to saying

the number $T_{ij}u^i v^j$ depends only on the vectors u^a and v^b ; it is independent of the choice of basis.

But this statement means that $(u^a, v^b) \rightarrow T_{ij}u^i v^j$ is a well defined map from vectors to numbers; in other words, giving a tensor in the sense of a set of numbers T_{ij} that transform by (1.38) is the same as giving a bilinear map T_{ab} from vectors to real numbers. The components of T_{ab} are the numbers T_{ij} .

Outer product

If S_{ab} and T^{ab}_c are tensors, so is $S_{ab}T^{cd}_e$ defined by $S_{ab}T^{cd}_e u^a v^b \sigma_c \tau_d w^e = (S_{ab}u^a v^b)(T^{cd}_e \sigma_c \tau_d w^e)$. $S_{ab}T^{cd}_e$ is the outer product of S_{ab} and T^{ab}_c , and its components in a basis are, of course, $S_{ij}T^{kl}_m$.

Any tensor T^{ab} , say, can be built up out of vectors (and covectors) by taking outer products. That is, if $(\hat{t}^\alpha, \hat{x}^\alpha, \hat{y}^\alpha, \hat{z}^\alpha)$ is a basis for V (think of $\hat{t}^\alpha$ as the unit vector along a t -axis in Minkowski space), any vector can be written in the form

$$A^\alpha = A^t \hat{t}^\alpha + A^x \hat{x}^\alpha + A^y \hat{y}^\alpha + A^z \hat{z}^\alpha$$

(where A^t, A^x, A^y , and A^z are just components of A^α in the basis $(\hat{t}^\alpha, \hat{x}^\alpha, \hat{y}^\alpha, \hat{z}^\alpha)$, and any tensor $T^{\alpha\beta}$ can be written

$$T^{\alpha\beta} = T^{tt} \hat{t}^\alpha \hat{t}^\beta + T^{tx} \hat{t}^\alpha \hat{x}^\beta + T^{xt} \hat{x}^\alpha \hat{t}^\beta + T^{ty} \hat{t}^\alpha \hat{y}^\beta + \cdots \quad (1.41)$$

For example, the Minkowski metric is

$$\eta^{\alpha\beta} = -\hat{t}^\alpha \hat{t}^\beta + \hat{x}^\alpha \hat{x}^\beta + \hat{y}^\alpha \hat{y}^\beta + \hat{z}^\alpha \hat{z}^\beta. \quad (1.42)$$

The index-free notation for the outer product of tensors S and T is $S \otimes T$.

Contraction

Given the tensor $T^a_b{}^{cd}_e = u^a \sigma_b v^c w^d \tau_e$, we can obtain a new tensor of lower rank by contracting on one lower and one upper index. That is, a tensor $\tilde{T}^{cd}_b \equiv T^a_b{}^{cd}_a$, with one down and two up indices is defined by

$$T^a_b{}^{cd}_a = u^a \sigma_b v^c w^d \tau_a \equiv \sigma_b v^c w^d \underbrace{(u^a \tau_a)}_{\text{a number}} \quad (1.43)$$

Since any tensor can be written as a sum of outer products of vectors,

$$A^{ab}_{cde} = u^a v^b \sigma_c \tau_d \omega_e + w^a x^b \zeta_c \eta_d \xi_e + \cdots,$$

one can define contraction between any up and any down index by

$$A^{ae}_{cde} = u^a v^e \sigma_c \tau_d \omega_e + w^a x^e \zeta_c \eta_d \xi_e + \cdots \quad (1.44)$$

The meaning of contraction is easy to understand in terms of components:

$$A^{im}{}_{klm} \equiv \sum_{m=1}^n A^{im}{}_{klm} \quad . \quad (1.45)$$

Exercise 13. Show the definition (1.44) implies (1.45) by looking first at tensors of the form $u^a \sigma_b$, then of the form $u^a \cdots v^b \sigma_c \cdots \tau_d$, and finally of the general form: linear combinations of these.

Index Symmetries

A tensor $T^{abc}{}_d$ is symmetric in the two upper indices a and c if $T^{abc}{}_d = T^{cba}{}_d$.

This means that the map, $T^{abc}{}_d$ is symmetric under interchange of its first and third arguments:

$$T^{abc}{}_d \sigma_a \tau_b \omega_c v^d = T^{cba}{}_d \sigma_a \tau_b \omega_c v^d = T^{abc}{}_d \omega_a \tau_b \sigma_d v^d.$$

(The first equality here is a change in the names of the dummy indices.)

A tensor $S_a{}^b{}_{cde}$ is similarly said to be symmetric in a pair of lower indices – in a and c , say – if

$$S_a{}^b{}_{cde} = S_c{}^b{}_{ade}$$

A tensor T^{abc} is antisymmetric under interchange of a and b if $T^{abc} = -T^{bac}$.

$T^{a\cdots b}$ is totally antisymmetric (totally symmetric) if it is antisymmetric (symmetric) under interchange of any pair of indices. If T^{ab} is any tensor

$$T^{(ab)} := \frac{1}{2}(T^{ab} + T^{ba}) \quad (1.46)$$

is a symmetric tensor. Similarly for r indices, $T^{a_1 \cdots a_r}$:

$$T^{(a_1 \cdots a_r)} := \frac{1}{r!} \sum_{\text{permutations } \pi \text{ of } r \text{ integers}} T^{a_{\pi(1)} \cdots a_{\pi(r)}} \quad (1.47)$$

(For example, if $r = 3$, one permutation π is $\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 1$; another is $\pi(1) = 3$, $\pi(2) = 2$, $\pi(3) = 1$.)

Thus

$$T^{(abc)} = \frac{1}{3!}(T^{abc} + T^{bca} + T^{cab} + T^{acb} + T^{bac} + T^{cba}) \quad (1.48)$$

Analogously, given any $T^{a_1 \cdots a_r}$, one constructs the totally antisymmetric tensor

$$T^{[a_1 \cdots a_r]} = \frac{1}{r!} \sum_{\pi} (\text{sign } \pi) T^{a_{\pi(1)} \cdots a_{\pi(r)}} , \quad (1.49)$$

where $(\text{sign } \pi)$ is $+1$ when π is an even permutation, -1 when π is odd. Eq. (1.49) is commonly written

$$T^{[a_1 \cdots a_r]} = \frac{1}{r!} \sum_{\pi} (-1)^{\pi} T^{a_{\pi(1)} \cdots a_{\pi(r)}} . \quad (1.50)$$

For example,

$$T^{[ab]} = \frac{1}{2}(T^{ab} - T^{ba}), \quad (1.51)$$

and

$$T^{[abc]} = \frac{1}{3!}(T^{abc} + T^{bca} + T^{cab} - T^{acb} - T^{bac} - T^{cba}). \quad (1.52)$$

Then $T^{a\cdots b}$ is totally symmetric if $T^{a\cdots b} = T^{(a\cdots b)}$, totally antisymmetric if $T^{a\cdots b} = T^{[a\cdots b]}$.

1.3.5 The tensors $\eta_{\alpha\beta}$, δ^α_β and $\epsilon_{\alpha\beta\gamma\delta}$

A Minkowski metric in a four dimensional vector space is a symmetric tensor $\eta_{\alpha\beta}$ with *signature* $-+++$. Because $\|\eta_{\mu\nu}\|$ is a symmetric matrix, there is a basis in which it is diagonal, and to say that its signature is $-+++$ means that in such a basis it has one negative and three positive diagonal elements (eigenvalues). For example:

$$\|\eta_{\mu\nu}\| = \begin{vmatrix} -\frac{1}{4} & & & \\ & 4 & & \\ & & 9 & \\ & & & 16 \end{vmatrix}$$

By redefining the basis $e_{0'} = 2e_0$, $e_{1'} = \frac{1}{2}e_1$, $e_{2'} = \frac{1}{3}e_2$, $e_{3'} = \frac{1}{4}e_3$, we have $\eta_{0'0'} = \eta_{\alpha\beta}e_{0'}^\alpha e_{0'}^\beta = \eta_{\alpha\beta}(2e_0^\alpha)(2e_0^\beta) = 4\eta_{00} = -1$ and $\eta_{1'1'} = \eta_{2'2'} = \eta_{3'3'} = 1$. That is,

$$\|\eta_{\mu'\nu'}\| = \begin{vmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix}$$

The tensor δ^α_β is defined by $\delta^\alpha_\beta \sigma_\alpha v^\beta = \sigma_\alpha v^\alpha$, and it has components δ^μ_ν in *every* basis. Moreover $\delta^\alpha_\beta v^\beta = v^\alpha$ and $\delta^\alpha_\beta \sigma_\alpha = \sigma_\beta$, all v^α, σ_β .

Given a Minkowski metric $\eta_{\alpha\beta}$, one defines a contravariant tensor $\eta^{\alpha\beta}$ by

(1) $\eta^{\alpha\beta}$ symmetric

(2) $\eta^{\alpha\gamma}\eta_{\beta\gamma} = \delta^\alpha_\beta$.

Then $\|\eta^{\mu\nu}\|$ is the matrix inverse of $\|\eta_{\mu\nu}\|$. In an orthonormal basis, a basis in which

$$\|\eta_{\mu\nu}\| = \begin{vmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix}, \text{ the inverse matrix is } \|\eta^{\mu\nu}\| = \begin{vmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix} \text{ as well.}$$

$\eta_{\alpha\beta}$ can be regarded as a map from vectors to covectors: If ξ^α is a vector, the map is $\xi^\alpha \rightarrow \eta_{\alpha\beta}\xi^\beta$. So given a metric $\eta_{\alpha\beta}$, we have a natural way to identify vectors and covectors, and the same letter ξ will be used for the covector as for the identified vector:

$$\xi_\alpha = \eta_{\alpha\beta}\xi^\beta. \quad (1.53)$$

Then $\eta^{\alpha\beta}\xi_\beta = \eta^{\alpha\beta}\eta_{\beta\gamma}\xi^\gamma = \delta^\alpha_\gamma\xi^\gamma = \xi^\alpha$. Similarly, if $T^{\alpha\beta}$ is a contravariant tensor, we write

$$T_{\alpha\beta} = \eta_{\alpha\gamma}\eta_{\beta\delta}T^{\gamma\delta}, \quad (1.54)$$

and it follows that

$$T^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta}T_{\gamma\delta}. \quad (1.55)$$

In a four dimensional vector space, there is – up to a constant – only one totally antisymmetric tensor $\epsilon_{\alpha\beta\gamma\delta} = \epsilon_{[\alpha\beta\gamma\delta]}$. This is easy to see in terms of components: $\epsilon_{0122} = -\epsilon_{0122}$, interchanging 2 and 2, whence $\epsilon_{0122} = 0$ and $\epsilon_{\mu\nu\sigma\tau}$ can have no nonvanishing components with repeated index. The only nonzero components are ϵ_{0123} , ϵ_{0132} , ϵ_{1203} , etc. and these are related by a permutation of $\{0, 1, 2, 3\}$.

$$\epsilon_{0123} = -\epsilon_{0132} = \epsilon_{1032} = \dots$$

Then $\epsilon_{\alpha\beta\gamma\delta}$ can be fixed by setting $\epsilon_{0123} = 1$ in a right handed orthonormal basis whose timelike basis vector points to the future. With this normalization, we have

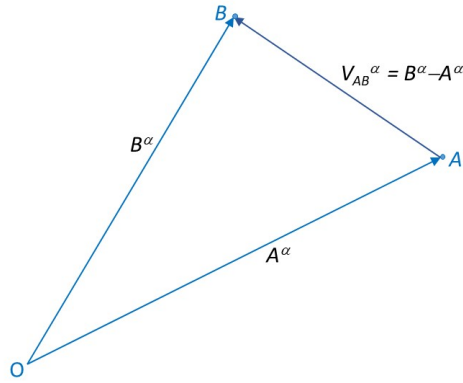
$$\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha\beta\gamma\delta} = -4! \quad (1.56)$$

Exercise.

- Show that (1.56) implies that $\epsilon_{0123} = \pm 1$ in an orthonormal basis.
- Suppose that, in a right-handed, future-pointing basis, $\epsilon_{0123} = 1$. Show that if $\hat{\epsilon}_{0123} = 1$ in a left handed future pointing basis, then $\hat{\epsilon}_{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\gamma\delta}$.

1.3.6 Tensor Fields and ∇_α

The arena of special relativity is Minkowski space, M , which we can now define as a four-dimensional vector space together with a Minkowski metric $\eta_{\alpha\beta}$. This definition singles out an origin – the zero vector $\mathbf{0}$, but physics doesn't hand us a preferred event so we can equally well regard another point A as the origin of a vector space, identifying any point B with the vector $V_{AB}^\alpha = B^\alpha - A^\alpha$.

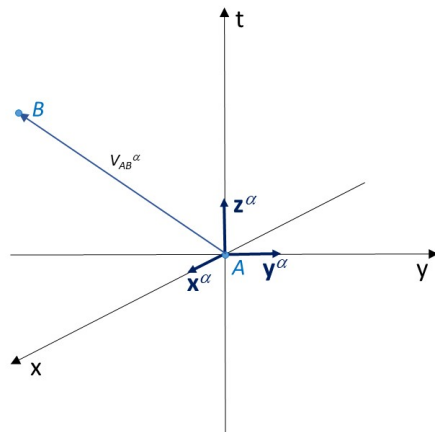


In this way, we regard the connecting vectors V_{AB}^α as vectors at A , and $\eta_{\alpha\beta}$ may be regarded as a metric for the vector space at each point A :

$$\eta_{\alpha\beta} V_{AB}^\alpha V_{AC}^\beta = \eta_{\alpha\beta} (B^\alpha - A^\alpha)(C^\beta - A^\beta). \quad (1.57)$$

If we choose A as the origin of the vector space, then $\mathbf{A} = \mathbf{0}$, and $V_{AB}^\alpha = B^\alpha$.

The natural Cartesian coordinate systems with A as origin assign as coordinates of the point B the components of B^α with respect to an orthonormal basis at A :



$$\mathbf{B} = \mathbf{V}_{AB} = B^t \mathbf{t} + B^x \mathbf{x} + B^y \mathbf{y}$$

The t, x, y coordinates of the point B are then $B^t = t, B^x = x, B^y = y$.

In curvilinear coordinates, the identification is gone: Suppose the point B has coordinates t, r, ϕ , with $x = r \cos \phi$, $y = r \sin \phi$, and let $\hat{r}$ and $\hat{\phi}$ be unit vectors in the r and ϕ directions. Because the vector B from the origin to the point B is radial, $B^{\hat{\phi}} = 0$, not ϕ . (Formally, $B^{\hat{\phi}} = -B^x \sin \phi + B^y \cos \phi = 0$.)

We'll call the Cartesian coordinates associated with an orthonormal frame and a choice of origin *natural* coordinates for M . Any two natural coordinate systems for M are related by a change of origin (translation) together with a change of orthonormal frame (Lorentz transformation) – i.e., two natural coordinate systems $\{x^\mu\}$ and $\{x'^\mu\}$ are related by a Poincaré transformation, $x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$.

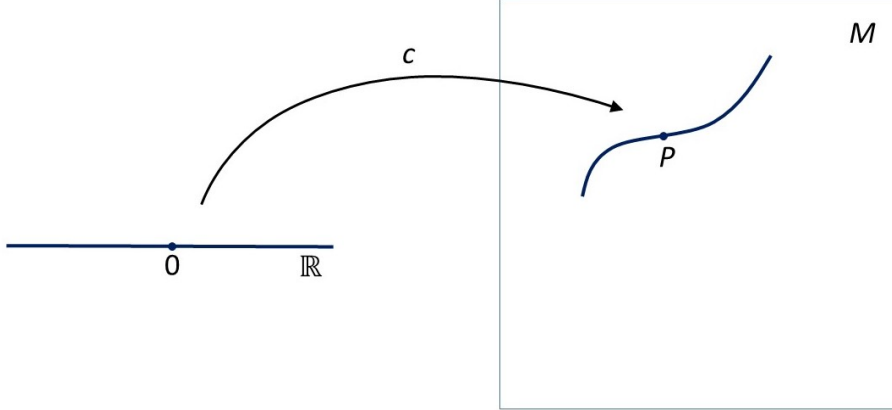
A scalar field on M is a map $f: M \rightarrow \mathbb{R}$ assigning to each point P in M a number $f(P)$.

A vector field on M assigns to each point P a vector $v^\alpha(P)$ at P .

If v^α is a vector at P , $\eta_{\alpha\beta} v^\beta = v_\alpha$ is a covector at P , and if u^α is another vector at P , $u^\alpha v^\beta$ is a tensor at P . In this way one constructs all tensors at P . A tensor field $T^{\alpha\beta}_\gamma$ on M assigns to each point P a tensor $T^{\alpha\beta}_\gamma(P)$ at P .

In a curved space, one no longer has a natural way to define a connecting vector between an arbitrary pair of points. But we will see that the tangent to a curve and the gradient of a function can naturally be identified with vectors and covectors, respectively. Here is the flat-space description, written in a way that will later let us use most of it in curved space.

A path through P is defined as a map $c: \mathbb{R} \rightarrow M$, whose image intersects P . Here we take $c(0) = P$.



Tangent vectors. In flat space, we can regard each point $c(\lambda)$ as a vector at P (the connecting vector from P to $c(\lambda)$) and so define the tangent to c at P as the vector

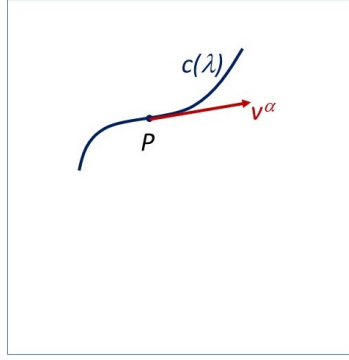
$$v^\alpha = \left. \frac{d}{d\lambda} c^\alpha(\lambda) \right|_{\lambda=0}. \quad (1.58)$$

Two curves are tangent at P if they have the same tangent vector.

A vector v^α at P maps functions on M to real numbers by taking their derivative in the direction of v^α . That is,

$$v(f) = \left. \frac{d}{d\lambda} f(c(\lambda)) \right|_{\lambda=0}, \quad (1.59)$$

where $c(\lambda)$ is a curve through P with tangent v^α . This will be used to *define* vectors in a curved space.



Example: Let t^α be the tangent to the path $\lambda \rightarrow (t + \lambda, x, y, z)$.
i.e. t^α is a unit vector in the t direction at the point (t, x, y, z) .

Then

$$t(f) = \frac{d}{d\lambda} f(t + \lambda, x, y, z) = \frac{\partial f}{\partial t}(t, x, y, z). \quad (1.60)$$

For this reason, the vector t^α is sometimes written ∂_t .

Gradients of functions are covectors.

A function f gives rise to a linear map from vectors at P to real numbers:

$$v^\alpha \rightarrow \left. \frac{d}{d\lambda} f(c(\lambda)) \right|_{\lambda=0}, \text{ where } c(\lambda) \text{ has tangent } v^\alpha.$$

In a natural coordinate system at P

$$\begin{aligned} \left. \frac{d}{d\lambda} f(c(\lambda)) \right|_{\lambda=0} &= \frac{\partial f}{\partial x^\mu} [c(\lambda)] \left. \frac{dc^\mu}{d\lambda} \right|_{\lambda=0} \\ &= \frac{\partial f}{\partial x^\mu}(P) v^\mu, \end{aligned} \quad (1.61)$$

and so the map is clearly linear:

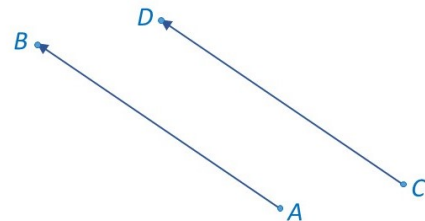
$$\begin{aligned} r u + s v &\rightarrow \frac{\partial f}{\partial x^\mu}(P) (r u^\mu + s v^\mu) \\ &= r \frac{\partial f}{\partial x^\mu}(P) u^\mu + s \frac{\partial f}{\partial x^\mu}(P) v^\mu. \end{aligned}$$

In other words, the map is a covector, and one calls it $\nabla_\alpha f$. Thus $\nabla_\alpha f : v^\alpha \rightarrow v^\alpha \nabla_\alpha f$ has components

$$\nabla_\mu f = \frac{\partial f}{\partial x^\mu} \quad (1.62)$$

Given a function f , one thereby acquires a covector field $\nabla_\alpha f$, the gradient of f .

To extend the action of ∇_α to vectors, we need to compare vectors at two different points. To do so, we use the natural parallel transport that identifies a vector v_{AB}^α at A with the vector v_{CD}^α at C when $v_{AB}^\alpha \equiv B^\alpha - A^\alpha = D^\alpha - C^\alpha \equiv v_{CD}^\alpha$.



A vector field v^α is said to be constant if, given its value $v^\alpha(A)$ at any point A , one obtains the vector $v^\alpha(B)$ by parallel transport of $v^\alpha(A)$ from A to B . Equivalently, the constant vector fields have constant components in a natural coordinate system: We can write

$$v^\alpha = v^t \hat{t}^\alpha + v^x \hat{x}^\alpha + v^y \hat{y}^\alpha + v^z \hat{z}^\alpha, \quad (1.63)$$

where

$\hat{t}^\alpha$ is tangent to the curve $(t + \lambda, x, y, z) = c_0(\lambda)$

$\hat{x}^\alpha$ is tangent to the curve $(t, x + \lambda, y, z) = c_1(\lambda)$,

...

Then the vector field v^α is a constant vector field when v^t, v^x, v^y , and v^z are constants. We set

$$\nabla_\alpha v^\beta = 0 \quad (1.64)$$

when v^α is a constant vector field and extend ∇_α to arbitrary vector fields by the Leibnitz rule

$$\nabla_\alpha (f v^\beta) = (\nabla_\alpha f) v^\beta + f \nabla_\alpha v^\beta, \quad (1.65)$$

v^α now not constant. Since any vector can be written in terms of constant vector fields $\hat{t}^\alpha, \hat{x}^\alpha, \hat{y}^\alpha$ and $\hat{z}^\alpha$ in the form (1.63), we have $\nabla_\alpha v^\beta = (\nabla_\alpha v^t) \hat{t}^\beta + (\nabla_\alpha v^x) \hat{x}^\beta + (\nabla_\alpha v^y) \hat{y}^\beta + (\nabla_\alpha v^z) \hat{z}^\beta$. (v^t, v^x, v^y and v^z are four scalar fields, the components of v^α in the basis $\hat{t}^\alpha, \hat{x}^\alpha, \hat{y}^\alpha, \hat{z}^\alpha$). Thus in the basis $\hat{t}^\alpha, \hat{x}^\alpha, \hat{y}^\alpha, \hat{z}^\alpha$, the components of $\nabla_\alpha v^\beta$ are

$$\begin{aligned} \nabla_\mu v^\nu &= \frac{\partial v^t}{\partial x^\mu} \delta_t^\nu + \frac{\partial v^x}{\partial x^\mu} \delta_x^\nu + \frac{\partial v^y}{\partial x^\mu} \delta_y^\nu + \frac{\partial v^z}{\partial x^\mu} \delta_z^\nu \\ &= \frac{\partial v^\nu}{\partial x^\mu} \quad (t^\nu = \delta_t^\nu, \hat{x}^\nu = \delta_x^\nu, \text{ etc.}) \end{aligned} \quad (1.66)$$

The operator is extended to covectors via

$$\nabla_\alpha \xi_\beta \equiv \eta_{\beta\gamma} \nabla_\alpha \xi^\gamma \quad (1.67)$$

and to arbitrary tensor fields by saying what tensor (multilinear map) $\nabla_\alpha T^{\beta\cdots\gamma}_{\delta\cdots\epsilon}$ is:

$$\begin{aligned} (\nabla_\alpha T^{\beta\cdots\gamma}_{\delta\cdots\epsilon}) w^\alpha \sigma_\beta \cdots \tau_\gamma u^\delta \cdots v^\epsilon &\equiv \nabla_\alpha (T^{\beta\cdots\gamma}_{\delta\cdots\epsilon} \sigma_\beta \cdots \tau_\gamma u^\delta \cdots v^\epsilon) w^\alpha \\ &\quad - T^{\beta\cdots\gamma}_{\delta\cdots\epsilon} (\nabla_\alpha \sigma_\beta) \cdots \tau_\gamma u^\delta \cdots v^\epsilon w^\alpha \\ &\quad - \cdots - T^{\beta\cdots\gamma}_{\delta\cdots\epsilon} \sigma_\beta \cdots \tau_\gamma u^\delta \cdots (\nabla_\alpha v^\epsilon) w^\alpha \end{aligned} \quad (1.68)$$

Exercise 14. Prove that the components of $\nabla_\alpha T^{\beta\cdots\gamma}_{\delta\cdots\epsilon}$ in a natural coordinate system (i.e. in the basis $\hat{t}^\alpha, \hat{x}^\alpha, \hat{y}^\alpha, \hat{z}^\alpha$ of vector fields tangent to the curves $(t + \lambda, x, y, z)$, $(t, x + \lambda, y, z)$ etc. of a natural coordinate system) are

$$\nabla_\lambda T^{\mu\cdots\nu}_{\iota\cdots\kappa} = \frac{\partial}{\partial x^\lambda} T^{\mu\cdots\nu}_{\iota\cdots\kappa}. \quad (1.69)$$

Note that in terms of the natural coordinate components, equation (1.68) is the Leibnitz rule

$$\begin{aligned} \frac{\partial}{\partial x^\lambda} (T^{\mu\cdots\nu}_{\iota\cdots\kappa} \sigma_\mu \cdots \tau_\nu u^\iota \cdots v^\kappa) &= \left(\frac{\partial}{\partial x^\lambda} T^{\mu\cdots\nu}_{\iota\cdots\kappa} \right) \sigma_\mu \cdots \tau_\nu u^\iota \cdots v^\kappa \\ &\quad + T^{\mu\cdots\nu}_{\iota\cdots\kappa} \frac{\partial \sigma_\mu}{\partial x^\lambda} \cdots \tau_\nu u^\iota \cdots v^\kappa + \cdots \end{aligned} \quad (1.70)$$

It follows immediately from (1.70) that

$$\nabla_\alpha \delta^\beta_\gamma = 0, \quad \nabla_\alpha \eta_{\beta\gamma} = 0, \quad \nabla_\alpha \eta^{\beta\gamma} = 0, \quad \nabla_\alpha \epsilon_{\beta\gamma\delta\epsilon} = 0. \quad (1.71)$$

(If all the components of a tensor vanish, the tensor itself vanishes.)

1.4 Particles

1.4.1 Particle Trajectories

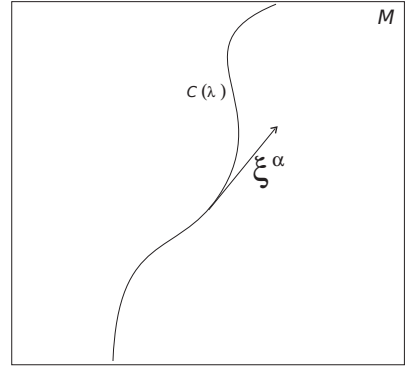
Our primary aim in the rather abstract treatment of tensors given in Sect. 1.3.4 was to develop a formalism that could be easily generalized to curved spacetimes. However, most of the standard calculations of special relativity are more easily performed in a coordinate-free framework - once one gets used to it and no longer feels the need to translate back to a more familiar language. Sect. 1.4 is therefore designed largely to restate concepts and calculations you already know in a somewhat different language.

The motion of a particle can be described as a path in spacetime – its trajectory or worldline – $c(\lambda)$.

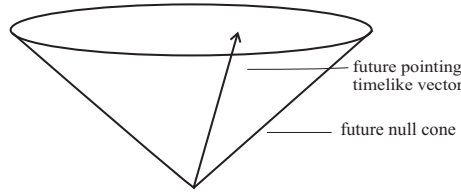
The tangent vector to the path,

$$\xi^\alpha = \frac{dc^\alpha(\lambda)}{d\lambda},$$

is timelike: That's what it means to say the particle travels slower than the speed of light.



The set of all points (spacetime events) that can be reached by a particle starting at a point P is the interior of the future light cone at P , and it can be identified with the set of all future pointing timelike vectors at P .



If the event P is an explosion so that an expanding spherical pulse of light is sent out from it, then the light emerging from this event forms a cone (in spacetime) with vertex P . (A slicing of the light's history is a sequence of spheres expanding from a point.) Formally the cone is the set of all future-pointing null vectors at P .

The velocity (four-velocity) of a particle is the unit vector tangent to its trajectory,

$$u^\alpha = \xi^\alpha (-\xi^\beta \xi_\beta)^{-1/2}, \quad (1.72)$$

with

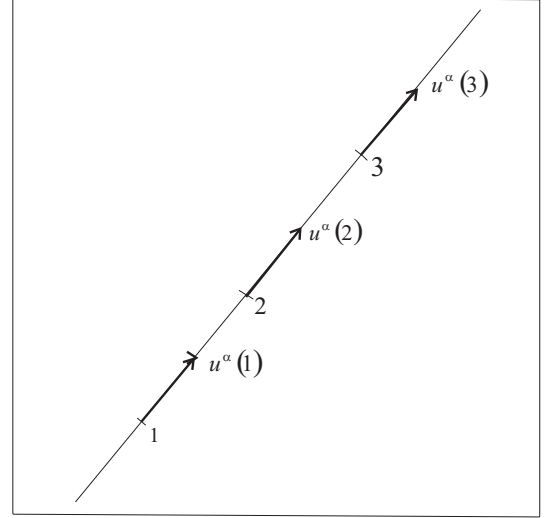
$$u^\alpha u_\alpha = \xi^\alpha (-\xi^\beta \xi_\beta)^{-1/2} \xi_\alpha (-\xi^\gamma \xi_\gamma)^{-1/2} = \xi^\alpha \xi_\alpha (-\xi^\beta \xi_\beta)^{-1} = -1. \quad (1.73)$$

A zero rest mass particle has no velocity vector in this sense; the tangent to its trajectory is a null vector k^α , $k^\alpha k_\alpha = 0$.

A free particle has a straight line trajectory - its tangent vector is parallel propagated along itself

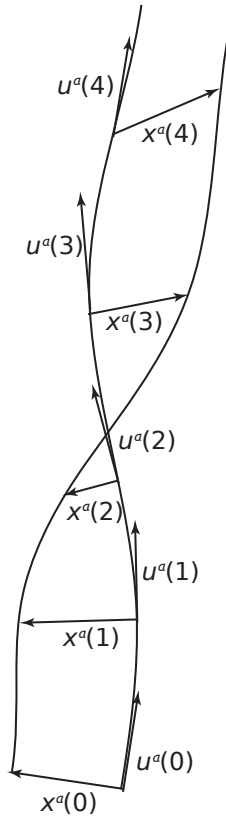
$$u^\beta \nabla_\beta u^\alpha = 0. \quad (1.74)$$

A preferred observer in Minkowski space (an inertial observer) will mean someone who moves along the trajectory of a free particle and who measures components of tensors along a basis $(u^\alpha, \hat{x}^\alpha, \hat{y}^\alpha, \hat{z}^\alpha)$ where u^α is his 4-velocity and $\hat{x}^\alpha, \hat{y}^\alpha, \hat{z}^\alpha$ are unit vectors orthogonal to each other and to u^α and parallel propagated along u^α . Such *inertial frames* are the preferred bases of constant vector fields introduced in Sect. 1.3.6.



In general, of course, particles and observers need not travel in straight lines. One can, for example, describe the measurements made by an accelerating, rotating, observer: At each point along a timelike trajectory (the observer's worldline) one chooses a basis whose timelike basis vector is the observer's u^α . The spacelike vectors are then unit, orthogonal to u^α , and have orientation fixed by the observer's rotation. Think of an observer as someone with three perpendicular rulers (thin rods) and a ticking watch. The connecting vector between ticks is the timelike basis vector; and the intersection of the slice orthogonal to u^α with the history of each ruler is a spacelike basis vector.

History of an observer with watch and one ruler; the observer rotates by π and then stops rotating.



The physical meaning of the metric $\eta_{\alpha\beta}$ is that it provides the measured distance between events. In particular, if a clock moves from A to B along a trajectory $c(\lambda)$, $c(0) = A$, $c(\lambda) = B$, it will tick off a time

$$\tau_c(\lambda) = \int_0^\lambda [-\xi^\alpha(\lambda)\xi^\beta(\lambda)\eta_{\alpha\beta}]^{1/2} d\lambda, \quad \text{where } \xi^\alpha(\lambda) = \frac{dc^\alpha(\lambda)}{d\lambda}. \quad (1.75)$$

τ_c is called the *proper time* along the path c .

Furthermore, there is a preferred time between two (timelike related) events A and B , namely

$$\tau = [-\eta_{\alpha\beta}(B^\alpha - A^\alpha)(B^\beta - A^\beta)]^{1/2},$$

and τ is called the proper time between the events. We will see in a minute that τ is the time measured by a freely moving clock whose trajectory is the straight line from A to B . But first note that if c is parameterized by proper time τ , then $\xi^\alpha = \frac{dc^\alpha(\tau)}{d\tau} = u^\alpha$, the unit vector tangent to c .

Proof: The proof is essentially immediate from the definition (1.75) of proper time. For any $\lambda(\tau)$, Eq. (1.75) implies $\frac{d\tau}{d\lambda} = [-\xi^\alpha \xi_\alpha]^{1/2}$. When $\tau = \lambda$, we have $d\tau/d\lambda = 1$ and $\xi^\alpha = dc^\alpha/d\tau$, implying

$$1 = [-\xi^\alpha \xi_\alpha]^{1/2}, \quad \xi^\alpha \xi_\alpha = \frac{dc^\alpha}{d\tau} \frac{dc_\alpha}{d\tau} = -1.$$

For a freely moving clock – the timepiece of an inertial observer – the trajectory will be

$$c^\alpha(\tau) = \tau u^\alpha, \quad (1.76)$$

parameterized by clock time, since $\frac{dc^\alpha}{d\tau}$ is the constant vector u^α . If τ_A is the clock's time at A and τ_B its time at B ,

$$c^\alpha(\tau_A) = A^\alpha, \quad c^\alpha(\tau_B) = B^\alpha, \quad \text{and}$$

$$\begin{aligned} (A^\alpha - B^\alpha)(A_\alpha - B_\alpha) &= (\tau_A u^\alpha - \tau_B u^\alpha)(\tau_A u_\alpha - \tau_B u_\alpha) \\ &= -(\tau_A - \tau_B)^2. \end{aligned}$$

That is, the proper time between the events A and B is the proper time along the straight line trajectory from A to B .

1.4.2 Boosts and “Addition of Velocity”

We will use “velocity” in this section to mean 4-velocity, and “speed” to mean the magnitude of the 3-velocity seen by an inertial observer. Suppose Alice and Bob are inertial observers with 4-velocities u^α and $\tilde{u}^\alpha$, moving relative to one another with speed v . To say that the boost is along Alice's x -axis means that Alice chooses her basis vector x^α to lie in the plane spanned by u^α and $\tilde{u}^\alpha$. Bob similarly takes his basis vector $\tilde{x}^\alpha$ to be in the same plane.

Bob's speed v is related to the components of his 4-velocity (along Alice's frame) by

$$v := \frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} = \frac{u^x}{u^t}. \quad (1.77)$$

This, together with the fact that u^α is a unit vector, $u^\alpha u_\alpha = -1$, implies

$$(\tilde{u}^\mu) = \left(\frac{1}{\sqrt{1-v^2}}, \frac{v}{\sqrt{1-v^2}}, 0, 0 \right) \equiv (\gamma, \eta, 0, 0). \quad (1.78)$$

Equivalently,

$$\tilde{u}^\alpha = \gamma u^\alpha + \eta x^\alpha, \quad (1.79)$$

where $\eta = \gamma v$. Notice that γ and v are defined in terms the dot-product of the 4-velocities u^α and $\tilde{u}^\alpha$:

$$\gamma := -u^\alpha \tilde{u}_\alpha, \quad v := \sqrt{1 - \gamma^{-2}}. \quad (1.80)$$

The vector $\tilde{x}^\alpha$ is a unit vector orthogonal to $\tilde{u}^\alpha$ that lies in the u^α - x^α plane, implying

$$\tilde{x}^\alpha = \eta u^\alpha + \gamma x^\alpha$$

We can phrase this in a more democratic way, writing

$$\tilde{u}^\alpha = \gamma u^\alpha + \eta x^\alpha, \quad u^\alpha = \gamma \tilde{u}^\alpha - \eta \tilde{x}^\alpha,$$

where $\tilde{x}^\alpha = -\eta u^\alpha + \gamma x^\alpha$. That is, the two sets of basis vectors, u^α, x^α and $\tilde{u}^\alpha, \tilde{x}^\alpha$, are related by a Lorentz transformation (a boost) and its inverse:

$$\tilde{u}^\alpha = \gamma u^\alpha + \eta x^\alpha, \quad u^\alpha = \gamma \tilde{u}^\alpha - \eta \tilde{x}^\alpha \quad (1.81a)$$

$$\tilde{x}^\alpha = \eta u^\alpha + \gamma x^\alpha, \quad x^\alpha = -\eta \tilde{u}^\alpha + \gamma \tilde{x}^\alpha. \quad (1.81b)$$

Addition of velocities.

Suppose now that three observers u^α , $\tilde{u}^\alpha$, and $\hat{u}^\alpha$ (Alice, Bob and Cat), move in straight lines in a plane in spacetime.

Problem: If $u^\alpha \tilde{u}_\alpha = -\tilde{\gamma}$ and $\tilde{u}^\alpha \hat{u}_\alpha = -\hat{\gamma}$, find $\gamma = -u^\alpha \hat{u}_\alpha$ in terms of $\tilde{\gamma}$ and $\hat{\gamma}$. This is the addition of velocity problem: If Bob moves at speed $\tilde{v}$ relative to Alice and Cat moves at speed $\hat{v}$ relative to Bob, what is the speed v of Cat relative to Alice?

The familiar derivations, in terms of components along Alice's frame are variants of

$$v = \frac{d\hat{x}}{d\hat{t}} = \frac{\hat{\gamma}(d\tilde{x} + \tilde{v}d\tilde{t})}{\hat{\gamma}(d\tilde{t} + \tilde{v}d\tilde{x})} = \frac{d\tilde{x}/d\tilde{t} + \tilde{v}}{1 + \tilde{v}d\tilde{x}/d\tilde{t}} = \frac{\tilde{v} + \hat{v}}{1 + \tilde{v}\hat{v}};$$

alternatively, using Eqs. (1.81a) and (1.81b) and their form (1.83) below

$$v = \frac{\hat{u}^x}{\hat{u}^t} = \frac{\hat{\gamma}(\tilde{u}^x + \tilde{v}\tilde{x}^x)}{\hat{\gamma}(\tilde{u}^t + \tilde{v}\tilde{x}^t)} = \frac{\tilde{u}^x/\tilde{x}^x + \tilde{v}}{1 + \tilde{v}\tilde{u}^t/\tilde{u}^x} = \frac{\tilde{v} + \hat{v}}{1 + \tilde{v}\hat{v}}. \quad (1.82)$$

But it's instructive to see how it goes in terms of the boosts.

Solution: We're given the speed $\hat{v}$ of Cat relative to Bob, implying

$$\hat{u}^\alpha = \hat{\gamma}\tilde{u}^\alpha + \hat{\eta}\tilde{x}^\alpha, \quad \hat{x}^\alpha = \hat{\eta}\tilde{u}^\alpha + \hat{\gamma}\tilde{x}^\alpha. \quad (1.83)$$

We already have Bob's 4-velocity in Alice's frame, Eq. (1.81a). We want the components of Cat's $\hat{u}^\alpha$ in Alice's frame: Substitute (1.83) in (1.81a):

$$\begin{aligned} \hat{u}^\alpha &= \hat{\gamma}(\gamma u^\alpha + \eta x^\alpha) + \hat{\eta}(\eta u^\alpha + \gamma x^\alpha) \\ &= (\hat{\gamma}\gamma + \hat{\eta}\eta)u^\alpha + (\hat{\gamma}\eta + \hat{\eta}\gamma)x^\alpha. \end{aligned}$$

Thus

$$\gamma = \tilde{\gamma}\tilde{\gamma} + \tilde{\eta}\tilde{\eta} \quad \text{and} \quad \eta = \tilde{\gamma}\tilde{\eta} + \tilde{\gamma}\tilde{\eta}. \quad (1.84)$$

We can rewrite this solution in terms of $v, \hat{v}, \tilde{v}$, using $\eta = v\gamma$, $\hat{\eta} = v\hat{\gamma}$, $\tilde{\eta} = \tilde{v}\tilde{\gamma}$ to recover the usual form:

$$v = \frac{\eta}{\gamma} = \frac{\hat{v} + \tilde{v}}{1 + \hat{v}\tilde{v}}. \quad (1.85)$$

But an equally useful form is obtained by noting that, by (1.84),

$$\begin{aligned} \gamma + \eta &= \tilde{\gamma}\tilde{\gamma} + \tilde{\eta}\tilde{\eta} + \tilde{\gamma}\tilde{\eta} + \tilde{\eta}\tilde{\gamma} \\ &= (\tilde{\gamma} + \tilde{\eta})(\tilde{\gamma} + \tilde{\eta}). \end{aligned}$$

In other words $\log(\gamma + \eta)$ is additive!

$$\log(\gamma + \eta) = \log(\tilde{\gamma} + \tilde{\eta}) + \log(\tilde{\gamma} + \tilde{\eta}). \quad (1.86)$$

Now

$$\begin{aligned} \gamma + \eta &= \frac{1 + v}{\sqrt{1 - v^2}} = \sqrt{\frac{1 + v}{1 - v}}, \quad \text{so} \\ \log\left(\frac{1 + v}{1 - v}\right) &= \log\left(\frac{1 + \hat{v}}{1 - \hat{v}}\right) + \log\left(\frac{1 + \tilde{v}}{1 - \tilde{v}}\right). \end{aligned} \quad (1.87)$$

The combination $\vartheta = \log\sqrt{\frac{1 + v}{1 - v}}$ is called the *velocity parameter*,⁴ and it's not hard to show that $v = \tanh \vartheta$.

If you remember that the Doppler shift in the frequency of light is given by

$$\tilde{\omega} = \omega \sqrt{\frac{1 + \tilde{v}}{1 - \tilde{v}}} = \omega e^{\vartheta}, \quad (1.88)$$

then the additivity of ϑ is not really new: Eq. (1.88) implies that successive boosts by $\tilde{v}$ and $\hat{v}$ change the frequency (and a photon's energy) by the product $e^{\vartheta} = e^{\tilde{\vartheta} + \hat{\vartheta}}$, implying $\vartheta = \tilde{\vartheta} + \hat{\vartheta}$. (If you haven't derived the Doppler shift, see *Exercise 28*.)

Exercise 15. (From the problems to Feynman v. I) A cart rolls on a table with speed v . A smaller cart rolls on the first in the same direction with speed v relative to the first cart, and so on up to n carts. What is the speed v_n of the n^{th} cart relative to the table? What is $\lim_{n \rightarrow \infty} v_n$?

Exercise. If you didn't do the last exercise using Eq. (1.87), do so now.

The next exercise is the calculation leading to (1.84), done as an explicit composition of boosts as matrices.

Exercise 16. The frames $\{e_\mu\} = \{\mathbf{u}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$, $\{\tilde{e}_\mu\} = (\tilde{\mathbf{u}}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}})$, and $\{\hat{e}_\mu\} = \{\hat{\mathbf{u}}, \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ are orthonormal and so are related by boosts

$$\tilde{e}_\mu = e_\nu \tilde{\Lambda}^\nu_\mu, \quad \hat{e}_\mu = \tilde{e}_\nu \hat{\Lambda}^\nu_\mu, \quad \text{and} \quad \hat{e}_\mu = e_\nu \Lambda^\nu_\mu, \quad \text{where} \quad \Lambda^\nu_\mu = \tilde{\Lambda}^\nu_\sigma \hat{\Lambda}^\sigma_\mu.$$

Find $\tilde{\Lambda}^\nu_\mu$ and $\hat{\Lambda}^\nu_\mu$, and compute Λ^ν_μ . In this way derive the addition-of-velocity law (1.84).

⁴There is no standard notation. Schutz calls the velocity parameter u , and these notes use Hartle's name for it.

Exercise 17. Consider the curve $c(\lambda)$ given in the natural coordinate system by

$$\begin{aligned} x(\lambda) &= \int^{\lambda} r \cos \theta \cos \phi d\lambda, & y(\lambda) &= \int^{\lambda} r \cos \theta \sin \phi d\lambda \\ z(\lambda) &= \int^{\lambda} r \sin \theta d\lambda, & t(\lambda) &= \int^{\lambda} r d\lambda, \end{aligned}$$

where r , θ , and ϕ are arbitrary functions of λ .

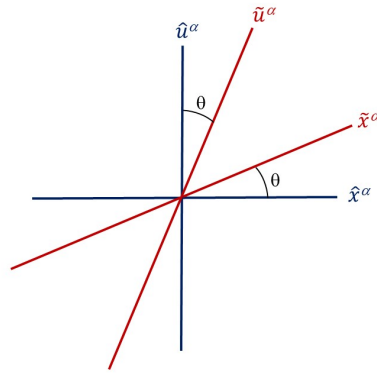
(a) Show that $c(\lambda)$ is a null curve.

(b) Under what conditions on the functions $r(\lambda)$, $\theta(\lambda)$ and $\phi(\lambda)$ will $c(\lambda)$ be a straight line?

1.4.3 Surfaces of simultaneity and the 3+1 decomposition of spacetime

An observer having velocity u^α can separate p^α into energy and three-momentum in a manner analogous to the decomposition of u^α in the last section into vectors along and orthogonal to the observer's velocity. The decomposition is frequently encountered and deserves some discussion. The surfaces of simultaneity of an observer with velocity $\hat{u}^\alpha$ are orthogonal to $\hat{u}^\alpha$, and so are spanned by three vectors $\hat{x}^\alpha, \hat{y}^\alpha, \hat{z}^\alpha$ perpendicular to $\hat{u}^\alpha$. In a natural coordinate system of an inertial observer (points are labeled by their components along the basis $\hat{u}^\alpha, \hat{x}^\alpha, \hat{y}^\alpha, \hat{z}^\alpha$) the surfaces of simultaneity are, of course, the $t = \text{constant}$ hyperplanes.

If $\tilde{u}^\alpha$ is moving at speed v relative to $\hat{u}^\alpha$, there is an easy way to draw the surfaces of simultaneity of $\tilde{u}^\alpha$ and $\tilde{x}^\alpha$.



With $\tan \theta = v$, the 2-D drawing looks like the figure above.

Here the angles labeled θ are angles of the Euclidean geometry of the screen (or paper), *not of the actual Lorentzian geometry of spacetime*.

Check that $\tan \theta = v$: In the Lorentzian geometry of spacetime, $\hat{u}^\alpha$ is perpendicular to $\hat{x}^\alpha$ and $\tilde{u}^\alpha$ is perpendicular to $\tilde{x}^\alpha$. By drawing the diagram with $\hat{u}^\alpha$ vertical and with light-rays at 45° , we have made $\hat{x}^\alpha$ perpendicular to $\hat{u}^\alpha$ in the Euclidean spatial geometry of the diagram. By (1.81a), $\tilde{u}^\alpha = \gamma \hat{u}^\alpha + \eta \hat{x}^\alpha$, so in that Euclidean geometry the angle between $\tilde{u}$ and $\hat{u}$ is given by $\tan \theta = \frac{\eta}{\gamma} = v$. Similarly, by (1.81b), $\tilde{x}^\alpha = \eta \hat{u}^\alpha + \gamma \hat{x}^\alpha$, so the angle between $\tilde{x}^\alpha$ and $\hat{x}^\alpha$ in the diagram is again given by $\tan \theta = \frac{\eta}{\gamma} = v$.

Had we drawn the diagram with $\tilde{u}^\alpha$ vertical, $\tilde{x}^\alpha$ would of course have been perpendicular to $\tilde{u}^\alpha$ in that diagram's spatial Euclidean geometry.

In the diagram's Euclidean metric $ds^2 = dt^2 + dx^2$, the dot product of $\tilde{x}$ and $\hat{x}$ is $\cos \theta$ with $\tan \theta = v$, while in the actual Lorentzian metric $ds^2 = -dt^2 + dx^2$, the dot product is $\tilde{x}^\alpha \hat{x}_\alpha = \gamma = \cosh \vartheta$, with $\tanh \vartheta = v$.

Exercise 18. Check that $\gamma = \cosh \vartheta$, with ϑ defined by $\tanh \vartheta = v$. Then check that the map $t \rightarrow it$ that takes the Lorentzian metric $-dt^2 + dx^2$ to the Euclidean metric $dt^2 + dx^2$ and takes $v = dx/dt$ to $v/i = dx/(i dt)$ also takes ϑ to $i\theta$; equivalently, if $T = it$, $X = x$, $V = v/i$, and $\theta = \vartheta/i$, then $V = \tan \theta$.

3+1 decomposition of tensors: Spacetime $\rightarrow$ space+time relative to an observer with 4-velocity u^α .

The tensor

$$\gamma^\alpha_\beta = \delta^\alpha_\beta + u^\alpha u_\beta \quad (1.89)$$

projects vectors into the surface orthogonal to u^α . In other words, if ξ^α is any vector,

$$\xi^\alpha_\perp := \gamma^\alpha_\beta \xi^\beta \quad (1.90)$$

is a vector orthogonal to u^α :

$$\gamma^\alpha_\beta u^\beta = \delta^\alpha_\beta u^\beta + u^\alpha u_\beta u^\beta = u^\alpha - u^\alpha = 0; \quad (1.91)$$

similarly, $u_\alpha \gamma^\alpha_\beta = 0$, whence

$$u_\alpha \xi^\alpha_\perp = u_\alpha \gamma^\alpha_\beta \xi^\beta = 0.$$

If ξ^α is already orthogonal to u^α , then

$$\xi^\alpha_\perp = \xi^\alpha.$$

Proof: $\xi^\alpha_\perp = \gamma^\alpha_\beta \xi^\beta = \delta^\alpha_\beta \xi^\beta + u^\alpha u_\beta \xi^\beta = \xi^\alpha$, because $u_\beta \xi^\beta = 0$.

Any vector can be written as the sum of a vector along u^α and one orthogonal to u^α :

$$\xi^\alpha = -(\xi^\beta u_\beta) u^\alpha + \xi^\alpha_\perp. \quad (1.92)$$

Similarly any tensor can be decomposed in such a way that each index is either along u^α or orthogonal to it: e.g.

$$\begin{aligned} T^{\alpha\beta} &= \delta^\alpha_\gamma \delta^\beta_\delta T^{\gamma\delta} = (\gamma^\alpha_\gamma - u^\alpha u_\gamma)(\gamma^\beta_\delta - u^\beta u_\delta) T^{\gamma\delta} \\ &= T^\alpha_\perp{}^\beta - u^\alpha \gamma^\beta_\delta u_\gamma T^{\gamma\delta} - u^\beta \gamma^\alpha_\delta u_\gamma T^{\gamma\delta} + u^\alpha u^\beta u_\gamma u_\delta T^{\gamma\delta}; \end{aligned} \quad (1.93)$$

here $T^\alpha_\perp{}^\beta = \gamma^\alpha_\gamma \gamma^\beta_\delta T^{\gamma\delta}$ is a tensor orthogonal to u^α (that is, $T^\alpha_\perp{}^\beta u_\beta = 0 = T^\alpha_\perp{}^\beta u_\alpha$); $u^\alpha \gamma^\beta_\delta u_\gamma T^{\gamma\delta}$ has one index (α) along u^α and one index (β) orthogonal to u^α ; and $u^\alpha u^\beta u_\gamma u_\delta T^{\gamma\delta}$ has both indices along u^α .

1.4.4 Energy, momentum and acceleration of a particle

A particle of rest mass m and velocity v^α is described by a momentum vector $p^\alpha = mv^\alpha$. Its rest mass is thus related to its momentum by $p^\alpha p_\alpha = -m^2$.

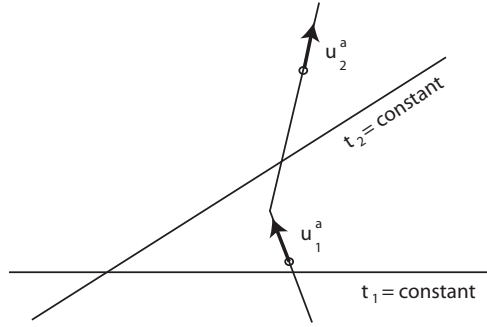
The momentum p^α of a particle can be decomposed by an observer u^α in the manner

$$p^\alpha = E u^\alpha + p^\alpha_\perp, \quad (1.94)$$

where $E = -p^\alpha u_\alpha$ and $p^\alpha_\perp = \gamma^\alpha_\beta p^\beta$ are the energy and 3-momentum of the particle. Then $p^\alpha p_\alpha = -E^2 + p_\perp^2 = -m^2$. If the observer is at rest with respect to the particle, $u^\alpha = v^\alpha$ so $p^\alpha = mv^\alpha$ is already parallel to u^α and $E = m$, $p^\alpha_\perp = 0$. When the relative speed of particle and observer is $v = (1 - \gamma^{-2})^{1/2}$ with $\gamma = -u^\alpha v_\alpha$,

$$E = m\gamma \quad p_\perp = m\eta. \quad (1.95)$$

For tensors *at* the position of the observer, such 3+1 decompositions of tensors relative to observers make sense even when the observer is accelerating. For tensor *fields*, however, they don't make much sense for non-inertial observers because "later" and "earlier" surfaces of simultaneity will intersect:



Acceleration: If $c(\tau)$ is a timelike path parameterized by proper time, we found that $u^\alpha = \frac{dc^\alpha(\tau)}{d\tau}$. The acceleration a^α is defined by

$$a^\alpha = \frac{du^\alpha(\tau)}{d\tau} = \frac{d^2}{d\tau^2} c^\alpha(\tau). \quad (1.96)$$

We can write this as $a^\alpha = u^\beta \nabla_\beta u^\alpha$ if we regard u^α as a vector field defined along the path $c(\tau)$:

$$\frac{d}{d\tau} u^\alpha(c(\tau)) = \nabla_\beta u^\alpha(c(\tau)) \frac{dc^\beta}{d\tau} = u^\beta \nabla_\beta u^\alpha \quad (1.97)$$

(Note that $\nabla_\beta u^\alpha$ alone is meaningless, however, because u^α is not defined off of $c(\tau)$ – only $u^\beta \nabla_\beta u^\alpha$, the derivative along $c(\tau)$, has meaning.) The acceleration a^α is a vector orthogonal to u^α :

$$u_\alpha a^\alpha = u_\alpha \frac{d}{d\tau} u^\alpha(\tau) = u^\alpha \frac{d}{d\tau} u_\alpha(\tau) \quad (\text{because } \eta_{\alpha\beta} \text{ is a constant tensor field}).$$

$$\Rightarrow u_\alpha \frac{d}{d\tau} u^\alpha = \frac{d}{d\tau} (u_\alpha u^\alpha) \frac{1}{2} = \frac{1}{2} \frac{d}{d\tau} (-1) = 0.$$

That is, a^α lies in the surface of simultaneity of an inertial observer at rest with respect to u^α . For such a *comoving* observer, the components of u^α are (instantaneously)

$$(u^\mu) = (1, 0, 0, 0),$$

from Eq. (1.78). A short time later,

$$(u^\mu) = (1, \mathbf{v}) + O(v^2)$$

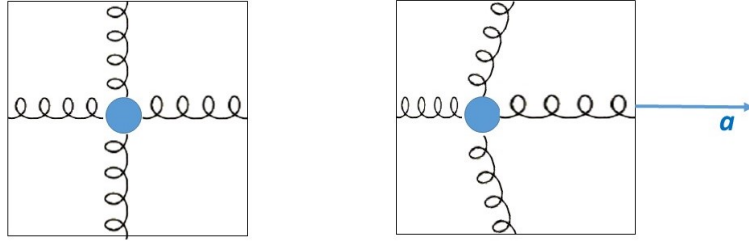
(use $\gamma = (1 - v^2)^{-1/2} = 1 + \frac{1}{2}v^2 + \dots$). Thus $a^\alpha = du^\alpha/dt$ has components

$$(a^\mu) = (0, \mathbf{a}_N) + O(v^2),$$

where $\mathbf{a}_N$ is the Newtonian acceleration and t the observer's proper time.

At this time, call it $t = 0$, when the particle is at rest relative to the observer, their trajectories are tangent and so $\frac{dt}{d\tau} = 1$, where τ is the particle's proper time as above. Thus $a^\mu = \frac{du^\mu}{d\tau} = (0, \mathbf{a}_N)$ at

$t = 0$, when $v = 0$; since $\frac{du^\alpha}{d\tau} = a^\alpha$, the vector a^α is the Newtonian acceleration measured by an inertial observer instantaneously at rest with respect to the particle. It can be measured either by using clocks and rulers or by transporting an “accelerometer” – a box with orthogonal springs supporting a mass – along the world line of the particle. Then a^α will be proportional to a vector orthogonal to u^α that joins the center of the box to the (in general) displaced position of the mass.



A particle is said to be uniformly accelerating if

$$a^\alpha a_\alpha = \text{constant}$$

and if a^α always lies in a plane (in spacetime). In other words, the acceleration is always in the x -direction of some inertial observer and its magnitude is constant as measured by an accelerometer or by an observer moving with the particle. Such a particle will appear, for example, to come in along the x -axis at nearly the speed of light, to slow down, stop, reverse direction and exit at a speed that increases to 1 (c) again. The trajectory is most simply obtained using not an orthonormal frame, but two null (lightlike) vectors along the ingoing and outgoing asymptotic null directions; these are constant vector fields, commonly written as ℓ^α and n^α . Because they are null,

$$\ell^\alpha \ell_\alpha = 0, \quad n^\alpha n_\alpha = 0. \quad (1.98)$$

We will normalize by

$$\ell^\alpha n_\alpha = -\frac{1}{2}. \quad (1.99)$$

For example, if $\hat{t}^\alpha$ and $\hat{x}^\alpha$ are unit orthogonal vectors in the plane of ℓ^α, n^α , one could take

$$\ell^\alpha = \frac{\hat{t}^\alpha + \hat{x}^\alpha}{2}, \quad n^\alpha = \frac{\hat{t}^\alpha - \hat{x}^\alpha}{2}. \quad (1.100)$$

Because u^α is in the plane of ℓ^α, n^α , we have $u^\alpha = \lambda \ell^\alpha + \mu n^\alpha$, $u^\alpha u_\alpha = -1$, implying $\lambda = \frac{1}{\mu}$ and

$$u^\alpha = \lambda \ell^\alpha + \lambda^{-1} n^\alpha. \quad (1.101)$$

Then

$$a^\alpha = \frac{du^\alpha}{d\tau} = \dot{\lambda} \ell^\alpha - \frac{\dot{\lambda}}{\lambda^2} n^\alpha, \quad \text{where } (\dot{}) := \frac{d}{d\tau}. \quad (1.102)$$

We have used the fact that ℓ^α and n^α are constant vector fields to infer $\frac{d}{d\tau} \ell^\alpha = 0 = \frac{d}{d\tau} n^\alpha$.

The relation $a^\alpha a_\alpha = \text{constant} \equiv a^2$ has the form

$$\begin{aligned} a^2 &= \left(\dot{\lambda} \ell_\alpha - \frac{\dot{\lambda}}{\lambda^2} n_\alpha \right) \left(\dot{\lambda} \ell^\alpha - \frac{\dot{\lambda}}{\lambda^2} n^\alpha \right) = -2 \frac{\dot{\lambda}^2}{\lambda^2} \ell_\alpha n^\alpha \quad (\text{use (1.98)}) \\ &= \frac{\dot{\lambda}^2}{\lambda^2} \quad (\text{use (1.99)}). \end{aligned}$$

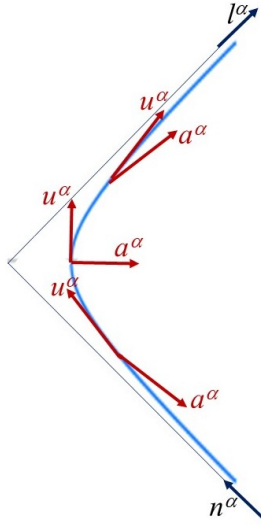


Figure 1.10: As shown below, a particle with constant acceleration travels along a hyperbola, and it remains at a fixed spacetime distance from an origin (the origin is the intersection of the null asymptotes).

Then

$$\frac{\dot{\lambda}}{\lambda} = \pm a, \quad \text{implying } \lambda = \lambda_+ e^{a\tau} + \lambda_- e^{-a\tau},$$

for some constants λ_+, λ_- . We want $u^\alpha \rightarrow k\ell^\alpha$ as $\tau \rightarrow \infty$, $u^\alpha \rightarrow k'n^\alpha$ as $\tau \rightarrow -\infty$, and so we pick $\lambda = \lambda_+ e^{a\tau}$, giving $u^\alpha = \lambda_+ e^{a\tau} \ell^\alpha + \lambda_+^{-1} e^{-a\tau} n^\alpha$. If, in addition, we fix the time τ at which $u^\alpha = \ell^\alpha + n^\alpha$ to be $\tau = 0$, then $\lambda_+ = 1$ and $u^\alpha(\tau) = e^{a\tau} \ell^\alpha + e^{-a\tau} n^\alpha$. After integrating and differentiating $u^\alpha(\tau)$, we have

$$c^\alpha(\tau) = \frac{1}{a}(e^{a\tau} \ell^\alpha - e^{-a\tau} n^\alpha), \quad u^\alpha(\tau) = e^{a\tau} \ell^\alpha + e^{-a\tau} n^\alpha, \quad a^\alpha(\tau) = a(e^{a\tau} \ell^\alpha - e^{-a\tau} n^\alpha), \quad (1.103)$$

as shown above. Notice that the acceleration vector a^α at each point $c(\tau)$ of the trajectory is proportional to the spacelike vector $c^\alpha(\tau)$ from the origin to that point.

In terms of the basis $\hat{t}^\alpha, \hat{x}^\alpha$ with

$$\ell^\alpha = \frac{\hat{t}^\alpha + \hat{x}^\alpha}{2}, \quad n^\alpha = \frac{\hat{t}^\alpha - \hat{x}^\alpha}{2},$$

$$c^\alpha = \frac{1}{a}[\sinh(a\tau) \hat{t}^\alpha + \cosh(a\tau) \hat{x}^\alpha], \quad u^\alpha = \cosh(a\tau) \hat{t}^\alpha + \sinh(a\tau) \hat{x}^\alpha, \quad a^\alpha = a^2 c^\alpha. \quad (1.104)$$

The path $c(\tau)$ is a hyperbola, whose x and t coordinates satisfy

$$x^2 - t^2 = \frac{1}{a^2}(\cosh^2 a\tau - \sinh^2 a\tau) = \frac{1}{a^2}. \quad (1.105)$$

That is, each point of the hyperbola in Fig. 1.10 is at the same spacetime distance $s = \sqrt{c^\alpha c_\alpha} = \frac{1}{a}$ from the origin! The hyperbola is the Lorentzian analogue of a circle. And the Euclidean analytic continuation to imaginary time $t = i\tau$ is a circle whose Euclidean distance from the origin is $t^2 + x^2 = 1/a^2$.

Exercise 19. Assuming that people can't endure a rocket acceleration much greater than g for years at a time, the distance a single generation of astronauts can travel is limited. If your (proper) acceleration is g for half the trip, and you then decelerate at g for the second half, how far can you go in 20 years (on your clock)? How long does it take to reach Andromeda ($d = 2.2 \times 10^6$ ly)?

Exercise 20. Prerequisite for the next section:

(a) Verify the identity

$$\epsilon_{ijm}\epsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

(b) Using $(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk}A_jB_k = \epsilon_{jki}A_jB_k$ and the identity of (a), show that

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{A} \cdot \mathbf{C} \mathbf{B} \cdot \mathbf{D} - \mathbf{A} \cdot \mathbf{D} \mathbf{B} \cdot \mathbf{C}.$$

(c) Similarly show that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \mathbf{A} \cdot \mathbf{C} - \mathbf{C} \mathbf{A} \cdot \mathbf{B}$.

1.5 Electromagnetism

To make the notation simpler, the unit vector $\hat{t}^\alpha$ will be written as t^α in this section.

1.5.1 $F^{\alpha\beta}, j^\alpha, E^\alpha, B^\alpha, \rho$

Conventions in this section agree with those of MTW and Wald. Jackson [50], however, uses a $+- - -$ signature, and his field tensor differs from ours by a sign: $F^{\alpha\beta} = -F_{\text{Jackson}}^{\alpha\beta}$.

An electromagnetic field in a flat spacetime is described by an antisymmetric tensor $F_{\alpha\beta}$. Maxwell's equations relate $F_{\alpha\beta}$ to the charged matter fields that produce it: They are

$$\nabla_{[\alpha} F_{\beta\gamma]} = 0 \quad (1.106)$$

and

$$\nabla_\beta F^{\alpha\beta} = 4\pi j^\alpha, \quad (1.107)$$

where j^α is the current density, a vector field that depends only on the charged matter. The motion of a charged particle is described by a timelike trajectory that satisfies the equation of motion (Lorentz force law)

$$u^\beta \nabla_\beta p^\alpha = q F^\alpha{}_\beta u^\beta, \quad (1.108)$$

where $p^\alpha = mu^\alpha$, and q is the particle's charge.

If an observer with velocity t^α measures the components of $F_{\alpha\beta}$ along vectors parallel or orthogonal to t^α (his natural basis vectors) he will be splitting $F_{\alpha\beta}$ in the way described by Eq. (1.93). Because $F_{\alpha\beta}$ is antisymmetric, its diagonal components vanish: In particular, $F_{\alpha\beta} t^\alpha t^\beta = F_{tt} = 0$. (That is, $F_{tt} = -F_{tt} \implies F_{tt} = 0$.)⁵

Furthermore, $\gamma_\alpha{}^\beta t^\gamma F_{\beta\gamma} = (\delta^\alpha{}_\beta + t^\alpha t^\beta) t^\gamma F_{\beta\gamma} = F_{\alpha\gamma} t^\gamma$ (again using $F_{\beta\gamma} t^\beta t^\gamma = 0$) and $\gamma_\alpha{}^\gamma t^\beta F_{\beta\gamma} = -F_{\alpha\gamma} t^\gamma$, so the only independent parts of the decomposed $F_{\alpha\beta}$ are

$$F_{\alpha\beta} t^\beta \quad \text{and} \quad \gamma_\alpha{}^\gamma \gamma_\beta{}^\delta F_{\gamma\delta} =: F_{\alpha\beta}^\perp. \quad (1.109)$$

The spatial vector

$$E_\alpha := F_{\alpha\beta} t^\beta \quad (1.110)$$

is called the electric field, and the antisymmetric spatial tensor $F_{\alpha\beta}^\perp$ corresponds to the magnetic field B^α , also a spatial vector:

$$\begin{aligned} B^\alpha &:= \frac{1}{2} \epsilon^{\delta\alpha\beta\gamma} t_\delta F_{\beta\gamma}^\perp \\ &= \frac{1}{2} \epsilon^{\delta\alpha\beta\gamma} t_\delta (\delta_\beta{}^\epsilon + t_\beta t^\epsilon) (\delta_\gamma{}^\zeta + t_\gamma t^\zeta) F_{\epsilon\zeta} \\ &= \frac{1}{2} \epsilon^{\delta\alpha\beta\gamma} t_\delta F_{\beta\gamma}, \end{aligned} \quad (1.111)$$

where we have again used the index antisymmetry, in this case of $\epsilon^{\alpha\beta\gamma\delta}$ to infer $\epsilon^{\delta\alpha\beta\gamma} t_\delta t_\beta = 0$.

⁵More generally, if $S^{\alpha\beta}$ is any symmetric tensor, and $A_{\alpha\beta}$ is antisymmetric, $A_{\alpha\beta} S^{\alpha\beta} = 0$. To see this, use $A_{\alpha\beta} = -A_{\beta\alpha}$ to write

$$A_{\alpha\beta} S^{\alpha\beta} = -A_{\beta\alpha} S^{\alpha\beta} = -A_{\beta\alpha} S^{\beta\alpha} = -A_{\alpha\beta} S^{\alpha\beta} \implies A_{\alpha\beta} S^{\alpha\beta} = 0.$$

The same manipulation shows that, in general, $A^{\alpha\beta\gamma\cdots\delta}$ antisymmetric in α and β and $S^{\alpha\beta\gamma\cdots\delta}$ symmetric in α and β together imply $A^{\alpha\beta\gamma\cdots\delta} S_{\alpha\beta\epsilon\cdots\zeta} = 0$.

Note that

$$\epsilon_{\alpha\beta\gamma} \equiv \epsilon_{\delta\alpha\beta\gamma} t^\delta \quad (1.112)$$

is an antisymmetric spatial tensor (each index orthogonal to t^α) and $\epsilon_{\alpha\beta\gamma}\epsilon^{\alpha\beta\gamma} = 3!$. In the orthonormal basis $t^\alpha, \hat{x}^\alpha, \hat{y}^\alpha, \hat{z}^\alpha$, $\epsilon_{123} = \epsilon_{0123} = 1$ so the spatial components of $\epsilon_{\alpha\beta\gamma}$ are

$$\epsilon_{\lambda\mu\nu} = \begin{cases} 1, & \lambda\mu\nu \text{ an even permutation of } 123 \\ -1, & \lambda\mu\nu \text{ an odd permutation of } 123, \end{cases}$$

and $\epsilon_{\lambda\mu\nu} = 0$ if any index is repeated or has the value 0.

The equation defining B^α in terms of $F^{\alpha\beta}$ is then

$$B^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}.$$

With respect to an orthonormal frame with timelike vector t^α , the field tensor has components

$$\|F_{\mu\nu}\| = \left\| \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \right\|. \quad (1.113)$$

This is MTW (3.7).

Some useful identities involving $\epsilon_{\alpha\beta\gamma}$ and $\epsilon_{\alpha\beta\gamma\delta}$ are

$$\begin{aligned} \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\epsilon\zeta\eta\theta} &= -4!0! \delta_\epsilon^{[\alpha} \delta_\zeta^\beta \delta_\eta^\gamma \delta_\theta^{\delta]} = -0! \sum_{\pi \in \mathbb{P}_4} (-1)^\pi \delta_\epsilon^{\pi(\alpha)} \dots \delta_\theta^{\pi(\delta)} \\ \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\epsilon\zeta\eta\delta} &= -3!0! \delta_\epsilon^{[\alpha} \delta_\zeta^\beta \delta_\eta^{\gamma]} = -1! \sum_{\pi \in \mathbb{P}_3} (-1)^\pi \delta_\epsilon^{\pi(\alpha)} \dots \delta_\eta^{\pi(\gamma)} \\ \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\epsilon\zeta\gamma\delta} &= -2!2! \delta_\epsilon^{[\alpha} \delta_\zeta^{\beta]} = -2! \sum_{\pi \in \mathbb{P}_2} (-1)^\pi \delta_\epsilon^{\pi(\alpha)} \delta_\zeta^{\pi(\beta)} \\ \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\epsilon\beta\gamma\delta} &= -1!3! \delta_\epsilon^\alpha = -3! \delta_\epsilon^\alpha \\ \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} &= -0!4! = -4! \end{aligned}$$

$$\begin{aligned} \epsilon^{\alpha\beta\gamma} \epsilon_{\delta\epsilon\zeta} &= 3!0! \gamma_\delta^{[\alpha} \gamma_\epsilon^\beta \gamma_\zeta^{\gamma]} = 0! \sum_{\pi} (-1)^\pi \gamma_\delta^{\pi(\alpha)} \gamma_\epsilon^{\pi(\beta)} \gamma_\zeta^{\pi(\gamma)} \\ \epsilon^{\alpha\beta\gamma} \epsilon_{\delta\epsilon\gamma} &= 2!1! \gamma_\delta^{[\alpha} \gamma_\epsilon^{\beta]} = 1! \sum_{\pi} (-1)^\pi \gamma_\delta^{\pi(\alpha)} \gamma_\epsilon^{\pi(\beta)} \\ \epsilon^{\alpha\beta\gamma} \epsilon_{\delta\beta\gamma} &= 1!2! \gamma_\delta^\alpha = 2! \gamma_\delta^\alpha \\ \epsilon^{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} &= 0!3! = 3! \end{aligned} \quad (1.114)$$

where $\pi \in \mathbb{P}_4$ is a permutation of the four letters $\alpha\beta\gamma\delta$, with $\pi \in \mathbb{P}_3$, $\pi \in \mathbb{P}_2$ defined analogously. (Here $\mathbb{P}_n$ is the group of permutation of n objects.)

The relations (1.110) and (1.111) for E^α and B^α in terms of $F^{\alpha\beta}$ can be inverted to give

$$F_{\alpha\beta} = t_\alpha E_\beta - t_\beta E_\alpha + \epsilon_{\alpha\beta\gamma} B^\gamma. \quad (1.115)$$

To show this, first note that

$$B^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} \quad (1.116)$$

can be inverted:

$$F_{\alpha\beta}^\perp = \epsilon_{\alpha\beta\gamma} B^\gamma \quad (1.117)$$

Proof.

$$\epsilon_{\alpha\beta\gamma}B^\gamma = \epsilon_{\alpha\beta\gamma}\left(\frac{1}{2}\epsilon^{\gamma\delta\epsilon}F_{\delta\epsilon}\right) = \frac{1}{2}(\gamma_\alpha^\delta\gamma_\beta^\epsilon - \gamma_\alpha^\epsilon\gamma_\beta^\delta)F_{\delta\epsilon} = \gamma_\alpha^\gamma\gamma_\beta^\delta F_{\gamma\delta},$$

where antisymmetry of $F_{\alpha\beta}$ was used to obtain the last equality.

Thus

$$\begin{aligned} t_\alpha E_\beta - t_\beta E_\alpha + \epsilon_{\alpha\beta\gamma}B^\gamma &= t_\alpha F_{\beta\gamma}t^\gamma - t_\beta F_{\alpha\gamma}t^\gamma + \gamma_\alpha^\gamma\gamma_\beta^\delta F_{\gamma\delta} \\ &= \underbrace{t_\alpha F_{\beta\gamma}t^\gamma}_{\text{cancel}} - \underbrace{t_\beta F_{\alpha\gamma}t^\gamma}_{\text{cancel}} + F_{\alpha\beta} + \underbrace{t_\alpha t^\gamma F_{\gamma\beta}}_{\text{cancel}} + \underbrace{t_\beta t^\delta F_{\alpha\delta}}_{\text{cancel}} + \underbrace{t_\alpha t^\gamma t_\beta t^\delta F_{\gamma\delta}}_0 \\ &= F_{\alpha\beta}. \end{aligned}$$

The fact that E_α and B_α carry the same information as $F_{\alpha\beta}$ could be expected at the outset by counting:

$$\begin{array}{ll} E^\alpha \text{ a spatial vector} & 3 \text{ independent components} \\ B^\alpha \text{ a spatial vector} & 3 \text{ independent components} \\ F^{\alpha\beta} \text{ an antisymmetric tensor} & 6 \text{ independent components} \end{array}$$

Finally, before writing Maxwell's equations and the Lorentz force law in terms of E^α and B^α , we decompose j^α :

$$j^\alpha = \rho_e t^\alpha + j_\perp^\alpha, \quad (1.118)$$

where ρ_e is the charge density and $j_\perp^\alpha$ the 3-current seen by the observer with velocity t^α .

For example, the current j^α associated with charges of density ρ_e in their own frame, whose 4-velocity is u^α , is $j^\alpha = \rho_e u^\alpha$. In terms of a frame with timelike unit vector $\tilde{t}^\alpha$, the 3+1 decomposition of u^α is $u^\alpha = \gamma \tilde{t}^\alpha + u_\perp^\alpha$, or

$$u^\alpha = \gamma(\tilde{t}^\alpha + \tilde{v}^\alpha), \text{ implying } \tilde{\rho}_e := -j_\alpha \tilde{t}^\alpha = \gamma \rho_e. \quad (1.119)$$

1.5.2 Maxwell and Lorentz

Let us now regard t^α as the constant timelike vector field obtained by parallel transporting the observer's velocity to all points of M , so that E^α and B^α have meaning everywhere.

The expression $F^\alpha_\beta u^\beta$ occurring in the Lorentz force law can now be written (using (1.115) and (1.119)) as

$$\begin{aligned} F^\alpha_\beta u^\beta &= [t^\alpha E_\beta - t_\beta E^\alpha + \epsilon^\alpha_{\beta\gamma} B^\gamma] \gamma (t^\beta + v^\beta) \\ &= \gamma (E^\alpha + \epsilon^\alpha_{\beta\gamma} v^\beta B^\gamma + v^\beta E_\beta t^\alpha) \end{aligned}$$

Project (1.108) along t^α , denoting the particle's energy, $-p_\alpha t^\alpha$, by E to distinguish it from the magnitude E of the electric field:

$$\begin{aligned} t_\alpha u^\beta \nabla_\beta p^\alpha &= t_\alpha q F^\alpha_\beta u^\beta \\ u^\beta \nabla_\beta (-E) &= -q \gamma E_\beta v^\beta, \quad 0 \text{ r} \\ \frac{dE}{dt} &= q E_\beta v^\beta, \end{aligned} \quad (1.120)$$

using $u \cdot \nabla E = \frac{dE}{d\tau} = \gamma \frac{dE}{dt}$. Eq. (1.120) says that the particle's energy E changes by the work $\mathbf{v} \cdot \mathbf{E}$ done per unit time by the field.

Project (1.108) orthogonal to t^α :

$$\gamma^\alpha{}_\gamma u^\beta \nabla_\beta p^\gamma = q \gamma^\alpha{}_\gamma F^\gamma{}_\beta u^\beta = q \gamma^\alpha{}_\gamma \gamma (E^\gamma + \epsilon^\gamma{}_{\beta\delta} v^\beta B^\delta)$$

Using $\gamma^\alpha{}_\gamma$ constant, $\nabla_\beta \gamma^\alpha{}_\gamma = 0$, we have

$$u^\beta \nabla_\beta p^\alpha_\perp = q \gamma (E^\alpha + \epsilon^\alpha{}_{\beta\gamma} v^\beta B^\gamma);$$

using $d/d\tau = \gamma d/dt$ then gives

$$\frac{d}{dt} p^\alpha_\perp = q (E^\alpha + \epsilon^\alpha{}_{\beta\gamma} v^\beta B^\gamma); \quad (1.121)$$

this is the usual Lorentz force law,

$$\frac{d}{dt} \mathbf{p}_\perp = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (1.122)$$

Maxwell's equations in terms of E^α and B^α emerge in an analogous way. The projection of $\nabla_\beta F^{\alpha\beta} = 4\pi j^\alpha$ along t^α is

$$\begin{aligned} t_\alpha \nabla_\beta F^{\alpha\beta} &= 4\pi t_\alpha j^\alpha, \quad \text{or, by (1.118)} \\ \nabla_\beta E^\beta &= 4\pi \rho_e \\ \text{i.e., } \nabla \cdot \mathbf{E} &= 4\pi \rho_e. \end{aligned} \quad (1.123)$$

The projection orthogonal to t^α is, using (1.115),

$$\nabla_\beta [\gamma^\alpha{}_\gamma (t^\gamma E^\beta - t^\beta E^\gamma + \epsilon^{\gamma\beta\delta} B_\delta)] = 4\pi j_\perp{}^\alpha \quad (1.124)$$

$$-t^\beta \nabla_\beta E^\alpha + \epsilon^{\alpha\beta\gamma} \nabla_\beta B_\gamma = 4\pi j_\perp{}^\alpha; \quad (1.125)$$

$$\text{i.e., } -\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = 4\pi \mathbf{j}.$$

Finally, the equation $\nabla_{[\alpha} F_{\beta\gamma]} = 0$ is equivalent to

$$\epsilon^{\alpha\beta\gamma\delta} \nabla_\beta F_{\gamma\delta} = 0. \quad (1.126)$$

Projecting (1.126) along t^α gives $\nabla \cdot \mathbf{B} = 0$:

$$\begin{aligned} 0 &= \nabla_\beta [t_\alpha \epsilon^{\alpha\beta\gamma\delta} (t_\gamma E_\delta - t_\delta E_\gamma + \epsilon_{\gamma\delta\epsilon} B^\epsilon)] \\ &= \nabla_\beta (\gamma^\beta{}_\epsilon B^\epsilon) \quad (\epsilon^{\beta\gamma\delta} \epsilon_{\epsilon\gamma\delta} = \gamma^\beta{}_\epsilon) \\ &= \nabla_\beta B^\beta. \end{aligned} \quad (1.127)$$

Projecting orthogonal to t^α gives $\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$:

$$\begin{aligned} 0 &= \epsilon^{\zeta\beta\gamma\delta} \nabla_\beta [t_\gamma E_\delta - t_\delta E_\gamma + \epsilon_{\gamma\delta\epsilon} B^\epsilon] \gamma^\alpha{}_\zeta \\ &= \epsilon^{\alpha\beta\delta} \nabla_\beta E_\delta + \epsilon^{\alpha\beta\gamma} \nabla_\beta E_\gamma - 2(\delta_\eta^\zeta \delta_\epsilon^\beta - \delta_\epsilon^\zeta \delta_\eta^\beta) \nabla_\beta B^\epsilon t^\eta \gamma^\alpha{}_\zeta \\ &= 2\epsilon^{\alpha\beta\gamma} \nabla_\beta E_\gamma + 2t^\beta \nabla_\beta B^\alpha \end{aligned}$$

or

$$t^\beta \nabla_\beta B^\alpha + \epsilon^{\alpha\beta\gamma} \nabla_\beta E_\gamma = 0. \quad (1.128)$$

Collecting the equations, we have

$$\begin{aligned} \nabla_\beta E^\beta &= 4\pi \rho_e & \nabla_\beta B^\beta &= 0 \\ -t^\beta \nabla_\beta E^\alpha + \epsilon^{\alpha\beta\gamma} \nabla_\beta B_\gamma &= 4\pi j_\perp{}^\alpha & t^\beta \nabla_\beta B^\alpha + \epsilon^{\alpha\beta\gamma} \nabla_\beta E_\gamma &= 0. \end{aligned} \quad (1.129)$$

Exercise 21. Let $*F^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$.

(a) Show that if $F_{\alpha\beta}$ satisfies the sourceless Maxwell equations, so does

$$\widehat{F}_{\alpha\beta} = F_{\alpha\beta} \cos \eta + *F_{\alpha\beta} \sin \eta. \quad (F \rightarrow \widehat{F} \text{ is called a duality rotation}).$$

(b) If $\eta = \frac{\pi}{2}$, show that (for any t^α) $\widehat{B}^\alpha = +E^\alpha$, $\widehat{E}^\alpha = -B^\alpha$.

(c) Show that, for any η ,

$$\widehat{B}^\alpha = B^\alpha \cos \eta + E^\alpha \sin \eta \quad \widehat{E}^\alpha = E^\alpha \cos \eta - B^\alpha \sin \eta$$

The energy momentum tensor of an electromagnetic field is

$$T^{\alpha\beta} = \frac{1}{4\pi} \left[F^{\alpha\gamma} F^\beta{}_\gamma - \frac{1}{4} \eta^{\alpha\beta} F^{\gamma\delta} F_{\gamma\delta} \right] \quad (1.130)$$

Exercise 22. Write $T^{\alpha\beta}$ in terms of E^α , B^α and t^α .

Exercise 23. Prove that if $F^{\alpha\beta}$ is a Maxwell field with source j^α ,

$$\nabla_\beta T^{\alpha\beta} = -F^{\alpha\beta} j_\beta.$$

Exercise. Show that $T^{\alpha\beta}$ is invariant under duality rotations: $\widehat{T}^{\alpha\beta} = T^{\alpha\beta}$. Hint:

(a) Write $*F^{\alpha\beta}$ in terms of E^α , B^α , and t^α .

(b) Show that $*F^{\alpha\gamma} F^\beta{}_\gamma = \eta^{\alpha\beta} E_\gamma B^\gamma$.

(c) Infer that $*F^{\alpha\gamma} F^\beta{}_\gamma + *F^{\beta\gamma} F^\alpha{}_\gamma - \frac{1}{2} \eta^{\alpha\beta} *F^{\gamma\delta} F_{\gamma\delta} = 0$.

1.5.3 Conservation of Charge

Because the tensor $F^{\alpha\beta}$ is antisymmetric,

$$\nabla_\alpha \nabla_\beta F^{\alpha\beta} = \frac{1}{2} [\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha] F^{\alpha\beta} = 0$$

That is, in natural coordinates on M ,

$$\nabla_\mu \nabla_\nu F^{\mu\nu} = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} F^{\mu\nu}, \quad \text{and} \quad (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) F^{\mu\nu} = 0,$$

for smooth $F^{\mu\nu}$. But

$$\begin{aligned} \nabla_\alpha \nabla_\beta F^{\alpha\beta} &= \nabla_\alpha (4\pi j^\alpha), \quad \text{whence} \\ \nabla_\alpha j^\alpha &= 0. \end{aligned} \quad (1.131)$$

The equation $\nabla_\alpha j^\alpha = 0$ is called conservation of charge. In terms of the charge ρ_e and current $j_\perp^\alpha$ of an observer with velocity t^α , it can be written

$$\nabla_\alpha (\rho_e u^\alpha + j_\perp^\alpha) = u^\alpha \nabla_\alpha \rho_e + \nabla_\alpha j_\perp^\alpha = 0, \quad (1.132)$$

or, more familiarly,

$$\partial_t \rho_e + \nabla \cdot j_\perp = 0. \quad (1.133)$$

The corresponding integral conservation law is obtained by integrating (1.131) ($\equiv$ (1.133)) over a 4-volume sandwiched between two $t = \text{constant}$ surfaces:

$$\begin{aligned} 0 &= \int_{t_1}^{t_2} dt \int dxdydz (\partial_t \rho_e + \nabla \cdot \mathbf{j}_\perp) \\ &= \int_{t=t_2} dxdydz \rho_e - \int_{t=t_1} dxdydz \rho_e + \int_{t_1}^{t_2} dt \int_{\text{spatial infinity}} \mathbf{j}_\perp \cdot d\mathbf{A}. \end{aligned}$$

Thus if $\mathbf{j}_\perp = 0$ at spatial infinity (e.g., if the matter vanishes outside some finite region) the total charge Q is conserved from one hypersurface $t = t_1$ to another ($t = t_2$):

$$Q = \int_{t=t_2} \rho_e dV = \int_{t=t_1} \rho_e dV.$$

1.6 Continuous Matter

1.6.1 The stress tensor and the stress-energy tensor

See Feynman, *Lectures on Physics v. II*, Chap. 31, 40.

https://www.feynmanlectures.caltech.edu/II_toc.html;

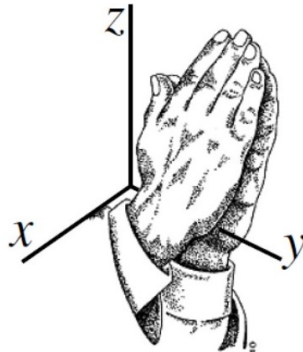
Schutz Chap. 4;

Blandford and Thorne, *Modern Classical Physics*[92], Chap. 1

<http://www.pmaweb.caltech.edu/Courses/ph136/yr2012/>.

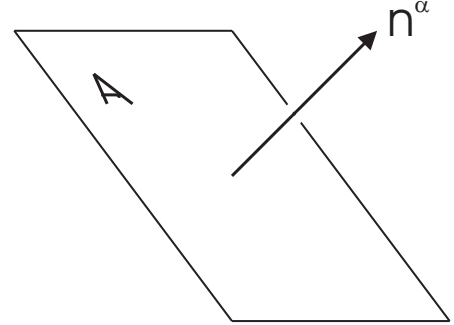
We start with the stress tensor. Here's the introduction to it from Blandford-Thorne:

Press your hands together in the y - z plane and feel the force that one hand exerts on the other across a tiny area A – say, one square millimeter of your hands' palms [figure below]. That force, of course, is a vector $\mathbf{F}$. It has a normal component (along the x direction). It also has a tangential component: if you try to slide your hands past each other, you feel a component of force along their surface, a “shear” force in the y and z directions. Not only is the force $\mathbf{F}$ vectorial; so is the 2-surface across which it acts, $\Sigma = Ae_x$. (Here e_x is the unit vector orthogonal to the tiny area A , and we have chosen the negative side of the surface to be the $-x$ side and the positive side to be $+x$. With this choice, the force $\mathbf{F}$ is that which the negative hand, on the $-x$ side, exerts on the positive hand.)



Suppose that inside a continuous material (solid or fluid) one makes a small cut perpendicular to a unit vector $\mathbf{n}$. The cut separates the matter on one side of a small plane from the matter on the other side.

Let $\mathbf{F}$ be the force needed to keep the matter on, say, the $+\mathbf{n}$ side in the state it would have been in had there been no cut. The force $\mathbf{F}$ is just the same force that had been exerted before the cut was made by the matter on the opposite side. And the force on the $-\mathbf{n}$ side is, of course, equal and opposite to the force on the $+\mathbf{n}$ side of the plane. Let $\mathbf{A} = \mathbf{n}A$, where A is the area of the cut.



Because the cut is small, $\mathbf{F}$ is linear in $\mathbf{A}$. That is, any smooth function $\mathbf{f}(\mathbf{x})$ has, about $\mathbf{x} = \mathbf{0}$, the Taylor expansion

$$f^i(\mathbf{x}) = f^i(\mathbf{0}) + \partial_j f^i(\mathbf{0})x^j + O(x^2).$$

Here the net force $\mathbf{F}$ on $\mathbf{A}$ vanishes for $\mathbf{A} = \mathbf{0}$, so for small A , $F^i(\mathbf{A}) = S^i_j A^j$, where $S^i_j := \frac{\partial F^i}{\partial A^j}(\mathbf{0})$. But a linear map from vectors to vectors is a tensor: What we have shown is that there is a tensor S^a_b , the *stress tensor*, for which

$$F^a = S^a_b A^b. \quad (1.134)$$

Equivalently,

$$S^a_b n^b \text{ is the force across a unit area orthogonal to } n^a.$$

Since $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, one also says that

$$S^a_b n^b \text{ is the rate of flow of momentum across a unit area orthogonal to } n^a.$$

Sign of the stress tensor: Let's go back to the Blandford-Thorne description with two hands pressing each other. The hand on the $-x$ side exerts a force in the $+x$ direction, so $S_{xx} > 0$. If your hands are stuck together with glue and you are trying to pull them apart, tension instead of pressure, then $S_{xx} < 0$. For shear, if the hand on the $-x$ side is pulling up on the other hand, then $S_{zx} > 0$: The force on the hand on the $+x$ side is in the $+z$ direction.

In a gas, if you make a cut, you need to exert a force PA perpendicular to the cut, where P is the gas pressure, to hold the gas in place. That is, the pressure exerts a force n^a , with the same magnitude P for every orientation of the cut:

$$S^a_b n^b = P n^a, \quad \text{all } n^a.$$

Then S^a_b/P is the identity map δ^a_b , and we have

$$S^a_b = P \delta^a_b.$$

Note that in a gas, the molecules flowing in the positive $\mathbf{n}$ direction ($v^a n_a > 0$) across a unit area orthogonal to n^a carry total momentum $P n^a$ across a unit area in unit time; so in this case the momentum flux $S^a_b n^b$ is tangible.

In physics, the word *fluid* refers both to liquids and gases. More generally, what distinguishes a fluid from a solid is that a fluid cannot maintain a shear stress: A fluid is continuous matter with stress tensor

$$S^a_b = P \delta^a_b. \quad (1.135)$$

(An *imperfect fluid*, a fluid with viscosity, is intermediate between a perfect fluid and a solid. It has nonzero shear stress proportional to the gradient of its velocity.)

For continuous matter in general the stress tensor is symmetric:

Claim: S^{ab} is a symmetric tensor: $S^{ab} = S^{ba}$.

Proof: We look at the torque on a small piece of the matter (called a *fluid element* in the case of a fluid). The torque τ_z about the z -axis involves the x - and y -projections of the forces on four faces, shown below for a small unit cube.

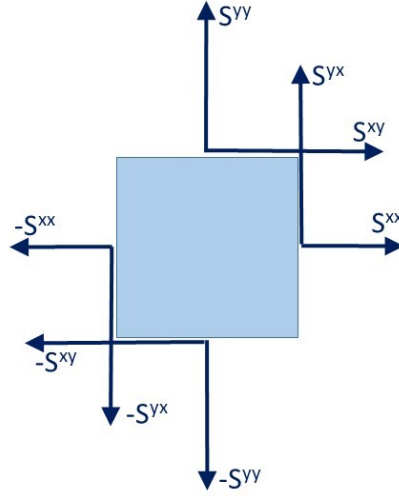


Figure 1.11: Figure adapted from Feynman Lectures, v. II, Fig. 31-9.

For a cube with sides Δx , Δy , Δz , we have

$$\begin{aligned}\tau_z &= 2(S^{xy}\Delta x\Delta z)\frac{\Delta y}{2} - 2(S^{yx}\Delta y\Delta z)\frac{\Delta x}{2} \\ &= (S^{xy} - S^{yx})\Delta V \quad \text{where } \Delta V = \Delta x\Delta y\Delta z.\end{aligned}$$

With ρ , M , I , and α the density, mass, moment of inertia, and angular acceleration of the cube,

$$\begin{aligned}\tau &= I\alpha = \frac{1}{12}M(\Delta x^2 + \Delta y^2)\alpha = \frac{1}{12}\rho\Delta V(\Delta x^2 + \Delta y^2)\alpha \\ \Rightarrow \alpha &= \frac{12}{\rho}(S^{xy} - S^{yx})\frac{1}{\Delta x^2 + \Delta y^2},\end{aligned}$$

and tiny cubes rotate with arbitrarily large angular acceleration α unless

$$S^{xy} = S^{yx}.$$

The stress tensor is generalized to a four dimensional tensor $T^{\alpha\beta}$ in the following way. An observer $\hat{t}^\alpha$ sees as her stress tensor the spatial part of $T^{\alpha\beta}$

$$T_{\perp}^{\alpha\beta} := \gamma^\alpha_\gamma \gamma^\beta_\delta T^{\gamma\delta} = S^{\alpha\beta},$$

where $\gamma_{\alpha\beta}$ is, as usual, the projection onto the space orthogonal to $\hat{t}^\alpha$.

Since the time component of momentum is energy, $T^{tx} \equiv -T^{\alpha\beta} \hat{t}_\alpha \hat{x}_\beta = T^{\alpha\beta} \nabla_\alpha \hat{t}_\beta$ is the rate of flow of energy across a unit spatial area orthogonal to $\hat{x}_\alpha$. And just as the 3-vector of flow (electric current $\mathbf{j}$, say) has as its time component the density (charge density ρ_e), here

$$\begin{aligned}T^{xt} &= \text{density of } x\text{-momentum and} \\ T^{tt} &= \text{energy density}\end{aligned}$$

In a frame moving with the material at a point P , $T^{it} = T^{ti} = 0$, $i = 1-3$, and $T^{\mu\nu}$ has the symmetric form

$$\|T^{\mu\nu}\| = \begin{vmatrix} T^{tt} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & S^{ij} & \\ 0 & & & \end{vmatrix}.$$

Consequently $T^{\alpha\beta}$ is symmetric in any frame.

Exercise 24. A rod has cross sectional area A and mass per unit length μ .

- What is its stress-energy tensor when it is under a tension F ? (assume F is uniformly distributed over the cross section)
- What is its stress-energy tensor when it is compressed?

Exercise 25. Find the stress-energy tensor at a point of a stretched drumhead, assuming mass per unit area σ and uniform stress. The thickness of the drumhead is h and the force per unit length along the edge of the drum is $\mathcal{F}$. Explain your choice of sign for components of the stress tensor.



Exercise 26. A rope of mass per unit length μ has a static breaking strength F . What is the maximum F can be without violating the “weak energy condition” $T_{\alpha\beta}\hat{t}^\alpha\hat{t}^\beta > 0$, all timelike $\hat{t}^\alpha$? How close is a steel cable to this maximum strength?

Two of these exercises are problems 5.5 and 5.6 of Lightman, Press, Price and Teukolsky[55].

1.6.2 Perfect fluids in a Newtonian context

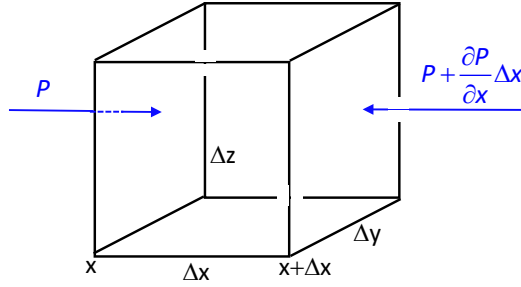
Continuous matter is a model for a large assembly of particles in which a continuous energy density ρ can reasonably describe the macroscopic distribution of mass. The statement that the density varies smoothly is the thermodynamic assumption that one can divide the dust into boxes large enough that many particles are in each box (so that the density $\rho = \frac{mN}{V}$ can be regarded as a continuous function of the number N of particles of mass m in a box of volume V) and small enough that the change in ρ from one box to an adjacent box is small: ρ can be regarded as a continuous function of position.

In fluids, one assumes that the microscopic particles are interacting and that they collide frequently enough that the mean free path is small compared with the scale on which the density ρ changes. One further assumes that a mean 3-velocity $\mathbf{v}$ can be defined in boxes small compared to the macroscopic length scale but large compared to the mean free path and that it will be continuous over the matter. An observer moving with this velocity $\mathbf{v}$ of the fluid, will see the collisions randomly distribute the nearby particle velocities so that the particles will look locally isotropic and will exhibit a macroscopic pressure P .

We'll begin in the Newtonian approximation to make things intuitively clear. Formally, however, the relativistic equations are simpler.

Euler equation: $\mathbf{F} = m\mathbf{a}$

The Euler equation is the Newtonian equation of motion, $\mathbf{F} = m\mathbf{a}$ for a fluid element, a small piece of fluid. Consider a fluid element with density ρ and velocity $\mathbf{v}$.



The pressure on the left face is $P(x)$; the pressure on the right face is $P(x + \Delta x) = P(x) + \frac{\partial P}{\partial x} \Delta x$. With $A = \Delta y \Delta z$ the area of the left and right faces of the box, the net force in the x -direction is

$$\begin{aligned} F_x &= P(x)A - P(x + \Delta x)A \\ &= -\frac{\partial P}{\partial x} V. \end{aligned}$$

where $V = \Delta x \Delta y \Delta z$ is the volume of the fluid element. Replacing the index x by y and z , we have

$$\mathbf{F} = -\nabla P V.$$

We want to write $\mathbf{F} = m\mathbf{a}$ or

$$-\nabla P V = \rho V \mathbf{a},$$

and we need to find $\mathbf{a}$ in terms of the velocity field $\mathbf{v}(t, \mathbf{x})$. The vector field $\mathbf{v}(t, \mathbf{x})$ has the meaning that at time t the fluid element at $\mathbf{x}$ has velocity $\mathbf{v}(t, \mathbf{x})$. Thus at time $t + \Delta t$ that same fluid element is at $\mathbf{x} + \mathbf{v}(t, \mathbf{x})\Delta t$ and has velocity $\mathbf{v}(t + \Delta t, \mathbf{x} + \mathbf{v}(t, \mathbf{x})\Delta t)$. The fluid element has changed its velocity by

$$\begin{aligned} \Delta \mathbf{v} &= \mathbf{v}(t + \Delta t, \mathbf{x} + \mathbf{v}(t, \mathbf{x})\Delta t) - \mathbf{v}(t, \mathbf{x}) \\ &= \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) \Delta t \end{aligned}$$

in time Δt , and its acceleration is therefore

$$\mathbf{a} = (\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v}. \quad (1.136)$$

In this way we obtain Euler's equation of motion ("Principes généraux du mouvement des fluides," Mémoires de l'Académie des Sciences de Berlin, 1757)

$$\rho (\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P. \quad (1.137)$$

In the presence of a gravitational field, with potential Φ satisfying $\nabla^2 \Phi = 4\pi G\rho$, there is an additional force $-\rho \nabla \Phi$ on each fluid element; and the Euler equation becomes

$$\rho (\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P - \rho \nabla \Phi. \quad (1.138)$$

Conservation of mass: The continuity equation

As a fluid element moves its volume changes. Because its mass is conserved (in a relativistic context use baryon mass) a fractional increase $\Delta V/V$ in its volume is equal to the fractional decrease $\Delta \rho/\rho$ in its density.

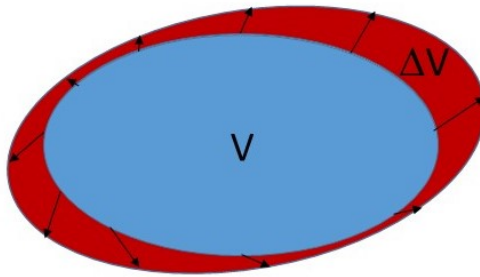
$$\rho V = \text{constant} \implies \frac{\Delta \rho}{\rho} = -\frac{\Delta V}{V} \quad \text{or} \quad \frac{d\rho/dt}{\rho} = -\frac{dV/dt}{V}, \quad (1.139)$$

where $\rho = \rho(t, \mathbf{x}(t))$, $V = V(t, \mathbf{x}(t))$.

We'll begin with the change in the volume of the fluid element as it moves. It is helpful first to recall or notice the geometrical meaning of the divergence of a vector field. If each point in a volume V moves by a small amount $\xi(x)$, from an initial position $\mathbf{x}$ to a final position $\bar{\mathbf{x}} = \mathbf{x} + \xi$, the volume of the box changes by $\Delta V = \bar{V} - V = \nabla \cdot \xi V$: That is, $\nabla \cdot \xi$ is the fractional change in volume. This is really Gauss's theorem: As illustrated by the figure below, moving V to $\bar{V}$ moves each point of the surface S of a volume V along ξ , changing the the volume of the box by

$$\Delta V = \int_S \xi \cdot d\mathbf{S},$$

to lowest order in ξ .



Gauss's theorem now implies

$$\Delta V = \int_V \nabla \cdot \xi dV. \quad (1.140)$$

For a small volume V the volume then changes by $\Delta V = V \nabla \cdot \xi$, or

$$\frac{\Delta V}{V} = \nabla \cdot \xi, \quad (1.141)$$

to lowest order in V and ξ . That is, as claimed, $\nabla \cdot \xi$ is the fractional change in volume.

Go back now to a fluid with a velocity field $\mathbf{v}(x, t)$. In a time Δt the fluid at $\mathbf{x}$ moves to $\mathbf{x} + \xi$, where $\xi = \mathbf{v}\Delta t$. Then, writing $\nabla \cdot \xi = \nabla \cdot (\mathbf{v} \Delta t)$, we have

$$\frac{dV/dt}{V} = \nabla \cdot \mathbf{v}. \quad (1.142)$$

The change in density is given by

$$\frac{d}{dt}\rho(t, \mathbf{x}(t)) = \partial_t \rho + \partial_i \rho \frac{dx_i}{dt} = (\partial_t + \mathbf{v} \cdot \nabla)\rho. \quad (1.143)$$

Finally, with these expressions for dV/dt and $d\rho/dt$, conservation of mass (1.139) is

$$\frac{1}{\rho}(\partial_t + \mathbf{v} \cdot \nabla)\rho = -\nabla \cdot \mathbf{v},$$

or

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (1.144)$$

This is commonly called the *continuity equation*.

To summarize: A fluid is characterized by its pressure P , density ρ and 3-velocity $\mathbf{v}$. In the Newtonian approximation, its motion is governed by the equations

$$\nabla^2 \Phi = 4\pi G \rho, \quad \lim_{r \rightarrow \infty} \Phi = 0, \quad (1.145a)$$

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1.145b)$$

$$\rho(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P - \rho \nabla \Phi. \quad (1.145c)$$

1.6.3 Relativistic equation of motion and conservation laws.

Mass conservation is the Newtonian limit of energy conservation, and the Newtonian equation of motion, $\mathbf{F} = m\mathbf{a}$, is conservation of momentum, when the field responsible for the force is included in the energy-momentum tensor $T^{\alpha\beta}$. In terms of $T^{\alpha\beta}$, energy conservation and the equation of motion (momentum conservation) are projections of the single equation,

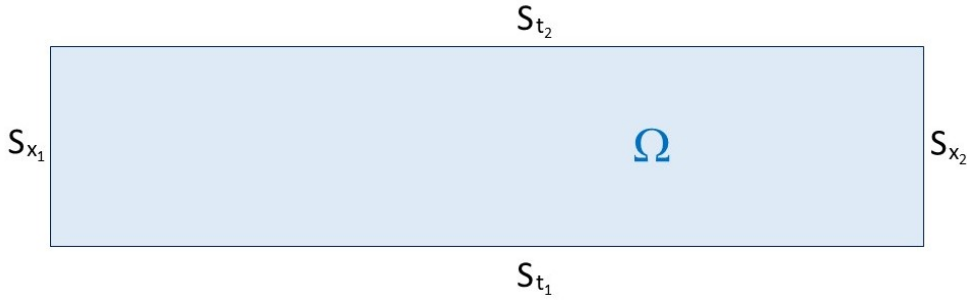
$$\nabla_\beta T^{\alpha\beta} = 0, \quad (1.146)$$

along and orthogonal to the 4-velocity $\hat{t}^\alpha$ of an observer.

We begin by showing that Eq. (1.146), the statement that $T^{\alpha\beta}$ is divergencefree, has the meaning of energy-momentum conservation.

Consider a four-dimensional box Ω whose boundary $\partial\Omega$ consists of the top and bottom volumes S_{t_1} and S_{t_2} together with six timelike sides, $S_{x_1}, \dots, S_{z_2}$. (A timelike hypersurface is one that has timelike vectors at each point. A spacelike hypersurface contains only spacelike vectors.) The total x -momentum in the volume S_{t_1} (i.e. in the box at $t = t_1$) is

$$p_1^x = \int_{S_{t_1}} (\text{density of } x\text{-momentum}) dx dy dz = \int_{S_{t_1}} T^{\alpha\beta} \hat{x}_\alpha \nabla_\beta t dx dy dz.$$



(Using $\nabla_\beta t$ instead of $\hat{t}_\beta$ avoids a minus sign.) Similarly, the total x -momentum in S_{t_2} is

$$p_2^x = \int_{S_{t_2}} T^{\alpha\beta} \hat{x}_\alpha \nabla_\beta t dx dy dz.$$

The total x -momentum that left through the timelike sides is

$$\begin{aligned} \Delta p^x &= \int (\text{rate of } x\text{-momentum flow per unit area}) dA dt \\ &= \int_{S_{x_1}} T^{\alpha\beta} \hat{x}_\alpha (-\nabla_\beta t) dy dz dt + \int_{S_{x_2}} T^{\alpha\beta} \hat{x}_\alpha \nabla_\beta x dy dz dt + \cdots + \int_{S_{z_2}} T^{\alpha\beta} \hat{x}_\alpha \nabla_\beta z dx dy dt \end{aligned}$$

If we now denote by n_α the unit outward normal at each point of the boundary $\partial\Omega$, and by dS each 3-dimensional volume element of the surfaces ($dx dy dz$, $dt dx dy$, etc.), conservation of x -momentum,

$$p_1^x - p_2^x + \Delta p^x = 0,$$

takes the form

$$\int_{\partial\Omega} T^{\alpha\beta} \hat{x}_\alpha n_\beta dS = 0. \quad (1.147)$$

Integrating over each coordinate x^μ establishes Gauss' theorem for our 4-dimensional box: For any vector field j^α

$$\int_{\Omega} \nabla_\alpha j^\alpha d^4V = \int_{\partial\Omega} j^\alpha n_\alpha dS. \quad (1.148)$$

($\partial\Omega$ means the boundary of Ω .) Thus if $\int_{\partial\Omega} j^\alpha n_\alpha dS = 0$ for arbitrarily small boxes Ω about a point $\mathcal{P}$, (1.148) $\Rightarrow \nabla_\alpha j^\alpha = 0$ at $\mathcal{P}$. And (1.147) means that for any constant vector field $\hat{x}^\alpha$,

$$\nabla_\alpha (T^{\alpha\beta} \hat{x}_\beta) = 0,$$

(x -momentum is conserved), implying Eq. (1.146)

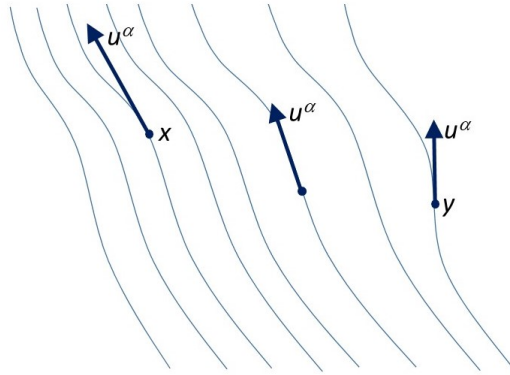
$$\nabla_\alpha T^{\alpha\beta} = 0.$$

In curved spacetime, this equation is unchanged, but there will in general be no constant vector fields like $\hat{x}^\alpha$, and, more generally, no vector fields v^α for which $\nabla_\alpha (T^{\alpha\beta} v_\beta) = 0$. In flat space, $\hat{x}^\alpha$ is along a translation symmetry of the geometry, and the current $j^\alpha = T^{\alpha\beta} \hat{x}_\beta$ is conserved because of that symmetry.

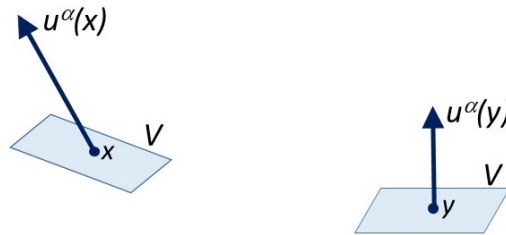
1.6.4 Dust

By dust is meant a collection of a large number of very small, very light particles which do not interact directly with each other, whose density varies smoothly, and whose velocity field (the unit tangent vectors to the particle trajectories) is also smooth. Because of its simplicity, dust is often used as a first, highly idealized, model for baryonic matter in the universe (the particles are galaxies); for stellar collapse (historically, for early studies of spherical and slightly nonspherical collapse); for the disk of spiral galaxies (the particles include the stars in the disk, as well as the interstellar medium of actual dust particles and gas); and for thin accretion disks around black holes and neutron stars. In the last example, a disk is thin if its pressure is small compared to the rotational energy of the disk; in general, ignoring pressure in comparison with the kinetic energy of the macroscopic fluid motion is the dust approximation.

Requiring the velocity field u^α to be smooth means that nearby dust particles travel along nearly parallel world lines.



In order to define density, one needs to know what surface of simultaneity is being used. The natural choice is to take at each point a small volume orthogonal to the fluid velocity at that point. Then the



rest-mass density ρ at a point x is the density measured at x by an observer moving with the dust at x —an observer with velocity $u^\alpha(x)$. Such an observer, for whom the dust appears to be locally at rest, is called *comoving*.

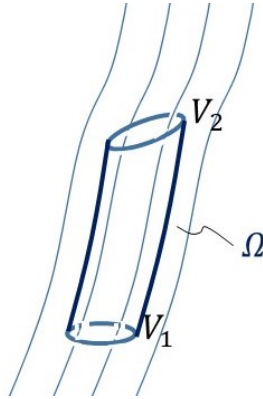
If there are no forces acting on the dust (if, for example, one can neglect not only the two-particle gravitational interactions, but also the interaction of each particle with the gravitational field of the dust as a whole) then each particle will travel on a straight line:

$$u^\beta \nabla_\beta u^\alpha = 0.$$

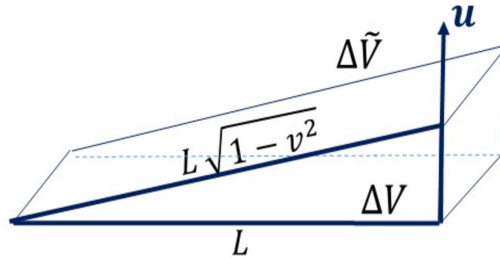
We will see that the same equation holds in curved spacetime, with ∇ the covariant derivative associated with a metric whose curvature encompasses gravity.

This equation, together with conservation of mass completely describes freely moving dust, and the equation describing conservation of mass can be obtained in the following way. Consider a 4-volume Ω with sides tangent to the trajectories and whose top and bottom faces are volumes V_1 and V_2 : Think of V_1 and V_2 as spacelike slices of the dust's history. Conservation of particles is the statement that if N particles “enter” (are in the volume) V_1 , N particles “leave” V_2 . If m is the mass of a particle (or the average mass per particle, if there are several species), mass conservation has the form

$$mN_1 = mN_2. \quad (1.149)$$



Our goal is to write this equation in terms of the density ρ . The rest-mass of the particles entering a small comoving volume ΔV that is orthogonal to u^α is just $M = \rho\Delta V$. Because u^α varies from point to point, however, even if we take V_1 and V_2 to be slices of constant time t , they will not, in general, be orthogonal to the 4-velocity. So we need to find the mass of a small slice $\Delta\tilde{V}$ of the dust that is not orthogonal to u^α . Boosting the small slice ΔV of the fluid by a velocity v gives a slice with volume



$$\Delta\tilde{V} = \Delta V \sqrt{1-v^2} \equiv \Delta V \tilde{\gamma}^{-1}, \quad (1.150)$$

by Eq. (1.11) – just the fact that lengths orthogonal to the boost are unchanged and the length along the boost is smaller than the proper length by the factor $\tilde{\gamma}^{-1}$. Because every dust particle that passes through ΔV also passes through $\Delta\tilde{V}$, the rest mass ΔM is the same in volume ΔV and in $\Delta\tilde{V}$,

$$\Delta M = \rho\Delta V = \rho\tilde{\gamma}\Delta\tilde{V}.$$

Let $n_\alpha = -u_\alpha$ be the normal to ΔV with sign chosen to make $u^\alpha n_\alpha = 1$, and let $\tilde{n}_\alpha$ be the corresponding unit normal to $\Delta\tilde{V}$. (That is, $\tilde{n}_\alpha = -\tilde{u}_\alpha$, with $\tilde{u}^\alpha$ the future pointing normal to $\Delta\tilde{V}$). By Eq. (1.80),

$$u^\alpha \tilde{n}_\alpha = \tilde{\gamma}.$$

Then

$$\Delta M = \rho u^\alpha n_\alpha \Delta V = \rho \tilde{u}^\alpha \tilde{n}_\alpha \Delta \tilde{V},$$

and the mass of dust in an arbitrary spacelike volume V is

$$M = \int_V \rho u^\alpha n_\alpha dV.$$

If, as in our previous integral conservation equations, we now take n_α to be the outward normal to the boundary of Ω , conservation of mass from V_1 to V_2 is

$$M_2 - M_1 = \int_{V_2} \rho u^\alpha n_\alpha dV + \int_{V_1} \rho u^\alpha n_\alpha dV = 0. \quad (1.151)$$

For example, if V_1 and V_2 were $t = \text{constant}$ surfaces n_α would be $\nabla_\alpha t$ on V_2 and $-\nabla_\alpha t$ on V_1 . This is the integral version of mass conservation. The corresponding differential equation again follows from Gauss' theorem,

$$\int_\Omega \nabla_\alpha (\rho u^\alpha) d\Omega = \int_{V_2} \rho u^\alpha n_\alpha dV + \int_{V_1} \rho u^\alpha n_\alpha dV.$$

By our construction, because our spatial volume of dust moves with the dust, no particles leave it and there is no contribution on the right side of the equation from the timelike part of the boundary of Ω . Formally, because u^α is tangent to the timelike sides of the boundary, $u^\alpha n_\alpha = 0$ there. Because we can choose an arbitrarily small volume of the form Ω about any point, (1.151) implies that $\nabla_\alpha (\rho u^\alpha) = 0$.

To summarize: Freely moving dust is described by a scalar field ρ and a unit timelike vector field u^α satisfying the equations

$$\nabla_\beta (\rho u^\beta) = 0, \quad u^\beta \nabla_\beta u^\alpha = 0. \quad (1.152)$$

Energy momentum tensor: A comoving observer sees the dust at rest and non-interacting, so there is no stress and $T^{ij} = 0$, $i, j = 1 - 3$. Furthermore, there is no momentum density (since $u^i = 0$), so $T^{0i} = 0$. Finally, he sees an energy density ρ : $T^{00} = \rho$.

$$\|T^{\mu\nu}\| = \begin{vmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{vmatrix}.$$

Noting that for the comoving observer $(u^\mu) = (1, 0, 0, 0)$, we have $T^{\mu\nu} = \rho u^\mu u^\nu$, and therefore

$$T^{\alpha\beta} = \rho u^\alpha u^\beta. \quad (1.153)$$

One expects that $\nabla_\beta T^{\alpha\beta} = 0$ will provide the equations that characterize the dust, and this is true:

Projecting along u_α gives conservation of mass:

$$\begin{aligned} 0 &= u_\alpha \nabla_\beta T^{\alpha\beta} = u_\alpha [u^\alpha \nabla_\beta (\rho u^\beta) + (\nabla_\beta u^\alpha) \rho u^\beta] \\ &= -\nabla_\beta (\rho u^\beta), \end{aligned}$$

where we used $u_\alpha \nabla_\beta u^\alpha = \frac{1}{2} \nabla_\beta (u^\alpha u_\alpha) = \frac{1}{2} \nabla_\beta (-1) = 0$ to eliminate the second term.

Projecting orthogonal to u^α gives the equation of motion:

$$0 = q^\alpha_\gamma \nabla_\beta T^{\beta\gamma} = q^\alpha_\gamma \nabla_\beta (\rho u^\beta u^\gamma) = \rho u^\beta q^\alpha_\gamma \nabla_\beta u^\gamma = \rho u^\beta \nabla_\beta u^\alpha,$$

again using $u_\gamma \nabla_\beta u^\gamma = 0$ to write $q^\alpha_\gamma \nabla_\beta u^\gamma = (\delta^\alpha_\gamma + u^\alpha u_\gamma) \nabla_\beta u^\gamma = \nabla_\beta u^\alpha$.

Charged dust: If the dust is charged, but the interparticle separation is large enough that we can ignore the electromagnetic interaction between any two dust particles (so that $F^{\alpha\beta}$ can be computed from an average current density $j^\alpha = \rho_e u^\alpha$, or from $j^\alpha = \rho_e u^\alpha + \text{external sources}$), then each dust particle will travel on a trajectory

$$u^\beta \nabla_\beta u^\alpha = \frac{q}{m} F^{\alpha\beta} u_\beta,$$

where q and m are the charge and mass of the dust particle. In the continuum limit, $q \rightarrow 0$ and $m \rightarrow 0$ but $\frac{eN}{V}$ and $\frac{mN}{V}$ remain finite and $\rho_e = \lim_{\substack{q \rightarrow 0 \\ N \rightarrow \infty}} \frac{qN}{V}$, $\rho = \rho_m = \lim_{\substack{m \rightarrow 0 \\ N \rightarrow \infty}} \frac{mN}{V}$ are the scalar fields that describe the dust. Thus the equation of motion is

$$\rho u^\beta \nabla_\beta u^\alpha = \rho_e F^{\alpha\beta} u_\beta. \quad (1.154)$$

When the dust is the only source of $F^{\alpha\beta}$, so that

$$\nabla_\beta F^{\alpha\beta} = 4\pi j^\alpha, \quad (1.155)$$

the system of field plus dust has the divergence-free energy momentum tensor

$$\begin{aligned} T^{\alpha\beta} &= T^{\alpha\beta}_{\text{dust}} + T^{\alpha\beta}_{\text{e-m}} \\ &= \rho u^\alpha u^\beta + \frac{1}{4\pi} \left[F^{\alpha\gamma} F^\beta_\gamma - \frac{1}{4} \eta^{\alpha\beta} F^{\gamma\delta} F_{\gamma\delta} \right] : \\ u_\alpha \nabla_\beta T^{\alpha\beta} &= -\nabla_\beta (\rho u^\beta) - F^{\alpha\gamma} u_\alpha j_\gamma \\ &= -\nabla_\beta (\rho u^\beta) - \rho_e \underbrace{F^{\alpha\gamma} u_\alpha u_\gamma}_0 \\ &= -\nabla_\beta (\rho u^\beta) \end{aligned} \quad (1.156)$$

and

$$q^\alpha_\gamma \nabla_\beta T^{\beta\gamma} = \rho u^\beta \nabla_\beta u^\alpha - F^{\alpha\beta} j_\beta \quad (\text{using Ex. 16 and Eq. (1.156)}):$$

that is, given Maxwell's equations, the equation of motion for the dust together with conservation of mass follows from $\nabla_\beta T^{\alpha\beta} = 0$:

$$\begin{aligned} u_\alpha \nabla_\beta T^{\alpha\beta} &= 0 \Rightarrow \nabla_\beta (\rho u^\beta) = 0 \\ q^\alpha_\gamma \nabla_\beta T^{\beta\gamma} &= 0 \Rightarrow \rho u^\beta \nabla_\beta u^\alpha = F^{\alpha\beta} j_\beta. \end{aligned} \quad (1.157)$$

1.6.5 Perfect Fluids

We will obtain the relativistic equations of motion and energy conservation for a fluid from Eq. (1.146), $\nabla_\beta T^{\alpha\beta} = 0$, and will then verify that they agree with the Newtonian equations in the limit $v \ll c$, $P \ll \rho c^2$. We again assume a fluid-element description, boxes large enough that pressure, density, and fluid velocity are defined, and small enough to neglect their change across the box. Saying that the fluid is shear-free is then equivalent to saying that a comoving observer sees the fluid as isotropic: In a box small enough to ignore changes in density and pressure there is no preferred spatial direction. That is, the components of the fluid's energy momentum tensor in the observer's frame must have no preferred

direction: $T^{\mu\nu}$ must be invariant under rotations. We can see this as follows: The components T^{0i} ($i = 1 - 3$) transform as a 3-vector under rotations and so can be invariant only if

$$T^{0i} = 0 \quad (1.158)$$

(for a comoving observer, the fluid element's 3-momentum vanishes). The spatial part of the tensor can be invariant under rotations only if it is a multiple of δ^{ij} : Pick a frame in which the symmetric matrix T^{ij} is diagonal. Unless all three eigenvalues are the same, the largest or smallest eigenvalue picks out a direction – along its eigenvector. It follows that the only nonzero components of $T^{\mu\nu}$ are the rotational scalars T^{00} and $\delta^{ij}T^k_k/3$:⁶

$$\|T^{\mu\nu}\| = \begin{vmatrix} \rho & & & \\ & P & & \\ & & P & \\ & & & P \end{vmatrix}, \quad (1.159)$$

where we have defined ρ and P by

$$\begin{aligned} \rho &= T^{00} \\ P &= \frac{1}{3}T^k_k. \end{aligned}$$

In the comoving frame, the fluid's four velocity has component $u^\mu = (1, 0, 0, 0)$, and the projection orthogonal to u^α ,

$$q^{\alpha\beta} = \eta^{\alpha\beta} + u^\alpha u^\beta, \quad (1.160)$$

has components

$$\|q^{\mu\nu}\| = \begin{vmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix}. \quad (1.161)$$

Then

$$T^{\mu\nu} = \rho u^\mu u^\nu + P q^{\mu\nu},$$

and we have

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + P q^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + P \eta^{\alpha\beta}. \quad (1.162)$$

Conservation of energy.

$$u_\alpha \nabla_\beta T^{\alpha\beta} = 0$$

$$\begin{aligned} 0 &= u_\alpha \nabla_\beta T^{\alpha\beta} = u_\alpha \nabla_\beta [\rho u^\alpha u^\beta + P q^{\alpha\beta}] \\ &= -\nabla_\beta (\rho u^\beta) + P u_\alpha \nabla_\beta (\eta^{\alpha\beta} + u^\alpha u^\beta) \\ &= -\nabla_\beta (\rho u^\beta) - P \nabla \cdot \mathbf{u} \end{aligned}$$

$$\nabla_\beta (\rho u^\beta) = -P \nabla \cdot \mathbf{u} \quad (1.163)$$

The equation means that the mass of a fluid element decreases by the work,

$$P dV = P V \nabla \cdot \mathbf{u} d\tau, \quad (1.164)$$

⁶Equivalently, we can decompose the spatial tensor T^{ab} into its symmetric tracefree part, the shear tensor, $S^{ab} = T^{ab} - \frac{1}{3}\delta_b^a T^c_c$, and its trace: $T^{ab} = S^{ab} + \frac{1}{3}\delta_b^a T^c_c$. A symmetric tracefree tensor transforms as an irreducible $j = 2$ representation of the rotation group and is therefore rotationally invariant only if it vanishes (as our quick eigenvalue argument shows).

it does in proper time $d\tau$. That is, in proper time $d\tau$, the fluid element is displaced by $\xi^\alpha = u^\alpha d\tau$. Eq. (1.141) implies, for a volume V orthogonal to u^α , the quantity dV/V is the spatial divergence of ξ^α , the divergence in the subspace orthogonal to u^α :

$$\frac{dV}{V} = q^{\alpha\beta} \nabla_\alpha \xi_\beta = q^{\alpha\beta} \nabla_\alpha u_\beta d\tau = \nabla_a u^\alpha d\tau.$$

This is Eq. (1.164) .

Equation of motion.

$$\begin{aligned} q^\alpha{}_\gamma \nabla_\beta T^{\beta\gamma} &= 0 \\ 0 &= q^\alpha{}_\gamma \nabla_\beta [\rho u^\beta u^\gamma + P q^{\beta\gamma}] \\ &= q^\alpha{}_\gamma \rho u^\beta \nabla_\beta u^\gamma + q^{\alpha\beta} \nabla_\beta P + q^\alpha{}_\gamma P \nabla_\beta (u^\beta u^\gamma) \\ &= \rho u^\beta \nabla_\beta u^\alpha + q^{\alpha\beta} \nabla_\beta P + P u^\beta \nabla_\beta u^\alpha \\ &\quad (\rho + P) u^\beta \nabla_\beta u^\alpha = -q^{\alpha\beta} \nabla_\beta P. \end{aligned} \tag{1.165}$$

Newtonian limit: Let e be a small parameter of order v/c or v_{sound}/c , whichever is larger.

$$\begin{aligned} u^\mu &= (1, v^i) + O(e^2), \quad P/\rho = O(e^2) \\ \rho &= \text{rest mass density} + O(e^2). \end{aligned}$$

Then Eq. (1.163) implies

$$\begin{aligned} \partial_t(\rho u^t) + \partial_i(\rho u^i) &= -P(\partial_t u^t + \partial_i u^i) \\ \partial_t \rho + \partial_i(\rho v^i) &= 0 + O(e^2), \end{aligned}$$

in agreement with the conservation of mass equation (1.145b). Similarly, Eq. (1.165) implies

$$\begin{aligned} \rho u^\mu \nabla_\mu u^i &= -\nabla^i P \\ \rho(\partial_t + v^j \nabla_j) v_i &= -\nabla_i P, \end{aligned}$$

in agreement with equations (1.145c).

Relativistic energy conservation, Eq. (1.163), also implies the Bernoulli equation, expressing energy conservation in a Newtonian flow. We have only looked at its lowest-order form, obtaining conservation of mass at order e^0 ; to extract Newtonian energy conservation, one must keep terms at the next nonvanishing order, order e^2 .

To summarize what we have just shown:

The fact that a perfect fluid is locally isotropic implies the form of its stress-energy tensor. The fact that the stress-energy tensor is divergencefree (true for any system) implies all of the equations governing a perfect fluid – the equations that we first obtained by following in detail the motion of a fluid element.

Exercise 27. Show that the entropy of a fluid element is conserved for a perfect fluid: The 2nd law of thermodynamics can be stated in the form $dE = TdS - PdV$, where

dV = the change in a volume $V \perp u^\alpha$ (i.e., V is a comoving volume)

dS = the change in entropy of the fluid element

dE = the change in energy of the fluid element as measured by a comoving observer.

a. Introduce a conserved baryon number density n :

Conservation of baryons is $\nabla_\alpha(nu^\alpha) = 0$. Show that if the number of baryons in a fluid element is constant, its volume satisfies $u^\alpha \nabla_\alpha V = V \nabla_\alpha u^\alpha$.

b. Show that $u_\beta \nabla_\alpha T^{\alpha\beta} = 0$ implies

$$u \cdot \nabla E = -Pu \cdot \nabla V.$$

c. Conclude from the 2nd law that the entropy of a fluid element is constant along its world line:

$$u \cdot \nabla S \equiv \frac{dS}{d\tau} = 0.$$

Exercise 28. Doppler Shift. Photons move along null geodesics and have null four-momenta: $p^\alpha p_\alpha = 0$.

An observer with 4-velocity t^α decomposes p^α into energy and 3-momentum in the usual way: $p^\alpha = Et^\alpha + p_\perp^\alpha$, where $t_\alpha p_\perp^\alpha = 0$. Because p^α is null, the decomposition has the form

$$p^\alpha = E(t^\alpha + n^\alpha),$$

where n^α is a unit vector along $p_\perp^\alpha$. We can identify the spatial 4-vector $p_\perp^\alpha$ with a 3-vector p^a that lies in the 3-space orthogonal to t^α : In an orthonormal frame with timelike vector t^α ,

$$p^\mu = (E, p^i) = E(1, n^i) \quad p_\perp^\mu = (0, p^i) \quad n^\mu = (0, n^i).$$

Photon energy and 3-momentum are related to frequency by $E = \hbar\omega$, $p^a = \hbar k^a$. If we define a wave 4-vector k^α by $\hbar k^\alpha = p^\alpha$, then the observer u^α sees frequency $\omega = -k_\alpha t^\alpha$ and wave 3-vector $\hbar k_\perp^\alpha = p_\perp^\alpha$, or $k^a = \omega n^a$.

Let u^α be the 4-velocity of a second observer, moving at 3-velocity $\mathbf{v}$ relative to the first. By computing the dot product $u^\alpha k_\alpha$, show that this observer sees a frequency ω' given by

$$\omega' = \omega\gamma(1 - \mathbf{v} \cdot \mathbf{n}) = \omega \frac{1 - v \cos \alpha}{\sqrt{1 - v^2}},$$

with α the angle between the velocity of the second observer and the direction of the photon, as seen by the first observer.

Exercise 29. Let u^α be the velocity field of a fluid in Minkowski space.

a. Show that $q^{\alpha\beta} \nabla_\alpha u_\beta = \nabla_\alpha u^\alpha$.

b. In a natural coordinate system for which the t -axis is parallel to u^α at some time t_0 , show that $\nabla_\alpha u^\alpha = \partial_i v^i$ at $t = t_0$, where $u^\alpha = \gamma(t^\alpha + v^\alpha)$, with v^α the 3-velocity ($v^\alpha t_\alpha = 0$). Conclude that conservation of energy, (1.50), has the form $\partial_t(\gamma\rho) = -P\partial_i v^i$.

c. Show that at $t = t_0$, the equation of motion (1.151) has the form

$$(\rho + P)(\partial_t + v^j \partial_j)v^i = -\partial^i P.$$

Exercise 30. (Similar to Hartle 5.17) *Relativistic beaming.* This is a challenging problem, and hints follow this set of exercises. Hartle's answer book, not written by Jim Hartle, gets part of it wrong. A body emits radiation of frequency ω , with j photons per unit time emitted isotropically in the source's rest frame. An observer sees the body move toward her with speed v along her x' axis.

- a. Let α' be the angle between a photon path and the source's velocity, measured in the observer's frame, and let α be the corresponding angle (between the photon path and the source's x -axis), measured in the source frame. Using the 4-vector k^α , show

$$\mu' = \frac{\mu + v}{1 + v\mu}, \text{ where } \mu = \cos \alpha. \quad (*)$$

- b. Find the number flux of photons $j'(\alpha')$ seen by the observer, at a fixed distance R from the source (as measured by the observer). Note that, if α' and α are related by $(*)$, the photons emitted by the source at angles between α and $\alpha + d\alpha$ are seen by the observer at angles between α' and $\alpha' + d\alpha'$.
- c. Find the intensity $f'(\alpha')$, the energy flux measured by the observer.
- d. Discuss the beaming of number flux and energy flux as $v \rightarrow 1$:

For $\gamma \gg 1$, show that $v = 1 - \frac{1}{2\gamma^2} + O(\gamma^{-4})$. Using this and $\cos \alpha' = 1 - \frac{1}{2}\alpha'^2 + O(\alpha'^4)$, show that $j' = \frac{8\gamma^3}{(1 + \gamma^2\alpha'^2)^3}j$, ignoring terms of higher than quadratic order in α' , $1/\gamma$.

For what angles is $j'/j \gg 1$? For what angles is $j'/j \ll 1$?

Exercise 31. GKZ cutoff (Also adapted from Hartle). The highest energy cosmic rays have energies of about 3×10^{20} eV. A celebrated argument by Greisen, Zatsepin, and Kuz'min shows that they cannot have traveled further than about 10^7 ly, not much farther than our Local Group of about 30 galaxies (Our local group extends slightly past Andromeda). GZK note that particles with high enough energy interact with photons of the cosmic microwave background (CMB) to produce pions. A typical cosmic ray is a proton, and the dominant reactions are

$$p + \gamma \rightarrow n + \pi^+ \quad p + \gamma \rightarrow p + \pi^0.$$

The reactions can proceed only if, in the center of mass frame, the energy is high enough to allow the final particles to emerge with zero speed.

$$p_n^\alpha = m_n u_{CM}^\alpha, \quad p_\pi^\alpha = m_\pi u_{CM}^\alpha.$$

(At higher energy, they emerge with nonzero relative velocity.) Find this threshold energy as follows:

- a. Write conservation of 4-momentum and square both sides of the equation to deduce

$$2p_{(\gamma)}^\alpha p_{p\alpha} - m_p^2 = -(m_n + m_\pi)^2.$$

- b. What fractional error does one make in $m_n + m_\pi$ if one replaces m_n by m_p ?
- c. Suppose that in a frame for which the CMB looks isotropic (e.g., the Earth's rest frame, to about 0.1% accuracy) the proton and photon are moving toward each other. Use the 3+1 decomposition of $p_{(\gamma)}^\alpha$ and p_p^α in this frame to show (with $m_p = m_n$ and $v = 1$)

$$E_p^{CMB} = \frac{m_p m_\pi}{2E_\gamma^{CMB}} \left(1 + \frac{m_\pi}{2m_p}\right) \approx 3 \times 10^{20} \text{ eV}.$$

- d. The mean free photon path is $\ell = \frac{1}{n_\gamma \sigma}$, with a cross section $\sigma \sim 2 \times 10^{-28} \text{ cm}^2$ and n_γ the number density of photons in 2.7° blackbody radiation, about $400/\text{cm}^3$. Check that ℓ is the 10^7 ly mentioned above. Protons with lower energy, however, will also interact if they encounter photons with higher energy in the thermal blackbody distribution. As a result, the interaction will also cut off protons of lower energy that come from larger distances. Use your knowledge of the energy distribution of particles (photons) at temperature T to estimate the minimum energy that a proton must have to have mean free path ℓ less than the radius of the visible universe.

Hint: First find the number density of photons for which ℓ is the size of the visible universe; next estimate the energy of photons with this number density, in a thermal distribution at about 3°K ; finally, for a photon of this energy find the threshold energy of the proton, from (c).

Hints for Exercise 30.

- a. Note that the components of k^α in the source frame are $(\omega(1, \cos \alpha, \sin \alpha, 0))$.
 b. Light emitted at time $t = t' = 0$ on the source's clock and a synchronized clock in the observer's frame reaches the observer after observer time $d/c = d'$, with d' the distance to the source in the observer's frame. At a later time $dt' = \gamma dt$, the distance between source and observer is $d - v\gamma dt'$ and a photon emitted at dt' reaches the observer after an observer time $d - v\gamma dt'$.
 The number of photons emitted into solid angle $d\Omega$ in time dt is $dN/d\Omega/dt$ and is those photons are observed in solid angle $d\Omega'$ in time dt' . To simplify, check and use the identity $\gamma(1 - v/c) = \gamma(1 - v/c) = \gamma(1 - v/c)$.
 c. $f'(v) = f(v)\gamma(1 - v/c)$. To simplify, check and use the identity $\gamma(1 - v/c) = \gamma(1 - v/c)$.
 d. $\frac{(n_\gamma + 1)\ell}{1} = (n_\gamma + 1)\ell$

1.7 Curvilinear Coordinates

A coordinate system (*chart*, in the mathematics literature) on an n -dimensional space M is a smooth map assigning to each point P of M n numbers $(x^1(P), \dots, x^n(P))$, called the coordinates of P . That is, a coordinate system is a map $x : M \rightarrow \mathbb{R}^n$ with x smooth and invertible. The natural coordinates

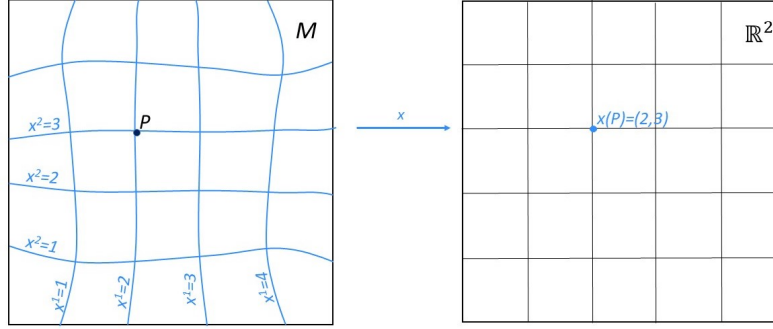
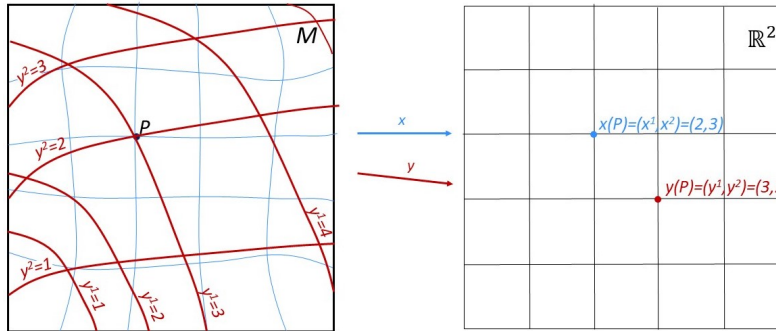


Figure 1.12: A coordinate system $x : M \rightarrow \mathbb{R}^2$ in two dimensions.

(t, x, y, z) on Minkowski space provide an obvious example. Another is spherical coordinates (t, r, θ, ϕ) , although strictly speaking these are coordinates on M with the t - z plane (the history of the symmetry axis) removed: $P \rightarrow (t(P), r(P), \theta(P), \phi(P))$ is neither smooth nor invertible on the z -axis.

The two-dimensional figure below shows coordinate systems $x : M \rightarrow \mathbb{R}^2$ and $y : M \rightarrow \mathbb{R}^2$. The



change in coordinates $(y^1, y^2) \mapsto (x^1, x^2)$ is the map $x \circ y^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, smooth and invertible if and only if the Jacobian matrix

$$\left\| \frac{\partial x^i}{\partial y^j} \right\|$$

of $x \circ y^{-1}$ is smooth and has nonzero determinant. In physicists' notation, the coordinate transformation is written $x(y)$ instead of $x \circ y^{-1}$. We will see that the Jacobian takes components of vectors and tensors from one coordinate system to another.

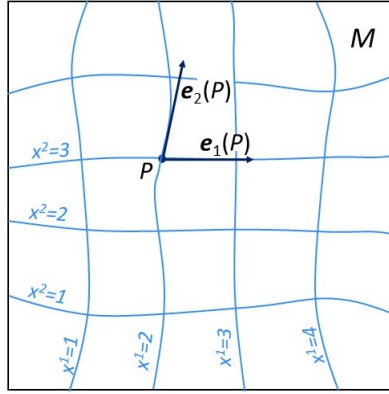
1.7.1 Components of Vectors and Tensors

A coordinate system on M provides a basis at each point P for the vectors at P . The radial basis vector $e_0 = \hat{r}$ at a point P in spherical coordinates, for example, is tangent to the path through P with constant

θ and ϕ coordinates, the path $c(\lambda)$ with coordinates $(x^i(\lambda)) = (r + \lambda, \theta, \phi)$. In general, for a coordinate system x , the path along which only x^1 changes is the path with coordinates

$$\lambda \rightarrow (x^1(P) + \lambda, x^2(P), \dots, x^n(P)) \quad (1.166)$$

and can be regarded as the x^1 -axis through P . The tangent to this path is the first basis vector e_1 at P .



Similarly, the tangent to the x^2 -axis through P , to the path with coordinates

$$\lambda \rightarrow (x^1(P), x^2(P) + \lambda, x^3(P), \dots, x^n(P)),$$

is the second basis vector e_2 .

If f is any function on M , the directional derivative of f in the direction e_1 is the derivative of f along the path (1.166) tangent to e_1 ,

$$\frac{\partial f}{\partial x^1}(P) = \left. \frac{d}{d\lambda} f(x^1(P) + \lambda, x^2(P), \dots, x^n(P)) \right|_{\lambda=0} = e_1 \cdot \nabla f.$$

In abstract index notation,

$$e_1 \cdot \nabla f = e_1^a \nabla_a f = \frac{\partial f}{\partial x^1}. \quad (1.167)$$

For this reason, as mentioned earlier, the vectors e_i of a coordinate basis are often written ∂_i . Regarded in this way as a directional derivative, the vector e_i is the map,

$$e_i(f) = e_i \cdot \nabla f = \partial_i f,$$

of Eqs. (1.59) and (1.60).

Return now in notation from a generic dimension to 4-dimensional Minkowski space. The basis dual to $\{e_\mu\}$ (following, e.g., MTW we will call it $\{\omega^\mu\}$) is just the set of covectors

$$\omega^0 = \nabla x^0, \omega^1 = \nabla x^1, \omega^2 = \nabla x^2, \omega^3 = \nabla x^3, \quad (1.168)$$

because

$$\omega^\mu(e_\nu) = \omega^\mu_\alpha e_\nu^\alpha = e_\nu^\alpha \nabla_\alpha x^\mu = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu. \quad (1.169)$$

use (1.167)

The components v^μ of a vector v with respect to the coordinate basis $\{e_\mu\}$ are called the components of v in the coordinate system x . If x' is another coordinate system, with $\{e_{\mu'}\}$ the associated basis, $e_{\mu'}$ can be related to e_μ in this way:

$$\mathbf{e}_{\mu'}(f) = \frac{\partial f}{\partial x^{\mu'}} = \frac{\partial f}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \mathbf{e}_\mu(f). \quad (1.170)$$

That is,

$$\mathbf{e}_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \mathbf{e}_\mu \quad (1.171)$$

(where $\left\| \frac{\partial x^\mu}{\partial x^{\mu'}} \right\|$ is the Jacobian matrix of the map $x \circ x'^{-1}$). Then

$$\begin{aligned} \mathbf{v} &= v^{\mu'} \mathbf{e}_{\mu'} = v^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \mathbf{e}_\mu = v^\mu \mathbf{e}_\mu, \\ \implies v^{\mu'} &= \frac{\partial x^{\mu'}}{\partial x^\mu} v^\mu, \end{aligned} \quad (1.172)$$

where we used (1.171) with primed $\leftrightarrow$ unprimed.

Similarly, the dual basis vectors $\boldsymbol{\omega}^{\mu'}$ are related to $\boldsymbol{\omega}^\mu$ by

$$\boldsymbol{\omega}^{\mu'} = \nabla x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \nabla x^\mu = \frac{\partial x^{\mu'}}{\partial x^\mu} \boldsymbol{\omega}^\mu, \quad (1.173)$$

whence the components σ_μ of a covector transform in the manner

$$\begin{aligned} \sigma_{\mu'} \boldsymbol{\omega}^{\mu'} &= \sigma_\mu \boldsymbol{\omega}^\mu = \sigma_\mu \frac{\partial x^\mu}{\partial x^{\mu'}} \boldsymbol{\omega}^{\mu'} \\ \implies \sigma_{\mu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \sigma_\mu. \end{aligned} \quad (1.174)$$

(This also follows from (1.173) together with $\sigma_\mu v^\mu = \sigma_{\mu'} v^{\mu'}$.)

For arbitrary tensors $T^{\alpha \dots \beta}_{\gamma \dots \delta}$, we have

$$T^{\mu' \dots \nu'}_{\sigma' \dots \tau'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \dots \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial x^\sigma}{\partial x^{\sigma'}} \dots \frac{\partial x^\tau}{\partial x^{\tau'}} T^{\mu \dots \nu}_{\sigma \dots \tau}. \quad (1.175)$$

This is just a special case of the transformation law (1.40), where, in the case of coordinate bases, the $a^{\mu'}_\mu$ and $a^\mu_{\mu'}$ matrices are the Jacobian matrices of the transformations $x' \circ x^{-1}$ and $x \circ x'^{-1}$:

$$a^{\mu'}_\mu = \frac{\partial x^{\mu'}}{\partial x^\mu}, \quad a^\mu_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}. \quad (1.176)$$

Because P has coordinates $x'(P)$ in the primed system and $x(P)$ in the unprimed system, Eq. (1.175) is sometimes written with explicit arguments x' and x :

$$T^{\mu' \dots \nu'}_{\sigma' \dots \tau'}(x') = \frac{\partial x^{\mu'}}{\partial x^\mu} \dots \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial x^\sigma}{\partial x^{\sigma'}} \dots \frac{\partial x^\tau}{\partial x^{\tau'}} T^{\mu \dots \nu}_{\sigma \dots \tau}(x). \quad (1.177)$$

1.7.2 Covariant Derivatives

We want to find the coordinate components of $\nabla_\alpha T^{\beta\cdots\gamma}_{\delta\cdots\epsilon}$ in terms of the partial derivatives

$$\frac{\partial}{\partial x^\lambda} T^{\mu\cdots\nu}_{\sigma\cdots\tau}.$$

In abstract-index notation, the covariant derivative of a vector v^α is unambiguously written $\nabla_\alpha v^\beta$. An ambiguity in notation arises when we write the components of $\nabla_\alpha v^\beta$ in a basis: Because the ν th component $v^\nu \equiv \omega^\nu(v)$ of a vector field v^α is a scalar field, the μ th component of its gradient is $\partial_\mu v^\nu$. Since we write the gradient of a scalar in the form $\nabla_\alpha f$, the μ th component of v^ν could be written $\nabla_\mu v^\nu$, but this is *not* the way the symbol $\nabla_\mu v^\nu$ is used by physicists. Instead, it is natural to write the μ - ν th component of the tensor $\nabla_\alpha v^\beta$ as $\nabla_\mu v^\nu$. When we use curvilinear coordinates (or a basis that is not a coordinate basis, or when we work in a curved spacetime) the two possibilities are not equivalent, because the basis vectors are not constant vector fields: The gradient of the scalar v^ν has as its μ th component

$$\nabla_\mu (v^\nu) = \nabla_\mu (v^\alpha \omega^\nu_\alpha) = \partial_\mu (v^\alpha \omega^\nu_\alpha) = \partial_\mu v^\nu;$$

while the μ - ν th component of the tensor $\nabla_\alpha v^\beta$ is

$$(\nabla_\mu v^\alpha) \omega^\nu_\alpha = \nabla_\mu (v^\alpha \omega^\nu_\alpha) - v^\alpha \nabla_\mu \omega^\nu_\alpha = \partial_\mu v^\nu - v^\alpha \nabla_\mu \omega^\nu_\alpha. \quad (1.178)$$

The second expression involves the covariant directional derivative of the basis vector ω^ν in the direction of e_μ . For the gradient of the scalar v^ν (rarely needed), these notes will use the expression $\nabla_\alpha (v^\nu)$, with μ th component

$$\nabla_\mu (v^\nu) := \partial_\mu v^\nu; \quad (1.179)$$

the parentheses means that the component is found *before* the derivative is evaluated. We could similarly write the μ - ν th component of $\nabla_\alpha v^\beta$ in the form $(\nabla_\mu v)^\nu$ to make clear that the component was computed *after* the covariant derivative was evaluated, but we will follow the standard physics notation, omitting the parentheses, and writing

$$\nabla_\mu v^\nu := (\nabla_\mu v)^\nu = (\nabla_\mu v^\alpha) \omega^\nu_\alpha = (e_\mu^\beta \nabla_\beta v^\alpha) \omega^\nu_\alpha. \quad (1.180)$$

The expression analogous to Eq. (1.178) for the covariant derivative of a covector σ_α is then

$$\begin{aligned} \nabla_\mu \sigma_\nu &= e_\mu^\alpha e_\nu^\beta \nabla_\alpha \sigma_\beta \\ &= \nabla_\mu (e_\nu^\beta \sigma_\beta) - (\nabla_\mu e_\nu^\beta) \sigma_\beta. \end{aligned}$$

Again, for a coordinate basis, $\nabla_\mu f = \partial_\mu f$, and we have

$$\nabla_\mu \sigma_\nu = \partial_\mu \sigma_\nu - (\nabla_\mu e_\nu^\beta) \sigma_\beta. \quad (1.181)$$

The vector $\nabla_\mu e_\nu$ is the covariant directional derivative of e_ν in the direction of e_μ . Because it is a vector, it can be written as a linear combination of basis vectors, namely

$$\nabla_\mu e_\nu = \Gamma^\lambda_{\nu\mu} e_\lambda \quad (1.182)$$

The coefficients $\Gamma^\lambda_{\nu\mu}$ are called Christoffel symbols (or, for an orthonormal basis, Ricci rotation coefficients). We have

$$\Gamma^\lambda_{\nu\mu} = e_\mu^\alpha \nabla_\alpha (e_\nu^\beta) \omega^\lambda_\beta. \quad (1.183)$$

Using the defining equation, (1.182), Eq. (1.181) for the components of $\nabla_\alpha \sigma_\beta$ becomes

$$\nabla_\mu \sigma_\nu = \partial_\mu \sigma_\nu - \Gamma^\lambda_{\nu\mu} e_\lambda^\beta \sigma_\beta = \partial_\mu \sigma_\nu - \Gamma^\lambda_{\nu\mu} \sigma_\lambda. \quad (1.184)$$

Observing that

$$\begin{aligned} \nabla_\mu (\omega^\nu_\beta) e^\beta_\lambda &= \nabla_\mu (\omega^\nu_\beta e^\beta_\lambda) - (\nabla_\mu e^\beta_\lambda) \omega^\nu_\beta \\ &= \partial_\mu \delta^\nu_\lambda - \Gamma^\nu_{\lambda\mu} \\ &= -\Gamma^\nu_{\lambda\mu}, \end{aligned}$$

we obtain from (1.178) the components of $\nabla_\beta v^\alpha$

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\lambda\mu} v^\lambda. \quad (1.185)$$

Equation (1.185) could also have been obtained from (1.184), using $\nabla_\mu (v^\nu \sigma_\nu) = \partial_\mu (v^\nu \sigma_\nu)$ and the Leibnitz rule, $\nabla_\mu (v^\nu \sigma_\nu) = (\nabla_\mu v^\nu) \sigma_\nu + v^\nu \nabla_\mu \sigma_\nu$.

With components of the covariant derivative of vectors and covectors computed, the components of $\nabla_\epsilon T^{\alpha\cdots\beta}_{\gamma\cdots\delta}$ follow quickly. One acquires one Γ with a + sign for each up index and one Γ with a – sign for each down index:

$$\begin{aligned} \nabla_\lambda T^{\mu\cdots\nu}_{\sigma\cdots\tau} &= \omega^\mu_\alpha \cdots \omega^\nu_\beta e_\sigma^\gamma \cdots e_\tau^\delta \nabla_\lambda T^{\alpha\cdots\beta}_{\gamma\cdots\delta} \\ &= \nabla_\lambda (T^{\alpha\cdots\beta}_{\gamma\cdots\delta} \omega^\mu_\alpha \cdots \omega^\nu_\beta e_\sigma^\gamma \cdots e_\tau^\delta) \\ &\quad - T^{\alpha\cdots\beta}_{\gamma\cdots\delta} \nabla_\lambda (\omega^\mu_\alpha) \cdots \omega^\nu_\beta e_\sigma^\gamma \cdots e_\tau^\delta - \cdots - T^{\alpha\cdots\beta}_{\gamma\cdots\delta} \omega^\mu_\alpha \cdots (\nabla_\lambda \omega^\nu_\beta) e_\sigma^\gamma \cdots e_\tau^\delta \\ &\quad - T^{\alpha\cdots\beta}_{\gamma\cdots\delta} \omega^\mu_\alpha \cdots \omega^\nu_\beta (\nabla_\lambda e_\sigma^\gamma) \cdots e_\tau^\delta - \cdots - T^{\alpha\cdots\beta}_{\gamma\cdots\delta} \omega^\mu_\alpha \cdots \omega^\nu_\beta e_\sigma^\gamma \cdots (\nabla_\lambda e_\tau^\delta) \\ \nabla_\lambda T^{\mu\cdots\nu}_{\sigma\cdots\tau} &= \partial_\lambda T^{\mu\cdots\nu}_{\sigma\cdots\tau} + \Gamma^\mu_{\lambda\alpha} T^{\alpha\cdots\nu}_{\sigma\cdots\tau} + \cdots + \Gamma^\nu_{\lambda\beta} T^{\mu\cdots\beta}_{\sigma\cdots\tau} \\ &\quad - \Gamma^\mu_{\sigma\lambda} T^{\mu\cdots\nu}_{\alpha\cdots\tau} - \cdots - \Gamma^\nu_{\tau\lambda} T^{\mu\cdots\nu}_{\sigma\cdots\alpha}. \end{aligned} \quad (1.186)$$

Eq. (1.186) could also have been obtained from Eqs. (1.184) and (1.185), using the Leibnitz rule and the fact that any $T^{\alpha\cdots\beta}_{\gamma\cdots\delta}$ is the sum of outer products of vectors and covectors. The derivation above is a little more direct. In this derivation, the only place where we used the fact that $\{e_\mu\}$ is a coordinate basis is in replacing the directional derivative $\nabla_\mu f$ by $\partial_\mu f$. In a general basis, Eq. (1.186) has the form

$$\begin{aligned} \nabla_\lambda T^{\mu\cdots\nu}_{\sigma\cdots\tau} &= \nabla_\lambda (T^{\mu\cdots\nu}_{\sigma\cdots\tau}) + \Gamma^\mu_{\lambda\alpha} T^{\alpha\cdots\nu}_{\sigma\cdots\tau} + \cdots + \Gamma^\nu_{\lambda\beta} T^{\mu\cdots\beta}_{\sigma\cdots\tau} \\ &\quad - \Gamma^\mu_{\sigma\lambda} T^{\mu\cdots\nu}_{\alpha\cdots\tau} - \cdots - \Gamma^\nu_{\tau\lambda} T^{\mu\cdots\nu}_{\sigma\cdots\alpha}. \end{aligned} \quad (1.187)$$

We next evaluate $\Gamma^\lambda_{\nu\mu}$ when $\{e_\mu\}$ is a coordinate basis.

First note that in a preferred coordinate system

$$\nabla_\mu \nabla_\nu f = \frac{\partial^2 f}{\partial x^\mu \partial x^\nu};$$

because partial derivatives commute for smooth scalars f ,

$$\nabla_{[\mu} \nabla_{\nu]} f = 0.$$

(In fact, because all components of $\nabla_{[\alpha} \nabla_{\beta]} f$ vanish in some basis, they must vanish in any basis: $\nabla_{[\alpha} \nabla_{\beta]} f = 0$.)

Now in a coordinate basis, we have $\nabla_\mu f = \partial_\mu f$, and our equation for the covariant derivative of a covector, (1.184), implies

$$\begin{aligned}\nabla_\mu \nabla_\nu f &= \partial_\mu \partial_\nu f - \Gamma^\lambda_{\nu\mu} \partial_\lambda f, \\ \nabla_{[\mu} \nabla_{\nu]} f &= 0 = \partial_{[\mu} \partial_{\nu]} f - \Gamma^\lambda_{[\nu\mu]} \partial_\lambda f = -\Gamma^\lambda_{[\nu\mu]} \partial_\lambda f\end{aligned}$$

That is, **for a coordinate basis**, $\Gamma^\lambda_{[\mu\nu]} = 0$: $\Gamma^\lambda_{\mu\nu}$ is symmetric in μ and ν :

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}. \quad (1.188)$$

Once the symmetry of $\Gamma^\lambda_{\mu\nu}$ is established, its value can be found in terms of the components $\eta_{\mu\nu}$ of the metric. To do this one uses the equation

$$\nabla_\alpha \eta_{\beta\gamma} = 0. \quad (1.189)$$

The components of (1.189) are

$$\nabla_\lambda \eta_{\mu\nu} = \partial_\lambda \eta_{\mu\nu} - \Gamma^\sigma_{\mu\lambda} \eta_{\sigma\nu} - \Gamma^\sigma_{\nu\lambda} \eta_{\mu\sigma} = 0.$$

Defining

$$\Gamma_{\lambda\mu\nu} := \Gamma^\sigma_{\mu\nu} \eta_{\sigma\lambda},$$

we have

$$\partial_\lambda \eta_{\mu\nu} = \Gamma_{\lambda\nu\mu} + \Gamma_{\nu\lambda\mu}.$$

To find $\Gamma_{\lambda\nu\mu}$ one cyclicly permutes μ, ν, λ :

$$\partial_\mu \eta_{\nu\lambda} = \Gamma_{\mu\lambda\nu} + \Gamma_{\lambda\mu\nu}$$

$$\partial_\nu \eta_{\lambda\mu} = \Gamma_{\nu\mu\lambda} + \Gamma_{\mu\nu\lambda}$$

Add the last two equations and subtract the first:

$$\begin{aligned}\partial_\mu \eta_{\nu\lambda} + \partial_\nu \eta_{\lambda\mu} - \partial_\lambda \eta_{\mu\nu} &= \Gamma_{\lambda\nu\mu} + \cancel{\Gamma_{\nu\lambda\mu}} + \cancel{\Gamma_{\mu\lambda\nu}} + \Gamma_{\lambda\mu\nu} - \cancel{\Gamma_{\nu\mu\lambda}} - \cancel{\Gamma_{\mu\nu\lambda}} \\ \partial_\mu \eta_{\nu\lambda} + \partial_\nu \eta_{\lambda\mu} - \partial_\lambda \eta_{\mu\nu} &= 2\Gamma_{\lambda\mu\nu}.\end{aligned}$$

(The symmetry of Γ was used to combine or cancel each pair of terms.) Then

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2} [\partial_\mu \eta_{\nu\lambda} + \partial_\nu \eta_{\lambda\mu} - \partial_\lambda \eta_{\mu\nu}] \quad (1.190)$$

and

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} \eta^{\lambda\epsilon} [\partial_\mu \eta_{\nu\epsilon} + \partial_\nu \eta_{\mu\epsilon} - \partial_\epsilon \eta_{\mu\nu}]. \quad (1.191)$$

As soon as we go to a curved spacetime, the same steps will give the same result in terms of an arbitrary metric $g_{\alpha\beta}$.

1.7.3 Examples

Example 1: Consider Minkowski space with two spatial dimensions in the coordinate system t, r, ϕ :

$$x = r \cos \phi, \quad y = r \sin \phi, \quad \phi = \tan^{-1} \frac{y}{x}, \quad r = (x^2 + y^2)^{1/2} \quad (1.192)$$

The covariant basis vectors are

$$\omega^t_\alpha = \nabla_\alpha t, \quad \omega^\phi_\alpha = \nabla_\alpha \phi \quad \text{and} \quad \omega^r_\alpha = \nabla_\alpha r. \quad (1.193)$$

with

$$\begin{aligned} \nabla_\alpha x &= \frac{\partial x}{\partial \phi} \nabla_\alpha \phi + \frac{\partial x}{\partial r} \nabla_\alpha r \\ &= \cos \phi \nabla_\alpha r - r \sin \phi \nabla_\alpha \phi \end{aligned} \quad (1.194)$$

$$\begin{aligned} \nabla_\alpha y &= \frac{\partial y}{\partial r} \nabla_\alpha r + \frac{\partial y}{\partial \phi} \nabla_\alpha \phi \\ &= \sin \phi \nabla_\alpha r + r \cos \phi \nabla_\alpha \phi. \end{aligned} \quad (1.195)$$

Then

$$\begin{aligned} \eta_{\alpha\beta} &= -\nabla_\alpha t \nabla_\beta t + \nabla_\alpha x \nabla_\beta x + \nabla_\alpha y \nabla_\beta y \\ &= -\nabla_\alpha t \nabla_\beta t + (\cos \phi \nabla_\alpha r - r \sin \phi \nabla_\alpha \phi)(\cos \phi \nabla_\beta r - r \sin \phi \nabla_\beta \phi) \\ &\quad + (\sin \phi \nabla_\alpha r + r \cos \phi \nabla_\alpha \phi)(\sin \phi \nabla_\beta r + r \cos \phi \nabla_\beta \phi) \\ &= -\nabla_\alpha t \nabla_\beta t + (\cos^2 \phi + \sin^2 \phi) \nabla_\alpha r \nabla_\beta r + \underbrace{(-r \sin \phi \cos \phi + \sin \phi r \cos \phi)}_0 \nabla_\alpha r \nabla_\beta \phi \\ &\quad + r^2 (\sin^2 \phi + \cos^2 \phi) \nabla_\alpha \phi \nabla_\beta \phi + \underbrace{(-r \sin \phi \cos \phi + r \sin \phi \cos \phi)}_0 \nabla_\alpha \phi \nabla_\beta r, \end{aligned} \quad (1.196)$$

giving

$$\eta_{\alpha\beta} = -\nabla_\alpha t \nabla_\beta t + \nabla_\alpha r \nabla_\beta r + r^2 \nabla_\alpha \phi \nabla_\beta \phi. \quad (1.197)$$

Thus

$$\eta_{tt} = -1, \quad \eta_{rr} = 1, \quad \eta_{\phi\phi} = r^2, \quad \text{and} \quad \eta_{\mu\nu} = 0, \quad \mu \neq \nu.$$

One could also have found $\eta_{\mu\nu}$ by, e.g.,

$$\begin{aligned} \eta_{\phi\phi} &= \frac{\partial t}{\partial \phi} \frac{\partial t}{\partial \phi} \eta_{tt} + \frac{\partial t}{\partial \phi} \frac{\partial x}{\partial \phi} \eta_{tx} + \cdots + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \phi} \eta_{yy} \\ &= \left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2 = r^2 \sin^2 \phi + r^2 \cos^2 \phi = r^2, \end{aligned}$$

but the calculation of (1.197) is more efficient because only nonzero components $\eta_{\mu\nu}$ appear.

The following notation makes coordinate calculations look simpler and more natural: Write

$$\text{Write } dx^\mu \quad \text{for} \quad \nabla_\alpha x^\mu = \omega^\mu_\alpha. \quad (1.198)$$

Then, writing ∂_μ for e_μ^α , we have $dx^\mu(\partial_\nu) = \delta_\nu^\mu$; the metric tensor $\eta_{\alpha\beta} = \eta_{\mu\nu} \nabla_\alpha x^\mu \nabla_\beta x^\nu$ is $\eta_{\mu\nu} dx^\mu dx^\nu$ and is abbreviated ds^2 :

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \quad (1.199)$$

The change of basis resulting from $x^\mu \rightarrow x^{\mu'}$ is written

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu \quad \partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu,$$

and the change in metric components is then

$$\begin{aligned} ds^2 &= \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = \eta_{\mu\nu} dx^\mu dx^\nu \\ &= \left(\eta_{\mu\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \right) dx^{\mu'} dx^{\nu'}. \end{aligned}$$

Eq. (1.196) takes the form

$$ds^2 = -dt^2 + dx^2 + dy^2, \quad (1.200)$$

Eqs. (1.194) and (1.195) become

$$\begin{aligned} dx &= \cos \phi \, dr - r \sin \phi \, d\phi \\ dy &= \sin \phi \, dr + r \cos \phi \, d\phi, \end{aligned} \quad (1.201)$$

and, by substituting (1.201) into (1.200), we obtain

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2. \quad (1.202)$$

Thus, by identifying the “infinitesimals” dx^i that had no precise meaning (by themselves) with the coordinate basis vectors $\nabla_\alpha x^i$ – the gradients of the scalar fields x^i – the intuitively clear but mathematically ill-defined equations (1.200)-(1.202) are given a precise meaning: they are the tensor equations (1.194)-(1.197).

An occasionally encountered notation for the contravariant metric is

$$\partial_s^2 = -\partial_t^2 + \partial_r^2 + \frac{1}{r^2} \partial_\phi^2.$$

Christoffel symbols: Because the only non-constant metric component in (1.202) is $\eta_{\phi\phi} = r^2$, the only nonzero components $\Gamma_{\lambda\nu\mu}$ are

$$\begin{aligned} \Gamma_{r\phi\phi} - \frac{1}{2} \partial_r \eta_{\phi\phi} &= -r, \\ \Gamma_{\phi r\phi} = \Gamma_{\phi\phi r} &= \frac{1}{2} \partial_r \eta_{\phi\phi} = r, \end{aligned} \quad (1.203)$$

and the only nonzero $\Gamma^\lambda_{\nu\mu}$ ’s are

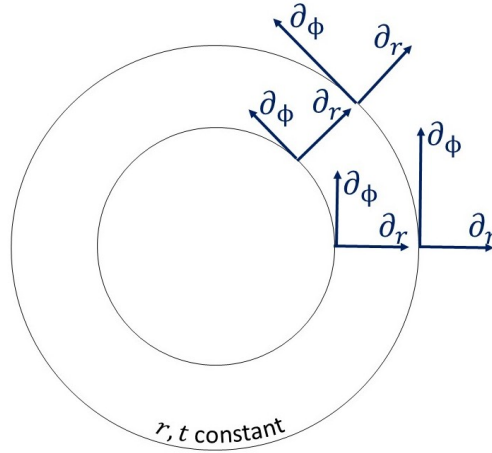
$$\Gamma^r_{\phi\phi} = -r \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \eta^{\phi\phi} \Gamma_{\phi r\phi} = \frac{1}{r}. \quad (1.204)$$

To see what (1.204) means, look at the basis vectors ∂_r and ∂_ϕ in an r - ϕ plane (a $t = \text{constant}$ plane):

$$\partial_r \text{ is a unit vector: } \sqrt{\eta(\partial_r, \partial_r)} = \sqrt{\eta_{rr}} = 1 \quad (1.205)$$

$$\partial_\phi \text{ has length } r: \sqrt{\eta(\partial_\phi, \partial_\phi)} = \sqrt{\eta_{\phi\phi}} = r \quad (1.206)$$

That is, if $\hat{\phi}$ is a unit vector in the ∂_ϕ direction, $\hat{\phi}(f) = \hat{\phi} \cdot \nabla f = \frac{1}{r} \partial_\phi f$, whereas $\partial_\phi(f) = \partial_\phi f$, so $\partial_\phi = r \hat{\phi}$.


 Figure 1.13: The basis vectors ∂_r and ∂_ϕ

Along a curve of constant r and ϕ , when only t changes, ∂_ϕ and ∂_r don't change, so $\nabla_t \partial_\phi = \nabla_t \partial_r = 0$, implying $\Gamma^\lambda_{\phi t} = 0$, $\Gamma^\lambda_{rt} = 0$.

The r - ϕ plane is more interesting: Along a curve of constant ϕ and t , ∂_r is constant but ∂_ϕ changes its length: Since the length of ∂_ϕ is r , and along a $\phi = \text{constant}$ curve the direction of ∂_ϕ doesn't change,

$$\partial_\phi|_{r+\Delta r} = \frac{r+\Delta r}{r} \partial_\phi|_r$$

$$\nabla_r \partial_\phi = \lim_{\Delta r \rightarrow 0} (\partial_\phi|_{r+\Delta r} - \partial_\phi|_r) \frac{1}{\Delta r} = \frac{1}{r} \partial_\phi.$$

Then

$$\Gamma^\phi_{r\phi} = \frac{1}{r} \quad \text{and} \quad \Gamma^r_{r\phi} = \Gamma^t_{r\phi} = 0.$$

We can similarly find $\nabla_\phi \partial_r$ and $\nabla_\phi \partial_\phi$: Along an $r, t = \text{constant}$ curve, the lengths of ∂_ϕ and ∂_r are constant but their directions change – see Fig. 1.13.

$$\partial_\phi|_{\phi+\Delta\phi} = \cos \Delta\phi \partial_\phi|_\phi - r \sin \Delta\phi \partial_r|_\phi$$

$$\partial_r|_{\phi+\Delta\phi} = \cos \Delta\phi \partial_r|_\phi + \frac{1}{r} \sin \Delta\phi \partial_\phi|_\phi$$

$$\nabla_\phi \partial_\phi = \lim_{\Delta\phi \rightarrow 0} \frac{1}{\Delta\phi} [\partial_\phi|_{\phi+\Delta\phi} - \partial_\phi|_\phi] = -r \partial_r$$

$$\nabla_\phi \partial_r = \lim_{\Delta\phi \rightarrow 0} \frac{1}{\Delta\phi} [\partial_r|_{\phi+\Delta\phi} - \partial_r|_\phi] = \frac{1}{r} \partial_\phi$$

$$\begin{aligned} \Rightarrow \Gamma^r_{\phi\phi} &= -r & \Gamma^t_{\phi\phi} &= \Gamma^\phi_{\phi\phi} = 0 \\ \Gamma^\phi_{\phi r} &= \frac{1}{r} & \Gamma^t_{\phi r} &= \Gamma^r = \Gamma^r_{\phi r} = 0. \end{aligned}$$

Example 2: A spherical null coordinate system for Minkowski space: The coordinates are u, r, θ, ϕ .

$$u = t - (x^2 + y^2 + z^2)^{1/2}$$

$$t = u + r$$

$$r = (x^2 + y^2 + z^2)^{1/2}$$

$$x = r \sin \theta \cos \phi$$

$$\theta = \tan^{-1} \left[\frac{(x^2 + y^2)^{1/2}}{z} \right]$$

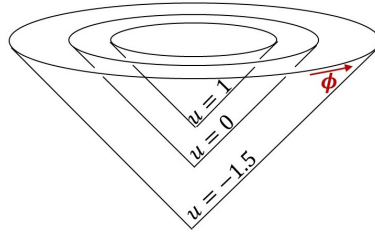
$$y = r \sin \theta \sin \phi$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$z = r \cos \theta,$$

where, of course, t, x, y, z is a natural coordinate system.

Here's a diagram of a $\theta = \text{constant}$ hypersurface with some constant- u light cones.



Notice that the $u = \text{constant}$ surfaces are not orthogonal to the $r = \text{constant}$ surfaces. This implies that the basis vectors ∂_u and ∂_r are not orthogonal, because ∂_u is tangent to a line of constant r, θ, ϕ and ∂_r is a null vector tangent to a line of constant u, θ, ϕ .

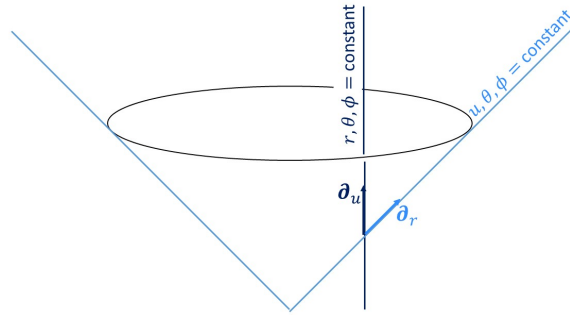


Figure 1.14: The basis vectors ∂_u and ∂_r

On the other hand, ∂_θ and ∂_ϕ are still orthogonal, because the $\theta = \text{constant}$ and $\phi = \text{constant}$ surfaces are orthogonal.

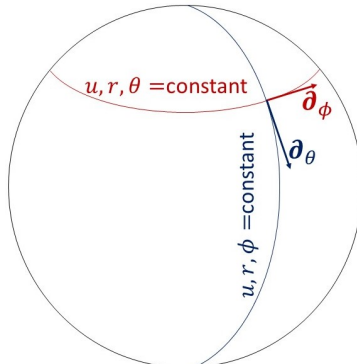


Figure 1.15: The basis vectors ∂_θ and ∂_ϕ

Because ∂_u and ∂_r are not orthogonal, the tr component of the metric is nonzero:

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\ &= -(du + dr)^2 + (dr \sin \theta \cos \phi + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi)^2 \\ &\quad + (dr \sin \theta \sin \phi + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi)^2 + (dr \cos \theta - r \sin \theta d\theta)^2; \end{aligned}$$

collecting terms and using $\cos^2 + \sin^2 = 1$ a few times gives

$$ds^2 = -du^2 - 2dudr + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (1.207)$$

or

$$\|\eta_{\mu\nu}\| = \begin{vmatrix} -1 & -1 & & \\ -1 & 0 & & \\ & & r^2 & \\ & & & r^2 \sin^2 \theta \end{vmatrix}. \quad (1.208)$$

Note that the off-diagonal components η_{ur} and η_{ru} are -1 , not -2 : Eq. (1.207) is a shorthand for

$$\eta_{\alpha\beta} = -\nabla_\alpha u \nabla_\beta u + \nabla_\alpha u \nabla_\beta r + \nabla_\alpha r \nabla_\beta u + r^2 \nabla_\alpha \theta \nabla_\beta \theta + r^2 \sin^2 \theta \nabla_\alpha \phi \nabla_\beta \phi; \quad (1.209)$$

that is, $dudr$ means the symmetric tensor $\frac{1}{2}(\nabla_\alpha u \nabla_\beta r + \nabla_\alpha r \nabla_\beta u)$. The contravariant metric components are

$$\|\eta^{\mu\nu}\| = \begin{vmatrix} 0 & -1 & & \\ -1 & 1 & & \\ & & r^{-2} & \\ & & & r^{-2} \sin^{-2} \theta \end{vmatrix}. \quad (1.210)$$

The basis vectors ∂_u , ∂_r , ∂_θ and ∂_ϕ have lengths $|\partial_u|^2 = |\eta(\partial_u, \partial_u)| = |\eta_{uu}| = 1$ or

$$\begin{aligned} |\partial_u| &= \sqrt{|\eta_{uu}|} = 1, \\ |\partial_r| &= \sqrt{|\eta_{rr}|} = 0, \\ |\partial_\theta| &= \sqrt{|\eta_{\theta\theta}|} = r, \\ |\partial_\phi| &= \sqrt{|\eta_{\phi\phi}|} = r \sin \theta. \end{aligned}$$

The covariant basis vectors have lengths

$$|du| = \sqrt{|\eta(du, du)|} = \sqrt{|\eta^{\alpha\beta} \nabla_\alpha u \nabla_\beta u|} = \sqrt{|\eta^{uu}|} = 0$$

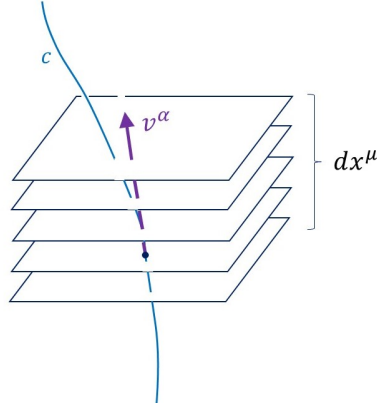
$$|dr| = \sqrt{\eta^{rr}} = 1$$

$$|d\theta| = \sqrt{\eta^{\theta\theta}} = \frac{1}{r}$$

$$|d\phi| = \sqrt{\eta^{\phi\phi}} = \frac{1}{r \sin \theta}$$

They can be visualized in either of two ways:

1. To picture dx^μ , draw a set of $x^\mu = \text{constant}$ surfaces a given coordinate distance Δx^μ apart. When dx^μ is large, the surfaces will be close together, and when dx^μ is small, far apart. If v^α is a vector tangent to some curve $c(\lambda)$, then $v^\mu = v^\alpha \nabla_\alpha x^\mu = \frac{d}{d\lambda} x^\mu(c(\lambda))$ is the rate at which a coordinate distance x^μ is crossed in parameter length λ along c . So if you draw v^α as an arrow with length proportional to the components v^μ , then its length will be proportional to the number of surfaces the arrow crosses.



The natural way to visualize the gradient $df = \nabla f$ of a function is then as a set of planes tangent to $f = \text{constant}$ surfaces, and it is presented that way in Schutz pp.61-2 and in MTW 2.5 and 2.6.

2. You can raise the index of the naturally covariant vector $\nabla_\alpha x^\mu$ and then draw the resulting contravariant vector $\nabla^\alpha x^\mu$ as an arrow. With an orthogonal basis and Euclidean metric, the result is an arrow that looks perpendicular to the surface in a diagram. Here, without some practice, the contravariant vector often looks peculiar, dramatizing the fact that, for any function f , $\nabla_\alpha f$ is naturally a covariant vector.

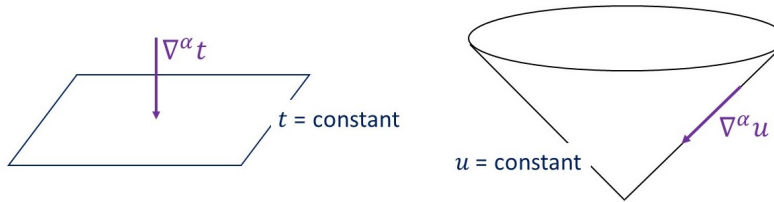


Figure 1.16: The vectors $\nabla^\alpha t$ and $\nabla^\alpha u$ acquired by raising the naturally covariant vectors $\nabla_\alpha t$ and $\nabla_\alpha u$.

In the first example, despite the fact that t increases to the future, $\nabla^\alpha t$ is a past-pointing vector. This is due to the $-+++$ signature of the Minkowski metric: The t, x, y, z components of $\nabla^\alpha t$ are $(\nabla^\mu t) = (-1, 0, 0, 0)$.

In the second example, $\nabla_\alpha u$ is null: $\eta^{\alpha\beta} \nabla_\alpha u \nabla_\beta u = 0$, and we can see as follows that the raised normal to a null surface, $\nabla^\alpha u$ is a vector that lies in the surface! Start with the fact that, for any scalar f , if v is a vector orthogonal to ∇f (i.e., $v^\alpha \nabla_\alpha f = 0$), then f is constant along v : That is, v^α lies in a surface of constant f . Here, with $f = u$ and $v^\alpha = \nabla^\alpha u$, the relation $\nabla^\alpha u \nabla_\alpha u = 0$ implies $\nabla^\alpha u$ is a vector lying in the $u = \text{constant}$ surface.

Check:

$$\nabla^\mu u = \eta^{\mu\nu} \nabla_\nu u = \eta^{\mu u} = -\delta_r^\mu,$$

where we have used Eq. (1.210) for the contravariant components $\eta^{\mu\nu}$ in u, r, θ, ϕ coordinates.

So, in these coordinates, $\nabla^\alpha u$ is $-\partial_r$.

Exercise 32. Consider 2-dimensional Minkowski space, with natural coordinates t and x . A family of uniformly accelerated observers follow the trajectories

$$t(T) = X \sinh T, \quad x(T) = X \cosh T,$$

with X fixed for a given observer.

- Show that, at each point, the curve with T fixed and X changing is orthogonal to the accelerated trajectory, the curve with X fixed and T changing.
- Show that the coordinates T, X cover only half of 2-dimensional Minkowski space. Show that the coordinates are not defined on the lines $x = \pm t$, so they really cover two disjoint quadrants. The right-hand quadrant in these coordinates is called *Rindler space*. Show that the metric in these coordinates, is $ds^2 = -X^2 dT^2 + dX^2$.
- The boundary line $x = t$ is analogous to a black hole horizon for uniformly accelerated observers: Show that signals from the far side of that line never cross it and therefore never reach observers who remain in Rindler space. (Observers at rest relative to a Schwarzschild black hole must uniformly accelerate to remain at rest; they are in this way analogous to our Rindler observers. The analogy is part of an equivalence-principle relation between Rindler space and a black hole spacetime.)

Exercise 33. a. Show that $\epsilon_{ijk} A^i B^j C^k = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \begin{vmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{vmatrix}$.

- Using part (a), write the determinant of a 3×3 matrix a_{ij} in terms of ϵ_{ijk} .
- Show for any covector A_a that $\nabla_a A_b - \nabla_b A_a$ has components $\partial_i A_j - \partial_j A_i$ in any chart.
- Similarly show, for any antisymmetric tensor $A_{a...b}$, that the components of $\nabla_{[a} A_{b...c]}$ are simply $\partial_{[i} A_{j...k]}$.

Exercise 34. Let g_{ab} be the metric of flat 3-space, $ds^2 = dx^2 + dy^2 + dz^2$. Let $\{e_{i'}\}$ be any basis, and write the relation between $\{e_{i'}\}$ and the natural coordinate basis $\{e_i\}$ in the usual way, $e_{i'} = e_i a^i_{i'}$.

- Show that the determinant g of the metric has the value $g = [\det(a)]^2$ in the primed basis.
- Show that $\epsilon_{i'j'k'} = \det(a) \epsilon_{ijk}$, and conclude that $\epsilon_{i'j'k'} = \sqrt{g}$. In other words, dropping the primes, in an arbitrary basis $\{e_i\}$, $\epsilon_{123} = \sqrt{g}$.
- What is ϵ_{ijk} in an arbitrary basis, in terms of g ?

Exercise 35. The metric of 3-dimensional Euclidean space in Cartesian coordinates is $\|g_{ij}\| = \|\delta_{ij}\|$ or $ds^2 = dx^2 + dy^2 + dz^2$.

- Find the metric in spherical coordinates r, θ, φ .
- Find the components of the contravariant metric g^{ab} in spherical coordinates.
- What is the value of $\epsilon_{r\theta\varphi}$? Of $\epsilon^{r\theta\varphi}$?
- Find all nonzero Christoffel symbols Γ^i_{jk} .
- Find the volume of a sphere by writing $V = \int \sqrt{g} dr d\theta d\varphi$.

Exercise 36. Divergence and Laplacian (still in 3-d).

- a. Show that the divergence $\nabla_a A^a$ of a vector can be written in the form

$$\frac{1}{2} \epsilon^{abc} \nabla_a (\epsilon_{bcd} A^d).$$

- b. Using the fact that this expression involves the derivative $\nabla_{[a} T_{bc]}$ of the antisymmetric tensor $T_{bc} = \epsilon_{bcd} A^d$, show that the divergence can be written in any chart in the form

$$\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} A^i).$$

- c. Conclude that the Laplacian $\nabla_\alpha \nabla^\alpha f$ can be written in any chart in the form

$$\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j f).$$

- d. Write the Laplacian in spherical and cylindrical coordinates. Cylindrical coordinates are often written ϖ, z, ϕ in astrophysics. “ ϖ ” is script π , used instead of ρ to avoid confusion with density. The relation to Cartesian coordinates is $x = \varpi \cos \phi$, $y = \varpi \sin \phi$.

Exercise 37. Short answer questions:

- a. What are the components of a flat 3-dimensional metric in the orthonormal frame $\mathbf{e}_r \equiv \hat{\mathbf{r}}$, $\mathbf{e}_\theta \equiv \hat{\boldsymbol{\theta}}$, $\mathbf{e}_\phi \equiv \hat{\boldsymbol{\phi}}$.
- b. What are the components $\epsilon_{\hat{i}\hat{j}\hat{k}}$ in that frame?
- c. Show that the redshift from a source of light moving away from you at speed v has the nonrelativistic limit $z \equiv \frac{\Delta\lambda}{\lambda} = v$.
- d. Show that the sourcefree Maxwell equation $\nabla_{[\alpha} F_{\beta\gamma]}$ is equivalent to

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0.$$

Chapter 2

Curved spacetime

2.1 Manifolds, Smooth Functions (Scalar Fields), Tensor Fields

A smooth n -dimensional manifold can be thought of as a smooth n -dimensional subset of some $\mathbb{R}^m$, $m \geq n$. For example a curve in $\mathbb{R}^m$ is a 1-dimensional manifold; any S^n is a smooth n -dimensional subset of $\mathbb{R}^{n+1}$; on the other hand, a self intersecting surface is not a manifold, and a curve with a cusp is a manifold, but not a smooth manifold. The embedding space $\mathbb{R}^m$, however, plays no real role here, and n -dimensional manifolds are defined intrinsically by piecing together open blocks of $\mathbb{R}^n$.

For example, a closed strip can be made from two pieces of $\mathbb{R}^2$, each one the piece $\begin{matrix} 0 < x < 4 \\ 0 < y < 1 \end{matrix}$, by gluing the squares of the same color with two different identifications:

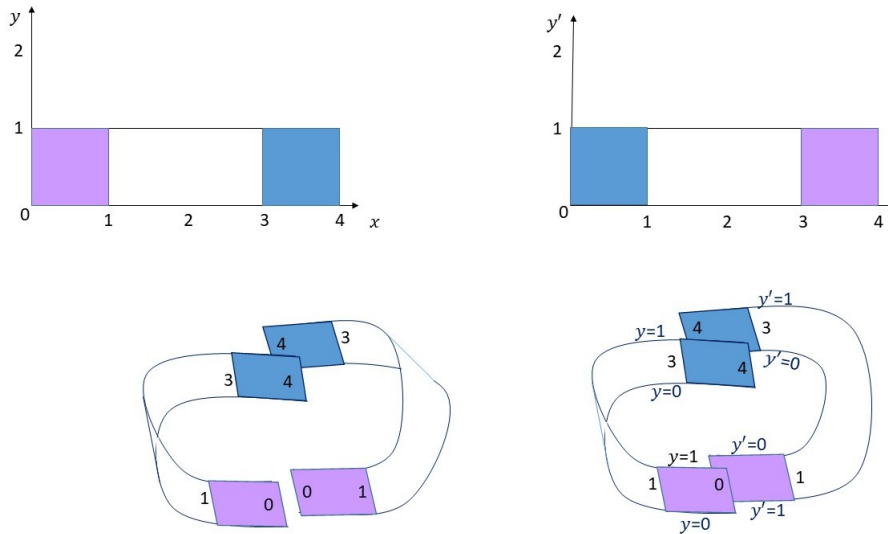


Figure 2.1: Two different gluings of the two pieces of $\mathbb{R}^2$ give an orientable strip and a Möbius strip.

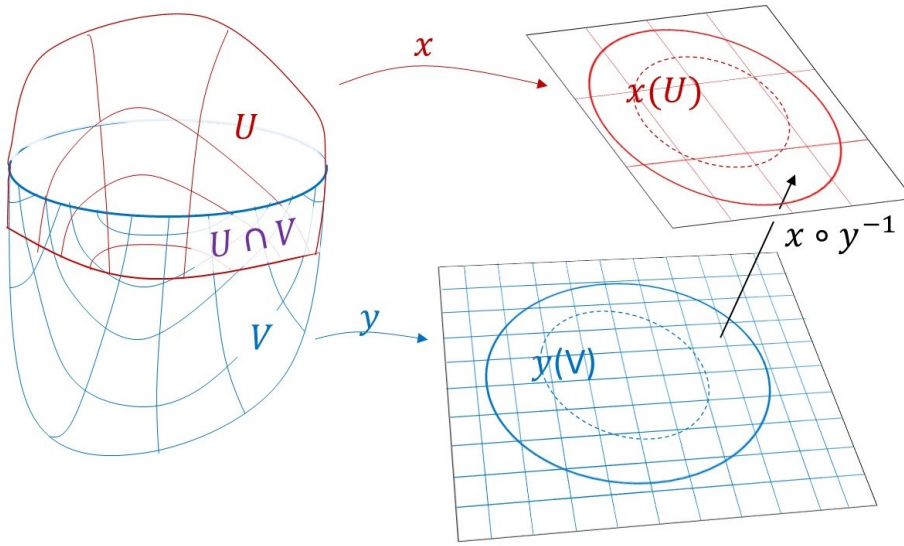
For the ordinary (orientable) strip, glue the point (x, y) to the point (x', y') where

$$\begin{aligned} (x', y') &= (1 - x, y), & 0 < x < 1, & 0 < y < 1 \\ (x', y') &= (7 - x, y), & 3 < x < 4, & 0 < y < 1 \end{aligned}$$

Similarly, a Möbius strip is made by the overlap maps (gluing instructions)

$$\begin{aligned} (x', y') &= (1 - x, -y), & 0 < x < 1, 0 < y < 1 \\ (x', y') &= (7 - x, y), & 3 < x < 4, 0 < y < 1 \end{aligned}$$

More formally, one defines a manifold M by requiring that each point P of M be in the domain U of some coordinate patch $x : U \rightarrow \mathbb{R}^n$ whose range $x(U)$ is an open set of $\mathbb{R}^n$, and that in the regions of $\mathbb{R}^n$ corresponding to the intersection $U \cap V$ of two coordinate patches $x : U \rightarrow \mathbb{R}^n$, $y : V \rightarrow \mathbb{R}^n$, the maps $x \circ y^{-1}$ and $y \circ x^{-1}$ are differentiable inverses of each other. A smooth function (scalar field) on



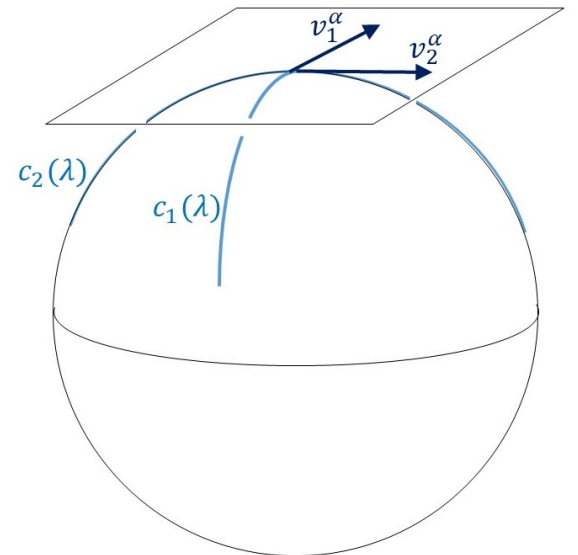
f is a map from $M \rightarrow \mathbb{R}$ smooth in each coordinate system. That is, $f : M \rightarrow \mathbb{R}$ is smooth at P if $f \circ x^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth at $x(P)$.

Mathematicians use the word “chart” instead of “coordinate system” because it has one instead of six syllables and because of the ordinary language/technical language statement that a chart is a map. “Chart” will be used in the notes because it’s shorter.

Vectors

In Minkowski space a vector v^α at a point P could simply be identified with the directed line from P to the point Q with $Q^\alpha - P^\alpha = v^\alpha$. In a curved space, if one embeds the manifold in $\mathbb{R}^m$ (see picture at right), the tangent to a curve in M , regarded as a curve in $\mathbb{R}^m$, is an arrow that sits not in M , but in the (hyper)plane tangent to M at P .

The figure shows a sphere embedded in $\mathbb{R}^3$. The tangent vectors v_1^α and v_2^α to the curves c_1 and c_2 (regarded as curves in $\mathbb{R}^3$) lie in the tangent plane $T_P S$ to S at P .



If we wanted to carry the embedding space along, we could identify the tangent vector with this arrow in $\mathbb{R}^m$; but the macroscopic universe in which we live is 4-dimensional, so we are handed M , not the embedding, and keeping $\mathbb{R}^m$ needlessly encumbers the mathematics.

Instead, as in Eq. (1.59), one identifies the tangent to a curve $c(\lambda)$ at P with the linear map from $\{\text{smooth functions on } M\}$ to $\mathbb{R}$ given by

$$\mathbf{v} : f \mapsto \mathbf{v}(f) = v^a \nabla_a f = \left. \frac{d}{d\lambda} f[c(\lambda)] \right|_{\lambda=0}. \quad (2.1)$$

As in flat space, the components of v^a in a chart $\{x^i\}$, are just $v^i(P) = \left. \frac{d}{d\lambda} x^i(\lambda) \right|_{\lambda=0}$.

Then $v^a \nabla_a f = v^i \partial_i f$, and the i^{th} component of v^a is

$$v^i = \mathbf{v}(x^i), \text{ because } \mathbf{v}(x^i) = v^j \partial_j x^i = v^j \delta_j^i = v^i. \quad (2.2)$$

A vector is a linear map from functions to $\mathbb{R}$, but the definition has one more piece to it, because if f and g are two scalar fields we have the Leibnitz rule

$$v^a \nabla_a (fg) = v^a \nabla_a f g(P) + f(P) v^a \nabla_a g. \quad (2.3)$$

With that included, we can define the space of vectors at a point of a smooth manifold M .

Definition. A vector V at a point P in M is a linear map from $\{\text{smooth functions on } M\}$ to $\mathbb{R}$ satisfying $V(fg) = V(f)g(P) + f(P)V(g)$ (the Leibnitz rule).

We had now better check that, with this definition, every vector can be written as the tangent to a curve.

Theorem. If V is any linear map from $\{\text{smooth functions on } M\}$ to $\mathbb{R}$ satisfying $V(fg) = V(f)g(P) + f(P)V(g)$ (the Leibnitz rule), then there is a curve $c(\lambda)$ through P whose tangent v^a is the map V .

Proof. First note that $V(k) = 0$ for any constant k (regarded as a constant function on M). This is quick: Given any f ,

$$\begin{aligned} V(kf) &= kV(f) \text{ by linearity} \quad \text{and} \\ V(kf) &= kV(f) + f(P)V(k) \text{ by Leibnitz} \\ \Rightarrow f(P)V(k) &= 0, \text{ all } f \Rightarrow V(k) = 0. \end{aligned}$$

Next, let x^i be a chart about P with $x^i(P) = 0$. Define n numbers v^i by

$$v^i = V(x^i). \quad (2.4)$$

We need first to write $V(f) = v^i \frac{\partial f}{\partial x^i} \Big|_P$, and then to show that the v^i are components of the tangent to some curve c . For the first part, we expand f in a Taylor series about P :

$$f = f(P) + \frac{\partial f}{\partial x^i} \Big|_P x^i + O(|x|^2), \quad (2.5)$$

where $O(|x|^2)$ means some function for which $|O(|x|^2)| \leq K|x|^2$, some constant K . (This is the formal way of stating the intuitive idea that $O(|x|^2)$ goes to zero as fast as $|x|^2$.) Because f is smooth, we can

write $O(|x|^2)$ as $f_{ij}x^ix^j$, with each $f_{ij}(x)$ a smooth function. Now the value $f(P)$ of f at the given point P is a constant, so

$$V(f) = \left. \frac{\partial f}{\partial x^i} \right|_P V(x^i) + V(O(|x|^2)). \quad (2.6)$$

The first term on the right is the term we want, because $V(x^i) = v^i$. We need to show that the other term vanishes at P , where $x^i = 0$. We have

$$V(O(|x|^2)) = V[f_{ij}x^ix^j] = V(f_{ij})x^i(P)x^j(P) + f_{ij}(P)V(x^i)x^j(P) + f_{ij}(P)x^i(P)V(x^j) = 0, \text{ because } x^i(P) = 0.$$

Thus (2.6) becomes

$$V(f) = \left. \frac{\partial f}{\partial x^i} \right|_P v^i.$$

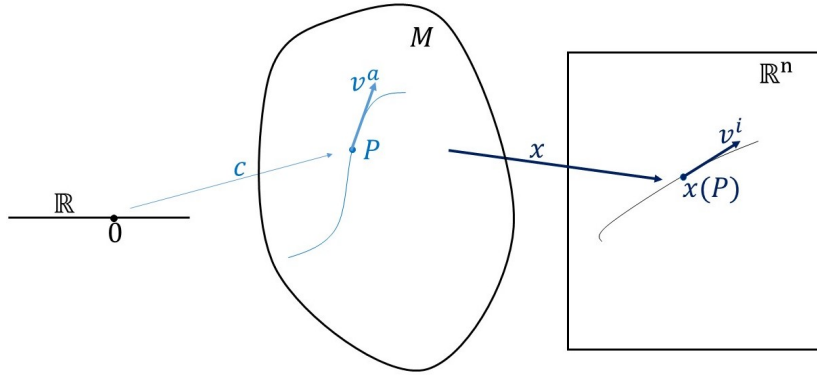
Finally, we need a curve c whose tangent v^a has components v^i , and this is easy: Let c be the path with coordinates

$$x^i(\lambda) = \lambda v^i. \quad (2.7)$$

Then

$$\left. \frac{d}{d\lambda} f(c(\lambda)) \right|_{\lambda=0} = \left. \frac{\partial f}{\partial x^i} \right|_P \left. \frac{dx^i}{d\lambda} \right|_{\lambda=0} = \left. \frac{\partial f}{\partial x^i} \right|_P v^i, \quad (2.8)$$

and we have proved that the tangent v^a to the path c is the map $v^a \nabla_a f = V(f)$.



Technical Note

As is endemic to physicists, we have sluffed over the distinction between the function $f : M \rightarrow \mathbb{R}$ and “ $f(x)$ ”, the function $f \circ x^{-1} : \mathbb{R}^4 \rightarrow \mathbb{R}$. To be more precise one should have said that $f \circ x^{-1}$ – call it $\bar{f}$ – can be Taylor expanded about 0 in $\mathbb{R}^n$. Denoting a point in $\mathbb{R}^n$ by r^i , we could write

$$\bar{f}(r) = \bar{f}(0) + \partial_i \bar{f}|_0 r^i + O(|r|^2).$$

Then $f = \bar{f} \circ x$ so if Q is a point in M near P ,

$$f(Q) = \bar{f}[x^i(Q)] + \partial_i \bar{f}|_0 x^i(Q) + O(|x(Q)|^2).$$

But $\bar{f}[x^i(Q)] = f(Q)$ and $\partial_i \bar{f}|_0$ is $e_i^a \nabla_a f|_P$, where e_i^a is tangent to the curve with coordinates $\lambda \rightarrow (x^1(P), \dots, x^i(P) + \lambda, \dots, x^n(P))$. So denoting $e_i^a \nabla_a f|_P$ by the shorthand $\partial_i f|_P$ we recover $\left. \frac{d}{d\lambda} f \circ c(\lambda) \right|_{\lambda=0} = v^i \partial_i f|_P$. In the same formal spirit, note that $x^i(\lambda) = x^i \circ c(\lambda)$ and $v^i = \frac{d}{d\lambda} x^i(\lambda)$ is the tangent in $\mathbb{R}^n$ to the curve $x^i(\lambda)$ in $\mathbb{R}^n$, as shown in the figure above. (Normally, we will use the convenient abuses of notation.)

We have shown that the set of tangent vectors to curves through a point P can be identified with the set of *all* linear maps at P satisfying Leibnitz. This space of tangent vectors at P with addition and scalar multiplication defined by

$$(\mathbf{u} + \mathbf{v})(f) = \mathbf{u}(f) + \mathbf{v}(f), \quad (k\mathbf{u})(f) = k\mathbf{u}(f)$$

is the *tangent space* to M at P . The tangent space is a vector space: We need only show closure under addition and scalar multiplication:

$$\begin{aligned} (\mathbf{u} + \mathbf{v})(fg) &= \mathbf{u}(fg) + \mathbf{v}(fg) \\ &= \mathbf{u}(f)g(P) + \mathbf{u}(g)f(P) + \mathbf{v}(f)g(P) + \mathbf{v}(g)f(P) \\ &= (\mathbf{u} + \mathbf{v})(f)g(P) + (\mathbf{u} + \mathbf{v})(g)f(P) \\ (k\mathbf{v})(fg) &= k(\mathbf{v}(fg)) = k(\mathbf{v}(f)g(P) + \mathbf{v}(g)f(P)) = (k\mathbf{v})(f)g(P) + (k\mathbf{v})(g)f(P). \end{aligned}$$

As in Minkowski space, a chart $x : U \rightarrow \mathbb{R}^n$ provides a natural basis for the tangent space at each point P of U , namely the tangent vectors e_i^a to the curves $\lambda \rightarrow (x^1, \dots, x^i + \lambda, \dots, x^n)$; again

$$e_i^a \nabla_a f = \frac{d}{d\lambda} f(x^1, \dots, x^i + \lambda, \dots, x^n) = \partial_i f|_P, \quad (2.9)$$

and a coordinate basis vector e_i is again written ∂_i .

The introduction of covectors and tensors at P and of tensor fields on M now proceeds in a manner identical to that used in Minkowski space. **Covectors at P are linear maps from vectors at P to numbers:** $\sigma_a : v^a \rightarrow \sigma_a v^a$. **Tensors are multilinear maps from covectors and vectors to numbers**

$$T^{a \dots b}_{c \dots d} : (\sigma_a, \dots, \tau_b, u^c, \dots, v^d) \rightarrow T^{a \dots b}_{c \dots d} \sigma_a \dots \tau_b u^c \dots v^d. \quad (2.10)$$

A tensor field assigns to each point P of M a tensor $T^{a \dots b}_{c \dots d}(P)$ at P .

δ^a_b is now the tensor field that takes

$$(\sigma_a, u^b) \text{ to } \delta^a_b \sigma_a u^b = \sigma_a u^a, \quad (2.11)$$

again, it has components δ^i_j in any basis for the tangent space at any point P . A basis at each P of M (a field of bases) will be called a basis on M .

Just as the tangent to a curve is naturally a contravariant vector, the gradient of a scalar is naturally a covariant vector. Just as any vector v^a at a point P can be written as the tangent vector to some curve with $c(0) = P$, any covector σ_a at P is the gradient of some function f defined near P : $\sigma_a(P) = \nabla_a f(P)$.

$$\text{Then } v^a \sigma_a = \left. \frac{d}{d\lambda} f[c(\lambda)] \right|_{\lambda=0}.$$

Smooth

A vector field is smooth (C^∞) if $v^a \nabla_a f$ is smooth for all smooth f . A tensor field is smooth if $T^{a \dots b}_{c \dots d} \sigma_a \dots \tau_b u^c \dots v^d$ is smooth for all smooth $\sigma_a, \dots, \tau_b, u^c, \dots, v^d$, where, in particular, σ_a is smooth if $\sigma_a v^a$ (i.e. the map $P \rightarrow \sigma_a(P) v^a(P)$) is smooth for all smooth v^a .

As we noted, a function f provides a covector field $\nabla_a f$ given by

$$\begin{aligned} \nabla_a f|_P v^a(P) &= \left. \frac{d}{d\lambda} f(c(\lambda)) \right|_{\lambda=0} \\ &= v^a \nabla_a f|_P \text{ where } c(\lambda) \text{ has tangent } v^a \text{ at } P. \end{aligned} \quad (2.12)$$

A chart x again provides the basis

$$\omega^i_a = \nabla_a x^i$$

dual to the coordinate basis $e_i = \partial_i$, of (2.9), and the discussion of coordinate components given for Minkowski space in Sect. 1.7.1 goes through as written.

Finally, there is an equivalent and sometimes more useful way to describe a tensor field:

A tensor field $T^{a\cdots b}_{c\cdots d}$ is a map from vector fields and covector fields to functions

$$T^{a\cdots b}_{c\cdots d} : (\sigma_a, \cdots, \tau_b, u^c \cdots v^d) \mapsto T^{a\cdots b}_{c\cdots d} \sigma_a \cdots \tau_b u^c \cdots v^d$$

that is linear in each argument under addition and under multiplication by scalar fields. That is, if $T^{a\cdots b}_{c\cdots d}$ is a tensor field, obviously

$$T^{a\cdots b}_{c\cdots d}(f\sigma_a + g\hat{\sigma}_a) \cdots \tau_b u^c \cdots v^d = fT^{a\cdots b}_{c\cdots d}\sigma_a \cdots \tau_b u^c \cdots v^d + gT^{a\cdots b}_{c\cdots d}\hat{\sigma}_a \cdots \tau_b u^c \cdots v^d. \quad (2.13)$$

And if T is any multilinear map (in this strong sense of scalar field multiplication), then T is a tensor field: e.g. let T be a map from vector fields to functions with $T(f\mathbf{u} + g\mathbf{v}) = fT(\mathbf{u}) + gT(\mathbf{v})$. Then the covector field T_a to be identified with T has the value $T_a(P)$ at P given by $T_a(P)v^a(P) = T(\mathbf{v})(P)$. The subtlety is to show that $T_a(P)$ is well defined: if v^a and $\hat{v}^a$ are any vector fields which agree at P , $v^a(P) = \hat{v}^a(P)$, then $T_a(P)v^a(P) = T_a(P)\hat{v}^a(P)$. Thus the definition will work only if $T(v)(P) = T(\hat{v})(P)$ whenever $v^a(P) = \hat{v}^a(P)$. And this follows from the strong linearity: write $v^a - \hat{v}^a$ in terms of a basis

$$\mathbf{v} - \hat{\mathbf{v}} = \underbrace{(v^i - \hat{v}^i)}_{\text{scalar field}} \mathbf{e}_i.$$

Then

$$\begin{aligned} T(\mathbf{v} - \hat{\mathbf{v}})(P) &= [(v^i - \hat{v}^i)T(\mathbf{e}_i)](P), \text{ by strong linearity,} \\ &= [v^i(P) - \hat{v}^i(P)]T(\mathbf{e}_i^a)(P) \\ &= 0, \end{aligned}$$

and since $T(\mathbf{v} - \hat{\mathbf{v}}) = T(\mathbf{v}) - T(\hat{\mathbf{v}})$, $T(\mathbf{v})(P) = T(\hat{\mathbf{v}})(P)$.

An example of a linear T that is not a tensor is $T(\mathbf{v})(P) = \int_M \sigma_a v^a d\tau$, where σ is a covector field on M . Here the value $T(\mathbf{v})$ is a constant scalar field that depends on what $\mathbf{v}$ looks like everywhere. Consequently $T(f\mathbf{v}) \neq fT(\mathbf{v})$ unless f is constant, and T cannot be represented by a tensor field.

Commutators (Lie brackets)

If u^a and v^a are vector fields, one can define the linear map $f \mapsto u^a \nabla_a [v^b \nabla_b f]$: that is, $v^b \nabla_b f$ is a function on M when f is a function, so we can take its directional derivative along u^a , $u^a \nabla_a (v^b \nabla_b f)$. But there is no vector w^a for which $w^a \nabla_a f = u^a \nabla_a (v^b \nabla_b f)$ because $f \mapsto u^a \nabla_a (v^b \nabla_b f)$ does not satisfy the Leibnitz rule:

$$\begin{aligned} fg \mapsto u^a \nabla_a [v^b \nabla_b (fg)] &= u^a \nabla_a [(v^b \nabla_b f)g + f v^b \nabla_b g] \\ &= u^a \nabla_a (v^b \nabla_b f)g + v^b \nabla_b f u^a \nabla_a g + u^a \nabla_a (v^b \nabla_b g)f + u^a \nabla_a f v^b \nabla_b g \\ &\neq u^a \nabla_a (v^b \nabla_b f)g + u^a \nabla_a (v^b \nabla_b g)f; \end{aligned}$$

the cross terms $v \cdot \nabla f u \cdot \nabla g + v \cdot \nabla g u \cdot \nabla f$ are not zero.

If, however, one takes the antisymmetric combination of u and v ,

$$[u, v]f := u \cdot \nabla(v \cdot \nabla f) - v \cdot \nabla(u \cdot \nabla f) \quad (2.14)$$

the symmetric cross terms will drop out:

$$\begin{aligned} u \cdot \nabla[v \cdot \nabla(fg)] - v \cdot \nabla[u \cdot \nabla(fg)] &= [u \cdot \nabla(v \cdot \nabla f) - v \cdot \nabla(u \cdot \nabla f)]g \\ &\quad + [u \cdot \nabla(v \cdot \nabla g) - v \cdot \nabla(u \cdot \nabla g)]f \\ &= ([u, v]f)g + ([u, v]g)f. \end{aligned} \quad (2.15)$$

Thus the map $f \mapsto [u, v]f$ is linear and satisfies Leibnitz, and so there is a vector field $[u, v]^a$ with

$$\nabla_a f \equiv [u, v]^a \nabla_a f = [u, v]f.$$

In a chart, $[u, v]^a$ has components $[u, v]^i$ given by

$$\begin{aligned} [u, v]^i \partial_i f &= u^j \partial_j (v^i \partial_i f) - v^j \partial_j (u^i \partial_i f) \\ &= (u^j \partial_j v^i) \partial_i f - (v^j \partial_j u^i) \partial_i f + \underbrace{u^j v^i (\partial_i \partial_j f - \partial_j \partial_i f)}_0 \\ &= (u^j \partial_j v^i - v^j \partial_j u^i) \partial_i f, \end{aligned}$$

whence

$$[u, v]^i = u^j \partial_j v^i - v^j \partial_j u^i. \quad (2.16)$$

Notice that if u^a and v^a are coordinate basis vectors, ∂_1 and ∂_2 , say, then $[u, v] = 0$:

$$[u, v]^a \nabla_a f = \partial_1 \partial_2 f - \partial_2 \partial_1 f = 0. \quad (2.17)$$

Or note that the components are constants, $u^i = \delta_1^i$, $v^i = \delta_2^i$, so $[u, v] = 0$ by Eq. (2.16).

The converse is that, if two nonvanishing vector fields commute, then one can choose a coordinate system for which the two vector fields are coordinate basis vectors ∂_1 and ∂_2 ; that is, the vector fields are tangent to a family of surfaces on which only x and y change. This is true as long as u and v are linearly independent. A cleaner statement is that, if u and v are linearly independent and $[u, v] = 0$, then the vector fields are *surface forming*, tangent to families of 2-surfaces.

We'll return later to the commutator as the Lie derivative of v along u (or of u along v after a sign change), giving it a geometric meaning and extending it to the Lie derivative of tensors.

2.2 Metric and Covariant Derivative

A *metric* on a manifold M is a symmetric covariant tensor g_{ab} . g_{ab} is nondegenerate (we may never look at any other kind) if $g_{ab}u^b = 0 \Rightarrow u^a = 0$. If nondegenerate, a metric defines an isomorphism from contravariant to covariant vectors $u^a \rightarrow g_{ab}u^b = u_a$; the inverse isomorphism is $g^{ab}u_b = u^a$, where g^{ab} is the contravariant metric given by

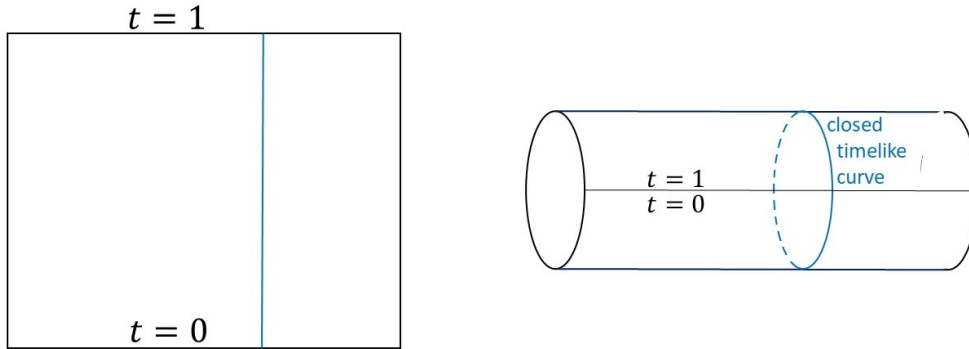
$$g^{ab}g_{bc} = \delta_c^a, \quad g^{ab} \text{ symmetric.} \quad (2.18)$$

A metric g_{ab} is positive definite if $g_{ab}u^a u^b > 0$, all $u^a \neq 0$. g_{ab} is a Lorentz metric on a 4-dimensional manifold M if the matrix $\|g_{ij}\|$ of the components of g_{ab} in a basis has 1 negative and 3 positive eigenvalues at each point of M . Then $g_{ab}(P)$ is a Minkowski metric for the tangent space at P .

A *spacetime* is a 4-dimensional manifold with a Lorentz metric.

Example 0. Minkowski space

Example 1. Take a slice of Minkowski space between $t = 0$ and $t = 1$ (t a natural time coordinate) and identify all points on the $t = 0$ surface with corresponding points on $t = 1$: $(0, x, y, z) \leftrightarrow (1, x, y, z)$



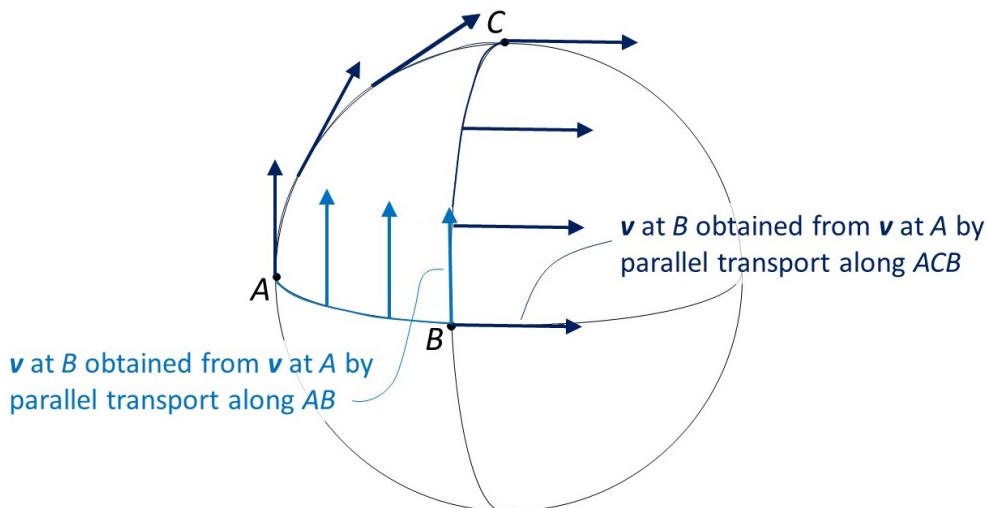
The resulting 4-dimensional cylinder $S^1 \times \mathbb{R}^3$ is a smooth (flat!) spacetime with same the flat Minkowski metric of the space before identifying points. It has closed timelike curves—e.g. the blue $x, y, z = \text{constant}$ curve running between the identified points at $t = 0$ and $t = 1$.

Given a metric, one wants to define a covariant derivative ∇_a . In Minkowski space, we used the natural constant vector fields and required that $\nabla_a v^b = 0$ for any constant vector field v^b . ∇_a was then extended to arbitrary tensors by $\nabla_a(f) = \nabla_a f$ on functions f and by Leibnitz

$$\nabla.(S^{\dots} \dots T^{\dots} \dots) = (\nabla.S^{\dots} \dots)T^{\dots} \dots + S^{\dots} \dots \nabla.T^{\dots} \dots \quad (2.19)$$

and linearity. We can still require $\nabla_a(f) = \nabla_a f$, the gradient of a scalar f , and also linearity and Leibnitz, but in curved manifolds there are no natural parallel vector fields; and, as we shall see, in a curved space one can't define (in general) vector fields for which $\nabla_a v^b = 0$ everywhere. The idea is that a vector v^a can be parallel transported around a curve, but the result of parallel transport from a point A to a point B depends on the curve.

Example.



It is intuitively clear how to parallel transport a vector v around each of the great quarter circles AC , CB and AB of the sphere shown here. The resulting vector at B depends on whether the path is ACB or AB . So given a vector v at A , there can be no vector field on S everywhere parallel to v . It would have to be both vertical and horizontal at B .

∇_a is to be defined so that if u^a is tangent to the curve $c(\lambda)$, then a vector v^a will be parallel transported along c when $u^b \nabla_b v^a = 0$. Parallel transport should preserve length and angles, so if v^a and w^a are parallel transported along c , the dot products $v^a v_a$, $v^a w_a$ and $w^a w_a$ should be constant along c : that is, if

$$u^b \nabla_b v^a = 0 \quad \text{and} \quad u^b \nabla_b w^a = 0, \quad (2.20)$$

then

$$u^a \nabla_a (g_{bc} v^b w^c) = 0.$$

By Leibnitz

$$\begin{aligned} u^a \nabla_a (g_{bc} v^b w^c) &= u^a (\nabla_a g_{bc}) v^b w^c + g_{bc} (u^a \nabla_a v^b) w^c + g_{bc} v^b u^a \nabla_a w^c \\ &= u^a (\nabla_a g_{bc}) v^b w^c \end{aligned}$$

by (2.20). So $\nabla_a g_{bc} u^a v^b w^c = 0$ for all u^a, v^a, w^a satisfying (2.20), and it follows¹ that

$$\nabla_a g_{bc} = 0. \quad (2.21)$$

We can now define our covariant derivative ∇_a by requiring, for all smooth functions f and tensor fields S, T ,

- (i) ∇_a linear
- (ii) $\nabla_a (S^{\dots} T^{\dots}) = S^{\dots} \nabla_a T^{\dots} + (\nabla_a S^{\dots}) T^{\dots}$ (Leibnitz)
- (iii) Commutes with contraction $\nabla_a (T^b_b) = \nabla_a T^b_b$
- (iv) $\nabla_a f$ on scalars is the gradient
- (v) $\nabla_{[a} \nabla_{b]} f = 0$
- (vi) $\nabla_a g_{bc} = 0$.

We saw that condition (v) holds in Minkowski space because, in a natural coordinate system,

$$\nabla_{[\mu} \nabla_{\nu]} f = \frac{1}{2} \left(\frac{\partial^2 f}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} \right) = 0. \quad (2.22)$$

Here we must put it in by hand. In the context of spacetime, the condition will turn out to be equivalent to demanding that the spacetime be *locally Minkowskian*, that there be inertial coordinates in the neighborhood of any point P , coordinates in which the equations of nongravitational fields take the same form at P they have in special relativity, with $\nabla_\mu v^\nu(P) = \partial_\mu v^\nu(P)$. This requirement, that there be coordinates in which all metric derivatives vanish at a point, is the way Schutz and Hartle impose this condition.

We will show that (i)–(vi) uniquely determine ∇_a , by explicitly constructing the components of $\nabla_a T^{b\dots c}{}_{d\dots e}$ in an arbitrary basis e_i^a . One could do this more elegantly, but we'll need the components

¹It follows because you can pick u^a, v^a, w^a arbitrarily at a point P and then parallel transport v and w along c (with tangent u^a) to get u, v and w satisfying (2.20). Then $\nabla_a g_{bc} u^a v^b w^c|_P = 0$ all vectors u, v, w at P , implying $\nabla_a g_{bc}(P) = 0$.

in any event. We'll begin by looking at the components of the covariant derivative of a covector σ_a . As in flat space, we define Γ^i_{jk} by

$$\nabla_k \mathbf{e}_j = \mathbf{e}_k \cdot \nabla \mathbf{e}_j = \Gamma^i_{jk} \mathbf{e}_i. \quad (2.23)$$

Terminology: For a coordinate basis, the coefficients Γ^i_{jk} are historically called Christoffel symbols; for an orthonormal basis, they are called Ricci rotation coefficients, because an orthonormal basis parallel-transported from point P to point Q differs from the value of the original orthonormal basis at Q by a rotation (or, for a Lorentzian signature metric, by a Lorentz transformation). In general, MTW calls the Γ 's *connection coefficients*.

Then Γ^i_{jk} will be expressed in terms of the partial derivatives of the metric components,

$$e_i^a \underbrace{\nabla_a (g_{jk})}_{\text{gradient of the scalar field } g_{jk}},$$

and of the commutators c^i_{jk} defined by

$$[\mathbf{e}_j, \mathbf{e}_k] = c^i_{jk} \mathbf{e}_i. \quad (2.24)$$

Note the antisymmetry $c^i_{jk} = c^i_{[jk]}$. For a coordinate basis, $[\mathbf{e}_j, \mathbf{e}_k] = [\partial_j, \partial_k] = 0$.

We have been through nearly the same calculation in Minkowski space. The components of $\nabla_a \sigma_b$ are

$$\begin{aligned} \nabla_i \sigma_j &:= e_i^a e_j^b \nabla_a \sigma_b = e_i^a \nabla_a (e_j^b \sigma_b) - e_i^a (\nabla_a e_j^b) \sigma_b \quad \text{by (iii)} \\ \nabla_i \sigma_j &= \mathbf{e}_i \cdot \partial \sigma_j - \Gamma^k_{ji} \sigma_k \quad \text{by (2.23)} \end{aligned} \quad (2.25)$$

To find Γ^k_{ji} we'll write the components of $\nabla_a g_{bc} = 0$, as we did in flat space. This time, however, we'll do the computation for an arbitrary basis, not restricting ourselves to a coordinate basis. We could have done the same thing for flat space, but there the object was to give a simpler computation for our first encounter with Christoffel symbols. Using the notation $e_i f \equiv e_i(f) = e_i^a \nabla_a f$, we'll denote the directional derivative of the scalars g_{jk} in the direction $\mathbf{e}_i$ by $e_i g_{jk}$. Then

$$\begin{aligned} 0 &= \nabla_i g_{jk} = e_i^a e_j^b e_k^c \nabla_a g_{bc} \quad \text{by (iv)} \\ &= e_i[e_j^b e_k^c g_{bc}] - (\nabla_i e_j^b) g_{bc} e_k^c - (\nabla_i e_k^c) e_j^b g_{bc} \quad \text{by (iii)} \\ &= e_i g_{jk} - \Gamma^l_{ji} g_{lk} - \Gamma^l_{ki} g_{jl} \quad \text{by (2.23)} \end{aligned}$$

Defining

$$\Gamma_{kji} = g_{kl} \Gamma^l_{ji}, \quad (2.26)$$

we have

$$0 = e_i g_{jk} - \Gamma_{kji} - \Gamma_{jki}. \quad (2.27)$$

For a coordinate basis, Γ_{ijk} is symmetric in j and k , but orthonormal bases are often more useful, and for a general basis Γ has no index symmetry. Instead, we'll write Γ_{ijk} as a sum of its symmetric and antisymmetric parts, $\Gamma_{ijk} = \Gamma_{i(jk)} + \Gamma_{i[jk]}$. The antisymmetric part is present when the basis vectors do not commute and is related to their commutator in a simple way:

$$\Gamma_{i[jk]} = -\frac{1}{2} c_{ijk} \equiv -\frac{1}{2} g_{il} c^l_{jk}. \quad (2.28)$$

To see (2.28), one notes that $[u, v]^a = u^b \nabla_b v^a - v^b \nabla_b u^a$:

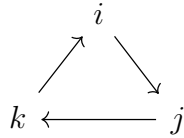
$$\begin{aligned} [u, v]^a \nabla_a f &= u^b \nabla_b (v^a \nabla_a f) - v^b \nabla_b (u^a \nabla_a f) \\ &= (u^b \nabla_b v^a) \nabla_a f - (v^b \nabla_b u^a) \nabla_a f + \underbrace{u^b v^a \nabla_b \nabla_a f - v^b u^a \nabla_b \nabla_a f}_0 \\ &= (u^b \nabla_b v^a - v^b \nabla_b u^a) \nabla_a f \implies \\ [u, v]^a &= u^b \nabla_b v^a - v^b \nabla_b u^a, \end{aligned} \quad (2.29)$$

where we used $(\nabla_b \nabla_a - \nabla_a \nabla_b)f = 0$ in the second line. In particular,

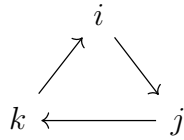
$$\begin{aligned} [e_j, e_k] &= e_j \cdot \nabla e_k - e_k \cdot \nabla e_j \\ c^i_{jk} e_i &= \Gamma^i_{kj} e_i - \Gamma^i_{jk} e_i \\ c^i_{jk} &= \Gamma^i_{kj} - \Gamma^i_{jk}, \end{aligned}$$

identical to (2.28). Finally we'll write (2.27) three times and add:

$$-e_i g_{jk} = -\Gamma_{k(ji)} - \Gamma_{j(ki)} - \Gamma_{k[ji]} - \Gamma_{j[ki]} \quad (2.30)$$



$$e_j g_{ki} = \Gamma_{i(kj)} + \Gamma_{k(ij)} + \Gamma_{i[kj]} + \Gamma_{k[ij]} \quad (2.31)$$



$$e_k g_{ij} = \Gamma_{j(ik)} + \Gamma_{i(jk)} + \Gamma_{j[ik]} + \Gamma_{i[jk]} \quad (2.32)$$

$$\begin{aligned} e_j g_{ki} + e_k g_{ij} - e_i g_{jk} &= 2\Gamma_{i(jk)} + 2\Gamma_{j[ik]} + 2\Gamma_{k[ij]} \\ &= 2\Gamma_{ijk} - 2\Gamma_{[ijk]} + 2\Gamma_{j[ik]} + 2\Gamma_{k[ij]} \\ 2\Gamma_{ijk} &= e_j g_{ik} + e_k g_{ij} - e_i g_{jk} + 2\Gamma_{[ijk]} - 2\Gamma_{j[ik]} - 2\Gamma_{k[ij]} \end{aligned}$$

$$\Gamma_{ijk} = \frac{1}{2} [e_j g_{ik} + e_k g_{ij} - e_i g_{jk} + c_{kij} - c_{jki} - c_{ijk}] \quad (2.33)$$

The covariant derivative of any tensor then follows from Leibnitz (iii), exactly as in Eq. (1.152). We'll keep the $e_i T^j_k$ notation instead of $\nabla_i(T^j_k)$, but they mean the same thing:

$$\nabla_i T^{j \dots k}_{m \dots n} = e_i T^{j \dots k}_{m \dots n} + \Gamma^j_{li} T^{l \dots k}_{m \dots n} + \dots + \Gamma^k_{li} T^{j \dots l}_{m \dots n} - \Gamma^l_{mi} T^{j \dots k}_{l \dots n} - \dots - \Gamma^l_{ni} T^{j \dots k}_{m \dots l}$$

Two special cases of Eq. (2.33) are all that one ordinarily encounters. With a coordinate basis,

$$[e_i, e_j] = [\partial_i, \partial_j] = 0,$$

implying $c^k_{ij} = 0$, and one recovers the usual Christoffel symbols; in greater detail,

$$e_i = \frac{\partial}{\partial x^i} \quad \text{and} \quad [e_i, e_j]f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} = 0. \quad (2.34)$$

Then

$$\Gamma_{ijk} = \frac{1}{2} (\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}) \quad (2.35)$$

With an orthonormal basis, the metric components are constant, with values 1, -1 or 0, so $e_i g_{jk} = 0$, and only the commutators contribute to Γ : For example, for a spacetime,

$$\|g_{\mu\nu}\| = \|e_\mu^\alpha e_\nu^\beta g_{\alpha\beta}\| = \left\| \begin{array}{cccc} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array} \right\|. \quad (2.36)$$

Thus, for an orthonormal basis or any basis with respect to which the metric has constant components,

$$\Gamma_{ijk} = \frac{1}{2}(c_{kij} - c_{jki} - c_{ijk}). \quad (2.37)$$

Cartan calculus (Appendix A.3), an efficient way to compute the Riemann tensor, and the Newman-Penrose formalism (Chap. 8) each use bases of this kind.

Local flatness: Locally inertial or “normal” coordinates at a point P

Spacetime near any point looks like flat Minkowski space to linear order in the distance from a point. In particular, one can always choose a chart about P in which the metric is the Minkowski metric at P and the first derivatives of the metric all vanish. In such a “locally inertial” chart,

$$\|g_{\mu\nu}(P)\| = \|\eta_{\mu\nu}\| \equiv \left\| \begin{array}{cccc} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array} \right\|, \quad \text{and} \quad \Gamma_{\lambda\mu\nu}(P) = 0.$$

Exercise 38. Show that $\Gamma_{\lambda\mu\nu}(P) = 0$ implies that all first derivatives of the metric components vanish, $\partial_\lambda g_{\mu\nu}(P) = 0$. (Hint: Add Eq. (2.35) to Eq. (2.35) with $i \leftrightarrow j$).

More generally, in an n -dimensional space with a Lorentzian (signature $- + \cdots +$) or Riemannian (signature $+ + \cdots +$) metric, one can always choose a chart for which the coordinate basis at a point P is orthonormal and for which all the derivatives of the metric components vanish at P . We’ll explicitly construct a chart of this form at the end of the next section.

2.2.1 Geodesics

Geodesic Equation:

Geodesics are paths of extremal length. Consider a curve $c(\lambda)$ with $c(\lambda_1) = P$ and $c(\lambda_2) = Q$. We show that the demand that the curve be of extremal length — that

$$s := \int_{\lambda_1}^{\lambda_2} |g_{ab}v^a v^b|^{1/2} d\lambda,$$

where v^a is tangent to $c(\lambda)$, be an extremum among all nearby curves from P to Q — is equivalent to demanding that the unit tangent vector u^a be parallel transported along c :

$$u^a \nabla_a u^b = 0 \Leftrightarrow \delta \int ds \equiv \delta \int |g_{ab}v^a v^b|^{1/2} d\lambda = 0. \quad (2.38)$$

Consider another curve $\bar{c}(\lambda)$ with the same endpoints P and Q as c . We again follow Wald in writing the coordinates of the curves c and $\bar{c}$ in a chart as $x^i(\lambda) := x^i \circ c(\lambda)$, $\bar{x}^i(\lambda) = x^i \circ \bar{c}(\lambda)$. Then $\bar{x}^i(\lambda) = x^i(\lambda) + \delta x^i(\lambda)$ where $\delta x^i(\lambda_0) = \delta x^i(\lambda_1) = 0$.

$$s = \int ds = \int_{\lambda_1}^{\lambda_2} |g_{ij}[x(\lambda)] \dot{x}^i(\lambda) \dot{x}^j(\lambda)|^{1/2} d\lambda, \text{ where } (\dot{}) \text{ means } \frac{d}{d\lambda}.$$

$$\delta \int ds = \int_{\lambda_1}^{\lambda_2} \left\{ |g_{ij}(x + \delta x)(\dot{x}^i + \delta \dot{x}^i)(\dot{x}^j + \delta \dot{x}^j)|^{1/2} - |g_{ij}(x)\dot{x}^i \dot{x}^j|^{1/2} \right\} d\lambda$$

Now $g_{ij}(x + \delta x) = g_{ij}(x) + \partial_k g_{ij}(x) \delta x^k + O(\delta x)^2$. Then, to $O(\delta x)$,

$$\delta \int ds = \int_{\lambda_1}^{\lambda_2} \frac{1}{2} |g_{ij}\dot{x}^i \dot{x}^j|^{-1/2} [\partial_k g_{ij}(x(\lambda)) \delta x^k \dot{x}^i \dot{x}^j + 2g_{ij}\dot{x}^i \delta \dot{x}^j] d\lambda$$

The last half of the calculation is easier if we take λ to be proper distance along the unperturbed curve c — so that the tangent vector v^a is the unit tangent u^a . We then have $\dot{x}^i = u^i$, $|g_{ij}\dot{x}^i \dot{x}^j| = 1$, and

$$\delta \int ds = \int_{\lambda_1}^{\lambda_2} \frac{1}{2} [\partial_k g_{ij} u^i u^j \delta x^k + 2g_{ij} u^i \delta \dot{x}^j] d\lambda.$$

Last term:

$$\begin{aligned} \int_{\lambda_1}^{\lambda_2} g_{ij}(x(\lambda)) u^i(x(\lambda)) \frac{d}{d\lambda} \delta x^j(\lambda) d\lambda &= - \int_{\lambda_1}^{\lambda_2} \frac{d}{d\lambda} [g_{ij}(x(\lambda)) u^i(x(\lambda))] \delta x^j d\lambda \\ &= - \int_{\lambda_1}^{\lambda_2} [g_{ij} \dot{u}^i + \partial_k g_{ij} \dot{x}^k u^i] \delta x^j d\lambda = - \int_{\lambda_1}^{\lambda_2} [g_{ik} \dot{u}^i + \partial_j g_{ik} u^i u^j] \delta x^k d\lambda \\ &= - \int_{\lambda_1}^{\lambda_2} [g_{ik} \dot{u}^i + \frac{1}{2} \partial_j g_{ik} u^i u^j + \frac{1}{2} \partial_i g_{jk} u^i u^j] \delta x^k d\lambda. \end{aligned}$$

Thus

$$\begin{aligned} \delta \int ds &= \int_{\lambda_1}^{\lambda_2} [-g_{ik} \dot{u}^i - \frac{1}{2} \partial_j g_{ik} u^i u^j - \frac{1}{2} \partial_i g_{jk} u^i u^j + \frac{1}{2} \partial_k g_{ij} u^i u^j] \delta x^k \\ &= - \int_{\lambda_1}^{\lambda_2} g_{kl} (\dot{u}^k + \Gamma^k_{ij} u^i u^j) \delta x^l. \end{aligned}$$

Then $\delta \int ds = 0$ for all $\delta c \Rightarrow$

$$\dot{u}^k + \Gamma^k_{ij} u^i u^j = 0, \quad \text{or} \quad (2.39a)$$

$$\frac{d^2 x^k}{d\tau^2} + \Gamma^k_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0 \quad \text{or} \quad (2.39b)$$

$$u^b \cdot \nabla_b u^a = 0. \quad (2.39c)$$

For spacelike separated points, the shortest path between them is a geodesic, but there can be more than one geodesic joining two points; for example, an infinite set of longitude lines join the north and south poles of a sphere. Each spacelike geodesic is a *local* minimum of length. Geodesics between timelike separated points, however, are local *maxima* of length (of proper time). As you have long known, time dilation means that clocks moving along accelerated paths run slow, registering shorter proper times, as in the figure on p. 19; the twin traveling on a roundtrip to a star returns home younger than the twin left behind.

Finally, light travels along geodesics $c(\lambda)$ whose tangent vector k^a is everywhere null, $k^a k_a = 0$, but one can choose the parameter λ so that the geodesic equation again has the form

$$k^b \nabla_b k^a = 0, \quad (2.40)$$

or

$$\frac{d^2 x^k}{d\lambda^2} + \Gamma^k_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 0. \quad (2.41)$$

The parameter λ is now not distance, but is called an *affine* parameter. It is not unique: If we replace λ by $\bar{\lambda} = C\lambda$ the geodesic equation keeps the same form. For timelike or spacelike geodesics, proper time τ and spatial distance s are affine parameters, because the geodesic equation has the form (2.39c), but so are Cs and $C\tau$. We'll use an affine parameter in the next section, as part of a construction of coordinates that exhibit the local flatness of spacetime.

The geodesic equation in the form (2.39b) is a coupled set of ordinary differential equations (ODEs) for the functions $x^i(\tau)$. The equations are second-order (involving only first and second derivatives), and second order ODE's have unique solutions for any given initial values $x^i(0)$ and $\dot{x}^i(0)$. That is, **given an initial starting point $c(0) = P$ and an initial tangent vector T^a at P , there is a unique geodesic $c(\lambda)$ through P .** (This is the fundamental existence theorem for solutions of ODE's. For a proof, see, for example, Coddington & Levinson *Theory of Ordinary Differential Equations*[29]. The statement of the theorem is in Wikipedia's ODE article.)

Exercise 39. Because a geodesic is a path of extremal length between two points, one can obtain the geodesic equation by requiring

$$0 = \delta \int_{\tau_1}^{\tau_2} d\tau = \delta \int_{\tau_1}^{\tau_2} L(\dot{x}, x) d\tau, \quad \text{where} \quad L = \sqrt{-g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu}$$

and where τ is proper time along the geodesic $x^i(\tau)$ (but not along the perturbed path $x^\mu(\tau) + \delta x^\mu(\tau)$).

- a. By using $L = 1$ for a geodesic, show that the geodesic equation also follows for

$$0 = \delta \int_{\tau_1}^{\tau_2} F[L(\dot{x}, x)] d\tau,$$

if $dF/dL \neq 0$ at the geodesic, where $F(L)$ is some smooth function of L . You will only need the fact that $L = 1$ along the geodesic (implying $F'(L) = F'(1)$), so use an arbitrary function $L(\dot{x}, x)$ rather than replacing L by its explicit form in terms of g .

- b. Derive the geodesic equation as the Euler-Lagrange equations of the Lagrangian

$$\tilde{L}(\dot{x}, x) = -L^2(\dot{x}, x) = g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu.$$

Exercise 40. a. Find the four components of the geodesic equations for flat space with polar coordinates as the Euler-Lagrange equations for the Lagrangian $-\dot{t}^2 + \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2$.

- b. This is an efficient way to compute the Christoffel symbols. Using your result for the last part, and the geodesic equation in the form $\ddot{x}^\mu + \Gamma_{\nu\lambda}^\mu \dot{x}^\nu \dot{x}^\lambda$, find the nonzero Christoffel symbols of the Minkowski metric in polar coordinates.

Exercise 41.

- a. Write the components of the equation $\mathbf{v} \cdot \nabla \mathbf{v} = 0$ for a geodesic in Euclidean 3-space in polar coordinates, where $v^i = \dot{x}^i$, with $(\cdot) := \frac{d}{ds}$. Replace $v^j \partial_j v^i$ by $\dot{v}^i$ or $\ddot{x}^i$. (Use the coordinate basis, as in the previous problem, not an orthonormal frame.)
- b. Specialize now to the $\theta = \pi/2$ plane, and use $ds^2 = dr^2 + r^2 d\phi^2$ to deduce $1 = \dot{r}^2 + r^2 \dot{\phi}^2$. Check that the derivative of this equation, with respect to s yields the r -component of the geodesic equation.
- c. Check that the ϕ -component of the geodesic equation has the first integral $r^2 \dot{\phi} = l$, $l = \text{constant}$.
- d. What equation for central force motion corresponds to the equation $r^2 \dot{\phi} = l$? Why does this agree for motion in a straight line, when t for central-force motion is replaced here by s ?

The exponential map and geodesic coordinate systems

As mentioned above, spacetime (and, in general, a manifold with a smooth metric) is locally flat. If you are a small bug on a surface with radius of curvature much larger than the bug, the surface will look flat to you. Our goal here is to find a coordinate system in which the metric components are as close as possible to $\eta_{\mu\nu}$, coordinates that are as close as possible to Cartesian.

In Minkowski space, the set of all geodesics—straight lines—through any point P exactly fills the entire space. Regarding a vector X^α as a connecting vector from P gives a one-one map from X^α to a point Q : $X^\alpha \mapsto Q^\alpha = P^\alpha + X^\alpha$. The components X^μ are then the coordinates of Q in a chart with origin at P . We now restate this construction in a way that can be carried over to a curved spacetime. The idea is again to identify each tangent vector X^a at P with the point Q a distance $|X|$ from P along the geodesic from P to Q with tangent X^a , but with a description that also works for light-like vectors.

Given a vector X^α , there is a unique geodesic through P with tangent X^α . The coordinates $x^\mu(\lambda)$ of the geodesic satisfy (still in Minkowski space)

$$\frac{dx^\mu}{d\lambda} = X^\mu, \quad \frac{d^2x^\mu}{d\lambda^2} = 0. \quad (2.42)$$

That is, the geodesic has the form

$$x^\mu(\lambda) = \lambda X^\mu, \quad (2.43)$$

and $x^\mu = X^\mu$ at $\lambda = 1$.

For spacelike or timelike geodesics parametrized by proper distance or proper time, the tangent vectors are unit vectors, but here the tangent vector has length equal to the distance to Q . The parameter λ is $s/|X|$ or $\tau/|X|$, differing from proper distance (or proper time) by the constant factor $1/|X|$. By definition, it is an affine parameter, and the map is defined for null vectors X^α where the proper distance vanishes. Like the original connecting-vector description, our construction of the map $X^\alpha \mapsto Q$ doesn't mention $|X|$ and doesn't care whether X^α is spacelike, timelike or null.

The map from vectors at a point P to points near P is called the *exponential map*. We can now say nearly the same words to define the exponential map in a curved space M . Let's start with a sphere and consider the tangent space at a point P , which we can call the north pole, as in Fig. 2.2. Given a vector

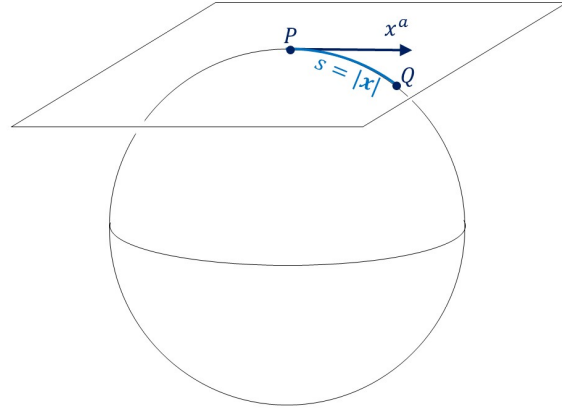


Figure 2.2: The exponential map takes a vector X^a at P to the point Q a parameter distance $\lambda = s/|X| = 1$ from P along the geodesic through P with tangent X^a .

X^a at P , there is a unique geodesic (longitude line) through P with tangent X^a . We map X^a to the point Q of the sphere at parameter distance $\lambda = 1$ along the geodesic (longitude line) through P . This gives a 1-1 map as long as the distance $|X|$ is less than half the circumference of the sphere. At the south pole, of course, all geodesics through P meet.

Here is the statement for spacetime (or any smooth manifold with a metric): In some finite neighborhood of a point P , geodesics from P do not cross each other. Let V_P be the space of vectors at P . The *exponential map* is the map $V_P \rightarrow M$ that takes each vector X^a at P to the point Q a parameter distance $\lambda = 1$ along the geodesic through P with tangent X^a .

Riemann normal coordinates.

We can now use the exponential map for a curved space M to construct a chart with origin at P in M , analogous to the chart $\{x^i\}$ of $\mathbb{R}^n$ with origin at a point P of $\mathbb{R}^n$. Let $\{e_i\}$ be an orthonormal basis at $P \in M$. Any vector x^a at P is a sum of the basis vectors, $x = x^i e_i$. The components x^i are the coordinates of the point Q associated with x^a . That is:

Each point Q in a neighborhood of P is joined to P by a unique geodesic $c(\lambda)$, and the coordinates of Q are $(x^1, \dots, x^n)$, where $x = x^i e_i$ is the tangent to c at P .

We now show that, in this chart for spacetime, the metric has components $g_{\mu\nu}(P) = \eta_{\mu\nu}$ and that its first derivatives vanish at P : $\partial_\lambda g_{\mu\nu} = 0$. To do this, we show that (1) the coordinates of geodesics starting from P with tangent X^μ have the exact flat space form (2.2), $x^\mu(\lambda) = \lambda X^\mu$; and (2) the geodesic equation (2.41) implies $\Gamma_{\mu\nu}^\lambda(P) = 0$ in this chart.

(1) Suppose $R = c(\lambda_R)$ is any point on the geodesic from P to Q with tangent X^α . R is the same direction as Q but is at $\lambda = \lambda_R$ instead of at $\lambda = 1$. The geodesic that meets R at $\lambda = 1$ is then $\bar{c}(\lambda) := c(\lambda_R \lambda)$ with coordinates $\bar{x}^\mu(\lambda) = x^\mu(\lambda_R \lambda)$. The tangent vector to that geodesic is $\bar{X}^\alpha = \lambda_R X^\alpha$. That is, by the chain rule,

$$\left. \frac{d\bar{x}^\mu(\lambda)}{d\lambda} \right|_{\lambda=0} = \lambda_R \left. \frac{dx^\mu(\lambda)}{d\lambda} \right|_{\lambda=0} = \lambda_R X^\mu. \quad (2.44)$$

By our definition of the coordinates, R has coordinates $x^\mu = \lambda_R X^\mu$. Since this is true for each point R on the geodesic, $x^\mu(\lambda) = \lambda X^\mu$ as claimed.

(2) For $x^\mu(\lambda) = \lambda X^\mu$, we have $\ddot{x}^\mu = 0$, where $(\dot{}) := d/d\lambda$. The geodesic equation (2.41) then implies $\Gamma_{\mu\nu}^\lambda \dot{x}^\mu \dot{x}^\nu = \Gamma_{\mu\nu}^\lambda X^\mu X^\nu = 0$. But this is true for all geodesics through P , implying that, at P it is true for all vectors X^α . Then $\Gamma_{\mu\nu}^\lambda(P) = 0$ as claimed, implying $\partial_\lambda g_{\mu\nu}(P) = 0$.

Finally, because the basis is orthonormal, the metric at P has components $g_{\mu\nu} = e_\mu \cdot e_\nu = \eta_{\mu\nu}$. $\square$

2.2.2 Curvature Tensor

In a flat spacetime, covariant derivatives commute. In curved spacetime, however,

$$\nabla_{[a} \nabla_{b]} v_c \neq 0$$

and as we will see, the fact is closely related to the statement that parallel transport from A to B depends on the path from A to B . Write

$$\frac{1}{2} R_{abc}(v) \equiv \nabla_{[a} \nabla_{b]} v_c;$$

so far, this just gives a new name to the tensor $\nabla_{[a} \nabla_{b]} v_c$. As a function of v , R_{abc} is linear under addition and multiplication by scalar *fields*:

$$\begin{aligned} \nabla_{[a} \nabla_{b]} (f v_c) &= \nabla_{[a} (\nabla_{b]} f v_c + f \nabla_{b]} v_c) \\ &= \underbrace{\nabla_{[a} \nabla_{b]} f}_{0} v_c + \nabla_{[b} f \nabla_{a]} v_c + \nabla_{[a} f \nabla_{b]} v_c + f \nabla_{[a} \nabla_{b]} v_c \\ &= f \nabla_{[a} \nabla_{b]} v_c \end{aligned}$$

Therefore $R_{abc}(v) = R_{abc}{}^d v_d$. The tensor $R_{abc}{}^d$ is called the *Riemann tensor* or the *curvature tensor*.

$$\nabla_{[a} \nabla_{b]} v_c = \frac{1}{2} R_{abc}{}^d v_d, \quad (2.45)$$

or

$$[\nabla_a, \nabla_b]v_c = R_{abc}{}^d v_d. \quad (2.46)$$

It has the symmetries

$$\left. \begin{aligned} R_{abcd} &= -R_{bacd} && \text{from its definition (2.46))} \\ R_{abcd} &= -R_{abdc} \\ R_{abcd} &= R_{cdab} \\ \text{and } R_{a[bcd]} &= 0 = R_{abcd} + R_{adbc} + R_{acdb}; \end{aligned} \right\} \quad (2.47)$$

these are not difficult to show, and are part of the problems at the end of Sec.2.2.3.

Acting on a vector v^a , the commutator of two covariant derivatives is again given in terms of the curvature tensor: Using $\nabla_a g^{bc} = 0$, we have

$$[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^d;$$

and if $T^{a\cdots b}{}_{c\cdots d}$ is any tensor,

$$\begin{aligned} [\nabla_e, \nabla_f]T^{a\cdots b}{}_{c\cdots d} &= R_{ef}{}^a{}_g T^{g\cdots b}{}_{c\cdots d} + \cdots + R_{ef}{}^b{}_g T^{a\cdots g}{}_{c\cdots d} \\ &\quad + R_{efc}{}^g T^{a\cdots b}{}_{g\cdots d} + R_{efd}{}^g T^{a\cdots b}{}_{c\cdots g}, \end{aligned} \quad (2.48)$$

which follows by Leibnitz, writing $T^{a\cdots b}{}_{c\cdots d}$ as a linear combination of outer products of vector and covector fields:

$$\begin{aligned} \text{e.g. } \nabla_{[a} \nabla_{b]}(\sigma_c \tau_d) &= (\nabla_{[a} \nabla_{b]} \sigma_c) \tau_d + \nabla_{[a} \sigma_{|c|} \nabla_{b]} \tau_d + \nabla_{[b} \sigma_{|c|} \nabla_{a]} \tau_d + \sigma_c \nabla_{[a} \nabla_{b]} \tau_d \\ &= \frac{1}{2} R_{abc}{}^e \sigma_e \tau_d + \frac{1}{2} R_{abd}{}^e \sigma_c \tau_e. \end{aligned}$$

We will write the coordinate components of R_{abcd} in terms of the Christoffel symbols Γ_{jk}^i and show that in a locally inertial chart about P , R_{ijkl} has a simple form in which the symmetries (2.47) are manifest. First the components in an arbitrary chart:

$$\begin{aligned} R_{ijk}{}^l v_l &= \nabla_i \nabla_j v_k - \nabla_j \nabla_i v_k \\ &= \partial_i \nabla_j v_k - \Gamma_{ji}^l \nabla_l v_k - \Gamma_{ki}^l \nabla_j v_l \\ &\quad - \partial_j \nabla_i v_k + \Gamma_{ij}^l \nabla_l v_k + \Gamma_{kj}^l \nabla_i v_l, \end{aligned}$$

where the symmetry $\Gamma_{ji}^l = \Gamma_{ij}^l$ for a coordinate basis was used in the cancellation of the second and fifth terms.

Then

$$\begin{aligned} R_{ijk}{}^l v_l &= \partial_i (\partial_j v_k - \Gamma_{kj}^l v_l) - \Gamma_{ki}^l (\partial_j v_l - \Gamma_{lj}^m v_m) \\ &\quad - \partial_j (\partial_i v_k - \Gamma_{ki}^l v_l) + \Gamma_{kj}^l (\partial_i v_l - \Gamma_{li}^m v_m) \\ &= -\partial_i \Gamma_{kj}^l v_l - \Gamma_{kj}^l \partial_i v_l - \Gamma_{ki}^l \partial_j v_l + \Gamma_{ki}^l \Gamma_{lj}^m v_m \\ &\quad + \partial_j \Gamma_{ki}^l v_l + \Gamma_{ki}^l \partial_j v_l + \Gamma_{kj}^l \partial_i v_l - \Gamma_{kj}^l \Gamma_{li}^m v_m \\ R_{ijk}{}^l &= \partial_j \Gamma_{ki}^l - \partial_i \Gamma_{kj}^l + \Gamma_{ki}^m \Gamma_{mj}^l - \Gamma_{kj}^m \Gamma_{mi}^l \end{aligned}$$

Using the index symmetries to interchange k and l and exchange the pairs i, j and k, l , we can order the indices in a way that allows comparison with, e.g., MTW's inside cover:

$$R_{lij}^k = \partial_i \Gamma_{lj}^k - \partial_j \Gamma_{li}^k + \Gamma_{mi}^k \Gamma_{lj}^m - \Gamma_{mj}^k \Gamma_{li}^m. \quad (2.49)$$

Components of the Riemann tensor in a general basis are given in Eq. (A.52) of the Appendix, where they are used for the efficient Cartan-calculus calculation of Riemann tensor components in an orthonormal basis.

In a locally inertial chart about a point P

$$\Gamma_{jk}^i|_P = 0, \quad \partial_i g_{jk}|_P = 0.$$

Then, at P , only derivatives of Γ and second derivatives of g_{ij} survive, and we have

$$R_{ijk}^l|_P = \partial_j \Gamma_{ki}^l|_P - \partial_i \Gamma_{kj}^l|_P, \quad R_{ijkl}|_P = \partial_j \Gamma_{lki}|_P - \partial_i \Gamma_{lkj}|_P; \quad (2.50)$$

$$\begin{aligned} \partial_j \Gamma_{ki}^l|_P &= \frac{1}{2} \partial_j [g^{lm} (\partial_k g_{mi} + \partial_i g_{mk} - \partial_m g_{ik})]|_P = \frac{1}{2} g^{lm} (\partial_j \partial_k g_{im} + \partial_i \partial_j g_{km} - \partial_m \partial_j g_{ik})|_P, \\ \partial_j \Gamma_{lki}|_P &= \frac{1}{2} (\partial_j \partial_k g_{il} + \partial_i \partial_j g_{kl} - \partial_l \partial_j g_{ik})|_P. \end{aligned} \quad (2.51)$$

Substituting this last expression and its ($i \leftrightarrow j$) counterpart into (2.50), we obtain (only at the single point P)

$$R_{ijkl}(P) = \frac{1}{2} [\partial_j \partial_k g_{il} + \partial_i \partial_l g_{jk} - \partial_i \partial_k g_{jl} - \partial_j \partial_l g_{ik}]|_P, \quad (2.52)$$

from which $R_{ijkl} = R_{[ij][kl]} = R_{klij}$ is evident, and $R_{ijkl} + R_{kijl} + R_{jkil} = 0$ follows as soon as it's written. If all the components of a tensor—e.g., the tensor $R_{abcd} - R_{[ab][cd]}$ —vanish in any basis, the tensor vanishes. Whence

$$R_{abcd} = R_{[ab][cd]} = R_{cdab}$$

and

$$R_{[abc]d} = 0.$$

From the Riemann tensor $R_{acb}{}^d$ one constructs the *Ricci* tensor by contracting:

$$R_{ab} := R_{acb}{}^c = R_b{}^c{}_{ac} = R_{bca}{}^c = R_{ba} \quad (2.53)$$

[the other contractions give the same tensor ($R_{cd} = R_{ac}{}^a{}_d$) or zero ($R_a{}^a{}_{cd} = 0$)]. A further contraction produces the *Ricci scalar*:

$$R = R^a{}_a = R^{ab}{}_{ab} \quad (2.54)$$

2.2.3 Bianchi Identities

From the coordinate components (2.49) of $R_{abc}{}^d$, it is easy to show the Bianchi identity

$$\nabla_{[a} R_{bc]de} = 0, \quad (2.55)$$

and (2.55) may also be proven directly—without reference to components (Exercise below). In geodesic coordinates at P , the covariant derivative $\nabla_a R_{bcd}{}^e$ has components

$$\begin{aligned} \nabla_i R_{jkl}{}^m|_P &= \partial_i R_{jkl}{}^m|_P \\ &= \partial_i \partial_k \Gamma_{jl}^m|_P - \partial_i \partial_j \Gamma_{kl}^m|_P, \end{aligned}$$

where we have used

$$\partial_i(\Gamma_{..}\Gamma_{..})|_P = (\partial_i\Gamma_{..}|_P) \underbrace{\Gamma_{..}|_P}_0 + \underbrace{\Gamma_{..}|_P}_0 \partial_i\Gamma_{..}|_P . \quad (2.56)$$

Thus, at P ,

$$\begin{aligned} \nabla_i R_{jkl}{}^m + \nabla_j R_{kil}{}^m + \nabla_k R_{ijl}{}^m &= \frac{\partial_i \partial_k \Gamma_{jl}^m}{+ \partial_j \partial_i \Gamma_{kl}^m} - \frac{\partial_i \partial_j \Gamma_{kl}^m}{- \partial_j \partial_k \Gamma_{il}^m} \\ &\quad + \frac{\partial_k \partial_j \Gamma_{il}^m}{- \partial_k \partial_i \Gamma_{jl}^m} \\ &= 0 \end{aligned}$$

By contracting $\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{baed} = 0$ on c and e we obtain

$$\nabla_a R_{bd} - \nabla_b R_{ad} + \nabla_c R_{abd}{}^c = 0$$

Contracting on a different pair of indices produces either the same equation (e.g. a and e) or $0 = 0$ (e.g., a and b).

Contracting on a further pair of indices b and d gives

$$\begin{aligned} 0 &= \nabla_a R - \nabla_b R_a{}^b - \nabla_c R_a{}^c \\ &= \nabla_a R - 2\nabla_b R_a{}^b . \end{aligned} \quad (2.57)$$

Then, defining the Einstein tensor G_{ab} by

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R, \quad (2.58)$$

one can write this last contracted Bianchi identity in the form

$$\nabla_b G^{ab} = 0. \quad (2.59)$$

The contracted identity (2.59) plays a major role in relativity—the field equation is $G_{\alpha\beta} = 8\pi GT_{\alpha\beta}$ and so the equation of motion $\nabla_\beta T^{\alpha\beta} = 0$ is a consequence of the field equation and the Bianchi identity (2.59).

Exercise 42. Use the definition of the curvature tensor,

$$\nabla_{[a}\nabla_{b]}\sigma_c = \frac{1}{2}R_{abc}{}^d\sigma_d,$$

the relation

$$\nabla_a g_{bc} = 0,$$

and the Leibnitz rule to show that for any tensor $T^{ab}{}_c$,

$$\nabla_{[a}\nabla_{b]}T^{cd}{}_e = \frac{1}{2}R_{ab}{}^c{}_f T^{fd}{}_e + \frac{1}{2}R_{ab}{}^d{}_f T^{cf}{}_e + \frac{1}{2}R_{abe}{}^f T^{cd}{}_f$$

Exercise 43. By writing $\nabla_{[a}\nabla_{b]}g_{cd} = 0$, show that $R_{ab(cd)} = 0$. Show that $R_{[abc]d} = 0$, by looking at $\nabla_{[a}\nabla_b\nabla_{c]}f$.

Exercise 44. Show that the symmetries $R_{abcd} = R_{[ab][cd]}$ and $R_{a[bcd]} = 0$ imply $R_{abcd} = R_{cdab}$.

Exercise 45. Prove the Bianchi identity $\nabla_{[a}R_{bc]de} = 0$ this way: The two equations,

$$\begin{aligned}\nabla_a \nabla_b \nabla_c v_d - \nabla_a \nabla_c \nabla_b v_d &= \nabla_a R_{bcd}{}^e v_e + R_{bcd}{}^e \nabla_a v_e \\ \nabla_a \nabla_b \nabla_c v_d - \nabla_b \nabla_a \nabla_c v_d &= R_{abc}{}^e \nabla_e v_d + R_{abd}{}^e \nabla_c v_e,\end{aligned}$$

allow you to write $\nabla_{[a} \nabla_b \nabla_c] v_d$ in two ways. Do so, and equate the resulting expressions to obtain $\nabla_{[a} R_{bc]de} = 0$.

Exercise 46. a. Eq. (2.29), $[u, v]^a = u^b \nabla_b v^a - v^b \nabla_b u^a$, is proved by applying $[u, v]^a$ to an arbitrary scalar f . Prove this relation in a different (shorter) way, by showing that the components of $u^b \nabla_b v^a - v^b \nabla_b u^a$ in a coordinate basis agree with Eq.(2.16), $[u, v]^i = u^j \partial_j v^i - v^j \partial_j u^i$.

b. Fill in the missing steps in the equations leading from Eqs. (2.50) to (2.52) in the notes, to obtain the components of the Riemann tensor in locally inertial coordinates.

Exercise 47. a. Calculate the Riemann tensor for a cylindrical surface, using cylindrical coordinates. You should find that it vanishes. In constructing a cylinder from a flat sheet, you don't stretch the sheet: All angles and intrinsic lengths (measured along the cylinder, not in the 3-dimensional space in which it's embedded) are unchanged.

b. Change to coordinates for which the metric has components δ_{ij} .

2.2.4 Meaning of $R^a{}_{bcd}$

We now carry through two calculations that show the geometrical and physical meaning of the Riemann curvature tensor $R^a{}_{bcd}$. The first relates the Riemann tensor to the path dependence of parallel transport: When the curvature is nonzero, a vector v^a parallel transported around a closed path returns not to its original value, but to a new value $v^a + \delta v^a$ related to v^a by a Lorentz transformation. For small loops, δv^a is proportional to the Riemann tensor and to the area of the loop. The second calculation shows that the relative acceleration of nearby free particles (geodesics) is proportional to $R^a{}_{bcd}$, which can thus be regarded as the gravitational force that can't be transformed away by making measurements in a freely falling frame.

First calculation.

We look at a loop in a two-dimensional surface. Following Wald's notation, we choose coordinates t, s (you could just as well call them x, y or t, x) and look at a loop with sides Δt and Δs as shown in Fig. 2.3. We denote by T^a and S^a the coordinate basis vectors ∂_t and ∂_s , so that $T^a \nabla_a f(t, s) = \partial_t f(t, s)$ for a function f on the surface. We are expecting to get a commutator $[T \cdot \nabla, S \cdot \nabla]$ of derivatives along S and T from going around the path, so look for antisymmetry under $S \leftrightarrow T$ as s and t first increase and then decrease along the path.

We start with a vector v^a at any point P and take the origin of the coordinates to be at P . The goal is to parallel-transport v^a counterclockwise around the loop, from $(0, 0)$ to $(\Delta t, 0)$ to $(\Delta t, \Delta s)$ to $(0, \Delta s)$ and back to the origin. We could find the equation for the components $v^i = v^a e_a^i$ along a basis but will follow Wald's more concise treatment, picking an arbitrary covector field w_a and computing $v^a w_a$. For

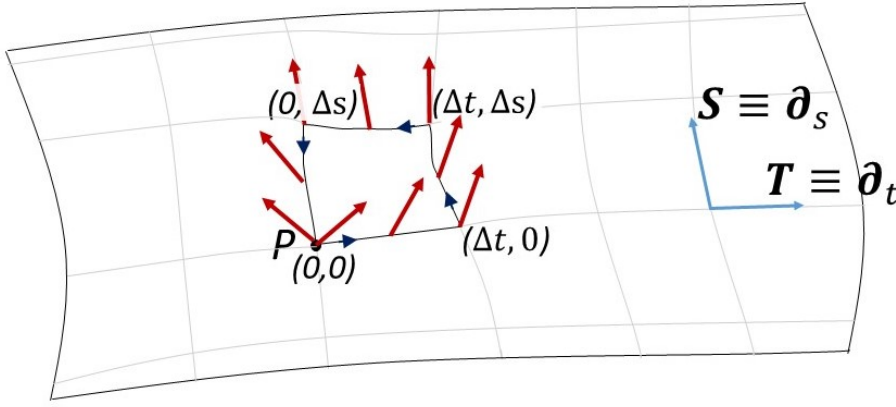


Figure 2.3: The red vector at P pointing rightward is parallel transported counterclockwise around the loop, coming back to P as the vector pointing leftward. Gray lines of constant coordinates t and s are shown with their coordinate basis vectors T^a and S^a (∂_t and ∂_s).

the first leg of the loop, $s = 0$, and t changes from 0 to Δt . We have²

$$(v^a w_a)(\Delta t, 0) = v^a w_a(P) + \partial_t(v^a w_a)|_{(0,0)} \Delta t + \frac{1}{2} \partial_t^2(v^a w_a)|_{(0,0)} (\Delta t)^2 + O(\Delta t^3). \quad (2.60)$$

Now

$$\partial_t(v^a w_a) = T^b \nabla_b(v^a w_a) = v^a T^b \nabla_b w_a$$

where the last equality is the requirement that v^a be parallel translated: $T^b \nabla_b v^a = 0$. The change δ_1 in $v^a w_a$ along the first part of the loop is then

$$\delta_1 = v^a T^b \nabla_b w_a|_{(0,0)} \Delta t + \frac{1}{2} \partial_t^2(v^a w_a)|_{(0,0)} (\Delta t)^2 + O((\Delta t)^3).$$

Along the second leg, $t = \Delta t$, and s changes from 0 to Δs . The change in $v^a w_a$ is then

$$\delta_2 = v^a S^b \nabla_b w_a|_{(\Delta t,0)} \Delta s + \frac{1}{2} \partial_s^2(v^a w_a)|_{(\Delta t,0)} (\Delta s)^2 + O((\Delta s)^3).$$

Similarly along the third leg, where $s = \Delta s$ and t decreases from Δt to 0, we expand around the final point of that leg at $(0, \Delta s)$ (go backwards from $t = 0$ to $t = \Delta t$ and change sign):

$$\delta_3 = -v^a T^b \nabla_b w_a|_{(0,\Delta s)} \Delta t - \frac{1}{2} \partial_t^2(v^a w_a)|_{(0,\Delta s)} (\Delta t)^2 + O((\Delta t)^3).$$

Along the fourth leg, where $t = 0$ and s decreases from Δs to 0, we expand about the final point at $(0, 0)$:

$$\delta_4 = -v^a S^b \nabla_b w_a|_{(0,0)} \Delta s - \frac{1}{2} \partial_s^2(v^a w_a)|_{(0,0)} (\Delta s)^2 + O((\Delta s)^3).$$

In the sum $\delta_1 + \delta_3$, the $O(\Delta t^2)$ terms cancel at second order (their sum is order $\Delta t^2 \Delta s$). The order Δt terms give, to quadratic order,

$$\delta_1 + \delta_3 = - \left[v^a T^b \nabla_b w_a|_{(0,\Delta s)} - v^a T^b \nabla_b w_a|_{(0,0)} \right] \Delta t = -v^a S^c \nabla_c (T^b \nabla_b w_a)|_{(0,0)} \Delta t \Delta s,$$

²Wald evaluates the first derivative on the right side of (2.60) at $(\Delta t/2, 0)$ to get rid of the order $(\Delta t)^2$ term. This is clever, but he pays for it later by having to imagine a second construction, parallel transporting v^a along another segment at $\Delta t/2$.

where we have used $S^c \nabla_c v^a = 0$ along the segment from $(0, 0)$ to $(0, \Delta s)$. Similarly,

$$\delta_2 + \delta_4 = \left[v^a S^b \nabla_b w_a \Big|_{(\Delta t, 0)} - v^a S^b \nabla_b w_a \Big|_{(0, 0)} \right] \Delta s = v^a T^c \nabla_c (S^b \nabla_b w_a) \Big|_{(0, 0)} \Delta t \Delta s.$$

Adding the last two results, gives us a total change that is antisymmetric under $S \leftrightarrow T$:

$$\delta(v^a w_a) \equiv \delta_1 + \delta_2 + \delta_3 + \delta_4 = v^a \left[T^c \nabla_c (S^b \nabla_b w_a) - S^c \nabla_c (T^b \nabla_b w_a) \right] \Delta t \Delta s. \quad (2.61)$$

Because T^a and S^a are the coordinate basis vectors, they commute: $T^c \nabla_c S^b - S^c \nabla_c T^b = 0$ —this is Eq. (2.17). Then

$$\delta(v^a w_a) = v^a T^c S^b (\nabla_c \nabla_b - \nabla_b \nabla_c) w_a \Delta t \Delta s,$$

and we have obtained the desired commutator of two covariant derivatives that defines the Riemann tensor in Eq. (2.46):

$$\delta(v^a w_a) = v^a T^c S^b R_{cba}{}^d w_d \Delta t \Delta s.$$

Changing the names of dummy indices on the right and using on the left the fact that $\delta w_a = 0$ (because w^a is continuous), we can write this equation as $w_a \delta v^a = w_a v^d T^c S^b R_{cbd}{}^a \Delta t \Delta s$. Finally, because this holds for all w_a , we have

$$\delta v^a = v^d T^c S^b R_{cbd}{}^a \Delta t \Delta s. \quad (2.62)$$

The area of the parallelogram spanned by two vectors $\mathbf{A}$ and $\mathbf{B}$ is the magnitude of their cross product or, equivalently, the magnitude of the antisymmetric combination $A^a B^b - A^b B^a$. Here, because $\mathbf{S}$ and $\mathbf{T}$ occur in this antisymmetric combination, $R_{cbd}{}^a (\Delta t T^b) (\Delta s S^c)$ is proportional to the area of the parallelogram spanned by $\Delta t \mathbf{T}$ and $\Delta s \mathbf{S}$ —proportional to the area of the loop, at this order, as claimed.

Because parallel transport preserves lengths and angles, the change in v^a after parallel transport around a loop is a rotation for a positive-definite metric and a Lorentz transformation for a Lorentzian metric. Then δv^a is the change in v^a under an infinitesimal rotation or Lorentz transformation, $\delta \Lambda^a{}_d$.

$$\delta v^a = \delta \Lambda^a{}_d v^d, \text{ where } \delta \Lambda^a{}_d = R_{cbd}{}^a (\Delta t T^b) (\Delta s S^c). \quad (2.63)$$

But $\delta \Lambda_{ad}$ (lowered by the metric) is an infinitesimal rotation or Lorentz transformation if and only if it is antisymmetric. **That's why the second pair of indices of the Riemann tensor is antisymmetric.**

Second Calculation.

In flat space free particles separate with constant speed. Because you can't tell the difference between flat space and a spacetime with a uniform gravitational field (constant gravitational acceleration $\mathbf{g}$ in a Newtonian context), the relative acceleration of free particles depends not on $\mathbf{g}$ but on the derivative of $\mathbf{g}$ – on the tidal force. In GR that means the relative acceleration of nearby geodesics is a measure of the curvature, proportional to the Riemann tensor.

Formally, let T^α be the unit tangent to a first geodesic. In flat space, let $S^\alpha(t)$ be a family of connecting vectors perpendicular to this first geodesic at proper time t , and meeting a second geodesic. The length of the connecting vector is its magnitude, $s = |S|$, and its acceleration vanishes

$$\ddot{S}^\alpha := \frac{d^2}{dt^2} S^\alpha = T^\gamma \nabla_\gamma (T^\beta \nabla_\beta S^\alpha) = 0. \quad (2.64)$$

In curved space nearby geodesics accelerate.

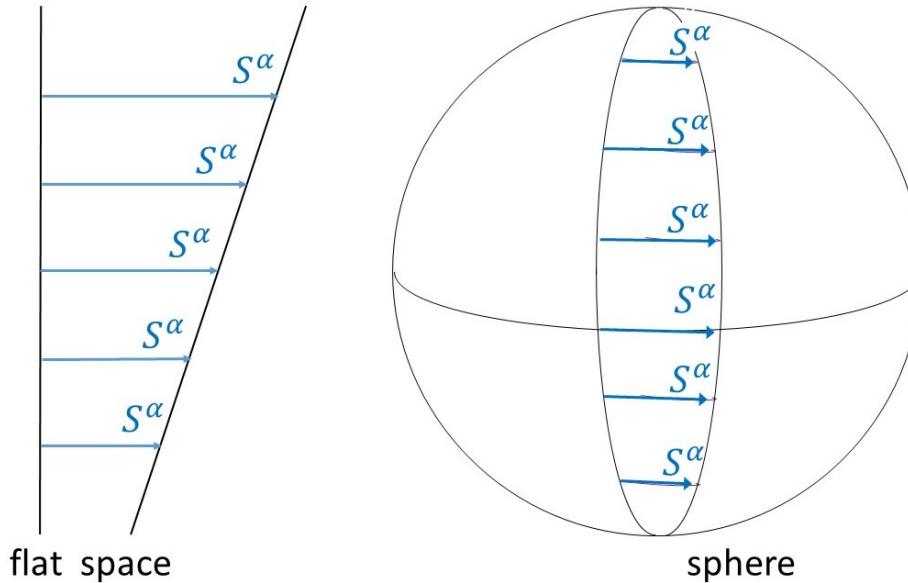


Figure 2.4: At left, geodesics in flat space have no relative acceleration: A connecting vector joining them increases its length at a constant rate. At right, on the surface of a sphere, the relative acceleration between geodesics is nonzero.

We want to define and calculate a relative acceleration between neighboring free particles in a curved spacetime. As in the first calculation, we'll use a two-dimensional surface with coordinates t, s . This time the curves $t \rightarrow c(t, s)$ for fixed s will be timelike geodesics. To construct the surface, start with a smooth spacelike curve $s \rightarrow c(0, s)$ and at each point of that curve, choose a future directed timelike geodesic perpendicular to the curve, each geodesic, $t \rightarrow c(t, s)$, parameterized by proper time.

Then, regarding each geodesic as the trajectory of a free particle, the set of particles at proper time t comprise the curve $s \rightarrow c(t, s)$. The vector S^α tangent to $s \rightarrow c(t, s)$ is called a connecting vector between neighboring particles.³ Nearby particles a parameter distance Δs apart remain at the same parameter distance, and their proper distance is then $|S|\Delta s$ at linear order in Δs . We will see that S remains orthogonal to T , so $|S|\Delta s$ is the proper distance between nearby particles measured orthogonal to either trajectory (to $O(\Delta s)$ it doesn't matter which).

³Wald changes his notation here and calls the connecting vector X instead of S . These notes keep S for consistency.

As in Eq. (2.17), because the vectors $\mathbf{S}$ and $\mathbf{T}$ are the coordinate basis vectors ∂_s and ∂_t , they commute

$$T^\beta \nabla_\beta S^\alpha = S^\beta \nabla_\beta T^\alpha, \quad (2.65)$$

and now T^α is the velocity u^α of the particles. Were we to choose spacelike geodesics, T^α would be a unit spacelike vector.

From (2.65), the statement that S^α remains orthogonal to T^α along the particle trajectories follows immediately from the geodesic equation for T^α and the fact that it is a unit vector:

$$\begin{aligned} T^\beta \nabla_\beta (T^\alpha S_\alpha) &= \underbrace{T^\beta \nabla_\beta T^\alpha}_{=0} S_\alpha + T^\beta T^\alpha \nabla_\beta S_\alpha \\ &= T^\alpha S^\beta \nabla_\beta T_\alpha \quad \text{by (2.65)} \\ &= 0, \quad \text{because } T^\alpha \text{ is a unit vector.} \end{aligned}$$

The separation vector S^α changes at the rate $\dot{S}^\alpha$ with $\dot{S}^\alpha = T^\beta \nabla_\beta S^\alpha$, and the relative acceleration of neighboring geodesics is then $\ddot{S}^\alpha \Delta s$. We have

$$\begin{aligned} \ddot{S}^\alpha &= T^\beta \nabla_\beta (T^\gamma \nabla_\gamma S^\alpha) \\ &= T^\beta \nabla_\beta (S^\gamma \nabla_\gamma T^\alpha) \quad \text{by (2.65)} \\ &= (T^\beta \nabla_\beta S^\gamma) \nabla_\gamma T^\alpha + T^\beta S^\gamma \nabla_\beta \nabla_\gamma T^\alpha \\ &= (S^\beta \nabla_\beta T^\gamma) \nabla_\gamma T^\alpha + T^\beta S^\gamma (\nabla_\gamma \nabla_\beta T^\alpha + [\nabla_\beta, \nabla_\gamma] T^\alpha) \\ &= (S^\beta \nabla_\beta T^\gamma) \nabla_\gamma T^\alpha + T^\beta S^\gamma \nabla_\gamma \nabla_\beta T^\alpha + T^\beta S^\gamma R_{\beta\gamma}{}^\alpha{}_\delta T^\delta \\ &= \underbrace{(S^\beta \nabla_\beta T^\gamma) \nabla_\gamma T^\alpha}_{=0} + S^\gamma \nabla_\gamma \underbrace{(T^\beta \nabla_\beta T^\alpha)}_{=0} - \underbrace{(S^\gamma \nabla_\gamma T^\beta) \nabla_\beta T^\alpha}_{=0} + T^\beta S^\gamma R_{\beta\gamma}{}^\alpha{}_\delta T^\delta \\ \ddot{S}^\alpha &= R^\alpha{}_{\beta\gamma\delta} T^\beta T^\gamma S^\delta \end{aligned} \quad (2.66)$$

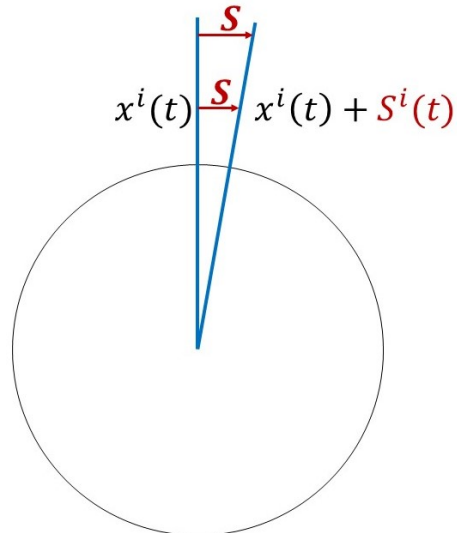
This is the *equation of geodesic deviation*, first obtained in 1926 by Levi-Civita [54] and by Synge.[85]

An observer moving along a geodesic (a freely falling observer) does not detect a uniform gravitational field — e.g. a man in a freely falling elevator in the Earth's field sees objects behaving almost as if there were no gravity. But because the Earth's field is not really uniform, he will see their relative acceleration, the relative acceleration of two particles falling toward the center of the Earth.

Newtonian approximation: For two nearby particles with separation vector $\mathbf{S}$ (here, for brevity, $\mathbf{S}$ is written instead of $\Delta s \mathbf{S}$),

$$\begin{aligned} \ddot{S}^i &= \ddot{x}^i + \ddot{S}^i - \ddot{x}^i \\ &= -\nabla^i \phi(\mathbf{x} + \mathbf{S}) + \nabla^i \phi(\mathbf{x}) \\ &= -\nabla^i \nabla_j \phi(\mathbf{x}) S^j. \end{aligned} \quad (2.67)$$

The magnitude of this relative acceleration is of order $\frac{|\mathbf{S}|}{R} \nabla \phi$, smaller than the “gravitational acceleration” $\nabla \phi$ by the ratio of the observer-particle distance to the radius of the Earth. It is this tidal force that alone reveals the presence of a gravitational field to a local observer.



In III these remarks will be made again in the context of a discussion of the equivalence principle and will motivate the form of the field equations via the correspondence between equations (2.66) and (2.67).

A note on interpretation: The calculation required only that the curves $s \rightarrow c(t, s)$ be smooth and spacelike. To interpret s as a distance between nearby particles, s should be proper distance along each curve, and each of these curves should be a spacelike geodesic orthogonal to the particle trajectory $t \rightarrow c(t, 0)$. We can always choose s to be proper distance along the curve $s \rightarrow (t, s)$, and we can choose the initial curve $s \rightarrow (0, s)$ to be a spacelike geodesic. But once we choose a set of timelike geodesics through each point of $c(0, s)$, the remaining curves $s \rightarrow c(t, s)$ are fixed: They are the position of the particles at proper time t ; equivalently, they are the result of Lie dragging $t \rightarrow c(0, s)$ a parameter distance t along T^α . It is only at zeroth order in s that these remaining curves are geodesics and only at zeroth order are they orthogonal to a particle trajectory.

A final comment on the Riemann tensor: If one chooses Riemann normal coordinates in a neighborhood of a point P , the components of the metric near P can be written in terms of the Riemann tensor and its derivatives at P . That is, the Taylor series of $g_{\mu\nu}(x)$ about P has the form

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\sigma\nu\tau}x^\sigma x^\tau + O(|x|^3), \quad (2.68)$$

where the higher order terms involve derivatives of $R_{\mu\sigma\nu\tau}$. (See, for example, MTW Sect. 11.6, or Leo Brewin's notes [brewin96](#).) The expansion is used in quantum field theory on curved spacetimes and in work on extreme mass-ratio inspiral (EMRI). In EMRI work, one uses point-particles moving in a background spacetime to approximate stellar size black holes spiraling into the vastly more massive black holes at the centers of galaxies.

Exercise 48. Wald's way of defining Christoffel symbols. The Christoffel symbols Γ^i_{jk} associated with a chart $\{x^i\}$ are not naturally the components of a tensor, and neither are the quantities $\partial_i v^j$. One can, however, simply *define* a tensor $\partial_a v^b$ to be the tensor whose components in the chart $\{x^i\}$, are $\partial_i v^j$. Then, in another chart x' , the components will *not* be $\partial_{i'} v^{j'}$. Check that defining $\partial_a v^b$ in this way is equivalent to defining ∂_a to be the covariant derivative of the artificial flat metric $(dx^1)^2 + \cdots + (dx^n)^2$ or $-(dx^0)^2 + \cdots + (dx^3)^2$. (This should be a one- or two-line argument.)

Exercise 49. Show that the difference $\nabla_a - \partial_a$ between the two derivative operators acts as a tensor, that it is linear under multiplication by scalar fields: $(\nabla_a - \partial_a)(f v^b) = f (\nabla_a - \partial_a)v^b$, for any smooth scalar field f . Then the action of $\nabla_a - \partial_a$ has the form

$$(\nabla_a - \partial_a)v^b = \Gamma^b_{ac} v^c,$$

where Γ^b_{ac} is a tensor field.

Exercise 50. Show that the components of Γ^b_{ac} in the coordinates $\{x^i\}$ are the usual Christoffel symbols Γ^j_{ik} . (This should be quick. You do not have to compute the form of Γ in terms of metric derivatives.)

Exercise 51. The notation has a problem if one wants to write components of $\partial_a v^b$ in in a new chart x' . In polar coordinates θ, ϕ for the unit sphere, let ∂_a be the covariant derivative associated with the flat metric $d\theta^2 + d\phi^2$. Show that the components of $\partial_a v^b$ in a chart x, z , where $z = a \cos \theta$, $x = a \sin \theta \cos \phi$ are not the partial derivatives: To avoid confusion, write $T^a_b \equiv \partial_a v^b$. Find T^x_x and T^x_z , and compare to $\frac{\partial v^x}{\partial x}, \frac{\partial v^x}{\partial z}$.

Exercise 52. The next two problems are simple Riemann tensor examples. If you have already done similar calculations, you would learn more by finding components in an orthonormal frame: See the starred version of the problems below.

- Find the independent components of the curvature tensor for a sphere of radius a , with metric $ds^2 = a^2(d\theta^2 + \sin^2 \theta d\phi^2)$. No need to recompute Christoffel symbols that you already know.
- Find all components of the Ricci tensor and Ricci scalar.

Exercise 53. (Essentially Problem 7 of Wald, Chap. 3). An arbitrary Lorentzian metric on a 2-manifold can always be written locally in the form $ds^2 = \Omega^2(-dt^2 + dx^2)$. Find the Riemann tensor of this metric.

Exercise 54. Show that the Riemann tensor in n dimensions has $\frac{n^2(n^2 - 1)}{12}$ independent components. First show that the symmetry $R_{abcd} = R_{cdab}$ is implied by the other index symmetries. Because of that, it is not used in the counting.

Variants of *Exercise 52.* and *Exercise 53.*:

Do *Exercise 52.* using the orthonormal frame $e_1 = \frac{1}{a}\partial_\theta, e_2 = \frac{1}{a \sin \theta}\partial_\phi$ or the dual basis $\omega^1 = a d\theta, \omega^2 = a \sin \theta d\phi$. You can either use commutators or Cartan calculus to find the Christoffel symbols.

Do *Exercise 53.* using the orthonormal frame $e_0 = \frac{1}{\Omega}\partial_t, e_1 = \frac{1}{\Omega}\partial_x$ or the dual basis $\omega^0 = \Omega dt, \omega^1 = \Omega dx$.

2.3 Lie Derivatives

This section and the sections on forms and integration that follow are largely taken from an appendix of the book *Rotating Relativistic Stars*, by JF and Nikolaos Stergioulas.[39]

Lie derivatives arise naturally in the context of fluid flow. They are needed to understand symmetries of spacetime and of matter in curved spacetimes, and they simplify calculations and aid one's understanding of relativistic fluids.

We have just seen that the connecting vector $\mathbf{S}$ between nearby particles with velocity $\mathbf{T}$ satisfies $[\mathbf{T}, \mathbf{S}] = 0$, and this will be the statement that the Lie derivative of $\mathbf{S}$ by the vector field $\mathbf{T}$ vanishes. We begin, for simplicity, in a Newtonian context, with a stationary fluid flow with 3-velocity $\mathbf{v}(\mathbf{r})$. A function f is said to be *dragged along* by the fluid flow, or *Lie-derived* by the vector field $\mathbf{v}$ that generates the flow, if the value of f is constant on a fluid element, that is, constant along a fluid trajectory $\mathbf{r}(t)$:

$$\frac{d}{dt}f[\mathbf{r}(t)] = \mathbf{v} \cdot \nabla f = 0. \quad (2.69)$$

The *Lie derivative* of a function f , defined by

$$\mathcal{L}_{\mathbf{v}}f = \mathbf{v} \cdot \nabla f, \quad (2.70)$$

is the directional derivative of f along $\mathbf{v}$, the rate of change of f measured by a comoving observer. If f is density or temperature, say, f is dragged along by the flow if the density or temperature of a fluid element remains constant as the fluid element moves with the flow.

For a vector, imagine an arrow painted on a river of sliding jello, the arrow dragged along as the jello moves and oscillates. That is, consider a vector that joins two nearby fluid elements, two points $\mathbf{r}(t)$ and $\bar{\mathbf{r}}(t)$ that move with the fluid: Call the connecting vector $\lambda \mathbf{w}$, so that for small λ the fluid elements are nearby: $\lambda \mathbf{w} = \bar{\mathbf{r}}(t) - \mathbf{r}(t)$. Then $\lambda \mathbf{w}$ is said to be *dragged along* by the fluid flow, as shown in Fig. (2.5). In the figure, the endpoints of $\mathbf{r}(t_i)$ and $\bar{\mathbf{r}}(t_i)$ are labeled $\mathbf{r}_i$ and $\bar{\mathbf{r}}_i$.

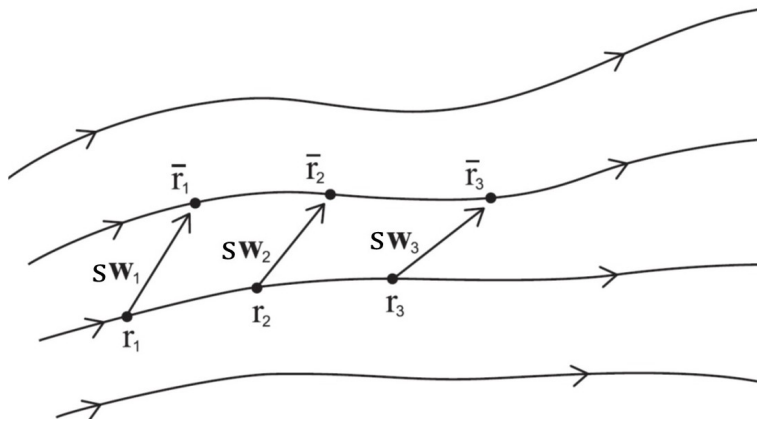


Figure 2.5: Two nearby fluid elements move along the flow lines, their successive positions labeled $\mathbf{r}_i$ and $\bar{\mathbf{r}}_i$. A vector field $s\mathbf{w}$ is said to be dragged along by the flow when, as shown here, it connects successive positions of two nearby fluid elements.

A vector field $\mathbf{w}$ is *Lie-derived* by $\mathbf{v}$ if, for small s , $s\mathbf{w}$ is dragged along by the fluid flow. To make this precise, we are requiring that the equation

$$\mathbf{r}(t) + s\mathbf{w}(\mathbf{r}(t)) = \bar{\mathbf{r}}(t) \quad (2.71)$$

be satisfied to $O(s)$. Taking the derivative of both sides of the equation with respect to t at $t = 0$, we have

$$\begin{aligned} \mathbf{v}(\mathbf{r}) + s\mathbf{v} \cdot \nabla \mathbf{w}(\mathbf{r}) &= \mathbf{v}(\bar{\mathbf{r}}) = \mathbf{v}[\mathbf{r} + s\mathbf{w}(\mathbf{r})] \\ &= \mathbf{v}(\mathbf{r}) + s\mathbf{w} \cdot \nabla \mathbf{v}(\mathbf{r}) + O(s^2), \end{aligned} \quad (2.72)$$

which holds if and only if

$$[\mathbf{v}, \mathbf{w}] \equiv \mathbf{v} \cdot \nabla \mathbf{w} - \mathbf{w} \cdot \nabla \mathbf{v} = 0. \quad (2.73)$$

The commutator $[\mathbf{v}, \mathbf{w}]$ is the *Lie derivative* of $\mathbf{w}$ with respect to $\mathbf{v}$, written

$$\mathcal{L}_{\mathbf{v}}\mathbf{w} = [\mathbf{v}, \mathbf{w}]. \quad (2.74)$$

Then $\mathbf{w}$ is Lie-derived by $\mathbf{v}$ when $\mathcal{L}_{\mathbf{v}}\mathbf{w} = 0$. The Lie derivative $\mathcal{L}_{\mathbf{v}}\mathbf{w}$ compares the change in the vector field $\mathbf{w}$ in the direction of $\mathbf{v}$ to the change that would occur if $\mathbf{w}$ were dragged along by the flow generated by $\mathbf{v}$.

In a curved spacetime the Lie derivative of a function f is again its directional derivative,

$$\mathcal{L}_{\mathbf{u}}f = u^\alpha \nabla_\alpha f. \quad (2.75)$$

If u^α is the 4-velocity of a fluid, generating the fluid trajectories in spacetime, $\mathcal{L}_{\mathbf{u}}f$ is commonly termed the convective derivative of f . The Newtonian limit of u^α is the 4-vector $\partial_t + \mathbf{v}$, and $\mathcal{L}_{\mathbf{u}}f$ has as its limit the Newtonian convective derivative $(\partial_t + \mathbf{v} \cdot \nabla)f$, again the rate of change of f measured by a comoving observer. (Now the flow is arbitrary, not the stationary flow of our earlier discussion.)

A connecting vector is naturally a contravariant vector, the tangent to a curve joining nearby points in a flow; and in a curved spacetime, the Lie derivative of a contravariant vector field is again defined by Eq. (2.74),

$$\mathcal{L}_{\mathbf{u}}w^\alpha = u^\beta \nabla_\beta w^\alpha - w^\beta \nabla_\beta u^\alpha. \quad (2.76)$$

We have used a fluid flow generated by a 4-velocity u^α to motivate a definition of Lie derivative; the definition, of course, is the same in any dimension and for any vector fields:

$$\mathcal{L}_{\mathbf{v}}w^a = v^b \nabla_b w^a - w^b \nabla_b v^a. \quad (2.77)$$

Although the covariant derivative operator ∇ appears in the above expression, we have already seen that the commutator is in fact independent of the choice of derivative operator. This was implied by our definition of the commutator (which did not involve covariant derivatives); it also follows from the symmetry $\Gamma_{jk}^i = \Gamma_{(jk)}^i$, which implies that the components have in any chart the form

$$\mathcal{L}_{\mathbf{v}}w^i = v^j \partial_j w^i - w^j \partial_j v^i. \quad (2.78)$$

We now extend the definition of Lie derivative to arbitrary tensors. This just involves using the definitions for vectors and scalars and the Leibnitz rule. For any covector σ_a and vector w^a , we have

$$\begin{aligned} \mathcal{L}_{\mathbf{v}}(\sigma_a w^a) &= (\mathcal{L}_{\mathbf{v}}\sigma_a)w^a + \sigma_a \mathcal{L}_{\mathbf{v}}w^a \\ v^b \nabla_b \sigma_a w^a + \cancel{\sigma_a v^b \nabla_b w^a} &= (\mathcal{L}_{\mathbf{v}}\sigma_a)w^a + \cancel{\sigma_a v^b \nabla_b w^a} - \sigma_a w^b \nabla_b v^a \implies \\ w^a \mathcal{L}_{\mathbf{v}}\sigma_a &= w^a (v^b \nabla_b \sigma_a + \sigma_b \nabla_a v^b). \end{aligned} \quad (2.79)$$

Because this holds for all w^a , we have

$$\mathcal{L}_{\mathbf{v}}\sigma_a = v^b \nabla_b \sigma_a + \sigma_b \nabla_a v^b. \quad (2.80)$$

Again the definition (2.80) is independent of the choice of derivative operator, because the Lie derivatives of the scalar $\sigma_a v^a$ and the vector v^a in Eq. (2.79) do not depend on ∇ . The components in any chart are given by

$$\mathcal{L}_{\mathbf{v}}\sigma_i = v^j \partial_j \sigma_i + \sigma_j \partial_i v^j. \quad (2.81)$$

Finally, the Lie derivative of an arbitrary tensor $T^{a\cdots b}_{c\cdots d}$ again follows from the Leibnitz rule (writing a tensor as a sum of products $u^a \cdots v^b \sigma_a \cdots \tau_d$ as usual). You can quickly see from Eqs. (2.77) and (2.80) that the result is

$$\begin{aligned} \mathcal{L}_{\mathbf{v}} T^{b\cdots c}_{a\cdots e} &= v^a \nabla_a T^{b\cdots c}_{d\cdots e} \\ &\quad - T^{a\cdots b}_{c\cdots d} \nabla_a v^b - \cdots - T^{b\cdots a}_{d\cdots e} \nabla_a v^c \\ &\quad + T^{b\cdots c}_{a\cdots e} \nabla_d v^a + \cdots + T^{b\cdots c}_{d\cdots a} \nabla_e v^a, \end{aligned} \quad (2.82)$$

independent of the derivative operator, and with components in a chart given by

$$\begin{aligned} \mathcal{L}_{\mathbf{v}} T^{j\cdots k}_{m\cdots n} &= v^k \partial_k T^{j\cdots k}_{m\cdots n} \\ &\quad - T^{k\cdots k}_{m\cdots n} \partial_k v^j - \cdots - T^{j\cdots k}_{m\cdots n} \partial_k v^k \\ &\quad + T^{j\cdots k}_{k\cdots n} \partial_m v^k + \cdots + T^{j\cdots k}_{m\cdots k} \partial_n v^k. \end{aligned} \quad (2.83)$$

Action of diffeos: A smooth 1-1- map of a manifold onto itself or onto another manifold is called a *diffeo* (or *diffeomorphism*) if its inverse is also smooth. We began by using the flow generated by a velocity field $\mathbf{v}$ to motivate the definition (2.74) of Lie derivative of a vector field $\mathbf{w}$. It is useful to see formally the way in which any vector field generates a flow and to use that flow to give a geometrical definition of Lie derivative. Note first that the trajectory of a fluid element is an *integral curve* of the vector field u^α , where:

Definition. An integral curve $c(\lambda)$ of a vector field v^a is a curve whose tangent vector at each point $P = c(\lambda_0)$ is $v^a(P)$.

In a chart $\{x^i\}$, the tangent v^i to an curve $c(\lambda)$ has components $\frac{d}{d\lambda}x^i(\lambda)$; and the statement that $c(\lambda)$ is an integral curve has the form $\frac{d}{d\lambda}x^i(\lambda) = v^i[c(\lambda)]$.

Proposition. Any smooth vector field v^a in an n -dimensional manifold M has a family of *integral curves*, one through each point of M .⁴

Example 1: As noted, the velocity field u^α of a fluid has as its integral curves the fluid trajectories parameterized by proper time. The 3-dimensional vector field $\mathbf{v}$ of a stationary Newtonian flow has as its integral curves the flow lines, parameterized by Newtonian time.

Example 2: The vector field $\partial_\phi = x\partial_y - y\partial_x$ has as integral curves the lines of constant t, r, θ ,

$$\lambda \rightarrow (t, r, \theta, \phi + \lambda)$$

Note that when a vector field vanishes at P (e.g., ∂_ϕ vanishes on the symmetry axis $x = y = 0$), the integral curve simply stays at P : $c(\lambda) = P$.

⁴This result is equivalent to the existence theorem for solutions to ordinary differential equations, proved, for example, in Coddington and Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, 1995.

We can view the 4-dimensional flow of a fluid as a family of smooth maps of the fluid to itself in the following way: In a given proper time τ each point P in the fluid moves along the fluid trajectory through P from $c(0) = P$ to the point $c(\tau)$. As in the case of the 4-velocity, we can use the integral curves of any vector field to define a family ψ_λ of diffeos of a manifold to itself (for a star, the fluid has a boundary, and the map ψ_τ is from the support of the fluid to itself):

For each point P , let $c(\lambda)$ be the integral curve of v^a for which $P = c(0)$. For a fixed value λ , define the map ψ_λ by

$$\psi_\lambda(P) = c(\lambda). \quad (2.84)$$

That is, ψ_λ maps each point P to the point a parameter distance λ from P along the integral curve through P . The vector field v^a is said to *generate* the family ψ_λ of diffeos. In a chart $\{x^i\}$, we have

$$v^i(x) = \left. \frac{d}{d\lambda} \psi_\lambda^i(x) \right|_{\lambda=0}. \quad (2.85)$$

Example: The vector field $\varphi := \partial_\phi$ generates the family of diffeos

$$(t, r, \theta, \phi) \rightarrow (t, r, \theta, \phi + \lambda),$$

rotations by λ in the x - y plane about the axis where ∂_ϕ vanishes. A spacetime is axisymmetric if its metric is invariant under rotations, if $\mathcal{L}_\varphi g_{\alpha\beta} = 0$ or, equivalently, $\psi_\lambda^* g_{\alpha\beta} = 0$.

We can now repeat for manifolds the relation with which we began this section, between the flow - the diffeos - generated by a vector field and the Lie derivative. We need the action of a smooth map ψ on a tensor, beginning with its action on a curve and on a function, and from that finding its action on a vector as a tangent to the curve, and on a covector as the gradient of the function. For Lie derivatives, we could look only at diffeos from a manifold to itself, but the action of ψ on vectors and covectors needs only a smooth map, and you'll use the more general description when you encounter the 3+1 formalism and the initial value problem in the GR literature or later in the notes.

Let $\psi : M \rightarrow N$ be a smooth map from one manifold to another. A curve on M is mapped by ψ to a curve on N . One says that the curve $c : \mathbb{R} \rightarrow M$ is *dragged* or *carried along* or *pushed forward* by ψ to the curve $\psi \circ c : \mathbb{R} \rightarrow N$.

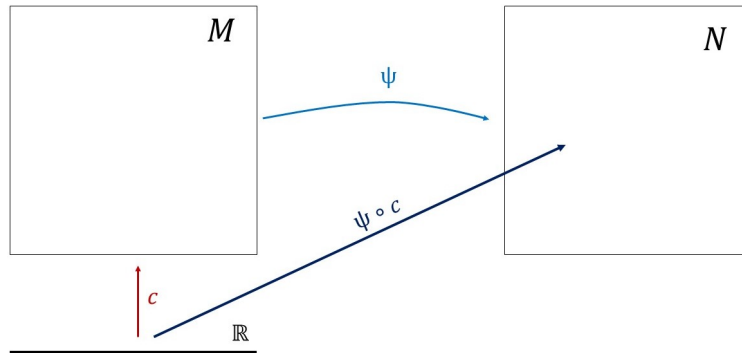


Figure 2.6

A function $f : N \rightarrow \mathbb{R}$ on N is *pulled back* to the function $f \circ \psi : M \rightarrow \mathbb{R}$ on M .

Then the tangent v to the curve c at a point $P = c(0)$ is dragged (or carried along) to the tangent ψ^*v to the curve $\psi \circ c$ at the point $\psi(P)$. And the gradient ∇f at a point $\psi(P)$ in N is pulled back to the gradient $\psi_* \nabla f := \nabla(f \circ \psi)$ of $f \circ \psi$ at P .⁵

⁵In general, ψ need not be 1-1, and this pullback gives a gradient at all points whose image is $\psi(P)$.

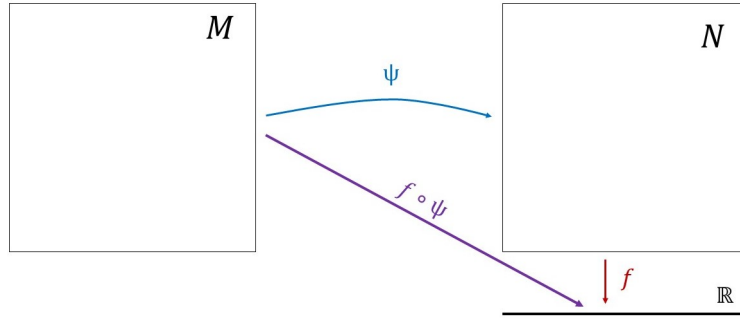


Figure 2.7

Because ψ^*v is tangent to $\psi \circ c$, the vector ψ^*v is the map $\{f\} \rightarrow \mathbb{R}$ given by

$$\psi^*v(f) = \frac{d}{d\lambda} f \circ \psi \circ c(\lambda) \Big|_{\lambda=0}. \quad (2.86)$$

Because $\psi_* \nabla f$ is the gradient of $f \circ \psi$, the right side of Eq. (2.86) is also $\psi_* \nabla f (v)$. That is, the action of the covector $\psi_* \nabla f$ on v is

$$\psi_* \nabla f (v) = \psi^*v(f) = \frac{d}{d\lambda} f \circ \psi \circ c(\lambda) \Big|_{\lambda=0}. \quad (2.87)$$

In charts x and y on M and N , the curve c has coordinates $x^i(\lambda)$, the dragged curve $\psi \circ c$ has coordinates $y^\mu[x(\lambda)]$, and we have

$$(\psi^*v)^\mu = \frac{d}{d\lambda} y^\mu[x(\lambda)] = \frac{\partial y^\mu}{\partial x^i} \frac{dx^i}{d\lambda} = \frac{\partial y^\mu}{\partial x^i} v^i,$$

or

$$(\psi^*v)^\mu|_{\psi(P)} = \frac{\partial y^\mu}{\partial x^i} v^i|_P. \quad (2.88)$$

The different indices here are used because M and N need not have the same dimension.

Similarly, the pulled-back gradient $\psi_* \nabla f = \nabla(f \circ \psi)$ has components given by

$$\nabla_i(f \circ \psi) = \frac{\partial}{\partial x^i} f[y(x)] = \frac{\partial f}{\partial y^\mu} \frac{\partial y^\mu}{\partial x^i} = \nabla_\mu f \frac{\partial y^\mu}{\partial x^i}. \quad (2.89)$$

Because any covector σ at a point $\psi(P)$ of N can be represented by a gradient on N , its components are

$$(\psi_*\sigma)_i = \sigma_\mu \frac{\partial y^\mu}{\partial x^i}. \quad (2.90)$$

The action of $\psi_*\sigma$ on the vector v is then

$$\psi_*\sigma(v) = \sigma(\psi^*v), \quad (2.91)$$

implied by Eq. (2.87) or by multiplying both sides of Eq. (2.88) by σ_μ . (Wald takes (2.91) as the definition of ψ_* .)

Summary:

A map $\psi : M \rightarrow N$ drags a curve c and its tangent vector v on M to the curve $\psi \circ c$ with tangent ψ^*v on N .

It pulls back a function f and its gradient ∇f on N to a function $f \circ \psi$ and its gradient $\psi_* \nabla f$ on M . Hence

ψ pulls back a covector σ on N to a covector $\psi_* \sigma$ on M .

Any contravariant tensor $T^{a \cdots b}$ is dragged by ψ from M to $(\psi^* T)^{\alpha \cdots \beta}$ on N because $T^{a \cdots b}$ can be written as a sum of products $u^a \cdots v^b$, each of which is dragged by ψ . Similarly, any covariant tensor $T_{\alpha \cdots \beta}$ is pulled back by ψ from N to $(\psi_* T)_{a \cdots b}$ on M because $T_{\alpha \cdots \beta}$ can be written as a sum of products $\sigma_\alpha \cdots \tau_\beta$, each of which is pulled back by ψ . Equivalently, acting on a covariant tensor $T_{\alpha \cdots \beta}$,

$$\psi_* T(u, \dots, v) = T(\psi^* u, \dots, \psi^* v), \quad (2.92)$$

and on a contravariant tensor $T^{a \cdots b}$,

$$\psi^* T(\sigma, \dots, \tau) = T(\psi_* \sigma, \dots, \psi_* \tau). \quad (2.93)$$

Notation. We have followed Wald in writing $\psi^* v$ for the dragged vector, and $\psi_* \sigma$ for the pulled back covector. This is opposite to the convention in most of the mathematics and mathematical physics literature.

Exercise. Find the components $(\psi_* T)_{i \cdots j}$ and $(\psi^* T)^{\mu \cdots \nu}$.

But ψ does not know how to drag a covector or pull back a vector. To talk about the action of a map on a general tensor, the map ψ must have an inverse – it must be a diffeomorphism, and we now consider only diffeomorphisms. By using $\psi^{-1} : N \rightarrow M$ to drag a covector from N to M , we can extend ψ^* to arbitrary tensors. Given a tensor $T^{a \cdots b}_{c \cdots d}$ on M , we define the dragged tensor $\psi^* T^{a \cdots b}_{c \cdots d}$ on N by

$$\psi^* T(\sigma, \dots, \tau, u, \dots, v) = T(\psi_*^{-1} \sigma, \dots, \psi_*^{-1} \tau, \psi^* u, \dots, \psi^* v). \quad (2.94)$$

Dragging a tensor from M to N by ψ^* is then equivalent to pulling it back from M to N by ψ_*^{-1} .

Exercise. Take $M = N$. Because there is only one coordinate system, denote by $\psi^i(P)$ the coordinates of $\psi(P)$ and write $\psi^i(x)$ for ψ regarded as a function of the coordinates: ψ^i maps $(x^1, \dots, x^n)$ to $(\psi^1(x), \dots, \psi^n(x))$. Show for a tensor T_b^a that the components of $\psi_* T$ are

$$(\psi_* T)_j^i = \frac{\partial \psi^i}{\partial x^k} \frac{\partial \psi^{-1l}}{\partial x^j} T_l^k. \quad (2.95)$$

Lie derivatives from a geometric standpoint.

The geometric definition of the Lie derivative $\mathcal{L}_u$ of functions and vectors just restates the way we introduced Lie derivatives for a fluid flow. For a fluid with velocity u^α , the Lie derivative $\mathcal{L}_u f$ of a function on the fluid is the directional derivative of f along u^α ,

$$\mathcal{L}_u f = \frac{d}{d\tau} f \circ \psi_\tau(P)|_{\tau=0} = u^\alpha \nabla_\alpha f. \quad (2.96)$$

Now $f \circ \psi_\tau$ is the pullback of f by ψ_τ or, equivalently, the result of dragging f by $\psi_\tau^{-1} = \psi_{-\tau}$. The Lie derivative subtracts the value of f at P from its value at $\psi_\tau(P)$, and we have stated this difference in a way that makes sense for vectors in curved space, where one knows only how to add and subtract vectors at the same point: That is, to define the Lie derivative $\mathcal{L}_u w^\alpha$ of a vector, we again subtract the value of w^α at P from the result of dragging w^α from $\psi_\tau(P)$ back to P by $\psi_{-\tau}$:

$$\mathcal{L}_u w^\alpha = \frac{d}{d\tau} \psi_{-\tau}^* w^\alpha. \quad (2.97)$$

And in general, for any for any smooth scalar f and two smooth vector fields v^a and w^a on a manifold,

$$\mathcal{L}_v f = \frac{d}{d\lambda} f \circ \psi_\lambda(P) \Big|_{\lambda=0}, \quad (2.98)$$

$$\mathcal{L}_v w^a = \frac{d}{d\lambda} \psi_{-\lambda}^* w^a, \quad (2.99)$$

where ψ_λ is the family of diffeos generated by v^a .

A vector field w^a is Lie-derived by a vector field v^a if the family of diffeos ψ_λ generated by v^a leave w^a unchanged, if $\psi_\lambda w^a = w^a$.

To check that our geometric definition (2.99) of the Lie derivative of a vector agrees with the commutator $[v, w]$, we compute the right side of Eq. (2.99), writing

$$\psi_{-\lambda}^* w^i = \frac{\partial \psi_{-\lambda}^i}{\partial x^j} w^j \circ \psi_\lambda(P). \quad (2.100)$$

When we take the derivative $\frac{d}{d\lambda}$ of this expression, we get two terms, with the derivative hitting each ψ_λ while the other ψ_λ is evaluated at $\lambda = 0$ where it is the identity:

$$\frac{d}{d\lambda} \left[\frac{\partial \psi_{-\lambda}^i}{\partial x^j} w^j \circ \psi_\lambda(P) \right]_{\lambda=0} = \frac{d}{d\lambda} \left(\frac{\partial \psi_{-\lambda}^i}{\partial x^j} \right) w^j(P) + \frac{\partial \psi_{-\lambda}^i}{\partial x^j} \Big|_{\lambda=0} \frac{d}{d\lambda} w^j(\psi_\lambda(P)) \quad (2.101)$$

Now $\frac{d}{d\lambda} w^j \circ \psi_\lambda(P) \Big|_{\lambda=0}$ is just the Lie derivative (2.98) of the scalar w^j :

$$\frac{d}{d\lambda} w^j \circ \psi_\lambda(P) \Big|_{\lambda=0} = v^k \partial_k w^j;$$

the first term is what distinguishes the Lie derivative of a vector from that of a scalar:

$$\frac{d}{d\lambda} \left[\frac{\partial \psi_{-\lambda}^i}{\partial x^j} \right]_{\lambda=0} = \frac{\partial}{\partial x^j} \left(\frac{d\psi_{-\lambda}^i}{d\lambda} \right)_{\lambda=0} = -\partial_j v^i.$$

So we have

$$\begin{aligned} \frac{d}{d\lambda} \psi_{-\lambda}^* w^i &= \frac{d}{d\lambda} \left(\frac{\partial \psi_{-\lambda}^i}{\partial x^j} \right) w^j(P) + \frac{\partial \psi_{-\lambda}^i}{\partial x^j} \Big|_{\lambda=0} \frac{d}{d\lambda} w^j(\psi_\lambda(P)) \\ &= -\partial_j v^i w^j + v^k \partial_k w^i, \end{aligned}$$

where we have used $\psi_0^i(x) = x^i$ to write $\frac{\partial \psi_{-\lambda}^i}{\partial x^j} \Big|_{\lambda=0} = \frac{\partial x^i}{\partial x^j} = \delta_j^i$ (the Jacobian of the identity map is the identity map). We have shown

$$\frac{d}{d\lambda} \psi_{-\lambda}^* w = [v, w],$$

as claimed.

A nearly identical calculation checks the components of the Lie derivative $\mathcal{L}_v \sigma_a$ of a covector field: Suppressing “ $\lambda = 0$,” we have

$$\begin{aligned} \frac{d}{d\lambda} \psi_{-\lambda}^* \sigma_i &= \frac{d}{d\lambda} \psi_{\lambda*} \sigma_i = \frac{d}{d\lambda} \left(\frac{\partial \psi_\lambda^j}{\partial x^i} \right) \sigma_j(P) + \frac{\partial \psi_\lambda^j}{\partial x^i} \frac{d}{d\lambda} \sigma_j(\psi_\lambda(P)) \\ &= \partial_i v^j \sigma_j + v^j \partial_j \sigma_i, \end{aligned}$$

in agreement with Eq. (2.81).

Finally, the Lie derivative of any tensor is given by

$$\mathcal{L}_v T^{a \cdots b}_{c \cdots d} = \frac{d}{d\lambda} \psi_{-\lambda}^* T^{a \cdots b}_{c \cdots d}, \quad (2.102)$$

and a calculation essentially identical to those we have just done for vectors and covectors verifies that the definition yields Eq. (2.83).

Killing Vectors

A common use of Lie derivatives in relativity arises from vector fields that generate symmetries of the spacetime. That is, a spacetime is symmetric under the diffeo ψ if $\psi^* g_{\alpha\beta} = g_{\alpha\beta}$. If a vector field ξ^α generates the diffeos ψ_λ ,

$$\psi_\lambda g_{\alpha\beta} = g_{\alpha\beta} \Rightarrow \mathcal{L}_\xi g_{\alpha\beta} = 0$$

By (2.82),

$$\begin{aligned} \mathcal{L}_\xi g_{\alpha\beta} &= \xi^\gamma \nabla_\gamma g_{\alpha\beta} + g_{\alpha\gamma} \nabla_\beta \xi^\gamma + g_{\gamma\beta} \nabla_\alpha \xi^\gamma \\ &= \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \end{aligned}$$

Definition: ξ^α is a Killing vector field if

$$\mathcal{L}_\xi g_{\alpha\beta} = 0, \quad e : kvdef \quad (2.103)$$

or

$$\nabla_{(\alpha} \xi_{\beta)} = 0. \quad (2.104)$$

If ξ^α is a Killing vector, one can choose coordinates in the following way:

Take a hypersurface S cutting the integral curves of ξ^α . x^0 is the scalar field with $x^0 = 0$ on S and $x^0(P)$ the parameter distance from S to P along the integral curve through P . x^1, x^2 and x^3 are picked arbitrarily on S and extended to M by requiring them to be constant along the integral curves.

$$\text{Then } \mathcal{L}_\xi x^0 = \frac{dx^0(\lambda)}{d\lambda} = \frac{d\lambda}{d\lambda} = 1$$

$$\mathcal{L}_\xi x^1 = \mathcal{L}_\xi x^2 = \mathcal{L}_\xi x^3 = 0$$

and

$$(\mathcal{L}_\xi g)_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu} \partial_\mu \xi^\lambda + g_{\mu\lambda} \partial_\nu \xi^\lambda$$

But

$$\begin{aligned} \xi^\mu &= \xi(x^\mu) = \mathcal{L}_\xi x^\mu = \delta_0^\mu \\ \Rightarrow (\mathcal{L}_\xi g)_{\mu\nu} &= \partial_0 g_{\mu\nu}. \end{aligned}$$

So in this coordinate system (where ξ^μ is just the tangent to $\lambda \rightarrow (x^0 + \lambda, x^1, x^2, x^3)$), the symmetry $(\mathcal{L}_\xi g)_{\mu\nu} = 0$ implies that $\partial_0 g_{\mu\nu} = 0$, that the components $g_{\mu\nu}$ are independent of the coordinate x^0 . Conversely, if there is a chart in which the components $g_{\mu\nu}$ of the metric are independent of a coordinate x^0 , then the vector with components $\xi^\mu = \delta_0^\mu$ in that chart is a Killing vector of the spacetime:

$$\begin{aligned} (\mathcal{L}_\xi g)_{\mu\nu} &= \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu} \partial_\mu \xi^\lambda + g_{\mu\lambda} \partial_\nu \xi^\lambda = \partial_0 g_{\mu\nu} = 0 \\ \Rightarrow (\mathcal{L}_\xi g)_{\mu\nu} &= 0. \end{aligned}$$

Example 1: Minkowski space has 10 independent Killing vector fields,

$$\begin{array}{ll} \partial_t, \partial_x, \partial_y, \partial_z & 4 \text{ translational} \\ x\partial_y - y\partial_x, z\partial_x - x\partial_z, y\partial_z - z\partial_y & 3 \text{ rotational} \\ x\partial_t + t\partial_x, y\partial_t + t\partial_y, z\partial_t + t\partial_z & 3 \text{ boosts,} \end{array}$$

and they generate 10 one-parameter groups of diffeos that leave $\eta_{\mu\nu}$ invariant: 4 translations, 3 rotations, 3 boosts, which together close under composition to form the Poincaré group. Here's the check that $\eta_{\alpha\beta}$ is invariant under boosts:

$$\begin{aligned} \xi^\mu &= (x, t, 0, 0) = x^1 \delta_0^\mu + x^0 \delta_1^\mu \\ (\mathcal{L}_\xi \eta)_{\mu\nu} &= \underbrace{\xi^\lambda \partial_\lambda \eta_{\mu\nu}}_0 + \partial_\mu \xi^\lambda \eta_{\lambda\nu} + \partial_\nu \xi^\lambda \eta_{\mu\lambda} \\ &= \partial_\mu (x^1 \delta_0^\lambda + x^0 \delta_1^\lambda) \eta_{\lambda\nu} + \partial_\nu (x^1 \delta_0^\lambda + x^0 \delta_1^\lambda) \eta_{\mu\lambda} \\ &= \frac{\delta_\mu^1 \eta_{0\nu}}{1} + \frac{\delta_\mu^0 \eta_{1\nu}}{1} + \frac{\delta_\nu^1 \eta_{\mu 0}}{1} + \frac{\delta_\nu^0 \eta_{\mu 1}}{1} \\ &= 0. \end{aligned}$$

That $\eta_{\alpha\beta}$ is invariant under translations is immediate: $\partial_t \eta_{\mu\nu} = 0$.

That $\eta_{\alpha\beta}$ is invariant under rotations is immediate in spherical coordinates with the axis of rotation taken to be the symmetry axis:

$$x\partial_y - y\partial_x = \partial_\phi, \quad \xi^\mu = \delta_\phi^\mu, \quad (\mathcal{L}_\xi \eta)_{\mu\nu} = \partial_\phi \eta_{\mu\nu} = 0.$$

Example 2: A rotating black hole has the (Kerr) metric

$$ds^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2)d\phi - a dt]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2,$$

where

$$\Delta = r^2 - 2Mr + a^2 + 0^2 \quad \rho^2 = r^2 + a^2 \cos^2 \theta,$$

with $a :=$ angular momentum/mass. Since $\|g_{\mu\nu}\|$ is independent of the coordinates t and ϕ , ∂_t and ∂_ϕ are Killing vectors and the spacetime is invariant under time translations and rotations about the $\theta = 0$ axis.

2.4 Introduction to integration on manifolds

In flat space, the area of a parallelogram spanned by the vectors $\mathbf{A}, \mathbf{B}$ is $|\mathbf{A} \times \mathbf{B}| = |\epsilon_{ab} A^a B^b|$; and the volume spanned the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is

$|\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}| = |\epsilon_{abc} A^a B^b C^c|$. Similarly, in Minkowski space, requiring that the volume spanned by four orthonormal vectors $t^\alpha, x^\alpha, y^\alpha, z^\alpha$ be 1 implies that a parallelepiped Ω spanned by any four vectors $A^\alpha, B^\alpha, C^\alpha, D^\alpha$ is

$$|\Omega| = |\epsilon_{\alpha\beta\gamma\delta} A^\alpha B^\beta C^\gamma D^\delta|.$$

The set of four vectors is positively oriented if $\epsilon_{\alpha\beta\gamma\delta} A^\alpha B^\beta C^\gamma D^\delta > 0$.

The volume of an arbitrary region Ω is obtained by adding volumes of infinitesimal parallelepipeds spanned by vectors along the coordinate axes, $e_0^\alpha, e_1^\alpha, e_2^\alpha, e_3^\alpha$ with lengths $\Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3$

$$\begin{aligned} \Delta^4 V &= \epsilon_{0123} \Delta x^0 \Delta x^1 \Delta x^2 \Delta x^3 \\ &= \frac{1}{4!} \epsilon_{\mu\nu\sigma\tau} \Delta x^\mu \Delta x^\nu \Delta x^\sigma \Delta x^\tau (-1)^\pi, \end{aligned}$$

where $(-1)^\pi = 1$ when μ, ν, σ, τ is an even permutation π of $0, 1, 2, 3$ and $(-1)^\pi = -1$ for an odd permutation. Because $\epsilon_{\alpha\beta\gamma\delta} A^\alpha B^\beta C^\gamma D^\delta$ is a scalar, the volume of a region of flat space is given in any oriented chart by

$$|\Omega| = \int_{\Omega} \epsilon_{0123} dx^0 dx^1 dx^2 dx^3 \equiv \int_{\Omega} d^4V.$$

The Jacobian, $\left| \frac{\partial x}{\partial x'} \right|$, that relates the volume element in two different coordinate systems arises from the coordinate transformation of the totally antisymmetric tensor $\epsilon_{\alpha\beta\gamma\delta}$:

$\epsilon_{0'1'2'3'} = \frac{\partial x^0}{\partial x'^0} \frac{\partial x^1}{\partial x'^1} \frac{\partial x^2}{\partial x'^2} \frac{\partial x^3}{\partial x'^3} \epsilon_{\mu\nu\sigma\tau} = \left| \frac{\partial x}{\partial x'} \right| \epsilon_{0123}$. In the mathematics literature the volume integral is written as

$$\int_{\Omega} \epsilon. \quad (2.105)$$

A curved space is locally flat in the sense that, in a locally inertial coordinate system, the metric components are flat up to quadratic order in the coordinates. By demanding that the volumes of small regions, to first order in the length of a side, are those measured by a locally inertial observer using her local Minkowski metric, one uniquely picks out the volume element

$$d^4V = \epsilon_{0123} dx^0 dx^1 dx^2 dx^3. \quad (2.106)$$

Because the totally antisymmetric tensor ϵ_{0123} has the value $\sqrt{|g|}$, the volume element can be written in the equivalent form

$$d^4V = \sqrt{|g|} dx^0 dx^1 dx^2 dx^3, \quad (2.107)$$

and in n-dimensions

$$d^nV = \epsilon_{1\dots n} dx^1 \dots dx^n = \sqrt{|g|} dx^1 \dots dx^n. \quad (2.108)$$

In index notation, one writes

$$d^4V = \epsilon_{a\dots b} dS^{a\dots b},$$

and thinks of $dS^{a\dots b}$ having components

$$“ dS^{\mu\nu\sigma\tau} = \pm \frac{1}{4!} dx^\mu dx^\nu dx^\sigma dx^\tau ”.$$

The integral,

$$\int_{\Omega} f d^nV = \int_{\Omega} f \epsilon_{1\dots n} dx^1 \dots dx^n = \int_{\Omega} f \sqrt{|g|} dx^1 \dots dx^n, \quad (2.109)$$

over a region Ω is well-defined (that is, its value is independent of the choice of coordinates), because, under a change of coordinates, the integrand on the right side is multiplied by the Jacobian $\left| \frac{\partial x}{\partial x'} \right|$.

2.4.1 Gauss's theorem

We'll need Gauss's theorem, whose form and proof for curved spacetime is nearly the same as that for flat space. The subsequent sections on forms, densities and Stokes's theorem are optional, useful but not needed for this first-semester GR course.

Gauss's theorem relates the volume integral of a divergence of a vector to the surface integral of its normal component. We have already seen in a flat-space context its meaning as a conservation law, relating the change in charge or mass in a volume to the flux leaving the volume.

In $\mathbb{R}^n$, in Cartesian coordinates, the integral of a divergence over an n -cube can be expressed as a surface integral after an integration by parts:

$$\begin{aligned}\int_V \partial_i A^i d^n x &= \int_V \partial_1 A^1 dx^1 dx^2 \cdots dx^n + \cdots + \int_V \partial_n A^n dx^n dx^1 \cdots dx^{n-1} \\ &= \int_{\partial_{1+} V} A^1 dx^2 \cdots dx^n - \int_{\partial_{1-} V} A^1 dx^2 \cdots dx^n + \cdots + \int_{\partial_{n+} V} A^n dx^1 \cdots dx^{n-1} - \int_{\partial_{n-} V} A^n dx^1 \cdots dx^{n-1} \\ &= \int_{\partial V} A^i dS_i \quad (\partial V \text{ means the boundary of } V)\end{aligned}$$

where $dS_i = \pm \epsilon_{ij \dots k} dx^j \cdots dx^k \frac{1}{(n-1)!}$, with

$$\begin{aligned}dS_1 &= +dx^2 \cdots dx^n \text{ for } x^1 \text{ increasing outward,} \\ dS_1 &= -dx^2 \cdots dx^n \text{ , for } x^1 \text{ increasing inward.}\end{aligned}$$

More generally the integral over any volume V in $\mathbb{R}^n$ of a divergence is related to a surface integral by

$$\int_V \partial_i A^i d^n V = \int_{\partial V} A^i dS_i = \int_{\partial V} A^i n_i dS$$

where n_i is the unit outward normal to S and dS the area element of S .

In curved space the analogous result is easily obtained:

$$\begin{aligned}\int_{\Omega} \nabla_a A^a d^n V &= \int_{\Omega} \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} A^i \right) \sqrt{|g|} d^4 x \\ &= \int_{\Omega} \partial_i \left(\sqrt{|g|} A^i \right) d^n x\end{aligned}$$

When Ω is a coordinate box,

$$\begin{aligned}\int_{\Omega} \partial_i \left(A^i \sqrt{|g|} \right) d^n x &= \int_{\partial_{1+} V} A^1 \sqrt{|g|} dx^2 \cdots dx^n - \int_{\partial_{1-} V} A^1 \sqrt{|g|} dx^2 \cdots dx^n \\ &\quad + \cdots + \int_{\partial_{n+} V} A^n \sqrt{|g|} dx^1 \cdots dx^{n-1} - \int_{\partial_{n-} V} A^n \sqrt{|g|} dx^1 \cdots dx^{n-1} \\ &= \int_{\partial V} A^i dS_i,\end{aligned}$$

where again each term on the right side has the form $\epsilon_{ij \dots k} dx^j \cdots dx^k \frac{1}{(n-1)!}$. We can then write

$$\int_{\Omega} \nabla_a A^a d^n V = \int_{\partial \Omega} A^a dS_a \quad (2.110)$$

where $dS_i = \pm \epsilon_{ij \dots k} dx^j \cdots dx^k \frac{1}{(n-1)!}$.

Equivalently, the integrand of each term on the right has the form $\pm A^k \nabla_k x^i \sqrt{|g|}$ for the part of the boundary that is a surface of constant x^i . To get the sign right, we need to choose the outward normal.

For example consider a 4-dimensional slab Ω whose boundary consists of two $t=\text{constant}$ slices, $t = t_1$ and $t = t_2$, and a part of the boundary at spatial infinity where A^α vanishes. On the future boundary,

$$\int_{t=t_2} A^\alpha dS_\alpha = \int_{t=t_2} A^t \sqrt{|g|} d^3x = \int_{t=t_2} A^\mu \nabla_\mu t \sqrt{|g|} d^3x, \quad t \text{ increasing outward},$$

so $dS_\mu = \nabla_\mu t \sqrt{|g|} d^3x$. On the past boundary,

$$\int_{t=t_2} A^\alpha dS_\alpha = - \int_{t=t_2} A^t \sqrt{|g|} d^3x = \int_{t=t_2} A^\mu \nabla_\mu (-t) \sqrt{|g|} d^3x, \quad -t \text{ increasing outward}.$$

In general, choose a scalar f that increases outward and is constant on a part of the surface. Then take f to be a coordinate $f = x^1$ and add coordinates on the surface, with $x^1 \equiv f, x^2, \dots, x^n$ a right handed chart, so that $\epsilon_{12\dots n} = \sqrt{|g|}$. We then have

$$dS_a = \nabla_a f \sqrt{|g|} dx^2 dx^3 \cdots dx^n. \quad (2.111)$$

If, as usual, we decompose the volume of integration into a set of cubes and take the limit as the size of each cube shrinks to zero, we find

$$\int_\Omega \nabla_a A^a d^n V = \int_{\partial\Omega} A^a dS_a. \quad (2.112)$$

As usual, surface terms from cubes that share a surface cancel, because the outward normal to one cube is the inward normal to the adjacent cube.

This form is correct for any (smooth, orientable) volume in a space with a metric, independent of the signature of the metric. When $\partial\Omega$ has a unit normal (when it is not a null surface), one can write dS_α in a more familiar form. With outward unit normal n_a (along the gradient of a scalar that increases outward) $dS_a = n_a dS$, with dS the $n - 1$ dimensional volume element, given by

$$dS = \sqrt{|^{(n-1)}g|} d^{n-1}x, \quad (2.113)$$

where $^{(n-1)}g_{ab}$ is the metric on the $n - 1$ -dimensional surface. We'll pick $n = 4$ and look at a $t = \text{constant}$ surface. You can always change t to f and 4 to another dimension. The unit normal (for t increasing outward) is related to $\nabla_\alpha t$ by

$$n_\alpha = \nabla_\alpha t \frac{1}{\sqrt{|\nabla_\beta t \nabla^\beta t|}} = \nabla_\alpha t \frac{1}{\sqrt{|g^{tt}|}}. \quad (2.114)$$

Now we need the metric $^3g_{ab}$ on a $t = \text{constant}$ surface. First, a simple-minded version using components. At the end, we'll circle back and define $^3g_{ab}$ as the pullback of $g_{\alpha\beta}$ to S . If v^α is any vector lying in a $t = \text{constant}$ surface, $v^t = v^\alpha \nabla_\alpha t = 0$. The metric in the surface is, by definition, the map from a pair of vectors v^α and w^α in the surface to $\mathbb{R}$, and that is already given by the original metric on M ,

$$g_{\alpha\beta} v^\alpha w^\beta = g_{ij} v^i w^j. \quad (2.115)$$

So the components of the 3-metric are the spatial components of the 4-metric.

$$^3g_{ij} = g_{ij}.$$

We can relate the determinant ${}^3g$ to g and the unit normal n_α as follows: The inverse metric has component g^{tt} given by $g^{tt} = |\text{minor of } g_{tt}|/g$, and the minor of g_{tt} is the determinant of the spatial components $g_{ij} = {}^3g_{ij}$ of the metric:

$$g^{tt} = \frac{{}^3g}{g}.$$

Then

$$\begin{aligned} dS_\mu &= \nabla_\mu t \sqrt{|g|} \, dx dy dz = \sqrt{|g^{tt}|} n_\mu \sqrt{|g|} \, dx dy dz = n_\mu \sqrt{{}^3g} \, dx dy dz \\ &= n_\mu dS, \end{aligned}$$

with $dS = \sqrt{{}^3g} \, d^3x$ in agreement with Eq. (2.113).

Finally, we show that ${}^3g_{ab}$ is the pullback of $g_{\alpha\beta}$ from the spacetime M to the submanifold S : That is, ${}^3g_{ab} = (\psi_* g)_{ab}$ where $\psi : S \rightarrow M$ takes each point in S to itself, the identity diffeo, but with its image regarded as a point of M . Then $(\psi^\mu(x)) = (0, x^i)$, and $\frac{\partial \psi^\mu}{\partial x^i} = \delta_i^\mu$, giving

$${}^3g_{ij} = \frac{\partial \psi^\mu}{\partial x^i} \frac{\partial \psi^\nu}{\partial x^j} g_{\mu\nu} = \delta_i^\mu \delta_j^\nu g_{\mu\nu} = g_{ij}. \quad (2.116)$$

Example: The integral form of mass conservation $\nabla_\alpha(\rho u^\alpha) = 0$ for dust is

$$\begin{aligned} 0 &= \int_\Omega \nabla_\alpha(\rho u^\alpha) d^4V = \int_{V_2} \rho u^\alpha dS_\alpha - \int_{V_1} \rho u^\alpha dS_\alpha \\ &\text{or } \int_V \rho u^\alpha dS_\alpha = \text{constant} \end{aligned}$$

Similarly, the number of baryons in a fluid is

$$N = \int_V n u^\alpha dS_\alpha = \int_V n u^\alpha n_\alpha dS$$

(and $N = \int_V n dS$ only if $u^\alpha \perp V$ so that $u^\alpha n_\alpha = 1$ —i.e. if the fluid is at rest relative to V .)

Note that in a chart with V a $t = \text{constant}$ surface,

$$\int_V A^\alpha dS_\alpha = \int_V A^t \sqrt{-g} \, d^3x,$$

so there is no need to introduce n_α and $\sqrt{{}^3g}$ to evaluate the integral. This fact is *essential* if one is evaluating an integral $\int j^\alpha dS_\alpha$ over a null surface, where there is no unit normal. The flux of energy or of baryons across the horizon, for example, can be computed in coordinates that are regular on the horizon: Consider coordinates v, r, θ, ϕ with the surface of constant r a null surface (think of $r = 2M$ in Schwarzschild). Then

$$\int j^\alpha dS_\alpha = \int j^r \sqrt{-g} \, dv d\theta d\phi.$$

There is no unit normal, but the flux across the surface doesn't care.

A more extensive treatment of forms, densities and integration is given in the Appendix.A

Chapter 3

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

3.1 “Derivation” of the Equation

To conform to the state of knowledge in 1910, a theory of gravity should satisfy:

I. With no gravity, special relativity is valid.

II. Principle of equivalence: “Locally” one can’t distinguish uniform acceleration from a uniform gravitational field – or free fall from no gravitational field.

III. Newtonian gravity is valid in the Newtonian regime, i.e. when the sources are nonrelativistic in character.

The observed solar-system departure from Newtonian gravity was the anomalous $43''$ precession of Mercury’s orbit each century, a correction of about one part in 10^7 . (Mercury’s year is 88 days, so $100/88$ orbits per century for a total of about $(100/88)3600 \times 360 \approx 5 \times 10^8$ arcseconds.)

I and II partly cease to make sense once the theory is accepted because there will be no gravitational “field” – just curved spacetime – and no distinction between a *uniform field observed by a stationary observer* and *no field observed by a uniformly accelerated observer*. In place of I and II will be the statement that free particles move on geodesics of the spacetime. One can, however, continue to test the equivalence principle prediction that the gravitational acceleration of small objects in a time independent gravitational field is independent of the objects’ internal structure and composition (Eötvös experiments), and that local Lorentz invariance holds.

Prior to writing the theory, I and II together are taken to imply that a freely falling, non-rotating observer, measuring components of tensors along her orthonormal frame finds that matter with energy momentum tensor $T^{\mu\nu}$ obeys $\partial_\nu T^{\mu\nu} = 0$ + terms of order $\ell^2 | \partial_i \partial_j \Phi |$, with Φ the gravitational potential, ℓ a length characteristic of the size of his laboratory. She will similarly find that electromagnetic fields satisfy

$$\begin{aligned} \partial_\nu F^{\mu\nu} &= 4\pi j^\mu + \text{terms smaller by a factor } \sim \ell^2 | \partial_i \partial_j \Phi | \\ \partial_{[\mu} F_{\nu\sigma]} &+ \text{terms smaller by } \ell^2 | \partial_i \partial_j \Phi | = 0 . \end{aligned}$$

Finally, III means that when source velocities are small compared to the speed of light, when

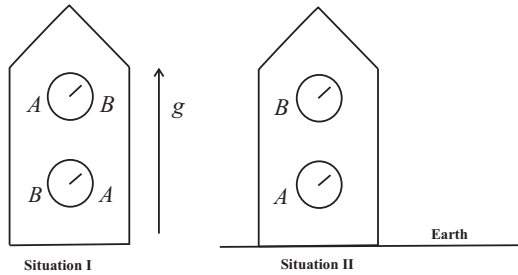
$$\begin{array}{l|l} v \ll 1 & \frac{v}{c} \ll 1 \\ \text{pressure } P \ll \rho & P \ll \rho c^2 \\ \text{and } \Phi \sim \frac{M}{R} \ll 1, & \frac{\Phi}{c^2} \sim \frac{GM}{Rc^2} \ll 1 \end{array}$$

then $\nabla^2\Phi = 4\pi G\rho$, and $\mathbf{g} = -\nabla\Phi$ is the acceleration due to gravity.

These are the constraints on the theory.

Why does gravity involve a metric? We have already mentioned Geroch's argument: Anything you use to decide what a straight line is— or, more generally to determine the geometry of spacetime—is curved by gravity. so it is perhaps not surprising that gravity is tied to geometry.

The more specific, equivalence-principle argument arises from the following experiment, an experiment that uses the first two assumptions above. Consider the following two situations. In the first, two identical clocks A and B are placed on the wall of a rocket that accelerates uniformly with acceleration g . In the second situation, the clocks are at rest, one above the other in the nearly uniform field of the Earth.

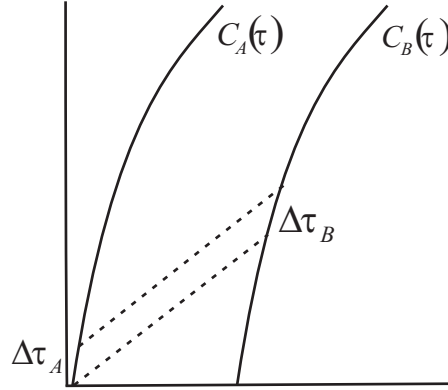


Technically, if each clock in situation I feels a uniform acceleration g , the spatial distance between them will not remain constant: two particles satisfying (I.66) have a time dependent separation defined as the length of a connecting line orthogonal to one of their trajectories. But the difference will be of order $\frac{v^2}{c^2} \sim \frac{(g\tau)^2}{c^2}$ and will be neglected: In this calculation we will work to $O\left(\frac{v}{c}\right)$.

In situation I, let h be the initial distance between clocks measured orthogonal to their trajectories. Situation II will then be “equivalent” if h is the distance measured by a meter stick at rest with respect to the clocks and if each clock feels constant acceleration g . For example, a pair of clocks at rest relative to the Earth are equivalent to a pair of uniformly accelerating clocks at fixed distance from each other. We will show that light signals sent a proper time $\Delta\tau_A$ apart from A to B are received at a different interval $\Delta\tau_B$ measured on B's clock. In the “equivalent” situation, where the clocks are at rest on Earth, the implication is that again $\Delta\tau_A \neq \Delta\tau_B$ - that the clocks move at different rates.

Exercise 55. Use the Newtonian trajectories of two uniformly accelerated clocks separated by a distance h and each at rest at $t = 0$. Show that photons emitted by clock A at $t = 0$ and at $t = \Delta t_A$ reach B at times differing by $\Delta t_B = \Delta t_A(1 + gh)$, neglecting terms of order $g(\Delta t)^2$ and order $\Delta t(gh)^2$.

Here is the same result using the relativistic trajectories and proper time measured by A and B .



In units with $c = 1$, the time for light to go from A to B is roughly h . In that time, A and B gain velocity v of order gh (i.e. $v \sim gh/c$), and we assume that this speed v is small: $gh \ll 1$ (i.e. $gh/c^2 \ll 1$).

The trajectory of clock B is

$$\begin{aligned} c_B^\alpha(\tau) &= \frac{1}{g} [\sinh(g\tau) t^\alpha + \cosh(g\tau) x^\alpha] + \left(h - \frac{1}{g}\right) x^\alpha \\ &= \tau t^\alpha + \left(h + \frac{1}{2}g\tau^2\right) x^\alpha + O[(g\tau)^2\tau] \end{aligned} \quad (3.1)$$

1st photon has path $\lambda(t^\alpha + x^\alpha)$ hitting $c_B(\tau)$ at the time τ_1 on clock B for which

$$\begin{aligned} \lambda(t^\alpha + x^\alpha) &= c_B^\alpha(\tau_1) \implies \lambda = \tau_1 \text{ and } \lambda = h + \frac{1}{2}g\tau_1^2 + O[(g\tau)^2\tau], \\ \tau_1 &= h + \frac{1}{2}g\tau_1^2 + O[(g\tau_1)^2\tau_1] \\ &= h + \frac{1}{2}gh^2 + O[(gh)^2h]. \end{aligned} \quad (3.2)$$

2nd photon has path $\lambda(t^\alpha + x^\alpha) + \Delta\tau_A t^\alpha$ hitting $c_B(\tau)$ at

$$\begin{aligned} \tau_2 &= \Delta\tau_A + h + \frac{1}{2}g\tau_2^2 + O[(gh)^2h], O[(\Delta\tau)^2] \\ &= \Delta\tau_A + \tau_1 + \frac{1}{2}g(\tau_2^2 - \tau_1^2) \\ &= \Delta\tau_A + \tau_1 + \frac{1}{2}g(\tau_2 - \tau_1)(2\tau_1) \\ &= \Delta\tau_A + \tau_1 + g\tau_1(\tau_2 - \tau_1) \end{aligned} \quad (3.3)$$

$$\begin{aligned} \Delta\tau_B &= \tau_2 - \tau_1 = \Delta\tau_A + g\tau_1 \Delta\tau_B \\ &= \Delta\tau_A(1 + gh) + O[(gh)^2\Delta\tau_A]. \end{aligned} \quad (3.4)$$

Then, for clocks on Earth, the equivalence principle implies

$$\Delta\tau(h) = \left(1 + \frac{\Phi}{c^2}\right) \Delta\tau(h=0) , \quad (3.5)$$

where $\Phi = gh$ is the gravitational potential.

From this result one infers that the metric of spacetime depends on Φ , because the proper time read on a clock is proportional to $\sqrt{g_{tt}}$. Here is the argument. Assume that the spacetime geometry of the two clocks and the Earth is time independent. Then there is a scalar t for which the metric components $g_{\mu\nu}$ are independent of t in a coordinate system $\{t, x^i\}$ for which the spatial coordinates of the clocks and the Earth are independent of t . By saying “there is a metric”, one means, in particular, that the proper time elapsed on a clock with trajectory $c(\lambda)$ is $\tau = \int^\lambda |g_{\alpha\beta} \dot{c}^\alpha \dot{c}^\beta|^{1/2} d\lambda$ (and, in general, the spacetime distance between two events P and Q is the geodesic distance with respect to $g_{\alpha\beta}$).

Consider now two photons sent from clock A at two times a coordinate interval Δt apart. Because the geometry is independent of t , the trajectory of the second photon is identical to that of the first photon, translated in time by Δt :

$$x_2^\mu(\lambda) = x_1^\mu(\lambda) + \delta_t^\mu \Delta t. \quad (3.6)$$

Therefore signals sent a coordinate time Δt apart are received at B a coordinate time Δt apart.

But $\Delta\tau_A = |g_{tt}(A)|^{1/2} \Delta t$, $\Delta\tau_B = |g_{tt}(B)|^{1/2} \Delta t$.

If the equivalence principle is to hold, these two time intervals must be related as they are in the uniformly accelerating spacecraft: They must satisfy Eq. (3.5). Thus

$$|g_{tt}(h)|^{1/2} = \left(1 + \frac{\Phi}{c^2}\right) |g_{tt}(0)|^{1/2}.$$

Normalizing t to make $\Delta\tau(h=0) = \Delta t$ gives

$$g_{tt} = - \left(1 + \frac{2\Phi}{c^2}\right). \quad (3.7)$$

In the more sophisticated language of the last section, one assumes that the metric has a timelike Killing vector, t^α , and that the Earth and clocks are simply time translated by its family of diffeos. Then, assuming each photon trajectory is similarly time translated, the parameter intervals Δt between transmitted photons are equal to the parameter intervals Δt between received photons, and (3.5) implies

$$g_{\alpha\beta} t^\alpha t^\beta(h) = \left(1 + \frac{2\Phi}{c^2}\right) g_{\alpha\beta} t^\alpha t^\beta(0). \quad (3.8)$$

The equivalence principle argument has led us to a metric having in the Newtonian limit a component g_{tt} of the form (3.7),

$$g_{tt} = - \left(1 + \frac{2\Phi}{c^2}\right),$$

where Φ is the Newtonian potential

$$\nabla^2 \Phi = 4\pi G \rho. \quad (3.9)$$

Now comes the kind of spectacular coincidence that happens when one is on the right track: If we suppose that

$$\frac{1}{c} \partial_t g_{\mu\nu} \sim \frac{v}{c} \partial_i g_{\mu\nu} \ll \partial_i g_{\mu\nu}, \quad i = 1-3, \quad (3.10)$$

as befits a field produced by slowly moving sources, then the geodesic equation

$$u^\nu \nabla_\nu u^\mu = 0, \quad \text{for} \quad u^\mu = (1, v^i) + O(v^2),$$

is

$$\frac{dv^i}{dt} + \Gamma^i_{\mu\nu} u^\mu u^\nu = 0 \quad \left(\frac{dv^i}{d\tau} = \frac{dv^i}{dt} + O(v^2) \right),$$

$$\frac{dv^i}{dt} + -\frac{1}{2}(\partial_i g_{tt} + 2\partial_t g_{ti}) = 0$$

or

$$\frac{dv^i}{dt} = -\partial_i \Phi, \quad (3.11)$$

the Newtonian equation of free particle motion! More generally, the relativistic equation of motion

$$0 = \nabla_\nu T^{i\nu} = \partial_\nu T^{i\nu} + \Gamma^i_{\sigma\nu} T^{\sigma\nu} + \Gamma^\nu_{\sigma\nu} T^{i\sigma}$$

has Newtonian limit

$$\partial_\nu T^{i\nu} + \Gamma^i_{tt} T^{tt},$$

using $T^{it}, T^{ij} \ll T^{tt}$ for nearly Newtonian matter. Then

$$\partial_\nu T^{i\nu} = -\rho \partial^i \Phi, \quad (T^{tt} \simeq \rho) \quad (3.12)$$

so a g_{tt} component of the metric of the form (3.7), $-(1 + 2\Phi/c^2)$, gives rise to the gravitational force, in the Newtonian regime.

Consequently, if one assumes that spacetime has a Lorentz metric $g_{\alpha\beta}$, that matter with energy momentum tensor $T^{\alpha\beta}$ satisfies $\nabla_\beta T^{\alpha\beta} = 0$, and that $g_{\alpha\beta}$ is related to $T_{\alpha\beta}$ by an equation whose Newtonian limit is

$$g_{tt} = -\left(1 + \frac{2\Phi}{c^2}\right),$$

with

$$\nabla^2 \Phi = 4\pi G \rho, \quad (3.13)$$

then virtually all relevant experimental knowledge (I-III) will be accounted for.

What remains is to find the relation between $g_{\alpha\beta}$ and $T^{\alpha\beta}$. One way to get it is to say one wants a scalar Lagrangian whose variation gives equations with no higher than 2nd derivatives of $g_{\alpha\beta}$ – because $\nabla^2 \Phi = \nabla^2 g_{tt}$ has 2nd derivatives of g . The only scalar that doesn't involve higher derivatives is R (really $R + \Lambda$ where Λ is a constant). We'll discuss this in VI – it's Hilbert's method. A different approach is as follows (due to Geroch).

From our discussion of the equivalence principle, it is clear that the form of g_{tt} obtained for a uniform gravitational field is the same g_{tt} that would be quoted by a system of uniformly accelerating observers (with no gravitational field) – that was how we found the “gravitational” g_{tt} . A real gravitational field, however, is nonuniform, and only by its nonuniformity can it be recognized. Thus to seek a relation between $g_{\alpha\beta}$ and matter, it is natural to look at the *relative* acceleration of free particles because it is that which depends on the fact that $\nabla \Phi$ is not constant – i.e. relative acceleration is given in the Newtonian regime by Eq. (2.67),

$$\ddot{S}^i = -\nabla^i \nabla_j \Phi S^j, \quad (3.14)$$

relating the separation vector $\ddot{S}^i$ to the tidal forces, the second derivatives of Φ . Because $S^t = 0$ and $\frac{1}{c} \partial_t \Phi \ll \partial_i \Phi$, this can be written

$$\ddot{S}^\alpha = -\nabla^\alpha \nabla_\beta \Phi S^\beta. \quad (3.15)$$

In relativity the same experiment is described by the deviation of neighboring geodesics (2.66),

$$\ddot{S}^\alpha = R^\alpha_{\beta\gamma\delta} u^\beta u^\gamma S^\delta \quad (3.16)$$

so that

$$R^\alpha_{\gamma\delta\beta} u^\gamma u^\delta \text{ must correspond in the Newtonian approximation to } -\nabla^\alpha \nabla_\beta \Phi .$$

Because Φ satisfies $\nabla^2 \Phi = 4\pi G\rho$, the curvature tensor must satisfy, in the Newtonian approximation,

$$R^\gamma_{\alpha\gamma\beta} u^\alpha u^\beta = R_{\alpha\beta} u^\alpha u^\beta = 4\pi G\rho. \quad (3.17)$$

There is, however some ambiguity in what to choose as the scalar that becomes ρ : From $T_{\alpha\beta}$ and u^α one can construct two scalars

$$T_{\alpha\beta} u^\alpha u^\beta \quad \text{and} \quad -T \equiv -T^\alpha_\alpha, \quad (3.18)$$

each of which has the mass density ρ as its Newtonian limit. Thus one could generalize (3.13) to any equation of the form

$$R_{\alpha\beta} u^\alpha u^\beta = 4\pi G[rT_{\alpha\beta} + (1-r)T g_{\alpha\beta}] u^\alpha u^\beta, \quad (3.19)$$

and in order that it hold for any timelike u^α , one has

$$R_{\alpha\beta} = 4\pi G[rT_{\alpha\beta} + (1-r)T g_{\alpha\beta}] \quad (3.20)$$

or

$$4\pi GrT_{\alpha\beta} = R_{\alpha\beta} - \frac{r-1}{3r-4} g_{\alpha\beta} R, \quad (3.21)$$

where (3.20) was contracted to give $R = 4\pi G(4-3r)T$. But (3.21) must be consistent with

$$\nabla_\beta T^{\alpha\beta} = 0 .$$

Recalling that the contracted Bianchi identity is

$$\nabla_\beta G^{\alpha\beta} \equiv \nabla_\beta (R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R) = 0$$

one has a unique choice of r which makes these equations consequences of each other, namely $r = 2$. Thus we are led to Einstein's equation

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = 8\pi G T^{\alpha\beta}. \quad (3.22)$$

Gravitational units. Beginning now, units are chosen to make $c = G = 1$. These are called *gravitational units* or *geometrized units*. With $c = 1$, length and time have the same dimension; if time is measured in seconds, length is measured in light-seconds, where 1 light-second = 3×10^{10} cm, the distance light travels in 1 second. MTW (Misner-Thorne-Wheeler) gives distance in cm, and the unit of time is then $1/(3 \times 10^{10})$ s. With $G = 1$, mass, length and time all have the same dimension, which MTW takes to be length with unit 1 cm. Then the unit of mass is $(1 \text{ cm}) \times c^2/G = 1.3468 \times 10^{28}$ g. Here is the MTW table:

Box 1.8 GEOMETRIZED UNITS

Throughout this book, we use “geometrized units,” in which the speed of light c , Newton’s gravitational constant G , and Boltzman’s constant k are all equal to unity. The following alternative ways to express the number 1.0 are of great value:

$$1.0 = c = 2.997930 \dots \times 10^{10} \text{ cm/sec}$$

$$1.0 = G/c^2 = 0.7425 \times 10^{-28} \text{ cm/g};$$

$$1.0 = G/c^4 = 0.826 \times 10^{-49} \text{ cm/erg};$$

$$1.0 = Gk/c^4 = 1.140 \times 10^{-65} \text{ cm/K};$$

$$1.0 = c^2/G^{1/2} = 3.48 \times 10^{24} \text{ cm/gauss}^{-1}.$$

One can multiply a factor of unity, expressed in any one of these ways, into any term in any equation without affecting the validity of the equation. Thereby one can convert one’s units of measure

from grams to centimeters to seconds to ergs to For example:

$$\begin{aligned} \text{Mass of sun} &= M_{\odot} = 1.989 \times 10^{33} \text{ g} \\ &= (1.989 \times 10^{33} \text{ g}) \times (G/c^2) \\ &= 1.477 \times 10^5 \text{ cm} \\ &= (1.989 \times 10^{33} \text{ g}) \times (c^2) \\ &= 1.788 \times 10^{54} \text{ ergs.} \end{aligned}$$

The standard unit, in terms of which everything is measured in this book, is centimeters. However, occasionally conventional units are used; in such cases a subscript “conv” is sometimes, but not always, appended to the quantity measured:

$$M_{\odot \text{conv}} = 1.989 \times 10^{33} \text{ g.}$$

Denoting by M, L, T dimensions of mass, length and time and denoting by $[Q]$ the dimension of a quantity Q , we have $[G] = M^{-1}L^3/T^2$, $[c] = L/T$. In units with $c = 1$, $L = T$; in gravitational units M also has the same dimension: $G = 1$ implies $M = L^3/T^2 = L = T$.

In particular, the gravitational potential Φ has conventional units of energy/mass because, by its definition, the gravitational potential energy of a small mass m is $m\Phi$. Then $[\Phi]_{\text{conventional}} = L^2/T^2$. With $c = 1$, length and time have the same dimension, and Φ is dimensionless. To convert back to conventional units, one must multiply by the factors of G and c that restore the conventional dimension, so in this case $[\Phi]_{\text{conventional}} = c^2[\Phi]_{\text{gravitational}}$. The gravitational potential outside a spherical object of mass M is $\Phi = -GM/R$ in conventional units and $\Phi = -M/R$ in gravitational units. In conventional units, the dimensionless quantity equal to M/R is $\frac{GM}{Rc^2}$.

Exercise 56. *Converting from conventional to gravitational to units.*

Consider a quantity Q with conventional dimension $M^m L^\ell T^t$ and hence dimension $L^{m+\ell+t}$ in gravitational units. Show that there is a unique product $G^i c^j$ for which $[G^i c^j] M^m L^\ell T^t = L^{m+\ell+t}$ and find i and j in terms of m, ℓ and t . (Using M or T as the single dimension for gravitational units, one could similarly find the associated unique powers of G and c that convert from conventional to these gravitational units.)

3.2 Newtonian Regime

The way we arrived at $G_{\alpha\beta} = 8\pi GT_{\alpha\beta}$ does not imply that this differential equation for $g_{\alpha\beta}$ will really lead in the Newtonian regime to a metric with $g_{tt} = -(1 + 2\Phi)$. That it does is the final miracle.

Newtonian limit. In a system for which the Newtonian approximation is valid, there is a frame in which the system's constituents move at speeds v slow compared to the speed of light. We define a small parameter e to be of order v (i.e., $e \sim v/c$, in conventional units). If the system's gravity does not produce larger speeds, the escape velocity from parts of the system must be of order e , implying the potential energy $m\Phi$ of particles of mass m must be of order mv^2 , or $\Phi \sim e^2$. In gravitational units, the potential outside a spherical mass M is $\Phi = M/r$, so $\frac{M}{r} \sim e^2$ (i.e. $GM/(rc^2) \sim e^2$).

gravitational units	conventional units
$v \sim e \ll 1$	$\frac{v}{c} \sim e \ll 1$
$\Phi \sim e^2$	$\frac{\Phi}{c^2} \sim e^2$
$\rho \sim nm_{\text{molecule}}, \quad p \sim \frac{1}{2} nm_{\text{molecule}} v^2 \Rightarrow \frac{P}{\rho} \sim e^2$	$\frac{P}{\rho c^2} \sim \frac{v^2}{c^2} \sim e^2$
If Q is a quantity that depends on the matter,	
$\frac{\partial_t Q}{\partial_x Q} \sim e$	$\frac{\frac{1}{c} \partial_t Q}{\partial_x Q} \sim \frac{1}{c} \frac{Q}{\frac{t}{x}} \sim \frac{v}{c} \sim e$

The departure of $g_{\alpha\beta}$ from $\eta_{\alpha\beta}$ will be of $O(\Phi)$, that is $O(e^2)$, so $\Gamma = O(e^2)$ and $\Gamma^2 = O(e^4)$.

The dominant contribution to $T^{\mu\nu}$ is $T^{tt} = \rho$. The field equation $R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = 8\pi T^{\alpha\beta}$ is equivalent to

$$R^{\alpha\beta} = 8\pi(T^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}T).$$

We will see that the R_{tt} equation is, to lowest order in e , an equation for g_{tt} . First evaluate the leading contribution to the right side, using

$$P \sim \rho e^2 \Rightarrow T = \rho u^\alpha u_\alpha (1 + O(e^2)) = -\rho (1 + O(e^2)).$$

$$\begin{aligned}
 R_{tt} &= 8\pi(T_{tt} - \frac{1}{2}g_{tt}T) \\
 &= 8\pi(\rho - \frac{1}{2}(-1)(-\rho)) \\
 &= 4\pi\rho.
 \end{aligned} \tag{3.23}$$

Next evaluate the left side, $R_{tt} = R_{it}{}^i{}_t$:

$$\begin{aligned}
 R_{itj}{}^t &= \partial_t \Gamma^t_{ij} - \partial_i \Gamma^t_{jt} + O(\Gamma^2) \\
 &= \partial_i \Gamma_{tjt} + O(e^3) \quad (\partial_t \Gamma \sim v \partial_x \Gamma \sim e^3) \\
 &= \partial_i [\frac{1}{2}(\partial_j g_{tt} + \partial_t g_{jt} - \partial_t g_{tj})] \\
 &= \frac{1}{2} \partial_i \partial_j g_{tt} + O(e^3).
 \end{aligned} \tag{3.24}$$

Thus

$$R_{tt} = -\frac{1}{2}\nabla^2 g_{tt}, \quad (3.25)$$

and the field equation becomes

$$\begin{aligned} \nabla^2 g_{tt} &= -8\pi\rho, \\ \text{implying } g_{tt} &= K - 2\Phi. \\ \text{Because } g_{tt} &= -1, \text{ when } \Phi = 0, \\ g_{tt} &= -1 - 2\Phi, \quad \text{as desired.} \end{aligned} \quad (3.26)$$

Our derivation implies that, with this g_{tt} , the equation of geodesic deviation has the right Newtonian form: That was our starting point. But here's a direct check:

$$\begin{aligned} \ddot{S}^i &= R^i{}_{ttj} S^j && \text{using } u^\mu = \delta_t^\mu + O(e) \\ &= \frac{1}{2}\partial^i\partial_j g_{tt} S^j && \text{from (3.24)} \\ &= -\partial^i\partial_j\Phi S^j. \end{aligned} \quad (3.27)$$

3.3 Pound-Rebka Experiment

A discussion of experimental verification of the theory beyond that discussed here will be deferred to discussions of precession in Sect. 4.4, of frame dragging in Sect. 5.5, and of gravitational radiation from binary inspiral in Sect. 6.3. Meanwhile, we can get cheaply the prediction that light traveling in a time-independent gravitational field has different frequencies as seen by observers “at rest” (moving along the Killing vector) at different spatial positions. This is essentially equivalent to our calculation of the time between two photons sent from A to B , with that time interval replaced by the time between two crests. In the new version, the result will come in the guise of a conservation law— energy conservation — associated with time-translation invariance.

Let t^α be the timelike Killing vector. Observers A and B are “at rest” if they move in spacetime along the timelike Killing vector. Their velocities are then

$$u_A^\alpha = C t^\alpha \quad u_B^\alpha = C' t^\alpha, \quad (3.28)$$

and $u^\alpha u_\alpha = -1 \Rightarrow C = (-g_{\alpha\beta} t^\alpha t^\beta)^{-1/2}$. Then

$$u_A^\alpha = (-g_{\beta\gamma} t^\beta t^\gamma)^{-1/2}(A) t^\alpha \quad u_B^\alpha = (-g_{\beta\gamma} t^\beta t^\gamma)^{-1/2}(B) t^\alpha. \quad (3.29)$$

A light ray sent from A to B has frequency

$$\begin{aligned} \omega_A &= -k_\alpha u_A^\alpha && \text{seen by } A, \text{ and} \\ \omega_B &= -k_\alpha u_B^\alpha && \text{seen by } B. \end{aligned}$$

k^α , the photon's wavevector, satisfies the geodesic equation

$$k^\beta \nabla_\beta k^\alpha = 0.$$

Because t^α is a Killing vector $\nabla_{(\alpha} t_{\beta)} = 0$, and we have

$$(k^\beta \nabla_\beta k^\alpha) t_\alpha = k^\beta \nabla_\beta (k^\alpha t_\alpha) - \underbrace{k^\beta k^\alpha \nabla_\beta t_\alpha}_0 \quad (3.30)$$

$$= k^\beta \nabla_\beta (k^\alpha t_\alpha) \quad (3.31)$$

Thus $k^\beta \nabla_\beta (k_\alpha t^\alpha) = 0$ and $k_\alpha t^\alpha$ is conserved along the trajectory. At ∞ , t_α is the 4-velocity of an inertial observer ($t^\alpha t_\alpha = -1$), so we write

$$-k_\alpha t^\alpha = \omega_\infty, \quad (3.32)$$

the frequency of the photon seen by an observer at ∞ . Because $k_\alpha t^\alpha$ is constant, the frequency seen by a stationary observer at a point A is, by Eq. (3.29),

$$\omega_A = (-t_\beta t^\beta)^{-1/2}(A) \omega_\infty = |g_{tt}(A)|^{-1/2} \omega_\infty. \quad (3.33)$$

Then

$$\omega_A - \omega_B = \omega_\infty [|g_{tt}(A)|^{-1/2} - |g_{tt}(B)|^{-1/2}]. \quad (3.34)$$

Nearly Newtonian: $g_{\alpha\beta} t^\alpha t^\beta = g_{tt} = -1 - 2\Phi$

$$|g_{tt}|^{-1/2}(A) - |g_{tt}|^{-1/2}(B) = \Phi_B - \Phi_A = |\Phi_A| - |\Phi_B| \quad (3.35)$$

$$\frac{\Delta\omega}{\omega} = \Delta\Phi \quad (3.36)$$

or $\frac{\Delta\omega}{\omega} = -gh$ on Earth. Reinstating c , we have $\frac{\Delta\omega}{\omega} = -\frac{gh}{c^2}$.

A photon sent from A up to B a height h above A loses energy in climbing up and is redshifted by $\frac{|\Delta\omega|}{\omega} = gh$. Because a photon's energy is $\hbar\omega$, conservation of $\hbar k_\alpha t^\alpha$ is called energy conservation, with $\mathcal{E} := -\hbar k_\alpha t^\alpha = \hbar\omega_\infty$ the conserved energy. Conservation of $\mathcal{E}$ gives the energy $-\hbar k_\alpha u^\alpha$ measured by a local observer with velocity u^α .

The redshift was first experimentally measured by Pound and Rebka <https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.3.439>[66]. To repeat: The experiment really tests only the equivalence principle. Two crests sent a proper time $\Delta\tau_A$ apart at A are received with proper time interval $\Delta\tau_B$, where $\frac{\Delta\tau_B - \Delta\tau_A}{\Delta\tau} = gh$.

Chapter 4

Spherical Relativistic Stars

4.1 Spherically symmetric spacetimes

A spacetime is spherically symmetric if the rotation group acts on it as a group of smooth maps from M to itself that preserve the metric: for each rotation R in O_3 there is a diffeo $\psi(R): M \rightarrow M$ that leaves the metric invariant,

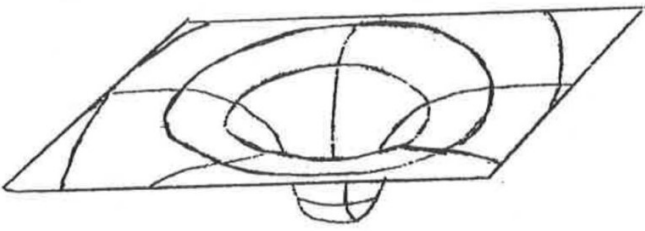
$$\psi(R)g_{\alpha\beta} = g_{\alpha\beta}, \quad (4.1)$$

such that $R \rightarrow \psi(R)$ is an action of O_3 :

$$\psi(\text{id}) = \text{id} \quad (4.2)$$

$$\psi(R_1 R_2) = \psi(R_1) \psi(R_2), \quad (4.3)$$

and such that almost every point of M sweeps out a sphere under the action of O_3 on $\mathbb{R}^3$. This last requirement eliminates the trivial action $\psi(R) = \text{id}$, all R . Picture a symmetric spacelike surface of M this way:



The circles represent symmetry spheres of M , orbits of points of M under the action of O_3 .

One acquires coordinates θ and ϕ on a symmetric sphere S in M this way: Denote by $\mathcal{R}(\psi\mathbf{n})$ the rotation of $\mathbb{R}^3$ by angle ψ about the axis $\mathbf{n}$. Acting on a symmetry sphere of M , the rotation $\mathcal{R}(\psi\mathbf{k})$ leaves two points fixed. One can call one of the fixed points $\theta = 0$ and give coordinates $(\theta, 0)$ to the image of $\theta = 0$ under $R(\theta\mathbf{j})$, just as one would do if the sphere were in flat space. Finally, the image of $(\theta, 0)$ under $R(\phi\mathbf{k})$ has coordinates (θ, ϕ) .

Let t and r be coordinates that label the symmetry spheres - i.e., t and r are scalars constant on each sphere. Then under a rotation, θ and ϕ change as they would in $\mathbb{R}^3$

$$\theta \rightarrow \theta'(\theta, \phi) \quad \phi \rightarrow \phi'(\theta, \phi) \quad (4.4)$$

or

$$\theta = \theta(\theta', \phi') \quad \phi = \phi(\theta', \phi'),$$

but t and r are constant

$$t' = t, \quad r' = r.$$

Thus

$$\begin{aligned} g_{t't'} &= \frac{\partial x^\mu}{\partial t'} \frac{\partial x^\nu}{\partial t'} g_{\mu\nu} = \delta_t^\mu \delta_t^\nu g_{\mu\nu} = g_{tt}, \\ g_{t'r'} &= g_{tr} \\ g_{r'r'} &= g_{rr}. \end{aligned} \tag{4.5}$$

In other words, g_{tt} , g_{tr} and g_{rr} are rotational scalars, while $g_{t\theta}$ and $g_{t\phi}$ transform as the θ and ϕ components of a vector in $\mathbb{R}^3$ under rotations, and $(g_{r\theta}, g_{r\phi})$ are again components of a rotational vector. Finally, the metric components

$$\begin{vmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{vmatrix}$$

transform under rotation as the $\theta\theta$, $\theta\phi$, and $\phi\phi$ components of a symmetric tensor on $\mathbb{R}^3$.

Because the only scalars on $\mathbb{R}^3$ invariant under rotations are functions independent of θ and ϕ , we have

$$g_{tr} = g_{tr}(t, r) \quad g_{tt} = g_{tt}(t, r) \quad g_{rr} = g_{rr}(t, r). \tag{4.6}$$

Because the only vector field on $\mathbb{R}^3$ independent of rotations is a radial vector field (whose r -component is independent of θ and ϕ) only the r -components of g_{ti} and g_{ri} are nonzero:

$$g_{t\theta} = g_{t\phi} = 0, \quad \text{and} \quad g_{r\theta} = g_{r\phi} = 0. \tag{4.7}$$

Finally any tensor $\begin{vmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{vmatrix}$ invariant under rotations is proportional to the metric on the 2-sphere:

$$f \begin{vmatrix} 1 & \\ & \sin^2 \theta \end{vmatrix}, \tag{4.8}$$

where f is independent of θ and ϕ . Thus

$$||g_{\mu\nu}|| = \begin{vmatrix} g_{tt}(t, r) & g_{tr} & & \\ g_{tr}(t, r) & g_{rr} & & \\ & & f(t, r) & \\ & & & f(t, r) \sin^2 \theta \end{vmatrix}.$$

Now pick a new radial coordinate $r' = \sqrt{f}$. Calling the new radial coordinate r again, we have for $||g_{\mu\nu}||$ the form

$$\begin{vmatrix} g_{tt} & g_{tr} & & \\ g_{tr} & g_{rr} & & \\ & & r^2 & \\ & & & r^2 \sin^2 \theta \end{vmatrix}$$

(Here g_{tt} , g_{tr} , and g_{rr} are the new components that result from transformation to the new radial coordinate.) With this new radial coordinate, $4\pi r^2$ is the area of a symmetry sphere.

Next, by a coordinate transformation $t \rightarrow T(r, t)$ we can make $g^{Tr} = 0$ (and so $g_{Tr} = 0$). This can be seen as follows. We seek a function T for which

$$0 = g^{Tr} = \frac{\partial T}{\partial t} g^{tr} + \frac{\partial T}{\partial r} g^{rr}.$$

Equivalently,

$$\xi \cdot \nabla T = 0, \quad \text{where } (\xi^\mu) = (g^{tr}, g^{rr}, 0, 0).$$

Let $c(\lambda)$ be the integral curve of the vector field ξ through a point $(t_0, r_0, \theta_0, \phi_0)$. If we set $T = \text{constant}$ along $c(\lambda)$, then $\xi \cdot \nabla T = 0$ along $c(\lambda)$. Extend this $T = \text{constant}$ curve to the 3-surface of revolution consisting of the rotated images of $c(\lambda)$ that pass through the points (t_0, r_0, θ, ϕ) , all values of θ, ϕ . Then $\xi \cdot \nabla T = 0$ on this surface, implying $g^{Tr} = 0$, as claimed.

Hence, changing the name “ T ” back to “ t ”, $\|g_{\mu\nu}\|$ can always be cast in the form

$$\|g_{\mu\nu}\| = \left\| \begin{array}{cccc} g_{tt} & & & \\ & g_{rr} & & \\ & & r^2 & \\ & & & r^2 \sin^2 \theta \end{array} \right\|$$

We will write $g_{tt} \equiv -e^{2\Phi}$, $g_{rr} \equiv e^{2\lambda}$, and $d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2$:

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\lambda} dr^2 + r^2 d\Omega^2. \quad (4.9)$$

For a collapsing star, Φ and λ depend on time, but we will be concerned with static equilibria.

Static, spherically symmetric spacetimes

A static spacetime has a timelike Killing vector t^α that is orthogonal to a family of spacelike hypersurfaces. There is then a natural time coordinate t that can be defined as follows. Let S be one of the hypersurfaces orthogonal to t^α . Set $t = 0$ on S , and define t elsewhere by $t^\alpha \nabla_\alpha t = 1$. Then t is the parameter distance from $t = 0$ along the integral curves of t^α . For example, the $t = 1$ surface is the result of dragging the $t = 0$ surface a parameter length 1 along the Killing vector. Any two constant t surfaces S_t are identical—they have the same geometry. Because t^α is perpendicular to S , it is perpendicular to S_t . If we extend t to a chart $t, \{x^i\}$, then orthogonality is equivalent to the vanishing of the time-space components of the metric:

$$g_{ti} = 0,$$

in a chart with $g_{\mu\nu}$ independent of t .

Note that a rotating star has a timelike Killing vector t^α , but t^α is not hypersurface orthogonal. In particular, natural spacelike hypersurfaces are axisymmetric, with the rotational Killing vector ϕ^α tangent to each hypersurface; but t^α is not perpendicular to ϕ^α : $g_{t\phi} = t^\alpha \phi_\alpha \neq 0$.

Our next objective is to write and solve the Einstein field equation for a static spacetime with the metric (4.9). First we need to compute $R_{\mu\nu}$. We'll sketch the derivation in a coordinate basis here, leaving as an exercise the somewhat quicker computation using Cartan calculus of the components along an orthonormal basis (see Appendix A.3).

We first find the nonzero components $\Gamma^\mu_{\nu\lambda}$, reading off Γ^μ 's from the Euler-Lagrange equation

$$\frac{d}{ds} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{\partial L}{\partial x^\mu} = 0,$$

with $L = \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$.

For example, writing $\Phi' := \partial_r \Phi$, we have

$$\begin{aligned} \frac{\partial L}{\partial \dot{t}} = -e^{2\Phi} \dot{t} \quad \frac{d}{ds} \frac{\partial L}{\partial \dot{t}} &= -e^{2\Phi} \ddot{t} - 2e^{2\Phi} \Phi' \dot{t} \dot{r} \quad \frac{\partial L}{\partial t} = 0 \\ \ddot{t} + 2\Phi' \dot{t} \dot{r} &= 0. \end{aligned}$$

Then

$\Gamma_{tr}^t = \Phi'$. Here are the remaining nonzero Γ 's:

$$\begin{aligned} \Gamma_{tt}^r &= \Phi' e^{2\Phi-2\lambda} & \Gamma_{rr}^r &= \lambda' & \Gamma_{\theta\theta}^r &= -r e^{-2\lambda} & \Gamma_{\phi\phi}^r &= -r \sin^2 \theta e^{-2\lambda} \\ \Gamma_{r\theta}^\theta &= \frac{1}{r} & \Gamma_{\phi\phi}^\theta &= -\sin \theta \cos \theta \\ \Gamma_{r\phi}^\phi &= \frac{1}{r} & \Gamma_{\theta\phi}^\phi &= \cot \theta. \end{aligned} \quad (4.10)$$

Next, the Ricci tensor components $R_{\mu\nu}$ are computed in terms of the Γ 's using the simplification

$$\Gamma_{\mu\lambda}^\lambda = \partial_\mu (\log \sqrt{-g}).$$

Here

$$\sqrt{-g} = r^2 \sin \theta e^{\Phi+\lambda}. \quad (4.11)$$

From (II.43)

$$R_{\mu\nu} = \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\mu \Gamma_{\lambda\nu}^\lambda + \Gamma_{\mu\nu}^\lambda \Gamma_{\lambda\sigma}^\sigma - \Gamma_{\sigma\nu}^\lambda \Gamma_{\lambda\mu}^\sigma \quad (4.12a)$$

$$= \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\mu \partial_\nu \log \sqrt{-g} + \Gamma_{\mu\nu}^\lambda \partial_\lambda \log \sqrt{-g} - \Gamma_{\sigma\nu}^\lambda \Gamma_{\lambda\mu}^\sigma \quad (4.12b)$$

Exercise: Find $G_{\mu\nu}$ for the static, spherically symmetric metric (4.9); start by computing the $\Gamma_{\mu\nu}^\lambda$ of Eq. (4.10) with $\lambda = r, \theta, \phi$.

The computation gives

$$G_t^t = e^{-2\lambda} \left(\frac{1}{r^2} - \frac{2}{r} \lambda' \right) - \frac{1}{r^2} \quad (4.13a)$$

$$= -\frac{1}{r^2} \frac{d}{dr} [r(1 - e^{-2\lambda})]$$

$$G_r^r = e^{-2\lambda} \left(\frac{1}{r^2} + \frac{2}{r} \Phi' \right) - \frac{1}{r^2} \quad (4.13b)$$

$$G_\theta^\theta = G_\phi^\phi = e^{-2\lambda} \left[\Phi'' + (\Phi')^2 + \frac{1}{r} (\Phi' - \lambda') - \Phi' \lambda' \right]. \quad (4.13c)$$

All other $G_{\mu\nu}$'s vanish (by the argument showing that off-diagonal $g_{\mu\nu}$'s are zero).

4.1.1 Schwarzschild solution

The Schwarzschild solution is the geometry outside a spherically symmetric star and the geometry of a black hole: It is the geometry of any asymptotically flat spherically symmetric vacuum spacetime.

In a vacuum, the field equations are

$$G_\nu^\mu = 0.$$

The G_t^t -equation (4.13b),

$$G_t^t = -\frac{1}{r^2} \frac{d}{dr} [r(1 - e^{-2\lambda})] = 0,$$

has the first integral

$$r(1 - e^{-2\lambda}) = 2M, \text{ for some constant } M, \quad (4.14)$$

implying

$$e^{2\lambda} = \left(1 - \frac{2M}{r}\right)^{-1}. \quad (4.15)$$

From the combination,

$$G^r_r - G^t_t = \frac{2}{r} e^{-2\lambda} (\Phi' + \lambda') = 0, \quad (4.16)$$

we have

$$\Phi' = -\lambda' \quad (4.17)$$

or

$$\Phi = -\lambda + k \quad e^{2\Phi} = k \left(1 - \frac{2M}{r}\right).$$

Reparametrizing the time by writing $\tilde{t} = \frac{1}{\sqrt{k}}t$, and changing the name of $\tilde{t}$ back to t , gives

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (4.18)$$

the *exterior Schwarzschild metric*. The geometry is asymptotically flat: For large r , the metric is the Minkowski metric:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2. \quad (4.19)$$

For $r = 2M$ the components $g_{\mu\nu}$ are singular, so the form (4.18) provides a metric for a spacetime with a hole in it: $\infty > r > 2M$, $-\infty < t < \infty$. When we discuss black holes, we'll see that this is a coordinate singularity, like the poles in spherical coordinates. Changing to coordinates that are smooth at $r = 2M$ reveals what you know: The surface $r = 2M$ is an event horizon. For large r , the metric (4.18) takes the post-Newtonian form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2. \quad (4.20)$$

For nearly Newtonian stars, g_{tt} determines the effect of the gravitational field on matter, and $-\frac{M}{r}$ is the Newtonian potential Φ . Because the trajectories of particles at large r are those of Newtonian particles about a mass M , one calls M the mass of the spacetime.

The equations, $G^\theta_\theta = 0$, $G^\phi_\phi = 0$, are automatically satisfied once $G^t_t = 0$ and $G^r_r = 0$ (not in general, just in this spherically symmetric case).

There is a different choice of radial coordinate for which the exterior Schwarzschild metric has the form

$$ds^2 = -\frac{\left(1 - \frac{M}{2r}\right)^2}{\left(1 + \frac{M}{2r}\right)^2} dt^2 + \left(1 + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\Omega^2). \quad (4.21)$$

The coordinates t, θ, ϕ (with t, θ, ϕ unchanged) are called *isotropic* coordinates. The next problem asks you to find the coordinate transformation that gives this form. A metric of the form $g_{ab} = H f_{ab}$, with f_{ab} flat is said to be conformally flat, and H is called the conformal factor. That the spatial part of the Schwarzschild metric can be written in isotropic form means that it is conformally flat, here with conformal factor $\left(1 + \frac{M}{2r}\right)^4$.

Exercise 57. (This is problem 1 of Wald, Chapter 6).

- a. Show that any spherically symmetric Riemannian 3-metric can be written in the form

$$ds^2 = [F(\bar{r})]^2(d\bar{r}^2 + \bar{r}^2 d\Omega^2).$$

From our earlier discussion, we already know that a metric of this kind can be written as the spatial part of (4.9), namely $ds^2 = e^{2\lambda} dr^2 + r^2 d\Omega^2$. So the problem is to find functions $\bar{r}(r)$ and $F(r)$ for which $[F(\bar{r})]^2(d\bar{r}^2 + \bar{r}^2 d\Omega^2) = e^{2\lambda(r)} dr^2 + r^2 d\Omega^2$.

- b. Specialize to the exterior Schwarzschild solution to recover Eq. (4.21) with $r \equiv \bar{r}$.

Stellar Interior

Equilibrium configurations of stars are accurately modeled as perfect fluids. For static, spherical stars, ρ and P depend only on r , while the 4-velocity u^α is along the Killing vector t^α :

$$u^\alpha = k t^\alpha.$$

But $t^\alpha t_\alpha = g_{tt} = -e^{+2\Phi}$, so $u^\alpha u_\alpha = -1$ implies

$$u^\alpha = e^{-\Phi} t^\alpha.$$

Then, from Eq. (1.162),

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + P g^{\alpha\beta},$$

we have

$$T^t_t = -\rho \quad T^r_r = P.$$

The field equation components are

$$G^t_t = 8\pi T^t_t:$$

$$\begin{aligned} \frac{1}{r^2} \frac{d}{dr} [r(1 - e^{-2\lambda})] &= 8\pi\rho \\ r(1 - e^{-2\lambda}) &= 2 \int_0^r 4\pi\rho r^2 dr =: 2m(r) \end{aligned} \quad (4.22)$$

$$e^{2\lambda} = \left(1 - \frac{2m(r)}{r}\right)^{-1}. \quad (4.23)$$

Here $m(r)$ is a kind of mass within a radius r , and $m(r) = M$, the mass measured at infinity, for $r \geq R$. $G^r_r = 8\pi T^r_r$:

$$\begin{aligned} e^{-2\lambda} \left(\frac{2\Phi'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} &= 8\pi P \\ \left(1 - \frac{2m}{r}\right) \left(\frac{2\Phi'}{r} + \frac{1}{r^2} \right) &= 8\pi P + \frac{1}{r^2} = \frac{8\pi P r^2 + 1}{r^2} \\ \frac{2\Phi'}{r} + \frac{1}{r^2} &= \frac{8\pi P r^2 + 1}{r(r - 2m)} \\ 2\Phi' &= \frac{8\pi P r^2 + 1}{r - 2m} - \frac{1}{r} = \frac{8\pi P r^3 + 2m}{r(r - 2m)} \\ \Phi' &= \frac{m + 4\pi P r^3}{r(r - 2m)} \end{aligned} \quad (4.24)$$

Note that the Newtonian limit ($P \ll \rho$, $r \ll m$) of (4.24) is

$$\Phi' = \frac{m}{r^2} ,$$

so that Φ is again the Newtonian potential.

The remaining field equation components, $G^\theta_\theta = 8\pi T^\theta_\theta$, $G^\phi_\phi = 8\pi T^\phi_\phi$, are identical to one another and are implied by (4.23), (4.24) and the equation of hydrostatic equilibrium $q^\alpha_\gamma \nabla_\beta T^{\gamma\beta}$, which we now obtain: Recall that $q^\alpha_\gamma \nabla_\beta T^{\gamma\beta} = 0$ has the form

$$u^\beta \nabla_\beta u_\alpha = -\frac{1}{\rho + P} q^\beta_\alpha \nabla_\beta P. \quad (4.25)$$

On the RHS, $q^\beta_\alpha \nabla_\beta P = (\delta_\alpha^\beta + u_\alpha u^\beta) \nabla_\beta P$. The second term vanishes because $\partial_t P = 0$. Then

$$q^\beta_\alpha \nabla_\beta P = \nabla_\alpha P.$$

Next, we have

$$\begin{aligned} u^\beta \nabla_\beta u_\alpha &= e^{-\Phi} t^\beta \nabla_\beta (e^{-\Phi} t_\alpha) \\ &= e^{-2\Phi} t^\beta \nabla_\beta t_\alpha \quad (\text{using } t^\beta \nabla_\beta \Phi = 0) \\ &= -e^{-2\Phi} t^\beta \nabla_\alpha t_\beta \quad (\nabla_\alpha t_\beta + \nabla_\beta t_\alpha = 0 \text{ — Killing vector eq.}) \\ &= -\frac{1}{2} e^{-2\Phi} \nabla_\alpha (t^\beta t_\beta) \\ &= \frac{1}{2} e^{-2\Phi} \nabla_\alpha (e^{2\Phi}) \\ &= \nabla_\alpha \Phi \\ \nabla_\alpha \Phi &= -\frac{1}{\rho + P} \nabla_\alpha P \end{aligned}$$

or

$$\Phi' = -\frac{1}{\rho + P} P'. \quad (4.26)$$

Eqs. (4.24) and (4.26) imply the equation of hydrostatic equilibrium - the TOV (Tolman-Oppenheimer-Volkov) equation:

$$P' = -(\rho + P) \frac{m + 4\pi r^3 P}{r(r - 2m)}. \quad (4.27)$$

A spherical relativistic star is a solution to equations (4.23), (4.24), and (4.27) together with an equation of state; the numerical models that have been constructed usually involve equations of state of the simplest form

$$P = P(\rho), \quad (4.28)$$

which are reasonably accurate for neutron stars. A general equation of state has the form $P = P(\rho, s, z_1, \dots, z_n)$, with s the entropy per baryon and z_i the concentration of the i th particle species. Neutron stars and white dwarfs are cold enough ($KT \ll$ Fermi energy) that they are nearly isentropic ($s = \text{constant}$), and their nuclear reactions have proceeded to completion, so each z_i is itself a function of ρ . That's why $P = P(\rho)$ is a good approximation. (At absolute zero of course, s is constant.)

One obtains a star by integrating eqs. (4.27) and (4.28) together with the defining equation 4.23 for m . That is, one integrates the system

$$m(r) = \int_0^r \rho \, 4\pi r^2 dr, \quad \frac{dP}{dr} = -(\rho + P) \frac{m + 4\pi r^3 P}{r(r - 2m)}, \quad P = P(\rho). \quad (4.29)$$

One begins with a central density ρ_c and integrates up to the radius R at which P drops to zero (P is a decreasing function of r). This is the boundary of the star. The metric inside the star is then given by

$$e^{2\lambda} = \left(1 - \frac{2m}{r}\right)^{-1}, \quad (4.30)$$

$$\Phi = \Phi(R) + \int_r^R \frac{m + 4\pi Pr^3}{r(r - 2m)}, \quad (4.31)$$

and outside by

$$e^{2\Phi} = e^{-2\lambda} = \left(1 - \frac{2M}{r}\right). \quad (4.32)$$

To restore G and c to these equations, one can multiply by the (unique) factors of G and c that allow each quantity to have the desired conventional units. For example, in Eq. (4.30), both sides are dimensionless, and the unique factor built from G and c that makes m/r dimensionless is G/c^2 . That is, $\frac{G}{c^2} \frac{m}{r}$ is dimensionless, so the right side of (4.30) becomes $\left(1 - \frac{2Gm}{c^2 r}\right)^{-1}$.

The Newtonian limit of the structure equations (4.27) of a spherical star are of course the Newtonian equations

$$m = \int_0^r \rho 4\pi r^2 dr, \quad \frac{dP}{dr} = -\frac{Gm\rho}{r^2}, \quad P = P(\rho). \quad (4.33)$$

The Newtonian potential Φ satisfies

$$\nabla^2 \Phi = 4\pi\rho, \quad \lim_{r \rightarrow \infty} \Phi = 0. \quad (4.34)$$

Gauss's law then implies $\Phi' = m/r^2$, for a spherical star.

4.1.2 Stellar models

The simplest stellar model is a uniform-density Newtonian star.

Exercise 58. Find the solutions $P(r)$ and $\Phi(r)$ to the Newtonian equations governing hydrostatic equilibrium, (4.33) and $\Phi' = m/r^2$, and show that they can be written in the form (4.35) below.

A star of radius R and constant density ρ has $m(r)$, $P(r)$ and $\Phi(r)$ given by

$$m(r) = \frac{4}{3}\pi r^3 \rho = M \frac{r^3}{R^3}, \quad (4.35a)$$

$$P(r) = P_c \left(1 - \frac{r^2}{R^2}\right), \quad \text{where } P_c = \frac{\rho M}{R}, \quad (4.35b)$$

$$\Phi(r) = \frac{1}{2} \frac{m}{r} - \frac{3}{2} \frac{M}{R}. \quad (4.35c)$$

Here P_c is the pressure at the center of the star. Outside the star,

$$\Phi = -\frac{M}{r}. \quad (4.36)$$

Inside the star, Φ' has the simple form $\frac{m}{r^2}$ given by Gauss's law, and a common error is to think of Gauss's theorem as meaning Φ itself is given by $-\frac{m}{r}$. But Φ involves the integral $\int_r^\infty \Phi' dr$.

Inside a uniform density spherical shell, for example, Φ is constant, like the electric potential inside a charged spherical shell.

Exercise 59. Relativistic uniform-density star. Find the pressure P inside and the metric potentials Φ and λ inside a relativistic star with uniform energy density ρ . Use Eq. (4.22) to find $m(r)$, the TOV equation (4.27) to find $P(r)$, and Eq. (4.26) to find Φ . Then show that they can be written in the form (4.37) below.

We have already found the solution 4.32 outside the star. Inside the star, it is given by

$$m(r) = \frac{4}{3}\pi r^3 \rho, \quad (4.37a)$$

$$P(r) = \rho \frac{\sqrt{1 - 2m/r} - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - \sqrt{1 - 2m/r}}, \quad (4.37b)$$

$$e^{\Phi(r)} = \frac{3}{2}\sqrt{1 - 2M/R} - \frac{1}{2}\sqrt{1 - 2m/r}, \quad (4.37c)$$

$$e^{\lambda(r)} = \sqrt{1 - 2m/r}. \quad (4.37d)$$

If you get stuck, a hint is given after *Exercise 62*.

Exercise 60. Write the TOV equation (4.27) and the solution 4.37 for a relativistic star in conventional units, restoring G and c to the expressions.

Exercise 61. Show that the Newtonian limit of the relativistic solution is the Newtonian solution for P and Φ . Either use the form in gravitational units, noting that $m/r \lesssim 1$, $M/R \lesssim 1$, and discarding terms of quadratic order in these quantities; or use the form with G and c restored, discarding terms with c^2 in the denominator. If you use conventional units, what is the small dimensionless quantity whose quadratic and higher orders you are discarding.

Exercise 62. The smallest possible value of $\frac{2M}{R}$ at the surface of a static (spherical) star is attained when the star is perfectly incompressible – when it is one of the uniform density models. (This is Buchdahl's Theorem[18], with details given in Sect. 4.3 below.

- Show for this sequence of stars that the central density becomes infinite when $\frac{2M}{R} = \frac{8}{9}$.
- Deduce from (a) the maximum redshift z ,

$$1 + z := \frac{\lambda_\infty}{\lambda_{\text{star}}} = \frac{\omega_{\text{star}}}{\omega_\infty},$$

from the surface of a spherical star. (Recall the redshift relation (3.33).)

Hint for Exercise 59: From the TOV equation, show $\frac{dP}{dr} = \frac{4}{3}\pi \frac{(\rho + P)(\rho + 3P)}{1 - \frac{8}{3}\pi \rho r^2}$, divide by $(\rho + P)(\rho + 3P)$, and integrate from P to 0 on the left side, and r to R on the right to obtain $P(r)$. Then show $\Phi' = [\ln(\rho + P)]'$ and integrate to find $\Phi(r)$, using the value of $\Phi(R)$ from the form (4.36) of Φ outside the star.

4.2 White dwarfs and neutron stars

4.2.1 Estimates

At the endpoint of stellar evolution, when nuclear reactions cease, stars with mass less than about $10 M_\odot$ die as white dwarfs.¹ In white dwarfs the major pressure supporting the star arises from the Fermi degeneracy pressure – the Pauli exclusion principle requiring the fermions to have nonzero momentum. Here is first an intuitive, then a more careful discussion of the inability of a degenerate Fermi gas to support more than about $1.4 M_\odot$ against its own gravity. (It starts with rocks for practice.)

Rocks

Intuitive: For small cold masses (rocks), gravity is unimportant and the Fermi pressure balances the electrostatic attraction of electrons to nuclei: We use this as follows to obtain a rough estimate of the ground-state size of atoms and hence of the density of rock. Let n be the number of electrons per unit volume, m_e the mass of an electron. Pauli exclusion restricts each fermion to its own volume $\mathfrak{v} = \frac{1}{n}$ and the uncertainty principle then implies a momentum,

$$p \sim \frac{\hbar}{\mathfrak{v}^{1/3}} = \hbar n^{1/3}, \quad (4.38)$$

and a corresponding pressure (momentum crossing unit area/unit time)

$$P \sim pvn. \quad (4.39)$$

For the nonrelativistic electrons of rock, $v = p/m_e$ and the pressure is

$$P \sim \frac{p^2}{m_e} n \sim \frac{\hbar^2 n^{5/3}}{m_e}. \quad (4.40)$$

This must balance the electrostatic energy density:

$$\frac{e^2}{\mathfrak{v}^{1/3}} n = e^2 n^{4/3}, \quad (4.41)$$

or

$$\frac{\hbar^2 n^{5/3}}{m_e} \sim e^2 n^{4/3}.$$

The corresponding number density of electrons is then

$$n \sim \frac{m_e^3 e^6}{\hbar^6}$$

With m_n the mass of a nucleon, we have

$$\rho \sim m_n n \sim \frac{e^6 m_e^3 m_n}{\hbar^6} \sim 10 \text{ g/cm}^3. \quad (4.42)$$

Gravity is dominant in objects massive enough that the gravitational binding energy density is greater than the electrostatic energy density. Writing $M \sim \rho R^3$, the gravitational energy density is

$$\frac{GM}{R} \rho \sim G \rho^{4/3} M^{2/3}.$$

¹The critical mass of the main-sequence progenitor star, above which it ends its life in a core-collapse supernova, depends on the initial *metallicity* of the star – the fraction of elements heavier than helium (see, e.g., Ibeling and Heger 2023).

Substituting $\rho = m_n n$ in this expression, and equating it to the RHS of Eq. (4.41) for the electrostatic energy density, we have

$$e^2 n^{4/3} \sim n^{4/3} m_n^{4/3} G M^{2/3},$$

when the electrostatic and gravitational energy densities are equal. Solving this equation for M , we find the mass

$$M \sim \frac{e^3}{G^{3/2} m_n^2} \sim 2 \times 10^{30} \text{ g} \sim M_{\text{Jupiter}}. \quad (4.43)$$

The critical size above which an asteroid or moon must be spherical is much smaller than this, because the molecular bonds that keep rock rigid have much lower energy than the binding energy (ionization energy) of their atoms. The next problem estimates that critical size, using the melting energy of quartz. This is an upper limit, ignoring weakness from, for example, grain boundaries and faults.

Exercise 63. The maximum height of mountains can be estimated by requiring that the energy needed to melt the rock in a layer of thickness h below the mountain is equal to the gravitational energy lost when the mountain sinks by the distance h .

- Using an energy of about 0.1 eV/molecule to melt quartz, estimate the maximum height of a mountain on the Earth and on Mars.
- Asteroids and rocky moons with large enough radius are spherical (if nonrotating). Estimate the radius above which a rocky moon is spherical (i.e., with bumps smaller than the moon's radius).

4.2.2 White Dwarfs

The cores of objects of mass greater than M_{Jupiter} are compressed beyond planetary densities and for masses $> 50 M_{\text{Jupiter}}$ balls of hydrogen begin nuclear reactions and are – by definition – stars. As noted above, a white dwarf is the final state of a star whose *initial* mass is less than about $10 M_{\odot}$. By the end of its evolution, the star has ejected its outer envelope of hydrogen, and the core that remains contracts and eventually cools to a dead ball of He, C and O, most commonly dominated by a C-O interior. (Although most stars have masses too small to burn He to C and O, the evolution time of stars with masses below about $0.8 M_{\odot}$ is longer than the age of the universe; and He will fuse in cores of stars with initial masses above about $0.5 M_{\odot}$.) The most massive stars that end as white dwarfs include heavier elements: Ne and Mg.

Structure: Mass-Radius Relation

Because its nuclear reactions have turned off, a dead star is held apart by its degeneracy pressure. The size of such a star turns out to *decrease* as its mass increases: adding baryons increases the gravitational attraction enough that more baryons are packed in a smaller total volume. This relation between mass and radius can be found from our equations of hydrostatic equilibrium and the equation of state of a degenerate gas. Here we'll again obtain an estimate based on the averaged form of the equation of hydrostatic equilibrium with dP/dr approximated by $-P/R$ and $m(r)$ by ρr^3 , with M the total mass. Then the Newtonian equation of hydrostatic equilibrium, $\frac{dP}{dr} = -\frac{GM\rho}{r^2}$, is roughly approximated by

$$\frac{P}{R} \sim \frac{GM\rho}{R^2} \quad (4.44)$$

Similarly dropping numerical factors, we have $\rho \sim \frac{M}{R^3}$, giving

$$P \sim \frac{GM^2}{R^4}. \quad (4.45)$$

Using the degeneracy pressure $P \approx \frac{\hbar^2}{m_e} n^{5/3}$ of (4.40) and

$$\rho = m_p n \sim \frac{M}{R^3} \Rightarrow n \sim \frac{M}{m_p R^3}$$

gives

$$\begin{aligned} \frac{GM^2}{R^4} &\sim P \sim \frac{\hbar^2}{m_e} \left(\frac{M}{m_p R^3} \right)^{5/3} \\ &\sim \frac{\hbar^2}{m_e} \frac{M^{5/3}}{m_p^{5/3} R^5} \\ R &\sim \frac{\hbar^2}{G m_e m_p^{5/3}} \cdot \frac{1}{M^{1/3}}. \end{aligned} \quad (4.46)$$

With the right numerical factors for a Helium dwarf,

$$\begin{aligned} R &= 1.4 \frac{\hbar^2}{G m_e m_p^{5/3}} M^{-1/3} \\ \text{or, } \frac{R}{R_\odot} &= 0.014 \left(\frac{M_\odot}{M} \right)^{1/3}. \end{aligned}$$

(When $M = M_\odot$, we have $R = .014 R_\odot$). So we obtain the relation $R \propto M^{-1/3}$, valid when the star is dense and cold enough for degeneracy and not so massive that it collapses.

Upper mass limit

The speed limit set by causality—the speed of light—also sets an upper limit on the mass of white dwarfs, on the dead iron cores of massive stars, and on the mass of neutron stars (this one takes more work).

The star has to go on radiating and radiating and contracting and contracting until, I suppose, it gets down to a few km. radius, when gravity becomes strong enough to hold in the radiation, and the star can at last find peace. Dr. Chandrasekhar had got this result before, but he has rubbed it in in his latest paper; and, when discussing it with him, I felt driven to the conclusion that this was almost a reductio ad absurdum of the relativistic degeneracy formula. A. S. Eddington (1935), published version of comments that followed a talk by Chandra on the upper mass limit. [25]

Notice that Chandrasekhar and Eddington recognized the possibility of collapse to a black hole resulting from the upper limit on a mass supported by degeneracy pressure, but they failed to make the connection between collapse and supernovae. Baade and Zwicky[10] had proposed in the previous year that a supernova was the result of collapse to a neutron star, but didn't relate collapse to the upper

mass limit. The connection between the limiting mass of a degenerate stellar core (or a white dwarf) and the collapse to a neutron star did not appear in print until 1939 articles by Gamow[40] and by Chandrasekhar[23].

Initially the gas will be nonrelativistic, but as the mass increases and the radius decreases, the density rises quadratically in M : Writing hydrostatic equilibrium in the form (4.45) with R replaced by $\left(\frac{M}{\rho}\right)^{1/3} = \left(\frac{M}{m_n n}\right)^{1/3}$ gives

$$P \sim GM^{2/3}(m_n n)^{4/3}, \quad (4.47)$$

implying

$$M \sim \frac{\hbar^3}{G^{3/2}} \frac{n^{1/2}}{m_n^2 m_e^{3/2}} \quad (\text{nonrelativistic}). \quad (4.48)$$

As the density rises, the Fermi momentum becomes relativistic when $p = \hbar n^{1/3}$ is of order $m_e c$. Because v is limited by c , it no longer grows as $n^{1/3}$, and that limit on v means the pressure is given by

$$\begin{aligned} P &\sim p v n < \hbar n^{1/3} c n \quad (v \sim c \text{ now, or}) \\ P &< \hbar c n^{4/3}. \end{aligned}$$

With P growing no faster than $n^{4/3}$ instead of its nonrelativistic $n^{5/3}$ behavior, hydrostatic equilibrium in the form (4.47) now implies

$$GM^{2/3} m_n^{4/3} n^{4/3} \sim P < \hbar c n^{4/3};$$

the $n^{4/3}$ cancels (!) and one finds a maximum mass

$$M \lesssim \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_n^2} = 1.8 M_\odot \quad (3.7 \times 10^{33} g). \quad (4.49)$$

For typical dwarfs (e.g., helium or helium-carbon dwarfs) the actual calculation gives

$$M_{\max} = 0.76 \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_n^2} = 1.4 M_\odot. \quad (4.50)$$

This is the Chandrasekhar limit – a collection of fermions supported by degeneracy pressure and with masses greater than this must collapse.

To summarize: As long as the gas is nonrelativistic, $P \propto n^{5/3}$, and as a star contracts, its pressure therefore rises faster than its gravitational energy density $\frac{GM\rho}{R}$, which is proportional to $n^{4/3}$. There will then always be a density, n , high enough that the pressure gradient, $\frac{P}{R}$, balances the gravitational force, $\frac{GM\rho}{R^2}$. At that density the star will cease to contract. If, however, the gas becomes relativistic before equilibrium is reached, then the pressure will be bounded by $\hbar c n^{4/3}$ because v is bounded by c ; and it will never catch up to gravity if the coefficient of $n^{4/3}$ in the pressure is smaller than the coefficient of $n^{4/3}$ in the gravitational energy density – if

$$M > \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_n^2}.$$

Some additional history: In 1931 shortly before Chadwick's discovery of the neutron and shortly after the first paper by Chandrasekhar [22] (following approximate computations by Anderson [3] and

Stoner [83]) on an upper mass limit of white dwarfs, Landau [53] submitted a paper that independently argued that there was an upper limit on the mass of a collection of degenerate fermions and speculated on the existence of stars with cores of nuclear density now called Thorne-Żytkow objects [91]. Léon Rosenfeld [70] gives a widely repeated but false description: “when the news on the neutron’s discovery reached Copenhagen, we had a lively discussion on the same evening about the prospects opened by this discovery. In the course of it Landau improvised the conception of neutron stars – “unheimliche Sterne,” weird stars, which would be invisible and unknown to us unless by colliding with visible stars they would originate explosions, which might be supernovae.” A careful historical study by Yakovlev *et al.* [98], however, finds that Landau was in Copenhagen in 1931 *before* Chadwick’s experiments – and just after Landau had submitted his paper on nuclear-density cores.

Exercise 64. Estimate the radius of a neutron star, assuming the degeneracy pressure of free nonrelativistic neutrons supplies the pressure that supports it.

Exercise 65. Suppose that a species of neutrino has mass $m_\nu = 10$ eV.

- Find the upper mass limit on a degenerate neutrino star.
- Consider a neutrino star with mass of 1/10 this upper limit. Estimate the radius of the star, as in the previous problem for for neutron stars.
- Use this radius to estimate the temperature above which the neutrinos in the neutrino star in part (b) would cease being degenerate.

4.2.3 Neutron Stars

A very short introduction is given here. Standard references are Shapiro-Teukolsky[76] and Camenzind[20].

The only stars in whose structure general relativity plays a major role are neutron stars. The first observations were made in two ways: First, in 1962, a team led by Giacconi, using an Aerobee rocket with an X-ray detector, found the first X-ray source outside the Solar System, Sco X-1 (the label means the brightest X-ray source in the direction of the constellation Scorpius). Shklovsky [80] suggested that Sco X-1 might be a neutron star and that synchrotron radiation from relativistic electrons might power the Crab nebula [79]). The “binary hypothesis” emerged from a discussion at the Noordwijk Symposium in August 1966 that included G. and M. Burbidge, Ginzburg, Shklovsky, Savedoff, Woltjer, Prendergast, and Herbig. After Rossi presented observations suggesting that Sco X-1 was a binary, Burbidge writes, “several of us then realized that . . . the very powerful x-ray source might naturally arise through mass exchange between the secondary and primary . . .” Articles with accretion from a companion to a neutron star, beginning with Shklovsky[81], quickly followed. Here’s what is happening: Old neutron stars in binary systems can accrete mass from their companion after the companion evolves off the main sequence (expanding from increased energy from a contracting core that gets hotter as it contracts and from the much faster reactions in a shell around the core that are the result of the hotter core). Matter falling onto the neutron star is moving fast enough to emit X-rays.

The second observation came within two years of Shklovsky’s first paper. In 1967, Jocelyn Bell, using as a radio telescope a field of wires designed by her advisor, Anthony Hewish, discovered[48] pulsing light signals (radio frequency) arriving $\sim$ every second, with fantastic regularity . The first one seen had period $P = 1.3373011$ s, another $P = .253065$ s. The regularity

$$\frac{1}{P} \frac{dP}{d\tau} \approx \frac{1}{2000 \text{ years}}$$

is much too great to be an oscillating star.

A rotating star would fly apart unless its density were great enough that (rotational kinetic energy) $< \frac{1}{2}$ (gravitational binding energy).

A rough estimate is enough to rule out rotating white dwarfs:

$$\begin{aligned}\frac{1}{2}I\Omega^2 &< \frac{1}{2} \text{ binding energy} \\ MR^2\Omega^2 &< \frac{M^2}{R} \quad (\text{ignoring numerical factors}) \\ \Omega^2 &< \rho \quad (\Omega^2 < G\rho) \\ \Omega = 60\pi \text{ s}^{-1} &\implies \rho > 10^{11} \text{ g/cm}^3\end{aligned}$$

Because $\rho_{\text{dwarf}} < 10^8 \text{ g/cm}^3$, only neutron stars are dense enough.

Nuclear density is

$$\rho = m_p n = 2.8 \times 10^{14} \text{ g cm}^{-3}, \quad (4.51)$$

corresponding to a separation between nucleons of about $\ell = 1.8 \times 10^{-13} \text{ cm}$. The average density of neutron stars is a few times greater than this, and their corresponding maximum spin is larger than 1000 Hz. As of 2023 the spin rate of the fastest observed pulsar is 716 Hz.

Additional strong evidence that pulsars are neutron stars was not long in coming. Just before the pulsar discovery, Pacini [62] proposed the magnetic field of a rotating neutron star as the energy source of the Crab and other supernova remnants; and, of course, searches for pulsars in the Crab and some of the other remnants were successful. The pulses can be reasonably ascribed to synchrotron radiation from particles spiraling in the magnetic field near the poles, with a contribution from radiation produced by a spinning magnetic field. The beam sweeps past you each rotation, and often one observes double pulses, as one would expect for a beam from each pole, if they are not close to the spin axis.

Pulsars spin down slowly; $\dot{P}/P \sim \frac{1}{2000 \text{ yrs}}$ for Crab, and the

loss of rotational energy = observed luminosity of the Crab nebula,

if one assumes that the Crab pulsar is a rotating neutron star, with

$$dE/dt \sim MR^2\omega\dot{\omega}.$$

This check, due to Tommy Gold[43]), together with the pulsar's location in a supernova remnant was the most compelling evidence that pulsars are neutron stars.

Exercise 66. Do Gold's calculation: Estimate the moment of inertia of a neutron star and the rotational energy of the Crab pulsar. Data: ATNF Catalogue: $P = 33 \text{ ms}$, $\dot{P} = 4.2 \times 10^{-13} \text{ Hz/s}$. You should roughly find the observed luminosity of the Crab Nebula, namely $L \sim 75,000 L_{\odot} = 3 \times 10^{38} \text{ erg/s}$ Wikipedia.

Here's a check that the gravitational field of a neutron star is strong enough to accelerate infalling electrons to X-ray energies, about $m_e/100$.

$$\frac{m_e M}{R} > m_e/100 \implies \frac{M}{R} > 1/100.$$

$$\left(\frac{M}{R}\right)_{\text{neutron star}} \approx 1.4 M_{\odot}/10 \text{ km} = \frac{1.4 \times 1.5 \text{ km}}{12 \text{ km}} = 0.18,$$

whereas

$$\left(\frac{M}{R}\right)_{\text{dwarf}} \sim \frac{M_{\odot}}{10^4 \text{ km}} \sim 10^{-4}.$$

So matter in an accretion disk around a neutron star can emit an intense beam of X-rays from radii up to more than 10 times the radius of the star, while the X-ray intensity from accreting white dwarfs is low.

Energy of a supernova

As mentioned above, Baade and Zwicky, armed with Landau's suggestion and with the observations of the previous ten years showing that observed supernova occurred in other galaxies, wrote in 1934 "With all reserve we advance the view that a super-nova represents the transition of an ordinary star into a neutron star, consisting mainly of neutrons." Their suggestion was due in part to the following computation of the energy that would be produced in such a collapse, although they were unaware that Chandrasekhar's upper limit on white dwarfs would imply that a collapse of this kind was in fact possible.

Supernovae are explosions that can outshine the entire galaxy in which they occur, with brightness as great as 10^{43} erg/s ($> 10^9 L_\odot$) and total energy emitted as large as 10^{53} erg. (Energy in light is $10^{51} - 10^{52}$ erg, in neutrinos about 10^{53} erg). If one could convert the entire mass of the Sun to light, the energy emitted would be $M_\odot c^2 = (2 \times 10^{33} \text{ g})(3 \times 10^{10} \text{ cm s}^{-1})^2 = 2 \times 10^{54}$ erg, so the total energy emitted by a supernova is nearly $\frac{1}{10} M_\odot c^2$.

The Baade-Zwicky calculation is simple: What is the gravitational energy released in the collapse of a star of mass $M_\odot$ to the size of a neutron star, 10 km?

$$\Delta E = E_{\text{initial}} - E_{\text{final}} = -\frac{GM_\odot^2}{R_{\text{initial}}} - \left(-\frac{GM_\odot^2}{10 \text{ km}}\right) \quad (4.52)$$

$$= \frac{GM_\odot^2}{10 \text{ km}}. \quad (4.53)$$

The first term is negligible compared to the second here, because the initial radius R_{initial} , is 1000 times the final 10 km radius for stellar core (at white dwarf size) or a white dwarf collapsing to a neutron star. Let's write ΔE as a fraction of $M_\odot c^2$:

$$\Delta E = \frac{GM_\odot}{10 \text{ km } c^2} M_\odot c^2 \quad (4.54)$$

$$\frac{GM_\odot}{10 \text{ km } c^2} = \frac{(6.7 \times 10^{-8} \text{ g}^{-1} \text{ cm}^3 \text{ s}^{-2})(2 \times 10^{33} \text{ g})}{(10^6 \text{ cm})(3 \times 10^{10} \text{ cm s}^{-1})^2} = 0.1 \implies \quad (4.55)$$

$$\Delta E = \frac{1}{10} M_\odot c^2. \quad (4.56)$$

Thus the collapse of an ordinary star (really a white-dwarf-like stellar core of iron) to a neutron star produces total energy equal to that observed in supernovae (most of the energy carried off by neutrinos).

Iron cores collapse to neutron stars instead of simply fusing to make heavier elements because iron is the stablest element - the element with the greatest binding energy per nucleon. White dwarfs, on the other hand, are the endpoints of evolution of stars too light to have formed iron. If they accrete enough matter from a companion to reach their upper mass limit (or if two degenerate stellar cores of elements lighter than iron merge) their initial collapse can lead to explosive nucleosynthesis, carbon fusion, well before neutron-star density is reached. As a result, much less energy is emitted: The total energy is about 10^{51} erg. Instead of collapsing to a neutron star, the star blows itself apart, and the event is identified with a type Ia supernova. Heavier C-O and O-Ne dwarfs, however, may again collapse to neutron stars (see, e.g., Nomoto and Kondo 1991 and citations to this paper for studies of initial conditions leading to accretion-induced collapse).

Maximum mass

The upper limit on the mass of neutron stars depends on an equation of state that is imperfectly understood. By ignoring nuclear interactions and treating the neutrons in a neutron star as gas of free fermions, Oppenheimer and Volkoff, in 1939, found an upper limit of $0.7M_\odot$ on the mass of a neutron star. Remarkably, they also considered the possibility of repulsive interactions at high density, but they restricted a stiffer EOS to $\rho > 10^{15} \text{ g/cm}^3$ and thereby enforced a maximum mass of order $M_\odot$, below the Chandrasekhar limit on white dwarfs. In fact it is easy to show (Hartle and Sabbadini[46]) that for *any* equation of state there is a maximum mass for a spherical star of average density $\geq$ nuclear density, $\rho_n = 2.7 \times 10^{14} \text{ g/cm}^3$. Here is the quick calculation:

With no horizon, $R < 2M$. Using Eq. (4.29) for $m(r)$, we have

$$\begin{aligned} \rho > \rho_n &\implies M > \int_0^R \rho_n 4\pi r^2 dr > \frac{4}{3}\pi r^3 \rho_n > \frac{4}{3}\pi \rho_n (2M)^3 \\ &\implies M < \left(\frac{3}{32\pi \rho_n} \right)^{1/2} \approx 8M_\odot. \end{aligned}$$

Matching to a known equation of state below a density ρ_{match} and imposing causality in the form (speed of sound) $< c$, Rhoades and Ruffini and Hartle and Chitre find

$$M < 4.1 \left(\frac{\rho_n}{\rho_{\text{match}}} \right)^{1/2}. \quad (4.57)$$

Candidate equations of state have maximum masses below $3 M_\odot$.

The four largest accurately measured neutron star masses have values near and slightly above $2.0M_\odot$.² The first observed inspiral and merger of a binary neutron star system, GW1701817 ended in a collapse to a black hole of matter with mass not larger than about $2.7 M_\odot$. Because the matter was hot and supported by differential rotation, the corresponding upper mass limit on cold nonrotating neutron stars is significantly smaller than this, with several authors giving estimates below $2.3M_\odot$. [57, 67, 77, 71, 1, 58].

4.3 Buchdahl's theorem

The upper mass limit on neutron stars is larger than that of white dwarfs because of repulsive nuclear interactions when nucleons are compressed above nuclear density. The equation of state that gives the most massive star with density above nuclear density is that of a completely incompressible star, with $\rho = \text{constant}$, and *Exercise 68* asks you to find M_{max} for that configuration. The incompressible model is also maximally compact: compactness is defined as the ratio M/R , and for this model, you show in this exercise that incompressible models can have M/R as large as $4/9$ and find the corresponding maximum redshift.

Buchdahl's theorem[18] states that any other equation of state gives a less compact star. Note that The speed of sound in a fluid is $v_{\text{sound}} = \sqrt{dP/d\rho}$ (true in both a Newtonian context and still true in relativity, with ρ the energy density). For $\rho = \text{constant}$, $d\rho/dP = 0$, implying an infinite speed of sound, so the equation of state is unphysically stiff. Causality, $v_{\text{sound}} < 1$, is enough to enforce compactness about twice the value from Buchdahl's theorem.

²The stars, J1614-2230[34], J0348+0432[4], J0740+6620[32], and J09520607[69] have, respectively, measured masses $1.97 \pm 0.04M_\odot$, 2.01 ± 0.04 , $2.14 \pm 0.1M_\odot$, and $2.35 \pm 0.17M_\odot$.

Proof of the theorem. The proof relies on the fact that the density ρ of a spherical star is maximum at the center, decreasing outward. This implies that the average density inside a radius r , namely $\frac{m(r)}{\frac{4}{3}\pi r^3}$, decreases outward, with minimum value at $r = R$:

$$\frac{m(r)}{r^3} \geq \frac{M}{R^3} > 0. \quad (4.58)$$

We use the G_θ^θ and G_r^r equations, (4.13c) and (4.13b). Because each have source $8\pi P$,

$$e^{2\lambda} G_\theta^\theta = e^{2\lambda} G_r^r \implies \Phi'' - \frac{1}{r}(\lambda' - \Phi') + \Phi'(\Phi' - \lambda') = \frac{2}{r}\Phi' - e^{2\lambda} \frac{1}{r^2} (1 - e^{-2\lambda})$$

Grouping all terms involving Φ on the left, we have

$$\begin{aligned} \Phi'' + \Phi' \left(-\frac{1}{r} + \Phi' - \lambda' \right) &= \frac{1}{r}\lambda' - e^{2\lambda} \frac{1}{r^2} (1 - e^{-2\lambda}) \quad \text{or} \\ r e^{\lambda - \Phi} \left(\frac{1}{r} \Phi' e^{\Phi - \lambda} \right)' &= \frac{r}{2} e^{2\lambda} \left[\frac{1}{r^2} (1 - e^{-2\lambda}) \right]' \end{aligned}$$

On the right side, $\frac{1}{r^2}(1 - e^{-2\lambda}) = \frac{2m}{r^3}$, proportional to the average density inside radius r . Because the average density decreases outward, the right side is negative (or zero in the limiting case of uniform density). Then

$$\left(\frac{1}{r} \Phi' e^{\Phi - \lambda} \right)' \leq 0.$$

Integrate from r to R :

$$\int_r^R \left(\frac{1}{r} \Phi' e^{\Phi - \lambda} \right)' dr = \frac{1}{R} \Phi' e^{\Phi - \lambda} \Big|_R - \frac{1}{r} \Phi' e^{\Phi - \lambda} \leq 0.$$

At the surface Φ and Φ' are continuous as long as the density and pressure are continuous, so we can evaluate $\Phi'(R)$ by taking the derivative outside the star and using its value $r = R$. Outside, because $\lambda = -\Phi$, we have $\Phi' e^{\Phi - \lambda} = \frac{1}{2}(e^{2\Phi})' = \frac{1}{2}(1 - 2M/r)' = \frac{M}{r^2}$. Our inequality now becomes

$$\frac{M}{R^3} \leq \frac{1}{r} \Phi' e^{\Phi - \lambda} \quad \text{or} \quad \Phi' e^{\Phi} \geq r e^{\lambda} \frac{M}{R^3}.$$

Using $e^{\lambda} = (1 - 2m(r)/r)^{-1/2}$, we have

$$(e^{\Phi})' \geq \frac{M}{R^3} r (1 - 2m/r)^{-1/2}.$$

We again use the fact that the average density decreases outward, this time in the form (4.58): Then $m/r \geq Mr^2/R^3$, so

$$(e^{\Phi})' \geq \frac{M}{R^3} r (1 - 2Mr^2/R^3)^{-1/2}.$$

We integrate once more, using

$$\int_0^R r (1 - 2Mr^2/R^3)^{-1/2} dr = -\frac{R^3}{2M} [(1 - 2M/R)^{1/2} - 1]$$

to get

$$e^{\Phi(R)} - e^{\Phi}|_0 \geq -\frac{1}{2} \left[\left(1 - \frac{2M}{R} \right)^{1/2} - 1 \right].$$

Finally, $e^{\Phi(R)} = (1 - 2M/R)^{1/2}$ and $e^{\Phi(0)} \geq 0$, giving

$$\sqrt{1 - 2M/R} \geq \frac{1}{3}, \quad \frac{M}{R} \leq \frac{4}{9},$$

as claimed.

4.4 Geodesics of Schwarzschild

As in the Newtonian central force problem, one reduces the orbit equation to a first-order ordinary differential equation for $r(\tau)$ by using conservation of energy and conservation of momentum, and by picking coordinates for which the orbit is in the $\theta = \pi/2$ plane. The Newtonian equation $E_{\text{newt}} = T + V$ comes from the relativistic equation $p^\alpha p_\alpha = -m^2$, or, equivalently, $u^\alpha u_\alpha = -1$.

Writing $(\cdot)$ for $\frac{d}{d\tau}$, we have $u^\alpha = \dot{t}t^\alpha + \dot{\phi}\phi^\alpha + \dot{r}r^\alpha$, or $u^\mu = \dot{x}^\mu$. Associated with the timelike Killing vector t^α is the conserved energy $-p_\alpha t^\alpha$; we'll use energy/rest-mass,

$$E := -u_\alpha t^\alpha = -u_t = \left(1 - \frac{2M}{r} \right) \dot{t}. \quad (4.59)$$

Associated with the rotational Killing vector ϕ^α is the conserved angular momentum $p_\alpha \phi^\alpha$; again we'll use angular momentum/rest-mass

$$L := u_\alpha \phi^\alpha = u_\phi = r^2 \dot{\phi}. \quad (4.60)$$

($\sin \theta = 1$, because we're in the $\theta = \pi/2$ plane).

The Newtonian equation $E = \frac{1}{2}mv^2 + V(r) = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$ is the Newtonian limit of the relativistic equation $p_\alpha p^\alpha = -m^2$ or $E_{\text{relativistic}}^2 = p_\perp^2 + m^2$. We again eliminate the rest mass, dividing by m^2 and using the equation in the form $u_\alpha u^\alpha = -1$, for a timelike geodesic.

$$\begin{aligned} -1 &= g_{\alpha\beta} u^\alpha u^\beta = g^{\alpha\beta} u_\alpha u_\beta = g^{tt} (u_t)^2 + g^{\phi\phi} (u_\phi)^2 + g^{rr} (u^r)^2 \\ &= - \left(1 - \frac{2M}{r} \right)^{-1} E^2 + \frac{L^2}{r^2} + \left(1 - \frac{2M}{r} \right)^{-1} \dot{r}^2 \\ \frac{1}{2} \dot{r}^2 &= \frac{1}{2} (E^2 - 1) - V_{\text{eff}} \end{aligned} \quad (4.61)$$

where $\dot{r} = dr/d\tau$ and

$$V_{\text{eff}} = -\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}. \quad (4.62)$$

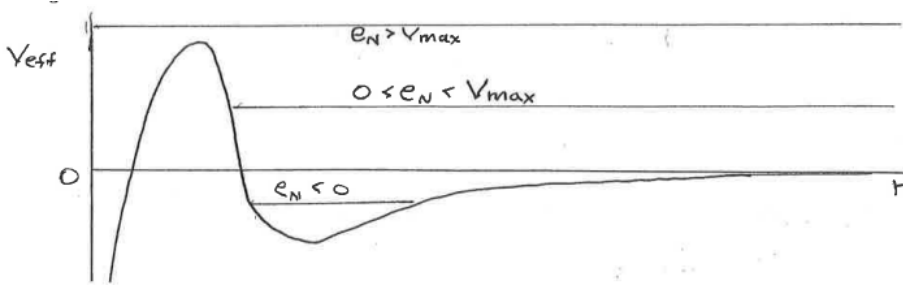
That is, the orbit is described as motion in an effective potential.

For $r \gg 2M$, V_{eff} is a Newtonian effective potential, the sum of a $1/r$ attractive potential and an L^2/r^2 centrifugal barrier. (The term ML^2/r^3 is of order $M/r \times L^2/r^2 \sim M/r(v^2)$, smaller than M/r by order $v^2 = v^2/c^2$.) At small r , however, relativistic gravity overcomes the centrifugal barrier. By $r = 2M$, V_{eff} has fallen to zero, and for $r < 2M$ it becomes increasingly negative. As a result, in addition to bound orbits and the analog of hyperbolic unbound orbits, the Schwarzschild geometry has a class of orbits with no Newtonian analog.

Write $e_N := \frac{1}{2}(E^2 - 1)$; asymptotically,

$$E = \frac{dt}{d\tau} = h_\infty = \sqrt{\frac{1}{1 - v_\infty^2}} \Rightarrow e_N = \frac{1}{2}v_\infty^2 \frac{1}{1 - v_\infty^2},$$

agreeing with the Newtonian kinetic energy/mass for small v . A particle moving in the effective potential V_{eff} is allowed to be in the region $e_N > V_{\text{eff}}$, so three types of orbits are possible from the figure below.



For $e_N < 0$, the orbits are bound – the particle moves between an inner and outer radius. For $e_N \geq 0$ but $e_N < V_{\text{max}}$, the particle is unbound and moves from ∞ to a minimum radius (B) and out again. These orbits are similar to the Newtonian hyperbolic orbits. But for $e_N > V_{\text{max}}$, the particle never turns around, and we will see that it reaches $r = 2M$ in *finite proper time*. In the extended geometry, which will include regions I and II, the particle falls to $r = 0$, where it hits an infinite curvature singularity. By making e_N larger for fixed L , you increase $\dot{r}$ and thus aim the particle more directly inward: You decrease the impact parameter.

Here's the effective potential for different values of L/M :

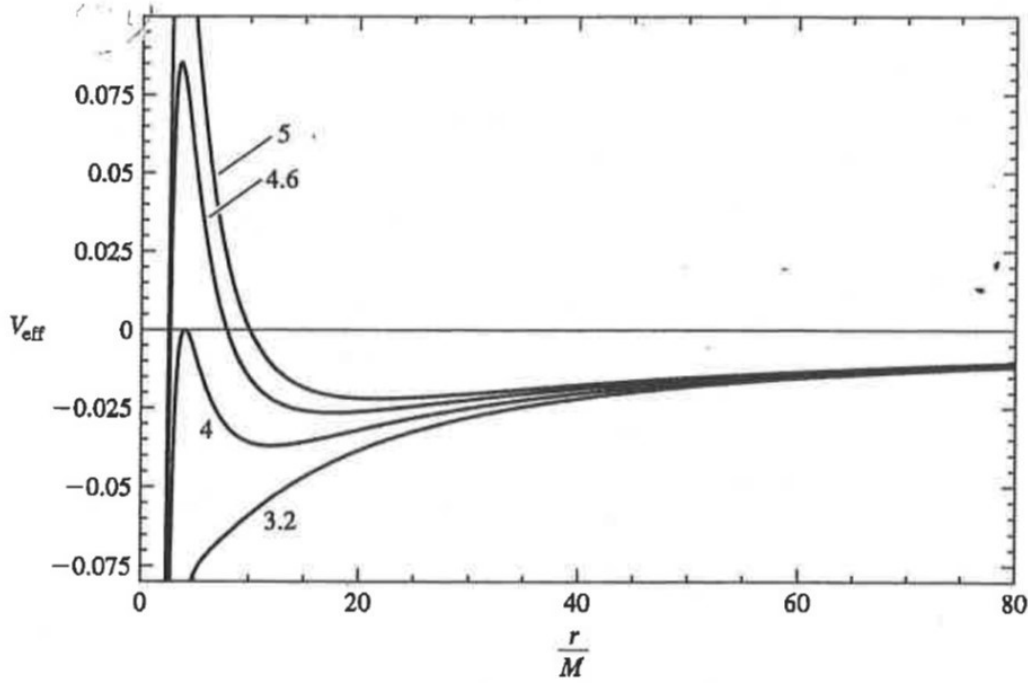


Figure 4.1: Figure from Hartle.

For $L^2 < 12M^2$, V_{eff} has no maxima or minima and all orbits eventually hit $r = 0$.

Circular orbits

For a circular orbit, $\dot{r} = 0$ implies $e_N \equiv \frac{1}{2}(E^2 - 1) = V_{\text{eff}}$, and $\ddot{r} = 0$ implies $V'_{\text{eff}} = 0$. Calling R the radius of the circular orbit, the second condition is

$$\begin{aligned} \frac{M}{r^2} - \frac{L^2}{R^3} - 3\frac{ML^2}{R^4} &= 0 \implies \\ L^2 &= \frac{MR^2}{R - 3M}. \end{aligned} \quad (4.63)$$

Then Eq. (4.62) and $e_N = V_{\text{eff}}$ implies

$$E^2 = \frac{(R - 2M)^2}{R(R - 3M)}, \quad (4.64)$$

and the Kepler relation between Ω and R is

$$\begin{aligned} \Omega = \frac{\dot{\phi}}{\dot{t}} &= g^{\phi\phi} L \frac{1}{-g^{tt} E} = \frac{1}{R^2} \sqrt{\frac{MR^2}{R - 3M}} (1 - 2M/R) \frac{\sqrt{R(R - 3M)}}{R - 2M} \\ &= \sqrt{\frac{M}{R^3}}, \end{aligned} \quad (4.65)$$

identical, with this choice of radial coordinate, to the Newtonian relation. For later reference, we will use $\omega_\phi := \dot{\phi}$, the orbital frequency measured in terms of the proper time τ of an observer in circular orbit:

$$\omega_\phi = \frac{1}{R^2} L = \sqrt{\frac{M}{R^2(R - 3M)}}. \quad (4.66)$$

Eqs. (4.63) and (4.64) allow circular orbits for every $r > 3M$. For $r < 6M$, however, the orbits are unstable: $V''_{\text{eff}} < 0$. Using $V'_{\text{eff}} = 0$ for a circular orbit, we have

$$\begin{aligned} r^4 V''_{\text{eff}} &= \frac{d}{dr} \left(r^4 \frac{dV_{\text{eff}}}{dr} \right) = \frac{d}{dr} (Mr^2 - L^2 r + 3ML^2) = 2Mr - L^2 = 2Mr - \frac{Mr^2}{r - 3M} \\ &= \frac{Mr(r - 6M)}{r - 3M}, \end{aligned} \quad (4.67)$$

positive for circular orbits only for $r > 6M$, as claimed. The innermost stable circular orbit is denoted by its acronym, ISCO:

$$r_{\text{ISCO}} = 6M. \quad (4.68)$$

For a $1.4 M_{\odot}$ neutron star, this is at $r = 6(1.4)(1.477 \text{ km}) = 12.4 \text{ km}$, either just outside or just inside the surface of the star: We haven't yet measured neutron star radii accurately enough to know.

A key goal of the future space-based gravitational-wave observatory LISA is to observe gravitational waves from stellar-size black holes ($< 100M_{\odot}$) spiraling in to supermassive black holes in the centers of galaxies, as each system loses energy to gravitational waves. An *extreme mass-ratio inspiral* (EMRI) is accurately modeled as a point particle orbiting the large black hole. For an eccentric orbit, more gravitational radiation is emitted when the particle is closest to the black hole, and that leads to a nearly circular orbit long before the particle reaches $r = 6M$. When it reaches $r = 6M$, it plunges into the black hole. As a result, a good approximation to the total energy emitted in gravitational waves during the inspiral is the decrease in the energy of a circular orbit between $E = 1$ at infinity and E at $r = 6M$. This is the binding energy E_B of the orbit at $6M$: From Eq.(4.64),

$$\begin{aligned} E_B &= 1 - E = 1 - \frac{r - 2M}{r(r - 3M)} = 1 - \frac{r - 2M}{\sqrt{r(r - 3M)}} = 1 - \frac{4M}{\sqrt{6M(6M - 3M)}} \\ &= 1 - \sqrt{\frac{8}{9}} \approx 0.06. \end{aligned} \quad (4.69)$$

That is, about 6% of the particle's rest mass is radiated in gravitational waves.

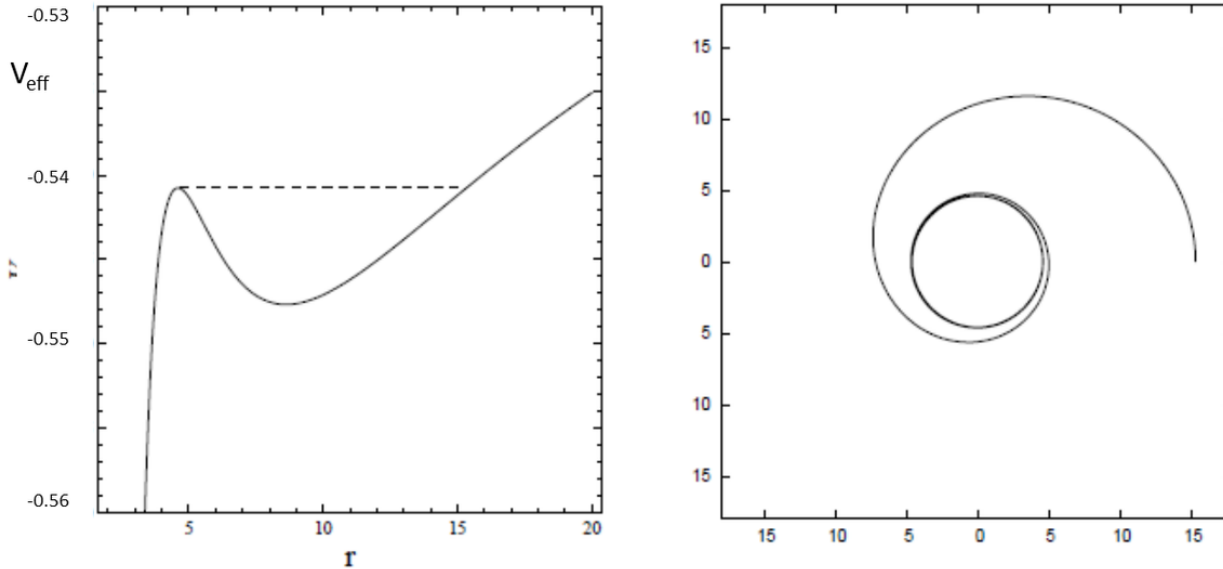
Particles do not move along the unstable orbits with $r < 6M$; they plunge into the black hole. Formally, however, along the family of unstable orbits, as the orbital radius decreases, the orbital speed increases, and the smallest circular orbit, at $r = 3M$, corresponds to the largest speed: As we will see, it is the unstable circular orbit of a photon.

General bound orbits

At large radii, it's clear from the effective potential for radial motion that a particle in bound orbit moves periodically between a minimum radius (periastron) and maximum radius (apastron) in its orbit about a star or black hole. (For orbits about the Sun the terms are perihelion and aphelion from Helios, the Greek god of the Sun). For Newtonian stars and at large distances from neutron stars or black holes, the orbits are nearly the Keplerian elliptical orbits, differing only by a slow precession, with ϕ increasing by an angle $\Delta\phi$ slightly larger than 2π between each periastron.

Zoom-whirl orbits

The unstable circular orbits at $r < 6M$ turn out to have an important implication for future observations of LISA and perhaps of LIGO. The figure on the left below shows the effective potential with a value of E corresponding to an unstable orbit at $r_c \approx 5M$. From the figure you can see that another orbit has this critical energy: A particle with energy E can move from a maximum value of r near $15M$ to a minimum value at r_c . Like any ball with exactly the energy needed to reach the top of a hill, the particle takes an infinite amount of time to reach r_c . But $\Omega = \frac{d\phi}{dt} \rightarrow \sqrt{M/r_c^3}$, a constant, finite value. So the particle spirals around the black hole an infinite number of times as it approaches r_c .



For E slightly below this limiting value, the particle spirals many times and then swings back out before repeating. The figure below shows the resulting *zoom-whirl* orbit for a single whirl between each zoom. These are particle orbits that graze the black hole, that have an impact parameter just large enough to avoid being swallowed. Typical comets in the solar system are the result of chance encounters of ice balls (in the Kuiper belt and Oort cloud) with passing stars that nudge them in a direction that removes enough of their orbital energy that they fall toward the Sun. Because orbital speed at that distance is already small, they start their infall with in nearly parabolic orbits with the small impact parameter associated with their small initial speed. Zoom-whirl orbits are most likely to arise from a similar scenario: Stars initially not too near a galactic black hole that are nudged into a trajectory that grazes the black hole have the best chance of being in the zoom-whirl parameter range.

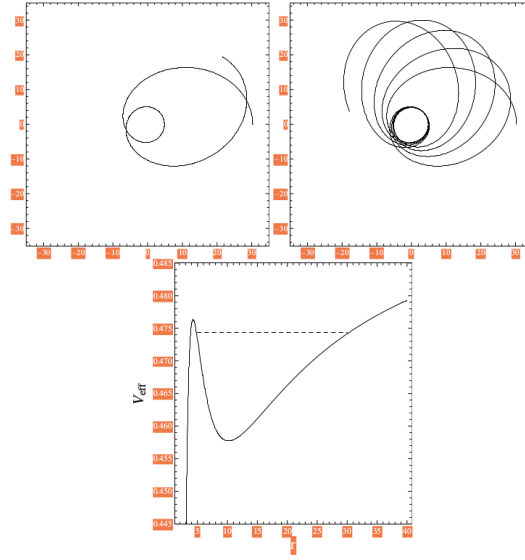


Figure 4.2: Source of figure not known to JF

Precession Angle

We look at the precession angle $\Delta\phi_P = \Delta\phi - 2\pi$ for nearly circular orbits, with $\Delta\phi$ the angle through which the particle moves between successive periastrons. The radial component of the geodesic equation has the form

$$\ddot{r} = -V'_{\text{eff}}. \quad (4.70)$$

If this is not obvious, note that, as in the Newtonian case, differentiating the expression for the conserved energy, Eq. (4.61), gives the equation of motion:

$$\dot{r}\ddot{r} = \frac{d}{d\tau}\left(\frac{1}{2}\dot{r}^2\right) = -\frac{d}{d\tau}V_{\text{eff}}(r(\tau)) = -V'_{\text{eff}}\dot{r} \implies \ddot{r} = -V'_{\text{eff}}.$$

The radial period is the coordinate time $t = T$ between periastrons; recall that t is Killing time and is the time measured by an observer at infinity, whose 4-velocity coincides with t^α . Let R be the radius of a circular orbit and $R + \delta r(\tau)$ the radius of a particle in a nearby orbit. For a Newtonian system, $R + \delta r(t)$ is the radial coordinate of a nearby elliptical orbit. Using $V'_{\text{eff}}(R) = 0$, we have

$$\begin{aligned} \delta\ddot{r} &= -V'_{\text{eff}}(R + \delta r) = -V''_{\text{eff}}(R)\delta r + O(\delta r^2), \\ \text{or} \quad \delta\ddot{r} + \omega_r^2 \delta r &= 0, \end{aligned}$$

with

$$\omega_r := \sqrt{V''_{\text{eff}}(R)} = \sqrt{\frac{M(R - 6M)}{R^3(R - 3M)}}. \quad (4.71)$$

Then δr oscillates with frequency ω_r defined in terms of proper time τ along the circular geodesic. The corresponding frequency Ω_r in terms of Killing time is $\Omega_r = \omega_r/\dot{t}$ and the radial period is $T = 2\pi/\Omega_r$.

In one radial period, the change in angle is

$$\begin{aligned}\Delta\phi &= \Omega T = 2\pi \frac{\Omega}{\Omega_r} = 2\pi \frac{\omega_\phi}{\omega_r} = 2\pi \sqrt{\frac{M}{R^2(R-3M)} \frac{R^3(R-3M)}{M(R-6M)}} \\ &= 2\pi \frac{1}{\sqrt{1-6M/R}}\end{aligned}$$

where we have used (4.66) and (4.71) for ω_ϕ and ω_r . The angle of precession is then

$$\Delta\phi_P = 2\pi \left(1 - \frac{1}{\sqrt{1-6M/R}} \right). \quad (4.72)$$

For M/R small, it becomes

$$\Delta\phi_P = 6\pi \frac{M}{r}. \quad (4.73)$$

In particular, Mercury's orbit is nearly circular with $r = 5.55 \times 10^7$ km. The mass in the formula is that of the sun, $M = M_\odot = 1.477$ km, giving

$$\begin{aligned}\Delta\phi_P &= 6\pi \frac{1.477 \text{ km}}{5.55 \times 10^7 \text{ km}} = 4.99 \times 10^{-7} \text{ radians/orbit}, \\ \Delta\phi_P &= \frac{4.99 \times 10^{-7}}{.24} \times \left(\frac{180}{\pi} \times 3600 \right) \\ &= 0.43''/\text{yr} = 43''/\text{century}.\end{aligned} \quad (4.74)$$

At the other extreme, for R near $6M$, the precession angle is large, and the particle follows a zoom-whirl orbit, circling the black hole many times in each radial period. In this case there's not much of a zoom, because of our $\delta r \ll R$ approximation.

Lightlike Orbits: Null Geodesics

The effective potential for null geodesics is simpler: Writing $(\cdot)$ for $d/d\lambda$ with λ an affine parameter, we have

$$0 = k^\alpha k_\alpha = -\frac{1}{1 - \frac{2M}{r}} E^2 + \frac{L^2}{r^2} + \frac{1}{1 - \frac{2M}{r}} \dot{r}^2$$

$$\frac{1}{2} \dot{r}^2 = \frac{1}{2} E^2 - V_{\text{eff}}, \quad \text{with} \quad V_{\text{eff}} = \left(1 - \frac{2M}{r}\right) \frac{L^2}{2r^2}. \quad (4.75)$$

The Photon Sphere

Like massive particles, photons moving inward with large E are trapped by the black hole, even if they have nonzero angular momentum. And there is a surprise: The potential has a maximum at $r = 3M$, implying an unstable circular orbit for photons at this radius.

Check:

$$0 = \frac{1}{L^2} \frac{d}{dr} V_{\text{eff}} = \frac{d}{dr} \left(\frac{1}{2r^2} - \frac{M}{r^3} \right) = -\frac{1}{r^3} + \frac{3M}{r^4} \implies r = 3M. \quad (4.76)$$

One calls $r = 3M$ the *photon sphere*.

Photons coming in from infinity hit the black hole when $\dot{r} > 0$ at the maximum of V_{eff} , and miss it when $\dot{r} = 0$ before they reach the top of the potential. At its $r = 3M$ peak,

$$V_{\text{eff,max}} = \frac{L^2}{54M^2},$$

and photons miss the black hole if and only if

$$V_{\text{eff,max}} > \frac{1}{2} E^2 \iff \frac{L^2}{54M^2} > \frac{1}{2} E^2 \iff b := \frac{L}{E} > 3\sqrt{3} M. \quad (4.77)$$

That is, the critical impact parameter for which photons just reach the top of the potential – spiraling an infinite number of times about the $r = 3M$ circular orbit – is $b = 3\sqrt{3} M$. The corresponding absorption cross section is then $\sigma = \pi b^2 = 27\pi M^2$.

The quantity $b = L/E$ is the natural generalization of the impact parameter b of a photon in flat space. In general, to define b for a particle moving in a central potential, one looks at the straight-line tangent to the particle's asymptotic trajectory. The impact parameter is the minimum distance b between that free-particle trajectory and the center of the potential. A photon with asymptotic momentum p has angular momentum $L = bpc$ and energy $E = pc$, so its impact parameter is $b = L/E$. Using this as the definition of b is appropriate because the spacetime is asymptotically flat, and L and E are the energy and angular momentum measured by a stationary observer at infinity.

Deflection of Light

Null geodesics that do not hit the black hole or star follow a symmetric trajectory that looks similar to a hyperbolic orbit, with the angle between the ingoing and outgoing asymptotes called the deflection angle $\Delta\phi$.

Mathematical Prerequisite

$$y'' + y = \sin^2 x \quad \text{has solution} \quad y = \frac{1}{3}(2 - \sin^2 x).$$

Spatial Path

The spatial path, $r = r(\phi)$ can, as usual be found by writing

$$r' \equiv \frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} = \frac{\dot{r}r^2}{L} \Rightarrow \dot{r} = \frac{L}{r^2} r'.$$

It is convenient to use

$$u = \frac{1}{r}, \quad u' = -\frac{r'}{r^2} = -\frac{\dot{r}}{L}.$$

In terms of u , the geodesic equation (4.75) is

$$u'^2 = \frac{E^2}{L^2} - u^2 + 2Mu^3 \quad (4.78)$$

Differentiate:

$$\begin{aligned} 2u'u'' &= -2u'u + 2u'3Mu^2 \\ u'' + u &= 3Mu^2 \end{aligned}$$

We'll solve this to order M/b , with $b = L/E$ the impact parameter. We first find the zeroth-order (flat space) solution:

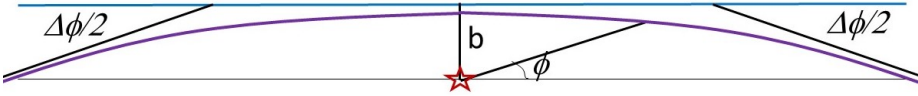


Figure 4.3: Flat-space trajectory in blue, actual trajectory in purple.

At zeroth order, with $M = 0$, we are in flat space, and we choose as the geodesic a straight line at constant $y = r \sin \phi$:

$$u'' + u = 0$$

has a solution

$$u = \frac{1}{b} \sin \phi, \quad \text{or} \quad y = r \sin \phi = b,$$

with r ranging from ∞ to $-\infty$ as ϕ runs from 0 to π , with minimum value b as claimed. Far from the star, space is flat, and we have just checked that the flat-space trajectory with $b = L/E$ has b as the distance of closest approach.

Perturb: Order M/b correction.

Let $u + \delta u$ be the perturbed solution, with δu of order M/b . Then to linear order in M/b we have

$$\begin{aligned} \delta u'' + \delta u &= \frac{3M}{b^2} \sin^2 \phi \\ \delta u &= \frac{M}{b^2} (2 - \sin^2 \phi) \quad \text{from the prerequisite above} \\ u + \delta u &= \frac{1}{b} \sin \phi + \frac{2M}{b^2} - \frac{M}{b^2} \sin^2 \phi. \end{aligned}$$

The perturbed orbit starts $\phi = -\Delta\phi/2$ with the particle at $r + \delta r = \infty$, $u + \delta u = 0$. As the photon moves from right to left, ϕ increases to $\pi/2$ at maximum $u + \delta u$ and then increases from $\pi/2$ to $\pi + \Delta\phi$ as the photon flies out to infinity and $u + \delta u$ symmetrically decreases to 0. To find $\delta\phi$ we just write

$$\begin{aligned} 0 = u + \delta u &= \frac{1}{b} \sin(-\Delta\phi/2) + \frac{M}{b^2} [2 - \sin^2(-\Delta\phi/2)] \\ &= -\frac{\Delta\phi}{2b} + \frac{2M}{b^2} + O((\Delta\phi)^2) \\ \implies \Delta\phi &= 4 \frac{M}{b}. \end{aligned} \quad (4.79)$$

For light grazing the Sun, $M = M_\odot = 1.477 \text{ km}$, $b = R_\odot = 6.96 \times 10^5 \text{ km}$, implying

$$\begin{aligned} \Delta\phi &= 4 \frac{1.477}{6.96 \times 10^5} \left(\frac{180}{\pi} \times 3600 \right) \\ &= 1.75''. \end{aligned}$$

This, of course, is the value first measured by the 1919 eclipse expedition led by Frank Dyson and Arthur Eddington. An interesting detailed history by José Lemos et al.: arXiv:1912.08354.

It is not too hard to find the exact deflection as an integral. Following Wald, we solve Eq. (4.78) for u' to write

$$\Delta\phi = 2 \int_0^{1/R_0} \frac{1}{\sqrt{b^{-2} - u^2 + 2Mu^3}}, \quad (4.80)$$

where R_0 is the minimum value of r along the exact trajectory, where $u' = 0$. To find R_0 , we have to solve the cubic equation $2Mu^3 - u^2 + b^{-2} = 0$ or $r^3 - b^2r + 2Mb^2 = 0$ to obtain

$$R_0 = \frac{2b}{\sqrt{3}} \cos \left[\frac{1}{3} \cos^{-1} \left(-\frac{3^{3/2}M}{b} \right) \right]. \quad (4.81)$$

If you have never solved a cubic, or never seen the solution in this form, here is how to do it: First change variables to get rid of any quadratic term – in our case it's already gone. The next step isn't necessary, but it gives a canonical form with a solution that's easy to understand: Rescale r to make the coefficient of the linear term -3: $y = \sqrt{3} \frac{r}{b}$, giving

$$y^3 - 3y + q = 0, \text{ with } q = \frac{3^{3/2}M}{b}.$$

Now, the key step: Write $y = a + b$, $y^3 = 3ab(a + b) + a^3 + b^3$. Our cubic then becomes

$3aby + a^3 + b^3 - 3y + 2q = 0$, satisfied when $ab = 1$, $a^3 + b^3 + 2q = 0$. Then $b = 1/a$, and $a^3 + b^3 + 2q = 0$ becomes $(a^3)^2 + 2qa^3 + 1 = 0$, a simple quadratic equation for a^3 : Here $q < 1$, and the roots are $a^3 = -q \pm i\sqrt{1 - q^2}$, with $|q \pm i\sqrt{1 - q^2}| = 1$. Because the roots have unit norm, we can write $a = e^{i\psi}$, $b = e^{-i\psi}$. Then $a^3 + b^3 + 2q = 0 \implies 2 \cos 3\psi + 2q = 0$, and we have

$$y = a + b = 2 \cos \psi = 2 \cos \left[\frac{1}{3} \cos^{-1}(-q) \right], \quad \text{giving the claimed expression for } R_0.$$

By the end of the last century, light deflection in the guise of gravitational lensing began to play an important role in astronomy, in particular for measurements of the mass of dark matter haloes that dominate the mass of galaxies and clusters of galaxies. DES (the Dark Energy Survey) uses lensing in a survey of 26 million galaxies to improve the precision of cosmological parameters (see, for example,

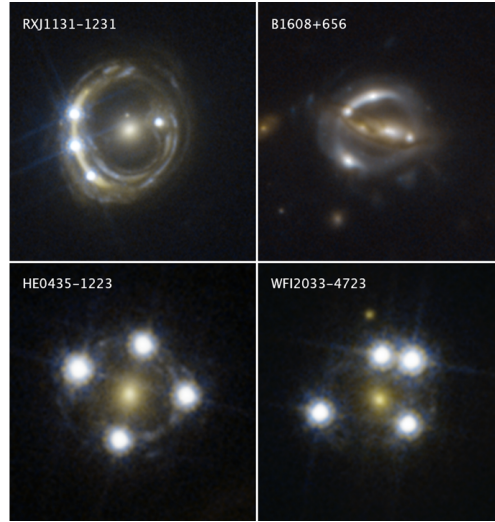


Figure 4.4: Each photo in the figure shows multiple images of a single quasar.

DES, DES papers).

Shapiro time delay

Many quasars are seen as more than one image, their light bent around intervening galaxies or clusters of galaxies. In the images, light from a quasar arrives at different times, depending on the path. Quasar intensity changes in less than a day, implying an emitting region less than a light-day across; that made accretion onto a black hole the leading candidate for the engine. The same bright flares can be identified in different images, arriving months or years apart: For example the same flare has been identified in 3 of the 6 images of SDSSJ1029+2623 (SDSS for a Sloan Digital Sky Survey object), with a 48-day and a 722-day delay between the brightest and two of the dimmer images. Dahle et al. 15

First computed and measured by Irwin Shapiro PRL[75], the delay has provided an accurate test of GR, first in the solar system and more recently in double pulsars. It is also used in measurements of neutron star masses. The presentation here uses *isotropic coordinates* for Schwarzschild. The derivation of these coordinates is a problem from Wald: It just involves changing to a new radial coordinate (still written here as r) for which the metric has the form

$$ds^2 = - \left(\frac{1 - M/2r}{1 + M/2r} \right)^2 dt^2 + (1 + M/2r)^4 (dx^2 + dy^2 + dz^2), \quad (4.82)$$

with $r^2 = x^2 + y^2 + z^2$. To get to this form starting from the standard radial coordinate r_S , write

$$(1 - 2M/r_S)^{-1} dr_S^2 + r_S^2 d\Omega^2 = F^2 (dr^2 + r^2 d\Omega^2);$$

then equate the coefficients of $d\Omega^2$ and require $(1 - 2M/r)^{-1} dr_S^2 = F^2 dr^2$.

We again look at the same geodesic with impact parameter b and calculate the correction to the travel time between the emitter (which could be a pulsar or a planet) at initial large radius r_p and the Earth at large radius r_e . The fractional correction to travel time will be of order $M/(r_e + r_p)$ and can be found from the flat-space path: The curved path deviates from the flat trajectory by an angle of order $O(b/M)$, its spatial length changing by $O((r_e + r_p)b^2/M^2)$: Think of the length of the hypotenuse of a right triangle of r_p and angle b/M . The corrected time delay from the change in path length is then of order $(b/M)^2(r_e + r_p)$, and can be ignored.

We take as the spatial trajectory the path $y = b$, with x running from $x = -r_p$ to $x = r_e$, with $x = 0$ the point of closest approach. The null path has $ds = 0$, with

$$0 = ds^2 = - \left(\frac{1 - M/2r}{1 + M/2r} \right)^2 dt^2 + (1 + M/2r)^4 dx^2. \quad (4.83)$$

Integrating to find the time from the source to the point of closest approach and the time from closest approach to Earth and adding them gives

$$\Delta t = r_p + r_e + 2M \log \frac{(r_p + \sqrt{r_p^2 + b^2})(r_e + \sqrt{r_e^2 + b^2})}{b^2}. \quad (4.84)$$

For $b \ll r_p, r_e$, the Shapiro delay (correction to $r_p + r_e$) is then

$$\Delta t = 2M \log \frac{4r_e r_p}{b^2}. \quad (4.85)$$

There is also a b -independent delay from the redshift of a photon leaving a pulsar and its binary system and a blueshift as it arrives at Earth, but only the b -dependent part is measured.

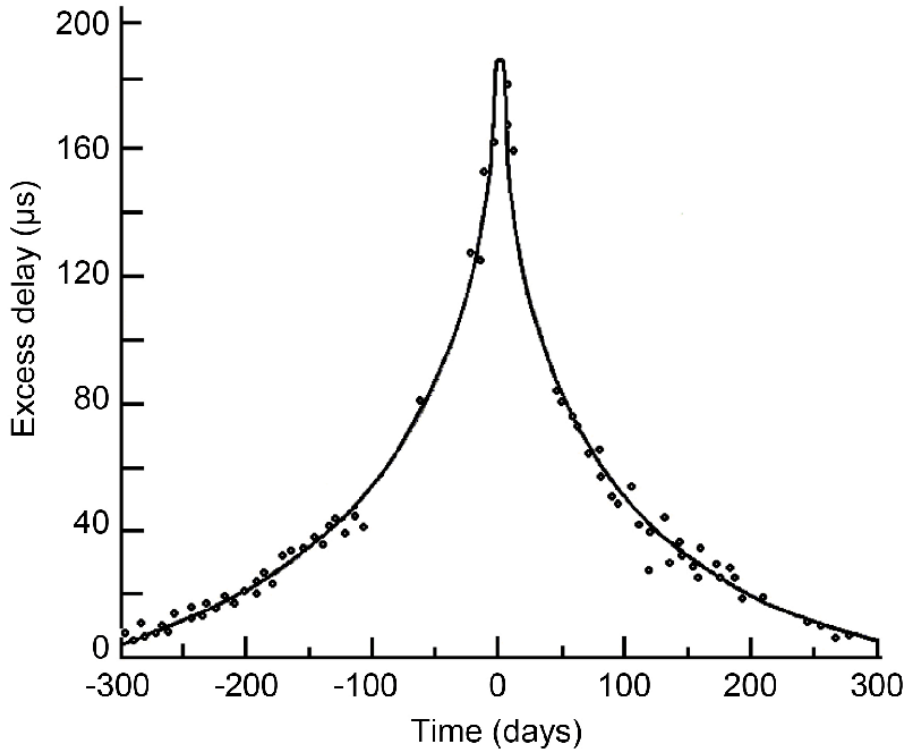


Figure 4.5: The measured delay of light traveling from Venus to Earth as Venus passes the Sun. From Jacob Schaf, *Journal of Modern Physics*, 2017[73]

A final photon orbit, that of a radially infalling photon, begins the black-hole section.

Exercise 67. Complete the computation of Eqs. (4.84) and (4.85) to show that the Shapiro time delay of a photon traveling from a pulsar to the earth and passing another star of mass M along the way has the form

$$\Delta t = 2M \log \frac{4r_e r_p}{b^2} + \text{terms smaller by factors } \frac{b}{r_e}, \frac{b}{r_p}, \frac{b}{M}.$$

Ignore the effect of the pulsar's and the Earth's gravity. M is the mass of the intervening star, and r_p and r_e are the distances from the pulsar to that star and from the Earth to that star.

An optional variant: Wald Chapter 6 Problem 5, finding the exact version.

Exercise 68. The smallest possible value of $\frac{2M}{R}$ at the surface of a static (spherical) star is attained when the star is perfectly incompressible – when it is one of the uniform density models you constructed.

- Show for this sequence of stars that the central density becomes infinite when $\frac{2M}{R} = \frac{8}{9}$.
- Deduce from (a) the maximum redshift z ,

$$1 + z := \frac{\lambda_\infty}{\lambda_{\text{star}}} = \frac{\omega_{\text{star}}}{\omega_\infty},$$

from the surface of a spherical star.

Exercise 69. For spherical (i.e. nonrotating) stars, there is an exact solution to the Newtonian equations of hydrostatic equilibrium

$$\nabla P = -\rho \nabla \Phi, \quad \nabla^2 \Phi = 4\pi \rho,$$

with the equation of state $P = K\rho^2$. Write $\nabla^2 \Phi$ in the form $\frac{1}{r} \frac{d^2}{dr^2}(r\Phi) = 4\pi\rho$ to show that $\rho = \rho_c \frac{\sin(\alpha r)}{\alpha r}$ satisfies the system of equations, find α , and find the radius of the star for $K = 150 \text{ km}$.

Exercise 70. Numerical model. A model of a neutron star more realistic than the uniform-density model is an equation of state of the form $P = K\rho_0^2$, where ρ_0 is the baryon mass density, called a polytropic equation of state. To make things simpler, use $P = K\rho^2$ as in the last problem, but this time for a general relativistic model. In gravitational units with km the only unit, take $K = 150 \text{ km}^2$.

- A typical central density of a neutron star is about 10^{-3} km^{-2} . What is that in g/cm^3 ?
- Starting with this central density, integrate the TOV equation from the center of the star to the surface, determined by the requirement that P vanish. Find the radius of the star and plot P and ρ as functions of r . The equation of hydrostatic equilibrium looks like $0 = 0/0$ at $r = 0$, so start near $r = 0$ and integrate outward, with $\rho = \rho_c$, $P = P(\rho_c)$ and $m = \frac{4}{3}\pi r_0^3 \rho_c$ as initial values.

Exercise 71. Upper limit on the mass of a star with $\rho > \rho_n$, where ρ_n is nuclear density, about $2.7 \times 10^{14} \text{ g/cm}^3$ (Hartle-Sabbadini). Use the inequality $\rho > \rho_n$ and the equation for the mass of a relativistic star in terms of ρ to show that $M > K\rho_n R^3$, and find K . Combine this inequality with the requirement that the star is not inside a black hole, $R > 2M$ to find an upper limit on M in terms of ρ_n . Evaluate $M_{\text{max}}/M_\odot$ for $\rho_n = 2.7 \times 10^{14} \text{ g/cm}^3$.

Chapter 5

Black Holes

5.1 Schwarzschild Black Holes

The Schwarzschild metric (4.18),

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (5.1)$$

provides a smooth spacetime in the region

I: $-\infty < t < \infty$ $2M < r < \infty$

and another (disconnected) spacetime corresponding to the coordinate values

II: $-\infty < t < \infty$ $0 < r < 2M$.

In region I, $t^\alpha = \partial_t$ is a timelike Killing vector, but in region II, $g_{tt} = t^\alpha t_\alpha > 0$, so that t^α , although still a Killing vector, is no longer timelike. Similarly, ∂_r is spacelike in I and timelike in II. In obtaining the form (5.1), we assumed the existence of a timelike Killing vector orthogonal to a family of spacelike ($t = \text{constant}$) hypersurfaces, but in the region $r < 2M$ the field is so strong that even light is constrained to move inward – no matter can remain at rest (i.e. at constant r), and everything follows trajectories that end at $r = 0$.

Remaining at rest means moving along t^α , so the fact that t^α is spacelike is exactly the statement that physical particles (which must move along timelike trajectories) cannot remain at rest. One might hope to join regions I and II by relaxing the condition that there be coordinates $\{t, x^i\}$, with ∂_t a Killing vector and $t = \text{constant}$ a spacelike surface. Before trying this, it is helpful to learn about the geometry outside $r = 2M$, and one way to do it is to look at the geodesics (another way is to embed the geometry in some $\mathbb{R}^n$ - we'll do that later).

Radial photons

We now show that radially ingoing light rays reach $r = 2M$ in finite affine parameter length, although the coordinate t becomes infinite along their trajectories. Similarly, radially ingoing massive particles reach $r = 2M$, in finite proper time. From (4.75) with $L = 0$, $dr = E d\lambda$, so r is itself an affine parameter. Thus photons travel from $8M$ to $2M$ in parameter length $\Delta r = 8M$, or $\Delta\lambda = \frac{8M}{E}$. The coordinate t , however, becomes infinite as $r \rightarrow 2M$:

$$\left(1 - \frac{2M}{r}\right) dt^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2.$$

$$dt = -\frac{dr}{1 - \frac{2M}{r}} \quad (5.2)$$

$$t = v_0 - r - 2M \log \left| \frac{r}{2M} - 1 \right|, \quad (5.3)$$

with v_0 the constant of integration. It is useful to introduce a coordinate r_* (called the “tortoise” coordinate), writing

$$t = v_0 - r_*, \quad (5.4)$$

$$r_* := \int^r \frac{dr}{1 - \frac{2M}{r}} = r + 2M \ln \left| \frac{r}{2M} - 1 \right|. \quad (5.5)$$

As $r \rightarrow 2M$, $r_* \rightarrow -\infty$, whence $t \rightarrow \infty$.

The same calculation shows that light emitted from $r = 2M(1 + \epsilon)$ reaches a distant observer after a time Δt that blows up like $|\log \epsilon|$ as $\epsilon \rightarrow 0$. From (5.4) with a + sign for outgoing photons,

$$\begin{aligned} \Delta t &= r_*(8M) - r_*[2M(1 + \epsilon)] \\ &= (8M - 2M \log 3) - [2M(1 + \epsilon) + 2M \log \epsilon] \\ &\cong 2M \log \epsilon \end{aligned} \quad (5.6)$$

Thus an observer who remains a finite distance outside $r = 2M$ never sees an infalling particle reach $r = 2M$. Moreover, the observed redshift becomes infinite: A photon’s frequency as measured by an observer moving along t^α (an observer at fixed r, θ, ϕ) is $-k_\alpha u^\alpha$, with

$$\begin{aligned} u^\alpha &= \frac{t^\alpha}{\sqrt{-g_{tt}}} = \frac{t^\alpha}{\sqrt{1 - 2M/r}}. \\ \omega &= \frac{E}{\sqrt{1 - 2M/r}} = \frac{\omega_\infty}{\sqrt{1 - 2M/r}} \end{aligned} \quad (5.7)$$

Thus

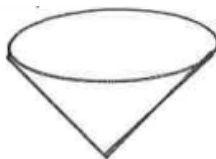
$$\frac{\omega(r = 2M(1 + \epsilon))}{\omega(r = R)} = \left[\frac{(1 + \epsilon)(1 - \frac{2M}{R})}{\epsilon} \right]^{1/2} \rightarrow \infty \text{ as } \epsilon \rightarrow 0. \quad (5.8)$$

5.2 Eddington-Finkelstein Coordinates

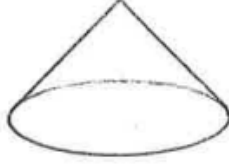
The fact that ingoing photons reach $r = 2M$ in finite value of their affine parameter and that ingoing finite rest mass particles reach $r = 2M$ in finite proper time suggests that one might be able smoothly to join regions I ($r > 2M$) and II ($r < 2M$) with coordinates tied to infalling particles. This is easier to do with photons, because they have simpler trajectories. Because the method leads to null coordinates, we’ll first revisit null coordinates in Minkowski space, to better understand their meaning.

Null coordinates in flat space:

Future null cones are $u = \text{constant}$ surfaces, where $u = t - r$



Past null cones are $v = \text{constant}$ surfaces, with $v = t + r$



Then

$$t = \frac{u + v}{2}, \quad r = \frac{v - u}{2}$$

Outgoing null coordinates: u, r, θ, ϕ

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2 d\Omega^2 = -(du + dr)^2 + dr^2 + r^2 d\Omega^2 \\ &= -du^2 - 2dudr + r^2 d\Omega^2 \end{aligned} \quad (5.9)$$

Ingoing null coordinates: v, r, θ, ϕ

$$ds^2 = -dv^2 + 2dvdr + r^2 d\Omega^2. \quad (5.10)$$

Double null coordinates: u, v, θ, ϕ

$$\begin{aligned} ds^2 &= -\left[\frac{1}{2}(du + dv)\right]^2 + \left[\frac{1}{2}(dv - du)\right]^2 + r^2(u, v) d\Omega^2 \\ ds^2 &= -dudv + r^2 d\Omega^2. \end{aligned} \quad (5.11)$$

In this chart, r is a function of u and v : $r^2 = \frac{1}{4}(u - v)^2$.

In the ingoing null coords v, r, θ, ϕ , the path $v, \theta, \phi = \text{const}$ is the trajectory of a radially ingoing photon, and r is an affine parameter along the path.

Ingoing and outgoing null coordinates in Schwarzschild

To mimic this in the Schwarzschild geometry, one must take

$$v = t + r_*, \quad (5.12)$$

because by (5.4) the trajectory of an ingoing photon is

$$t + r_* = \text{constant}, \quad \theta, \phi \text{ constant.}$$

Again, r is an affine parameter along the geodesic so we try a (v, r, θ, ϕ) chart:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \\ &= -\left(1 - \frac{2M}{r}\right) (dv - \frac{dr_*}{dr} dr)^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \\ &= -\left(1 - \frac{2M}{r}\right) \left[dv - \left(1 - \frac{2M}{r}\right)^{-1} dr \right]^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \end{aligned}$$

where we used Eq. (5.5) to write $\frac{dr_*}{dr} = \left(1 - \frac{2M}{r}\right)^{-1}$. The form of the metric is surprisingly simple:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2. \quad (5.13)$$

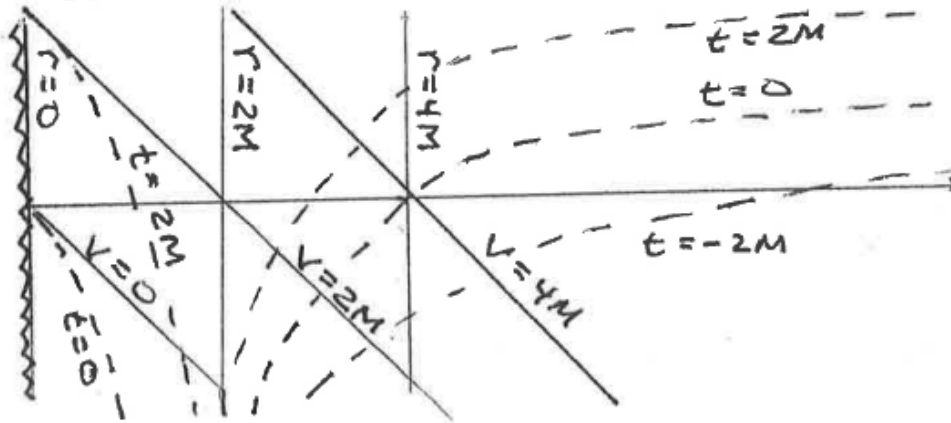
The metric components are well behaved for $0 < r < \infty$. For $r \gg 2M$, (5.13) looks like the flat space form (5.10) for a single null coordinate; for $r = 2M$ it is

$$ds^2 = 2dvdr + r^2 d\Omega^2, \quad (5.14)$$

similar to the form (5.11) for two null coordinates; and for $r < 2M$, the coefficient of dv^2 is positive.

The sign change in g_{vv} shows that the “timelike” Killing vector, $t^\alpha = \partial_v$, becomes null at $r = 2M$ and spacelike for $r < 2M$. Note that, at $r = 2M$, the completely null form ($g_{vv} = g_{rr} = 0$) reflects the fact that both ∂_v and ∂_r are null there.

How do regions I and II in the old (t, r, θ, ϕ) charts fit into the new extended geometry in the (v, r, θ, ϕ) chart? It’s easier to visualize the v, r, θ, ϕ description with v drawn so that the $v = \text{constant}$ ($\theta = \frac{\pi}{2}, \phi = 0$) null geodesics are 45° lines (representing ingoing photons).



The images of some $t = \text{constant}$ lines are shown in the v, r diagram as dotted lines.

Note that the $r = 2M$ line in t, r, θ, ϕ coordinates has been pushed to $v = -\infty$.

Light cones:

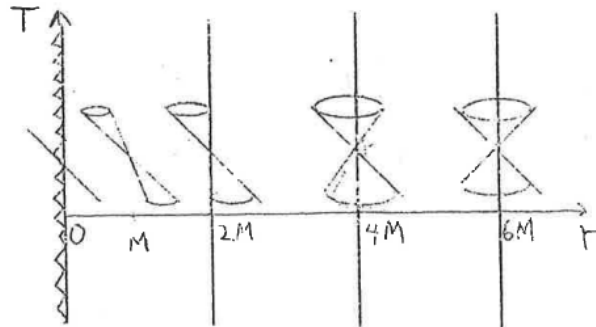
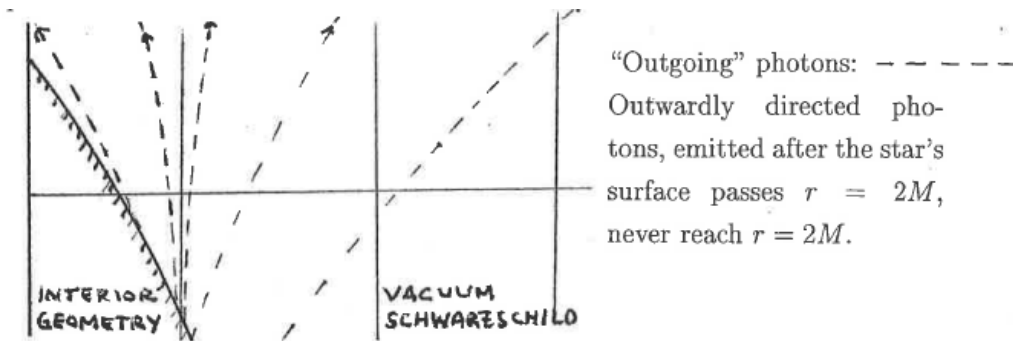


Figure 5.1: Light cones in an Eddington-Finkelstein diagram, with $T = v - r$.

Ingoing radial photons move along $v = \text{constant}$ lines. Outgoing photons move along $u = t - r_* = \text{constant}$ lines. Then $u = v - 2r_* = \text{constant}$, and along the $u = \text{constant}$ lines (i.e., along the outward radial null geodesics) $dv = 2dr_* = 2(1 - 2M/r)^{-1} dr$. Or, with $T = v - r$, ingoing rays are given by $dT = -dr$, outgoing rays by

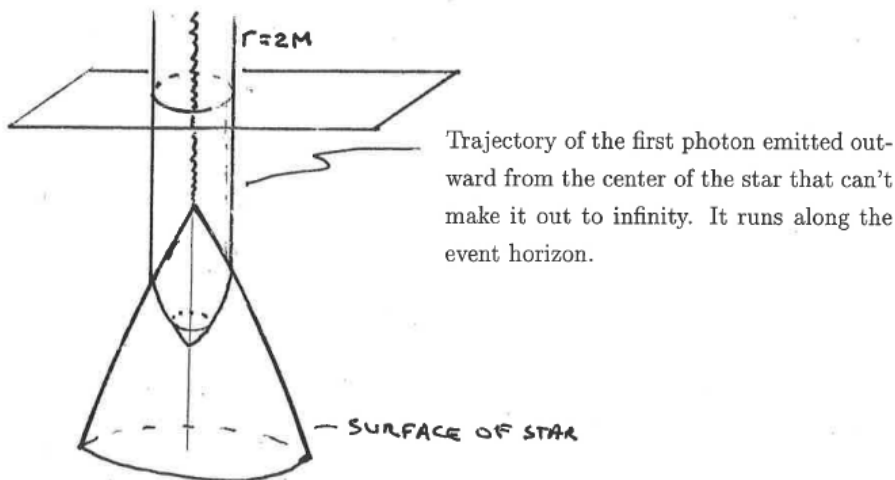
$$dT = \frac{r + 2M}{r - 2M} dr.$$

Stellar collapse can be described using Eddington-Finkelstein coordinates: Inside the star the geometry is regular initially, and outside the geometry is vacuum Schwarzschild. The surface of the star follows a timelike trajectory and so once inside $r = 2M$, it is constrained to go to $r = 0$ in finite proper time.



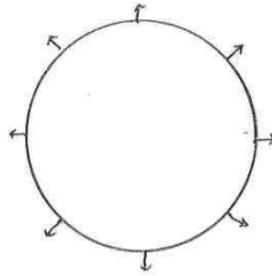
Finkelstein, after reinventing Eddington's coordinates, realized the the $r = 2M$ surface was an event horizon. Observers outside it are unable to receive signals from within. A spacelike slice of the region inside an event horizon is a black hole.

In a general asymptotically flat spacetime, event horizons and black holes are characterized this way: The set of all future directed null geodesics that make it out to infinity (that eventually leave any compact region) is called the past of future null infinity (in Schwarzschild this is in the region outside $r = 2M$). The boundary of this region, *the boundary of the past of future null infinity*, is called the event horizon; and a spacelike section of the spacetime inside the event horizon is called a black hole.



Trapped Surfaces:

From the light cone diagram, Fig. 5.1, it is apparent that any photon emitted within $r = 2M$ eventually hits $r = 0$. Because the radial coordinate r has the meaning that $4\pi r^2$ is the area of a $v, r = \text{constant}$ surface, the following peculiar phenomenon occurs. Consider the set of all outgoing photons emitted from an $r = \text{constant}$ sphere.



For large r , the emitted photons comprise a sphere of increasing area – they are diverging. For $r < 2M$ however, even the “outgoing” photons are moving to smaller values of r , so the sphere of outgoing photons is shrinking in area – the photons are converging, trapped.

Note, however, that an outgoing photon, emitted by a flashlight pointing radially outward, moves away from the flashlight in the direction that it is emitted at the speed of light, according to an observer at rest relative to the flashlight. This is consistent, because the flashlight itself is falling inward faster than the emitted photons. In general, a *trapped surface* S is one whose outward directed null geodesics ($\perp$ to S) are all converging. Thus in Schwarzschild, all r, v constant surfaces with $r < 2M$ are trapped.

It used to be thought likely that the singularity at $r = 0$ in Schwarzschild represented the end of spherical collapse only – that nonspherical collapsing stars would never encounter singularities. In 1965, however, Penrose proved that if energy dominates pressure in the sense that

$$(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)u^\alpha u^\beta \geq 0 \quad (5.15)$$

all timelike u^α , and if there is a trapped surface, then the spacetime is singular. For a fluid this is satisfied unless $P < -\frac{1}{3}\rho$ (attractive pressure exceeding the relativistic limit); for an electromagnetic field it's always true; for a massive scalar field the condition can fail in regions on the order of the Compton wavelength, $\sim 10^{-16}$ cm for the Higgs boson. Eq. (5.15) is called the *strong energy condition*. The singularity may not be as violent as the $r = 0$ singularity of Schwarzschild where $R_{\alpha\beta\gamma\delta}$ becomes singular ($R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, for example, blows up). The theorem only says that at least one null geodesic is incomplete – the geodesic leaves the spacetime in finite affine parameter length. The coordinate singularity at $r = 2M$ had that character, but that was the result of picking coordinates that cover only a piece of the spacetime – in effect, by choosing Schwarzschild coordinates, one is cutting a hole out of the spacetime at $r = 2M$. In that case, one can smoothly extend the truncated geometry. Penrose's theorem guarantees that there is no way to avoid the singularity by, in particular, extending the spacetime. As we have already seen, null geodesics hit $r = 0$ in finite parameter length, and this is the singularity implied by Penrose's theorem. Penrose's theorem is proved in Chapter 9 of Wald, where it is Theorem 9.5.3; a somewhat stronger version due to Hawking and Penrose is 9.5.4.

The theorems only guarantee geodesic incompleteness. Timelike geodesics within the horizon of the analytically extended Kerr geometry do not generically encounter infinite curvature; whether they do so in the actual geometries of rotating black holes is not yet known.

5.3 Embedding Diagrams

If you are handed the components of a metric in some coordinates, it is often difficult to understand what the metric means physically or geometrically. Commonly one draws light cones and constructs the geodesics, as we have done for the Schwarzschild geometry. Another, more powerful, technique, when it can be used, is to construct an embedding diagram. The idea is to regard a section of the spacetime (e.g., a $t = \text{constant}$ surface) as a curved surface in flat space (some $\mathbb{R}^n$). For example, a sphere can be characterized intrinsically by the metric

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

or pictured as an $r = \text{constant}$ subset of $\mathbb{R}^3$. In other words, one looks for a submanifold of $\mathbb{R}^n$ whose metric is that of the surface you want to understand.

Let us first embed a $t = \text{constant}$ surface of a spherical star

$$ds^2 = \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (5.16)$$

To visualize, we can't look at more than a 2-dimensional surface embedded in $\mathbb{R}^3$, so we'll take a $\theta = \frac{\pi}{2}$ plane:

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\phi^2. \quad (5.17)$$

The object is to find a surface $z = z(r, \phi)$ in $\mathbb{R}^3$ with this metric. The flat metric of $\mathbb{R}^3$ has z, r, ϕ components

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 \quad (5.18)$$

The metric of a $z = z(r, \phi)$ surface is thus

$$ds^2 = \left(\frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \phi} d\phi\right)^2 + dr^2 + r^2 d\phi^2. \quad (5.19)$$

Axisymmetry $\Rightarrow z = z(r)$,

$$ds^2 = (z'^2 + 1)dr^2 + r^2 d\phi^2.$$

To agree with (5.17) we need

$$z'^2 + 1 = \left(1 - \frac{2m}{r}\right)^{-1}.$$

$$z = \int_0^r \frac{dr}{\left[\frac{r}{2m(r)} - 1\right]^{1/2}}, \quad \text{when there is no black hole -- when } 2m(r) \text{ is everywhere greater than } r. \quad (5.20)$$

Near the center of the star

$$\begin{aligned} m &= \frac{4}{3}\pi \rho_c r^3 + O(r^4) \implies \\ z &= \int_0^r \left(\frac{8\pi\rho_c}{3}\right)^{1/2} r dr + O(r^3) \\ &= \left(\frac{2\pi\rho_c}{3}\right)^{1/2} r^2 + O(r^3). \end{aligned} \quad (5.21)$$

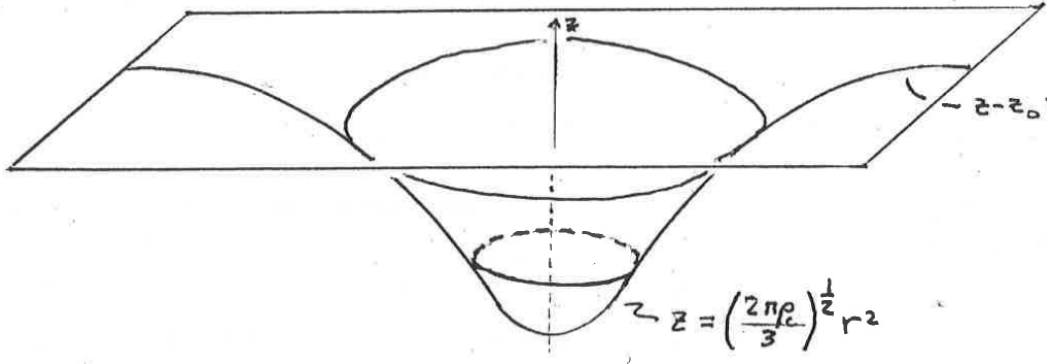
Outside the star

$$z = z_0 + [8M(r - 2M)]^{1/2} \text{ or } (r - 2M) = \frac{1}{8M}(z - z_0)^2. \quad (5.22)$$

Check:

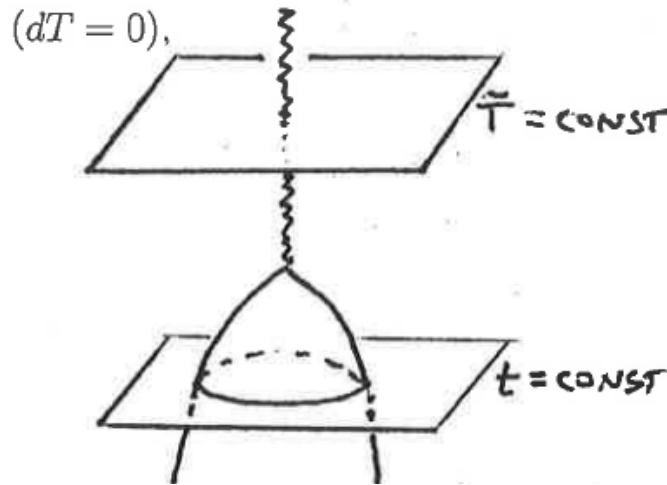
$$\frac{dz}{dr} = (8M)^{1/2} \frac{1}{2}(r - 2M)^{-1/2} = \frac{1}{\left(\frac{r}{2M} - 1\right)^{1/2}}.$$

Note that the surface is coordinate independent – specified by demanding that its metric inherited from



the flat metric on $\mathbb{R}^3$ be the same as the metric of an equatorial plane in the Schwarzschild geometry.

After the star collapses one can embed a $T = \text{constant}$ surface: From (5.13), with $dv = dr$ ($dT = 0$),

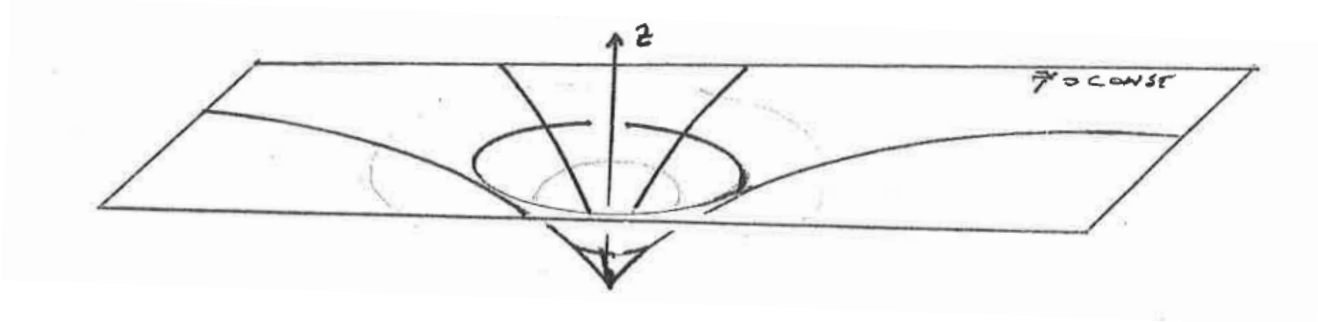


$$ds^2 = \left[-\left(1 - \frac{2M}{r}\right) + 2 \right] dr^2 + r^2 d\Omega^2 = \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2$$

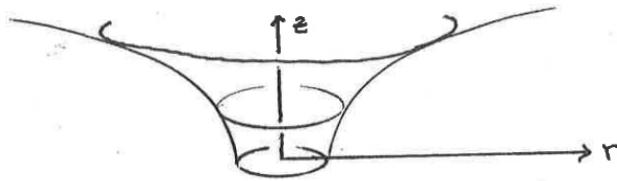
Embed:

$$\begin{aligned} (z')^2 + 1 &= 1 + \frac{2M}{r} \\ z' &= \sqrt{\frac{2M}{r}} \\ z &= (8Mr)^{1/2} \text{ or } r = \frac{1}{8M} z^2. \end{aligned} \quad (5.23)$$

After the collapse the surface is no longer smooth at $r = 0$; there is a cusp, implying infinite curvature. (Old figure has $\tilde{T}$ instead of T .)



Although Eddington-Finkelstein ingoing coordinates are fine for describing collapse, and provide a completion for the future-directed null and timelike geodesics that are cut off in the $\{t, r, \theta, \phi\}$ chart, the $\{v, r, \theta, \phi\}$ coordinates are not good for describing past-directed geodesics that move inward (equivalently, the past of future-directed outward geodesics). These run off the manifold in finite length as before. When the past was a star, the problem disappears, but if one wants to look formally at the vacuum Schwarzschild geometry a further completion is needed. One way to see what to expect is to look at the embedding of a $t = \text{constant}$ surface of vacuum Schwarzschild. From (5.22), the embedding looks like $r - 2M = \frac{1}{8M} (z - z_0)^2$, and it ends abruptly at $r = 2M$, $z = 0$ (where, without loss of generality, we have taken $z_0 = 0$).

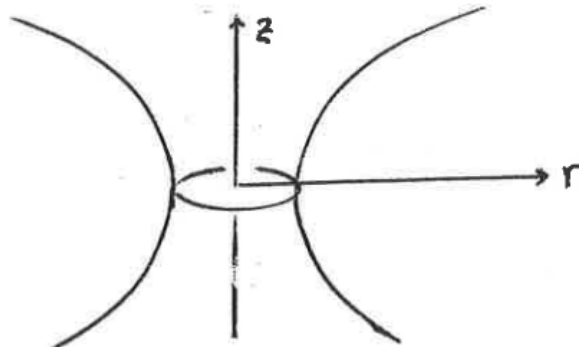


But the bottom half of the paraboloid is identical to the top half and is also described by the metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (5.24)$$

of a $t = \text{constant}$ surface.

Thus the full paraboloid is a smooth extension of the $t = \text{constant}$ surface, and it has the spatial Schwarzschild metric (5.24) everywhere.



This suggests that one can extend the spacetime in such a way as to get two identical asymptotically flat regions. One way is to patch together different charts. But it turns out to be possible to find a single chart that covers the complete extension of the geometry.

5.4 Kruskal-Szekeres Coordinates

The papers: Kruskal[52], Szekeres[86, 87].

Part of the problem with Eddington-Finkelstein is that, by choosing ingoing coordinates, one imposes a preferred time direction (future) on a geometry that doesn't know future from past. As in flat space, one could have picked outgoing null coordinates $u = t - r_*$, r, θ, ϕ and found a metric $ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2 d\Omega^2$ for a spacetime with complete past directed geodesics, but incomplete future geodesics.

So to regain past-future symmetry one could try null coordinates u, v, θ, ϕ . Then, using

$$t = \frac{u+v}{2}, \quad r_* = \frac{v-u}{2}, \quad \text{and} \quad \frac{dr}{dr_*} = \left(1 - \frac{2M}{r}\right), \quad (5.25)$$

we have

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right) \frac{1}{4}(du + dv)^2 + \left(1 - \frac{2M}{r}\right)^{-1} \frac{1}{4}(dv - du)^2 \left(1 - \frac{2M}{r}\right)^2 + r^2 d\Omega^2 \\ &= -\left(1 - \frac{2M}{r}\right) dudv + r^2 d\Omega^2 \end{aligned} \quad (5.26)$$

Unfortunately, at $r = 2M$ the 4-dimensional metric is degenerate (not invertible): $ds^2 = r^2 d\Omega^2$, implying $\sqrt{-g} = 0$. This, however, is the only remaining obstacle, and it is easily overcome by a change of scale:

$$\begin{aligned} U &= -e^{-u/4M} & V &= e^{v/4M} \\ dU &= \frac{1}{4M} e^{-u/4M} du & dV &= \frac{1}{4M} e^{v/4M} dv \end{aligned} \quad (5.27)$$

$$\begin{aligned} du dv &= 16M^2 e^{(-v+u)/4M} dU dV = 16M^2 e^{-r_*/2M} dU dV \\ &= 16M^2 \exp \left[-\frac{r}{2M} - \log \left| \frac{r}{2M} - 1 \right| \right] dU dV \\ &= 16M^2 e^{-r/2M} \frac{1}{\frac{r}{2M} - 1} dU dV, \quad r > 2M. \end{aligned} \quad (5.28)$$

Finally,

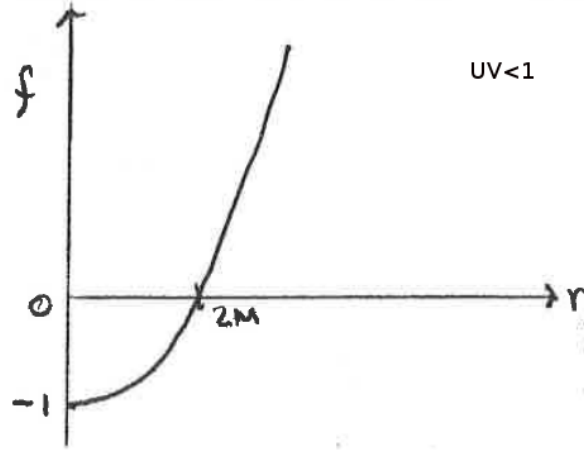
$$\begin{aligned} ds^2 &= -16M^2 e^{-r/2M} \frac{2M}{r} dU dV + r^2 (U, V) d\Omega^2, \\ ds^2 &= -\frac{32M^3}{r} e^{-r/2M} dU dV + r^2 d\Omega^2. \end{aligned} \quad (5.29)$$

This looks good for $r > 0$, but we still need to find out when $r(U, V)$ is well behaved. From (5.5), (5.25) and (5.27), r is given implicitly for $r > 2M$ by

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = -UV \equiv f(r), \quad (5.30)$$

and we *define* $r(U, V)$ by this relation for all $r > 0$. $f(r)$ is invertible to $r > 0$ when $-UV$ is in the range of f :

$$UV < 1. \quad (5.31)$$



One commonly introduces the spacelike and timelike coordinates

$$X = \frac{1}{2}(V - U) \quad T = \frac{1}{2}(V + U). \quad (5.32)$$

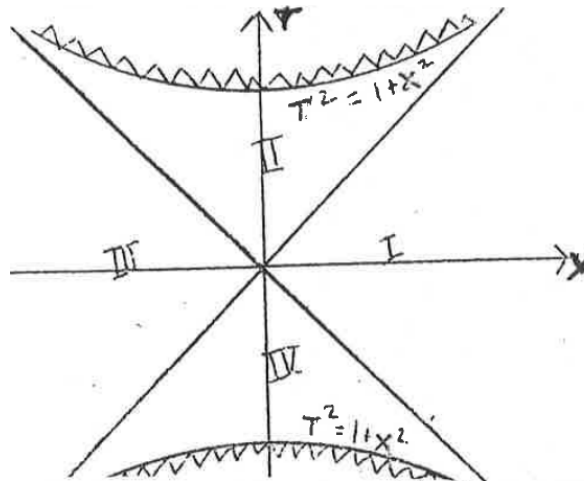
Then

$$f(r) = X^2 - T^2 \quad (5.33)$$

and the metric has components

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} (-dT^2 + dX^2) + r^2 d\Omega^2.$$

Eq. (5.33) is invertible to a unique $r > 0$ when $-X^2 + T^2 < 1$, or $T^2 < 1 + X^2$, a region between two hyperbolae; it has four parts, I, II, III, IV.



In I, one obtains a t, r patch with $r > 2M$ by setting

$$\begin{aligned} \text{I } (r > 2M) : \quad X &= \left(\frac{r}{2M} - 1 \right)^{1/2} e^{r/4M} \cosh \frac{t}{4M} \\ T &= \left(\frac{r}{2M} - 1 \right)^{1/2} e^{r/4M} \sinh \frac{t}{4M} \end{aligned}$$

Region II provides a t, r patch with $r < 2M$

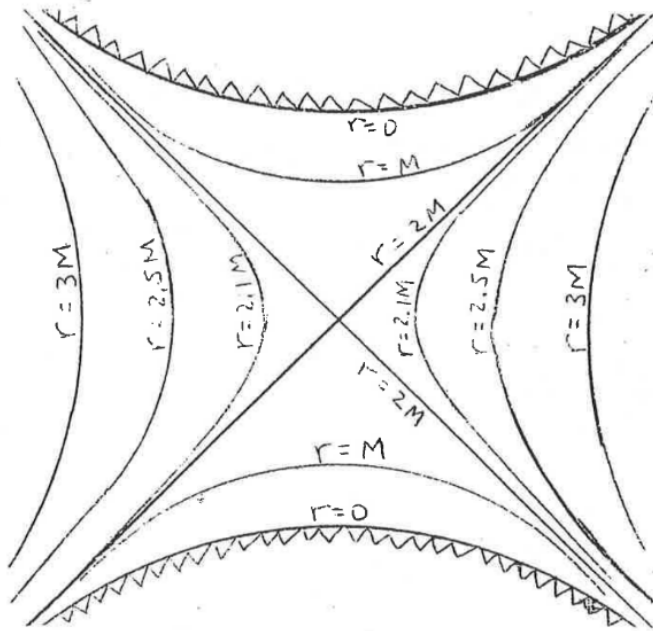
$$\begin{aligned} \text{II } (r < 2M) : \quad X &= \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh \frac{t}{4M} \\ T &= \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \cosh \frac{t}{4M} \end{aligned}$$

Similarly regions III and IV give t, r patches isometric to I and II, respectively, with

$$\begin{aligned} \text{III } (r > 2M) : \quad X &= -\left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh \frac{t}{4M} \\ T &= -\left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh \frac{t}{4M} \end{aligned}$$

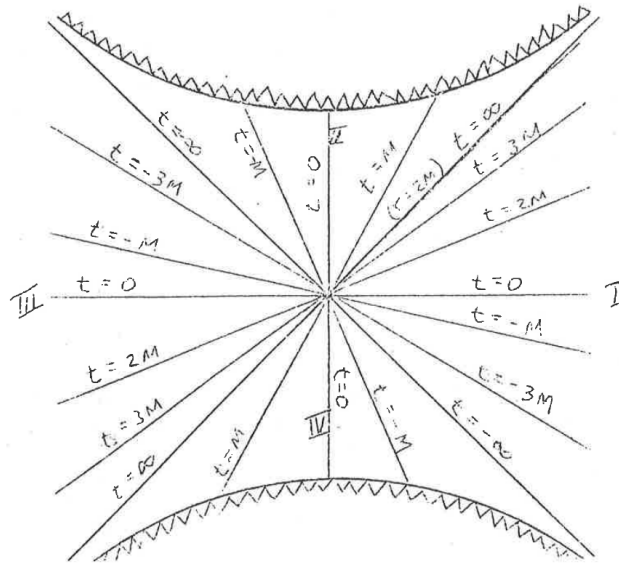
$$\begin{aligned} \text{IV } (r < 2M) : \quad X &= -\left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh \frac{t}{4M} \\ T &= -\left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \cosh \frac{t}{4M} \end{aligned}$$

As we assumed to begin with in (5.30) and (5.33), r is given everywhere by $X^2 - T^2 = \left(\frac{r}{2M} - 1\right) e^{r/2M}$, and the $r = \text{constant}$ curves are then hyperbolae:



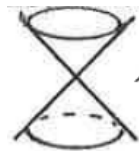
The $t = \text{constant}$ curves are straight lines:

$$\text{In regions I and III, } \tanh \frac{t}{4M} = \frac{T}{X}. \quad \text{In regions II and IV, } \tanh \frac{t}{4M} = \frac{X}{T}.$$



Features of the complete Schwarzschild geometry in Kruskal coordinates: All values of t for $r = 2M$ are mapped to a single point, the center point of the figure above. With the angular dimensions included, this is a sphere of radius $r = 2M$ in spacetime. This problem with the coordinate t is analogous to the coordinate singularity in polar coordinates where all values of ϕ at $\theta = 0$ are mapped to a single point (a line, the z -axis, when one includes r).

In the Kruskal diagram, the radial null geodesics are 45° lines: The light cones are 45° cones.



There are two asymptotically flat regions. Spacelike surfaces, for example the $t = \text{constant}$ surfaces of regions I and III, extend from one asymptotic region to another, via a wormhole (initially called an Einstein-Rosen bridge). It is immediately clear from the Kruskal diagram that no particle can go through through the wormhole and reach the other asymptotically flat region: Only spacelike lines do that – timelike ($> 45^\circ$ angle with horizontal) lines all hit the $r = 0$ singularity and don't make it through from I to III or III to I.

This is a generic property of asymptotically flat spacetimes with non-Euclidean topology (there are a countably infinite number of such topologies). No null geodesic that starts and ends at infinity can thread a topological structure: Positive energy implies that any topological structure collapses too quickly for observers to find out that it's there (JF, Schleich, Witt) topological censorship[38]. A corollary is a strengthened version of a theorem by Hawking that any spacelike slice of the event horizon of a stationary black hole has spherical topology (Chrusciel, Wald)[27].

It is also apparent from the diagram that no particle in II, in the black-hole interior, can escape from the black hole to regions I or III.

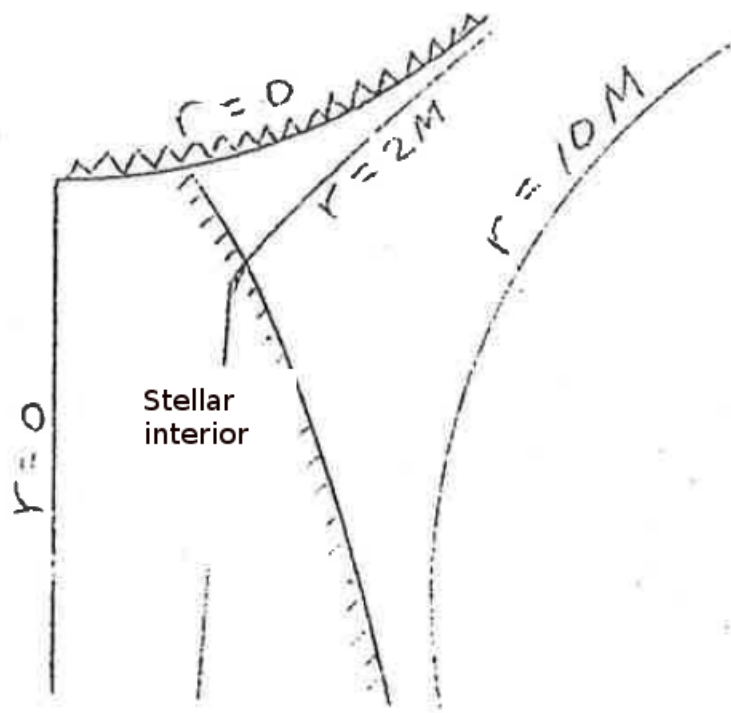
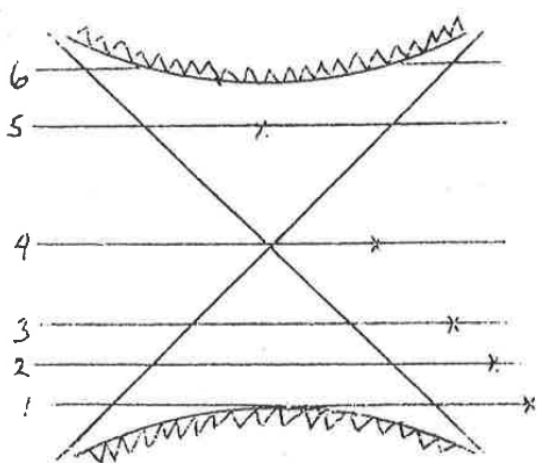


Figure 5.2: A collapsing star in a Kruskal diagram looks like this figure. The left half of the Kruskal spacetime doesn't exist here. Because the Eddington-Finkelstein ingoing chart is mapped to regions II and III, the Eddington-Finkelstein chart also covers the full collapse spacetime, if one continues the coordinates v, r to the star's interior.

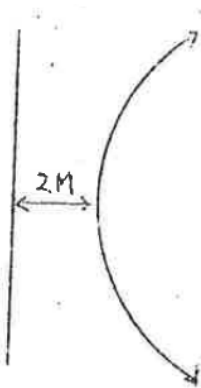
The Schwarzschild geometry is time dependent: the Killing vector that is timelike outside $r = 2M$ becomes spacelike inside. Here are two ways to slice the history (the spacetime) given à la MTW as sequences of embedding diagrams.



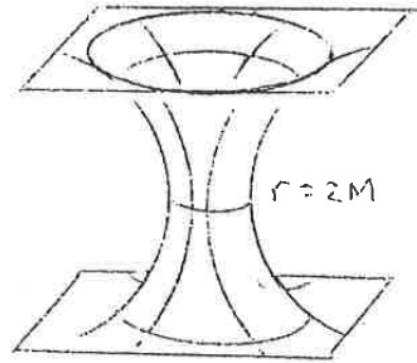
In figuring out what the embedding of each slice should look like qualitatively, the key point is that distances on the embedded surface are proper distances in the spacetime. In particular, a circle with radial coordinate r has proper circumference $2\pi r$ and thus is always at a distance r from the z -axis in the embedding diagram. And distance along a radial curve in an embedding diagram is proper spacetime distance.

Figure 5.3: First slicing

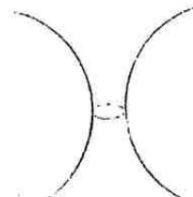
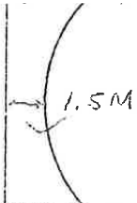
Slice 4: r begins on the right at ∞ , then goes to $r = 2M$ (the central point in the diagram), and then out to ∞ again:



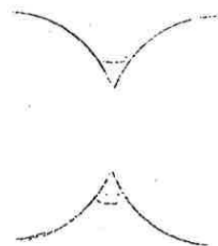
rotate: each
point of Kruskal
is a sphere;
here each point
goes to a circle



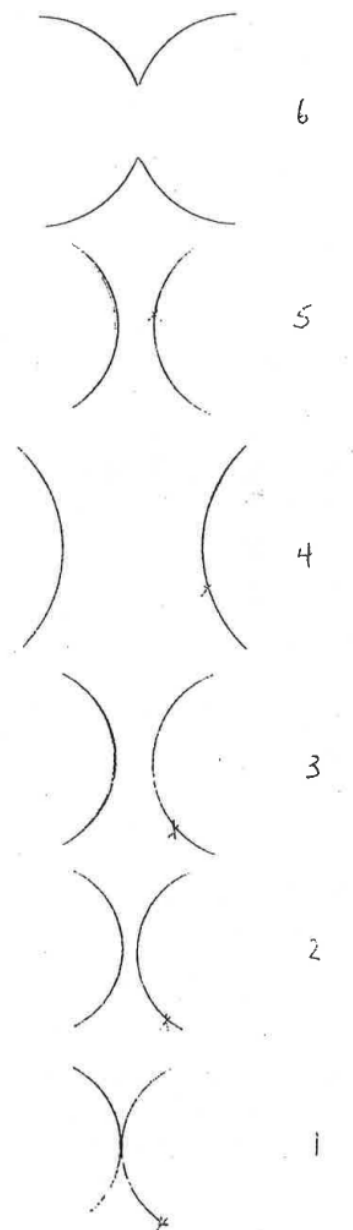
Slice 5: r begins at ∞ , goes to $r = 1.5$, then out to ∞ :



Slice 6: r begins at ∞ , goes to $r = 0$; a disconnected piece starts at $r = 0$ and goes to ∞ :

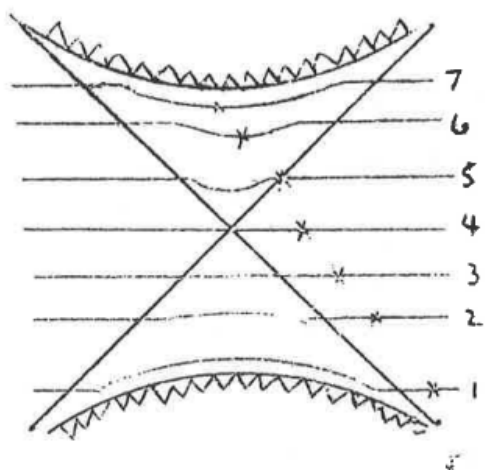


So the slicing 1-6 of the history looks like this.

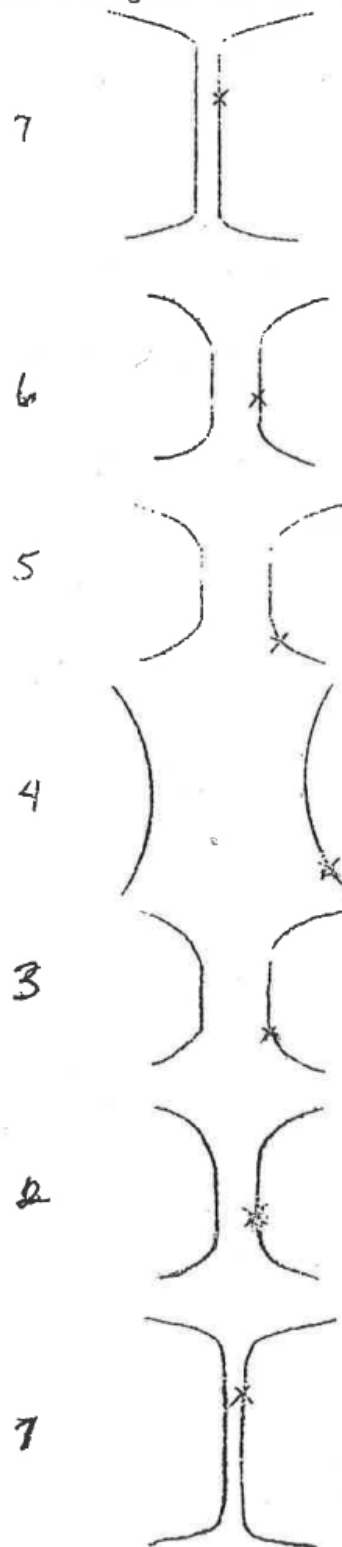


The crosses mark the trajectory of an ingoing particle that fails to make it through the wormhole before the asymptotic regions pinch off.

Another slicing



You can also slice (foliate) the entire spacetime with spacelike slices that never touch the singularities. They then spend a long stretch at $r \approx 0$:



Exercise 72. Calculate the cross section for capture of particles by a Schwarzschild black hole for $v \ll 1$. Hint: First show that $V_{\text{eff}} \ll 1$ at the largest impact parameter for which the particle can avoid capture.

Exercise 73. (Hartle 12-12) Working in Eddington-Finkelstein coordinates, check that the normal vector to the horizon 3-surface of a Schwarzschild black hole is a null vector.

Exercise 74. (Adapted from Hartle) An observer falls feet first into a Schwarzschild black hole looking down at her feet. Is there a radius at which she cannot see her feet? In particular,

- When her eyes are at the horizon, can she see her feet? If so, at what radius does she see them?
- Does she see her feet hit the singularity, assuming she remains sentient until her head hits the singularity?
- Is it dark inside a black hole?* An observer outside sees a star become dark as it collapses to a black hole. But would it be dark inside a black hole, assuming a collapsing star continues to radiate at a steady rate, measured by an observer on its surface?
- Finally, if the observer looks up from within a black hole in the disk of the Milky Way, would she see stars above her?

This exercise is most naturally done in Kruskal coordinates, using a Kruskal diagram.

Exercise 75. Recall that conservation of baryons has the form $\nabla_\alpha j^\alpha = 0$, where $j^\alpha = nu^\alpha$, with n = baryon density of a fluid with 4-velocity u^α .

- Using ingoing Eddington-Finkelstein coordinates, find the number of baryons per unit time entering a Schwarzschild black hole. That is, find dN/dv , with dN the number of baryons that cross the horizon between successive slices at $v = \text{constant}$ and $v + dv = \text{constant}$.
- The Killing energy flux of a scalar field into the Schwarzschild horizon is positive: Show that

$$\int j^\alpha dS_\alpha \geq 0,$$

where

$$j^\alpha = T^\alpha_\beta t^\beta, \quad T_{\alpha\beta} = \nabla_\alpha \Phi \nabla_\beta \Phi - \frac{1}{2} g_{\alpha\beta} \nabla_\gamma \Phi \nabla^\gamma \Phi.$$

Note that the Killing vector t^α is ∂_v in this chart. See p. A.2.1 for both parts of the problem.

Exercise 76. An observer falls radially into a spherical black hole of mass M . She starts from rest relative to a stationary observer at $r = 8M$. What time does her watch read just before she reaches the singularity at $r = 0$?

5.5 Kerr Black Holes

Rotating stars and black holes

The metric of a rotating star or a black hole has a timelike Killing vector t^α and a rotational Killing vector ϕ^α . We have seen that a system with stress-energy tensor $T^{\alpha\beta}$ has a conserved current associated with each Killing vector. In the case of a rotational Killing vector, the current $j^\alpha = T^\alpha_\beta \phi^\beta$ is associated with angular momentum: $j^t = T^t_\phi$ is the angular momentum density, and the integral $J = \int j^\alpha dS_\alpha$ is the total angular momentum of the spacetime. In a nearly Newtonian star, this has the form

$$J = \int \rho v_i \phi^i dV = \int \rho (x v_y - y v_x).$$

For uniform rotation, $\mathbf{v} = \Omega \phi$, with

$$\phi = \hat{\mathbf{z}} \times \mathbf{r} = x \hat{\mathbf{y}} - y \hat{\mathbf{x}}, \quad \text{and} \quad J = \int \rho v_i \phi^i = \int \rho (x^2 + y^2) \Omega dV. \quad (5.34)$$

The Coulomb potential of a ball of charge of electric charge density ρ_e is, up to a constant, identical in form to the Newtonian potential Φ of a mass with density ρ , each satisfying $\nabla^2 \Phi = C\rho$ for some constant C . Similarly the magnetic field of a stationary current density $\mathbf{j}_e = \rho_e \mathbf{v}$ is identical up to a constant to components of the metric associated with the mass current $\mathbf{j} = \rho \mathbf{v}$ of a nearly Newtonian fluid.

Let's start with the magnetic field. In a Lorenz gauge, a time-independent vector potential satisfies $\nabla \cdot \mathbf{A} = 0$, and the magnetic field of a stationary current is given by

$$\nabla \times \mathbf{B} = 4\pi \mathbf{j}_e, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad \implies \nabla^2 A_i = -4\pi j_{ei},$$

with solution for the Cartesian components

$$A_i = \int \frac{\rho_e v_i}{|\mathbf{r} - \mathbf{r}'|} dV'. \quad (5.35)$$

Now consider the stress tensor of nearly Newtonian fluid with velocity field $\mathbf{v}$. It has the form

$$(T^{\mu\nu}) = \begin{pmatrix} \rho & \rho \mathbf{v} \\ \rho \mathbf{v} & 0 \end{pmatrix} + O(e^2),$$

with e as before a small parameter of order the largest of v and v_{sound} . (Recall that $P = O(\rho v_{\text{sound}}^2)$.) For a nearly Newtonian source, the metric has the form $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, and we can ignore terms quadratic in the small correction $h_{\alpha\beta}$ to the flat metric. The Ricci and Einstein tensors are then linear in $h_{\alpha\beta}$; because the $\Gamma\Gamma$ terms in the Riemann tensor are quadratic in $h_{\alpha\beta}$, we can use Eq. (2.52), valid when terms quadratic in Γ are discarded. Then, using $\partial_\sigma g_{\mu\nu} = \partial_\sigma h_{\mu\nu}$, we have

$$R_{\mu\nu\sigma\tau}(P) = \frac{1}{2} [\partial_\nu \partial_\sigma h_{\mu\tau} + \partial_\mu \partial_\tau h_{\nu\sigma} - \partial_\mu \partial_\sigma h_{\nu\tau} - \partial_\nu \partial_\tau h_{\mu\sigma}]. \quad (5.36)$$

In the nearly-Newtonian regime, the diagonal components of $h_{\mu\nu}$ are given by the diagonal components of $T^{\mu\nu}$, as in Eq. (3.24), and we can quickly see that the off-diagonal components of the Einstein equation are satisfied when h_{ti} satisfies

$$\nabla^2 h_{ti} = 16\pi \rho v_i \quad (5.37)$$

with conservation of mass, $\partial_i(\rho v^i)$, implying $\partial^i h_{ti} = 0$: For $\mu \neq \nu$, $G_{\mu\nu} = R_{\mu\nu} + O(\gamma^2)$, and contracting (5.36) on ν and τ gives

$$R_{ti} = \frac{1}{2}[\partial^\nu \partial_i h_{t\nu} + \partial_t \partial^\nu h_{\nu i} - \partial_t \partial_i h_\nu^\nu - \partial^\nu \partial_\nu h_{ti}].$$

Because the star is stationary, the time derivatives vanish; where we used $\partial_t h_{\mu\nu} = 0$, $\partial^i h_{\mu i} = 0$.

$$R_{ti} = -\frac{1}{2}\nabla^2 h_{ti} = 8\pi T_{ti},$$

implying (5.37) – notice the sign: $T_{ti} = -\rho v_i$. The off-diagonal spatial components $R_{ij} = 8\pi T_{ij} = 0$, are automatically satisfied by $h_{ij} = 0$, $i \neq j$. Then

$$h_{ti} = -4 \int \frac{\rho v_i(x')}{|\mathbf{r} - \mathbf{r}'|} dV' \quad (5.38)$$

For a rotating star, $v = \Omega\phi$, and the asymptotic form of h_{ti} is proportional to the total angular momentum J . This follows from the expansion

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{|r^2 - 2\mathbf{r} \cdot \mathbf{r}' + r'^2|^{1/2}} = \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + O(r^{-3}). \quad (5.39)$$

Because the total linear momentum vanishes, $\int \rho v_i dV = 0$, and the leading asymptotic term is the dipole contribution,

$$h_{ti} = -4 \frac{1}{r^3} \mathbf{r} \cdot \int \mathbf{r}' \rho v_i dV' + O(r^{-3}). \quad (5.40)$$

Now $g_{ti} = h_{ti} + O(h^2)$. For a rotating star, $v^i = \Omega\phi^i$, and Eq. (5.40) implies

$$g_{ti} = h_{ti} = 2\epsilon_{ijk} \frac{x^j J^k}{r^3} + O(r^{-3}). \quad (5.41)$$

Exercise 77. Asymptotic behavior of a rotating star.

- Show Eq. (5.40) implies (5.41).
- The equations assume Cartesian coordinates. Convert (5.41) to spherical coordinates, showing that the asymptotic g_{ti} has the single nonzero component

$$g_{t\phi} = -\frac{2J}{r} \sin^2 \theta.$$

The Kerr spacetime

Had anyone taken seriously the idea that stars might really collapse to within their Schwarzschild radius, the Kerr solution might have been found in the 1920s by asking what would happen if a rotating star collapsed. Instead, Roy Kerr found it forty years later by looking for exact vacuum solutions whose Riemann tensor had a particularly simple form.¹

¹Ezra Newman nearly found it – he, Unti and Tambourino used the Newman-Penrose formalism to classify all solutions of that kind but made a error that Kerr caught; Newman was not happy that Kerr didn't share credit with him. In Kerr's version of the history, he and Alan Thompson had written an equivalent set of equations and that allowed Kerr to immediately spot the error <https://arxiv.org/abs/0706.1109>. Newman and Janis found a simple way to find the Kerr metric <https://aip.scitation.org/doi/10.1063/1.1704350> from a complex coordinate transformation of Schwarzschild, and Newman generalized that to obtain the charged, rotating black hole solutions that generalize the charged Schwarzschild solutions of Riessner and Nordstrom.

Kerr black holes are the unique stationary asymptotically flat vacuum spacetimes with event horizons. For each mass M , there is a 1-parameter family of Kerr black holes with that mass; the parameter is the angular momentum J about the symmetry axis, and at $J = 0$ the family begins with Schwarzschild. There is a maximum angular momentum for each mass M : No star can contract past the point where its rotational speed exceeds the speed of light. This limits the angular momentum of a black hole that can be physically generated. The physical limitation is mirrored in the solution set: For $J > M^2$ (think of $J_{\max} = Mv_{\max}R = McM = M^2$), Kerr vacuum metrics exist, but they have no horizons. Instead a naked singularity deprives these spacetimes of physical meaning.

Several features of a rotating black hole could have been predicted at the outset: Its two Killing vectors ϕ^α and t^α and its asymptotic form (5.41). Note that the t - ϕ part of the metric just involves the Killing vectors:

$$g_{t\phi} = g_{\alpha\beta} t^\alpha \phi^\beta = t^\alpha \phi_\alpha, \quad g_{tt} = t^\alpha t_\alpha, \quad g_{\phi\phi} = \phi^\alpha \phi_\alpha.$$

Furthermore, the geometry should be symmetric under a simultaneous reversal of time and the direction of rotation, under

$$t \rightarrow -t, \quad \phi \rightarrow -\phi. \quad (5.42)$$

This condition eliminates the tr , $t\theta$, ϕr , and $\phi\theta$ components of $g_{\alpha\beta}$ and $G_{\alpha\beta}$; and one can choose coordinates to eliminate $g_{r\theta}$.

The Kerr metric, in a chart that becomes the (t, r, θ, ϕ) chart of Schwarzschild when $J = 0$, is

$$ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} dt^2 - 2 \frac{2Mar \sin^2 \theta}{\rho^2} dt d\phi + \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2} \sin^2 \theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2, \quad (5.43)$$

where

$$a = J/M, \quad \rho^2 = r^2 + a^2 \cos^2 \theta = \Sigma_{\text{Wald}}, \quad \Delta = r^2 - 2Mr + a^2. \quad (5.44)$$

Notation: The notes follow MTW and Chandrasekhar's *Mathematical Theory of Black Holes*[24] in using ρ^2 , while Wald and Shapiro-Teukolsky [76] use Σ .

Here (t, r, θ, ϕ) are called Boyer-Lindquist coordinates, after their discoverers. The metric (5.43) can also be written in the factored form

$$ds^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2)d\phi - a dt]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \quad (5.45)$$

Its determinant is

$$\sqrt{-g} = \rho^2 \sin \theta, \quad (5.46)$$

and the determinant of the t - ϕ part of the metric is

$${}^2g = \mathbf{t} \cdot \mathbf{t} \phi \cdot \phi - (\mathbf{t} \cdot \phi)^2 = g_{tt}g_{\phi\phi} - (g_{t\phi})^2 = \Delta \sin^2 \theta. \quad (5.47)$$

The inverse metric is then

$$(g^{\mu\nu}) = \begin{pmatrix} -\frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2 \Delta} & 0 & 0 & -2\frac{aMr}{\rho^2 \Delta} \sin^2 \theta \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -2\frac{aMr}{\rho^2 \Delta} \sin^2 \theta & 0 & 0 & \frac{\Delta - a^2 \sin^2 \theta}{\rho^2 \Delta \sin^2 \theta} \end{pmatrix} \quad (5.48)$$

When $a = 0$ and $M \neq 0$, $\rho^2 = r^2$, $\Delta = r^2(1 - \frac{2M}{r})$, and the metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + r^2 \sin^2 \theta d\phi^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2,$$

the Schwarzschild metric in Schwarzschild coordinates.

The most important difference between the geometry of a rotating star or black hole and the geometry of a stationary object is the *dragging of inertial frames* (or just *frame dragging*), a nonzero t - ϕ metric component, whose meaning is that time-translation is not orthogonal to rotation, $t^\alpha \phi_\alpha \neq 0$. The angular velocity

$$\omega := -\frac{t^\alpha \phi_\alpha}{\phi^\beta \phi_\beta}, \quad (5.49)$$

measures the frame dragging in the sense that particles with zero angular momentum move along trajectories whose angular velocity relative to infinity is $d\phi/dt = \omega$. To see this, note that the angular momentum of a particle is $p_\alpha \phi^\alpha = p_\phi = mu_\phi$; we'll use the angular momentum per unit rest mass, $L = u_\alpha \phi^\alpha$. For a freely falling particle (and hence for an inertial observer), L is conserved. But the angular velocity of an observer, measured from infinity is $\Omega = d\phi/dt = u^\phi/u^t$. Now

$$L = u_\phi = g_{\phi t} u^t + g_{\phi\phi} u^\phi = u^t \phi \cdot t + u^\phi \phi \cdot \phi, \quad (5.50)$$

so a particle with zero angular momentum has angular velocity

$$\Omega \equiv \frac{u^\phi}{u^t} = -\frac{t \cdot \phi}{\phi \cdot \phi} = \omega. \quad (5.51)$$

A zero-angular momentum observer (ZAMO) at constant radius r has 4-velocity u^α normal to a $t = \text{constant}$ surface: That is, the vectors $\partial_\phi = \phi$, ∂_r and ∂_θ span the vectors tangent to a constant t surface; u^α is proportional to $t^\alpha + \omega \phi^\alpha$, which is obviously orthogonal to the vectors in the r and θ directions, and it is orthogonal to ϕ^α because

$$(t^\alpha + \omega \phi^\alpha) \phi_\alpha = t^\alpha \phi_\alpha - t^\alpha \phi_\alpha = 0. \quad (5.52)$$

For Kerr, from its definition and the form (5.43) of the metric, ω is

$$\omega = \frac{2Mar}{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}. \quad (5.53)$$

Frame dragging was measured to about 20% accuracy by gravity probe B. <https://ui.adsabs.harvard.edu/abs/2011PhRvL.106v1101E/abstract> More on frame dragging when we get to particle orbits.

The horizon

Because the spacetime is stationary and axisymmetric, rotations and time-translations map the horizon to itself. This implies that t^α and ϕ^α are tangent to the horizon H . (This is because ϕ^α is tangent to the circular trajectories of rotated points, and t^α is tangent to the trajectories of time-translated points.) We now show that the horizon of the Kerr geometry is the set of points where $\Delta = 0$. We begin with an outline and then fill in the argument for each step.

Here's the outline:

1. The event horizon is a null surface; that is, it is a hypersurface whose tangent space includes one null vector and no timelike vectors.

2. The null vector field of the Kerr horizon (and of any stationary axisymmetric horizon) is a linear combination, $t^\alpha + f\phi^\alpha$, of the two Killing vectors.
3. The metric on a null surface has vanishing determinant.
4. Because the space of vectors spanned by t^α and ϕ^α is a plane and that plane is null on the horizon, the determinant ${}^2g = \Delta \sin^2 \theta = 0$ on the horizon.

Here are the details:

1. The fact that the horizon H is a null surface follows from its definition as boundary of the set of points from which timelike curves can reach null infinity. From H and from points inside H , no null or timelike curves reach null infinity. From each point outside H , outwardly directed null rays do reach null infinity. Outwardly directed null rays beginning at points of the horizon, however, remain on the horizon. Thus a null geodesic passes through each point of H . Because one can regard H as formed by these rays, the horizon is said to be *generated* by its null geodesics or by the null vector field ℓ^α tangent to them. A horizon then is a null hypersurface, a hypersurface whose tangent space includes one null direction and no timelike vectors.

Why can H have no timelike vectors? If u^α is a timelike vector on H and v^α a spacelike vector pointing out from H , then for small enough λ , $u^\alpha + \lambda v^\alpha$ is still timelike, and it points out of H . Then a timelike curve tangent to $u^\alpha + \lambda v^\alpha$ reaches a point outside H . Because all points outside H are joined by timelike curves to null infinity, H itself would be joined by a timelike curve to null infinity, contradicting its definition.

2. We next show that the generators ℓ^α of the horizon are linear combinations of t^α and ϕ^α . Because $t^\alpha + k\phi^\alpha$ is a Killing vector for every constant k , it must be tangent to the horizon. (Combinations of rotations and time translations map the horizon to itself.) Because H is a null surface, $t^\alpha + k\phi^\alpha$ is either spacelike or null. Suppose that the null generator at a point of H has a part along θ or r : $\ell^\alpha = (t^\alpha + f\phi^\alpha) + v^\alpha$, with v^α orthogonal to the t - ϕ plane. Because the r and θ directions are spacelike, $v_\alpha v^\alpha \geq 0$, vanishing only if $v^\alpha = 0$. But

$$0 = \ell_\alpha \ell^\alpha = (t_\alpha + f\phi_\alpha)(t^\alpha + f\phi^\alpha) + v_\alpha v^\alpha.$$

Because each term on the right is non-negative, $v^\alpha = 0$ and $\ell^\alpha = t^\alpha + f\phi^\alpha$ as claimed.

3. Consider an element of area in a null plane with one segment along the null direction, and one in a perpendicular direction (call them the 1 and 2 directions). Because the null segment has proper length $d\ell_1 = 0$, the area is $dA = d\ell_1 d\ell_2 = 0$. Thus a null plane has ${}^2g = 0$.

4. At each point of the horizon of Kerr, the t - ϕ plane is null, whence Eq. (5.47) implies $\Delta = 0$. $\square$

The argument holds for distorted black holes, black holes with stationary, axisymmetric matter (e.g., an accretion disk) outside the horizon: H is again composed of t - ϕ surfaces for which

$${}^2g = \mathbf{t} \cdot \mathbf{t} \, \phi \cdot \phi - (\mathbf{t} \cdot \phi)^2 = 0.$$

For a Schwarzschild black hole, the asymptotically timelike Killing vector t^α becomes null on the horizon and is therefore its null generator. For Kerr, the vector field $t^\alpha + \omega\phi^\alpha$ is null on the horizon,

because

$$\begin{aligned}
 (t^\alpha + \omega\phi^\alpha)(t_\alpha + \omega\phi_\alpha) &= t^\alpha t_\alpha + 2\omega t^\alpha \phi_\alpha + \omega^2 \phi^\alpha \phi_\alpha \\
 &= (\phi^\beta \phi_\beta)^{-1} [(t^\alpha t_\alpha)(\phi_\beta \phi_\beta) - 2(t^\alpha \phi_\alpha)^2 + (t^\alpha \phi_\alpha)^2] \\
 &= (\phi^\beta \phi_\beta)^{-1} [(t^\alpha t_\alpha)(\phi^\beta \phi_\beta) - (t^\alpha \phi_\alpha)^2] \\
 &= 0 \text{ at } \Delta = 0.
 \end{aligned} \tag{5.54}$$

We have shown that the horizon is at the radius r where $\Delta = 0$ and that it is generated by the null vector field $t^\alpha + \omega_H \phi^\alpha$, where ω_H is the value of the frame-dragging angular velocity ω on H . Now $\Delta \equiv r^2 - 2Mr + a^2 = 0$ has the roots

$$r_\pm = M \pm (M^2 - a^2)^{1/2}. \tag{5.55}$$

The solution $r = r_+$ is the horizon.² At the horizon, we then have

$$\omega_H = - \left. \frac{g_{t\phi}}{g_{\phi\phi}} \right|_{r_+} = \frac{a}{2Mr_+}. \tag{5.56}$$

Because the generators of the horizon rotate relative to infinity with angular velocity ω_H , one regards the horizon itself as an object rotating with the angular velocity ω_H , constant on H .

Within the horizon (for $r < r_+$), there are no nonspacelike constant r paths, and $g_{rr} < 0 \Rightarrow \partial_r$ is a timelike vector; future directed timelike curves are then constrained to have $u^r < 0$. (One can analytically extend Kerr across $\Delta = 0$, as in the Kruskal or Eddington-Finkelstein extensions of Schwarzschild.)

Asymptotically, the metric (5.43) has the form

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 - \frac{4Ma}{r} \sin^2 \theta dt d\phi + \left(1 + \frac{2M}{r} \right) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

This is also the asymptotic form of the metric of a rotating star. In the case of a rotating star, the star's angular momentum is $J = Ma$, and that is defined to be the angular momentum of a black hole as well.³ Note however, that the rest of the geometry of Kerr – the quadrupole and higher multipole moments of the metric – does not agree with the geometry of any rotating star. And rotating stars with different equations of state have different shapes and different gravitational fields (in both GR and in the Newtonian approximation).

The asymptotically timelike Killing vector becomes null on a surface that lies outside the horizon (except at $\theta = 0, \pi$ where it touches the horizon). t^α is spacelike ($t^\alpha t_\alpha > 0$) inside the surface $t^\alpha t_\alpha = 0$:

$$\begin{aligned}
 0 = t^\alpha t_\alpha &= g_{tt} = \frac{\Delta - a^2 \sin^2 \theta}{\rho^2} \implies r^2 - 2Mr + a^2 - a^2 \sin^2 \theta = 0 \\
 r^2 - 2Mr + a^2 \cos^2 \theta &= 0.
 \end{aligned} \tag{5.57}$$

This region, where t^α is spacelike, is called the *ergosphere* and is discussed in some detail below.

The rotational Killing vector is spacelike everywhere (at least for $r > 0$):

$$\phi^\alpha \phi_\alpha = g_{\phi\phi} = \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2} \sin^2 \theta.$$

²The second solution $r = r_-$ is also a null surface, lying inside the horizon and is called the inner horizon. It is not part of the spacetime of a collapsing star.

³One can also define a symmetry group of rotations and translations at spatial infinity and use the symmetries to define the associated asymptotic conserved quantities.

Then

$$\rho^2 > 0, \quad r^2 + a^2 > a^2, \quad r^2 + a^2 > r^2 - 2Mr + a^2 = \Delta \quad \text{and} \quad 1 > \sin^2 \theta$$

$\Rightarrow$

$$\frac{(r^2 + a^2)^2}{\rho^2} > \frac{a^2 \Delta \sin^2 \theta}{\rho^2} \Rightarrow \phi^\alpha \phi_\alpha > 0.$$

[If $\phi^\alpha \phi_\alpha < 0$ as happens if we allow $r < 0$, there are closed timelike curves – namely the curves of constant t, r, θ , the circular trajectories along the vector field ϕ^α .]

When $a > M$, $(M^2 - a^2)^{1/2}$ is imaginary and thus Δ is nowhere zero. The components of the Kerr metric (5.43) are singular at $r = 0$, and this singularity cannot be avoided by extending Kerr via a coordinate change: The scalar $R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$, for example, blows up as $r \rightarrow 0$, ($\theta = \pi/2$). In this case, unlike Schwarzschild, or Kerr with $a < M$, null geodesics from all points outside $r = 0$ escape to infinity, so the singularity is not shrouded by a horizon and is said to be naked. As mentioned above, however, it appears that naked singularities cannot, in fact, form from nonsingular initial data of physical fields, and, in particular, attempts to find processes that take a black hole with $a < M$ and add enough angular momentum to make $a > M$ have failed (see e.g. Wald, *Annals of Physics*, **82**, 548 (1974)). One reason they fail is this: A typical attempt is to drop a gyroscope with $a_{\text{gyro}} \equiv \frac{J_{\text{gyro}}}{M_{\text{gyro}}} \gg M_{\text{gyro}}$ into the $a < M$ black hole. But there is a spin-spin repulsion, and in order to push the gyroscope into the black hole, you have to do so much work that $(W + M_{\text{gyro}} + M)^2 > J_{\text{gyro}} + J$, so the new mass of the hole, $W + M + M_{\text{gyro}}$ exceeds its new angular momentum per unit mass. The statement that naked singularities cannot evolve from nonsingular initial data of physical fields is called the cosmic censorship hypothesis and its proof is a fundamental unsolved problem of classical relativity. We henceforth assume $a < m$.

Particle trajectories: General features

The appearance of accreting black holes depends in part on the trajectories of photons and orbits of massive particles. We'll restrict ourselves here to orbits in the equatorial plane. If, at ∞ , $u^\phi = 0$, then $L = u_\phi = 0$ as well, because the metric is asymptotically Minkowski; but, as we have seen, zero angular momentum particles have nonzero angular velocity

$$\Omega = \omega. \tag{5.58}$$

Because $\omega \sim r^{-3}$ as $r \rightarrow \infty$ this dragging is hard to measure unless one observes objects near a dense, rapidly rotating object – in this case, a Kerr black hole.

Consider an arbitrary timelike or null trajectory:

$$u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha + v^\alpha)$$

where $v^\alpha \perp t^\alpha, \phi^\alpha$. In Minkowski space, a particle can have a timelike or null trajectory only if $r \sin \theta \Omega < 1$, implying $-1/r < \Omega < 1/r$. Here the angular velocity Ω is limited to a range of values that is not symmetric as seen by an observer at infinity. (The range is symmetric as seen locally by an observer with zero angular momentum). From $u_\alpha u^\alpha = -1 < 0$, we have

$$t^\alpha t_\alpha + 2\Omega t^\alpha \phi_\alpha + \Omega^2 \phi^\alpha \phi_\alpha \leq 0.$$

Then $g_{rr} > 0, g_{\theta\theta} > 0 \Rightarrow v^\alpha v_\alpha > 0, \Rightarrow \Omega_- \leq \Omega \leq \Omega_+$, where

$$\Omega_\pm = -\frac{\mathbf{t} \cdot \boldsymbol{\phi}}{\boldsymbol{\phi} \cdot \boldsymbol{\phi}} \pm \left[\left(\frac{\mathbf{t} \cdot \boldsymbol{\phi}}{\boldsymbol{\phi} \cdot \boldsymbol{\phi}} \right)^2 - \frac{\mathbf{t} \cdot \mathbf{t}}{\boldsymbol{\phi} \cdot \boldsymbol{\phi}} \right]^{1/2}$$

or

$$\Omega_{\pm} = \omega \pm \left(\omega^2 - \frac{\mathbf{t} \cdot \mathbf{t}}{\boldsymbol{\phi} \cdot \boldsymbol{\phi}} \right)^{1/2}. \quad (5.59)$$

These extrema are reached when $v^\alpha = 0$, that is, for motion in the equatorial plane at constant r .

For large r ,

$$\omega \sim \frac{2J}{r^3} \quad t^\alpha t_\alpha = g_{tt} \sim -1 \quad \phi^\alpha \phi_\alpha = g_{\phi\phi} \sim r^2,$$

and $\Omega_{\pm} \approx \pm \frac{1}{r}$, as in Minkowski space. As long as $t^\alpha t_\alpha < 0$, $\Omega_- < 0$ and $\Omega_+ > 0$ and particles can rotate with or opposite to the black hole rotation. But when $t^\alpha t_\alpha > 0$, both Ω_+ and Ω_- are greater than zero: *All timelike or null trajectories rotate with the geometry* – no physical particle can remain at rest as seen by an observer at infinity. In other words, when t^α is spacelike, no physical particle can move along this Killing vector. In contrast to the Schwarzschild case, however, particles within the region where t^α is spacelike can escape to infinity because for $\Omega_- < \Omega < \Omega_+$, u^r can be positive.

The region in which t^α is spacelike is called the ergosphere (The term is similar to *atmosphere*; the ergosphere of a dense, rapidly rotating stellar model is a solid torus, not a topological sphere.) The boundary, where $t^\alpha t_\alpha = 0$ is called the static limit by MTW (stationary limit by most other people). Observers inside the static limit can remain at constant distance from the black hole (constant r) as long as they rotate in the positive direction with angular velocity Ω in the range $\Omega_- < \Omega < \Omega_+$. But as $\Delta \rightarrow 0$, $\Omega_- \rightarrow \Omega_+$:

$$\begin{aligned} \Omega_+ - \Omega_- &= \left(\omega^2 - \frac{t^\alpha t_\alpha}{\phi^\beta \phi_\beta} \right)^{1/2} \\ &= (\phi^\alpha \phi_\alpha)^{-1} [(t^\beta \phi_\beta)^2 - (t^\beta t_\beta)(\phi^\gamma \phi_\gamma)]^{1/2}. \end{aligned} \quad (5.60)$$

From eq. (5.47), namely $(t^\beta \phi_\beta)^2 - (t^\beta t_\beta)(\phi^\gamma \phi_\gamma) = \Delta \sin^2 \theta$,

$$\Omega_+ - \Omega_- \rightarrow 0 \text{ as } \Delta \rightarrow 0.$$

So the allowed range of rotation shrinks to zero as $\Delta \rightarrow 0$, and for $\Delta = 0$ the only nonspacelike constant r path is the null trajectory along the generator of the horizon, with $\Omega = \Omega_+ = \Omega_- = \omega_H$.

5.6 Ergospheres: The Penrose process, superradiant scattering, and ergosphere instabilities

Ergospheres are regions in which an asymptotically timelike Killing vector t^α becomes spacelike. They are present not only in rotating black holes, but also in models of dense, rapidly rotating stars (perfect fluids) which have been constructed numerically by Ipser & Butterworth[19] (earlier, for dust disks, by Bardeen & Wagoner). If a star is compact enough, and it spins fast enough, its rotation of a star can force all physical particles within a region to rotate with respect to a distant observer. For later references, see Tsokaros et al. [94], who look at the rotation laws and equations of state that give rise to these *ergostars*.

It is possible to extract rotational energy from a geometry with an ergosphere in the following way. A particle (1) sent from infinity to the ergosphere can decay into particles (2) and (3), leaving (3) in the ergosphere and sending (2) back out to ∞ with greater energy than (1) had to begin with. Let particle

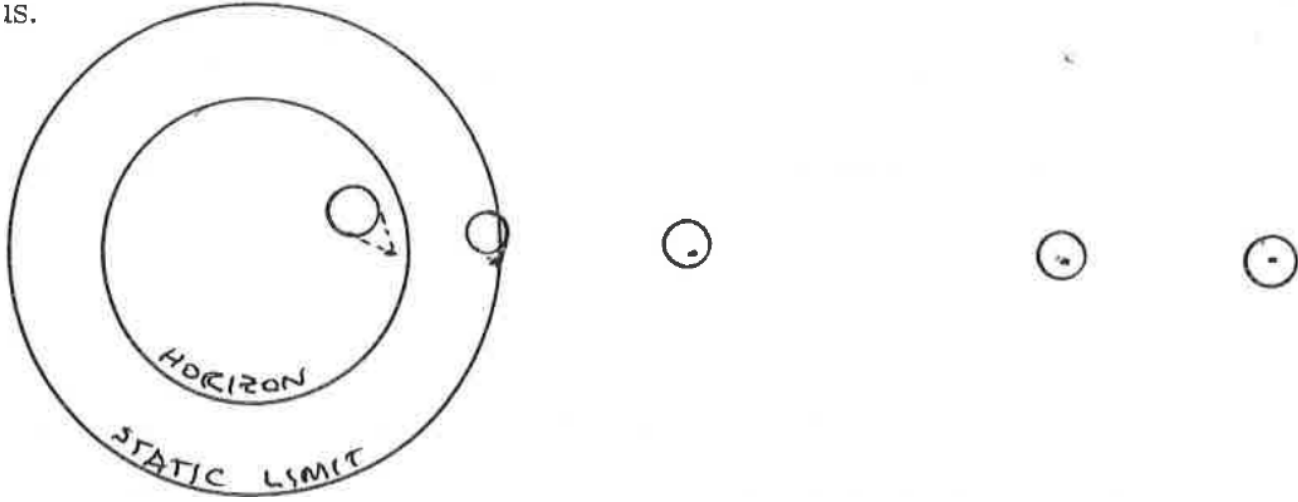


Figure 5.4: A $t = \text{const}$, $\theta = \frac{\pi}{2}$ plane showing the horizon, static limit, and ergosphere. Light cones are represented by dots for their vertices and circles for the cone - a circle represents the position of flash of light a short time after it was emitted from the dot. Picture the light cone by imagining time out of the paper, so the circles sit just above the paper.

(1) begin with momentum p_1^α . Along its trajectory, $-p_{1\alpha}t^\alpha = E_1$ is constant; E_1 is its energy measured by an observer at infinity who sends the particle in (at infinity, t^α is a unit timelike vector and can be chosen as the velocity of our observer).

Inside the ergosphere, t^α is spacelike, so $-p_{1\alpha}t^\alpha$ is not the energy measured by any observer. In particular, consider a ZAMO, an observer with velocity $u^\alpha = u^t(t^\alpha + \omega\phi^\alpha)$. Let (1) decay into (2) + (3). The observer u^α must see $u^\alpha p_{2\alpha} < 0$ and $u^\alpha p_{3\alpha} < 0$ because the locally measured energy is positive:

$$0 > u^\alpha p_{3\alpha} = u^t(t^\alpha p_{3\alpha} + \omega\phi^\alpha p_{3\alpha}) = u^t(t^\alpha p_{3\alpha} + \omega L),$$

L the angular momentum of (3).

Since t^α is spacelike, $t^\alpha p_{3\alpha}$ can be positive. If we choose $L < 0$, the momentum $t^\alpha p_{3\alpha}$ need only satisfy $t^\alpha p_{3\alpha} < |\omega L|$; because all decays can occur that satisfy conservation of 4-momentum, we can choose a decay for which $t^\alpha p_{3\alpha} > 0$.

Momentum conservation in the decay is

$$p_{1\alpha} = p_{2\alpha} + p_{3\alpha}.$$

Its component along t^α ,

$$p_{1\alpha}t^\alpha = p_{2\alpha}t^\alpha + p_{3\alpha}t^\alpha,$$

implies

$$E_2 = -p_{2\alpha}t^\alpha = -p_{1\alpha}t^\alpha + p_{3\alpha}t^\alpha > E_1,$$

and particle (2) reaches infinity with energy greater than that particle (1) had. Note that particle (3) cannot escape to infinity, or even to a region where t^α is timelike because then the physical energy of (3) measured by an observer with velocity along t^α would be negative. In the case of an ergosphere about a Kerr black hole, particle (3) would spiral into the black hole. When a geometry has an ergosphere and no horizon, however, a particle with $L < 0$ (or a field in the ergosphere with negative “energy”) will radiate and its “energy,” $-p_\alpha t^\alpha$, will grow increasingly negative until ultimately the enough angular momentum is radiated to ∞ that no ergosphere remains. This growth of a particle’s negative energy,

defined as $-p_\alpha t^\alpha$, is a simple example of the instability of an ergostar, described below.

Superradiant scattering and ergosphere instabilities.

The counterpart of the Penrose process for a classical field is called superradiant scattering. A scalar, electromagnetic or linearized gravitational field can radiate negative energy across the horizon, here meaning that the current $J^\alpha = -T^\alpha_\beta t^\beta$ has negative flux across the horizon, that $\int_H J^\alpha dS_\alpha < 0$. For a massless scalar field, an electromagnetic field or a gravitational wave, this implies that the outgoing field carries more energy to future null infinity than the energy of the initial incoming field.

In the case of an ergostar (with no horizon), the conserved energy of radiation fields on a background with negative values in the ergosphere leads to instability Friedman '78[37]: Because the negative energy part of the field is again trapped within the ergosphere, and because a time-dependent, nonaxisymmetric field radiates positive energy to infinity, fields of this kind are unstable in spacetimes with an ergosphere and no horizon. As linear fields on a background spacetime, they grow without bound; in the full theory, they radiate away the angular momentum of the ergostar until no ergosphere remains. The associated growth time, however, is comparable to the age of the universe [30].

There is a related instability of rotating black holes, associated in this case with a *massive* scalar field (or any massive boson field) outside the ergosphere. Because a massive scalar field falls off exponentially at spatial and null infinity, scalar radiation flows only into the horizon, not to infinity. If the field is superradiant, radiating negative energy across the horizon, the field outside the horizon again grows without bound on a fixed background spacetime (Damour '76)[33], Zouros & Eardley 79[101], Detweiler '80[35].

With spacetime dynamics included, the black hole spins down and is surrounded by a condensate of the massive boson field. For very small scalar masses, whose Compton wavelength is larger than the black hole's radius, the growth time is small compared to the age of the universe. As pointed out by Arvanitaki and Dubovsky[8], this means that any observation of the mass and spin of a black hole can set an upper bound on the mass of light bosons. Because ultralight bosons (axions, for example) are a candidate for dark matter, a ballooning literature has followed the Arvanitaki-Dubovsky paper.

Exercise 78. An observer is stationed at fixed values of r and ϕ in the equatorial plane of a rotating black hole of mass M and angular momentum Ma . A proton is moving in the ϕ -direction as it traverses the laboratory of the observer, who measures its energy E and momentum $p_\perp$.

- What is the relation between E and $p_\perp$?
- What are the components of the proton's 4-momentum in the Kerr coordinate basis in terms of E and $p_\perp$?
- What is the proton's speed v as measured by the observer? (Give v in terms of E , $p_\perp$, and rest mass m ?)
- What are the components of the proton's 4-velocity along an orthonormal frame aligned with the coordinate directions in terms of v ?
- What are the values of the two conserved quantities $E = -u_\alpha t^\alpha$ and $L = u_\alpha \phi^\alpha$?

5.7 Orbits in Kerr

This section partly follows corresponding discussions in Chandrasekhar [24] and in Shapiro-Teukolsky[76] 12.7.

5.7.1 Photon orbits

We will look at orbits in the equatorial plane. Massive particles have orbits through any point for each value of the particle's speed. Photon orbits have only one speed and so only one orbit for a given spatial direction. Where massive particles can adjust their speed to have circular orbits for all radii greater than some minimum value, photons have only a single circular orbit outside the horizon.

As in Schwarzschild, we write $k^\alpha k_\alpha = 0$ in terms of $\dot{r}$ and the conserved energy/ $\hbar$ and angular momentum/ $\hbar$,

$$E = -k_\alpha t^\alpha = -k_t, \quad L = k_\alpha \phi^\alpha = k_\phi. \quad (5.61)$$

At $r = \infty$, a stationary observer has 4-velocity t^α , so $-\hbar k_t$ and $\hbar k_\phi$ are the energy and angular momentum of a photon, if the affine parameter is chosen to equal t at infinity. The orbit of a photon moving in the equatorial plane is determined by its impact parameter⁴

$$b := \frac{L}{E}.$$

The equatorial plane is the $\theta = \pi/2$ surface, with

$$\sin \theta = 1, \quad \rho^2 = r^2.$$

We have

$$\begin{aligned} 0 &= g^{tt} E^2 - 2g^{t\phi} EL + g^{\phi\phi} L^2 + g_{rr} \dot{r}^2 \\ &= -\frac{(r^2 + a^2)^2 - a^2 \Delta}{r^2 \Delta} E^2 + 4\frac{aM}{\Delta r} EL + \frac{\Delta - a^2}{r^2 \Delta} L^2 + \frac{r^2 \dot{r}^2}{\Delta} \\ r^2 \dot{r}^2 &= \left(r^2 + a^2 + \frac{2M}{r} a^2 \right) E^2 - 4\frac{aM}{r} EL - \left(1 - \frac{2M}{r} \right) L^2. \end{aligned} \quad (5.62)$$

Dividing by r^2 and grouping terms with the same power of r gives

$$\dot{r}^2 = E^2 + \frac{2M}{r^3} (L - aE)^2 - \frac{L^2 - a^2 E^2}{r^2}, \quad (5.63)$$

the equation of a particle moving in an effective potential $U = -\frac{2M}{r^3} (L - aE)^2 + \frac{L^2 - a^2 E^2}{r^2}$.

⁴The term is accurate because L/E is the impact parameter in flat space, the spacetime is asymptotically flat, and $\hbar L$ and $\hbar E$ are the energy and angular momentum measured by a stationary observer at infinity. So it is a natural definition of the impact parameter in our curved spacetime. Note that changing E and L to kE and kL in (5.62) changes $\dot{r} = dr/d\lambda$ to $k\dot{r}$: This corresponds to a change $\lambda \rightarrow \lambda/k$ in the affine parameter, and it leaves $dr/dt = \dot{r}/\dot{t}$ unchanged. Adopting our convention that $\dot{t} = 1$ at infinity fixes $\dot{r}$.

Ingoing and outgoing photons: Principal null geodesics

Eq. (5.63) has its simplest form when $L = aE$:

$$\dot{r} \equiv \frac{dr}{d\lambda} = \pm E. \quad (5.64)$$

This means r is an affine parameter, as it was for Schwarzschild. Without loss of generality, we can take $\lambda = \pm r$, $E = 1$.

To complete the solution, we need $\dot{\phi} = k^\phi$ and $\dot{t} = k^t$ as functions of r :

$$\begin{aligned} \dot{\phi} = k^\phi &= g^{\phi\phi}k_\phi + g^{\phi t}k_t = \frac{\Delta - a^2 \sin^2 \theta}{\rho^2 \Delta \sin^2 \theta} L + 2 \frac{aMr}{\rho^2 \Delta} \sin^2 \theta E \\ &= \frac{1}{\Delta} \left[\left(1 - \frac{2M}{r} \right) L + a \frac{2M}{r} E \right], \quad \text{at } \theta = \pi/2, \end{aligned} \quad (5.65)$$

$$\dot{t} = k^t = \frac{1}{\Delta} \left[\left(r^2 + a^2 + 2 \frac{a^2 M}{r} \right) E - a \frac{2M}{r} L \right], \quad (5.66)$$

Here, with $L = aE = a$, the relations are simply

$$\dot{\phi} = \frac{a}{\Delta}, \quad \dot{t} = \frac{r^2 + a^2}{\Delta}. \quad (5.67)$$

Eqs. (5.67) now give the coordinates t and ϕ of outgoing and ingoing geodesics as functions of r : Up to a choice of initial values, we have

$$\phi = \pm \int \frac{a}{\Delta} dr = \pm \frac{a}{r_+ - r_-} \ln \left| \frac{r - r_+}{r - r_-} \right| \quad (5.68a)$$

$$t = \pm \int \frac{r^2 + a^2}{\Delta} dr = \pm r_*, \quad \text{with } r_* = r + \frac{Mr_+}{r_+ - r_-} \ln \left| \frac{r - r_+}{r_+} \right| - \frac{Mr_-}{r_+ - r_-} \ln \left| \frac{r - r_-}{r_-} \right|. \quad (5.68b)$$

The corresponding null vectors tangent to the outgoing and ingoing geodesics then have t, r, θ, ϕ components

$$(\ell^\mu) = \left(\frac{r^2 + a^2}{\Delta}, 1, \frac{a}{\Delta}, 0 \right), \quad (5.69a)$$

$$(n^\mu) = \left(\frac{r^2 + a^2}{\Delta}, -1, \frac{a}{\Delta}, 0 \right). \quad (5.69b)$$

These vectors will reappear in Sect. 8.2.3 (with $\mathbf{n}$ rescaled to make $\ell \cdot \mathbf{n} = 1$) as two of the Newman-Penrose tetrad vectors for the Kerr geometry; and the ingoing null geodesics will, as in Schwarzschild, be used in Sect. 5.8 to construct a chart regular on the future horizon.

Although we have found these geodesics by assuming $\theta = \pi/2$, the equations hold for any value of θ : The ingoing and outgoing solutions describe null geodesics with $\dot{\theta} = 0$.

Circular orbits

For a circular orbit, $\dot{r} = 0$, implying $E^2 = U$, and $dU/dr = 0$:

$$E^2 + \frac{2M}{r^3}(L - aE)^2 - \frac{L^2 - a^2E^2}{r^2} = 0, \quad (5.70)$$

$$-\frac{6M}{r^4}(L - aE)^2 + 2\frac{L^2 - a^2E^2}{r^3} = 0. \quad (5.71)$$

These are two equations relating r , E and L ; notice, however, that they are homogeneous in (E, L) , that every term is proportional to E^2 , L^2 or EL . Dividing each equation by L^2 then gives two equations for the two variables r and $b = L/E$ and so determines the values of each, giving r and the angular velocity of the orbit. The second equation immediately gives

$$r = 3M \frac{L - aE}{L + aE} = 3M \frac{b - a}{b + a}. \quad (5.72)$$

Substituting this value of r in the first equation (5.70) yields for b the cubic equation

$$1 = U/E^2 = \frac{1}{27M^2} \frac{(b + a)^3}{b - a} \quad (5.73)$$

To solve, we simplify as usual by appropriately picking dimensionless variables

$$y := \frac{b + a}{3M}, \quad \chi = \frac{a}{M}. \quad (5.74)$$

Then $\frac{b - a}{M} = 3y - 2\chi$, and the equation is surprisingly simple:

$$y^3 - 3y + 2\chi = 0, \quad (5.75)$$

with a real solution that depends on whether the orbit is corotating or counter-rotating with the black hole. (See the discussion of the solution to a cubic after Eq. (4.81)). Assuming $\chi = a/M > 0$, the corotating orbit is the solution given by

$$y = 2 \cos \left[\frac{1}{3} \cos^{-1}(-\chi) \right] =: 2 \cos \psi.$$

and, from (5.74) and (5.72), respectively,

$$b = 6M \cos \psi - a, \quad r = 3M(1 - 2\chi/y) = M \left(3 - \frac{\chi}{\cos \psi} \right). \quad (5.76)$$

As χ increases from 0 to 1 (as a increases from 0 to M), ψ increases from $\pi/6$ to $\pi/3$ and r decreases from

$$r = 3M \text{ at } \chi = 0, \quad \text{to} \quad r = M \text{ at } \chi = 1. \quad (5.77)$$

(When $\chi = 1$, the horizon itself has shrunk to $r = M$.)

Counter-rotating orbits have Ω and a with opposite signs. We can just switch the sign of a (of χ) to write

$$r = M \left(3 + \frac{|\chi|}{\cos \psi} \right), \quad \cos 3\psi = |\chi| \quad (\text{counter-rotating}), \quad (5.78)$$

with r increasing from $3M$ to $4M$ as $|\chi|$ increases from 0 to 1.

Eqs. (5.65) and (5.66) again give Ω in terms of b and r ,

$$\Omega = \frac{(1 - 2M/r)b + 2aM/r}{(r^2 + a^2 + 2a^2M/r) - (2aM/r)b}. \quad (5.79)$$

5.7.2 Particle orbits

Finding the circular orbits in the equatorial plane is done in essentially the same way as for photons, and the basic equations are easy to write down, just changing $k^\alpha k_\alpha = 0$ to $u^\alpha u_\alpha = -1$: Again multiplying by Δ/r^2 gives Eq. (5.63) with the additional term $-\Delta/r^2$ on the right side:

$$\dot{r}^2 = E^2 - U, \quad U = -\frac{2M}{r^3}(L - aE)^2 + \frac{L^2 - a^2E^2}{r^2} + \frac{\Delta}{r^2}. \quad (5.80)$$

The conditions for a circular orbit are again $\dot{r} = 0$, $\ddot{r} = 0$, or $E^2 - U = 0$, $dU/dr = 0$. Again grouping powers of r , we write the first relation as

$$E^2 - 1 + \frac{2M}{r^3}(L - aE)^2 - \frac{L^2 - a^2(E^2 - 1)}{r^2} + \frac{2M}{r} = 0. \quad (5.81)$$

Because $E^2 - U = 0$, we can write the second relation, $U' = 0$, in the form $\frac{1}{2r}[r^2(E^2 - U)]' = 0$:

$$E^2 - 1 - \frac{M}{r^3}(L - aE)^2 + \frac{M}{r} = 0. \quad (5.82)$$

Our goal is to use the two equations to find the generalized Kepler relation $\Omega(r)$ and to find the innermost stable circular orbit.

We have two equations for the three variables r , E and L . The strategy is to solve them for E and L in terms of r , to write Ω in terms of E and L , and so to obtain $\Omega(r)$. First the suprisingly simple result, then the calculation. The Newtonian Keplerian angular velocity is $\Omega_N = \sqrt{\frac{M}{r^3}}$, and we have seen the form of Ω is identical for Schwarzschild. For Kerr it has the form

$$\begin{aligned} \Omega &= \frac{\Omega_N}{1 + a\Omega_N}, & \text{corotating,} \\ \Omega &= -\frac{\Omega_N}{1 - a\Omega_N}, & \text{counter-rotating.} \end{aligned} \quad (5.83)$$

At order a , the change is due to frame dragging because the order a change in the metric is the nonzero value of $g_{t\phi}$. From Eq. (5.53), $\omega = 2a\Omega_N^2 + O(a^2)$, implying $\Omega = \pm(\Omega_N \mp \omega/2) + O(a^2)$.

The calculation:

This time use the variable⁵ $x = L - aE$ and write the equations in terms of x , E and r , using $L^2 - a^2E^2 = x^2 + 2aEx$ in the first equation. The equations are now

$$\begin{aligned} E^2 &= -\frac{2M}{r^3}x^2 + \frac{x^2 + 2aEx}{r^2} + \frac{\Delta}{r^2} \\ E^2 &= \frac{Mx^2}{r^3} - \frac{M}{r} + 1. \end{aligned} \quad (5.84)$$

We now eliminate E to get an equation for x and r alone and then solve for x in terms of r . First get rid of E^2 by subtracting the first equation from the second. This lets us find E in terms of x and r without having to solve a quadratic equation:

$$2axE = -\left(1 - \frac{3M}{r}\right)x^2 + Mr - a^2 \quad (5.85)$$

⁵If you're going to try the calculation, you might want to set $M = 1$ or divide variables by M to make everything dimensionless. On the other hand, keeping M has the advantage that you can spot errors by checking that every term has the right dimension as a power of $L = [M]$.

We now have independent expressions (5.85) for E and (5.84) for E^2 . So we get an equation for x and r alone by

[right side of (5.85)]² = [right side of (5.84)] $\times(2ax)^2$: Notice that, although it's quartic in x , it's quadratic in x^2 ! So we're almost done. The equation has the form

$$\alpha x^4 - 2\beta x^2 + \gamma = 0, \quad (5.86)$$

with

$$\begin{aligned} \alpha &= \left(1 - \frac{3M}{r}\right)^2 - 4a^2 \frac{M}{r^3} = (1 - 3M/r - 2a\Omega_N)(1 - 3M/r + 2a\Omega_N), \\ \beta &= \left(1 - \frac{3M}{r}\right)(Mr - a^2) + 2a^2 \left(1 - \frac{M}{r}\right), \\ \gamma &= (Mr - a^2)^2, \end{aligned} \quad (5.87)$$

and the next happy fact is that, although α and β are somewhat long, the discriminant is short

$$\beta^2 - \alpha\gamma = 4a^2\Delta^2 M/r^3 = (2a\Delta\Omega_N)^2. \quad (5.88)$$

(This is one line in Mathematica, using `Factor[]`. By hand it took a page.) The solution is then ⁶

$$x = \frac{r^2\Omega_N - a}{\sqrt{1 - 3M/r + 2a\Omega_N}}. \quad (5.89)$$

Everything has this same denominator, and it just gets carried along, so we'll call it Q :

$$Q = \sqrt{1 - 3M/r + 2a\Omega_N}. \quad (5.90)$$

Eq. (5.84) gives

$$E = \frac{1 - 2M/r + a\Omega_N}{Q}, \quad (5.91)$$

and we have

$$L = x + aE = \frac{(r^2 + a^2)\Omega_N - 2aM/r}{Q}. \quad (5.92)$$

Finally, we find $\Omega(r)$ from Eq. (5.65) and (5.66) for $\dot{\phi}$ and $\dot{t}$. This is quick:

$$\begin{aligned} \dot{\phi} &= \frac{\Omega_N}{Q}, \\ \dot{t} &= \frac{1 + a\Omega_N}{Q}, \end{aligned} \quad (5.93)$$

whence

$$\Omega = \frac{\Omega_N}{1 + a\Omega_N}, \quad (5.94)$$

as claimed. This is the solution for a corotating orbit, with $a > 0$. For a counter-rotating (retrograde) orbit, change a to $-a$ with the convention that $\Omega > 0$, or replace a by $-a$ and change the sign of Ω for the standard convention that $a > 0$ and $\Omega < 0$ for counter-rotating orbits, $\Omega = -\Omega_N/(1 - a\Omega_N) < 0$.

⁶Intermediate steps are $x^2 = (\beta - 2a\Delta\Omega_N)/\alpha$, $\beta - 2a\Delta\Omega_N = (1 - 3M/r - 2a\Omega_N)(r^2\Omega_N - a)^2$.

The condition for stability of the circular orbit, $U'' > 0$ can be written as $[r^3(E^2 - U)]'' = 0$, because $E^2 - U = 0$ and $(E^2 - U)' = -U' = 0$. From Eq. (5.80), we have at the ISCO (innermost stable circular orbit),

$$E^2 = 1 - \frac{2}{3} \frac{M}{r}. \quad (5.95)$$

The radius of the ISCO decreases from $3M$ to M as a increases from 0 to M for corotating orbits, and it increases from $3M$ to $9M$ for counter-rotating orbits. Its explicit form is given by Bardeen, Press and Teukolsky 1972:

$$\begin{aligned} r_{\text{ISCO}} &= M \left\{ 3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2} \right\}, \\ Z_1 &= 1 + \left(1 - \frac{a^2}{M^2} \right)^{1/2} \left[\left(1 + \frac{a}{M} \right)^{1/3} + \left(1 - \frac{a}{M} \right)^{1/3} \right], \\ Z_2 &= \left(3 \frac{a^2}{M^2} + Z_1^2 \right)^{1/2}. \end{aligned}$$

5.8 Kerr coordinates: Coordinates regular on the future (or past) horizon

Kerr coordinates are the analog for rotating black holes of the Eddington-Finkelstein coordinates for spherical black holes (Sect. 5.2). As in Eq. (5.12), we introduce a null coordinate v for which the $v = \text{constant}$ surfaces are null surfaces generated by ingoing null geodesics. From Eqs. (5.69b), we see that along the principal ingoing null geodesics of Kerr, $d\phi + \frac{a}{\Delta} dr = 0$ and $dt + \frac{r^2 + a^2}{\Delta} dr = 0$. Then, if we define v and a new angular coordinate $\tilde{\phi}$ by

$$d\tilde{\phi} = d\phi + \frac{a}{\Delta} dr, \quad (5.96a)$$

$$dv = dt + dr_* = dt + \frac{r^2 + a^2}{\Delta} dr, \quad (5.96b)$$

$\tilde{\phi}$ and v will be constant along the geodesics. Replacing $d\phi$ and dt in the factored form (5.45) of the metric by $d\tilde{\phi} - \frac{a}{\Delta} dr$ and $dv - \frac{r^2 + a^2}{\Delta} dr$ gives

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^2} \left[(dv - \frac{\rho^2}{\Delta} dr - a \sin^2 \theta d\tilde{\phi})^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a dv]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \right] \\ &= -\frac{\Delta}{\rho^2} (dv - a \sin^2 \theta d\tilde{\phi})^2 + 2dv dr + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a dv]^2 - 2a \sin^2 \theta dr d\tilde{\phi} + \rho^2 d\theta^2, \end{aligned} \quad (5.97)$$

smooth at the horizon where $\Delta = 0$. With a set to zero, the metric in Kerr coordinates is the Eddington-Finkelstein form (5.13). By construction, the surfaces of constant v are null, and v and $\tilde{\phi}$ are constant along the ingoing null geodesics. In the $(v, r, \tilde{\phi})$ chart, the ingoing null vector n^α is then simply

$$n = -\partial_r. \quad (5.98)$$

Direct check:

For the change of coordinates $(v, r, \tilde{\phi}) \rightarrow (t, r, \phi)$

$$\frac{\partial t}{\partial r} = -\frac{r^2 + a^2}{\Delta}, \quad \frac{\partial r}{\partial r} = 1, \quad \frac{\partial \phi}{\partial r} = -\frac{a}{\Delta},$$

implying

$$-\partial_r(\text{with } v, \tilde{\phi} \text{ fixed}) = \frac{r^2 + a^2}{\Delta} \partial_t - \partial_r(\text{with } t, \phi \text{ fixed}) + \frac{a}{\Delta} \partial_\phi. \quad (5.99)$$

For treatments of the analytically extended Kerr solution, analogous to but more complex than the full Schwarzschild solution in a Kruskal-Szekeres chart, see, for example, Hawking and Ellis[47], Wald[95], or Poisson [64].

Chapter 6

Linearized gravity and gravitational waves

Some recent treatments of gravitational waves:

Jolien Creighton & Warren Anderson, Gravitational-Wave Physics and Astronomy,[31]

Michele Maggiore, Gravitational Waves [56]

Thorne&Blandford [92]Chap. 27

Schutz [74], Chap. 9

6.1 Linearized field equations

6.1.1 Preview

Before doing any calculation, we can quickly estimate the maximum amplitude of a gravitational wave from the inspiral of two black holes: When black holes of mass M are about to merge, the fractional change in length near the black holes is of order 1: That is, at a distance of order M from the black holes, a length of order M changes by order M . Because the amplitude of a gravitational wave is proportional to $1/d$, with d the distance to the source, and that amplitude is the magnitude of the metric perturbation, the maximum fractional change in length at the observer is of order M/d . Using $M_{\odot} = 1.5$ km, $d = 100$ Mpc $= 3 \times 10^{21}$ km, we have

$$\frac{M}{d} \approx \frac{1}{2} \times 10^{-21} \frac{M}{M_{\odot}} \frac{100 \text{ Mpc}}{d}. \quad (6.1)$$

Black hole masses in detected inspirals (LIGO graveyard) are larger than this and the distances to them have been greater than 100 Mpc. A detailed estimate of the amplitude of gravitational waves from GW170817, the closest observed NS-NS event as of 2022, at 30 Mpc, is *Exercise 84*.

With the exception of spacetime in or near black holes and neutron stars, the geometry of the universe is nearly flat. We can write the metric in the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad (6.2)$$

where the components $h_{\mu\nu}$ are small compared to 1 in a chart for which $(\eta_{\mu\nu})$ is $\text{diag}(-1,1,1,1)$. It is the magnitude of this metric perturbation at the observer that we just estimated in Eq. (6.1) and thus the magnitude of the corresponding change in distance between the test masses of a gravitational-wave detector (see Eq. (6.63)).

Terms quadratic in $h_{\alpha\beta}$ can be neglected compared to linear terms, and we similarly assume that terms quadratic in $\partial_{\lambda}h_{\mu\nu}$ are small compared to $\partial_{\sigma}\partial_{\tau}h_{\mu\nu}$. In the Einstein equation, we will then ignore

terms of quadratic and higher orders in $h_{\alpha\beta}$ and its derivatives. The inverse metric to linear order in $h_{\alpha\beta}$ is

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} \quad (6.3)$$

where indices are raised using $\eta^{\alpha\beta}$. We will denote by ∂_α the flat covariant derivative operator, the covariant derivative associated with $\eta_{\alpha\beta}$, and we denote by $R_{\alpha\beta\gamma\delta}^{(1)}$ the Riemann tensor at linear order in $h_{\alpha\beta}$.

The linearized Riemann tensor $R_{\alpha\beta\gamma\delta}^{(1)}$ has the form we found in Eq. (5.36) by discarding terms quadratic in derivatives of $h_{\alpha\beta}$,

$$R_{\alpha\gamma\beta\delta}^{(1)} = \frac{1}{2}(\partial_\beta\partial_\gamma h_{\alpha\delta} + \partial_\alpha\partial_\delta h_{\beta\gamma} - \partial_\alpha\partial_\beta h_{\gamma\delta} - \partial_\gamma\partial_\delta h_{\alpha\beta}), \quad (6.4)$$

The corresponding linearized Ricci tensor is then

$$R_{\alpha\beta}^{(1)} = R_{\alpha\gamma\beta}^{(1)\gamma} = \frac{1}{2}(-\partial_\gamma\partial^\gamma h_{\alpha\beta} + \partial_\beta\partial^\gamma h_{\alpha\gamma} + \partial_\alpha\partial^\gamma h_{\beta\gamma} - \partial_\alpha\partial_\beta h), \quad \text{where } h := h_\alpha{}^\alpha, \quad (6.5)$$

and the resulting $G_{\alpha\beta}^{(1)}$ looks complicated. It can be greatly simplified by a choice of *gauge*. Preview: An infinitesimal diffeo (or an infinitesimal coordinate transformation) gives a physically equivalent $h_{\alpha\beta}$ satisfying an analogue of the Lorenz gauge for electromagnetism: Outside the matter, we can choose $\partial^\gamma h_{\alpha\gamma} = 0$, $h = 0$. Then $R^{(1)} = 0$, and we immediately get the wave equation,

$$G_{\alpha\beta}^{(1)} = R_{\alpha\beta}^{(1)} = -\frac{1}{2}\square h_{\alpha\beta}. \quad (6.6)$$

Now a diffeo has only four components ψ^μ , and when matter is present, we can only satisfy four of the five conditions $\partial^\nu h_{\mu\nu} = 0$, $h = 0$. Inside the matter, one instead imposes the 4-component condition $\partial^\beta \bar{h}_{\alpha\beta} = 0$ with $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h$. Again one obtains a wave equation, this time for $\bar{h}_{\alpha\beta}$:

$$\square \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}.$$

Here are the details.

6.1.2 Gauge transformations

A less abstract treatment using coordinate transformations is given on p. 231.

We describe the universe and its inhabitants by a set of tensors (and spinors, which we'll ignore). The geometry and physics do not depend on the choice of coordinates we choose, and this statement is equivalent to saying that the geometry and physics are unchanged by a diffeomorphism ψ that acts on all the tensors. Let's start more familiarly with rotations of ordinary 3-space, with each point P a vector from the origin: Rotating the space by a rotation R and simultaneously rotating the coordinates leaves the components of the vector unchanged. That is, if $R : M \rightarrow M$ is a rotation and $x : M \rightarrow \mathbb{R}^3$ is a coordinate system, the coordinates $x(P)$ are the same as the coordinates $\bar{x}$ of the point $R(P)$ if $\bar{x}[R(P)] = x(P)$ or $\bar{x} = x \circ R^{-1}$.

Here is a more formal way of saying the same thing using our notation for dragging a vector by a map, in this case by the coordinate map $x : M \rightarrow \mathbb{R}^3$. The components of a vector v are just $x^*v = (v^1, v^2, v^3)$. The rotated vector is R^*v and its components in the chart $\bar{x}$ are $\bar{x}^*(R^*v) = (x \circ R^{-1})^*R^*v = (x \circ R^{-1} \circ R)^*v = x^*v$.

Written in this way, the statement can be immediately generalized to any tensor field and any diffeo ψ : The coordinates $x(P)$ of a point P are the same as the coordinates $\bar{x}$ of the point $\psi(P)$ if $\bar{x} = x \circ \psi^{-1}$.

Then the components of a tensor field T in the chart $x : M \rightarrow \mathbb{R}^n$ are the same as the components of the dragged along tensor ψ^*T in the dragged along chart $x \circ \psi^{-1}$:

$$\bar{x}^*(\psi^*T) = (x \circ \psi^{-1})^*\psi^*T = (x \circ \psi^{-1} \circ \psi)^*T = x^*T \quad (6.7)$$

Now to linear order in the vector field ξ^a generating a family of diffeos, the change in a tensor T is its Lie derivative: Let ψ_λ be generated by ξ^a . We have

$$\psi_\lambda T_{\dots} = \psi_0 T_{\dots} + \lambda \frac{d}{d\lambda} \psi_\lambda^* T_{\dots} |_{\lambda=0} + O(\lambda^2) = T_{\dots} - \lambda \mathcal{L}_\xi T_{\dots} + O(\lambda^2). \quad (6.8)$$

Let's change notation from $\lambda \xi^a$ to $-\xi^a$; then ξ^a itself is small, and we get a plus sign for the Lie derivative. If $h_{\alpha\beta}$ is a perturbation of a metric $g_{\alpha\beta}$, then the metric perturbation $h_{\alpha\beta}$ is physically equivalent to

$$h_{\alpha\beta} + \mathcal{L}_\xi g_{\alpha\beta} = h_{\alpha\beta} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha. \quad (6.9)$$

And if $(h_{\alpha\beta}, T_{\alpha\beta}^{(1)})$ is a solution to the linearized field equations

$$G_{\alpha\beta}^{(1)} = 8\pi T_{\alpha\beta}^{(1)},$$

then a physically equivalent solution is

$$h_{\alpha\beta} + \mathcal{L}_\xi g_{\alpha\beta} \quad T_{\alpha\beta}^{(1)} + \mathcal{L}_\xi T_{\alpha\beta}, \quad (6.10)$$

where ξ^α is any vector field. In our case, the unperturbed metric is a flat vacuum spacetime. Because the background (i.e., unperturbed) $T^{\alpha\beta}$ is zero for us, $T_{\alpha\beta}^{(1)}$ is *gauge invariant*. Similarly, because the background Riemann tensor vanishes, $R_{\alpha\beta\gamma\delta}^{(1)}$ is gauge invariant.

Finally, we can choose a gauge to simplify the field equation. It will be analogous to the Lorenz gauge for electromagnetism, $\partial_\beta A^\beta = 0$. Define

$$\bar{h}_{\alpha\beta} := h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h.$$

Claim: One can always pick a gauge in which

$$\partial_\beta \bar{h}^{\alpha\beta} = 0. \quad (6.11)$$

This is called the deDonder (or transverse or harmonic or Lorenz) gauge.

Proof: Under a gauge transformation generated by ξ^α ,

$$\begin{aligned} \bar{h}_{\alpha\beta} &= \bar{h}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \bar{h} \rightarrow \bar{h}_{\alpha\beta} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - \frac{1}{2} g_{\alpha\beta} 2\partial_\gamma \xi^\gamma \\ &= \bar{h}_{\alpha\beta} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - g_{\alpha\beta} \partial_\gamma \xi^\gamma \end{aligned} \quad (6.12)$$

$$\begin{aligned} \partial_\beta \bar{h}_\alpha{}^\beta &\rightarrow \partial_\beta \bar{h}_\alpha{}^\beta + \partial_\alpha \partial_\beta \xi^\beta + \partial_\beta \partial^\beta \xi_\alpha - \partial_\alpha \partial_\gamma \xi^\gamma \\ &= \partial_\beta \bar{h}_\alpha{}^\beta + \partial_\beta \partial^\beta \xi_\alpha \end{aligned} \quad (6.13)$$

Thus by choosing as ξ_α a solution to the wave equation with source $-\partial_\beta \bar{h}_\alpha{}^\beta$,

$$\square \xi_\alpha \equiv \partial_\beta \partial^\beta \xi_\alpha = -\partial_\beta \bar{h}_\alpha{}^\beta, \quad (6.14)$$

one obtains a new gauge in which

$$\partial_\beta \bar{h}_\alpha{}^\beta = 0.$$

In this deDonder gauge, $G_{\alpha\beta}^{(1)}$ is simple: From Eq. (6.5), with $\partial^\gamma h_{\alpha\gamma} = \frac{1}{2}\partial_\alpha h$, we have

$$R_{\alpha\beta}^{(1)} = -\frac{1}{2}\partial_\gamma \partial^\gamma h_{\alpha\beta} \quad (6.15a)$$

$$R^{(1)} = -\frac{1}{2}\partial_\gamma \partial^\gamma h, \quad (6.15b)$$

$$G_{\alpha\beta}^{(1)} = -\frac{1}{2}\partial_\gamma \partial^\gamma \bar{h}_{\alpha\beta}. \quad (6.15c)$$

As claimed, the linearized field equation is then

$$\square \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}, \quad (6.16)$$

and the Cartesian components of $\bar{h}_{\alpha\beta}$ are

$$\bar{h}_{\mu\nu}(x) = 4 \int \frac{T_{\mu\nu}|_{ret}}{|\mathbf{r} - \mathbf{r}'|} dV' = 4 \int \frac{T_{\mu\nu}(t - |\mathbf{r} - \mathbf{r}'|, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV', \quad (6.17)$$

In observing gravitational waves, one sees only the $1/r$ term in the asymptotic expansion. We will call the size of the source $\mathcal{R}$, taking $T^{\mu\nu}(t, \mathbf{r}')$ to be nonzero only when $\mathbf{r}' < \mathcal{R}$. Then

$$\bar{h}_{\mu\nu}(x) = 4 \int \frac{T_{\mu\nu}|_{ret}}{|\mathbf{r} - \mathbf{r}'|} dV' = 4 \int \frac{T_{\mu\nu}(t - |\mathbf{r} - \mathbf{r}'|, \mathbf{r}')}{r} dV' [1 + O(\mathcal{R}/r)]. \quad (6.18)$$

This $1/r$ term also includes the time-independent M/r part of the asymptotic metric, which always agrees with Schwarzschild at this order.

We now turn to a slow-motion approximation, in which successively higher multipoles (values of ℓ) are smaller by factors of $\omega\mathcal{R}$. A scalar field has all multipoles, starting with monopole radiation, $\ell = 0$, corresponding to loss of scalar charge. For an electromagnetic field, charge is conserved, and the radiation starts with a dipole ($\ell = 1$) field. For gravity, there is again no monopole radiation and now there is no dipole radiation (this is exact, not just for slow motion). The first nonzero multipole is quadrupole, $\ell = 2$. We will see that this follows from asymptotic conservation of momentum as well as mass at linear order in $h_{\alpha\beta}$. In a Newtonian context, the center of charge of a rotating dipole has the acceleration of the circular motion of its opposite charges, with $\ddot{\mathbf{d}} = \frac{d^2}{dt^2} \int \rho_e \mathbf{x} dV \neq 0$; but in a binary system, the masses have the same sign, and the center of mass does not accelerate, $\frac{d^2}{dt^2} \int \rho \mathbf{x} dV = 0$.

6.1.3 Slow-motion source

A source with an oscillation frequency or an orbital frequency ω has velocity of order $v = \mathcal{R}\omega$. When the source moves slowly, $v/c = \mathcal{R}\omega \ll 1$. Then

$$\mathcal{R} \partial_t T_{\mu\nu} \ll T_{\mu\nu}.$$

The multipole expansion of the radiation field involves an expansion of $T_{\mu\nu}(t - |\mathbf{r} - \mathbf{r}'|, \mathbf{r}')$ in powers of $\mathcal{R}\omega$, i.e., in powers of the post-Newtonian small parameter $e \sim v/c$. We start with

$$|\mathbf{r} - \mathbf{r}'| = [r^2 - 2\mathbf{r} \cdot \mathbf{r}' + r'^2]^{1/2} = r - \hat{\mathbf{r}} \cdot \mathbf{r}' + O\left(\frac{\mathcal{R}}{r}\right). \quad (6.19)$$

Then

$$\begin{aligned} T_{\mu\nu}(t - |\mathbf{r} - \mathbf{r}'|, \mathbf{r}') &= T_{\mu\nu}(t - r, \mathbf{r}') - \hat{\mathbf{r}} \cdot \mathbf{r}' \partial_t T_{\mu\nu} - \frac{1}{2} (\hat{\mathbf{r}} \cdot \mathbf{r}')^2 \partial_t^2 T_{\mu\nu} - \dots \\ &= T_{\mu\nu}(t - r, \mathbf{r}') [1 - O(\mathcal{R}\omega) - O((\mathcal{R}\omega)^2) - \dots]. \end{aligned} \quad (6.20)$$

Now for a nearly Newtonian source, $T^{00} = \rho$, $T^{0i} = \rho v^i$, $T^{ij} = \rho v^i v^j$, so T^{ij} is already smaller than T^{00} by order v^2 . But the leading terms from T^{00} and T^{0i} are just the components of the 4-momentum, and the equation of motion

$$\partial_0 T^{0\mu} = -\partial_j T^{j\mu},$$

implies that they are conserved:

$$\partial_0 \int T^{0\mu}(t - r, \mathbf{r}') dV' = - \int \frac{\partial}{\partial x'^j} T^{j\mu} dV' = 0. \quad (6.21)$$

The next-order (dipole) term in T^{00} from Eq. (6.20) is again time independent, because the center of mass $\int \rho \mathbf{r}' dV'$ is fixed:

$$\int [-\hat{\mathbf{r}} \cdot \mathbf{r}' \partial_t T^{00}] dV' = -\hat{\mathbf{r}} \cdot \partial_t \int \rho \mathbf{r}' dV'.$$

Ordinarily, one picks a frame in which the center of mass is at the origin. It stays there because of conservation of momentum.

With the monopole and dipole parts of T^{00} time-independent, we are left for its radiative part with the quadrupole contribution

$$\int \left[\frac{1}{2} (\hat{\mathbf{r}} \cdot \mathbf{r}')^2 \partial_t^2 T^{00} \right] dV' = -\frac{1}{2} \hat{x}^i \hat{x}^j \ddot{I}_{ij}(t - r), \text{ where } I_{ij}(t) := \int T^{00} x_i x_j dV. \quad (6.22)$$

Now the leading term T^{ij} is already quadrupole and as we now show, its value is related to the radiative part of T^{00} . (We'll soon see that the trace I_i^i does not contribute to the radiation field, and the quadrupole contribution is the tracefree part of I_{ij} , namely $I_{ij} - \frac{1}{3} \delta_{ij} I_k^k$.)

Again using the equation of motion $\partial_\nu T^{\mu\nu} = 0$, we have

$$\begin{aligned} \partial_0 \int (T^{i0} x^j + T^{j0} x^i) dV &= - \int (x^j \partial_k T^{ik} + x^i \partial_k T^{jk}) dV = 2 \int T^{ij} dV \quad \text{after integration by parts,} \\ \partial_0 \int T^{00} x^i x^j dV &= - \int x^i x^j \partial_k T^{0k} dV = \int (x^i T^{0j} + x^j T^{0i}) dV \quad \text{after integration by parts,} \end{aligned}$$

whence

$$\int T^{ij}(t - r, \mathbf{r}') dV' = \frac{1}{2} \partial_0^2 \int T^{00}(t - r, \mathbf{r}') x^i x^j dV' =: \frac{1}{2} \ddot{I}^{ij}(t - r).$$

Then at leading order in $1/r$,

$$\bar{h}_{ij} = \frac{4}{r} \left(\frac{1}{2} \ddot{I}_{ij} \right) = \frac{2}{r} \ddot{I}_{ij}(t - r). \quad (6.23)$$

We now show that a choice of gauge reduces the asymptotic form of the radiation field $h_{\mu\nu}$ to two free functions.

6.1.4 Transverse tracefree gauge

Requiring that the gauge vector satisfy Eq. (6.14) does not completely specify it: We can add to ξ^α any vector ζ^α satisfying $\square\zeta^\alpha = 0$. Outside the source, this allows us to further restrict the gauge to make $h_{\alpha\beta}$ tracefree and to require $h_{t\mu} = 0$.

We follow a simple standard derivation using plane waves, waves of the form $h_{\alpha\beta} = A_{\alpha\beta}e^{ik_\mu x^\mu}$. Because any solution to $\square\Phi = 0$ is a superposition of plane waves, it follows that any vacuum $h_{\alpha\beta}$ can be written in TT gauge. Wald avoids using plane waves in favor of an initial value argument; for readers following Wald, his argument, with details filled in, is given below.²¹⁸

In the transverse gauge (6.11), $h_{\mu\nu}$ is given by the retarded solution to the wave equation. To obtain that gauge, we required the gauge vector ξ^α to satisfy the wave equation (6.14), and the new gauge will satisfy both that condition and the additional conditions. Then, in the new gauge, $\bar{h}_{\mu\nu}^{NEW} = h_{\mu\nu}^{NEW}$. The new $h_{\mu\nu}$ is denoted by $h_{\mu\nu}^{TT}$, “TT” abbreviating “transverse tracefree.” For a wave of the form $h_{\mu\nu} \propto e^{ik_\mu x^\mu}$, with k_μ along the null direction $\hat{t}^\mu + \hat{z}^\mu$, the transverse condition is

$$0 = k^\mu h_{\mu\nu} \implies 0 = h_{t\mu} + h_{z\mu} = h_{z\mu}.$$

The nonzero components of $h_{\mu\nu}^{TT}$ are then

$$h_+ := h_{xx}^{TT} = -h_{yy}^{TT}, \quad h_\times := h_{xy}^{TT}. \quad (6.24)$$

The gauge conditions have thus reduced the wave to the two degrees of freedom, h_+ and $h_\times$.

In spherical coordinates t, r, θ, ϕ , the nonzero components of $h_{\mu\nu}^{TT}$ are

$$h_{\hat{\theta}\hat{\theta}}^{TT} = -h_{\hat{\phi}\hat{\phi}}^{TT}, \quad h_{\hat{\theta}\hat{\phi}}^{TT}, \quad (6.25)$$

with k^α along $\hat{r}^\alpha$. The wave looks to a distant observer at r_0, θ_0, ϕ_0 like a plane wave of the form (6.24) with $\hat{x}, \hat{y}$ and $\hat{z}$ taken along $\hat{\theta}, \hat{\phi}$, and $\hat{r}$ at the observer.

Constructing the gauge:

Let ζ^α be the gauge vector from our transverse $\bar{h}_{\alpha\beta}$ to $h_{\alpha\beta}^{NEW} = h_{\alpha\beta}^{TT}$:

$$h_{\mu\nu}^{TT} = h_{\mu\nu} + \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu. \quad (6.26)$$

We are to find $\zeta^\alpha = B^\alpha e^{ik_\mu x^\mu}$ to make

$$h^{TT} = 0, \quad h_{t\mu}^{TT} = 0.$$

Impose 4 conditions to get the 4 components ζ^μ :

$$(1) \quad 0 = h^{TT} = h + 2\partial_\mu \zeta^\mu \Rightarrow ik_\mu B^\mu = -\frac{1}{2}A_\mu^\mu.$$

$$(2-4) \quad 0 = h_{ti}^{TT} = h_{ti} + \partial_t \zeta_i + \partial_i \zeta_t \Rightarrow$$

$$i\omega \zeta_x = h_{tx},$$

$$i\omega \zeta_y = h_{ty}$$

$$i\omega \zeta_z - i\omega \zeta_t = h_{tz}$$

Then $k^\nu h_{\mu\nu}^{TT} = 0$ implies $h_{tt}^{TT} = 0$.

In our TT gauge, the only nonvanishing components of the perturbed metric are h_{xx}^{TT} , h_{xy}^{TT} , and h_{yy}^{TT} . That is, $h_{t\mu}^{TT} = 0$ and $\partial^\nu h_{\mu\nu} = 0$ imply $0 = -\partial_t h_{t\mu}^{TT} + \partial_z h_{z\mu}^{TT} = \partial_z h_{z\mu}^{TT} \rightarrow h_{z\mu} = 0$. These nonvanishing components are just the tracefree parts of our original transverse gauge:

With the gauge vector in hand, we can quickly write the two independent components h_+ and $h_\times$ of (6.24) of $h_{\mu\nu}^{TT}$ in terms of $\bar{h}_{\mu\nu}$, the retarded solution (6.17) to the wave equation. Because $h_{\mu\nu}^{TT} = h_{\mu\nu} + \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu$ and $\zeta_m u = \zeta_m u(t - z)$, we have

$$h_{xx}^{TT} = h_{xx} = \bar{h}_{xx} - \frac{1}{2}h. \quad (6.27)$$

Then

$$h_+ = \frac{1}{2}(h_{xx}^{TT} - h_{yy}^{TT}) = \frac{1}{2}(h_{xx} - h_{yy}) = \frac{1}{2}(\bar{h}_{xx} - \bar{h}_{yy}), \quad (6.28a)$$

$$h_\times = h_{xy}^{TT} = \bar{h}_{xy}. \quad (6.28b)$$

Finally, returning the names of components along x and y back to $\hat{\theta}$ and $\hat{\phi}$,

$$h_{\hat{\theta}\hat{\theta}}^{TT} = -h_{\hat{\phi}\hat{\phi}}^{TT} = h_+, \quad h_{\hat{\theta}\hat{\phi}}^{TT} = h_\times. \quad (6.29)$$

In particular, in the slow-motion approximation, the gravitational waves described by $h_{\mu\nu}^{TT}$ have as their source the *quadrupole tensor*, the tracefree part of the tensor I_{ij} ,

$$\mathcal{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}I^k{}_k. \quad (6.30)$$

From Eq.(6.23), at leading order in $1/r$,

$$h_+ = \frac{1}{r}(\ddot{\mathcal{I}}_{\hat{\theta}\hat{\theta}} - \ddot{\mathcal{I}}_{\hat{\phi}\hat{\phi}}), \quad h_\times = \frac{2}{r}\ddot{\mathcal{I}}_{\hat{\theta}\hat{\phi}}. \quad (6.31)$$

Far from a periodic source, the wave looks like a plane wave. That is, our wave has the asymptotic behavior $h_{\mu\nu} = \frac{1}{r_0}A_{\mu\nu}(t - r_0 - z) + O(r^{-2}) = h_{\mu\nu}(t - z) + O(r^{-2})$, and $\partial_r h_{\mu\nu} = \partial_z h_{\mu\nu} + O(r^{-2})$. Treating the wave as a plane wave $h_{\mu\nu}(t - z)$ is then equivalent to ignoring terms of order r^{-2} , terms smaller than $h_{\mu\nu}$ itself by a factor of order $\mathcal{R}/r$. This is a spectacularly good approximation: For a source of size 10^3 km at a distance of $30 \text{ Mpc} \sim 10^{15}$ km, the error is one part in 10^{12} .

The waveform of an inspiraling binary, however, has a time-dependent frequency; in late inspiral and merger, it is far from periodic. To avoid introducing plane waves and making the argument that any vacuum solution can be written as a superposition,¹ Wald avoids plane waves in favor of an initial-value argument. If you are not following Wald, you can go ahead to the next section, 6.2.

¹The superposition argument has a subtlety, because the spacetime is not everywhere vacuum. Outside any finite neighborhood of the source, one can duplicate the actual solution for a finite time by a sourcefree solution that includes incoming waves from past null infinity. To obtain the sourcefree solution, one uses initial data between slices of future null infinity and evolves the data backwards in time. The incoming and outgoing waves overlap only in the neighborhood of what was the source. The simplicity of the plane-wave derivation justifies ignoring the subtlety.

Wald's derivation

Given the values of a scalar Φ and its time derivative $\partial_t \Phi$ on an initial surface, say at $t = 0$, the wave equation $\square \Phi = 0$ uniquely determines $\Phi(t, x)$ everywhere. Our job is to find a gauge vector for which $h_{\mu\nu}^{TT} = \partial_t h_{\mu\nu}^{TT} = 0$ and $h_{0\mu}^{TT} = \partial_t h_{0\mu}^{TT} = 0$, at $t = 0$. The wave equation, $\square h_{\mu\nu}^{TT} = 0$, will then force each component to be zero at all times.

To obtain a transverse gauge, we required the gauge vector ξ^α to satisfy the wave equation (6.14). This does not quite exhaust our gauge freedom, because we can choose initial conditions – the values of ξ^α and $\partial_t \xi^\alpha$ on an initial spacelike surface – at, say, $t = 0$. This allows us to make $h_{\alpha\beta}$ tracefree: $h = h_\alpha^\alpha = 0$ and to choose $h_{0\mu} = 0$ at $t = 0$. Inside the source, the evolution of h is given by the wave equation with source $-16\pi T_\alpha^\alpha$, so h does not remain tracefree, nor does $h_{0\mu}$ remain zero. Outside the source, however, because $h_{\alpha\beta}$ satisfies the homogeneous wave equation,

$$\square h_{\alpha\beta} = 0,$$

they remain zero.

Because the old $\bar{h}_{\alpha\beta}$ is already transverse, ζ^α keeps it transverse when ζ^μ satisfies $\square \zeta^\alpha = 0$. The tracefree condition means $\bar{h}_{\alpha\beta}^{TT} = h_{\alpha\beta}^{TT}$, so the gauge will have $\partial^\beta h_{\alpha\beta}^{TT} = 0$. Our gauge conditions are then

$$0 = \square \zeta^\mu \quad \text{transverse,} \quad (6.32a)$$

$$0 = h_\mu^{TT\mu} = h + 2\partial_\mu \zeta^\mu, \quad \text{tracefree} \quad (6.32b)$$

$$0 = h_{t\mu}^{TT} = h_{t\mu} + \partial_t \zeta_\mu + \partial_\mu \zeta_t \quad \text{no } t \text{ component.} \quad (6.32c)$$

Any gauge vector of the form $\zeta^\mu = \zeta^\mu(t - z)$ satisfies the first condition, $\square \zeta^\mu = (-\partial_t^2 + \partial_z^2)\zeta^\mu = 0$.

The tracefree condition gives

$$0 = h + 2(\partial_t \zeta^t + \partial_z \zeta^z) \implies \dot{\zeta}^t - \dot{\zeta}^z = -\frac{1}{2}h.$$

The spatial components of the last condition determine $\partial_t \zeta^i$ and $\dot{\zeta}^t$

$$\begin{aligned} -h_{tz} &= \partial_t \zeta_z + \partial_z \zeta_t = \dot{\zeta}^z + \dot{\zeta}^t \implies \\ \dot{\zeta}^t &= -\frac{1}{2}h_{tz} - \frac{1}{4}h, \quad \dot{\zeta}^z = -\frac{1}{2}h_{tz} + \frac{1}{4}h. \end{aligned} \quad (6.33)$$

The tx and ty components of the no- t -component condition are immediate:

$$\dot{\zeta}_x = -h_{tx}, \quad \dot{\zeta}_y = -h_{ty}. \quad (6.34)$$

The tt component gives no additional condition on $\dot{\zeta}^t$, because $\partial^\mu \bar{h}_{t\mu} = 0$ implies $\partial_t(h_{tt} + \frac{1}{2}h + h_{tz}) = 0 = -\partial_z(h_{tt} + \frac{1}{2}h + h_{tz})$, whence

$$h_{tt} + \frac{1}{2}h + h_{tz} = 0.$$

The $h_{tt}^{TT} = 0$ condition is then automatically satisfied:

$$\frac{1}{2}h_{tt} - 2\dot{\zeta}^t = h_{tt} + \frac{1}{2}h + h_{tz} = 0.$$

The gauge conditions (6.33) and (6.34) are now satisfied, and we have completely determined $\dot{\zeta}^\mu|_{t=0}$. We now need to use our freedom to choose ζ^μ itself at $t = 0$ to enforce $\dot{h}^{TT} = 0$ and $\dot{h}_{0\mu}^{TT} = 0$. To write these conditions, replace ζ^μ by $\dot{\zeta}^\mu$ and $h_{\mu\nu}$ by $\dot{h}_{\mu\nu}$ in Eqs. (6.33) and (6.34):

$$\ddot{\zeta}^t = -\frac{1}{2}\dot{h}_{tz} - \frac{1}{4}\dot{h}, \quad \ddot{\zeta}^z = -\frac{1}{2}\dot{h}_{tz} + \frac{1}{4}\dot{h}, \quad \ddot{\zeta}_x = -\dot{h}_{tx}, \quad \ddot{\zeta}_y = -\dot{h}_{ty} \quad (6.35)$$

But these are conditions on $\ddot{\zeta}$ at $t = 0$, not on ζ itself. What are we to do? We use the wave equation for ζ^μ , and simply replace $\ddot{\zeta}^\mu|_{t=0}$ by $\nabla^2\zeta^\mu$, writing

$$\nabla^2\zeta^t = -\frac{1}{2}\dot{h}_{tz} - \frac{1}{4}\dot{h}, \quad \nabla^2\zeta^z = -\frac{1}{2}\dot{h}_{tz} + \frac{1}{4}\dot{h}, \quad \nabla^2\zeta_x = -\dot{h}_{tx}, \quad \nabla^2\zeta_y = -\dot{h}_{ty}. \quad (6.36)$$

Each Poisson equation has a unique solution (vanishing at spatial infinity) for ζ^μ , and with this initial data, the wave equation, $\ddot{\zeta}^\mu = \nabla^2\zeta^\mu$, for ζ^μ enforces the initial conditions (6.35) for $\dot{h}_{\mu\nu}^{TT}$.

6.2 The quadrupole formula and gravitational-wave energy

Effective stress-energy tensor for gravitational waves

The radiated energy is obtained from an effective stress-energy tensor for the gravitational wave. Like the stress-energy tensor of the electromagnetic field, which is quadratic in first derivatives of the vector potential A_α , the effective stress-energy of gravitational waves is quadratic in first derivatives of $h_{\alpha\beta}$. The asymptotic radiated flux of the electromagnetic field in a Lorenz gauge is proportional to $\dot{\mathbf{A}}^2$, involving only components orthogonal to the direction of propagation. Similarly, in our analogous TT gauge for $h_{\alpha\beta}$, the radiated energy is quadratic in $\dot{h}_{ij}$, involving only components orthogonal to the direction of propagation.

We now find an effective stress-energy tensor $T_{GW}^{\alpha\beta}$ for gravitational waves and use it to find the flux of energy through an $r = \text{constant}$ surface at large r . That is, given a stress-energy tensor satisfying $\partial_\beta T_{GW}^{\alpha\beta} = 0$ on a flat background, we have the conservation law

$$\frac{dE_{GW}}{dt} = \partial_t \int_V T^{tt} d^3x = - \int_V \partial_i T^{ti} d^3x = - \int_{r=\text{constant}} T^{tr} r^2 d\Omega, \quad (6.37)$$

where V is bounded by an $r = \text{constant}$ sphere.

To find $T_{GW}^{\alpha\beta}$, we will use the fact that the field equation $G^{\alpha\beta} = 8\pi T^{\alpha\beta}$ can be obtained by varying an action $I = I_G + I_M$, where I_G is the gravitational action and I_M the action for the matter fields. In fact, as we will see in Sect.10.1,

$$I_G = (16\pi)^{-1} \int R \sqrt{-g} d^4x,$$

but everything follows just from the existence of an action, from the fact that varying an action with respect to the metric $g_{\alpha\beta}$ gives

$$\delta I = \delta I_G + \delta I_M = C \int \left(\frac{1}{8\pi} G^{\alpha\beta} - T^{\alpha\beta} \right) h_{\alpha\beta} \sqrt{-g} d^4x, \quad h_{\alpha\beta} := \delta g_{\alpha\beta}. \quad (6.38)$$

Here I_G depends only on $g_{\alpha\beta}$, while I_M is a function of both the matter fields and the metric, and Eq. (6.38) implies $T^{\alpha\beta} = -\frac{1}{C} \frac{\delta I_M}{\delta g_{\alpha\beta}}$. The form of $T_{GW}^{\alpha\beta}$ will follow from this equation; all that is missing

is the factor $C = -1/2$, and this follows from the explicit form of any matter action. Let's use a free particle (the sign of the action is determined by the sign of the Newtonian kinetic energy, $mv^2/2$)

$$I_M = - \int d\lambda \, m \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu},$$

$$\frac{\delta I_M}{\delta g_{\mu\nu}} = \frac{1}{2} m u^\mu u^\nu = \frac{1}{2} T^{\mu\nu} \implies C = -\frac{1}{2}. \quad (6.39)$$

With C known, we have for any matter action,

$$T^{\mu\nu} = 2 \frac{\delta I_M}{\delta g_{\mu\nu}}. \quad (6.40)$$

Then from Eq.6.38,

$$\delta I_G = - \int \frac{1}{16\pi} G^{\mu\nu} h_{\mu\nu} \sqrt{-g} d^4x. \quad (6.41)$$

Now $G^{\mu\nu}$ vanishes for the background spacetime, and at first order in $h_{\mu\nu}$, it is given by Eq. (6.15c). An action for the perturbed field equations is then

$$I_G^{(2)} = - \int \frac{1}{16\pi} \left(-\frac{1}{2} \nabla_\lambda \nabla^\lambda h^{\mu\nu} \right) h_{\mu\nu} \sqrt{-g} d^4x = - \frac{1}{32\pi} \int \nabla_\lambda h_{\mu\nu} \nabla^\lambda h^{\mu\nu} \sqrt{-g} d^4x, \quad (6.42)$$

and in the TT gauge we have

$$I_G^{(2)}(h, g) = - \frac{1}{32\pi} \int \nabla_\lambda h_{\mu\nu}^{TT} \nabla^\lambda h^{TT\mu\nu} \sqrt{-g} d^4x = - \frac{1}{32\pi} \int \nabla_\kappa h_{\mu\nu}^{TT} \nabla_\lambda h_{\sigma\tau}^{TT} g^{\kappa\lambda} g^{\mu\sigma} g^{\nu\tau} \sqrt{-g} d^4x. \quad (6.43)$$

We need the stress-energy tensor $\delta I^{(2)}/\delta g_{\alpha\beta}$ at large r , where it is simplest to evaluate. We once again use the relation $\partial_t h_{\mu\nu}^{TT} = -\partial_r h_{\mu\nu}^{TT} + O(1/r^2)$: Dropping the TT and using dots to represent arbitrary indices, we have

$$\partial_\gamma h_{..} \partial^\gamma h_{..} = -\partial_t h_{..} \partial_t h_{..} + \partial_r h_{..} \partial_r h_{..} = O(1/r^3). \quad (6.44)$$

Now unless $\frac{\delta}{\delta g_{\alpha\beta}}$ hits $g^{\kappa\lambda}$ on the right side of (6.43), the resulting term in the stress-energy tensor will be proportional to $\partial_\gamma h_{..} \partial^\gamma h_{..}$ and so will not contribute to the radiated energy.² The only surviving term is then

$$T_G^{\alpha\beta} = 2 \frac{\delta I^{(2)}}{\delta g_{\alpha\beta}} = \frac{1}{16\pi} \nabla^\alpha h_{\mu\nu}^{TT} \nabla^\beta h^{TT\mu\nu}. \quad (6.45)$$

The radiated energy flux is T_G^{tr} and the asymptotic form of the radiated energy is then

$$\begin{aligned} \frac{dE}{dt d\Omega} &= r^2 T_G^{tr} = -\frac{r^2}{16\pi} \left[(\dot{h}_{\hat{\theta}\hat{\theta}}^{TT})^2 + (\dot{h}_{\hat{\phi}\hat{\phi}}^{TT})^2 + 2(\dot{h}_{\hat{\theta}\hat{\phi}}^{TT})^2 \right] + O\left(\frac{1}{r}\right) \\ &= \frac{r^2}{16\pi} (\dot{h}_+^2 + \dot{h}_\times^2) \\ &= \frac{1}{4\pi} \left[\left(\frac{\ddot{I}_{22} - \ddot{I}_{33}}{2} \right)^2 + (\ddot{I}_{23})^2 \right], \end{aligned} \quad (6.46)$$

²There are also terms from $\delta/\delta g_{\alpha\beta}$ hitting the covariant derivative ∇_κ and ∇_λ . Each of these terms is either proportional to $\nabla_\lambda h^\lambda_\mu$ or to $\nabla_\nu \nabla_\lambda h_{\sigma\tau} h^\lambda_\mu$. In our TT gauge, $\nabla_\lambda h^\lambda_\mu = 0$; and, at leading order in $1/r$, $\nabla_\lambda h_{\sigma\tau}$ has only $\lambda = t$ and $\lambda = r$ components, while h^λ_μ has only components orthogonal to t and r , so there is no contribution from these terms at $O(r^{-2})$. (For asymptotic behavior $h_{..} \sim \frac{1}{r} e^{ik_\mu x^\mu}$, the argument is simpler: $k_\lambda h^\lambda_\mu = 0$ and $k_\lambda k^\lambda = 0$.)

where, for ease of notation in the next part, we write the indices as $\hat{\theta} = 2$ and $\hat{\phi} = 3$. (I_{ij} is written here instead of $\ddot{I}_{ij}$ to emphasize that $I_{22} - I_{33}$ and I_{23} involve only tracefree part of I_{ij} .)

Then (6.46) implies

$$\frac{dE}{dt d\Omega} = \frac{1}{4\pi} \left[\left(\frac{\ddot{I}_{22} - \ddot{I}_{33}}{2} \right)^2 + (\ddot{I}_{23})^2 \right]$$

Denote by P_{ij} the projection operator orthogonal to the direction of $\hat{\mathbf{r}}$: $P_{ij} = \delta_{ij} - \hat{r}_i \hat{r}_j$. Then

$$\begin{aligned} 4\pi \frac{dE}{dt d\Omega} &= \frac{1}{2} (\ddot{I}_{22} \ddot{I}_{22} + 2 \ddot{I}_{23} \ddot{I}_{23} + \ddot{I}_{33} \ddot{I}_{33}) - \frac{1}{4} (\ddot{I}_{22} + \ddot{I}_{33})^2 \\ &= \frac{1}{2} P^{i\ell} P^{jm} \ddot{I}_{ij} \ddot{I}_{\ell m} - \frac{1}{4} (P^{ij} \ddot{I}_{ij})^2 \\ &= \frac{1}{2} [\ddot{I}_{ij} \ddot{I}^{ij} - 2 \hat{r}^j \ddot{I}_{ij} \ddot{I}^i{}_k \hat{r}^k + \frac{1}{2} (\hat{r}^i \ddot{I}_{ij} \hat{r}^j)^2] \end{aligned} \quad (6.47)$$

To find the total power radiated we need to evaluate

$$\int \hat{r}_i \hat{r}_j d\Omega \quad \text{and} \quad \int \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_\ell d\Omega.$$

Because the integral over angles sums over all directions of $\mathbf{r}$, $\langle \hat{r}_i \hat{r}_j \rangle := \frac{1}{4\pi} \int \hat{r}_i \hat{r}_j d\Omega$ is a symmetric tensor that knows no direction. So it must be constructed from δ_{ij} :

$$\langle \hat{r}_i \hat{r}_j \rangle = k \delta_{ij}$$

Take its trace:

$$\begin{aligned} \delta^{ij} \langle \hat{r}_i \hat{r}_j \rangle &= \langle 1 \rangle = 1 = 3k \implies k = \frac{1}{3}, \\ \langle \hat{r}_i \hat{r}_j \rangle &= \frac{1}{3} \delta_{ij}. \end{aligned} \quad (6.48)$$

Similarly

$$\begin{aligned} \langle \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_\ell \rangle &= k' \delta_{(ij} \delta_{kl)} = k'' (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\ \delta^{ij} \delta^{kl} \langle \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_\ell \rangle &= 1 = k'' (9 + 3 + 3) \\ k'' &= \frac{1}{15} \\ \langle \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_\ell \rangle &= \frac{1}{15} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \end{aligned} \quad (6.49)$$

Finally Eqs. (6.47)-(6.49) $\implies$

$$\frac{dE}{dt} = \frac{1}{2} \left[\ddot{I}_{ij} \ddot{I}^{ij} - 2 \left(\frac{1}{3} \delta^{kl} \right) \ddot{I}_{ik} \ddot{I}_\ell^i + \frac{1}{2} \left(\frac{1}{15} \right) (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) \ddot{I}_{ij} \ddot{I}_{kl} \right],$$

giving

$$\frac{dE}{dt} = \frac{1}{5} \ddot{I}_{ij} \ddot{I}^{ij}. \quad (6.50)$$

Restore G and c to the right side to recover the conventional dimension $[E/T] = ML^2 T^{-3}$: Because $\ddot{I}_{ij} \ddot{I}^{ij}$ has conventional dimension $M^2 L^4 T^{-6}$, we multiply by the combination of c and G that has dimension $M^{-1} L^{-2} T^3$, namely $[G/c^5]$:

$$\frac{dE}{dt} = \frac{G}{c^5} \frac{1}{5} \ddot{I}_{ij} \ddot{I}^{ij}. \quad (6.51)$$

6.3 Binary Inspiral

Consider now a binary system with masses m_A and m_B , in circular orbits about their center of mass, and with separation a between the masses. The distance from each mass to the center of mass is $r_A = a m_B/M$, $r_B = a m_A/M$. For an orbit in the x - y plane, we have

$$\begin{aligned} x_A &= r_A \cos \Omega t, & x_B &= -r_B \cos \Omega t, \\ y_A &= r_A \sin \Omega t, & y_B &= -r_B \sin \Omega t, \end{aligned} \quad (6.52)$$

$$\begin{aligned} I_{xx} &= m_A x_A^2 + m_B x_B^2 = \frac{1}{2} \mu a^2 \cos 2\Omega t + \text{constant}, \\ I_{yy} &= -\frac{1}{2} \mu a^2 \cos 2\Omega t + \text{constant}, \\ I_{xy} &= \frac{1}{2} \mu a^2 \sin 2\Omega t, \end{aligned} \quad (6.53)$$

implying that the wave has frequency $\omega = 2\Omega$.

Because $I_{xx} + I_{yy} = 0$, we have $I_{ij} = I_{ij}$ and ³

$$\ddot{I}_{ij} \ddot{I}^{ij} = \ddot{I}_{xx}^2 + \ddot{I}_{yy}^2 + 2 \ddot{I}_{xy}^2 = 32(\mu a^2 \Omega^3)^2. \quad (6.54)$$

The energy of the binary is

$$E = -T = -\frac{1}{2} \mu a^2 \Omega^2, \quad \text{or} \quad E = \frac{1}{2} U_G = -\frac{\mu M}{2a}. \quad (6.55)$$

From Eq. (6.50)

$$\frac{dE}{dt} = -\frac{32}{5} \mu^2 a^4 \Omega^6. \quad (6.56)$$

Because the frequency of the wave is directly measurable, while the binary separation is not, it is helpful to use Kepler's law, $\Omega^2 = M/a^3$ to replace a by $M^{1/3} \Omega^{-2/3}$:

$$\frac{dE}{dt} = -\frac{32}{5} (\mathcal{M} \Omega)^{10/3}, \quad \mathcal{M} \equiv \text{chirp mass} := \mu^{3/5} M^{2/5}, \quad (6.57)$$

or, with G and c restored,

$$\frac{dE}{dt} = -\frac{32}{5} \frac{G^{7/3}}{c^5} (\mathcal{M} \Omega)^{10/3}.$$

To find an equation for the change in the orbital radius as the binary loses energy to gravitational waves, use the second version of E in (6.55) to write

$$\frac{\dot{E}}{E} = -\frac{\dot{a}}{a}.$$

Then replace Ω^2 by M/a^3 in dE/dt of (6.56):

$$\frac{dE}{dt} = -\frac{32}{5} \frac{\mu^2 M^3}{a^5},$$

³For a general periodic source, one would take a time average over one cycle. In our case, however, the expression $\ddot{I}_{ij} \ddot{I}^{ij}$ is already time independent.

giving

$$\begin{aligned}\frac{\dot{a}}{a} &= -\frac{64}{5} \frac{\mu M^2}{a^4} \implies a^3 \dot{a} = -\frac{64}{5} \mu M^2 \\ a^4 &= a_0^4 - \frac{256}{5} \mu M^2 t,\end{aligned}\tag{6.58}$$

with $a = a_0$ at $t = 0$. Alternatively, with $t = 0$ at coalescence,

$$a \propto |t|^{1/4}, \quad |t| \text{ the time until coalescence},\tag{6.59}$$

and, from Kepler's law, $\Omega \propto |t|^{-3/8}$.

Exercise 79. a. Find the number of gravitons emitted per unit time by a spinning rod of length ℓ and negligible mass joining two point masses, each of mass m . Assume each graviton has energy $\hbar\omega$, with ω the frequency of classical gravitational waves.

b. Estimate this rate of emission for $m = 1$ kg, $\ell = 1$ m, and rotational frequency 500 rad/s. What is the probability that the system will radiate at all if it spins for a time $t = 1/\omega$? Weirdly, you should find Einstein's spinning rod rarely radiates gravitational waves!

Exercise 80. Estimate the number of gravitons emitted per second by two equal-mass neutron stars, each of mass $m = 1.4M_\odot$, when the distance between their centers of mass is 20 m .

Exercise 81. a. Write the quadrupole tensor I_{ij} in terms of the moment of inertia tensor $\mathcal{I}_{ij}$.

b. In a principal-axis frame (where $\mathcal{I}_{ij}$ is diagonal), show that I_{ij} is diagonal and relate the principal moments of inertia to the eigenvalues of I_{ij}

c. Using the relations

$$r^2 Y_{2\pm 2} = \sqrt{\frac{15}{32\pi}} (x \pm iy)^2, \quad r^2 Y_{21} = \mp \sqrt{\frac{15}{8\pi}} z(x \pm iy), \quad r^2 Y_{20} = \sqrt{\frac{5}{16\pi}} (3z^2 - r^2),$$

relate the components of I_{ij} along the complex basis $\{\hat{e}_\pm = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y}), \hat{z}\}$ to the quantities

$$I_{2m} = \int \rho Y_{\ell m} dV.$$

d. Write h_+ and $h_\times$ in terms of Y_{2m} , with the z -axis along the direction from source to observer.

6.4 Strain and detection of gravitational waves

Because of the difficulty in defining energy in a curved spacetime, parts of the physics community were uncertain whether gravitational waves carried energy from a system. Pirani and days later (but apparently independently) Feynman gave equivalent arguments at the same 1957 Chapel Hill conference to resolve the issue. Pirani had just published a paper using the equation of geodesic deviation (2.66),

$$\ddot{S}^\alpha = R^\alpha_{\beta\gamma\delta} T^\beta T^\gamma S^\delta,$$

to describe the relative acceleration of free particles as a gravitational wave passes.⁴

Here is Pirani's argument, with the idea of energy absorption put in by Bondi. Feynman's is at the end of the same volume:

Felix Pirani, *Measurement of Classical Gravitational Fields*, in *Report from the 1957 Chapel Hill Conference*[36], Chap. 14.

PIRANI: Because of the principle of equivalence, one cannot ascribe a direct physical interpretation to the gravitational field insofar as it is characterized by Christoffel symbols $\Gamma^\mu_{\nu\rho}$. One can, however, give an invariant interpretation to the variation of the gravitational field. These variations are described by the Riemann tensor; therefore, measurements of the relative acceleration of neighboring free particles, which yield information about the variation of the field, will also yield information about the Riemann tensor.

Now the relative motion of free particles is given by the equation of geodesic deviation

$$\frac{\partial^2 \eta^\mu}{\partial \tau^2} + R^\mu_{\nu\rho\sigma} v^\nu \eta^\rho v^\sigma = 0 \dots$$

BONDI: Can one construct in this way an absorber for gravitational energy by inserting a $d\eta/d\tau$ term to learn what part of the Riemann tensor would be the energy producing one, because it is that part that we want to isolate to study gravitational waves?

PIRANI: I have not put in an absorption term, but I have put in a “spring.” You can invent a system with such a term quite easily.

We will look at such a system, a damped oscillator driven by gravitational waves, in Eq. (6.66) below.

⁴The question also led to a study of asymptotic flatness, with an isolated system modeled as an asymptotically flat spacetime. The total mass of the spacetime can then be defined, and that mass decreases by the energy radiated to null infinity. That is, along a sequence of asymptotically null slices (like the $u = \text{constant}$ slices of Schwarzschild), the mass between successive slices decreases by the energy radiated to null infinity between the two slices. On asymptotically *spacelike* slices of the spacetime, the mass of the spacetime has a fixed, constant value. The radiation is simply at successively larger distances from the source along a sequence of spacelike slices. See references on p. 303

First, let's use the equation of geodesic deviation to get the standard picture of a circle of free particles stretched and compressed in orthogonal directions to become an oscillating ellipse. We look at the Riemann tensor of a plane wave with wave vector k^α orthogonal to the plane of the circle of particles.

Here's the Thorne-Blandford figure:

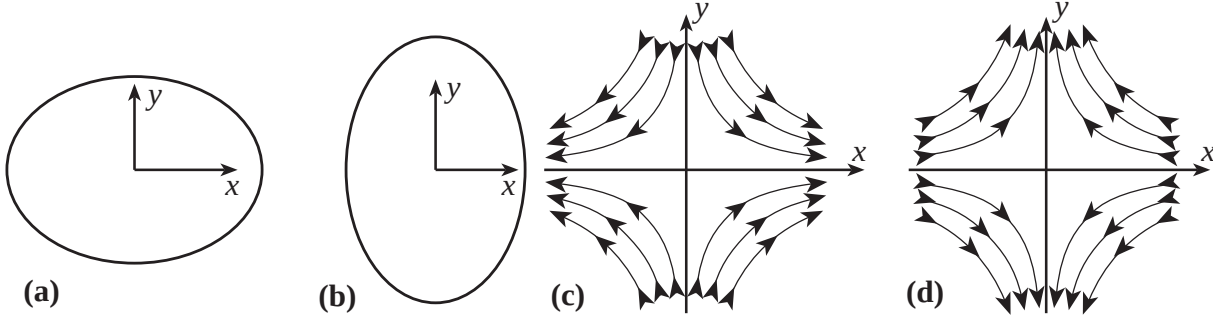


Figure 6.1: Physical manifestations, in a particle's local Lorentz frame, of gravitational waves with polarization in the x - y plane. (a) Transverse deformation of an initially spherical cloud of test particles in a transverse plane at a phase of the wave when $h_+ > 0$. (b) Deformation of the cloud when $h_+ < 0$. (c)-(d) Field lines representing the acceleration field $\delta \ddot{x}$ when $\ddot{h}_+ > 0$ and $\ddot{h}_+ < 0$, respectively.

A plane wave with wavevector $k^\alpha = \omega(\hat{t}^\alpha + \hat{z}^\alpha)$ and + polarization in the x - y plane has the form

$$h_{\alpha\beta} = (\hat{x}_\alpha \hat{x}_\beta - \hat{y}_\alpha \hat{y}_\beta) A \cos[\omega(t - z)] = (\hat{x}_\alpha \hat{x}_\beta - \hat{y}_\alpha \hat{y}_\beta) h_+, \quad (6.60)$$

following the notation of Eq.(6.24). The linearized Riemann tensor (6.4) is then given by

$$R_{\alpha\beta\gamma\delta}^{(1)} = \frac{1}{2}(-k_\beta k_\gamma h_{\alpha\delta} - k_\alpha k_\delta h_{\beta\gamma} + k_\alpha k_\beta h_{\gamma\delta} + k_\gamma k_\delta h_{\alpha\beta}). \quad (6.61)$$

In the geodesic deviation equation, we take $T^\alpha = \hat{t}^\alpha$, the 4-velocity of a particle prior to the arrival of the wave, and consider particles lying in the $z = 0$ plane. First consider two particles with separation vector S^α of the form

$$S^\alpha = \ell \hat{x}^\alpha + \xi^\alpha,$$

where $\xi^\alpha = 0$ before the wave arrives. The geodesic deviation equation then has the form

$$\begin{aligned} \ddot{\xi}^\alpha &= R^\alpha_{\beta\gamma\delta} t^\beta t^\gamma x^\delta \ell + O(A\xi) = \hat{x}^\alpha R_{xttx} \ell + O(A\xi) = -\hat{x}^\alpha k_t^2 h_+ \ell + O(A\xi) \\ &= -\hat{x}^\alpha \frac{1}{2} \omega^2 \ell A \cos \omega t + O(A\xi). \end{aligned} \quad (6.62)$$

At linear order in the amplitude of the wave, the solution is

$$\xi^\alpha = \frac{1}{2} \ell h_+ \hat{x}^\alpha = \frac{1}{2} A \ell \cos \omega t \hat{x}^\alpha, \quad (6.63)$$

where we have chosen $t = 0$ as a time of maximum displacement. For particles with displacement $\ell \hat{y}^\alpha$ instead of $\ell \hat{x}^\alpha$ prior to arrival of the wave, it is easy to see that the corresponding solution is $\xi^\alpha = -\frac{1}{2} \ell h_+ \hat{y}^\alpha$. Then for a displacement of particles initially along a line at arbitrary angle ϕ from the x -axis, we have

$$\xi^\alpha(\phi) = (\cos \phi \hat{x}^\alpha - \sin \phi \hat{y}^\alpha) \frac{1}{2} \ell h_+, \quad (6.64)$$

and

$$S^\alpha(\phi) = (\cos \phi \hat{x}^\alpha + \sin \phi \hat{y}^\alpha) \ell + \xi^\alpha(\phi).$$

Particles initially in a circle in the $z = 0$ plane then comprise an oscillating ellipse, with particle separation $S^\alpha(\phi)$, as shown in Fig.6.1(a)-(b). Their relative acceleration $\ddot{\xi}^\alpha$ agrees with the quadrupole pattern of Fig.6.1(c)-(d).

We can now return to Pirani's argument, with two particles joined by a spring of natural length ℓ , frequency ω , and damping coefficient γ . In flat space, with no gravitational waves, the relative acceleration of the particles with connecting vector $S^\alpha = \ell \hat{x}^\alpha + \xi^\alpha$ is given by

$$\ddot{\xi}^\alpha = -\omega^2 \xi^\alpha - 2\gamma \dot{\xi}^\alpha. \quad (6.65)$$

Intuitively, then, in the equation of geodesic deviation, one should add to the relative acceleration of the particles the forcing terms, giving (at linear order in the gravitational wave amplitude)

$$\ddot{\xi}^\alpha = R^\alpha{}_{\beta\gamma\delta} t^\beta t^\gamma x^\delta \ell - \omega^2 \xi^\alpha - 2\gamma \dot{\xi}^\alpha. \quad (6.66)$$

To obtain a formal derivation, notice that, in our derivation of geodesic deviation before Eq. (2.66), only the last line uses the geodesic equation $T^\beta \nabla_\beta T^\alpha = 0$. We again consider a family of timelike trajectories, parameterized by proper time, but this time with an external force acting. Each particle's velocity T^α then satisfies $T^\beta \nabla_\beta T^\alpha = F^\alpha/m \equiv a^\alpha$. Using this in the last line of the derivation gives the relative acceleration as a function of particle position,

$$\ddot{S}^\alpha = R^\alpha{}_{\beta\gamma\delta} T^\beta T^\gamma S^\delta + S^\beta \nabla_\beta a^\alpha. \quad (6.67)$$

For the Pirani-Bondi-Feynman example of a damped spring consider two small masses joined by a massless rod initially along the x -axis, again taking the rod to have natural length ℓ , frequency $m\omega^2$, and damping coefficient γ . The masses at the left and right ends of the rod have trajectories given by $T^\beta \nabla_\beta T^\alpha = \pm \frac{1}{2} \omega^2 \xi^\alpha$. Now choose a coordinate x for which $|x|$ is proper distance from the center of the rod. The acceleration of a particle of the rod at coordinate x is then proportional to x :

$$a^\alpha = -\frac{x}{\ell} (\omega^2 \xi^\alpha - 2\gamma \dot{\xi}^\alpha),$$

and Eq. (6.67) implies (6.66) at linear order in ξ^α because $S^\alpha = \ell \hat{x}^\alpha$ at zeroth order in ξ^α .

Exercise 82. a. Check that the separation vector S^α , with ξ^α given by the solution (6.64), gives the stretching and contracting ellipse of Fig.6.1.

b. Check that the acceleration vector conforms to parts (c)-(d) of the Figure.

c. Find the form of ξ^α for the polarization $h_\times$ and sketch the corresponding figures.

Exercise 83. *Energy transferred by a gravitational wave to an oscillator.*

a. For two particles connected by a spring, Eq. (6.62) is changed by the additional terms of Eq. (6.66). Find the corresponding steady-state solution to this equation for a driven oscillator.

b. Find the rate at which the oscillator absorbs energy from the gravitational wave. What is the time-average of this rate?

Hint: The rate at which the oscillator absorbs energy is the rate at which its energy would decrease if there were no gravitational wave.

Interferometric detectors (See, for example, the Creighton-Anderson text [31] on which this section is based.)

If, when you think of optics, the first thing that comes to mind is the angular resolution λ/d , it seems impossible that an interferometer using light with wavelength of roughly 10^{-4} cm can detect a gravitational wave that changes the distance between mirrors by less than 10^{-15} cm. Obtaining angular resolution better than λ/d means measuring a difference in intensity corresponding to a fraction of the distance between dark fringes. In an interferometer, one is again trying to measure the difference in intensity corresponding to a fraction of a fringe, but it turns out to be possible to find that difference to very high accuracy. How small a difference in intensity can be measured by a photodiode? We begin with the usual diagram of a simple interferometer and then look at a limit on the accuracy with which it can measure a change in length.

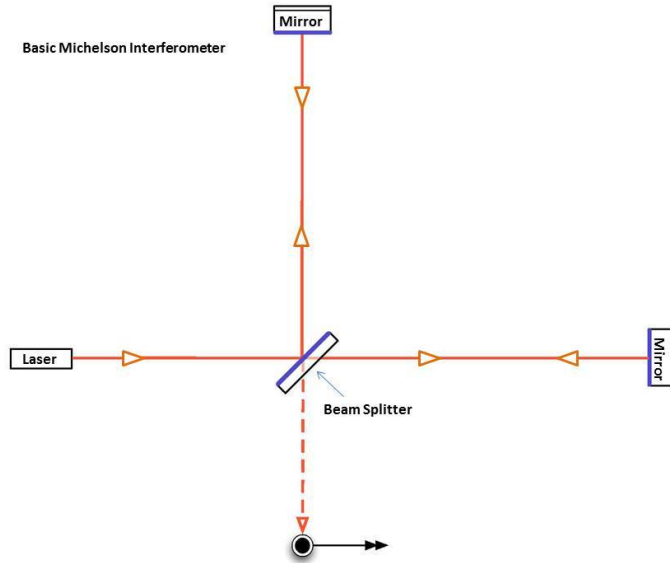


Figure 6.2: Diagram from a LIGO webpage

Photons from the laser can traverse two alternative paths to the detector. Along the first path, light reflects upward from the beam splitter, then reflects off the top mirror, and finally passes through the beam splitter to the detector. Along the second, light passes through the beam splitter, reflects off the mirror on the right, and then reflects off the beam splitter to the detector. The number of photons reaching the detector depends on the phase difference between the two paths; by changing the arm lengths (the distances between mirrors and beam splitter), a gravitational wave changes the number of detected photons.

Detector accuracy is limited by noise. At frequencies above about 200 Hz, the limit is set by *shot noise*, random fluctuations in the flux of photons. Roughly stated, a gravitational wave is detectable if the change δN_{GW} in the number of photons reaching the detector is larger than the random fluctuation δN_{random} . Let N be the average number of photons that reach the detector in time T in the absence of a gravitational wave. Random fluctuations in photon number change that by⁵

$$\delta N_{\text{random}} \approx \sqrt{N}. \quad (6.68)$$

⁵The random-walk relation $\delta N_{\text{random}} = \sqrt{N}$ holds for macroscopic incandescent and florescent light sources because a large number of independent contributing degrees of freedom yield a Poisson distribution of photons. For a laser, it holds

From Eq. (6.64), a wave of amplitude h changes the distance ℓ between mirrors by a length $\delta\ell \sim h\ell$, implying a change of phase

$$\delta\phi \sim h \frac{\ell}{\lambda_{\text{laser}}}, \quad (6.69)$$

λ_{laser} the wavelength of the light. Now the amplitude of interfering beams with phase difference ϕ is proportional to

$$\sin(\omega_{\text{laser}}t + \phi) + \sin(\omega_{\text{laser}}t) = 2 \sin(\omega_{\text{laser}}t + \phi/2) \cos(\phi/2). \quad (6.70)$$

The time-averaged power P reaching the detector is then proportional to $\cos^2 \frac{\phi}{2}$. Because the number of photons reaching the detector is proportional to P ,

$$N \propto \cos^2 \frac{\phi}{2}, \quad \text{implying} \quad \delta N_{GW} \propto \delta\phi \frac{d}{d\phi} \cos^2 \frac{\phi}{2} = \delta\phi \sin \phi, \quad (6.71)$$

taking its maximum value at $\phi = \pi/2$, $\sin \phi = 1$. Then, for a detector operating with unperturbed phase difference $\pi/2$, we have $N(\phi) = 2N \cos^2 \frac{\phi}{2}$, and

$$\delta N_{GW} = N \delta\phi \sim Nh \frac{\ell}{\lambda_{\text{laser}}}. \quad (6.72)$$

From Eqs.(6.68) and (6.72), the detector can hear a wave when

$$1 \lesssim \frac{\delta N_{GW}}{\delta N_{\text{random}}} \sim \sqrt{N} h \frac{\ell}{\lambda_{\text{laser}}},$$

or

$$h \gtrsim \frac{1}{\sqrt{N}} \frac{\lambda_{\text{laser}}}{\ell}. \quad (6.73)$$

For a laser of power P ,

$$N = \frac{PT}{\hbar\omega_{\text{laser}}}. \quad (6.74)$$

The laser power and wavelength of the LIGO laser and interferometer arm length are

$$P = 20 \text{ W}, \quad \lambda_{\text{laser}} = 1.064 \times 10^{-4} \text{ cm} \quad (\omega_{\text{laser}} = 1.8 \times 10^{15} \text{ s}^{-1}), \quad \ell = 4 \text{ km}.$$

The binary black hole systems with individual masses $\gtrsim 30M_{\odot}$ that have dominated the LIGO/VIRGO observations have frequencies above 200 Hz for less than 10 cycles, implying an observation time $T \lesssim 10/f$. The number of photons in this time is, by Eq. (6.74),

$$N \approx \frac{(20 \times 10^7 \text{ erg/s})(10/200 \text{ s}^{-1})}{\hbar(1.8 \times 10^{15} \text{ s}^{-1})} = 5 \times 10^{18}. \quad (6.75)$$

If nothing further were done, the amplitude of an observable 300 Hz wave ($\lambda_{GW} = 10^8 \text{ cm}$) would, by Eq. (6.73), be

$$h \gtrsim \frac{1}{\sqrt{5 \times 10^{18}}} \frac{10^{-4} \text{ cm}}{4 \times 10^5 \text{ cm}} \approx \times 10^{-19}, \quad (6.76)$$

because the quantum state of the light is a coherent state (or a sum of coherent states of different phases), having the form $\exp(\alpha a^\dagger)|0\rangle$, with $a^\dagger$ the creation operator associated with a particular polarization and frequency and α a complex number. For a state of this form, the probability of n photons is the Poisson distribution

$$P(n) = e^{-\langle n \rangle} \frac{\langle n \rangle^n}{n!},$$

where $n = aa^\dagger$ is the number operator and $\langle n \rangle = |\alpha|^2$ is its expectation value. Finally, for a Poisson distribution with large $\langle n \rangle$, the rms deviation is $\sqrt{\langle (n - \langle n \rangle)^2 \rangle} = \sqrt{\langle n^2 \rangle}$.

a few orders of magnitude larger than any observed source.

Two of the improvements that dramatically enhance the sensitivity are

1. The effective length of each arm is greatly increased by adding an extra mirror along the arm. After entering the instrument via the beam splitter, the light in each arm bounces between its two mirrors about 300 times before leaving from the beam splitter and merging with the light from the other arm. The result is an average photon path of about 1200 km and a decrease by a factor of 300 in the smallest observable h .⁶ (Each pair of mirrors with light bouncing between them constitutes a *Fabry-Perot cavity*.)

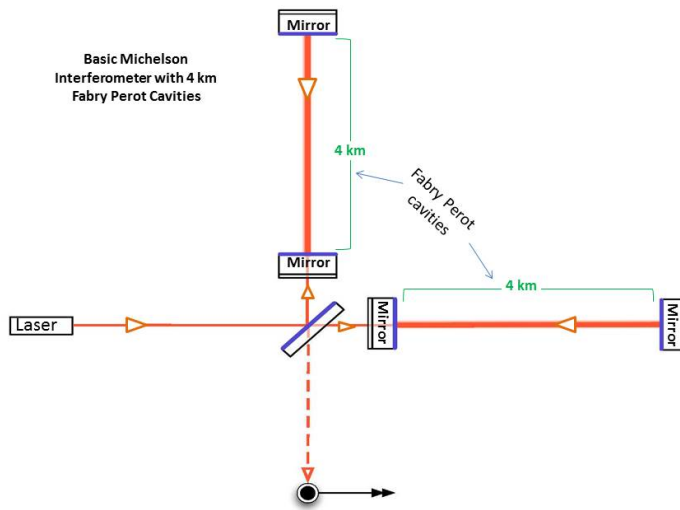


Figure 6.3: Diagram from the same LIGO webpage as that of Fig. 6.2

2. A power recycling mirror, with high reflectivity, is placed between the laser and the interferometer. The light that would reflect off the beam splitter in the direction of the laser is thus recycled, and the net power entering the interferometer is enhanced by a factor of about 35.

Our estimate for the minimum observable amplitude then decreases by a factor of 300 from (1) and a factor of $\sqrt{35}$ from (2), giving

$$h \gtrsim 6 \times 10^{-23}. \quad (6.77)$$

This value is now smaller than h for the first observed source, GW150917.

Below about 200 Hz, other sources of noise are larger than shot noise. Sensitivity below about 100 Hz is limited by fluctuations in radiation pressure on the mirrors; like shot noise this is due to random fluctuations in the number of photons, this time in the number of photons hitting each mirror. Below about 10 Hz, seismic noise is dominant, and between about 100 Hz and 200 Hz, thermal noise in the mirror coatings is important. Shot noise and radiation-pressure noise are together called quantum noise because they both due to the Poisson distribution of photon number that describes the quantum state of laser light (see footnote 5).

⁶This effective optical path length is deliberately kept below the wavelength corresponding to detectable frequencies. A greater path length decreases sensitivity because the distance between mirrors changes in the time a photon traverses the path.

- Exercise 84.* a. Find the maximum value of h_+ or $h_\times$ for a binary neutron star system, with stars of mass $1.4M_\odot$, at a distance 100 Mpc. You can use Eq.(6.53) or the preceding equations for I_{ij} ; assume that the maximum amplitude is reached just before the stars merge, with a distance of about 30 km between their centers.
- b. Check that the maximum amplitude for a distance of 30 Mpc agrees with that reported by LIGO/Virgo for GW170817.

Gauge transformations as infinitesimal coordinate transformations

In Sect. 6.1.2, we defined a gauge transformation using an active transformation, an infinitesimal diffeo that drags along all tensor fields to give an physically equivalent system. One can equivalently keep the tensor fields fixed and make an infinitesimal coordinate transformation. This is the more common approach, and it is given here for reference.

We consider a coordinate transformation from x to $\bar{x}(x)$. The equations in this section will be more transparent if the bar is on g instead of on its indices. We will write the barred components of the tensor $g_{\alpha\beta}$ as

$$\bar{g}_{\mu\nu}(\bar{x}) = \frac{\partial x^\sigma}{\partial \bar{x}^\mu} \frac{\partial x^\tau}{\partial \bar{x}^\nu} g_{\sigma\tau}(x). \quad (6.78)$$

Now write $x^\mu(\bar{x}) = \bar{x}^\mu + \xi^\mu$ with ξ small. To first order in ξ , we have

$$\bar{g}_{\mu\nu}(\bar{x}) = \bar{g}_{\mu\nu}(x - \xi) = \bar{g}_{\mu\nu}(x) - \xi^\sigma \partial_\sigma g_{\mu\nu}(x). \quad (6.79)$$

On the right side of (6.78), we have

$$\frac{\partial x^\sigma}{\partial \bar{x}^\mu} = \frac{\partial \bar{x}^\sigma}{\partial \bar{x}^\mu} + \frac{\partial \xi^\sigma}{\partial \bar{x}^\mu} = \delta_\mu^\sigma + \frac{\partial \xi^\sigma}{\partial x^\mu} + O(\xi^2),$$

giving

$$\frac{\partial x^\sigma}{\partial \bar{x}^\mu} \frac{\partial x^\tau}{\partial \bar{x}^\nu} g_{\sigma\tau}(x) = g_{\mu\nu}(x) + \partial_\mu \xi^\sigma g_{\sigma\nu} + \partial_\nu \xi^\sigma g_{\mu\sigma}. \quad (6.80)$$

Then, replacing the left and right sides of Eq. (6.78) by the expressions in Eq. (6.79) and (6.80), respectively, we obtain

$$\begin{aligned} \bar{g}_{\mu\nu}(x) &= g_{\mu\nu}(x) + (\xi^\sigma \partial_\sigma g_{\mu\nu} + \partial_\mu \xi^\sigma g_{\sigma\nu} + \partial_\nu \xi^\sigma g_{\mu\sigma}) \\ &= g_{\mu\nu}(x) + \mathcal{L}_\xi g_{\mu\nu}(x). \end{aligned} \quad (6.81)$$

That is, to linear order in ξ a change of coordinates changes the metric components by the components of its Lie derivative $\mathcal{L}_\xi g_{\alpha\beta}$.

Equivalently, given a family of coordinate transformations $\bar{x}(\lambda, x)$ with

$$\bar{x}^\mu(0, x) = x^\mu, \quad \frac{d}{d\lambda} \bar{x}^\mu(\lambda, x)|_{\lambda=0} = -\xi^\mu, \quad (6.82)$$

i.e., $x^\mu = \bar{x}^\mu + \lambda \xi^\mu + O(\lambda^2)$, we have

$$\frac{d}{d\lambda} \bar{g}_{\mu\nu}(\lambda)|_{\lambda=0} = \mathcal{L}_\xi g_{\mu\nu}. \quad (6.83)$$

The family $\bar{g}_{\mu\nu}(\lambda)$ differs only by a coordinate transformation from $g_{\mu\nu}(\lambda)$, and at linear order in λ ,

$$\delta \bar{g}_{\mu\nu} = \frac{d}{d\lambda} \bar{g}_{\mu\nu}(\lambda)|_{\lambda=0} = h_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}, \quad (6.84)$$

where

$$h_{\mu\nu} = \frac{d}{d\lambda} g_{\mu\nu}(\lambda)|_{\lambda=0}. \quad (6.85)$$

More generally, if $S_{\sigma\cdots\tau}^{\mu\cdots\nu}$ are the components of any tensor, then its components in the chart $\bar{x}(\lambda)$ differ from the original components by $\mathcal{L}_\xi S$ at linear order in λ :

$$\frac{d}{d\lambda} \bar{S}_{\sigma\cdots\tau}^{\mu\cdots\nu}|_{\lambda=0} = \mathcal{L}_\xi S_{\sigma\cdots\tau}^{\mu\cdots\nu} \quad (6.86)$$

and a family of tensors, e.g., $g_{\alpha\beta}(\lambda, x)$, $T_{\alpha\beta}(\lambda, x)$, has components in the family of charts $\bar{x}(\lambda, x)$ given at linear order in λ by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}, \quad \bar{\delta}T_{\mu\nu} = \delta T_{\mu\nu} + \mathcal{L}_\xi T_{\mu\nu}. \quad (6.87)$$

In particular, for a flat background, the perturbation $h_{\alpha\beta}$ is physically equivalent to

$h_{\alpha\beta} + \mathcal{L}_\xi \eta_{\alpha\beta} = h_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha$, where ∇_α is the flat covariant derivative, with components ∂_μ in Cartesian coordinates.

Chapter 7

3+1 Split and Initial Value Equations

7.1 Notation

A detailed table is given here in case of confusion from differences between notation in the notes, in the text by Baumgarte & Shapiro[15] and in the text by Wald[95].

Item	These notes	Baumgarte & Shapiro	Wald
Signature	$- + ++$	$- + ++$	$- + ++$
Indices $\alpha, \beta, \dots$	spacetime abstract	not used	spacetime concrete
Indices μ, ν	spacetime concrete	not used	spacetime concrete
Indices $a, b, \dots$	space and n -dimensional abstract	spacetime	spacetime abstract
Indices $i, j, \dots$	space and n -dimensional concrete	space	not used
Spacetime	M	not named	M
Spacelike hypersurface	Σ_t	not named	Σ_t
3-dimensional surface	Σ	not named	Σ
Future pointing normal	n^α	n^a	n^a
Projection $\perp \mathbf{n}$	γ_α^β	γ_a^b	h_a^b
Spatial metric on M	$\gamma_{\alpha\beta}$	γ_{ab}	h_{ab}
Pullback from M to Σ	γ_a^α	γ_i^a	not used
Extrinsic curvature	$K_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_\mathbf{n}\gamma_{\alpha\beta}$	$K_{ab} = -\frac{1}{2}\mathcal{L}_\mathbf{n}\gamma_{ab}$	$K_{ab} = \frac{1}{2}\mathcal{L}_\mathbf{n}\gamma_{ab}$
Riemann tensor on M	$R_{\alpha\beta\gamma\delta}$	${}^{(4)}R_{abcd}$	R_{abcd}
Spatial Riemann on M	${}^3R_{\alpha\beta\gamma\delta}$	R_{ijkl}	${}^{(3)}R_{abcd}$
Lapse	α	α	N
Shift	β^α	β^a	N^a
∂_t	t^α	t^a	t^a
Energy density $T_{\alpha\beta}u^\alpha u^\beta$	ρ	not used	ρ
Energy density $T_{a\beta}n^\alpha n^\beta$	ρ_E	ρ	not used

7.2 Introduction

Like the equations of ordinary mechanics, the equations governing classical fields – scalar, electromagnetic, and gravitational – allow one to predict the future (or retrodict the past) from initial data at a given time. For a system of particles, one specifies their initial positions q^i and momenta p_i (or velocities $\dot{q}^i$)

and finds their time evolution from equations for $\dot{q}^i$ and $\dot{p}_i$ in terms of q^i and p_i :

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad (7.1)$$

where $H(p, q)$ depends on q and p , not on their time derivatives. For example, with $H = \frac{1}{2}g^{ij}p_i p_j + V(q)$, where g^{ij} is constant,

$$\dot{q}^i = p^i, \quad \dot{p}_i = -\frac{\partial V}{\partial q^i}. \quad (7.2)$$

Here $p^i := g^{ij}p_j$. Notice that, with $\dot{q}^i$ regarded as a vector, the momentum p_i is naturally a dual vector (covariant vector).

Exercise 85. What is (g^{ij}) for a two particle system with masses m and M ? Assume Cartesian coordinates.

Exercise 86. What is (g^{ij}) for one particle of mass m in spherical coordinates? Find the equations of motion that generalize Eqs. (7.2).

Scalar field

We consider first a scalar field in flat space, a field satisfying the scalar wave equation

$$\square \Phi := \nabla_\alpha \nabla^\alpha \Phi = 0. \quad (7.3)$$

We begin with a coordinate-based description of the initial value problem and then rephrase it in a way that can be carried over to curved spacetime. The coordinate description uses natural coordinates $\{t, x^i\}$ on Minkowski space. As in the case of a system of particles, one freely specifies the initial values of the field Φ and its conjugate momentum $p = \dot{\Phi} = \partial_t \Phi$ at a fixed time t_0 , now at each point (t_0, x^i) . The time evolution of the field is given by

$$\dot{\Phi} = p, \quad \dot{p} = \nabla^2 \Phi = \partial_i \partial^i \Phi. \quad (7.4)$$

Exercise 87. With Hamiltonian $H = \frac{1}{2} \int d^3r [p^2 + (\nabla \Phi)^2]$, show that

$$\frac{\delta H}{\delta p(\mathbf{r})} = p(\mathbf{r}), \quad \frac{\delta H}{\delta \Phi(\mathbf{r})} = -\nabla^2 \Phi(\mathbf{r}), \quad (7.5)$$

for $\delta \Phi$ vanishing as $\mathbf{r} \rightarrow \infty$. That is, Eqs. (7.4) are Hamilton's equations for the scalar field.

Here's the rephrasing, without introducing spatial coordinates. A choice t of Minkowski time slices spacetime into a set of $t = \text{constant}$ surfaces, which we call Σ_t . The future pointing unit normal to the surface is $n^\alpha = -\nabla^\alpha t$. Using t and the flat metric, we can decompose spacetime into a product $\mathbb{R} \times \Sigma$ of time $\mathbb{R}$ and space $\Sigma = \mathbb{R}^3$, with $\Sigma_t = \{t\} \times \Sigma$; here, if $\mathbf{r}$ is a point of Σ , the lines of fixed $\mathbf{r}$ are orthogonal to Σ_t , with tangent $n^\alpha = -\nabla^\alpha t$.

We can similarly decompose the 4-dimensional gradient $\nabla_\alpha \Phi$ into the time derivative $\dot{\Phi} = n^\alpha \nabla_\alpha \Phi$ and spatial gradient $\nabla \Phi$. The spatial gradient $\nabla \Phi$ is the gradient of Φ regarded as a function on the hypersurface Σ .

One freely specifies the initial values of the field Φ and its conjugate momentum $p = n^\alpha \nabla_\alpha \Phi$ at a fixed time t_0 at each point $\mathbf{r}$ of Σ . The time evolution of the field is given by

$$\dot{\Phi} = p, \quad \dot{p} = \nabla^2 \Phi. \quad (7.6)$$

We can then regard the time evolution as giving successive values of $\Phi(\mathbf{r}), p(\mathbf{r})$ on the fixed space Σ .

Equivalently, one can maintain a spacetime view, in which Φ and p are functions on spacetime and the spatial gradient of Φ is the spatial projection of the 4-dimensional gradient: One subtracts from the gradient $\nabla_\alpha f$ of a function the part of the gradient along $\nabla_\alpha t$. That is, we define the spatial gradient D_α of a function f on spacetime by

$$D_\alpha f = \nabla_\alpha f - \nabla_\alpha t n^\beta \nabla_\beta f = (\delta_\alpha^\beta - \nabla_\alpha t n^\beta) \nabla_\beta f. \quad (7.7)$$

(We will from now on use D instead of ∇ for the spatial gradient, to distinguish it from the spacetime gradient.) We will repeatedly use this spatial projection. Using $n_\alpha = -\nabla_\alpha t$, we write

$$\gamma_\alpha^\beta := \delta_\alpha^\beta + n_\alpha n^\beta = \delta_\alpha^\beta + n_\alpha n^\beta, \quad D_\alpha f = \gamma_\alpha^\beta \nabla_\beta f. \quad (7.8)$$

One says that γ_α^β projects vectors and covectors tangent to Σ_t and orthogonal to its normal n^α . In this spacetime description, one is specifying Φ and $\dot{\Phi}$ on a given Σ_t and finding Φ everywhere else in spacetime from the scalar wave equation, $\square \Phi = 0$.

Notice that the nonzero part of $\gamma_{\alpha\beta}$ is the spatial part of the Minkowski metric, the Euclidean metric on each Σ_t . In the chart $\{t, x^i\}$, it has components

$$[\gamma_{ij}] = \begin{bmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{bmatrix}. \quad (\text{Minkowski space}) \quad (7.9)$$

In Wald's textbook, the projection operator is denoted by h_α^β instead of γ_α^β . These days, however, you are most likely to encounter the initial value equations in numerical relativity papers. In this part of the recent literature, the spatial metric on $t = \text{constant}$ surfaces is generally written as γ_{ab} , with γ_α^β the projection operator.

Electromagnetism

For electromagnetism and gravity, an additional feature arises: The equations are now analogous to those governing a system of particles with a constraint that is preserved by the time evolution— for example, a set of particles constrained to move on a fixed surface.

We follow the treatment of electromagnetism in Sect. 1.5, with $j^\alpha = 0$. In that section, the future pointing unit normal $t^\alpha \equiv \partial_t$ was denoted by t^α instead of n^α . Here we will stick with n^α for the future pointing normal, because in curved space, the normal n^α to a $t = \text{constant}$ surface will not in general be along ∂_t .

The 3+1 split of spacetime is extended to a 3+1 split of each index of the electromagnetic tensor $F_{\alpha\beta}$ and of the Maxwell equations

$$\nabla_\beta F^{\alpha\beta} = 0, \quad \nabla_{[\alpha} F_{\beta\gamma]} = 0. \quad (7.10)$$

We can decompose each index of a tensor into temporal and spatial parts, parts along and orthogonal to n^α , by writing

$$\delta_\alpha^\beta = \gamma_\alpha^\beta - n_\alpha n^\beta; \quad (7.11)$$

for example, for a two-index tensor,

$$\begin{aligned} T_{\alpha\beta} &= \delta_\alpha^\delta \delta_\beta^\epsilon T_{\delta\epsilon} = (\gamma_\alpha^\delta - n_\alpha n^\delta)(\gamma_\beta^\epsilon - n_\beta n^\epsilon) T_{\delta\epsilon} \\ &= \gamma_\alpha^\delta \gamma_\beta^\epsilon T_{\delta\epsilon} - n_\alpha n^\delta \gamma_\beta^\epsilon T_{\delta\epsilon} - \gamma_\alpha^\delta n_\beta n^\epsilon T_{\delta\epsilon} + n_\alpha n^\delta n_\beta n^\epsilon T_{\delta\epsilon} \end{aligned} \quad (7.12)$$

Because of the antisymmetry of $F_{\alpha\beta}$, the time projection of both indices vanishes:

$$F_{\alpha\beta} n^\alpha n^\beta = 0. \quad (7.13)$$

Projecting one index along n^α gives a 4-vector that is already spatial:

$$E^\alpha := F^\alpha{}_\beta n^\beta, \quad (7.14)$$

with

$$E_\alpha n^\alpha = 0 \implies E_\alpha = \gamma_\alpha^\delta E_\delta = \gamma_\alpha^\delta n^\beta F_{\delta\beta}. \quad (7.15)$$

We can regard E^α as the electric field seen by an observer whose 4-velocity is n^α : In the chart $\{t, x^i\}$, E^α has components $(E^\mu) = (0, E^i)$. E^α is the unique 4-vector E^α orthogonal to n^α whose spatial part is the usual 3-vector electric field of an observer with 4-velocity n^α .

The magnetic field is the spatial part of $F_{\alpha\beta}$; its natural form is a 2-index antisymmetric tensor,

$$B_{\alpha\beta} = \gamma_\alpha^\delta \gamma_\beta^\epsilon F_{\delta\epsilon}. \quad (7.16)$$

To define the corresponding vector, one needs to use $\epsilon_{\alpha\beta\gamma\delta}$,

$$B^\alpha = \frac{1}{2} \epsilon^{\beta\alpha\gamma\delta} n_\beta B_{\gamma\delta}. \quad (7.17)$$

Exercise 88. Using Eq. (7.17) and the antisymmetry of $\epsilon^{\alpha\beta\gamma\delta}$, show that

$$B^\alpha = n_\beta {}^*F^{\beta\alpha}, \text{ where } {}^*F^{\alpha\beta} := \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}. \quad (7.18)$$

It is because $\epsilon_{\alpha\beta\gamma\delta}$ changes sign under reflection that B^α behaves as a pseudovector – that its image under reflection is opposite to that of a vector like E^α . In terms of E_α and B_α , the 3+1 split (7.12) of $F_{\alpha\beta}$ has the form

$$F_{\alpha\beta} = n_\alpha E_\beta - n_\beta E_\alpha + \epsilon_{\alpha\beta\gamma} B^\gamma. \quad (7.19)$$

where $\epsilon_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma\delta} n^\delta$.

In the 3+1 split of the Maxwell equations, write the second equation in (7.10) in the form $0 = \nabla_\beta {}^*F^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\nabla_\beta F_{\gamma\delta}$. The projections along n_α of the expressions in Maxwell's equations are

$$n_\alpha \nabla_\beta F^{\alpha\beta} = \nabla_\beta E^\beta, \quad n_\alpha \nabla_\beta {}^*F^{\alpha\beta} = \nabla_\beta B^\beta, \quad (7.20)$$

giving the two equations

$$\nabla_\beta E^\beta = 0, \quad \nabla_\beta B^\beta = 0.$$

These equations involve only spatial derivatives. In Minkowski space, they can be written

$$D_\beta E^\beta = 0, \quad D_\beta B^\beta = 0, \quad (7.21)$$

with 3-dimensional forms on the 3-surface Σ

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0. \quad (7.22)$$

They are *constraint equations*, constraints on the initial values of E^α and B^α that can be specified on Σ .

The spatial projections of the two Maxwell equations are the *dynamical equations*, giving the time evolution of the initial data:

$$\gamma_\delta^\alpha \nabla_\beta F^{\delta\beta} = -n^\beta \nabla_\beta E^\alpha + \epsilon^{\alpha\beta\gamma} D_\beta B_\gamma = 0, \quad (7.23a)$$

$$\gamma_\delta^\alpha \nabla_\beta {}^*F^{\delta\beta} = n^\beta \nabla_\beta B^\alpha + \epsilon^{\alpha\beta\gamma} D_\beta E_\gamma = 0. \quad (7.23b)$$

or, in 3-dimensional form on Σ ,

$$\dot{\mathbf{E}} = \nabla \times \mathbf{B}, \quad (7.24a)$$

$$\dot{\mathbf{B}} = -\nabla \times \mathbf{E}. \quad (7.24b)$$

We have been using the one-one correspondence between vectors v^a on Σ and vectors v^α at each point of a given Σ_t that are orthogonal to n^α . The right way to describe the correspondence uses the definitions of *pullback* and *push forward* (drag); see, e.g., p. 125. The map $\psi : \Sigma \rightarrow \Sigma_t \subset M$ takes each point $\mathbf{r}$ of Σ to the point $\psi(\mathbf{r}) = (t, \mathbf{r})$ of $M = \mathbb{R} \times \Sigma$. Then ψ drags a vector v^a at the point $\mathbf{r}$ to a vector $\psi^*v = v^\alpha$ at $(t, \mathbf{r})$ of M . Because t is constant on Σ_t , $v^\alpha \nabla_\alpha t = 0$. There is similarly a one-one correspondence between tensors $T^{a\dots b}$ on Σ and tensors $T^{\alpha\dots\beta}$ at each point of a given Σ_t that are orthogonal in each index to n^α ,

$$T^{\alpha\dots\beta} n_\alpha = 0, \quad \dots, \quad T^{\alpha\dots\beta} n_\beta = 0. \quad (7.25)$$

And the map ψ pulls back covariant vectors and tensors on M to tensors on Σ . We will revisit this correspondence after introducing the 3+1 split in curved spacetimes.

7.3 GR: The 3+1 Split

The initial value problem in general relativity is associated with a slicing of spacetime by a family of spacelike hypersurfaces. For detailed accounts on the historical development of the subject, see for, example, [2, 16, 13, 45]. The formalism we and much of the literature uses was introduced about sixty years ago by Arnowitt, Deser and Misner[5] (out of print and republished as [7]).

The Einstein equation (and the associated conservation of energy and momentum), conservation of baryons, and the equation of state of the fluid, determine the evolution of the geometry and the fluid variables from one hypersurface to another. We will see that the 3+1 decomposition of the spacetime is mirrored in a 3+1 decomposition of the equations and of the tensors whose time-evolution they describe.

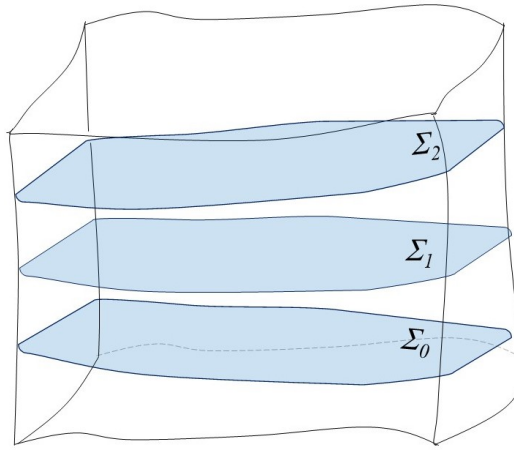


Figure 7.1: Three of the hypersurfaces that slice a spacetime

A choice of time coordinate t again gives a slicing by $t = \text{constant}$ surfaces Σ_t whose normal $\nabla_\alpha t$ is everywhere timelike. The spacetimes we consider have the form $\mathbb{R} \times \Sigma$, with each hypersurface Σ_t a copy $\{t\} \times \Sigma$ of Σ .¹ We call the future direction the direction in which t increases: That is, a timelike path is future directed if t increases along the path. The $-+++$ metric signature implies that the future pointing contravariant unit vector normal to each slice Σ_t is

$$n^\alpha = -\frac{\nabla^\alpha t}{\sqrt{-\nabla_\beta t \nabla^\beta t}} = -\alpha \nabla^\alpha t, \quad (7.26)$$

where

$$\alpha := (-\nabla_\beta t \nabla^\beta t)^{-1/2} = |g^{tt}|^{-1/2}, \quad (7.27)$$

in a chart of the form $\{t, x^i\}$. The components of $\nabla_\alpha t$ and n_α are then

$$\nabla_\mu t = \delta_\mu^t, \quad n_\mu = -\alpha \delta_\mu^t. \quad (7.28)$$

The scalar α is called the *lapse*, because αdt is the proper time elapsed in a normal direction between coordinate times t and $t + dt$. It is the proper time, measured normal to the hypersurfaces between slices Σ_t and Σ_{t+dt} .

¹The mathematical literature calls a decomposition of an n -dimensional manifold M into a set $\mathbb{R} \times \Sigma$ of $(n-1)$ -dimensional manifolds a *foliation*, with each copy of Σ a *leaf*; following the physics literature, we use the word *slicing*, and each copy Σ_t of Σ is a *slice*.

Exercise 89. Check that αdt is the proper time between slices by showing that, if the chart $\{t, x^i\}$ is chosen to have lines of constant x^i perpendicular to Σ_t , then $g_{ti} = 0$ and $g_{tt} = \frac{1}{g^{tt}}$. Conclude that proper time elapsed normal to Σ_t is $d\tau = \alpha dt$.

We now extend the 3+1 decomposition $\mathbb{R} \times \Sigma$ of spacetime into space and time to a decomposition of vectors and tensors. Call a vector v^α a *spatial* vector if it is tangent to a path $c(\lambda)$ that lies in a spacelike slice Σ_t ; equivalently $v^\alpha n_\alpha = 0$. We say such a vector v^α is tangent to Σ_t . If v^α is tangent to Σ_t , then $v^t = 0$ in a chart $\{t, x^i\}$. That is, if $\{t, x^i\}$ are the coordinates of $c(0)$, then the coordinates of the path c are $(x^\mu(\lambda)) = (t, x^i(\lambda))$, and the components of v^α are

$$v^\mu = \left. \frac{d}{d\lambda} c^\mu(\lambda) \right|_{\lambda=0},$$

implying

$$(v^\mu) = (0, v^i). \quad (7.29)$$

We again introduce the projection operator

$$\gamma^\alpha{}_\beta := \delta^\alpha{}_\beta + n^\alpha n_\beta. \quad (7.30)$$

onto each Σ_t : $\gamma^\alpha{}_\beta$ kills vectors $v^\alpha = v n^\alpha$ normal to Σ_t , and it leaves invariant vectors v^α tangent to Σ_t :

$$\gamma^\alpha{}_\beta n^\beta = 0, \quad \gamma^\alpha{}_\beta v^\beta = v^\alpha \text{ for } v_\alpha n^\alpha = 0. \quad (7.31)$$

Any vector v^α (or any tensor index) can be decomposed into a spatial vector (one that is tangent to Σ_t) and a vector orthogonal to Σ_t :

$$v^\alpha = \delta^\alpha{}_\beta v^\beta = (\gamma^\alpha{}_\beta - n^\alpha n_\beta) v^\beta = \gamma^\alpha{}_\beta v^\beta - n_\beta v^\beta n^\alpha. \quad (7.32)$$

The corresponding decomposition of the field equation $E^{\alpha\beta} := G^{\alpha\beta} - 8\pi T^{\alpha\beta} = 0$, is

$$E^{\alpha\beta} = \gamma^\alpha{}_\gamma \gamma^\beta{}_\delta E^{\gamma\delta} - 2n_\delta E^{\gamma\delta} \gamma_\gamma^{(\alpha} n^{\beta)} + n_\gamma n_\delta E^{\gamma\delta} n^\alpha n^\beta. \quad (7.33)$$

We will see that the projection $\gamma^\alpha{}_\gamma \gamma^\beta{}_\delta E^{\gamma\delta}$ of $E^{\alpha\beta} = 0$ onto the hypersurface Σ_t is a dynamical equation for the metric (involving its second time derivative), while the remaining parts, $G_{\alpha\beta} n^\beta = 8\pi T_{\alpha\beta} n^\beta$, of the field equation serve as constraints (equations involving only the values of the metric and its first time derivative on Σ_t).

The 3+1 decomposition of the metric is just

$$g_{\alpha\beta} = \gamma_{\alpha\beta} - n_\alpha n_\beta, \quad (7.34)$$

the result of lowering the index α in Eq. (7.31). In particular, $g_{\perp\alpha} = g_{\sigma\tau} \gamma^\sigma{}_\alpha n^\tau = \gamma_{\alpha\tau} n^\tau = 0$.

The tensor $\gamma_{\alpha\beta}$ can be regarded as the 3-metric on Σ_t , written with spacetime indices: Acting on vectors tangent to Σ_t , it gives their dot product, and it kills vectors orthogonal to Σ_t . More formally, at each time t , the spacetime metric $g_{\alpha\beta}$ induces a metric γ_{ab} , on Σ . The map $\psi : \Sigma \rightarrow \Sigma_t$ pulls back $g_{\alpha\beta}$ to $\gamma_{ab} = (\psi_* g)_{ab}$. In a chart of the form $\{t, x^i\}$, the components of γ_{ab} and the spatial components of $\gamma_{\alpha\beta}$ and $g_{\alpha\beta}$ coincide:

$$\gamma_{ij} = g_{ij}. \quad (7.35)$$

We use the same symbol γ for the tensor γ_{ab} on Σ and $\gamma_{\alpha\beta}$ on Σ_t : $\gamma_{\alpha\beta}$ is the unique spacetime tensor orthogonal to n^α whose pullback is γ_{ab} .

It will simplify covariant derivations of the initial value equations to stick with spacetime indices, instead of using both tensors on Σ and tensors on M . We do this as follows:

Definition. A tensor $T^{\alpha\cdots\beta}_{\gamma\cdots\delta}$ *spatial* if it is orthogonal in all its indices to n_α :

$$T^{\alpha\cdots\beta}_{\gamma\cdots\delta}n_\alpha = 0, \dots, T^{\alpha\cdots\beta}_{\gamma\cdots\delta}n^\delta = 0. \quad (7.36)$$

We then note:

There is a one-one correspondence between spatial tensors $T^{\alpha\cdots\beta}_{\gamma\cdots\delta}$ at a point (t, x) in M and tensors $T^{a\cdots b}_{c\cdots d}$ at x in Σ .

This is easy to show. We'll again use a chart $\{t, x^i\}$ on M with corresponding chart $\{x^i\}$ on Σ . We've already seen that a vector on Σ with components v^i is dragged to a vector on M with components $(v^\mu) = (0, v^i)$, or

$$v^\mu = \frac{\partial x^\mu}{\partial x^i} v^i = \delta_i^\mu v^i, \quad (7.37)$$

A contravariant tensor on Σ with components $T^{i\cdots j}$ is then dragged to the spatial tensor with vanishing t components and with spatial components $T^{i\cdots j}$.

$$T^{\mu\cdots\nu} = \delta_i^\mu \cdots \delta_j^\nu T^{i\cdots j}. \quad (7.38)$$

Conversely, any contravariant spatial tensor $T^{\alpha\cdots\beta}$ has vanishing t components and corresponds to the unique tensor $T^{a\cdots b}$ on Σ with components $T^{i\cdots j}$.

While spatial vectors are naturally defined as tangents to curves that lie in Σ_t or vectors for which $v^\alpha \nabla_\alpha t = 0$, a *spatial covector* σ_α uses the metric on M for its definition. That is, one needs the metric to define the dot product $\sigma^\alpha \nabla_\alpha t = g^{\alpha\beta} \sigma_\beta \nabla_\alpha t = 0$. To define the tensor $T^{\alpha\cdots\beta}_{\gamma\cdots\delta}$ corresponding to $T^{a\cdots b}_{c\cdots d}$, one raises the indices of each tensor and requires $T^{\alpha\cdots\beta\gamma\cdots\delta}$ to be the unique spatial tensor corresponding to $T^{a\cdots b}_{c\cdots d}$. Then the spatial components of corresponding tensors—contravariant or covariant—agree. But be careful: Covariant t components need not vanish.

$$\sigma_\alpha \text{ spatial does not imply } \sigma_t = 0.$$

Exercise 90. Let σ_a be a covector at a point $x \in \Sigma$. Show that the corresponding vector σ_α at (t, x) has spatial components σ_i , but that σ_t is nonzero if g_{ti} is nonzero. To avoid confusion in this exercise, write σ_a for the 3-vector on Σ , $\tilde{\sigma}_\alpha$ for the 4-vector on M .

Finally, note that the pullback of a tensor $T_{\alpha\cdots\beta}$ on M at $\{t, x^i\}$ is a tensor $T_{a\cdots b}$ at $\{x^i\}$ with the same spatial components $T_{i\cdots j}$: The pullback of $T_{\alpha\cdots\beta}$ has components

$$T_{i\cdots j} = \frac{\partial x^\mu}{\partial x^i} \cdots \frac{\partial x^\nu}{\partial x^j} T_{\mu\nu} = \delta_i^\mu \cdots \delta_j^\nu T_{\mu\cdots\nu}. \quad (7.39)$$

That means the pullback of a spatial covariant tensor on M is the corresponding tensor on Σ . Because $\gamma_i^\mu = \delta_i^\mu + n_\mu n_i$, and $n_i = 0$ in a chart (t, x^i) , we have

$$\gamma_i^\mu = \delta_i^\mu, \quad (\Psi^* T)_{i\cdots j} = \gamma_i^\mu \cdots \gamma_j^\nu T_{\mu\cdots\nu},$$

where $\Psi : \Sigma \rightarrow \Sigma_t \subset M$.

We need two final definitions to cast the field equation in the standard initial-value form, the covariant derivative of γ_{ab} and the extrinsic curvature of a hypersurface Σ_t .

Covariant derivative, D_a, D_α

Denote by D_a the covariant derivative operator on Σ associated at a time t with the metric $\gamma_{ab}(t)$. If v^α is the spatial vector corresponding to the vector v^a on Σ , we will see that its spatial covariant derivative is

$$D_\beta v^\alpha := \gamma_\beta^\sigma \gamma_\tau^\alpha \nabla_\sigma v^\tau. \quad (7.40)$$

That is, with this definition, $D_\alpha v^\beta$ is the unique spatial tensor corresponding to $D_a v^b$ on Σ . On a general spatial tensor $T_{\gamma \dots \delta}^{\alpha \dots \beta}$,

$$D_\epsilon T_{\gamma \dots \delta}^{\alpha \dots \beta} := \gamma_\epsilon^\lambda \gamma_\mu^\alpha \dots \gamma_\nu^\beta \gamma_\gamma^\sigma \dots \gamma_\delta^\tau \nabla_\lambda T_{\sigma \dots \tau}^{\mu \dots \nu}. \quad (7.41)$$

It is easy to see that D_α is a derivative operator, satisfying the conditions following Eq. 2.21 (linear, Leibnitz rule, torsion free). The covariant derivative D_a on Σ is the unique derivative operator for which $D_a \gamma_{ab} = 0$; that D_α corresponds to D_a on Σ follows from the fact that $D_\delta \gamma_{\alpha\beta} = 0$:

$$D_\delta \gamma_{\alpha\beta} = \gamma_\delta^\lambda \gamma_\alpha^\sigma \gamma_\beta^\tau \nabla_\lambda (g_{\sigma\tau} + n_\sigma n_\tau) = \gamma_\delta^\lambda \gamma_\alpha^\sigma \gamma_\beta^\tau (0 + n_\sigma \nabla_\lambda n_\tau + n_\tau \nabla_\lambda n_\sigma) = 0,$$

where we used $\gamma_\alpha^\beta n_\beta = 0$. We will generally adopt spacetime indices.

Extrinsic curvature

Although one ordinarily chooses coordinates for which t^α is not normal to Σ_t , it is the time derivative normal to the surface that naturally enters the projections of the field equation, (7.48,7.46,7.45) below. This time derivative of the 3-metric is (up to a factor -1/2) the *extrinsic curvature* $K_{\alpha\beta}$ of the hypersurface Σ_t .²

$$K_{\alpha\beta} := -\frac{1}{2} \mathcal{L}_n \gamma_{\alpha\beta}. \quad (7.42)$$

$K_{\alpha\beta}$ is spatial, orthogonal to n^α on each index. This is immediate from $\mathcal{L}_n n^\alpha = 0$ and $\gamma_{\alpha\beta} n^\beta = 0$:

$$K_{\alpha\beta} n^\beta = -\frac{1}{2} n^\beta \mathcal{L}_n \gamma_{\alpha\beta} = -\frac{1}{2} [\mathcal{L}_n (n^\beta \gamma_{\alpha\beta}) - (\mathcal{L}_n n^\beta) \gamma_{\alpha\beta}] = 0.$$

We denote by K its contraction

$$K := K_\alpha^\alpha = \gamma^{\alpha\beta} K_{\alpha\beta}. \quad (7.43)$$

The extrinsic curvature is thus the time derivative of the 3-metric for an observer with 4-velocity n^α ; it describes the change in the geometry of Σ_t as one moves a small proper distance orthogonal to the hypersurface.

The *intrinsic* curvature, the Riemann tensor of the a surface, depends only on the metric γ_{ab} on that surface. It does not care how the surface is embedded in a higher dimensional manifold. The *extrinsic* curvature looks at the embedding. We'll see below in Eq. (7.60) that $K_{\alpha\beta}$ measures the rate at which the normal to the surface changes as one moves along the surface. First an example.

Example. Consider the flat cylinder obtained by gluing two opposite edges of a sheet of paper. Because the paper is not stretched or sheared, its intrinsic geometry is still flat. (Think of the universe of Ms. Pac Man or Asteroids with the top and bottom edges of the screen identified. The screen is still flat after the identification.) But the extrinsic curvature is not zero: The flat Euclidean 3-metric is

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2.$$

²There is no consistent convention in the literature for the sign of the extrinsic curvature of a spacelike hypersurface. Our convention agrees with that of MTW [60] and has sign opposite to Wald's [95].

A cylinder of radius $r = a$ has the flat 2-metric

$$d\sigma^2 = dz^2 + a^2 d\phi^2 = dz^2 + dx^2, \text{ with } x := a\phi.$$

($\phi = 0$ is identified with $\phi = 2\pi$ and $x = 0$ with $x = 2\pi a$.) The unit normal to the cylinder is $n^a = \nabla^a r$, with components $n^i = \delta_r^i$. Then, because $\partial_k n^i = 0$, we have $\mathcal{L}_n g_{ij} = n^k \partial_k g_{ij} = 2r \nabla_i \phi \nabla_j \phi$, where, in our cylindrical coordinates, $\nabla_i \phi = \delta_i^\phi$. The extrinsic curvature of the $r = a$ cylinder is then

$$K_{ab} = \mp a \nabla_a \phi \nabla_b \phi, \quad (7.44)$$

the sign depending on which unit normal is chosen.

The 3-metric and extrinsic curvature on an initial hypersurface Σ_t will constitute our initial data. They are constrained by the projections $E^{\alpha\beta} n_\alpha n_\beta = 0$ and $E^{\delta\beta} \gamma^\alpha_\delta n_\beta = 0$ of the Einstein equation, called, respectively, the *Hamiltonian constraint* and the *momentum constraint*. The terminology is related to the fact that

$$\rho_E := T^{\alpha\beta} n_\alpha n_\beta \quad \text{and} \quad j^\alpha := -T^{\delta\beta} \gamma^\alpha_\delta n_\beta$$

are, respectively, the densities of energy and momentum for an observer with 4-velocity n^α . We will show that the constraint equations are

Momentum constraint:

$$D_\beta (K^{\alpha\beta} - \gamma^{\alpha\beta} K) = 8\pi j^\alpha; \quad (7.45)$$

Hamiltonian constraint:

$${}^3R - K^{\alpha\beta} K_{\alpha\beta} + K^2 = 16\pi \rho_E. \quad (7.46)$$

The time evolution of the initial data is a pair of equations for $\mathcal{L}_n \gamma_{\alpha\beta}$ and $\mathcal{L}_n K_{\alpha\beta}$. The evolution equation for $\gamma_{\alpha\beta}$, analogous to $\dot{q} = p$, is simply Eq. (7.42),

$$\mathcal{L}_n \gamma_{\alpha\beta} = -2K_{\alpha\beta}. \quad (7.47)$$

The time evolution of the extrinsic curvature is given by the remaining, spatial, part of the Einstein equation, written using the Ricci tensor, namely

$$\gamma_\alpha{}^\sigma \gamma_\beta{}^\tau [R_{\sigma\tau} - 8\pi (T_{\sigma\tau} - \frac{1}{2} g_{\sigma\tau} T)] = 0, \quad \text{with } T = T_\gamma{}^\gamma.$$

It has the form

$$\mathcal{L}_n K_{\alpha\beta} = -\frac{1}{\alpha} D_\alpha D_\beta \alpha - 2K_{\alpha\gamma} K_\beta{}^\gamma + K K_{\alpha\beta} + {}^3R_{\alpha\beta} - 8\pi (\gamma_\alpha{}^\delta \gamma_\beta{}^\epsilon T_{\delta\epsilon} - \frac{1}{2} \gamma_{\alpha\beta} T). \quad (7.48)$$

Here ${}^3R_{\alpha\beta}$ is the spatial Ricci tensor of the spatial metric $\gamma_{\alpha\beta}$, and we will denote by

$${}^3R = \gamma^{\alpha\beta} {}^3R_{\alpha\beta} \quad (7.49)$$

the corresponding 3-dimensional Ricci scalar.

Our main job in this section is to derive these equations: the momentum and Hamiltonian constraints and the evolution equation for $K_{\alpha\beta}$. We begin by showing that the projections $n_\alpha G^{\alpha\beta} n_\beta$ and $\gamma^\alpha_\delta G^{\delta\beta} n_\beta$ of the Einstein tensor are given in terms of the 3-metric and extrinsic curvature by

$$n_\alpha G^{\alpha\beta} n_\beta = {}^3R + K^2 - K_{\alpha\beta} K^{\alpha\beta}, \quad (7.50)$$

and

$$-\gamma^\alpha_\delta G^{\delta\beta} n_\beta = D_\beta (K^{\alpha\beta} - \gamma^{\alpha\beta} K). \quad (7.51)$$

Because $\gamma^\alpha_\beta n^\beta = 0$, this last equation involves only the Ricci tensor:

$$-\gamma^\alpha_\delta R^{\delta\beta} n_\beta = D_\beta (K^{\alpha\beta} - \gamma^{\alpha\beta} K). \quad (7.52)$$

It is one of two *Gauss-Codazzi equations*.

The second Gauss-Codazzi equation relates the Riemann tensor of the 3-metric $\gamma_{\alpha\beta}$ to the Riemann tensor of the 4-metric, and its contraction has the form

$${}^3R_{\alpha\beta} = \gamma_\alpha^\sigma \gamma_\beta^\tau \gamma^{\delta\epsilon} R_{\sigma\delta\tau\epsilon} + K K_{\alpha\beta} - K_{\alpha\gamma} K^\gamma_\beta. \quad (7.53)$$

We now derive the Gauss-Codazzi relations for a hypersurface Σ_t . We want to relate its 3-dimensional Riemann tensor to the 4-dimensional Riemann tensor of spacetime and the extrinsic curvature $K_{\alpha\beta}$ that describes the way Σ_t is embedded in M . Here are two derivations; the first uses an appropriate choice of chart, while the second is covariant and uses the projection operator γ^α_β .

Derivation by a choice of chart

The quickest derivation uses the fact that an equation involving tensors holds if holds in one chart. We choose a chart (x^0, x^i) for which x^0 is proper time from Σ_t along curves normal to Σ_t and x^i is constant along the curves, so that $n^i = n^\alpha \nabla_\alpha x^i = 0$. Then

$$g_{00} = -1, \quad g_{0i} = 0, \quad n^\mu = \delta^\mu_0, \quad n_\mu = -\delta^\mu_0, \quad (7.54)$$

($g_{0i} = 0$ because ∂_i lies in an $x^0 = \text{constant}$ surface and is therefore orthogonal to $\partial_0 = n^\alpha$), implying

$$\gamma_{ij} = g_{ij} \text{ (true for any } \{x^i\} \text{ on } \Sigma), \quad \gamma^{ij} = g^{ij} \text{ (true because } n^i = 0), \quad g^{0i} = 0. \quad (7.55)$$

Eqs. 7.55 imply that the 3-dimensional Christoffel symbols are the spatial parts of the 4-dimensional Christoffel symbols

$${}^3\Gamma^i_{jk} = \Gamma^i_{jk},$$

and the 3-dimensional Riemann tensor on Σ_t therefore has components

$${}^3R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{mk} \Gamma^m_{jl} - \Gamma^i_{ml} \Gamma^m_{jk}. \quad (7.56)$$

Because $n^\mu = \delta^\mu_0$, the components of K_{ab} are

$$K_{ij} = -\frac{1}{2} \mathcal{L}_n g_{ij} = -\frac{1}{2} \partial_0 g_{ij}.$$

Then

$$\Gamma^0_{ij} = -K_{ij}, \quad \Gamma^i_{0j} = -K^i_j.$$

The spatial components of the 4-dimensional Riemann tensor at P are now

$$\begin{aligned} R^i_{jkl} &= \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{\mu k} \Gamma^\mu_{jl} - \Gamma^i_{\mu l} \Gamma^\mu_{jk} \\ &= {}^3R^i_{jkl} + \Gamma^i_{0k} \Gamma^0_{jl} - \Gamma^i_{0l} \Gamma^0_{jk} \\ &= {}^3R^i_{jkl} + K^i_k K_{jl} - K^i_l K_{jk} \end{aligned}$$

and these are the nonzero components of the second Gauss-Codazzi relation (Gauss's Theorema Egregium)

$${}^3R^\alpha{}_{\beta\gamma\delta} = \gamma^\alpha_\epsilon \gamma^\zeta_\beta \gamma^\sigma_\gamma \gamma^\tau_\delta R^\epsilon{}_{\zeta\sigma\tau} - K^\alpha{}_\gamma K_{\beta\delta} + K^\alpha{}_\delta K_{\beta\gamma}$$

Equivalently, denoting by R_{abcd} the pullback of $R_{\alpha\beta\gamma\delta}$ to Σ_t , we have

$${}^3R_{abcd} = R_{abcd} - K_{ac}K_{bd} + K_{ad}K_{bc}. \quad (7.57)$$

Exercise 91. Prove the first Gauss-Codazzi relation,

$$n_\delta R^\delta{}_{cab} = D_a K_{bc} - D_b K_{ac},$$

where $n_\delta R^\delta{}_{cab}$ is the pullback of $n_\delta R^\delta{}_{\gamma\alpha\beta}$ to Σ_t . It is simplest to note that one can pick the chart $\{x^i\}$ on Σ_t to make $\Gamma^i{}_{jk} = 0$ at a point P ; then show that the relation holds at P . Because P is arbitrary, it holds everywhere.

Covariant derivations

We turn now to covariant proofs of the relations, with no coordinates introduced. In the coordinate-based derivation, because n_α was the timelike basis vector, the Christoffel symbol $\Gamma^0{}_{ij}$ had the meaning of the spatial part of $\nabla_\alpha n_\beta$, so that is the form we will find for $K_{\alpha\beta}$. The derivations end with the paragraph containing Eq. (7.68).

The Gauss-Codacci equations are true in any dimension; our convention is to use Latin indices for arbitrary dimensions, and that may make the derivations easier to read and follow. The derivations work in the following way. The Riemann tensor in any dimension n is defined by the commutator of two covariant derivatives (2.46):

$$R_{abc}{}^d \xi_d = [\nabla_a, \nabla_b] \xi_c. \quad (7.58)$$

Then the Riemann tensor on an $(n-1)$ -dimensional submanifold is given by

$${}^{(n-1)}R_{abc}{}^d \xi_d = [D_a, D_b] \xi_c \quad (7.59)$$

where ξ_c is tangent to the submanifold ($\xi_c n^c = 0$). As in the earlier covariant sections, we work with n -dimensional tensors, instead of introducing additional indices for tensors on an $(n-1)$ -dimensional submanifold, and this is useful in the covariant derivations. And, as before, $D_a \xi_b := \gamma_a^c \gamma_b^d \nabla_c \xi_d$. In the derivations, we multiply covariant derivatives ∇_b by projection operators γ_a^b . We then bring the projection operators inside ∇ , using

$$\nabla_a \gamma_b^c = \nabla_a (\delta_b^c + n_b n^c) = \nabla_a (n_b n^c).$$

All that remains is to compute the spatial projection of $\nabla_a n^b$. It is given by

$$K_{ab} = -\gamma_a^c \nabla_c n_b. \quad (7.60)$$

To show this, use the definition (7.42) and the relation $\mathcal{L}_n g_{ab} = \nabla_a n_b + \nabla_b n_a$ (see, e.g., p. 129). We have

$$\begin{aligned} \mathcal{L}_n \gamma_{ab} &= \mathcal{L}_n (g_{ab} + n_a n_b) = 2\nabla_{(a} n_{b)} + n_b \mathcal{L}_n n_a + n_a \mathcal{L}_n n_b \Rightarrow \\ K_{ab} &= -\frac{1}{2} \gamma_a^c \gamma_b^d \mathcal{L}_n \gamma_{cd} = \gamma_a^c \gamma_b^d \nabla_{(c} n_{d)}, \quad \text{where we used } \gamma_a^b n_b = 0. \end{aligned}$$

Now use $n_a = -\alpha \nabla_a t$ to show that $\gamma_a^c \gamma_b^d \nabla_c n_d$ is already a symmetric tensor:

$$\gamma_a^c \gamma_b^d \nabla_{[c} n_{d]} = -\gamma_a^c \gamma_b^d \nabla_{[c} (\alpha \nabla_{d]} t) = -\gamma_a^c \gamma_b^d \alpha \nabla_{[c} \nabla_{d]} t = 0.$$

Then $K_{ab} = -\gamma_a^c \gamma_b^d \nabla_c n_d$. Finally, the second index of $\nabla_a n_b$ is already orthogonal to n^b ($n^b \nabla_a n_b = 0$), so the second projection γ_b^d is not needed, implying Eq. (7.60). $\square$

First Gauss-Codazzi equation: $\gamma_b^a R^{bc} n_c = -D_b (K^{ab} - \gamma^{ab} K)$. (This is Wald's problem 10.4.)

We start with the definition of the Riemann tensor, writing $[\nabla_b, \nabla_a] n_c = R_{bac}{}^e n_e$. We hit all indices with γ 's, contracting the indices b and c with γ^{bc} :

$$\gamma^{ad} \gamma^{bc} [\nabla_b, \nabla_d] n_c = \gamma^{ad} \gamma^{bc} R_{bdc}{}^e n_e = \gamma^{ad} R_d{}^e n_e, \quad \text{where we used } R_{bdc}{}^e n_e = 0. \quad (7.61)$$

$$\begin{aligned} \text{LHS} &= (\gamma^{ad} \gamma^{bc} - \gamma^{ab} \gamma^{cd}) \nabla_b \nabla_d n_c = \gamma_e^a \gamma_f^b \gamma_g^c (\gamma^{ed} \gamma^{fg} - \gamma^{ef} \gamma^{gd}) \nabla_b \nabla_d n_c \\ &= \gamma_e^a \gamma_f^b \gamma_g^c \nabla_b [(\gamma^{ed} \gamma^{fg} - \gamma^{ef} \gamma^{gd}) \nabla_d n_c] - \gamma_e^a \gamma_f^b \gamma_g^c \nabla_b (\gamma^{ed} \gamma^{fg} - \gamma^{ef} \gamma^{gd}) \nabla_d n_c. \end{aligned} \quad (7.62)$$

The first term is what we want, and the second term vanishes: Use the form (7.60) for K_{ab} and then the definition of D_a to write the first term as

$$\begin{aligned} \gamma_e^a \gamma_f^b \gamma_g^c \nabla_b [(\gamma^{ed} \gamma^{fg} - \gamma^{ef} \gamma^{gd}) \nabla_d n_c] &= \gamma_e^a \gamma_f^b \gamma_g^c \nabla_b (-K^e{}_c \gamma^{fg} + K^g{}_c \gamma^{ef}) \\ &= -(D_b K^{ab} - D^a K). \end{aligned} \quad (7.63)$$

To get the last equality, we used $D_f \gamma^{fg} = 0$, $D_f \gamma^{ef} = 0$.

To show that the second term in (7.62) vanishes, we again use Eq. (7.60) as well as the relations

$$n^b \nabla_a n_b = 0, \quad \gamma_{ab} n^b = 0, \quad \nabla_a \gamma_{bc} = \nabla_a (n_b n_c). \quad (7.64)$$

We have

$$\begin{aligned} -\gamma_e^a \gamma_f^b \gamma_g^c \nabla_b (\gamma^{ed} \gamma^{fg} - \gamma^{ef} \gamma^{gd}) \nabla_d n_c &= -\gamma_e^a \gamma_f^b \gamma_g^c [\nabla_b (n^e n^d) \gamma^{fg} + \gamma^{ed} \nabla_b (n^f n^g) - \nabla_b (n^e n^f) \gamma^{gd} - \gamma^{ef} \nabla_b (n^g n^d)] \nabla_d n_c \\ &= -\gamma_e^a \gamma_f^b \gamma_g^c [\nabla_b (n^e) n^d \gamma^{fg} - \gamma^{ef} \nabla_b (n^g) n^d] \nabla_d n_c \\ &= K^{ca} n^d \nabla_d n_c - K^{ac} = n^d \nabla_d n_c = 0, \end{aligned}$$

Then the right side of (7.61) is equal to the expression (7.63), giving the first Gauss-Codazzi relation (in its contracted form), Eq. (7.52). $\square$

The second Gauss-Codazzi relation relates the Riemann tensor R_{abcd} on an n -dimensional manifold with metric g_{ab} to the Riemann tensor ${}^{(n-1)}R_{abcd}$ on an $(n-1)$ -dimensional hypersurface:

$${}^{(n-1)}R_{abc}{}^d = \gamma_a^l \gamma_b^m \gamma_c^n \gamma_p^d R_{lmnp} + K_a{}^d K_{bc} - K_{ac} K_b{}^d. \quad (7.65)$$

To prove this, we again write the Riemann tensor in terms of the commutator,

$${}^{(n-1)}R_{abc}{}^d \xi_d = [D_a, D_b] \xi_c \quad (7.66)$$

this time with ξ_c a vector tangent to the submanifold ($\xi_c n^c = 0$). We have

$$D_a D_b \xi_c = \gamma_a^l \gamma_b^m \gamma_c^n \nabla_l (D_m \xi_n) = \gamma_a^l \gamma_b^m \gamma_c^n \nabla_l (\gamma_m^r \gamma_n^s \nabla_r \xi_s).$$

In the derivation, we once again use Eqs. (7.60) and (7.64). In particular, each time ∇ hits n , we get $K_{..}$. We also need $n^b \nabla_a \xi_b = -\xi_b \nabla_a n^b + \nabla_a (\xi_b n^b) = -\xi_b \nabla_a n^b$.

We have

$$\begin{aligned} D_a D_b \xi_c &= \gamma_a^l \gamma_b^m \gamma_c^n [\nabla_l (n_m n^r) \gamma_n^s \nabla_r \xi_s + \gamma_m^r \nabla_l (n_n n^s) \nabla_r \xi_s + \gamma_m^r \gamma_n^s \nabla_l \nabla_r \xi_s] \\ &= -K_{ab} n^r \gamma_c^s \nabla_r \xi_s - \gamma_b^r K_{ac} n^s \nabla_r \xi_s + \gamma_a^l \gamma_b^m \gamma_c^n \nabla_l \nabla_m \xi_n. \end{aligned}$$

Because the first term is symmetric in a and b , its contribution to $[D_a, D_b] \xi_c$ vanishes. The second term is

$$\gamma_b^r K_{ac} \xi_s \nabla_r n^s = -K_{ac} K_b^d \xi_d.$$

Finally, antisymmetrizing on a and b gives the n -dimensional Riemann tensor in the last term:

$$[D_a, D_b] \xi_c = (-K_{ac} K_b^d + K_{bc} K_a^d) \xi_d + \gamma_a^l \gamma_b^m \gamma_c^n \gamma_p^d R_{lmn}{}^p \xi_d, \quad (7.67)$$

where we have used $\xi_a = \gamma_a^b \xi_b$ to get the fourth γ in the last term. Because this equation holds for all spatial ξ_d , we recover the second Gauss-Codazzi relation (7.65).

Contracting Eq. (7.65) on b and d and then changing the index c back to b , gives us the contracted identity,

$${}^{(n-1)}R_{ab} = \gamma_a^c \gamma_b^d \gamma^{ef} R_{cedf} + K_a^c K_{bc} - K_{ab} K, \quad (7.68)$$

where $K := K_c^c$. $\square$

We now return to four dimensions and the Einstein equation. From the first Gauss-Codazzi equation (7.52) we can immediately write the momentum constraint in terms of $\gamma_{\alpha\beta}$ and $K_{\alpha\beta}$, recovering Eq. (7.45):

$$D_\beta (K^{\alpha\beta} - \gamma^{\alpha\beta} K) = -8\pi \gamma^\alpha{}_\beta T^{\beta\gamma} n_\gamma.$$

To obtain the Hamiltonian constraint, we need another contraction of the second Gauss-Codazzi equation (7.68), namely

$${}^3R = \gamma^{\alpha\beta} {}^3R_{\alpha\beta} = \gamma^{\alpha\beta} \gamma^{\gamma\delta} R_{\alpha\gamma\beta\delta} + K^{\alpha\beta} K_{\alpha\beta} - K^2 \quad (7.69)$$

The first term on the right is

$$(g^{\alpha\beta} + n^\alpha n^\beta)(g^{\gamma\delta} + n^\gamma n^\delta) R_{\alpha\gamma\beta\delta} = R + 2n^\alpha n^\beta R_{\alpha\beta} = 2G_{\alpha\beta} n^\alpha n^\beta \quad (7.70)$$

whence

$$2G_{\alpha\beta} n^\alpha n^\beta = {}^3R - K^{\alpha\beta} K_{\alpha\beta} + K^2,$$

and the Hamiltonian constraint, Eq. (7.46), is

$${}^3R - K^{\alpha\beta} K_{\alpha\beta} + K^2 = 16\pi T^{\alpha\beta} n_\alpha n_\beta,$$

as claimed.

The derivation of the evolution equation (7.47) for $K_{\alpha\beta}$ again involves Gauss-Codazzi and needs only one additional relation, the derivative of n^α along itself:

$$n^\beta \nabla_\beta n_\alpha = D_\alpha \ln \alpha. \quad (7.71)$$

To see this, use $n_\alpha = -\alpha \nabla_\alpha t$ to write

$$n^\beta \nabla_\beta n_\alpha = -n^\beta \nabla_\beta \alpha \nabla_\alpha t - \alpha n^\beta \nabla_\beta \nabla_\alpha t. \quad (7.72)$$

From the definition (7.27) of α , we have

$$\nabla_\alpha \alpha = \alpha^3 \nabla^\beta t \nabla_\beta \nabla_\alpha t = -\alpha^2 n^\beta \nabla_\beta \nabla_\alpha t.$$

Using this in the second term of Eq. (7.72) and replacing $-\nabla_\alpha t$ by $\alpha^{-1}n_\alpha$ in the first term gives

$$n^\beta \nabla_\beta n_\alpha = \alpha^{-1} (n_\alpha n^\beta \nabla_\beta \alpha + \nabla_\alpha \alpha) = \alpha^{-1} \gamma_\alpha^\beta \nabla_\beta \alpha,$$

implying Eq. (7.71).

We can now compute $\mathcal{L}_n K_{\alpha\beta}$, noting first that it is spatial:

$$n^\beta \mathcal{L}_n K_{\alpha\beta} = \mathcal{L}_n (n^\beta K_{\alpha\beta}) - (\mathcal{L}_n n^\beta) K_{\alpha\beta} = 0 - 0 = 0.$$

Then

$$\mathcal{L}_n K_{\alpha\beta} = -\gamma_\alpha^\sigma \gamma_\beta^\tau \mathcal{L}_n (\gamma_\sigma^\delta \nabla_\delta n_\tau) = -\gamma_\alpha^\sigma \gamma_\beta^\tau \mathcal{L}_n (\gamma_\sigma^\delta) \nabla_\delta n_\tau - \gamma_\alpha^\sigma \gamma_\beta^\tau \gamma_\sigma^\delta \mathcal{L}_n \nabla_\delta n_\tau$$

In the red term on the right, because $\mathcal{L}_n n^\delta = 0$,

$$\mathcal{L}_n (\gamma_\sigma^\delta) = \mathcal{L}_n (n_\sigma) n^\delta = (n^\gamma \nabla_\gamma n_\sigma + n_\gamma \nabla_\sigma n^\gamma) n^\delta = n^\gamma (\nabla_\gamma n_\sigma) n^\delta = D_\sigma \ln \alpha n^\delta,$$

and the red term becomes

$$-\gamma_\alpha^\sigma \gamma_\beta^\tau \mathcal{L}_n (\gamma_\sigma^\delta) \nabla_\delta n_\tau = -D_\alpha \ln \alpha D_\beta \ln \alpha.$$

The blue term is

$$\begin{aligned} -\gamma_\alpha^\sigma \gamma_\beta^\tau \mathcal{L}_n \nabla_\sigma n_\tau &= -\gamma_\alpha^\sigma \gamma_\beta^\tau (n^\gamma \nabla_\gamma \nabla_\sigma n_\tau + \nabla_\gamma n_\tau \nabla_\sigma n^\gamma + \nabla_\sigma n_\gamma \nabla_\tau n^\gamma) \\ &= -\gamma_\alpha^\sigma \gamma_\beta^\tau n^\gamma \nabla_\gamma \nabla_\sigma n_\tau - K_{\gamma\beta} K_\alpha^\gamma - K_{\alpha\gamma} K_\beta^\gamma \\ \text{1st term} &= -\gamma_\alpha^\sigma \gamma_\beta^\tau n^\gamma ([\nabla_\gamma, \nabla_\sigma] n_\tau + \nabla_\sigma \nabla_\gamma n_\tau) \\ &= -\gamma_\alpha^\sigma \gamma_\beta^\tau [n^\gamma R_{\gamma\sigma\tau}^\delta n_\delta + \nabla_\sigma (n^\gamma \nabla_\gamma n_\tau) - \nabla_\sigma n^\gamma \nabla_\gamma n_\tau] \\ &= R_{\alpha\gamma\beta\delta} n^\gamma n^\delta - D_\alpha D_\beta \ln \alpha + K_\alpha^\gamma K_{\beta\gamma} \end{aligned} \quad (7.73)$$

Then

$$-\gamma_\alpha^\sigma \gamma_\beta^\tau \mathcal{L}_n \nabla_\sigma n_\tau = R_{\alpha\gamma\beta\delta} n^\gamma n^\delta - D_\alpha D_\beta \ln \alpha - K_\alpha^\gamma K_{\beta\gamma}$$

Combining red and blue terms, we have

$$\mathcal{L}_n K_{\alpha\beta} = R_{\alpha\gamma\beta\delta} n^\gamma n^\delta - \frac{1}{\alpha} D_\alpha D_\beta \alpha - K_\alpha^\gamma K_{\beta\gamma}. \quad (7.74)$$

Finally, we use the contracted Gauss-Codazzi equation (7.68) and the remaining part of the Einstein equation to write the Riemann-tensor term as follows:

$$\begin{aligned} R_{\alpha\gamma\beta\delta} n^\gamma n^\delta &= \gamma_\alpha^\sigma \gamma_\beta^\tau n^\gamma n^\delta R_{\gamma\sigma\delta\tau} = \gamma_\alpha^\sigma \gamma_\beta^\tau (\gamma^{\gamma\delta} - g^{\gamma\delta}) R_{\gamma\sigma\delta\tau} \\ \gamma_\alpha^\sigma \gamma_\beta^\tau \gamma^{\gamma\delta} R_{\gamma\sigma\delta\tau} &= {}^3R_{\alpha\beta} - K_{\alpha\gamma} K_\beta^\gamma + K K_{\alpha\beta}, \quad \text{by (7.68);} \\ -\gamma_\alpha^\sigma \gamma_\beta^\tau g^{\gamma\delta} R_{\gamma\sigma\delta\tau} &= -\gamma_\alpha^\sigma \gamma_\beta^\tau R_{\sigma\tau} = -8\pi \gamma_\alpha^\sigma \gamma_\beta^\tau (T_{\sigma\tau} - \frac{1}{2} g_{\sigma\tau} T) = -8\pi (\gamma_\alpha^\sigma \gamma_\beta^\tau T_{\sigma\tau} - \frac{1}{2} \gamma_{\alpha\beta} T) \\ \implies R_{\alpha\gamma\beta\delta} n^\gamma n^\delta &= {}^3R_{\alpha\beta} - K_{\alpha\gamma} K_\beta^\gamma + K K_{\alpha\beta} - 8\pi (\gamma_\alpha^\delta \gamma_\beta^\epsilon T_{\delta\epsilon} - \frac{1}{2} \gamma_{\alpha\beta} T). \end{aligned}$$

Using this equation to replace $R_{\alpha\gamma\beta\delta} n^\gamma n^\delta$ in Eq. (7.74), we obtain the second evolution equation (7.48). $\square$

This completes our derivations of the initial value equations. To make contact with the numerical relativity literature, two additional pieces are needed. Because derivatives are computed in a general timelike direction t^α that is not along n^α , we need to express $\mathcal{L}_n$ in terms of $\mathcal{L}_t$. And we need explicit forms of the projections of the stress-energy tensor for a fluid.

7.4 Initial value equations in the NR literature.

Because of the dragging of frames, the timelike Killing vector t^α of a rotating star or a rotating black hole is not in general orthogonal to the surfaces Σ_t . Not surprisingly, in the more general spacetimes that describe oscillating stars and binary systems, convenient choices of coordinates $\{t, x^i\}$ again have a time direction $t^\alpha \equiv \partial_t$ that is not along the normal. That is, a chart $\{t, x^i\}$ picks out the direction of time ∂_t along which the coordinates x^i are fixed and only t changes. We write t^α for ∂_t . Then, in the chart $\{t, x^i\}$,

$$t^\mu = \delta_t^\mu, \quad (7.75a)$$

$$\nabla_\mu t = \delta_\mu^t. \quad (7.75b)$$

The vector field t^α is the sum of orthogonal components, αn^α along the unit normal n^α and a vector β^α tangent to the slices:

$$t^\alpha = \alpha n^\alpha + \beta^\alpha, \text{ where } \beta^\alpha n_\alpha = 0. \quad (7.76)$$

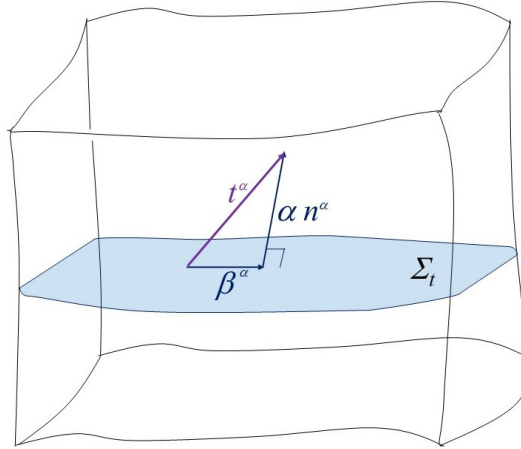


Figure 7.2: The decomposition of t^α into lapse and shift.

As before, the component α of t^α along the unit normal is called the *lapse*, because αdt is the proper time elapsed in a normal direction between coordinate times t and $t + dt$. The spatial vector β^α is called the *shift*, because $\beta^\alpha dt$ is the spatial shift in the position from a path along the normal n^α to the actual path along t^α .

One regards the choice of the vector t^α as the direction in which the time-evolution proceeds, noting that the time derivative of a function f , defined by

$$\dot{f} := \mathcal{L}_t f = t^\alpha \nabla_\alpha f, \quad (7.77)$$

is the partial derivative $\dot{f} = \partial_t f$ in the chart $\{t, x^i\}$. For any covariant spatial tensor $S_{\alpha\cdots\beta}$,

$$\mathcal{L}_t S_{\alpha\cdots\beta} = (\alpha \mathcal{L}_n + \mathcal{L}_\beta) S_{\alpha\cdots\beta}. \quad (7.78)$$

Exercise 92. Use the fact that $S_{\alpha\cdots\beta}$ is spatial to check this relation. Show that it is not true for a contravariant tensor by checking that it does not hold for a contravariant spatial vector v^α , and show

$$\mathcal{L}_t v^\beta = (\alpha \mathcal{L}_n + \mathcal{L}_\beta) v^\beta + n^\beta v^\gamma \nabla_\gamma \alpha.$$

Because $t^\alpha = \partial_t$ has components δ_t^μ in the chart $\{t, x^i\}$, the Lie derivative of any tensor $T^{\alpha\cdots\beta}_{\gamma\cdots\delta}$, has, in this chart, components

$$\mathcal{L}_t T^{\mu\cdots\nu}_{\kappa\cdots\lambda} = \partial_t T^{\mu\cdots\nu}_{\kappa\cdots\lambda}. \quad (7.79)$$

More generally, for any nonzero vector field ξ^a on a manifold, if one chooses the parameter distance along its integral curves as a coordinate, say x^1 , then in a chart $x^1, x^2, \dots, x^n$, we have $\xi^\mu = \delta_1^\mu$, implying

$$\mathcal{L}_\xi T^{i\cdots j}_{k\cdots l} = \partial_1 T^{i\cdots j}_{k\cdots l}.$$

In particular, the derivatives $\mathcal{L}_t \gamma_{\alpha\beta}$ and $\mathcal{L}_t K_{\alpha\beta}$ have spatial components

$$\mathcal{L}_t \gamma_{ij} = \partial_t \gamma_{ij}, \quad \mathcal{L}_t K_{ij} = \partial_t K_{ij}, \quad (7.80)$$

implying the pullbacks of $\gamma_{\alpha\beta}$ by the family of maps $\Sigma \rightarrow \Sigma_t$ is the time-dependent tensor $g_{ab}(t)$ on Σ , and the family of pullbacks of $\mathcal{L}_t \gamma_{\alpha\beta}$ is $\partial_t g_{ab}(t)$.

The spacetime metric at a time t can be written in terms of the lapse, the shift, and $\gamma_{\alpha\beta}$. A small coordinate displacement $(\Delta t, \Delta x^i)$ gives a change in proper time $\alpha \Delta t$ perpendicular to Σ_t and a net spatial coordinate change $\Delta x^i + \beta^i \Delta t$ tangent to Σ_t . The proper length of the displacement is then given by

$$\Delta s^2 = -\alpha^2 \Delta t^2 + \gamma_{ij}(\Delta x^i + \beta^i \Delta t)(\Delta x^j + \beta^j \Delta t)$$

implying

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \quad (7.81)$$

More formally, in the coordinates $\{t, x^i\}$, the 3+1 decomposition of the 4-metric, $g_{\alpha\beta} = \gamma_{\alpha\beta} - n_\alpha n_\beta$, has, from Eq. (7.76) and the relations $\beta^t = 0$ and $n_i = 0$, components

$$g_{tt} = t_\alpha t^\alpha = -\alpha^2 + \beta_i \beta^i, \quad g_{ij} = \gamma_{ij}, \quad g_{ti} = g_{\mu i} t^\mu = \alpha n_i + \beta_i = \beta_i, \quad (7.82)$$

or

$$[g_{\mu\nu}] = \begin{bmatrix} -\alpha^2 + \beta_i \beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{bmatrix}.$$

The corresponding decomposition of the volume element is given by

$$\sqrt{|g|} = \alpha \sqrt{\gamma}, \quad (7.83)$$

where $\gamma = \det(\gamma_{ij})$. This follows from the fact that the determinant γ is the minor of g_{tt} in the matrix $g_{\mu\nu}$, so $g^{tt} = \frac{\gamma}{g}$. Because $g^{tt} = -\alpha^{-2}$, we have

$$g = -\alpha^2 \gamma, \quad \sqrt{|g|} = \alpha \sqrt{\gamma}. \quad (7.84)$$

The inverse metric has components

$$g^{tt} = -\frac{1}{\alpha^2}, \quad g^{ti} = \frac{\beta^i}{\alpha^2}, \quad g^{ij} = \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2}, \quad (7.85)$$

or

$$[g^{\mu\nu}] = \begin{bmatrix} -\alpha^{-2} & \beta^i \alpha^{-2} \\ \beta^i \alpha^{-2} & \gamma^{ij} - \beta^i \beta^j \alpha^{-2} \end{bmatrix}. \quad (7.86)$$

Equivalently,³

$$g^{\mu\nu}\partial_\mu\partial_\nu = -\frac{1}{\alpha^2}(\partial_t - \beta^i\partial_i)^2 + \gamma^{ij}\partial_i\partial_j. \quad (7.87)$$

The next two exercises give a quick way to invert the metric $g_{\alpha\beta}$, showing why a metric written in the form (7.81) has an inverse of the form (7.87).

Exercise 93. Let ω^μ and e_μ be dual bases, i.e., $\omega^\mu(e_\nu) = \delta^\mu_\nu$. Show that a metric $g_{\mu\nu}\omega^\mu\omega^\nu$ has inverse $g^{\mu\nu}e_\mu e_\nu$, where, as usual, $[g^{\mu\nu}]$ is the matrix inverse of $[g_{\mu\nu}]$.

Exercise 94. Invert the metric (7.81) to obtain (7.87) by showing that the basis $\omega^0 = dt$, $\omega^i = dx^i + \beta^i dt$ has dual $e_0 = \partial_t - \beta^i\partial_i$, $e_i = \partial_i$ and using the previous exercise.

Finally, for those of you familiar with wedge products:

Exercise 95. Show that a metric of the form $g_{\mu\nu}dx^\mu dx^\nu = \bar{g}_{\mu\nu}\omega^\mu\omega^\nu$ has determinant satisfying $\sqrt{g} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \sqrt{\bar{g}} \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3$. Compute the expression on the right for the metric (7.81), with the basis ω^μ of the last problem to show $\sqrt{g} = \alpha\sqrt{\bar{g}}$.

Using Eq. (7.78), we can write the dynamical equations (7.47) and (7.48) for $\gamma_{\alpha\beta}$ and $K_{\alpha\beta}$ in the forms

$$\dot{\gamma}_{\alpha\beta} = -2\alpha K_{\alpha\beta} + \mathcal{L}_\beta \gamma_{\alpha\beta}, \quad (7.88)$$

$$\dot{K}_{\alpha\beta} = \alpha \left({}^3R_{\alpha\beta} - 2K_{\alpha\gamma}K_{\beta}{}^\gamma + K K_{\alpha\beta} \right) - D_\alpha D_\beta \alpha - 8\pi\alpha(\gamma_\alpha^\delta \gamma_\beta^\epsilon T_{\delta\epsilon} - \frac{1}{2}\gamma_{\alpha\beta}T) + \mathcal{L}_\beta K_{\alpha\beta}. \quad (7.89)$$

The system (7.88), (7.89) describes the evolution of the spatial tensors $\gamma_{\alpha\beta}$ and $K_{\alpha\beta}$. Because the equations do not contain time derivatives of the lapse function α or of the shift vector β^α , these metric functions are not dynamical variables. One can regard the four degrees of gauge freedom associated with the choice of coordinates (t, x^i) as the freedom to choose α and β^α . From this point of view, once α and β^α are prescribed, initial data for the geometry consist of initial values $\gamma_{\alpha\beta}(0)$ and $K_{\alpha\beta}(0)$ of the metric and extrinsic curvature satisfying the constraint equations 7.45), (7.46).

For black hole spacetimes, there is no matter, and the 3+1 decomposition of the Einstein equation, the set 7.45)-(7.48) (or (7.45),(7.46),(7.88), (7.89)), determines the spacetime metric given initial data and a choice of gauge. For spacetimes with matter, one must add equations determining the evolution of the matter.

³This is a shorthand for $g^{\mu\nu}\partial_\mu \otimes \partial_\nu = -\frac{1}{\alpha^2}(\partial_t - \beta^i\partial_i) \otimes (\partial_t - \beta^i\partial_i) + \gamma^{ij}\partial_i \otimes \partial_j$.

Matter

One can accurately model stationary neutron stars, oscillating neutron stars and neutron stars in binary inspiral as perfect fluids, with stress-energy tensor

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + P g^{\alpha\beta} \quad (7.90)$$

and an equation of state of the form $P = P(\rho)$.

Here $\rho = T_{\alpha\beta}u^\alpha u^\beta$ is the energy density seen by a family of *comoving observers*, observers with 4-velocity u^α . The projection operator orthogonal to u^α is $q_\alpha^\beta := \delta_\alpha^\beta + u_\alpha u^\beta$, and we can decompose the stress-energy tensor in the manner

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + P q^{\alpha\beta}. \quad (7.91)$$

A comoving observer sees stress tensor $P q^{\alpha\beta}$.

The evolution of the matter variables u^α , ρ , P is given by the energy-momentum conservation law,

$$\nabla_\beta T^{\alpha\beta} = 0. \quad (7.92)$$

For analytic problems, the equation is ordinarily decomposed into parts along and orthogonal to the 4-velocity u^α , giving respectively equations expressing conservation of energy,

$$\nabla_\beta(\rho u^\beta) = -P \nabla_\beta u^\beta, \quad (7.93)$$

and momentum (the relativistic Euler equation)

$$(\rho + P)u^\beta \nabla_\beta u^\alpha = -q^{\alpha\beta} \nabla_\beta P. \quad (7.94)$$

Exercise 96. Derive these equations, starting from $u_\alpha \nabla_\beta T^{\alpha\beta} = 0$ and $q_\gamma^\alpha \nabla_\beta T^{\gamma\beta} = 0$. If you get stuck, the derivations are identical to those given in Eqs. (1.163) and before (1.165) for Minkowski space, with $\eta_{\alpha\beta}$ replaced by $g_{\alpha\beta}$.

The first equation gives the time derivative $u^\alpha \nabla_\alpha \rho$ of the energy density; the second gives the time derivative of the velocity, and the equation of state completes the system.

For numerical work, one generally uses the 3+1 decomposition along and orthogonal to n_α . The energy and momentum densities and spatial stress tensor seen by an observer with velocity n^α are

$$\rho_E = T_{\alpha\beta} n^\alpha n^\beta, \quad j^\alpha = -\gamma_\delta^\alpha n_\beta T^{\delta\beta}, \quad S^{\alpha\beta} = \gamma_\delta^\alpha \gamma_\epsilon^\beta T^{\delta\epsilon}, \quad (7.95a)$$

with corresponding decomposition

$$T^{\alpha\beta} = \rho_E n^\alpha n^\beta + j^\alpha n^\beta + n^\alpha j^\beta + S^{\alpha\beta}. \quad (7.96)$$

Before writing the decomposition of $\nabla_\beta T^{\alpha\beta} = 0$, it is helpful to look at the simpler conservation law $\nabla_\alpha J^\alpha = 0$, where J^α is a conserved current: Examples are the currents associated with conservation of baryons and conservation of charge. Decompose the baryon current J^α into baryon density and spatial current j^α ,

$$J^\alpha = \rho_0 n^\alpha + j^\alpha, \quad \text{with} \quad \rho_0 = -J^\alpha n_\alpha, \quad j^\alpha = \gamma_\beta^\alpha J^\beta, \quad (7.97)$$

(Note that ρ_0 here is the baryon density seen by an observer with velocity n^α , not that of a comoving observer.) Next, the contraction of Eq. (7.60) gives

$$K = -\gamma^{\alpha\beta} \nabla_\alpha n_\beta = -\nabla_\beta n^\beta. \quad (7.98)$$

Then

$$\begin{aligned}\nabla_\beta J^\beta &= \nabla_\beta(\rho_0 n^\beta + j^\beta) = n^\beta \nabla_\beta \rho_0 + \rho_0 \nabla_\beta n^\beta + (\gamma^{\alpha\beta} - n^\alpha n^\beta) \nabla_\alpha j_\beta \\ &= n^\beta \nabla_\beta \rho_0 - K \rho_0 + D_\beta j^\beta + j^\beta D_\beta \ln \alpha,\end{aligned}$$

where we use $n^\beta \nabla_\alpha j_\beta = -j_\beta \nabla_\alpha n^\beta$ and $n^\alpha \nabla_\alpha n_\beta = \nabla_\beta \ln \alpha$ to get the last term. Equivalently,

$$\mathcal{L}_t \rho_0 = \mathcal{L}_\beta \rho_0 + \alpha K \rho_0 - D_\beta (\alpha j^\beta). \quad (7.99)$$

We can now follow essentially the same pattern in writing the projections of $\nabla_\beta T^{\alpha\beta} = 0$ along and orthogonal to n^α . These equations for the evolution of energy density and momentum density are

$$\mathcal{L}_t \rho_E = \mathcal{L}_\beta \rho_E + \alpha K \rho_E - \frac{1}{\alpha} D_\beta (\alpha^2 j^\beta) + \alpha K_{\alpha\beta} S^{\alpha\beta}, \quad (7.100a)$$

$$\mathcal{L}_t j_\alpha = \mathcal{L}_\beta j_\alpha + \alpha K j_\alpha - D_\beta (\alpha S_\alpha^\beta) - \rho_E D_\alpha \alpha. \quad (7.100b)$$

Exercise 97. Derive Eq. (7.100a).

Here's the derivation of Eq. (7.100b):

The time derivative of j_α comes from the blue term below where n^β hits ∇_β .

$$\begin{aligned}\gamma_\alpha^\delta \nabla_\beta T^\beta_\delta &= \gamma_\alpha^\delta \nabla_\beta (\rho_E n^\beta n_\delta + j^\beta n_\delta + j_\delta n^\beta + S^\beta_\delta) \\ &= \rho_E n^\beta \nabla_\beta n_\alpha + j^\beta \gamma_\alpha^\delta \nabla_\beta n_\delta + j_\alpha \nabla_\beta n^\beta + \gamma_\alpha^\delta n^\beta \nabla_\beta j_\delta + \gamma_\alpha^\delta \nabla_\beta S^\beta_\delta \\ &= \rho_E D_\alpha \ln \alpha - \frac{K_\alpha^\beta j_\beta}{\alpha} - K j_\alpha + \gamma_\alpha^\delta (\mathcal{L}_n j_\delta - \frac{j_\beta \nabla_\delta n^\beta}{\alpha}) + \gamma_\alpha^\delta (\gamma^{\beta\epsilon} - n^\beta n^\epsilon) \nabla_\beta S_{\delta\epsilon} \\ &= \rho_E D_\alpha \ln \alpha - K j_\alpha + \gamma_\alpha^\delta \mathcal{L}_n j_\delta + D_\beta S_\alpha^\beta + \gamma_\alpha^\delta n^\beta \nabla_\beta n^\epsilon S_{\delta\epsilon} \\ &= \rho_E D_\alpha \ln \alpha - K j_\alpha + \gamma_\alpha^\beta \mathcal{L}_n j_\beta + D_\beta S_\alpha^\beta + S_\alpha^\beta \nabla_\beta \ln \alpha.\end{aligned}$$

The equation of state and local thermodynamics of the fluid involve the pressure and comoving energy density ρ , so one needs the relations between ρ_E, j^α and ρ, u^α . These involve the 3+1 decomposition of u^α ,

$$u^\alpha = \Upsilon(n^\alpha + v^\alpha), \text{ where } \Upsilon = -u^\alpha n_\alpha, \quad v_\alpha = \Upsilon^{-1} \gamma_{\alpha\beta} u^\beta. \quad (7.101)$$

Then v^α is the 3-velocity of an observer with 4-velocity u^α , and $u^\alpha u_\alpha = -1$ implies

$$\Upsilon = \frac{1}{\sqrt{1 - v^2}}.$$

We easily find

$$\rho_E = \frac{\rho + P v^2}{1 - v^2}, \quad j^\alpha = (\rho + P) \frac{v^\alpha}{1 - v^2}, \quad (7.102a)$$

$$S_{\alpha\beta} = j_\alpha v_\beta - P \gamma_{\alpha\beta}. \quad (7.102b)$$

In numerical evolutions that use variables j^α and ρ_E , one must numerically invert Eqs. (7.102a), using the equation of state $P = P(\rho)$, to obtain the *primitive* fluid variables ρ, P and u^α .

3-dimensional form of the equations

By sticking to 4-dimensional indices we simplified the covariant derivation of the 3+1 form of the field equation. To write the four equations, (7.45), (7.46), (7.88), (7.89), using 3-dimensional indices is simple. Because every spatial tensor $T^{\alpha\cdots\beta}_{\gamma\cdots\delta}$ at a time t (at a point of $\Sigma_t \subset M$) corresponds to a unique tensor $T^{a\cdots b}_{c\cdots d}$ on Σ , we can write the constraint equations as

$$D_b(K^{ab} - \gamma^{ab}K) = 8\pi j^a, \quad (7.103)$$

$${}^3R - K^{ab}K_{ab} + K^2 = 16\pi\rho_E. \quad (7.104)$$

The components of the momentum constraint in a chart $\{x^i\}$ on Σ are the same as its nonzero components in the chart $\{t, x^i\}$ on M ,

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi j^i.$$

Similarly, using Eq. (7.78) to replace $\mathcal{L}_t$ on M by ∂_t on Σ , we obtain the dynamical equations on Σ corresponding to (7.88) and (7.89) on M :

$$\partial_t \gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta \gamma_{ab}, \quad (7.105a)$$

$$\partial_t K_{ab} = \alpha ({}^3R_{ab} - 2K_{ac}K_b^c + K K_{ab}) - D_a D_b \alpha - 8\pi\alpha(T_{ab} - \frac{1}{2}\gamma_{ab}T) + \mathcal{L}_\beta K_{ab}. \quad (7.105b)$$

We have already seen that a spatial tensor $K_{\alpha\beta}$ is a tensor K_{ab} whose components in a chart $\{t, x^i\}$ are the spatial components K_{ij} of $K_{\alpha\beta}$. The components of the 3+1 Einstein equations on Σ in the chart x^i are then just the spatial components of the corresponding equations on M in the chart t, x^i .

Harmonic gauge (A.K.A. Lorenz gauge, DeDonder gauge, Fock gauge)

Electromagnetism

The equations of electromagnetism more closely resemble those of gravity when the tensor $F_{\alpha\beta}$ is written in terms of a vector potential,

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha. \quad (7.106)$$

Then the first Maxwell equation $\nabla_{[\alpha} F_{\beta\gamma]} = 0$ is automatically satisfied,

$$\nabla_{[\alpha} \nabla_\beta A_{\gamma]} = 0,$$

because partial derivatives commute and the antisymmetry gets rid of Christoffel symbols (or, use $d\omega = 0$ for any form ω , here with $\omega = A$).

The remaining equation, $\nabla_\beta F^{\alpha\beta} = 4\pi j^\alpha$, takes the form of the wave equation for the vector potential if we use the Lorenz gauge condition:

$$\nabla_\alpha A^\alpha = 0. \quad (7.107)$$

(This was introduced by Hedvig Lorenz, not the famous Hendrik Lorentz.) Then

$$\nabla_\beta \nabla^\alpha A^\beta - \nabla_\beta \nabla^\beta A^\alpha = 4\pi j^\alpha.$$

Writing

$$\nabla_\beta \nabla^\alpha A^\beta = [\nabla_\beta, \nabla^\alpha] A^\beta + \underbrace{\nabla^\alpha \nabla_\beta A^\beta}_0 = R_{\alpha\beta} A^\beta, \quad (7.108)$$

we obtain a wave equation for A^α on a curved background spacetime:

$$\nabla_\beta \nabla^\beta A^\alpha - R^\alpha{}_\beta A^\beta = -4\pi j^\alpha. \quad (7.109)$$

Once a vector potential is introduced, the constraint $\nabla_\alpha B^\alpha = 0$ is automatically satisfied. In its place is the gauge condition, here a relation between $\dot{A}^0$ and A^i that again must be satisfied by the initial data, now the values of A^α and $\dot{A}^\alpha$ on Σ_t .

It is the 2nd derivatives that determine the character of the dynamical equation (7.109): As in flat space, they have the form of a wave equation, a hyperbolic equation that gives a unique solution for initial data satisfying the gauge condition.

Gravity

The harmonic coordinate condition for gravity is a nonlinear analog of the Lorenz gauge. It has the form

$$\partial_\nu(\sqrt{-g}g^{\mu\nu}) = 0, \quad (7.110)$$

and is equivalent to requiring that each coordinate x^μ satisfy the scalar wave equation:

$$\nabla_\alpha \nabla^\alpha (x^\mu) = 0. \quad (7.111)$$

Note that each x^μ is a scalar. This condition was first used by Fock and deDonder. Fock called it *harmonic*, presumably because in the mathematics literature a harmonic function is a solution to $\nabla_a \nabla^a f = 0$ for a Riemannian metric (signature $+\cdots+$).

The condition is also equivalent to

$$g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0, \quad (7.112)$$

and this relation lets us write $R_{\mu\nu} = 0$ as a nonlinear wave equation, an equation of the form

$$\partial_\lambda \partial^\lambda g_{\mu\nu} + F(g, \partial g) = 0, \quad (7.113)$$

where F involves only the metric and its first derivatives.

To show all of this, we begin by checking the equivalence of our three forms of the harmonic condition. The divergence of a vector field v^α can be written as

$$\nabla_\nu v^\nu = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} v^\nu). \quad (7.114)$$

This follows from

$$\Gamma^\nu_{\mu\nu} = \frac{1}{2} g^{\sigma\tau} \partial_\mu g_{\sigma\tau} = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g}, \quad (7.115)$$

where we have used $\partial_\lambda g = g g^{\mu\nu} \partial_\lambda g_{\mu\nu}$ (If this is not familiar to you, jump to the end of this section on page 256 for a quick derivation of the derivative of a determinant.)

For each μ , $\nabla^\alpha x^\mu$ is a vector, and we have

$$\begin{aligned} \nabla_\nu \nabla^\nu (x^\mu) &= \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} \partial^\nu x^\mu) = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} g^{\nu\lambda} \partial_\lambda x^\mu) \\ &= \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} g^{\mu\nu}). \end{aligned}$$

For the third form, write

$$\begin{aligned} g^{\mu\nu}\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}g^{\mu\nu}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) = g^{\mu\nu}g^{\lambda\sigma}\partial_\mu g_{\nu\sigma} - \frac{1}{2}g^{\mu\nu}g^{\lambda\sigma}\partial_\sigma g_{\mu\nu} \\ &= -\partial_\mu g^{\mu\lambda} - \frac{1}{\sqrt{-g}}\frac{1}{2}\partial^\lambda\sqrt{-g} = -\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\lambda}\sqrt{-g}). \end{aligned}$$

In the last line we used, for a matrix M (here $(g_{\mu\nu})$), $\partial_\mu M^{-1} = -M^{-1}(\partial_\mu M)M^{-1}$.

Check: Write $0 = \partial_\mu(M^{-1}M) = \partial_\mu(M^{-1})M + M^{-1}\partial_\mu M$, and multiply on the right by M^{-1} .

From the usual coordinate expression for the Riemann tensor,

$$\begin{aligned} R^\mu_{\sigma\nu\tau} &= \partial_\nu\Gamma^\mu_{\sigma\tau} - \partial_\tau\Gamma^\mu_{\sigma\nu} + \Gamma^\mu_{\lambda\nu}\Gamma^\lambda_{\sigma\tau} - \Gamma^\mu_{\lambda\tau}\Gamma^\lambda_{\sigma\nu} \\ &= \partial_\nu\Gamma^\mu_{\sigma\tau} - \partial_\tau\Gamma^\mu_{\sigma\nu} + F(g, \partial g), \end{aligned} \tag{7.116}$$

where $F(g, \partial g)$ here and in later equations will mean some function of $g_{\mu\nu}$ and $\partial_\lambda g_{\mu\nu}$, but not the same function each time the symbol is used. Contracting on σ and τ and using our gauge condition in the form $\Gamma^\mu_{\sigma\tau}g^{\sigma\tau} = 0$, we have

$$R^\mu_{\nu} = -\partial^\sigma\Gamma^\mu_{\sigma\nu} + F(g, \partial g).$$

Now lowering the index μ and noting that $g_{\mu\lambda}\partial^\sigma\Gamma^\lambda_{\sigma\nu} = \partial^\sigma\Gamma_{\mu\sigma\nu} + F(g, \partial g)$, we have

$$R_{\mu\nu} = -\partial^\sigma\Gamma_{\mu\sigma\nu} + F(g, \partial g). \tag{7.117}$$

Because $R_{\mu\nu}$ is a symmetric tensor, we can write this as

$$R_{\mu\nu} = -\frac{1}{2}\partial^\sigma(\Gamma_{\mu\sigma\nu} + \Gamma_{\nu\sigma\mu}) + F(g, \partial g). \tag{7.118}$$

Finally, in the sum of the two Γ 's, the derivatives with respect to μ and ν cancel, leaving

$$\Gamma_{\mu\sigma\nu} + \Gamma_{\nu\sigma\mu} = \partial_\sigma g_{\mu\nu},$$

whence

$$R_{\mu\nu} = -\frac{1}{2}\partial^\sigma\partial_\sigma g_{\mu\nu} + F(g, \partial g), \tag{7.119}$$

as claimed.

Exercise 98. a. Use our expressions for $\partial_i g^{jk}$ and $\partial_i \sqrt{-g}$ to show

$$G^{\mu\nu} = \frac{1}{2}\partial^\sigma\partial_\sigma(g^{\mu\nu}\sqrt{-g}) + F(g, \partial g). \tag{7.120}$$

b. Now write $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ and show that, to $O(h..)$, $g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$, $\sqrt{-g} = 1 + \frac{1}{2}h$, and

$$g^{\alpha\beta}\sqrt{-g} = -(h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h),$$

where $h^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta}h_{\gamma\delta}$ and $h := h_\alpha{}^\alpha$. Show that the linearized harmonic condition is then

$$\nabla_\beta(h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h) = 0, \tag{7.121}$$

with ∇_α the flat space derivative operator.

This gauge condition on a metric perturbation $h_{\alpha\beta}$, introduced earlier in Eq. (6.11), was first used by Hilbert and then Einstein. It is referred to in the literature by at least seven different names: the Hilbert, Einstein, Fock, deDonder, Lorenz, Lorentz (mistakenly), and harmonic gauge. Now $F(g, \partial g)$ is quadratic in ∂g (this follows from the fact that every term has two derivatives and they can't both hit the same $g_{..}$), so at linear order in $h_{..}$, $F(g, \partial g)$ vanishes and Eq. (7.120) has the form

$$\delta G^{\alpha\beta} = -\frac{1}{2}\nabla_\gamma \nabla^\gamma \bar{h}^{\alpha\beta}, \quad \text{where } \bar{h}_{\alpha\beta} := h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h. \quad (7.122)$$

The field equation linearized about flat space is then

$$\nabla_\gamma \nabla^\gamma \bar{h}^{\alpha\beta} = -16\pi \delta T^{\alpha\beta}. \quad (7.123)$$

Derivative of a determinant

We want the derivative $dg/d\lambda$ of a determinant g when the matrix elements g_{ij} are functions of λ . (λ could, for example, be a coordinate with g_{ij} the components of a metric). We have

$$\frac{dg}{d\lambda} = \frac{\partial g}{\partial g_{ij}} \frac{dg_{ij}}{d\lambda}.$$

To find $\partial g/\partial g_{ij}$, write the determinant as the sum of the matrix elements g_{ik} in the i^{th} row times their minors Δ^{ik} :

$$g = \sum_k g_{ik} \Delta^{ik} \text{ (no sum over } i\text{)}.$$

Because none of these minors involve g_{ij} , we have

$$\partial g/\partial g_{ij} = \Delta^{ij}. \quad (7.124)$$

Finally, the inverse of a matrix is given in terms of its minors by

$$g^{ij} = \frac{\Delta^{ji}}{g},$$

whence $\Delta^{ij} = gg^{ji}$, and for our case, with g_{ab} a symmetric tensor, we have

$$\frac{dg}{d\lambda} = gg^{ab} \frac{dg_{ab}}{d\lambda}, \quad \frac{d}{d\lambda} \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{ab} \frac{dg_{ab}}{d\lambda}. \quad (7.125)$$

Chapter 8

Newman-Penrose (NP) formalism and the Teukolsky equation

8.1 NP formalism

The Newman-Penrose paper[61] uses a signature $+ - - -$, and this convention is followed by the papers by Teukolsky[88, 89], and by Chandrasekhar's detailed derivation of the Teukolsky equation in *Mathematical Theory of Black Holes*[24]. Because of that we will switch (in this section only) to the -2 signature. The definition of the Riemann tensor $R^a{}_{bcd}$ is unchanged, and the conventions for the Ricci tensor and scalar are still $R_{ab} = R^c{}_{acb}$, $R = R^a{}_a$.

Item	These notes	Teukolsky	Chandrasekhar	NP
Signature	$+ - - -$	$+ - - -$	$+ - - -$	$+ - - -$
Coordinate indices	$\mu, \nu, \dots$	$\mu, \nu, \dots$	$i, j, \dots$	$\mu, \nu, \dots$
Tetrad indices	$m, n, p, \dots$	not used	$(a), (b), \dots$	m, n, p, ...
Numbered tetrad vectors	$e_m{}^\alpha$	not used	$e_{(a)}{}^i$	$z_{\mathbf{m}}{}^\mu$
$r^2 + a^2 \cos^2 \theta$	ρ^2	Σ	ρ^2	not used
$r^2 - 2Mr + a^2$	Δ	Δ	Δ	not used
$n^\mu \partial_\mu$	Δ	Δ	Δ	Δ
Christoffel symbol	$\Gamma^m{}_{np}$	not used	$\gamma^{(a)}{}_{(b)(c)}$	$-\gamma^{\mathbf{m}}{}_{\mathbf{np}}$
$\Gamma_{314} = \overline{m}^\beta \nabla_\beta \ell_\alpha m^\alpha$	ϱ	ρ	$\bar{\rho}$	ρ

A radiation field far from its source and the Kerr family of geometries have something in common: a preferred null direction. In the case of radiation, the direction is a radial null vector ℓ_α whose asymptotic form is $\ell_\alpha = \nabla_\alpha u = \nabla_\alpha(t - r)$. An outgoing radiation field has asymptotic form proportional to $\frac{f(\ell_\mu x^\mu)}{r} = \frac{f(u)}{r}$. The corresponding contravariant vector, ℓ^α is along $\hat{t}^\alpha + \hat{r}^\alpha$. The polarization of the field is perpendicular to the null direction, so an electromagnetic field in a Lorenz gauge has

$$A_\alpha = \epsilon_\alpha \frac{f(u)}{r} + O(r^{-2}), \quad F_{\alpha\beta} = (\ell_\alpha \epsilon_\beta - \ell_\beta \epsilon_\alpha) \frac{f'(u)}{r} + O(r^{-2}),$$

with ϵ_α in the $\hat{\theta}$ - $\hat{\phi}$ plane. A weak gravitational wave is described by a metric perturbation

$$h_{\alpha\beta} = \epsilon_{\alpha\beta} \frac{f(u)}{r} + O(r^{-2})$$

with corresponding Riemann tensor given by Eq. (5.36).

$$\begin{aligned} R_{\alpha\gamma\beta\delta} &= \frac{1}{2}(\partial_\beta \partial_\gamma h_{\alpha\delta} + \partial_\alpha \partial_\delta h_{\beta\gamma} - \partial_\alpha \partial_\beta h_{\gamma\delta} - \partial_\gamma \partial_\delta h_{\alpha\beta}) \\ &= \frac{1}{2} \frac{f''}{r} (\ell_\beta \ell_\gamma \epsilon_{\alpha\delta} + \ell_\alpha \ell_\delta \epsilon_{\beta\gamma} - \ell_\alpha \ell_\beta \epsilon_{\gamma\delta} - \ell_\gamma \ell_\delta \epsilon_{\alpha\beta}) + O(r^{-2}) \end{aligned} \quad (8.1)$$

The null vector ℓ_α is called a *principal null direction* of the geometry. The NP formalism is designed to exploit symmetries of the geometry tied to principal null directions. Although their 1962 paper emphasized the asymptotic behavior of gravitational radiation, Newman and Penrose knew the vacuum Schwarzschild geometry has two principal null directions, tangent to ingoing and outgoing radial null geodesics n_α and ℓ_α , with each term in the Riemann tensor proportional to ℓ_α and n_α .

The NP formalism completes the basis (tetrad) by using complex combinations m and $\bar{m}$ of unit vectors orthogonal to the null vectors n^α and ℓ^α . Like the combinations $x \pm iy$ that you've encountered in spherical harmonics, m and $\bar{m}$ change by a phase under rotations about the z -axis. For null vectors

$$\ell^\alpha = \frac{1}{\sqrt{2}}(\hat{t}^\alpha + \hat{z}^\alpha), \quad n^\alpha = \frac{1}{\sqrt{2}}(\hat{t}^\alpha - \hat{z}^\alpha),$$

a corresponding choice of m and $\bar{m}$ is

$$m^\alpha = \frac{1}{\sqrt{2}}(\hat{x}^\alpha + i\hat{y}^\alpha), \quad \bar{m}^\alpha = \frac{1}{\sqrt{2}}(\hat{x}^\alpha - i\hat{y}^\alpha),$$

Then m^α and $\bar{m}^\alpha$ are complex null vectors, and the tetrad satisfies

$$\ell_\alpha n^\alpha = 1, \quad m_\alpha \bar{m}^\alpha = -1, \quad \text{all other dot products vanishing.} \quad (8.2)$$

An NP tetrad is then a null tetrad $\{e_m\} = \{e_m^\alpha\}$, $m = 1 - 4$,

$$e_1^\alpha = \ell^\alpha, \quad e_2^\alpha = n^\alpha, \quad e_3^\alpha = m^\alpha, \quad e_4^\alpha = \bar{m}^\alpha, \quad (8.3)$$

for which the spacetime metric has components $g_{mn} = e_m^\alpha e_{n\alpha}$ given by

$$[g_{mn}] = [\eta_{mn}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (8.4)$$

Then $g^{\alpha\beta}$ has the same components

$$[\eta^{mn}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (8.5)$$

and

$$g_{\alpha\beta} = \eta^{mn} e_{m\alpha} e_{n\beta} = \ell_\alpha n_\beta + \ell_\beta n_\alpha - m_\alpha \bar{m}_\beta - m_\beta \bar{m}_\alpha. \quad (8.6)$$

Examples:

For the Schwarzschild geometry, with

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \quad (8.7a)$$

$$= \frac{\Delta}{r^2} dt^2 - \frac{r^2}{\Delta} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad \Delta := r^2 - 2Mr, \quad (8.7b)$$

an orthonormal frame is

$$\hat{t}^\alpha = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \partial_t, \quad \hat{r}^\alpha = \sqrt{1 - \frac{2M}{r}} \partial_r, \quad \hat{\theta}^\alpha = \frac{1}{r} \partial_\theta, \quad \hat{\phi}^\alpha = \frac{1}{r \sin \theta} \partial_\phi. \quad (8.8)$$

We could take

$$\ell^\alpha = \frac{1}{\sqrt{2}} (\hat{t}^\alpha + \hat{r}^\alpha), \quad n^\alpha = \frac{1}{\sqrt{2}} (\hat{t}^\alpha - \hat{r}^\alpha), \quad m^\alpha = \frac{1}{\sqrt{2}} (\hat{\theta}^\alpha + i \hat{\phi}^\alpha). \quad (8.9)$$

with contravariant components

$$\ell^\mu = \frac{1}{\sqrt{2}} \left(\frac{r}{\sqrt{\Delta}}, \frac{\sqrt{\Delta}}{r}, 0, 0 \right), \quad n^\mu = \frac{1}{\sqrt{2}} \left(\frac{r}{\sqrt{\Delta}}, -\frac{\sqrt{\Delta}}{r}, 0, 0 \right), \quad m^\mu = \frac{1}{\sqrt{2} r} \left(0, 0, 1, \frac{i}{\sin \theta} \right) \quad (8.10)$$

Notice that multiplying ℓ^α by any function f and dividing n_α by f gives another null tetrad with the same null directions and the same metric components η_{mn} . Kinnersley, one of Kip Thorne's first students, used this freedom to get rid of the square roots in ℓ and n , making the commutators and the Christoffel symbols (Ricci rotation coefficients) simpler, at the cost of breaking the symmetry associated with time reversal. The Kinnersley tetrad is

$$(\ell^\mu) = (r^2/\Delta, 1, 0, 0), \quad (n^\mu) = \frac{1}{2}(1, -\Delta/r^2, 0, 0), \quad (m^\mu) = \frac{1}{\sqrt{2} r} (0, 0, 1, \frac{i}{\sin \theta}) \quad (8.11)$$

The Kerr metric has the factored form

$$ds^2 = \frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 - \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a dt]^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2, \quad (8.12)$$

where

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta. \quad (8.13)$$

An orthonormal covariant (dual) frame is

$$\Omega^0 = \frac{\sqrt{\Delta}}{\rho}(dt - a \sin^2 \theta d\phi), \quad \Omega^1 = \frac{\rho}{\sqrt{\Delta}}dr, \quad \Omega^2 = \rho d\theta, \quad \Omega^3 = \frac{\sin \theta}{\rho}[(r^2 + a^2)d\phi - a dt],$$

and the corresponding contravariant frame to which it is dual is

$$\mathbf{E}_0 = \frac{1}{\rho\sqrt{\Delta}}((r^2 + a^2)\partial_t + a\partial_\phi), \quad \mathbf{E}_3 = \frac{1}{\rho \sin \theta}(\partial_\phi + a \sin^2 \theta \partial_t), \quad \mathbf{E}_1 = \frac{\sqrt{\Delta}}{\rho}\partial_r, \quad \mathbf{E}_2 = \frac{1}{\rho}\partial_\theta. \quad (8.14)$$

The associated Kinnersley tetrad is

$$\begin{aligned} e_1 = \ell &= \frac{\rho}{\sqrt{\Delta}}(\mathbf{E}_0 + \mathbf{E}_1) \\ e_2 = n &= \frac{1}{2} \frac{\sqrt{\Delta}}{\rho}(\mathbf{E}_0 - \mathbf{E}_1) \\ e_3 = m &= \frac{\rho}{\sqrt{2} \tilde{\rho}}(\mathbf{E}_2 + i\mathbf{E}_3) \\ e_4 = \bar{m} &= \frac{\rho}{\sqrt{2} \tilde{\rho}^*}(\mathbf{E}_2 - i\mathbf{E}_3) \end{aligned}$$

where

$$\tilde{\rho} := r + ia \cos \theta \quad (8.15)$$

is a kind of complex square root of ρ^2 : $\tilde{\rho}\tilde{\rho}^* = \rho^2$. Introducing it gets rid of the square root, $\rho = \sqrt{r^2 + a^2 \cos^2 \theta}$, while keeping $m^\alpha \bar{m}_\alpha = -1$:

$$(e_1^\mu) = (\ell^\mu) = \left(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta} \right), \quad (8.16a)$$

$$(e_2^\mu) = (n^\mu) = \frac{1}{2} \left(\frac{r^2 + a^2}{\rho^2}, -\frac{\Delta}{\rho^2}, 0, \frac{a}{\rho^2} \right), \quad (8.16b)$$

$$(e_3^\mu) = (m^\mu) = \frac{1}{\sqrt{2} \tilde{\rho}} \left(ia \sin \theta, 0, 1, \frac{i}{\sin \theta} \right). \quad (8.16c)$$

When $a = 0$, we recover the Kinnersley tetrad for Schwarzschild (8.10).

Weyl tensor

The tracefree part of the Riemann tensor is called the Weyl tensor and written C_{abcd} . In a vacuum spacetime the two coincide. In general the decomposition of the Riemann tensor into its tracefree part and parts involving the Ricci tensor and Ricci scalar is

$$R_{abcd} = C_{abcd} + \frac{1}{2}(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{ad}R_{bc} - g_{bc}R_{ad}) - \frac{1}{6}(g_{ac}g_{bd} - g_{ad}g_{bc})R. \quad (8.17)$$

Up to the constant factor $1/2$, the Ricci tensor part is the only combination of g_{ab} and R_{ab} with Riemann tensor index symmetries (and linear in R_{ab}); and the Ricci scalar part is the only combination of R and g_{ab} with those symmetries (linear in R). All that remains is to take the trace of both sides to get the factors $1/2$ and $1/6$. We will be working with vacuum spacetimes, and for those $R_{abcd} = C_{abcd}$.

Exercise 99. Show that, for an n -dimensional spacetime, the decomposition of the Riemann tensor is given by

$$R_{abcd} = C_{abcd} + \frac{1}{n-2}(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{ad}R_{bc} - g_{bc}R_{ad}) - \frac{1}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{bc})R.$$

The Teukolsky equation is an equation for a single tetrad component of the Weyl tensor. The NP formalism gives names to each independent Christoffel symbol (Ricci rotation coefficient) and to each independent tetrad component of the Weyl tensor. We begin by noting that the Weyl tensor has 10 independent real components and that these correspond to 5 independent complex components along the NP tetrad.

First: The Riemann tensor in n dimensions has $\frac{n^2(n^2-1)}{12}$ independent components. This is Prob. 3b of Wald, and if you are already familiar with it, skip to the end of this paragraph. The index symmetries of R_{ijkl} are

$$R_{ijkl} = R_{[ij][kl]}, \quad R_{i[jkl]} = 0, \quad R_{ijkl} = R_{klij}.$$

The last of these is implied by the others, so we do not include it in our counting. Because the index pair ij is antisymmetric, the independent pairs have $i < j$, and there are $n(n-1)/2$ such pairs. There are similarly $n(n-1)/2$ pairs kl with $k < l$. The number of pairs $(ij)(kl)$ is then $n^2(n-1)^2/4$. These are not all independent because we have the relations $R_{i[jkl]} = 0$. Here we can choose i in n different ways; and for $[jkl]$ we have n objects taken 3 at a time or $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$. The number of relations is then $n^2(n-1)(n-2)/6$, and each relation reduces the number of independent components by one, leaving

$$\frac{n^2(n-1)^2}{4} - \frac{n^2(n-1)(n-2)}{6} = \frac{n^2(n^2-1)}{12}$$

independent components of the Riemann tensor in n dimensions.

To find the number of independent components of C_{abcd} , we subtract the number of independent components of R_{ab} . Here the only symmetry is $R_{ab} = R_{ba}$, so there are $n(n+1)/2$ independent pairs (ab) . Then the Weyl tensor has

$$\frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2} = \frac{(n-3)n(n+1)(n+2)}{12} \quad (8.18)$$

independent components, 10 in four dimensions. In fewer than four dimensions, the Weyl tensor vanishes.

Spin weight

A rotation of the tetrad in the plane spanned by $\mathbf{m}$ and $\overline{\mathbf{m}}$ has the form

$$\mathbf{m} \rightarrow e^{i\eta}\mathbf{m}, \quad \overline{\mathbf{m}} \rightarrow e^{-i\eta}\overline{\mathbf{m}} \quad (8.19)$$

Exercise 100. Two-line problem: Write $\mathbf{m} = \hat{\mathbf{x}} + i\hat{\mathbf{y}}$ and check that a clockwise rotation of the frame by η in the $\hat{\mathbf{x}}\text{-}\hat{\mathbf{y}}$ plane gives the above equation.

Then a component of a tensor with p indices along $\mathbf{m}$ and q indices along $\overline{\mathbf{m}}$ (and remaining indices along $\mathbf{n}$ and ℓ) changes under a tetrad rotation by the factor $e^{i(p-q)\eta}$, and is said to have *spin weight* $p - q$. This is, of course, analogous to an eigenvalue of L_z in quantum mechanics: A wavefunction ψ with eigenvalue $m\hbar$ changes under a rotation by η about the z -axis by the factor $e^{im\eta}$.

In particular, NP order the five independent complex components of the Weyl tensor by their spin weight, from 2 to -2 :

$$\Psi_0 = -C_{1313} = -C_{\alpha\beta\gamma\delta}\ell^\alpha m^\beta \ell^\gamma m^\delta \quad (8.20a)$$

$$\Psi_1 = -C_{1213} = -C_{\alpha\beta\gamma\delta}\ell^\alpha n^\beta \ell^\gamma m^\delta \quad (8.20b)$$

$$\Psi_2 = -C_{1342} = -C_{\alpha\beta\gamma\delta}\ell^\alpha m^\beta \bar{m}^\gamma n^\delta \quad (8.20c)$$

$$\Psi_3 = -C_{1242} = -C_{\alpha\beta\gamma\delta}\ell^\alpha n^\beta \bar{m}^\gamma n^\delta \quad (8.20d)$$

$$\Psi_4 = -C_{2424} = -C_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta. \quad (8.20e)$$

It would have made more sense to write these as $\Psi_2, \Psi_1, \dots, \Psi_{-2}$, and Press and Teukolsky tried that, but there were too many other papers with the NP convention, and the change didn't stick.

For Schwarzschild and Kerr the only nonzero component of the Weyl tensor is Ψ_2 , the spin-weight zero part. Geometries with this character are Petrov type D . They have the two principal null directions ℓ_α and n_α .

Exercise 101. NP components of $F_{\alpha\beta}$.

- How many independent components does the electromagnetic tensor $F_{\alpha\beta}$ have?
- In analogy with the components of the Weyl tensor above, write these components of $F_{\alpha\beta}$ as complex components with definite spin weight, taking as the spin-weight 0 component $F_{\alpha\beta}(\ell^\alpha n^\beta + \bar{m}^\alpha m^\beta)$.
- Using the flat space tetrad

$$\ell = \frac{1}{\sqrt{2}}(\hat{t} + \hat{z}), \quad n = \frac{1}{\sqrt{2}}(\hat{t} - \hat{z}), \quad m = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y}), \quad \bar{m} = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y}),$$

write each of these complex components in terms of the t, x, y, z components of E and B .

We have already seen in Eq. (8.1) that outgoing waves have, at leading order in $1/r$, a Weyl tensor that is quadratic in ℓ_α and is constructed from ℓ_α and a polarization tensor in the $m\text{-}\bar{m}$ plane. From $\ell_\alpha \ell^\alpha = \ell_\alpha m^\alpha = 0$, it immediately follows that the only nonvanishing Ψ_i is Ψ_4 . Similarly, purely ingoing radiation has Ψ_0 as its leading asymptotic term. And it is Ψ_0 and Ψ_4 that appear in the decoupled wave equations that describe perturbations of Schwarzschild and Kerr spacetimes.

Exercise 102. Writing the asymptotic form of m as $\frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})$ (corresponding to a chart near an observer with z in the direction of propagation of the wave), show that the two polarizations of the wave are given by the real and imaginary parts of Ψ_4 . (+ and $\times$ polarizations are proportional to $h_{xx} - h_{yy}$ and h_{xy} , respectively, for a given choice of x and y .) Use Eq. (8.1).

Christoffel symbols (Ricci rotation coefficients)

We looked in Section 2.2 at the Christoffel symbols (2.37) and Riemann tensor components for a frame field $\{e_n\}$ in which the components g_{mn} are constant. As we saw, the quantity $\Gamma^m_{n\alpha}v^\alpha$ is the derivative of the family of rotations or Lorentz transformations that relate a frame parallel transported along v^α to the original frame field. That's why Γ^m_{np} is called a rotation coefficient and why Γ_{mnp} , defined by Eq. (2.23), is antisymmetric in m and n . Chandrasekhar's convention is

$$\gamma^m_{np} = \Gamma^m_{np},$$

while NP take the opposite sign for $\gamma^{\mathbf{m}}_{\mathbf{np}}$. Chandra also uses $(a), (b), \dots$ for concrete tetrad indices, instead of the $\mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}$ of NP.

The NP naming of independent symbols is

$$\begin{array}{lll}
 \kappa = \Gamma_{311} & \varrho = \Gamma_{314} & \epsilon = \frac{1}{2}(\Gamma_{211} + \Gamma_{341}) \\
 \sigma = \Gamma_{313} & \mu = \Gamma_{243} & \gamma = \frac{1}{2}(\Gamma_{212} + \Gamma_{342}) \\
 \lambda = \Gamma_{244} & \tau = \Gamma_{312} & \alpha = \frac{1}{2}(\Gamma_{214} + \Gamma_{344}) \\
 \nu = \Gamma_{242} & \pi = \Gamma_{241} & \beta = \frac{1}{2}(\Gamma_{213} + \Gamma_{343})
 \end{array}$$

$$\Gamma_{211} = \epsilon + \bar{\epsilon} \quad \Gamma_{341} = \epsilon - \bar{\epsilon} \quad \Gamma_{212} = \gamma + \bar{\gamma} \quad \Gamma_{342} = \gamma - \bar{\gamma} \quad \Gamma_{214} = \alpha + \bar{\beta} \quad \Gamma_{344} = \alpha - \bar{\beta}. \quad (8.21)$$

Each of these has a fixed spin weight. The quantities $\Gamma_{21\alpha}$ and $\Gamma_{34\alpha}$ are respectively real and imaginary, and the third column lists the four tetrad components of the complex vector $\Gamma_{21\alpha} + \Gamma_{34\alpha}$.

Quick question: Why is $\Gamma_{34\alpha}$ a purely imaginary covector?

The signs are the same for Chandra and NP if one writes the NP symbols directly as derivatives of tetrad vectors: That is,

$$\kappa = e_{3\alpha} \nabla_1 e_1^\alpha = m_\alpha \ell^\beta \nabla_\beta \ell^\alpha, \quad \pi = e_{2\alpha} \nabla_1 e_4^\alpha = n_\alpha \ell^\beta \nabla_\beta \bar{m}^\alpha = -\bar{m}^\alpha \ell^\beta \nabla_\beta n_\alpha.$$

for everyone.

Symbols for derivatives

A final set of individual symbols introduced by NP are the names of the basis vectors regarded as derivatives (as we have defined vectors):

$$\ell^\mu \partial_\mu = D, \quad n^\mu \partial_\mu = \Delta, \quad m^\mu \partial_\mu = \delta, \quad \bar{m}^\mu \partial_\mu = \bar{\delta}. \quad (8.22)$$

8.2 Bardeen-Press and Teukolsky equations

The presentation here illustrates how you might find your way to the Teukolsky equation starting from the wave equation for the perturbed Weyl tensor on a flat background and then moving on to Schwarzschild (the Bardeen-Press equation) and finally to Kerr. For a concise derivation of the Teukolsky equation using the NP formalism, you can skip to the section 8.2.3 on Kerr. Everything uses the equations in the NP paper that express the Weyl tensor components, the commutators of basis vectors, and the Bianchi identities in terms of Christoffel symbols (spin coefficients).

8.2.1 Flat space

This second section of the notes will eventually follow NP equations for commutators and for the Weyl tensor in terms of the NP symbols of the last section. For now, to see where we're going, it will be helpful to look at the wave equation in its simplest context – flat space with basis constructed from a t, x, y, z chart (instead of spherical coordinates).

We start with the scalar wave equation with partial derivatives along the null basis vectors,

$$\begin{aligned} e_1 &= \ell = \hat{t} + \hat{z} \\ e_2 &= n = \frac{1}{2}(\hat{t} - \hat{z}) \\ e_3 &= m = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y}) \\ e_4 &= \bar{m} = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y}). \end{aligned}$$

Then

$$\nabla_\alpha \nabla^\alpha \Phi = \eta^{mn} \partial_m \partial_n \Phi = (\partial_1 \partial_2 + \partial_2 \partial_1 - \partial_3 \partial_4 - \partial_4 \partial_3) \Phi = 2(\partial_1 \partial_2 - \partial_3 \partial_4) \Phi,$$

and the scalar wave equation is

$$(\partial_1 \partial_2 - \partial_3 \partial_4) \Phi = 0. \quad (8.23)$$

In NP notation, this is

$$(D\Delta - \delta\bar{\delta})\Phi = 0. \quad (8.24)$$

Exercise 103. a. Two-line exercise: Replace the null basis vectors by their expressions in terms of the Cartesian basis to recover the usual form of the scalar wave equation.

b. Another two-line exercise in flat space. Show that Maxwell's equations, $\nabla_{[\alpha} F_{\beta\gamma]} = 0$ and $\nabla_\beta F^{\alpha\beta} = 0$, imply $\nabla_\gamma \nabla^\gamma F_{\alpha\beta} = 0$. (Hint: Take the divergence of the first Maxwell equation and then use the second.)

The wave equations satisfied by Ψ_0 and Ψ_4 are easily derived as a perturbation of flat space. In curved space, nonzero Christoffel symbols mix together Weyl tensor parts with different spin weights and leave only these extreme spin components with their own decoupled equations. We'll find the equation for $\Psi_0 = -C_{1313}$. Begin with the Bianchi identity,

$$\nabla_{[\alpha} C_{\beta\gamma]\delta\epsilon} = 0, \quad (8.25)$$

writing the two nontrivial components that involve $C_{1313} = -\Psi_0$. These are

$$\nabla_{[4} C_{13]13} = 0 \text{ and } \nabla_{[2} C_{13]13} = 0. \quad (8.26)$$

Next, we'll use the Riemann-tensor symmetries and the fact that $C_{\dots}$ is tracefree to show

$$C_{4113} = 0, \quad C_{3213} = 0, \quad C_{3413} = -C_{1213} = \Psi_1. \quad (8.27)$$

Proof: The Riemann-tensor symmetries imply $C_{4113} = C_{3114}$, and $C_{1112} = 0 = C_{2111}$. Then

$$\begin{aligned} C_{4113} &= \frac{1}{2}(C_{4113} + C_{3114} - C_{1112} - C_{2111}) \\ &= -\frac{1}{2}\eta^{mn} C_{m11n} = 0, \end{aligned}$$

where the last equality is the statement that $C_{\alpha\beta\gamma\delta}$ is tracefree. The same steps, using $\eta^{mn} C_{3mn3} = 0$, show $C_{3213} = 0$. Finally, again using the tracefree condition, we have

$$\begin{aligned} 0 &= \eta^{mn} C_{1mn3} = C_{1123} + C_{1213} - C_{1343} - C_{1433} = C_{1213} - C_{1343} \implies \\ \Psi_1 &:= -C_{1213} = -C_{1343} = C_{3413}. \end{aligned}$$

Now we're ready to use the Bianchi identity: Eqs. (8.26) and (8.27) give

$$\nabla_4 C_{1313} + \nabla_1 C_{3413} = 0, \quad \nabla_2 C_{1313} + \nabla_3 C_{2113} = 0. \quad (8.28)$$

Now linearize about flat space and write ψ_i for the first-order change in Ψ_i . Because the Weyl tensor and all Christoffel symbols vanish in the background spacetime for a Cartesian frame, we have, for example, $\nabla_4 C_{1313} = -\bar{\delta}\psi_0$, and the equations take the form,

$$-\bar{\delta}\psi_0 + D\psi_1 = 0, \quad -\Delta\psi_0 + \delta\psi_1 = 0.$$

To eliminate ψ_1 , act with δ on the first equation and with D on the second and subtract to obtain

$$(D\Delta - \delta\bar{\delta})\psi_0 = 0. \quad (8.29)$$

That is, ψ_0 satisfies the scalar wave equation (8.24).

Flat space, spherical coordinates.

We look next at the key equations (8.28) for Ψ_0 in the Kinnersley tetrad associated with spherical coordinates but still in flat space:

$$e_1 = \ell = \partial_t + \partial_r, \quad (8.30a)$$

$$e_2 = n = \frac{1}{2}(\partial_t - \partial_r), \quad (8.30b)$$

$$e_3 = m = \frac{1}{\sqrt{2}r} \left(\partial_\theta + \frac{i}{\sin\theta} \partial_\phi \right) \quad (8.30c)$$

$$e_4 = \bar{m} \quad (8.30d)$$

The independent nonzero commutators are

$$[e_1, e_3] = \left[\partial_r, \frac{1}{\sqrt{2}r} \left(\partial_\theta + \frac{i}{\sin\theta} \partial_\phi \right) \right] = -\frac{1}{\sqrt{2}r^2} \left(\partial_\theta + \frac{i}{\sin\theta} \partial_\phi \right) \quad (8.31a)$$

$$= -\frac{1}{r} e_3, \quad (8.31b)$$

$$[e_1, e_4] = -\frac{1}{r} e_4, \quad (8.31c)$$

$$[e_2, e_3] = \frac{1}{2r} e_3, \quad (8.31d)$$

$$[e_2, e_4] = \frac{1}{2r} e_4, \quad (8.31e)$$

$$[e_3, e_4] = \frac{\cot\theta}{\sqrt{2}r} (e_3 - e_4) \quad (8.31f)$$

By inspection, the corresponding nonzero coefficients c^m_{np} (and then with lowered indices) are

$$\begin{aligned} c^3_{13} = c^4_{14} = -\frac{1}{r}, \quad c^3_{23} = c^4_{24} = \frac{1}{2r}, \quad c^3_{34} = -c^4_{34} = \frac{\cot\theta}{\sqrt{2}r}, \\ c_{413} = c_{314} = \frac{1}{r}, \quad c_{423} = c_{324} = -\frac{1}{2r}, \quad c_{434} = -c_{334} = -\frac{\cot\theta}{\sqrt{2}r}, \end{aligned} \quad (8.32)$$

together with those obtained by exchanging the last two indices: $c_{pmn} = -c_{pnm}$.

Exercise 104. Check the last four commutators of Eqs. (8.31).

In NP notation for derivative operators, Eqs. (8.31) are

$$[D, \delta] = -\frac{1}{r}\delta, \quad [D, \bar{\delta}] = -\frac{1}{r}\bar{\delta}, \quad [\Delta, \delta] = -\frac{1}{2r}\delta, \quad [\Delta, \bar{\delta}] = -\frac{1}{2r}\bar{\delta}, \quad [\delta, \bar{\delta}] = -\frac{\cot \theta}{\sqrt{2}r}(\delta - \bar{\delta}). \quad (8.33)$$

The nonzero Christoffel symbols and corresponding spin coefficients are now obtained from Eq. (2.37)

$$\Gamma_{mnp} = \frac{1}{2}(c_{pmn} - c_{npm} - c_{mnp}). \quad (8.34)$$

Notice that the nonzero c_{mnp} have at most one index in $\{1, 2\}$ and the first index is 3 or 4. Because Γ_{341} and Γ_{342} are imaginary while all c_{mnp} are real $\Gamma_{341} = \Gamma_{342} = 0$. We are left with

$$\Gamma_{134} = \frac{1}{r} = -\varrho, \quad \Gamma_{234} = \Gamma_{243} = -\frac{1}{2r} = \mu, \quad \Gamma_{343} = \frac{1}{\sqrt{2}} \frac{\cot \theta}{r} = \beta - \bar{\alpha}, \quad \Gamma_{434} = \frac{1}{\sqrt{2}} \frac{\cot \theta}{r} = \alpha - \bar{\beta}, \quad (8.35)$$

together with those obtained by exchanging the first two indices: $\Gamma_{mnp} = -\Gamma_{nmp}$. Notice also that taking the complex conjugate exchanges 3 and 4: $\Gamma_{243} = \bar{\Gamma}_{234}$, and in this flat-space example, the Γ 's are real.

For reference in computing the Bianchi identities, here are the corresponding raised Γ 's and the nonzero spin coefficients:

$$\begin{aligned} \Gamma^1_{34} = \Gamma^1_{43} = \mu, \quad \Gamma^2_{34} = \Gamma^2_{43} = -\varrho, \quad \Gamma^3_{33} = \Gamma^4_{44} = 2\beta = -2\alpha, \\ \Gamma^3_{13} = \Gamma^4_{14} = -\varrho, \quad \Gamma^4_{24} = \Gamma^3_{23} = \mu, \quad \Gamma^3_{34} = \Gamma^4_{43} = -2\beta = 2\alpha. \end{aligned} \quad (8.36)$$

$$\alpha = -\frac{1}{2\sqrt{2}} \frac{\cot \theta}{r}, \quad \beta = \frac{1}{2\sqrt{2}} \frac{\cot \theta}{r}, \quad \varrho = -\frac{1}{r}, \quad \mu = -\frac{1}{2r}. \quad (8.37)$$

We'll see later that Schwarzschild is only slightly different.

For the two components of the Bianchi identities, we'll need to show that the component $\nabla_\alpha C_{4113} \equiv \nabla_\epsilon C_{\alpha\beta\gamma\delta} \bar{m}^\alpha \ell^\beta \ell^\gamma m^\delta$ is zero. We already showed that $C_{4113} = 0$, but the covariant derivative involves other components of $C_{\alpha\beta\gamma\delta}$ multiplied by Christoffel symbols. The proof, however, is essentially identical if we keep the basis vectors outside the derivative operator

$$\begin{aligned} \nabla_\epsilon C_{\alpha\beta\gamma\delta} \bar{m}^\alpha \ell^\beta \ell^\gamma m^\delta &= \nabla_\epsilon C_{\delta\gamma\beta\alpha} \bar{m}^\alpha \ell^\beta \ell^\gamma m^\delta = \nabla_\epsilon C_{\alpha\beta\gamma\delta} m^\alpha \ell^\beta \ell^\gamma \bar{m}^\delta \\ &= \frac{1}{2} \nabla_\epsilon C_{\alpha\beta\gamma\delta} \ell^\beta \ell^\gamma (\bar{m}^\alpha m^\delta + m^\alpha \bar{m}^\delta - \ell^\alpha n^\delta - \ell^\delta n^\alpha) = \frac{1}{2} \nabla_\epsilon C_{\alpha\beta\gamma\delta} \ell^\beta \ell^\gamma (-g^{\alpha\delta}) \\ &= 0. \end{aligned}$$

The equivalent steps show $\nabla_\alpha C_{3213} = 0$. Set off and numbered for future reference:

$$\nabla_\alpha C_{4113} \equiv \nabla_\alpha C_{\beta\gamma\delta\epsilon} \bar{m}^\beta \ell^\gamma \ell^\delta m^\epsilon = 0, \quad \nabla_\alpha C_{3213} \equiv \nabla_\alpha C_{\beta\gamma\delta\epsilon} m^\beta n^\gamma \ell^\delta m^\epsilon = 0. \quad (8.38)$$

Naming of parts

Again for reference, here's a list of the Weyl tensor components in terms of Ψ_i ordered top to bottom and then left to right. Those in blue are, up to exchanging antisymmetric indices, the definitions (8.20).

$$\begin{array}{llll} C_{1212} = -(\Psi_2 + \bar{\Psi}_2) & C_{1313} = -\Psi_0 & C_{1414} = -\bar{\Psi}_0 & C_{2323} = -\bar{\Psi}_4 \quad C_{3434} = -(\Psi_2 + \bar{\Psi}_2) \\ C_{1213} = -\Psi_1 & C_{1314} = 0 & C_{1423} = \bar{\Psi}_2 & C_{2324} = 0 \\ C_{1214} = -\bar{\Psi}_1 & C_{1323} = 0 & C_{1424} = 0 & C_{2334} = \bar{\Psi}_3 \\ C_{1223} = \bar{\Psi}_3 & C_{1324} = \Psi_2 & C_{1434} = -\bar{\Psi}_1 & C_{2424} = -\Psi_4 \\ C_{1224} = \Psi_3 & C_{1334} = \Psi_1 & & C_{2434} = -\Psi_3 \\ C_{1234} = \Psi_2 - \bar{\Psi}_2 & & & \end{array}$$

Every term comes from the definition via either complex conjugation or the tracefree condition or both. For example, to get C_{1212} , start from $C_{1342} = -\Psi_2$. Weyl tracefree implies

$$0 = \eta^{mn} C_{1mn2} = \underbrace{C_{1122}}_0 + C_{1212} - C_{1342} - C_{1432} \implies C_{1212} = C_{1342} + \bar{C}_{1342} = -(\Psi_2 + \bar{\Psi}_2).$$

All vanishing components also come from the tracefree condition, as in the lines following Eq. (8.27).

Up to sign everything is determined by spin-weight and:

More ℓ 's than n 's $\Rightarrow \Psi_0, \Psi_1$;

more n 's than ℓ 's $\Rightarrow \Psi_3, \Psi_4$;

if the component has the same number of ℓ 's and n 's, it involves Ψ_2 and if real or imaginary is twice the real or imaginary part of Ψ_2 .

The two Bianchi identities involving ψ_0 , (8.28), now involve the nonzero Γ 's in Eq. (8.36).

First identity:

$$\begin{aligned} \nabla_4 C_{1313} &= -\bar{\delta}\psi_0 - 2\Gamma^m_{14} C_{m313} - 2\Gamma^m_{34} C_{1m13} \\ &= -\bar{\delta}\psi_0 - 2\Gamma^4_{14} C_{4313} - 2\Gamma^3_{34} C_{1313} \\ &= -\bar{\delta}\psi_0 - 4\rho\psi_1 + 4\alpha\psi_0 \\ \nabla_1 C_{3413} &= D\psi_1 \quad \text{because no } \Gamma^m_{n1} \text{ is nonzero} \\ 0 &= 3\nabla_{[4} C_{13]13} = -(\bar{\delta} - 4\alpha)\psi_0 + (D - 4\rho)\psi_1. \end{aligned} \tag{8.39}$$

Second identity:

$$\begin{aligned} \nabla_2 C_{1313} &= -\Delta\psi_0 \quad \text{because no } \Gamma^m_{n2} \text{ is nonzero} \\ \nabla_3 C_{2113} &= \delta\psi_1 - \Gamma^m_{23} C_{m113} - \Gamma^m_{13} C_{2m13} - \Gamma^m_{13} C_{21m3} - \Gamma^m_{33} C_{211m} \\ &= \delta\psi_1 - \Gamma^3_{23} C_{3113} - \Gamma^3_{13} \underbrace{C_{2313}}_0 - \Gamma^3_{13} \underbrace{C_{2133}}_0 - \Gamma^3_{33} C_{2113} \\ &= \delta\psi_1 - \mu\psi_0 - 2\beta\psi_1 \end{aligned} \tag{8.40}$$

$$0 = 3\nabla_{[2} C_{13]13} = -(\Delta + \mu)\psi_0 + (\delta - 2\beta)\psi_1 \tag{8.41}$$

Wave equation: Again we need to eliminate ψ_1 , but this time $\delta - 2\beta$ doesn't commute with $D - 4\rho$. This is easy to rectify because the operator

$$r(\delta - 2\beta) = \frac{1}{\sqrt{2}} \left(\partial_\theta + \frac{i}{\sin\theta} \partial_\phi - 2 \cot\theta \right)$$

is independent of r . So we multiply Eq. 8.41 by r before applying $D - 4\rho$: Using $[D - 4\rho, r(\delta - 4\beta)] = 0$, we write

$$\frac{1}{r} \left(D + \frac{4}{r} \right) (r \times \text{Eq. (8.41)}) - (\delta - 2\beta) (\text{Eq. (8.39)}) = 0,$$

and obtain our decoupled equation for ψ_0 :

$$\frac{1}{r} (D - 4\rho) [r(\Delta + \mu)\psi_0] - (\delta - 2\beta)(\bar{\delta} - 4\alpha)\psi_0 = 0, \tag{8.42}$$

or

$$\left[\left(D + \frac{5}{r} \right) \left(\Delta + \frac{1}{2r} \right) - \left(\delta - \frac{1}{\sqrt{2}} \frac{\cot\theta}{r} \right) \left(\bar{\delta} + \sqrt{2} \frac{\cot\theta}{r} \right) \right] \psi_0 = 0. \tag{8.43}$$

In generalizing the derivation of our wave equation (8.42) for ψ_0 to Kerr, it will be helpful to write the commutation relation we used to eliminate ψ_1 in terms of spin coefficients and operators D and δ . Replacing r by $-\frac{1}{\varrho}$, we have

$$[(D - 5\varrho)(\Delta + \mu) - (\delta - 2\beta)(\bar{\delta} - 4\alpha)]\psi_0 = 0. \quad (8.44)$$

Exercise 105. Fill in the steps to obtain Eq. (8.43) and then write it in terms of $\partial_t, \partial_r, \partial_\theta, \partial_\phi$, multiplying by $2r^2$ to get rid of the factors of $1/(2r^2)$.

Exercise 106. Check that NP Eqs. (4.2e), and the second equation in NP (4.4) are, respectively

$$D\beta = \rho\beta, \quad [D, \delta] = \rho\delta, \quad (8.45)$$

and use these to go from Eqs. (8.39) and (8.41) to the wave equation in the form (8.44).

8.2.2 Schwarzschild

We start with the Christoffel symbols and Weyl tensor for Schwarzschild. Because the $m\text{-}\bar{m}$ part of the metric is unchanged from flat space, only the commutators involving D and Δ are different. From Eq. (8.11), we have

$$e_1 = D = \frac{r^2}{\Delta}\partial_t + \partial_r, \quad e_2 = n = \Delta = \frac{1}{2}\left(\partial_t - \frac{\Delta}{r^2}\partial_r\right), \quad e_3 = m = \delta = \frac{1}{\sqrt{2}r}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right), \quad (8.46)$$

and

$$\partial_t = \frac{1}{2}\frac{\Delta}{r^2}e_1 + e_2, \quad \partial_r = \frac{1}{2}e_1 - \frac{r^2}{\Delta}e_2.$$

Then

$$\begin{aligned} [e_1, e_2] &= -\frac{1}{2}\partial_r(1 - 2M/r)\partial_r + \frac{1}{2}(1 - 2M/r)\partial_r(1 - 2M/r)^{-1}\partial_t \\ &= -\frac{M}{r^2}\partial_r - \frac{1}{2}(1 - 2M/r)^{-1}\frac{2M}{r^2}\partial_t \\ &= -\frac{M}{r^2}\left(\frac{1}{2}e_1 - \frac{r^2}{\Delta}e_2\right) - \frac{M}{\Delta}\left(\frac{1}{2}\frac{\Delta}{r^2}e_1 + e_2\right) \\ &= -\frac{M}{r^2}e_1. \\ [e_1, e_3] &= \partial_r\left(\frac{1}{r}\right)re_3 = -\frac{1}{r}e_3, \text{ (This one didn't change)} \\ [e_2, e_3] &= -\frac{1}{2}\frac{\Delta}{r^2}\partial_r\left(\frac{1}{r}\right)re_3 = \frac{1}{2}\frac{\Delta}{r^3}e_3 \end{aligned}$$

The flat-space c^m_{np} and c_{mnp} of Eq. (8.32), with changed coefficients in blue, are

$$\begin{aligned} c^1_{12} &= -\frac{M}{r^2} & c^3_{13} &= c^4_{14} = -\frac{1}{r}, & c^3_{23} &= c^4_{24} = \frac{\Delta}{2r^3}, & c^3_{34} &= -c^4_{34} = \frac{\cot\theta}{\sqrt{2}r}, \\ c_{212} &= -\frac{M}{r^2} & c_{413} &= c_{314} = \frac{1}{r}, & c_{423} &= c_{324} = -\frac{\Delta}{2r^3}, & c_{434} &= -c_{334} = -\frac{\cot\theta}{\sqrt{2}r}. \end{aligned}$$

With c_{212} nonzero, we have a new nonzero Γ , and Eq. (8.35) becomes

$$\Gamma_{212} = \frac{M}{r^2} \quad \Gamma_{134} = \frac{1}{r} = -\varrho, \quad \Gamma_{234} = \Gamma_{243} = -\frac{\Delta}{2r^3} = \mu, \quad \Gamma_{343} = \frac{1}{\sqrt{2}} \frac{\cot \theta}{r} = \alpha - \bar{\beta}, \quad (8.47)$$

$$\begin{aligned} \Gamma^1_{12} &= \gamma + \bar{\gamma} & \Gamma^1_{34} &= \Gamma^1_{43} = \mu, & \Gamma^2_{34} &= \Gamma^2_{43} = -\varrho, & \Gamma^3_{33} &= \Gamma^4_{44} = 2\beta = -2\alpha, \\ & & \Gamma^3_{13} &= \Gamma^4_{14} = -\varrho, & \Gamma^4_{24} &= \Gamma^3_{23} = \mu, & \Gamma^3_{34} &= \Gamma^4_{43} = -2\beta = 2\alpha. \end{aligned} \quad (8.48)$$

$$\gamma = \frac{M}{2r^2}, \quad \alpha = -\frac{1}{2\sqrt{2}} \frac{\cot \theta}{r}, \quad \beta = \frac{1}{2\sqrt{2}} \frac{\cot \theta}{r}, \quad \varrho = -\frac{1}{r}, \quad \mu = -\frac{\Delta}{2r^3}. \quad (8.49)$$

We next show that the Schwarzschild Weyl tensor is type D: Only Ψ_2 is nonzero. Instead of writing out the equation for each component of the Riemann tensor in terms of the Γ 's, let's take advantage of the fact that Newman and Penrose have already done this for us: Eqs. (4.2) list the expressions for components of the Weyl and Ricci tensors in terms of the spin coefficients. The Ricci tensor components are given as Φ_{mn} , so for us $\Phi_{mn} = 0$ in these equations. The equations for the relevant components of the Weyl tensor are (4.2b) for Ψ_0 , (4.2c) for Ψ_1 and (4.2f) for Ψ_2 . Because of the $\ell \leftrightarrow n$, $m \leftrightarrow \bar{m}$ symmetry, we need not check that Ψ_3 and Ψ_4 vanish.

First (4.2b) for $\Psi_0 = -C_{1313} = C^4_{131}$. This one time we'll check the NP equation from scratch. There's no subtlety. Begin with the Riemann tensor written in terms of Γ 's and then write the NP symbol for each Γ (listed on the last page of these notes). In the second line below, we omit terms that vanish by $\Gamma_{ijk} = -\Gamma_{jik}$, (like Γ^1_{23}). For a general frame, the components of the Riemann tensor are given by Eq. (A.52):

$$\begin{aligned} R^i_{jkl} &= e_k \Gamma^i_{jl} - e_l \Gamma^i_{jk} + \Gamma^i_{mk} \Gamma^m_{jl} - \Gamma^i_{ml} \Gamma^m_{jk} + 2\Gamma^i_{jm} \Gamma^m_{[kl]} \\ C^4_{131} &= e_3 \Gamma^4_{11} - e_1 \Gamma^4_{13} + \Gamma^4_{m3} \Gamma^m_{11} - \Gamma^4_{m1} \Gamma^m_{13} + 2\Gamma^4_{1m} \Gamma^m_{[31]} \\ &= -\delta\kappa + D\sigma + (\Gamma^4_{13} \Gamma^1_{11} + \Gamma^4_{43} \Gamma^4_{11}) - (\Gamma^4_{11} \Gamma^1_{13} + \Gamma^4_{41} \Gamma^4_{13}) + 2(\Gamma^4_{11} \Gamma^1_{[31]} + \Gamma^4_{12} \Gamma^2_{[31]}) \\ &= -\delta\kappa + D\sigma + [-\sigma(\epsilon + \bar{\epsilon}) + (\bar{\alpha} - \beta)(-\kappa)] - [(-\kappa)(-\sigma) + (\bar{\epsilon} - \epsilon)(-\sigma)] \\ &\quad + [(-\kappa)\bar{\pi} + (-\tau)(-\kappa) + (-\sigma)(\epsilon - \bar{\epsilon}) - (-\sigma)(-\bar{\varrho}) - (-\varrho)(-\sigma)] \implies \\ &\quad D\sigma - \delta\kappa = (\varrho + \bar{\varrho} + 3\epsilon - \bar{\epsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0 \end{aligned} \quad (\text{NP 4.2b})$$

For Schwarzschild, only $\alpha, \beta, \gamma, \mu$ and ϱ are nonzero. Because σ and κ vanish, we have

$$\Psi_0 = 0. \quad (8.50)$$

The next Weyl-tensor equation we need is (4.2c) for $\Psi_1 = -C_{1213} = -C^2_{312}$:

$$D\tau - \Delta\kappa = (\tau + \bar{\pi})\varrho + (\bar{\tau} + \pi)\sigma + (\epsilon - \bar{\epsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 \quad (\text{NP 4.2c})$$

In this case, because τ, π, σ , and κ vanish, we have

$$\Psi_1 = 0. \quad (8.51)$$

The symmetry of Schwarzschild under $t \rightarrow -t$ and under $\phi \rightarrow -\phi$ implies a symmetry of the NP equations under $\ell \leftrightarrow n$ and $m \leftrightarrow \bar{m}$, giving $\Psi_3 = \Psi_4 = 0$, and leaving only Ψ_2 nonzero as claimed. To find its value, we use a third Weyl-tensor equation, (4.2f): The only nonzero spin coefficient in that equation is γ and the calculation is simple:

$$\Psi_2 = D\gamma = \partial_r \left(\frac{M}{2r^2} \right) = -\frac{M}{r^3}. \quad (8.52)$$

We now turn to the wave equation for Schwarzschild, again looking at the two components of the Bianchi identity that gave us the wave equation for ψ_0 . The Bianchi identity components are NP (4.5). $\nabla_{[4}C_{13]13}$ is the first of these equations,

$$(D - 2\epsilon - 4\rho)\Psi_1 - (\bar{\delta} - 4\alpha + \pi)\Psi_0 = -3\kappa\Psi_2, \quad (8.53)$$

and the fifth is $\nabla_{[2}C_{13]13}$,

$$(\Delta - 4\gamma + \mu)\Psi_0 - (\delta - 2\beta - 4\tau)\Psi_1 = 3\sigma\Psi_2. \quad (8.54)$$

We need to linearize the equations about the Schwarzschild background. That is, we consider a family of metrics, $g_{\alpha\beta}(\lambda) = g_{\alpha\beta}(0) + \lambda\dot{g}_{\alpha\beta} + O(\lambda^2)$, with $g_{ab}(0)$ the unperturbed spacetime, and we write

$$\dot{g}_{\alpha\beta} := \left. \frac{d}{d\lambda} g_{\alpha\beta}(\lambda) \right|_{\lambda=0}. \quad (8.55)$$

We'll write the unperturbed quantity, like $\Psi_2(0)$ or the spin-coefficient $\alpha(0)$ as simply Ψ_2 and α . We'll again use $\psi_i := \dot{\Psi}_i$, and will just write $\dot{\sigma}, \dot{\kappa}$ for the perturbed spin coefficients.

Then, because $\epsilon, \kappa, \mu, \Psi_0$ and Ψ_1 vanish at $\lambda = 0$, $\left. \frac{d}{d\lambda} \text{Eq. (8.53)}(\lambda) \right|_{\lambda=0}$ is

$$(D - 4\rho)\psi_1 - (\bar{\delta} - 4\alpha)\psi_0 = -3\dot{\kappa}\Psi_2. \quad (8.56)$$

Similarly, the linearized equation (8.54) is

$$(\Delta - 4\gamma + \mu)\psi_0 - (\delta - 2\beta)\psi_1 = 3\dot{\sigma}\Psi_2. \quad (8.57)$$

Again we need to eliminate ψ_1 , and again just multiplying (8.57) by r gets rid of the r dependence in $(\delta - 2\beta)$, allowing it to commute with D . But now there's a problem. The right sides of the two equations involve $\dot{\kappa}$ and $\dot{\sigma}$, so we don't succeed in getting a decoupled equation for ψ_0 .

We are saved by a miracle that will look more miraculous for Kerr. The equation for ψ_0 in terms of the Γ 's is the NP equation (NP 4.2b) that we derived above, and its linearized version is

$$(D - \rho - \bar{\rho})\dot{\sigma} - (\delta - \bar{\alpha} - 3\beta)\dot{\kappa} = \psi_0, \quad (8.58)$$

or, using $\rho = \bar{\rho}$, $\bar{\alpha} = -\beta$ for Schwarzschild),

$$(D - 2\rho)\dot{\sigma} - (\delta - 2\beta)\dot{\kappa} = \psi_0. \quad (8.59)$$

We will simply follow our path to the equation in flat space, leaving terms involving $\dot{\sigma}$ and $\dot{\kappa}$ on the right side of the wave equation. And those terms will be the above expression for ψ_0 .

As in Exercise 8, we use NP equations (4.2e) and (4.4); and we also use the second Bianchi identity in (4.5), all for the unperturbed Schwarzschild spacetime:

$$D\beta = \rho\beta, \quad [D, \delta] = \rho\delta, \quad D\Psi_2 = 3\rho\Psi_2. \quad (8.60)$$

We just follow the steps for flat space, writing $(\delta - 2\beta)[\text{Eq. (8.56)}] + (D - 5\rho)[\text{Eq. (8.57)}]$

$$\begin{aligned} & (\delta - 2\beta)(D - 4\rho)\psi_1 - (\delta - 2\beta)(\bar{\delta} - 4\alpha)\psi_0 \\ & + (D - 5\rho)(\Delta - 4\gamma + \mu)\psi_0 - (D - 5\rho)(\delta - 2\beta)\psi_1 = -3\Psi_2(\delta - 2\beta)\dot{\kappa} + 3(D - 5\rho)(\Psi_2\dot{\sigma}). \end{aligned} \quad (8.61)$$

The commutation relation that eliminates ψ_1 is unchanged from the flat-space case: Using the first two relations in Eq. (8.60), we have

$$\begin{aligned} [\delta - 2\beta, D - 4\rho] &= [\delta, D] + 2D\beta = -\rho(D - 2\beta) \\ \implies (\delta - 2\beta)(D - 4\rho)\psi_1 &= (D - 5\rho)(D - 2\beta)\psi_1, \end{aligned}$$

and the terms involving ψ_1 cancel exactly as before. On the right side, the term involving $\dot{\sigma}$ is

$$\begin{aligned} 3(D - 5\rho)(\Psi_2\dot{\sigma}) &= 3[(D\Psi_2)\dot{\sigma} + \Psi_2(D - 5\rho)\dot{\sigma}] = 3[3\rho\Psi_2\dot{\sigma} + \Psi_2(D - 5\rho)\dot{\sigma}] \\ &= 3\Psi_2(D - 2\rho)\dot{\sigma}. \end{aligned}$$

Finally, Eq. (8.61) becomes

$$[(D - 5\rho)(\Delta - 4\gamma + \mu) - (\delta - 2\beta)(\bar{\delta} - 4\alpha)]\psi_0 = 3\Psi_2[-(\delta - 2\beta)\dot{\kappa} + (D - 2\rho)\dot{\sigma}] = 3\Psi_2\psi_0,$$

or

$$[(D - 5\rho)(\Delta - 4\gamma + \mu) - 3\Psi_2 - (\delta - 2\beta)(\bar{\delta} - 4\alpha)]\psi_0 = 0. \quad (8.62)$$

This is the Bardeen-Press equation[12]. Multiplying by r^2 (or ρ^{-2}) gives an equation with all radial dependence and derivatives in the first terms and all angular dependence and derivatives in the last: The separation of variables is essentially already here.

Explicit equation:

Multiply by $-2r^2$ and bring the r^2 inside $(D - 5\rho)$, changing it to $(D - 3\rho) = (D + 3/r)$. Then, using Eqs. (8.46) for the derivative operators and Eqs. (8.49) for the spin coefficients, we obtain the explicit form

$$\begin{aligned} &\left\{ \left(\frac{r^2}{\Delta} \partial_t + \partial_r + \frac{3}{r} \right) \left[\Delta \left(-\frac{r^2}{\Delta} \partial_t + \partial_r + \frac{1}{r} + \frac{4M}{\Delta} \right) \right] \right. \\ &\quad \left. + \left(\partial_\theta + \frac{i}{\sin\theta} \partial_\phi - \cot\theta \right) \left(\partial_\theta - \frac{i}{\sin\theta} \partial_\phi + 2 \cot\theta \right) \right\} \psi_0 = 0. \end{aligned}$$

In our derivation, the input was the set of six equations in blue on the last page:

- two perturbed Bianchi identities,
- the equation for ψ_1 in terms of Christoffel symbols (spin coefficients),
- the vanishing of Ψ_1 in Schwarzschild (giving $D\beta = \rho\beta$),
- the commutator $[D, \delta] = [e_1, e_3]$ in Schwarzschild,
- a Bianchi identity for Ψ_2 in Schwarzschild.

We also used the fact that all spin coefficients are real. We'll follow the same path for Kerr, with a couple of additional relations associated with θ derivatives that vanish for Schwarzschild and with spin coefficients that are complex.

8.2.3 Kerr

Although the explicit forms of the spin coefficients is given below, with a sample calculation, they are not used to derive the Teukolsky equation in the form (8.71) below. You need only the NP equations and the fact that the other spin coefficients vanish. The vanishing of $\kappa, \sigma, \lambda, \nu, \tau$ is automatic for a type D Weyl tensor: Five components of the Bianchi identities—i.e., five of NP Eqs. (4.5)—are simply

$$3\kappa\Psi_2 = 0, \quad -3\lambda\Psi_2 = 0, \quad 3\sigma\Psi_2 = 0, \quad -3\tau\Psi_2 = 0, \quad 3\nu\Psi_2 = 0.$$

The choice of Kinnersley tetrad also makes $\epsilon = 0$ and $\alpha = \pi - \beta$. The derivation of Eq. (8.71) using only this information, begins after Eq. (8.67) below.

From Eq. (8.16), we have

$$\mathbf{e}_1 = \boldsymbol{\ell} = \frac{1}{\Delta}[(r^2 + a^2)\partial_t + \Delta\partial_r + a\partial_\phi], \quad (8.63a)$$

$$\mathbf{e}_2 = \mathbf{n} = \frac{1}{2\rho^2}[(r^2 + a^2)\partial_t - \Delta\partial_r + a\partial_\phi], \quad (8.63b)$$

$$\mathbf{e}_3 = \mathbf{m} = \frac{1}{\sqrt{2}(r + ia\cos\theta)} \left(ia\sin\theta\partial_t + \partial_\theta + \frac{i}{\sin\theta}\partial_\phi \right) \quad (8.63c)$$

In calculating the commutation coefficients c^p_{mn} , you'll need to invert these expressions – to invert $\mathbf{e}_m = e_m^\mu \partial_\mu$ in order to express ∂_μ in terms of $\mathbf{e}_m$. The easy way to do this is to use the dual frame ω^m . That is, because $\omega^m(\mathbf{e}_n) = \omega^m_\mu e_n^\mu = \delta_n^m$, ω^m_μ is the inverse of e_n^μ : $\omega^m_\mu e_m^\nu = \delta_\mu^\nu$. Then

$$\partial_\mu = \omega^m_\mu \mathbf{e}_m.$$

From the orthonormal dual frame vectors listed after Eq. (8.13), we quickly see

$$\omega^1 = \frac{\sqrt{\Delta}}{2\rho}(\Omega^0 + \Omega^1), \quad \omega^2 = \frac{\rho}{\sqrt{\Delta}}(\Omega^0 - \Omega^1), \quad \omega^3 = \frac{r + ia\cos\theta}{\sqrt{2}\rho}(\Omega^2 - i\Omega^3)$$

giving

$$\omega^1 = \frac{\Delta}{2\rho^2}(dt - a\sin^2\theta d\phi) + \frac{1}{2}dr, \quad (8.64a)$$

$$\omega^2 = dt - a\sin^2\theta d\phi - \frac{\rho^2}{\Delta}dr, \quad (8.64b)$$

$$\omega^3 = \frac{1}{\sqrt{2}(r - ia\cos\theta)} [ia\sin\theta dt + \rho^2 d\theta - (r^2 + a^2)i\sin\theta d\phi]. \quad (8.64c)$$

Here are the nonzero spin coefficients, followed by a sample calculation of one of them:

$$\begin{aligned} \varrho &= -\frac{1}{r - ia\cos\theta} & \beta &= \frac{1}{2\sqrt{2}(r + ia\cos\theta)} \cot\theta, & \pi &= \frac{ia}{2\sqrt{2}(r - ia\cos\theta)^2} \sin\theta, & \alpha &= \pi - \bar{\beta}, \\ \mu &= -\frac{\Delta}{2\rho^2(r - ia\cos\theta)}, & \gamma &= \mu + \frac{r - M}{2\rho^2}, & \tau &= -\frac{ia}{2\rho^2} \sin\theta. \end{aligned} \quad (8.65)$$

Calculation of $\varrho = \Gamma_{314} = \frac{1}{2}(c_{431} - c_{143} - c_{314})$:

$$c_{143} = c^2_{43} = -\omega^2([\mathbf{e}_3, \mathbf{e}_4]), \quad (8.66a)$$

$$c_{431} = -c^3_{31} = \omega^3([\mathbf{e}_1, \mathbf{e}_3]), \quad c_{314} = -c_{341} = \bar{c}_{431}. \quad (8.66b)$$

Next, use Eqs. (8.63) for e_m to write

$$\begin{aligned}
 [e_3, e_4] &= \frac{1}{\sqrt{2}(r + ia \cos \theta)} \partial_\theta \left[\frac{1}{\sqrt{2}(r - ia \cos \theta)} \left(-ia \sin \theta \partial_t + \partial_\theta - \frac{i}{\sin \theta} \partial_\phi \right) \right] - \text{c.c.} \\
 &= -\frac{\partial_\theta(r - ia \cos \theta)}{\sqrt{2}\rho^2} e_4 + \frac{\partial_\theta(r + ia \cos \theta)}{\sqrt{2}\rho^2} e_3 + \frac{2}{2\rho^2} \left(-ia \cos \theta \partial_t + \frac{i \cot \theta}{\sin \theta} \partial_\phi \right) \\
 &= -\frac{ia \sin \theta}{\sqrt{2}\rho^2} (e_3 + e_4) + \frac{i}{\rho^2} \cot \theta \left(-a \sin \theta \partial_t + \frac{1}{\sin \theta} \partial_\phi \right), \\
 [e_1, e_3] &= -2 \frac{\partial_r(r + ia \cos \theta)}{r + ia \cos \theta} e_3 = -\frac{1}{r + ia \cos \theta} e_3
 \end{aligned}$$

Using Eqs. (8.64b) and (8.64c), we have

$$\begin{aligned}
 c_{143} &= -\omega^2([e_3, e_4]) = -\frac{i}{\rho^2} \cot \theta (-a \sin \theta + a \sin \theta) = 2 \frac{ia \cos \theta}{\rho^2} \\
 c_{413} &= \omega^3([e_1, e_3]) = -\frac{1}{r + ia \cos \theta},
 \end{aligned}$$

and

$$\varrho = \Gamma_{314} = \frac{1}{2} \left(-\frac{2r}{\rho^2} - 2 \frac{ia \cos \theta}{\rho^2} \right) = -\frac{2r}{\rho^2}. \quad (8.67)$$

We'll use the following equations from NP, most of which we've already used: Starting from Eq. (NP 4.2b) for Ψ_0 in terms of spin coefficients, they are

$$(D - \varrho - \bar{\varrho})\sigma - (\delta - \tau - 2\beta)\kappa = \Psi_0, \quad (4.2b)$$

$$(D - 4\varrho)\Psi_1 - (\bar{\delta} - 4\alpha + \pi)\Psi_0 = -3\kappa\Psi_2, \quad (4.5\text{-first})$$

$$(\Delta - 4\gamma + \mu)\Psi_0 - (\delta - 2\beta - 4\tau)\Psi_1 = -3\sigma\Psi_2 \quad (4.5\text{-fifth})$$

$$D\Psi_2 = 3\varrho\Psi_2 \quad (4.5\text{-second})$$

$$\delta\Psi_2 = 3\tau\Psi_2 \quad (4.5\text{-sixth})$$

$$D\tau = \varrho\tau + \varrho\bar{\pi} + \Psi_1 \quad (4.2c)$$

$$D\beta = \bar{\varrho}\beta \quad (4.2e)$$

$$\delta\varrho = (\bar{\alpha} + \beta)\varrho + (\varrho - \bar{\varrho})\tau \quad (4.2k)$$

$$[D, \delta] = \bar{\varrho}\delta \quad (4.4)$$

In the first paragraph, we used the vanishing of Ψ_i , $i \neq 2$ to infer the vanishing of a set of spin coefficients. Conversely, if one uses the explicit tetrad to show $\kappa = \sigma = 0$, then, from the NP equation for Ψ_0 in terms of spin coefficients and (NP 4.2b), we immediately have $\Psi_0 = 0$ for the background spacetime. Eq. (NP 4.2c) implies $\Psi_1 = 0$, after one checks that $D\tau = (\tau + \bar{\pi})$.¹

We use the remaining equations in their perturbed forms to get the Teukolsky equation. As usual, we start with the Bianchi identities for ψ_0 ,

$$(D - 4\varrho)\psi_1 - (\delta - 4\alpha + \pi)\psi_0 = -3\dot{\kappa}\Psi_2, \quad (8.68)$$

$$(\Delta - 4\gamma + \mu)\psi_0 - (\delta - 4\tau - 2\beta)\psi_1 = -3\dot{\sigma}\Psi_2. \quad (8.69)$$

¹The relations $\kappa = \sigma = 0 \iff \Psi_0 = \Psi_1 = 0$ and $\nu = \lambda = 0 \iff \Psi_3 = \Psi_4 = 0$ are two instances of the Goldberg-Sachs theorem. The vanishing of κ and σ is the statement that ℓ^α generates a congruence of shear-free null geodesics and the theorem states that such a congruence exists if and only if the Weyl tensor is algebraically special, $\Psi_0 = \Psi_1 = 0$. Similarly, $\nu = \lambda = 0$ means that n^α generates a second family of shear-free null geodesics.

From the right sides, we infer the form of the operators that should act on each equation to eliminate $\dot{\sigma}$ and $\dot{\tau}$ and hope that they will also eliminate ψ_1 : From (NP 4.2b), we need $(D - \varrho - \bar{\varrho})$ to act on $\dot{\sigma}$, and $(\delta - \tau - 2\beta)$ to act on $\dot{\kappa}$. By (4.5-second) and (4.5-sixth), we have

$$\begin{aligned}(D - 4\varrho - \bar{\varrho})(\Psi_2 \dot{\sigma}) &= \Psi_2(D - \varrho - \bar{\varrho})\dot{\sigma}, \\ (\delta - 4\tau - 2\beta)(\Psi_2 \dot{\kappa}) &= \Psi_2(\delta - \tau - 2\beta)\dot{\kappa},\end{aligned}$$

So we use $(D - 4\varrho - \bar{\varrho})$ [Eq.(8.69)] $-(\delta - 4\tau - 2\beta)$ [Eq. (8.68)]:

$$\begin{aligned}& (D - 4\varrho - \bar{\varrho})[(\Delta - 4\gamma + \mu)\psi_0 - (\delta - 4\tau - 2\beta)\psi_1] \\ & - (\delta - 4\tau - 2\beta)[(D - 4\varrho)\psi_1 - (\delta - 4\alpha + \pi)\psi_0] = -3\Psi_2\psi_0.\end{aligned}$$

Then what remains is to show

$$[(D - 4\varrho - \bar{\varrho})(\delta - 4\tau - 2\beta) - (\delta - 4\tau - 2\beta)(D - 4\varrho)]\psi_1 = 0.$$

The operator on the left side is

$$-\bar{\varrho}(\delta - 4\tau - 2\beta) + [D - 4\varrho, \delta - 4\tau - 2\beta]. \quad (8.70)$$

Expanding the commutator, and using the last set of NP equations above, we have

$$\begin{aligned}[D - 4\varrho, \delta - 4\tau - 2\beta] &= [D, \delta] - 4D\tau - 2D\beta + 4\delta\varrho \\ &= \bar{\varrho}\delta - 4\tau\varrho - 4\bar{\pi}\varrho - 2\bar{\varrho}\beta + 4(\bar{\alpha} + \beta)\varrho + 4(\varrho - \bar{\varrho})\tau \\ &= \bar{\varrho}(\delta - 4\tau - 2\beta),\end{aligned}$$

canceling the first term in (8.70) ! (Use $\bar{\pi} = \bar{\alpha} + \beta$ to get the last equality.)

The Teukolsky equation is then

$$[(D - 4\varrho - \bar{\varrho})(\Delta - 4\gamma + \mu) + 3\Psi_2 - (\delta - 4\tau - 2\beta)(\delta - 4\alpha + \pi)]\psi_0 = 0. \quad (8.71)$$

8.3 Appendix to Chapter 8

Naming of parts

$$\ell^\mu \partial_\mu = D, \quad n^\mu \partial_\mu = \Delta, \quad m^\mu \partial_\mu = \delta, \quad \bar{m}^\mu \partial_\mu = \bar{\delta}.$$

$$\begin{array}{lll} \kappa = \Gamma_{311} & \varrho = \Gamma_{314} & \epsilon = \frac{1}{2}(\Gamma_{211} + \Gamma_{341}) \\ \sigma = \Gamma_{313} & \mu = \Gamma_{243} & \gamma = \frac{1}{2}(\Gamma_{212} + \Gamma_{342}) \\ \lambda = \Gamma_{244} & \tau = \Gamma_{312} & \alpha = \frac{1}{2}(\Gamma_{214} + \Gamma_{344}) \\ \nu = \Gamma_{242} & \pi = \Gamma_{241} & \beta = \frac{1}{2}(\Gamma_{213} + \Gamma_{343}) \end{array}$$

$$\Gamma_{211} = \epsilon + \bar{\epsilon} \quad \Gamma_{341} = \epsilon - \bar{\epsilon} \quad \Gamma_{212} = \gamma + \bar{\gamma} \quad \Gamma_{342} = \gamma - \bar{\gamma} \quad \Gamma_{214} = \alpha + \bar{\beta} \quad \Gamma_{344} = \alpha - \bar{\beta}.$$

$$\begin{array}{llll} \Gamma^1_{11} = \epsilon + \bar{\epsilon} & \Gamma^2_{21} = -(\epsilon + \bar{\epsilon}) & \Gamma^3_{11} = -\bar{\kappa} & \Gamma^4_{11} = -\kappa \\ \Gamma^1_{12} = \gamma + \bar{\gamma} & \Gamma^2_{22} = -(\gamma + \bar{\gamma}) & \Gamma^3_{12} = -\bar{\tau} & \Gamma^4_{12} = -\tau \\ \Gamma^1_{13} = \bar{\alpha} + \beta & \Gamma^2_{23} = -(\bar{\alpha} + \beta) & \Gamma^3_{13} = -\bar{\varrho} & \Gamma^4_{13} = -\sigma \\ \Gamma^1_{14} = \alpha + \bar{\beta} & \Gamma^2_{24} = -(\alpha + \bar{\beta}) & \Gamma^3_{14} = -\bar{\sigma} & \Gamma^4_{14} = -\varrho \\ \Gamma^1_{31} = \bar{\pi} & \Gamma^2_{31} = -\kappa & \Gamma^3_{21} = \pi & \Gamma^4_{21} = \bar{\pi} \\ \Gamma^1_{32} = \bar{\nu} & \Gamma^2_{32} = -\tau & \Gamma^3_{22} = \nu & \Gamma^4_{22} = \bar{\nu} \\ \Gamma^1_{33} = \bar{\lambda} & \Gamma^2_{33} = -\sigma & \Gamma^3_{23} = \mu & \Gamma^4_{23} = \bar{\lambda} \\ \Gamma^1_{34} = \bar{\mu} & \Gamma^2_{34} = -\varrho & \Gamma^3_{24} = \lambda & \Gamma^4_{24} = \bar{\mu} \\ \Gamma^1_{41} = \pi & \Gamma^2_{41} = -\bar{\kappa} & \Gamma^3_{31} = \epsilon - \bar{\epsilon} & \Gamma^4_{41} = \bar{\epsilon} - \epsilon \\ \Gamma^1_{42} = \nu & \Gamma^2_{42} = -\bar{\tau} & \Gamma^3_{32} = \gamma - \bar{\gamma} & \Gamma^4_{42} = \bar{\gamma} - \gamma \\ \Gamma^1_{43} = \mu & \Gamma^2_{43} = -\bar{\varrho} & \Gamma^3_{33} = \beta - \bar{\alpha} & \Gamma^4_{43} = \bar{\alpha} - \beta \\ \Gamma^1_{44} = \lambda & \Gamma^2_{44} = -\bar{\sigma} & \Gamma^3_{34} = \alpha - \bar{\beta} & \Gamma^4_{44} = \bar{\beta} - \alpha \end{array}$$

$$\begin{array}{llll} C_{1212} = -(\Psi_2 + \bar{\Psi}_2) & C_{1313} = -\Psi_0 & C_{1414} = -\bar{\Psi}_0 & C_{2323} = -\bar{\Psi}_4 \quad C_{3434} = -(\Psi_2 + \bar{\Psi}_2) \\ C_{1213} = -\Psi_1 & C_{1314} = 0 & C_{1423} = \bar{\Psi}_2 & C_{2324} = 0 \\ C_{1214} = -\bar{\Psi}_1 & C_{1323} = 0 & C_{1424} = 0 & C_{2334} = \bar{\Psi}_3 \\ C_{1223} = \bar{\Psi}_3 & C_{1324} = \Psi_2 & C_{1434} = -\bar{\Psi}_1 & C_{2424} = -\Psi_4 \\ C_{1224} = \Psi_3 & C_{1334} = \Psi_1 & & C_{2434} = -\Psi_3 \\ C_{1234} = \Psi_2 - \bar{\Psi}_2 & & & \end{array}$$

Chapter 9

Four Laws of Black Hole Mechanics

We're back to $- + ++$ and are roughly following the Bardeen-Carter-Hawking paper, *The Four Laws of Black Hole Mechanics*, Comm. Math. Phys. **31**, 161-170 (1973). These analogues of four laws of thermodynamics are presented below in the order

2nd law: Area increase theorem, corresponding to entropy increase.

0th law: The surface gravity κ of a black hole is constant, corresponding to the constant temperature of a system in thermodynamic equilibrium.

3rd law: One cannot reduce κ to zero, corresponding to the inability to reach $T = 0$ in classical thermodynamics.

1st law: $\delta M = \frac{1}{8\pi} \kappa \delta A + \Omega_H \delta J$.

9.1 The 2nd Law: The Area of the Event Horizon Can Never Decrease

Recall that the event horizon, H , is the boundary of the past of future null infinity. From any point outside H there is a future directed timelike line that reaches infinity—an inextendible timelike line without future endpoint that remains forever outside H . By considering a family of such lines that begin from points $P_1, P_2, \dots$ converging to P on H , one acquires at each P in H a future directed null line l that is inextendible and has no future endpoint.

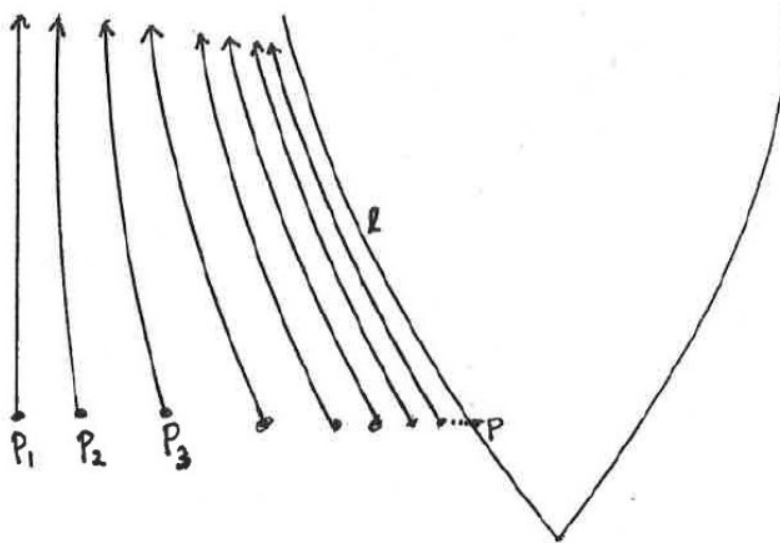


Figure 9.1: Similar to figures in a Geroch-Horowitz paper[41]

These null lines are called the null generators of H .

Let ℓ^α be a nonvanishing vectorfield that is tangent on H to the null generators.

Claim: The null generators of a null hypersurface (e.g., H) are geodesics.

Proof: Call the null hypersurface H . Let u be a scalar that is constant on H and with $\nabla_\alpha u \neq 0$. Note first that $\nabla_\alpha u$ is null. This follows from the fact that the tangent space at a point P of H is spanned by a null vector ℓ^α and two orthogonal spacelike vectors. Then ℓ^α is orthogonal to every vector in H ; and any vector orthogonal to all other vectors in H must be along ℓ^α (other vectors are not null and so not orthogonal to themselves). So $\ell_\alpha = f \nabla_\alpha u$, some scalar f . We have

$$\begin{aligned}
 \ell^\beta \nabla_\beta \ell^\alpha &= \ell^\beta \nabla_\beta (f \nabla^\alpha u) = (\ell^\beta \nabla_\beta f) \nabla^\alpha u + f \ell^\beta \nabla_\beta \nabla^\alpha u \\
 &= \frac{\ell^\beta \nabla_\beta f}{f} \ell^\alpha + f^2 \nabla^\beta u \nabla_\alpha \nabla_\beta u = \frac{\ell^\beta \nabla_\beta f}{f} \ell^\alpha + f^2 \frac{1}{2} \nabla^\alpha \underbrace{(\nabla^\beta u \nabla_\beta u)}_0 \\
 &= \frac{\ell^\beta \nabla_\beta f}{f} \ell^\alpha.
 \end{aligned}$$

Thus ℓ^α generates a geodesic congruence. $\square$

When we get to stationary, axisymmetric black holes, we will see that ℓ^α is along a Killing vector, and we can set it equal to the Killing vector if we do not use affine parametrization. Following Bardeen-Carter-Hawking, we write

$$\ell^\beta \nabla_\beta \ell^\alpha = \kappa \ell^\alpha. \quad (9.1)$$

This κ is *not* the NP spin coefficient, which we will denote by κ_{NP} on the one occasion when we need it.

We can now state the area increase law. Let S be a spacelike slice of H orthogonal to ℓ^α ; and let ψ_λ be the family of diffeos generated by ℓ^α , so ψ_λ moves you a parameter distance λ along the null generators of H . Let $S_\lambda = \psi_\lambda(S)$ and let A_λ be the area of the black hole with boundary S_λ : $A_\lambda = \int_{S_\lambda} \epsilon_{\alpha\beta} dS^{\alpha\beta}$.

2nd Law:

$$\frac{d}{d\lambda} A_\lambda \geq 0. \quad (9.2)$$

This theorem, due to Hawking, follows quickly from the *Raychaudhuri equation*, the general relativistic form of the *optical scalar equation*. This equation describes the cross section of a bundle of nearby null rays – called a *congruence* – and it is applied to the generators of the horizon. Let ℓ^α be a null vector field tangent to a family of affinely parametrized null geodesics. (For this proof we use the affinely parametrized ℓ^α .) A circular cross section of the family distorts as it evolves, and the distortion can be regarded as a sum of three parts, a change in its area, a rotation of the circle and a shearing of the circle, taking the circle to an ellipse. In terms of ℓ^α this describes a decomposition of $\nabla_\alpha \ell_\beta$ or, more accurately, of the projection of $\nabla_\alpha \ell_\beta$ onto the cross section orthogonal to ℓ^α .

Let n^α be a null vector with $\ell^\alpha n_\alpha = -1$. Let $e_{\alpha\beta}$ be the projection $\perp$ to ℓ^α and n^α :

$$e_{\alpha\beta} = g_{\alpha\beta} + \ell_\alpha n_\beta + \ell_\beta n_\alpha = m_\alpha \bar{m}_\beta + \bar{m}_\alpha m_\beta. \quad (9.3)$$

We can regard $e_{\alpha\beta}$ as the Euclidean 2-metric on a cross section of the congruence of null geodesics.

The divergence, shear and rotation of the congruence are defined by

$$\begin{aligned} e_\alpha^\gamma e_\beta^\delta \nabla_\gamma \ell_\delta &= \sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta + \omega_{\alpha\beta}, \\ \theta &= e^{\alpha\beta} \nabla_\alpha \ell_\beta, \quad \sigma_{\alpha\beta} = e_\alpha^\gamma e_\beta^\delta \nabla_{(\gamma} \ell_{\delta)} - \frac{1}{2} e_{\alpha\beta} \theta, \quad \omega_{\alpha\beta} = e_\alpha^\gamma e_\beta^\delta \nabla_{[\gamma} \ell_{\delta]} \end{aligned} \quad (9.4)$$

Because $\ell_\alpha = f \nabla_\alpha u$, $\omega_{\alpha\beta}$ vanishes:

$$\begin{aligned} \omega_{\alpha\beta} &= e_{[\alpha}^\gamma e_{\beta]}^\delta \nabla_\gamma \ell_\delta \\ &= e_{[\alpha}^\gamma e_{\beta]}^\delta \nabla_\gamma f \nabla_\delta u \\ &= 0, \quad \text{using } e_\alpha^\beta \nabla_\beta u = 0. \end{aligned}$$

Then, using $\ell^\beta \nabla_\beta \ell_\alpha = \kappa \ell_\alpha$, $\ell^\beta \nabla_\alpha \ell_\beta = 0$, we have

$$\begin{aligned} \nabla_\alpha \ell_\beta &= (e_\alpha^\gamma - \ell_\alpha n^\gamma - \ell^\gamma n_\alpha)(e_\beta^\delta - \ell_\beta n^\delta - \ell^\delta n_\beta) \nabla_\gamma \ell_\delta \\ &= e_\alpha^\gamma e_\beta^\delta \nabla_\gamma \ell_\delta + \ell_\alpha \ell_\beta n^\gamma n^\delta \nabla_\gamma \ell_\delta - e_\alpha^\gamma \ell_\beta n^\delta \nabla_\gamma \ell_\delta - e_\beta^\delta \ell_\alpha n^\gamma \nabla_\gamma \ell_\delta - \kappa n_\alpha \ell_\beta. \\ &= \sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta + \ell_\alpha \ell_\beta n^\gamma n^\delta \nabla_\gamma \ell_\delta - e_\alpha^\gamma \ell_\beta n^\delta \nabla_\gamma \ell_\delta - e_\beta^\delta \ell_\alpha n^\gamma \nabla_\gamma \ell_\delta - \kappa n_\alpha \ell_\beta. \end{aligned} \quad (9.5)$$

Proof of 2nd Law: We have

$$\frac{d}{d\lambda} A_\lambda = \int_\Sigma \mathfrak{L}_\ell \epsilon_{\alpha\beta} dS^{\alpha\beta},$$

where $\epsilon_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} \ell^\gamma n^\delta$. We will, for this proof, choose ℓ^α to be affinely parametrized ($\kappa = 0$), and it will be helpful to extend ℓ^α to a null vector field on a neighborhood of H . Then

$$\mathcal{L}_\ell \ell_\alpha = \ell^\beta \nabla_\beta \ell_\alpha + \ell_\beta \nabla_\alpha \ell^\beta = 0 + \frac{1}{2} \nabla_\alpha (\ell_\beta \ell^\beta) = 0.$$

Using

$$\mathcal{L}_\ell \epsilon_{\alpha\beta\gamma\delta} = \nabla \cdot \ell \epsilon_{\alpha\beta\gamma\delta}, \quad \mathcal{L}_\ell \ell^\gamma = 0, \quad \ell_\delta \mathcal{L}_\ell n^\delta = -n^\delta \mathcal{L}_\ell \ell_\delta = 0,$$

we obtain

$$\mathcal{L}_\ell \epsilon_{\alpha\beta} dS^{\alpha\beta} = \nabla_\gamma \ell^\gamma \epsilon_{\alpha\beta} dS^{\alpha\beta}.$$

Let us now use (9.5) to obtain an equation for $\ell^\beta \nabla_\beta \theta = \dot{\theta}$ — we will show that $\dot{\theta} \leq 0$.

$$\begin{aligned} \nabla_\gamma \nabla_\delta \ell_\alpha - \nabla_\delta \nabla_\gamma \ell_\alpha &= R_{\alpha\beta\gamma\delta} \ell^\beta \\ \Rightarrow \ell^\delta e^{\alpha\gamma} (\nabla_\gamma \nabla_\delta \ell_\alpha - \nabla_\delta \nabla_\gamma \ell_\alpha) &= R_{\alpha\beta} \ell^\alpha \ell^\beta \end{aligned} \quad (9.6)$$

$$\begin{aligned} LHS &= (\ell^\alpha e^{\beta\epsilon} - \ell^\epsilon e^{\alpha\beta}) \nabla_\epsilon \nabla_\alpha \ell_\beta \\ &= (\ell^\alpha e^{\beta\epsilon} - \ell^\epsilon e^{\alpha\beta}) \nabla_\epsilon (\sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta + \ell_\alpha \ell_\beta n^\gamma n^\delta \nabla_\alpha \ell_\delta - e_\alpha^\gamma \ell_\beta n^\delta \nabla_\gamma \ell_\delta - e_\beta^\delta \ell_\alpha n^\gamma \nabla_\gamma \ell_\delta) \end{aligned} \quad (9.7)$$

The last three terms in parentheses all die by virtue of the relations

$$e^{\alpha\beta} \ell_\beta = 0, \quad \ell^\beta \nabla_\beta \ell_\alpha = -0, \quad \ell^\beta \nabla_\alpha \ell_\beta = 0, \quad \ell^\alpha \ell_\alpha = 0,$$

and we are left with

$$LHS = \ell^\alpha e^{\beta\gamma} \nabla_\gamma (\sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta) - \ell^\gamma e^{\alpha\beta} \nabla_\gamma (\sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta). \quad (9.8)$$

Considering each term in turn, we have

$$\begin{aligned} \ell^\alpha e^{\beta\gamma} \nabla_\gamma (\sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta) &= e^{\beta\gamma} \nabla_\gamma \underbrace{[\ell^\alpha (\sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta)]}_0 - e^{\beta\gamma} \nabla_\gamma \ell^\alpha (\sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta) \\ &= -e^{\beta\gamma} \nabla_\gamma \ell_\delta e^{\alpha\delta} (\sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta) \\ &= -(\sigma^{\alpha\beta} + \frac{1}{2} e^{\alpha\beta} \theta) (\sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta) \\ &= -\sigma^{\alpha\beta} \sigma_{\alpha\beta} - \frac{1}{2} \theta^2 \end{aligned} \quad (9.9)$$

$$\begin{aligned} -\ell^\gamma e^{\alpha\beta} \nabla_\gamma (\sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta) &= -\ell^\gamma \nabla_\gamma [e^{\alpha\beta} (\sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta)] + \ell^\gamma \nabla_\gamma e^{\alpha\beta} (\sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta) \\ &= -\ell^\gamma \nabla_\gamma \theta + \ell^\gamma \nabla_\gamma (\ell^\alpha n^\beta + \ell^\beta n^\alpha) (\sigma_{\alpha\beta} + \frac{1}{2} e_{\alpha\beta} \theta) \\ &= -\ell^\gamma \nabla_\gamma \theta. \end{aligned} \quad (9.10)$$

From Eqs. (9.6), (9.8), (9.9), and (9.10), we obtain the Raychaudhuri equation,

$$-\sigma^{\alpha\beta} \sigma_{\alpha\beta} - \frac{1}{2} \theta^2 - \dot{\theta} = R_{\beta\alpha} \ell^\beta \ell^\alpha$$

or

$$\dot{\theta} = -\frac{1}{2} \theta^2 - \sigma^{\alpha\beta} \sigma_{\alpha\beta} - R_{\alpha\beta} \ell^\beta \ell^\alpha. \quad (9.11)$$

Then the energy condition

$$R_{\alpha\beta} \ell^\alpha \ell^\beta \geq 0, \quad \text{all null } \ell^\alpha,$$

implies

$$\dot{\theta} \leq -\frac{1}{2}\theta^2. \quad (9.12)$$

This last equation is the key to the area theorem. If $\theta = \theta_0 < 0$ anywhere on H , then (9.12) implies

$$\begin{aligned} \theta^{-2} \frac{d\theta}{d\lambda} \leq -\frac{1}{2} &\implies \int_0^\lambda \theta^{-2} \frac{d\theta}{d\lambda} d\lambda \leq -\frac{1}{2}\lambda \\ &\quad -\frac{1}{\theta} + \frac{1}{\theta_0} \leq -\frac{\lambda}{2} \\ \implies \theta &\leq \frac{2}{\lambda - \left(-\frac{2}{\theta_0}\right)} \end{aligned} \quad (9.13)$$

Eq. (9.13) implies $\theta \rightarrow -\infty$ within an affine parameter length $\lambda < \left|\frac{2}{\theta_0}\right|$ from the point where $\theta = \theta_0$.

This means that two future-directed neighboring null generators of H cross in finite affine parameter length—at some point P of H . But then their tangents will be *two* null vectors at P : ℓ^α and $\hat{\ell}^\alpha$ (!), and a linear combination will be a timelike vector on H , contradicting the fact that H is null. Therefore $\theta \geq 0$ on H and $\frac{dA}{d\lambda} \geq 0$. $\square$

9.1.1 Area of a Kerr black hole and irreducible mass

The area increase theorem gives an upper limit on the energy that can be extracted from a black hole, by, for example, the Penrose process or in the inspiral and merger of two black holes. In the Penrose process, the area of the final black hole must be larger than that of the initial black hole; in inspiral and merger, the final black hole must have an area larger than the sum of the initial areas. (The horizon of a spacetime of two merging black holes is a continuous 3-surface; an early spacelike slice is the disjoint union of two spheres, and each late slice is a single sphere.)

In the Kerr geometry, the horizon area A is the area of a θ, ϕ sphere with $\Delta = 0$. Although the Boyer-Lindquist form (5.43) of the metric gives the correct induced metric on the horizon 2-spheres spanned by θ and ϕ , these are surfaces of constant t and ϕ , and t is undefined on the horizon. So we should use the metric in Kerr coordinates (5.97), setting $r = r_+$, $\Delta = 0$, and $v = \text{constant}$, to read off the metric on a slice of the future horizon:

$$ds^2 = \frac{\sin^2 \theta}{\rho^2} (r_+^2 + a^2) d\phi^2 + \rho^2 d\theta^2, \quad \sqrt{^2g} = (r_+^2 + a^2) \sin \theta. \quad (9.14)$$

Then, writing $A = \int \sqrt{^2g} d\theta d\phi$ and using $r_+^2 + a^2 = 2Mr_+$, we have

$$A = 8\pi Mr_+ = 8\pi M(M + \sqrt{M^2 - a^2}). \quad (9.15)$$

The maximum energy that can be extracted is the difference in mass between this rotating black hole and the smallest-mass black hole with an area this large, namely a Schwarzschild black hole, with

$$A = 16\pi M_{\text{final}}^2. \quad (9.16)$$

The *irreducible mass* of the original black hole is this smallest final mass,

$$M_{\text{irreducible}} = \sqrt{\frac{A}{16\pi}} = \frac{M}{\sqrt{2}} \sqrt{1 + \sqrt{1 - (a/M)^2}}. \quad (9.17)$$

Then the largest amount of available rotational energy is $M - M_{\text{irreducible}}$ for an initial maximally rotating black hole; from Eq. (9.17) $a < M$ implies $M_{\text{irreducible}} > 1/\sqrt{2}$, and the extracted energy is limited by

$$\frac{\Delta M}{M} < 1 - \frac{1}{\sqrt{2}} \approx 0.29. \quad (9.18)$$

It is easy to check that the limit for collision of two Schwarzschild black holes with negligible initial kinetic energy is again

$$\frac{\Delta M}{M_{\text{total}}} < 1 - \frac{1}{\sqrt{2}}, \quad (9.19)$$

much larger than the $\sim 0.05 M_{\text{total}}$ actually radiated in binary inspiral (see, e.g., Eq. (4.69) or GW150914).

Exercise 107. A 2-line exercise: Show for equal-mass Kerr black holes with opposite spins, area increase imposes only the weaker limit

$$\frac{\Delta M}{M_{\text{total}}} < \frac{1}{2}. \quad (9.20)$$

Stationary, axisymmetric horizons

We look now at equilibrium configurations—stationary, axisymmetric black holes, with Killing vectors t^α and ϕ^α . Because the spacetime is stationary and axisymmetric, rotations and time-translations map the horizon to itself. It follows that t^α and ϕ^α are tangent to the horizon H . This is because ϕ^α is tangent to the circular trajectories of rotated points, and t^α is tangent to the trajectories of time-translated points.

We have seen (beginning on pp. 197) that the horizon is the set of points where the determinant of the 2-metric in the t - ϕ plane vanishes $g_{tt}g_{\phi\phi} - g_{t\phi}^2 = t \cdot t \phi \cdot \phi - (t \cdot \phi)^2 = 0$.

Frame dragging is measured by $\omega = -t \cdot \phi / \phi \cdot \phi$, the angular velocity of freely falling observers with zero angular momentum. And, as shown in Eqs. (5.54), the vanishing of the 2-metric determinant implies that it is the vector field $t^\alpha + \omega \phi^\alpha$ that is null on the horizon. That is, the generators of the horizon are the paths of zero-angular-momentum photons.

Exercise 108. The tangent space at each point of spacetime is a copy of Minkowski space. Show that the metric in a null hyperplane in Minkowski space has vanishing determinant. Pick your favorite null hyperplane and show that the induced metric on it has vanishing determinant. This means that the 3-metric is degenerate, mapping some vector to 0. Show that ${}^3g_{\alpha\beta}\ell^\beta = 0$, where ℓ^α is a null vector in the null plane. Finally, defining n^α as the null vector orthogonal to the horizon for which $n^\alpha \ell_\alpha = -1$, show that ${}^3g_{\alpha\beta} = e_{\alpha\beta} := g_{\alpha\beta} + \ell_\alpha n_\beta + \ell_\beta n_\alpha$.

9.2 Mass and angular momentum of stationary axisymmetric black-hole spacetimes

The asymptotic metric of rotating stars and black holes has the form that we saw for Kerr black holes, agreeing with Schwarzschild at order $1/r$ and with Kerr to order $1/r^2$ in an asymptotically Cartesian chart. In particular, the mass appears in the asymptotic form of $t^\alpha t_\alpha$,

$$g_{tt} = -(1 - 2M/r) + O(r^{-2}); \quad (9.21)$$

and the angular momentum appears in the leading term in $t^\alpha \phi_\alpha$: As in (5.40),

$$g_{t\phi} = -2\frac{J}{r} \sin^2 \theta [1 + O(r^{-1})], \quad (9.22)$$

corresponding to the form 5.41, $g_{ti} = 2\epsilon_{ijk} \frac{x^j J^k}{r^3} + O(r^{-3})$, in the associated Cartesian chart.

We will see that the asymptotic behavior is tied to expressions for mass and angular momentum as integrals at infinity involving the timelike and rotational Killing vectors, respectively. Gauss's theorem then relates the asymptotic integrals to volume integrals over the matter. (Underlying all this is the general relation between symmetries and conserved quantities, discussed in the Noether part of Sect. 10.) We begin with surface integral expressions for M and J , introduced by Komar, showing

$$M = -\frac{1}{4\pi} \int_\infty \nabla^\alpha t^\beta dS_{\alpha\beta}, \quad J = \frac{1}{8\pi} \int_\infty \nabla^\alpha t^\beta dS_{\alpha\beta}. \quad (9.23)$$

When $\partial\Sigma$ is a $t = \text{constant}$, $r = \text{constant}$ surface, $dS_{\mu\nu} = \sqrt{-g} \nabla_{[\mu} t \nabla_{\nu]} r d\theta d\phi$.

For our asymptotic calculations, we'll use a chart in which, as in Boyer-Lindquist coordinates for Kerr,

$$\begin{aligned} ds^2 = & - \left[1 - \frac{2M}{r} + O(r^{-2}) \right] dt^2 - \frac{2J}{r} \sin^2 \theta dt d\phi [1 + O(r^{-1})] + r^2 \sin^2 \theta d\phi^2 [1 + O(r^{-2})] \\ & + \left[1 + \frac{2M}{r} + O(r^{-2}) \right] dr^2 + r^2 d\theta^2 [1 + O(r^{-2})]. \end{aligned} \quad (9.24)$$

Then, recalling $t_\mu = g_{\mu t}$, $\phi_\mu = g_{\mu\phi}$, we have

$$\begin{aligned} \int_\infty \nabla^\alpha t^\beta dS_{\alpha\beta} &= \lim_{r \rightarrow \infty} \int g^{tt} g^{rr} \frac{1}{2} (\partial_t t_r - \partial_r t_t) r^2 d\Omega = -\frac{1}{2} \lim_{r \rightarrow \infty} \int \partial_r (1 - 2M/r) r^2 d\Omega \\ &= -4\pi M. \end{aligned} \quad (9.25)$$

Similarly,

$$\begin{aligned} \int_\infty \nabla^\alpha \phi^\beta dS_{\alpha\beta} &= \lim_{r \rightarrow \infty} \int g^{t\mu} g^{r\nu} \partial_{[\mu} \phi_{\nu]} r^2 d\Omega = \lim_{r \rightarrow \infty} \int \frac{1}{2} [g^{tt} g^{rr} (-\partial_r g_{t\phi}) + g^{t\phi} g^{rr} (-\partial_r g_{\phi\phi})] r^2 d\Omega \\ &= \lim_{r \rightarrow \infty} \int \frac{1}{2} \left[\partial_r \left(-\frac{2J}{r} \sin^2 \theta \right) + \frac{2J}{r^3} \partial_r (r^2 \sin^2 \theta) \right] d\Omega \\ &= 8\pi J. \end{aligned} \quad (9.26)$$

Now, from the identity (9.39) for a Killing vector ξ^α ,

$$\nabla_\beta \nabla^\alpha \xi^\beta = R^\alpha{}_\beta \xi^\beta, \quad (9.27)$$

Gauss's theorem gives the integral $\int_\Sigma \nabla_\beta \nabla^\alpha \xi^\beta dS_\alpha$ over a 3-dimensional hypersurface in terms of the integral over its boundary,

$$\int_\Sigma \nabla_\beta \nabla^\alpha \xi^\beta dS_\alpha = \int_{\partial\Sigma} \nabla^\alpha \xi^\beta dS_{\alpha\beta}. \quad (9.28)$$

(See Sect. 9.5 below for a quick proof of Gauss's theorem in this form or read section A.2.1 of the integration appendix A.)

For a rotating star (or, more generally, a stationary, axisymmetric, asymptotically flat orientable spacetime with no interior boundary), the only boundary is at infinity, and Gauss's theorem relates M and J , given as integrals at infinity, to volume integrals over the matter:

$$M = -\frac{1}{4\pi} \int_{\infty} \nabla^{\alpha} t^{\beta} dS_{\alpha\beta} = -\frac{1}{4\pi} \int_{\Sigma} R^{\alpha}_{\beta} t^{\beta} dS_{\alpha}, \quad (9.29a)$$

$$J = \frac{1}{8\pi} \int_{\infty} \nabla^{\alpha} \phi^{\beta} dS_{\alpha\beta} = \frac{1}{8\pi} \int_{\Sigma} R^{\alpha}_{\beta} \phi^{\beta} dS_{\alpha}. \quad (9.29b)$$

Exercise 109. Using the field equation in the form $R_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)$, check that these expressions have the Newtonian limits $M = \int \rho dV$, $J = \int \rho \varpi^2 \Omega dV$, for a rotating fluid with density ρ and velocity $v = \varpi\omega$, with ϖ the cylindrical radius.

Eqs. (9.29) show that a stationary, asymptotically flat, orientable vacuum spacetime with no interior boundary and a single asymptotic region has vanishing mass and angular momentum: It is flat space.

For a black hole spacetime, the boundary $\partial\Sigma$ is the disjoint union of the sphere at infinity and a slice S of the horizon. Gauss's theorem now relates the surface integrals M and J at infinity to the surface integrals over the black hole:

$$M = -\frac{1}{4\pi} \int_S \nabla^{\alpha} t^{\beta} dS_{\alpha\beta} - \frac{1}{4\pi} \int_{\Sigma} R^{\alpha}_{\beta} t^{\beta} dS_{\alpha} \equiv M_H + M_{\text{matter}}, \quad (9.30a)$$

$$J = \frac{1}{8\pi} \int_S \nabla^{\alpha} \phi^{\beta} dS_{\alpha\beta} = \frac{1}{8\pi} \int_{\Sigma} R^{\alpha}_{\beta} \phi^{\beta} dS_{\alpha} \equiv J_H + J_{\text{matter}}. \quad (9.30b)$$

Here, the surface integrals over the horizon, M_H and J_H , can be regarded as the horizon's mass and angular momentum; the volume integrals M_{matter} and J_{matter} are the contributions of the matter to the total M and J . Note, however, that the contribution of the matter to the total mass includes its gravitational binding energy.

9.3 Two Constant Scalars on H

To summarize: The horizon H is the surface on which $t^{[\alpha}\varphi^{\beta]}$ is null:

$$0 = 2t^{[\alpha}\varphi^{\beta]}t_{\alpha}\varphi_{\beta} = t \cdot t \varphi \cdot \varphi - (t \cdot \varphi)^2; \quad (9.31)$$

and the linear combination $t^{\alpha} + \Omega_H \varphi^{\alpha}$ is null, where

$$\Omega_H \equiv \omega_H = - \left. \frac{t \cdot \varphi}{\varphi \cdot \varphi} \right|_H. \quad (9.32)$$

The null generators of the horizon then have tangent $\ell^{\alpha} = (t^{\alpha} + \Omega_H \varphi^{\alpha})f$, where f is any nonzero scalar. If one chooses $f = 1$, then

$$\ell^{\alpha} = t^{\alpha} + \Omega_H \varphi^{\alpha}. \quad (9.33)$$

By setting the coefficient of t^{α} to 1, one sacrifices the affine parametrization of ℓ^{α} . Away from the horizon, we will take ℓ^{α} to be tangent to outgoing null rays and denote by k^{α} the Killing vector

$$k^{\alpha} = t^{\alpha} + \Omega_H \phi^{\alpha} \quad \text{everywhere}, \quad (9.34)$$

coinciding with ℓ^α only on the horizon:¹

Ω_H is the angular velocity of H in the sense that a point moving along the generators of H (along the integral curves of ℓ^α) has angular velocity Ω_H as seen by an observer at infinity; well, a point on the horizon can't be seen from infinity, so Ω_H is the limiting angular velocity as seen from infinity of particle trajectories just outside the horizon. The amazing thing is that the horizon rotates as a rigid body: Ω_H is constant. For Kerr black holes, this is already implied by the explicit form (5.56), namely $\Omega_H = a/2Mr_+$. For distorted black holes – e.g., stationary axisymmetric black holes surrounded by an accretion disk, the result is still valid:

Ω_H is constant on H .

Proof. We already know that the component of $\nabla\Omega_H$ along ℓ^α vanishes; that is, since t^α and ϕ^α are Killing vectors, $t^\alpha\nabla_\alpha\Omega_H = 0 = \phi^\alpha\nabla_\alpha\Omega_H$, implying

$$\ell^\alpha\nabla_\alpha\Omega_H = 0.$$

Our job is to show that $e^\beta_\alpha\nabla_\beta\Omega_H = 0$, or, equivalently, $m^\alpha\nabla_\alpha\Omega_H = 0$. We begin by using the same argument to write

$$\dot{\theta} \equiv \ell^\alpha\nabla_\alpha\theta = 0.$$

When the null energy condition is satisfied, all terms on the RHS of the Raychaudhuri equation, (9.11), are negative, and $\dot{\theta} = 0$ then implies that each term must vanish,

$$\sigma_{\alpha\beta} = 0, \quad \theta = 0, \quad R_{\alpha\beta}l^\alpha l^\beta = 0. \quad (9.35)$$

(Because the Raychaudhuri equation was written for an affinely parametrized ℓ^α , we have shown only that $\sigma_{\alpha\beta}$ and θ vanish for an affinely parametrized ℓ^α ; but it is easy to check that they then vanish for any parametrization of ℓ^α .)

Exercise 110. Check this.

We next use the vanishing of the shear and divergence on H :

$$\sigma_{\alpha\beta} = 0 = \theta \Rightarrow 0 = m^\alpha m^\beta \nabla_\alpha \ell_\beta = m^\alpha m^\beta \nabla_\alpha k_\beta \quad (\text{this is the NP } \sigma). \quad (9.36)$$

Using $\mathcal{L}_t g_{\alpha\beta} = 0 = \mathcal{L}_\phi g_{\alpha\beta}$, we have

$$\mathcal{L}_k g_{\alpha\beta} = 2\nabla_{(\alpha} k_{\beta)} = 2\nabla_{(\alpha} \Omega_H \phi_{\beta)} \quad (9.37)$$

Contracting with $m^\alpha m^\beta$, already symmetric in the indices $\alpha\beta$, gives

$$m^\alpha m^\beta \phi_\alpha \nabla_\beta \Omega_H = 0.$$

Now $m_\alpha \phi^\alpha$ is nonzero; otherwise ϕ^α would be orthogonal to m^α and $\bar{m}^\alpha$, implying it would be parallel to ℓ^α . This is impossible because the trajectories of ϕ^α are circles, closed curves with finite affine parameter length, contradicting the infinite affine parameter length of the horizon's generators. (It would also violate causality - photons would follow closed curves, looping backward in time.) Thus

$$m^\beta \nabla_\beta \Omega_H = 0. \quad \square \quad (9.38)$$

¹Bardeen-Carter-Hawking paper[11] use ℓ^α to mean the Killing vector, so for them, ℓ^α is null only on the horizon. We keep ℓ^α null to conform to NP formalism, which we'll use in the calculations.

Ω_H constant on the horizon means that $\ell^\alpha = t^\alpha + \Omega_H \phi^\alpha$ is a Killing vector on the horizon. A horizon whose null generator is a Killing vector is called a *Killing horizon*; so we have just shown that the horizon of a stationary, axisymmetric spacetime is a Killing horizon.

Note that ℓ^α is a Killing vector on the horizon for the parametrization of the null generators that gives ℓ^α the form $t^\alpha + \Omega_H \phi^\alpha$. As mentioned above, with this choice, ℓ^α is not affinely parametrized:

$$\ell^\beta \nabla_\beta \ell^\alpha = \kappa \ell^\alpha,$$

with κ nonzero. κ is called the “surface gravity,” and it measures the acceleration of ℓ^α . It will play the role of temperature. (This is *not* the NP spin-coefficient labeled κ ; it is the real part of the NP ϵ in a tetrad for which ℓ is not affinely parametrized.) Once the QFTCST calculation is done, the identification is $T = \frac{\kappa}{2\pi}$, entropy $S = \frac{A}{4}$, in units with $G = c = \hbar = k_B = 1$, where k_B is Boltzmann’s constant. (Keeping the constants and taking κ to have dimension of acceleration, L/T^2 , we have $k_B T = \kappa \hbar / (2\pi c)$, $S = k_B A / (4\ell_P^2)$, with $\ell_P = \sqrt{G\hbar/c^3} = \text{Planck length}$.)

Exercise 111. To prove the 0^{th} law, that κ is constant on the horizon, we’ll need the following identity for a Killing vector ξ^α :

$$\nabla_\alpha \nabla_\beta \xi_\gamma = -R_{\beta\gamma\alpha\delta} \xi^\delta. \quad (9.39)$$

Prove this, starting from the relation $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$, taking its derivative, $\nabla_\gamma (\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) = 0$, writing down the two cyclic permutations, and adding the three equations with one minus sign to get the relation. (You’ll also need $R_{[\alpha\beta\gamma]\delta} = 0$.)

0^{th} Law: κ is constant on H (for vacuum or for matter satisfying the dominant energy condition).

This time, we’ll first give the general proof and then explicitly find κ for Kerr.

Proof. Again, to prove κ constant on H , it suffices to show

$$m^\beta \nabla_\beta \kappa = 0. \quad (9.40)$$

The definition (9.1) of κ is equivalent to

$$\kappa = -n_\alpha \ell^\beta \nabla_\beta \ell^\alpha. \quad (9.41)$$

We’ll use the vanishing of shear, divergence and twist of ℓ^α in Eq. (9.5) for $\nabla_\alpha \ell_\beta$:

$$\nabla_\alpha \ell_\beta = \ell_\alpha \ell_\beta n^\gamma n^\delta \nabla_\gamma \ell_\delta - e_\alpha^\gamma \ell_\beta n^\delta \nabla_\gamma \ell_\delta - e_\beta^\delta \ell_\alpha n^\gamma \nabla_\gamma \ell_\delta - \kappa n_\alpha \ell_\beta. \quad (9.42)$$

In particular, we’ll need the result of dotting this with m^α . Only the second term on the right survives:

$$m^\alpha \nabla_\alpha \ell_\beta = -\ell_\beta m^\gamma n^\delta \nabla_\gamma \ell_\delta, \quad (9.43)$$

implying that the NP spin coefficients κ_{NP} and ϱ , as well as σ , vanish on H .² (With the Kinnersley tetrad, ℓ^α blows up on the horizon. Our ℓ^α is $l_{\text{Kinnersley}}^\alpha \Delta$ evaluated on the horizon where $\Delta = 0$. The rescaling gives $\varrho = 0$, although it is nonzero on H for Kinnersley.)

Now on to $\nabla \kappa$:

$$\begin{aligned} -m^\delta \nabla_\delta \kappa &= m^\delta \nabla_\delta (n^\gamma \ell^\beta \nabla_\beta \ell_\gamma) \\ &= m^\delta \nabla_\delta n^\gamma \ell^\beta \nabla_\beta \ell_\gamma + m^\delta \nabla_\delta \ell^\beta n^\gamma \nabla_\beta \ell_\gamma + m^\delta \ell^\beta \nabla_\delta \nabla_\beta \ell_\gamma n^\gamma \end{aligned}$$

²From the fact that ℓ^α is a Killing vector and Eq. (9.43), we have (with our $-++$ convention) $\kappa_{NP} := -\ell^\beta m^\alpha \nabla_\beta \ell_\alpha = \ell^\beta m^\alpha \nabla_\alpha \ell_\beta = \ell^\beta (-\ell_\beta m^\gamma n^\delta \nabla_\gamma \ell_\delta) = 0$ and $\varrho := -m^\beta \bar{m}^\alpha \nabla_\alpha \ell_\beta = -m^\beta (-\ell_\beta \bar{m}^\gamma n^\delta \nabla_\gamma \ell_\delta) = 0$.

Use (9.43) in 2nd term and the Killing vector identity (9.39) in the 3rd:

$$\begin{aligned}
 -m^\delta \nabla_\delta \kappa &= m^\delta \nabla_\delta n^\gamma \kappa \ell_\gamma - \ell^\beta m^\delta n^\epsilon \nabla_\delta \ell_\epsilon n^\gamma \nabla_\beta \ell_\gamma + R_{\beta\gamma\delta\epsilon} m^\beta \ell^\gamma n^\delta \ell^\epsilon \\
 \text{2nd term} &= -m^\delta n^\epsilon \nabla_\delta \ell_\epsilon \underbrace{n^\gamma \ell^\beta \nabla_\beta \ell_\gamma}_{-\kappa} = \kappa m^\delta n^\epsilon \nabla_\delta \ell_\epsilon = -\text{1st term}.
 \end{aligned} \tag{9.44}$$

Then only the 3rd term remains:

$$\begin{aligned}
 -m^\delta \nabla_\delta \kappa &= m^\delta \ell^\beta R^\epsilon_{\delta\beta\gamma} \ell_\epsilon n^\gamma = R_{\alpha\beta\gamma\delta} \ell^\alpha m^\beta \ell^\gamma n^\delta \\
 &= \left[C_{\alpha\beta\gamma\delta} + \frac{1}{2} (g_{\alpha\gamma} R_{\beta\delta} + g_{\beta\delta} R_{\alpha\gamma} - g_{\beta\gamma} R_{\alpha\delta} - g_{\alpha\delta} R_{\beta\gamma}) \right] \ell^\alpha m^\beta \ell^\gamma n^\delta \\
 &= -\Psi_1 + R_{\beta\gamma} m^\beta \ell^\gamma \quad (\text{coefficient of } R \text{ vanishes}).
 \end{aligned}$$

Now Ψ_1 vanishes: Because $\sigma = 0 = \kappa_{NP}$, NP (4.2k) implies $\Psi_0 = 0$ as usual. With $D\Psi_1 = \ell^\alpha \nabla_\alpha \Psi_1 = 0$ and $\varrho = 0$, there is only one remaining term in NP (4.5, first equation), which now has the form

$$2\epsilon\Psi_1 = 0.$$

But $\epsilon = \frac{1}{2}(\Gamma_{121} + \Gamma_{431})$, with nonzero real part $-\frac{1}{2}\ell^\beta \nabla_\beta \ell_\alpha n^\alpha = \kappa$. So $\Psi_1 = 0$.

Finally, we show $R_{\beta\gamma} e_\alpha^\beta \ell^\gamma = 0$. We have already seen that $R_{\beta\gamma} \ell^\beta \ell^\gamma = 0$. The *dominant energy condition* is the condition that $J^\alpha := T^\alpha_\beta \ell^\beta$ is nonspacelike. Now

$$\begin{aligned}
 T_{\beta\gamma} \ell^\beta \ell^\gamma = J_\beta \ell^\beta = 0 &\Rightarrow J_\beta = f \ell_\beta + e_\beta^\gamma J_\gamma \\
 &\Rightarrow J_\beta J^\beta = e^{\beta\gamma} J_\beta J_\gamma > 0 \text{ unless } e^{\beta\gamma} J_\gamma = 0.
 \end{aligned} \tag{9.45}$$

Then J^α nonspacelike implies

$$\begin{aligned}
 0 &= e^{\alpha\beta} J_\beta = \frac{1}{8\pi} e^{\alpha\beta} (R_{\beta\gamma} - \frac{1}{2} \underbrace{g_{\beta\gamma} R}_0) \ell^\gamma \\
 &= e^{\alpha\beta} R_{\beta\gamma} \ell^\gamma.
 \end{aligned} \tag{9.46}$$

$\Rightarrow \kappa$ constant on H . $\square$

Surface gravity κ for Kerr

To find κ for a Kerr black hole, we'll evaluate $\ell^\beta \nabla_\beta \ell^\alpha$ on the horizon, with ℓ^α the null generator normalized to agree with the Killing vector $k^\alpha = t^\alpha + \Omega_H \phi^\alpha$ on the horizon. We use the fact that the generator ℓ^α of the horizon is along a principal null direction, agreeing with the affinely parametrized null vector $\ell^\alpha_{\text{affine}}$ of Eq. (5.69a) up to an overall factor f :

$$\ell^\alpha|_H = \lim_{r \rightarrow r_+} (f \ell^\alpha_{\text{affine}}). \tag{9.47}$$

where ℓ_{affine} satisfies the affinely parametrized geodesic equation,

$$\ell^\beta_{\text{affine}} \nabla_\beta \ell^\alpha_{\text{affine}} = 0. \tag{9.48}$$

Although $\ell^\alpha_{\text{affine}}$ is not defined on H , ℓ^α is smooth everywhere, so its value on H is the limit as $r \rightarrow r_+$ of its value for $r > r_+$. Then

$$\kappa \ell^\alpha|_H = \lim_{r \rightarrow r_+} \ell^\beta \nabla_\beta (f \ell^\alpha_{\text{affine}}) = \lim_{r \rightarrow r_+} (\ell^\beta \nabla_\beta f) \ell^\alpha_{\text{affine}}. \tag{9.49}$$

To find f , write $\ell_{\text{affine}}^\alpha$ in terms of the basis vectors of the Kerr coordinates $v, r, \tilde{\phi}, \theta$: From Eqs.(5.96) we have

$$\partial_t = \partial_v, \quad \partial_\phi = \partial_{\tilde{\phi}}, \quad \partial_r(\text{with } t, \phi \text{ fixed}) = \frac{r^2 + a^2}{\Delta} \partial_v + \partial_r(\text{with } v, \tilde{\phi} \text{ fixed}) + \frac{a}{\Delta} \partial_{\tilde{\phi}}. \quad (9.50)$$

(the last equality is (5.99) with ∂_v and $\partial_{\tilde{\phi}}$ replacing ∂_t and ∂_ϕ). In particular, in a Kerr basis, the generator of the horizon is

$$\ell|_H = \partial_v + \Omega_H \partial_{\tilde{\phi}}. \quad (9.51)$$

From Eq. (8.63a), in the Boyer-Lindquist coordinate basis, $\ell_{\text{affine}} = \frac{r^2 + a^2}{\Delta} \partial_t + \partial_r + \frac{a}{\Delta} \partial_\phi$, so in the Kerr basis,

$$\begin{aligned} \ell_{\text{affine}} &= \frac{r^2 + a^2}{\Delta} \partial_v + \left(\frac{r^2 + a^2}{\Delta} \partial_v + \partial_r + \frac{a}{\Delta} \partial_{\tilde{\phi}} \right) + \frac{a}{\Delta} \partial_{\tilde{\phi}} \\ &= 2 \frac{r^2 + a^2}{\Delta} \left(\partial_v + \frac{\Delta}{2(r^2 + a^2)} \partial_r + \frac{a}{r^2 + a^2} \partial_{\tilde{\phi}} \right). \end{aligned} \quad (9.52)$$

Then,

$$\lim_{r \rightarrow r_+} \left[\frac{\Delta}{2(r^2 + a^2)} \ell_{\text{affine}} \right] = \partial_v + \frac{a}{r_+^2 + a^2} \partial_{\tilde{\phi}} = \partial_v + \Omega_H \partial_{\tilde{\phi}} = \ell|_H. \quad (9.53)$$

Thus $f = \frac{\Delta}{2(r^2 + a^2)}$,

$$\ell = \partial_v + \frac{\Delta}{2(r^2 + a^2)} \partial_r + \frac{a}{r^2 + a^2} \partial_{\tilde{\phi}}, \quad (9.54)$$

and Eq. (9.49) gives

$$\begin{aligned} \kappa \ell^\alpha|_H &= \lim_{r \rightarrow r_+} \ell_{\text{affine}}^\alpha \frac{\Delta}{2(r^2 + a^2)} \partial_r \frac{\Delta}{2(r^2 + a^2)} \Big|_H = \frac{r_+ - M}{r_+^2 + a^2} \ell^\alpha, \\ \kappa &= \frac{r_+ - M}{r_+^2 + a^2} = \frac{\sqrt{M^2 - a^2}}{2Mr_+}. \end{aligned} \quad (9.55)$$

3rd Law: κ cannot be reduced to zero.

The third law of thermodynamics is the statement: *It is impossible by any process, no matter how idealized, to reduce the temperature to zero in a finite sequence of operations.*

Its black hole analog is:

It is impossible by any process, no matter how idealized, to reduce κ to zero in a finite sequence of operations.

As one spins up a Kerr black hole to its maximum value, $a = M$ (an *extremal* black hole), its surface gravity drops to zero. Reaching or surpassing $a = M$ means creating a naked singularity, and the cosmic censorship hypothesis is the statement this is impossible. There is no formal proof of cosmic censorship, but all calculations that attempt to add enough angular momentum to a black hole by sending in particles or fields end up adding enough mass to keep κ positive, as long as the matter satisfies a positive energy condition. An extremal black hole has no trapped surface, and Israel proves a version of the third law that prevents a spacetime with trapped surfaces from evolving to one with no trapped surface.

9.4 The 1st Law

The first law of thermodynamics gives the difference in energy between nearby equilibrium configurations. We will begin with the first law for vacuum, asymptotically flat, stationary black hole spacetimes. The Kerr family of geometries are the only such spacetimes, so the first calculation uses explicit expressions for the energy (mass) of Kerr black holes.

We then present a more general derivation for stationary axisymmetric black holes surrounded by matter (e.g., an accretion disk) following BCH, the original Bardeen-Carter-Hawking paper[11], and filling in steps. They use ℓ^α to mean the Killing vector

$$\ell_{\text{BCH}}^\alpha = t^\alpha + \Omega_H \phi^\alpha, \quad (9.56)$$

so ℓ_{BCH}^α is null only on the horizon. To avoid confusion, these notes will denote the Killing vector by

$$k^\alpha = t^\alpha + \Omega_H \phi^\alpha, \quad (9.57)$$

with ℓ^α a null vector for which $\ell^\alpha = k^\alpha$ only on H .

Consider a family of stationary axisymmetric black holes with masses, angular momenta, horizon areas, and horizon angular velocities $M(\lambda)$, $J(\lambda)$, $A(\lambda)$, $\Omega_H(\lambda)$. The first law of thermodynamics expresses the change in the mass (energy) along such a family of equilibria in terms of the thermodynamic variables that describe the system. For the black hole, these will be entropy and angular momentum, with the entropy identified with the black hole's area (more precisely, after the BH temperature was discovered, with $A/8\pi$): With δ denoting $d/d\lambda$ at $\lambda = 0$, it has the form

$$\delta M = \frac{1}{8\pi} \kappa \delta A + \Omega_H \delta J, \quad (9.58)$$

analogous to

$$\delta M = T \delta S + \Omega \delta J, \quad (9.59)$$

for a uniformly rotating system. For the unperturbed quantities, the black-hole relation is

$$M = \frac{1}{4\pi} \kappa A + 2\Omega_H J, \quad (9.60)$$

reminiscent of the $M = TS - PV$ thermodynamic relation.

9.4.1 First law for Kerr black holes

To obtain Eq. (9.58) for Kerr black holes, we first need to find the surface gravity κ , for a black hole of mass M and angular momentum J . We already know A and Ω_H from Eqs. (9.15), (5.56) and (5.55),

$$A = 8\pi M r_+, \quad \Omega_H = \frac{a}{2Mr_+}, \quad \text{where } r_+ = M + \sqrt{M^2 - a^2}. \quad (9.61)$$

We can now easily check the first law (9.58) and the corresponding Smarr relation (9.60). We'll start with the Smarr relation. Using Eqs.(5.56), (9.15) and (9.55), for A , Ω_H and κ , together with $J = aM$, we have

$$\frac{A}{4\pi} \kappa + 2\Omega_H J = \frac{8\pi M r_+}{4\pi} \frac{r_+ - M}{2Mr_+} + 2 \frac{a}{2Mr_+} aM = \frac{r_+^2 - Mr_+ + a^2}{r_+} = M. \quad (9.62)$$

The last equality used $r_+^2 + a^2 = 2Mr_+$ once again.

To obtain the first law, we express the Smarr relation in terms of the variables M , A and J . Writing $r_+ = \frac{A}{8\pi M}$, we have

$$\kappa = \frac{r_+ - M}{2Mr_+} = \frac{1}{2M} - \frac{4\pi M}{A}, \quad \Omega_H = \frac{J}{2M^2 r_+} = \frac{4\pi J}{AM}, \quad (9.63)$$

and Eq. (9.60) takes the form

$$M = \frac{A}{8\pi M} - M + \frac{8\pi J^2}{AM},$$

or

$$M^2 = \frac{A}{16\pi} + \frac{4\pi J^2}{A}. \quad (9.64)$$

Then

$$2M\delta M = \left(\frac{1}{16\pi} - \frac{4\pi J^2}{A^2} \right) \delta A + \frac{8\pi J}{A} \delta J. \quad (9.65)$$

Finally, divide by $2M$, replace $4\pi J^2/A$ by $M^2 - A/(16\pi)$, and then use Eq. (9.63) in reverse to restore κ and Ω_H , to get the first law.

$$\delta M = \frac{\kappa}{8\pi} \delta A + 2\Omega_H \delta J. \quad \square \quad (\text{e:first1})$$

9.4.2 First law for relativistic stars and matter outside a black hole

We show in next section, Sect. 9.4.3 below, that the first law differs from its vacuum form, (9.58), only by a contribution δM_{matter} from the matter. For an axisymmetric disk outside a black hole and for a rotating star with no black hole, modeled in each case by a perfect fluid, the contribution of the matter to change in total energy is a consequence of the local form of the first law: The change in mass is given in terms of the change in the entropy S , baryon number N . The change in the energy of a fluid element, measured by an comoving observer (an observer whose velocity u^α is the velocity of the fluid element), is $dE = TdS + \mu dN$. The corresponding change dM in the mass of the spacetime involves a redshift factor and a term ΩdJ , with Ω the star's angular velocity (which may depend on position), and dJ the angular momentum of the fluid element.

The derivation in this section is drawn from previous discussions by Thorne [90], Zel'dovich and Novikov [100] (with a section on injection energy by Thorne), and by Carter[21]. The text below is largely taken from Friedman and Stergioulas [39].

Injection energy: First law deduced by dropping a group of baryons from infinity

Consider a stationary, axisymmetric star (or fluid outside a black hole) with fluid velocity

$$u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha). \quad (9.66)$$

The contents of a box at infinity are to be injected into the star at a point P .³ The box holds a collection of baryons having the same composition as the matter at the point P . The collection has baryon number δN and entropy δS ; the box, including its contents, has energy δM_1 and angular momentum δJ . In order that the matter can be injected in a thermodynamically reversible process, the entropy per baryon

³To preserve axisymmetry, we can take the box to be an axisymmetric ring of fluid to be inserted at an axisymmetric circle in the star.

s in the box is to be that of the star at the point P , so that $\delta S = s\delta N$ is the entropy of the δN baryons. Denoting by $p_{1\alpha}$ the initial four-momentum of the box, we have

$$\delta M_1 = -p_{1\alpha}t^\alpha, \quad \delta J = p_{1\alpha}\phi^\alpha. \quad (9.67)$$

Imagine the box falling freely to the point P of the star. An observer at rest with respect to the fluid has velocity $u^\alpha(P)$ given by Eq. (9.66), and when the box reaches P , the observer measures an energy

$$\begin{aligned} \delta E_1 &= -p_{1\alpha}u^\alpha = -u^t(p_{1\alpha}t^\alpha + \Omega p_{1\alpha}\phi^\alpha) \\ &= u^t(\delta M_1 - \Omega\delta J). \end{aligned} \quad (9.68)$$

Now, following Thorne, we suppose the observer catches the box at P and reversibly injects its contents into the fluid, imparting to the fluid an energy

$$\delta E = T\delta S + \mu\delta N. \quad (9.69)$$

Because the initial entropy per baryon was already s , not all of the available energy is used: Our active observer uses the remaining energy to throw the empty box back up to infinity, on a trajectory with zero angular momentum, so that the angular momentum δJ is retained by the fluid. The returning box then has momentum $p_{2\alpha}$ with vanishing angular momentum, $p_{2\alpha}\phi^\alpha = 0$. Its locally measured energy is $\delta E_2 = \delta E_1 - \delta E = p_{2\alpha}u^\alpha = p_{2\alpha}t^\alpha u^t$. Because its free trajectory conserves $p_{2\alpha}t^\alpha$, the box reaches infinity with redshifted energy $\delta M_2 = p_{2\alpha}t^\alpha = \frac{\delta E_2}{u^t}$. The change in mass of the star is then related to the change in baryon number, entropy, and angular momentum of the fluid by

$$\begin{aligned} \delta M &= \delta M_1 - \delta M_2 = \frac{1}{u^t}(\delta E_1 - \delta E_2) + \Omega\delta J = \frac{\delta E}{u^t} + \Omega\delta J \\ &= \frac{\mu}{u^t}\delta N + \frac{T}{u^t}\delta S + \Omega\delta J. \end{aligned} \quad (9.70)$$

The first law for generic black holes surrounded by an axisymmetric configuration of fluid or for a relativistic star has the form

$$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega_H\delta J + \int_\Sigma \left(\frac{\mu}{u^t}\delta dN + \frac{T}{u^t}\delta dS + \Omega\delta dJ \right). \quad (9.71)$$

where $dJ = T_\beta^\alpha\phi^\beta dS_\alpha$, $dN = nu^\alpha dS_\alpha$, and $dS = sdN$.

Although the reversible injection of a ring of matter preserves the local thermodynamic equilibrium, the star itself is no longer in gravitational equilibrium after the process described in our derivation of the first law. But the ignored global correction to the mass is second order in the perturbation: Readjusting to the new equilibrium is a perturbation that preserves entropy, baryon number and angular momentum, and an equilibrium configuration is an extremum of the mass for perturbations that conserve entropy, baryon number and angular momentum.

9.4.3 First law for generic stationary axisymmetric black holes.

We show here that, as noted above, the vacuum terms in the first law have the same form for perturbations of a stationary axisymmetric black hole spacetime with matter. We first show that the surface terms at the horizon and at infinity again give $\frac{\kappa}{8\pi}\delta A + 2\Omega\delta J$ as the black-hole contribution to δM . We keep the contribution from the matter for an arbitrary stress-energy tensor.

We start by generalizing the Smarr relation (9.60). We can assume that the Killing vectors t^α and ϕ^α are the same for all λ : This uses our freedom to make an arbitrary diffeo or, in coordinate language, to say that in a fixed chart, each metric $g_{\alpha\beta}(\lambda)$ has components independent of t and ϕ . Consistent with this, we can also require that the set of points comprising the horizon is the same for all metrics $g_{\alpha\beta}(\lambda)$.⁴

From the Killing vector identity (9.27), we have

$$\nabla_\beta \nabla^\alpha k^\beta = R^\alpha{}_\beta k^\beta, \quad (9.72)$$

vanishing for our vacuum spacetimes. Gauss's theorem now gives the integral $\int_\Sigma \nabla_\beta \nabla^\alpha k^\beta dS_\alpha$ over a 3-dimensional hypersurface in terms of the integral over its boundary,

$$\int_\Sigma \nabla_\beta \nabla^\alpha k^\beta dS_\alpha = \int_{\partial\Sigma} \nabla^\alpha k^\beta dS_{\alpha\beta}. \quad (9.73)$$

When $\partial\Sigma$ is a $t = \text{constant}$, $r = \text{constant}$ surface, $dS_{\mu\nu} = \sqrt{-g} \nabla_{[\mu} t \nabla_{\nu]} r d\theta d\phi$. (See Sect. 9.5 below for a quick proof of Gauss's theorem in this form or read section A.2.1 of the integration appendix A.) With k^α replaced by t^α or ϕ^α , the identity shows that a stationary, asymptotically flat, orientable vacuum spacetime with no interior boundary and a single asymptotic region has vanishing mass and angular momentum: It is flat space. In our case, denoting by S the 2-dimensional slice of the horizon, $S := \Sigma \cap H$, we have

$$\int_\infty \nabla^\alpha k^\beta dS_{\alpha\beta} = \int_S \nabla^\alpha k^\beta dS_{\alpha\beta} + \int_\Sigma R^\alpha{}_\beta k^\beta dS_\alpha \quad (9.74)$$

where we have used Eq. (9.30a). From Eqs. (9.23), we have

$$\frac{1}{4\pi} \int_\infty \nabla^\alpha k^\beta dS_{\alpha\beta} = -M + 2\Omega_H J. \quad (9.75)$$

Now to the integral over the horizon: We can take our slice S of H to be orthogonal to ℓ^α and n^α , spanned by m^α and $\bar{m}^\alpha$. On H , $k^\alpha = \ell^\alpha$, and the area element of S is

$$dS_{\alpha\beta} = \frac{1}{2}(\ell_\alpha n_\beta - \ell_\beta n_\alpha) dA, \quad (9.76)$$

where the orientation is chosen in the same way as at the sphere at infinity: $dS_{tr} = \epsilon_{tr\theta\phi} d\theta d\phi$ for t increasing to the future and r increasing in the usual way for a radial coordinate.

Exercise 112. Quick exercise: Check this, starting with the area element in the form $\omega^0_{[\alpha} \omega^1_{\beta]} dA$, for an orthonormal frame $\{\omega^\mu\}$ with ω^0 and ω^1 normal to $\mathbf{m}$. Then for some constant a , $\ell_\alpha = a(\omega^0_\alpha - \omega^1_\beta)$, $n_\alpha = \frac{1}{2a}(\omega^0_\alpha + \omega^1_\beta)$.

Next, using the Killing equation, $\nabla^\alpha k^\beta = \nabla^{[\alpha} k^{\beta]}$, we have on S

$$\nabla^\alpha k^\beta \frac{1}{2}(k_\alpha n_\beta - k_\beta n_\alpha) = k^\alpha \nabla_\alpha k^\beta n_\beta = \ell^\alpha \nabla_\alpha \ell^\beta n_\beta = -\kappa, \quad (9.77)$$

⁴If the new and old horizons do not coincide, one can always make a t - and ϕ -independent diffeo that maps the new horizon to the old one. Keeping t^α and ϕ^α fixed was implicit in our derivation of the first law for Kerr, and fixing the horizon as well is a sensible choice for proving the first law in general. Although this way of describing the family of black holes is natural for proving the first law, it is not the way one would ordinarily describe changes in a black hole that alter its orientation: Galactic black holes that accrete matter by swallowing stars change from one stationary axisymmetric configuration to another, and the changes in area, mass, and total angular momentum are related by the first law. But to describe the time-dependent accretion or the change in spin direction, one would typically pick a chart in which the coordinate direction of the rotation axis changes.

implying

$$\frac{1}{4\pi} \int_S \nabla^\alpha k^\beta dS_{\alpha\beta} = -\frac{1}{4\pi} \kappa A. \quad (9.78)$$

With this expression for the surface integral at the horizon and Eq. (9.75) for the integral at infinity, Eq. (9.74) gives the generalized Smarr relation

$$M = \frac{1}{4\pi} \kappa A + 2\Omega_H J - \frac{1}{4\pi} \int_\Sigma R^\alpha_\beta k^\beta dS_\alpha, \quad (9.79)$$

with the last term on the right the contribution from the matter.

To obtain the first law, we will use this relation, in the perturbed form

$$\delta M = \frac{1}{4\pi} \delta(\kappa A) + 2\delta(\Omega_H J) + \delta_1 M_{\text{matter}}, \quad (9.80)$$

$$\delta_1 M_{\text{matter}} = -\frac{1}{4\pi} \int_\Sigma R^\alpha_\beta k^\beta dS_\alpha, \quad (9.81)$$

together with a relation

$$\delta M = -\frac{1}{4\pi} \delta\kappa A - 2\delta\Omega_H J + \delta_2 M_{\text{matter}}, \quad (9.82)$$

that comes from varying the Hilbert action, $I = \frac{1}{16\pi} \int R\sqrt{-g} d^4x$. Adding these two equations immediately gives the first law.

The variation of the action is given in Sect. 10.1 below. To avoid factors of 1/2 and conform to BCH, we'll look at the change in $\frac{1}{8\pi} R\sqrt{-g}$ from an arbitrary perturbation $h_{\alpha\beta}$ of the metric. From Eq.(10.4), we have

$$\frac{1}{\sqrt{-g}} \delta(R\sqrt{-g}) = -G^{\alpha\beta} h_{\alpha\beta} + \nabla_\alpha A^\alpha, \quad (9.83)$$

where

$$A^\alpha = \nabla_\beta h^{\alpha\beta} - \nabla^\alpha h. \quad (9.84)$$

One ordinarily obtains the field equation by varying the action, defined as an integral over a region of spacetime between an initial and final hypersurface. Here, because we are looking at stationary solutions, we can omit the integral over time and write the action as an integral over Σ : Eq. (9.83) gives

$$\frac{1}{8\pi} \int_\Sigma \delta(R\sqrt{-g}) d^3x = -\frac{1}{8\pi} \int_\Sigma G^{\alpha\beta} h_{\alpha\beta} \sqrt{-g} d^3x + \frac{1}{8\pi} \int_\Sigma \nabla_\alpha A^\alpha \sqrt{-g} d^3x. \quad (9.85)$$

We now show that the second integral on the right is the sum of surface terms at infinity and at the horizon, having the form

$$\begin{aligned} \frac{1}{8\pi} \int_\Sigma \nabla_\alpha A^\alpha \sqrt{-g} d^3x &= \text{surface integral at infinity} - \text{surface integral at horizon} \\ &= \delta M + \frac{1}{4\pi} \delta\kappa A + \delta\Omega_H J_H, \end{aligned} \quad (9.86)$$

where the minus sign in the first equality corresponds to using a normal that is radially outward at both the horizon and at spatial infinity. In a vacuum spacetime, only the surface terms are nonzero, and the

change in mass is given by $\delta M = -\frac{1}{4\pi}\delta\kappa A - \delta\Omega_H J$. When matter is present, the remaining terms in Eq. (9.85) are the contributions from the matter to δM that we have called $\delta_2 M_{\text{matter}}$ in Eq. (9.82)

$$\delta_2 M_{\text{matter}} = \frac{1}{8\pi} \int_{\Sigma} \delta(R\sqrt{-g}) d^3x + \frac{1}{8\pi} \int_{\Sigma} G^{\alpha\beta} h_{\alpha\beta} \sqrt{-g} d^3x. \quad (9.87)$$

Surface integrals.

Choose an axisymmetric scalar t for which Σ is a $t = \text{constant}$ surface and with $t^\alpha \nabla_\alpha t = 1$. (For Kerr or Schwarzschild, a natural choice is a scalar agreeing with v near the future horizon and with the usual coordinate t near infinity.) In a chart $\{t, x^i\}$, $dS_\alpha = \nabla_\alpha t \sqrt{-g} d^3x$, and $t^\alpha dS_\alpha = \sqrt{-g} d^3x$. Then

$$\delta \int_{\Sigma} R t^\alpha dS_\alpha = \delta \int_{\Sigma} R \sqrt{-g} d^3x = - \int_{\Sigma} G_{\beta\gamma} h^{\beta\gamma} t^\alpha dS_\alpha + \int_{\Sigma} \nabla_\beta A^\beta t^\alpha dS_\alpha. \quad (9.88)$$

Because $h_{\alpha\beta}$ and $g_{\alpha\beta}$ are time independent, we have

$$\sqrt{-g} \nabla_\alpha A^\alpha = \partial_\mu (A^\mu \sqrt{-g}) = \partial_i (A^i \sqrt{-g}).$$

The surface term at infinity from $\nabla_\alpha A^\alpha$ in Eq. (9.88) can be written as

$$\begin{aligned} \int_{\infty} A^\alpha dS_\alpha &= \lim_{r \rightarrow \infty} \int A^\alpha \nabla_\alpha r r^2 d\Omega = \lim_{r \rightarrow \infty} \int [\partial_\mu (r^2 h^{r\mu}) - r^2 \partial^r h] d\Omega \\ &= \lim_{r \rightarrow \infty} \int [\partial_r (r^2 h^{rr}) - r^2 \partial_r h] d\Omega. \end{aligned}$$

For our family of asymptotic metrics (9.24), to order $1/r$,

$$h^{rr} = h_{rr} = \frac{2\delta M}{r}, \quad h^t_t + h^r_r = -h_{tt} + h_{rr} = 0,$$

and we have

$$\frac{1}{8\pi} \int_{\infty} A^\alpha dS_\alpha = \delta M. \quad (9.89)$$

Finally, to compute the surface integral at the horizon, where what we understand is $dS_{\alpha\beta}$ in terms of the two normals n_α and ℓ_α , it will be helpful to rewrite the surface term. Start with

$$\begin{aligned} \nabla_\beta A^\beta t^\alpha &= \nabla_\beta (A^\beta t^\alpha - A^\alpha t^\beta) - \underbrace{A^\beta \nabla_\beta t^\alpha + t^\beta \nabla_\beta A^\alpha}_{\mathfrak{L}_t A^\alpha=0} - \underbrace{A^\alpha \nabla_\beta t^\beta}_0 \\ &= \nabla_\beta (A^\beta t^\alpha - A^\alpha t^\beta). \end{aligned}$$

Then the divergence term in the varied action (9.88) can be written as

$$\begin{aligned} \int_{\Sigma} \nabla_\beta A^\beta t^\alpha dS_\alpha &= \int_{\Sigma} \nabla_\beta (A^\beta t^\alpha - A^\alpha t^\beta) dS_\alpha \\ &= - \int_{\partial\Sigma} (A^\beta t^\alpha - A^\alpha t^\beta) dS_{\alpha\beta}, \end{aligned} \quad (9.90)$$

and the surface term at the horizon is

$$\begin{aligned}
 \int_S (A^\beta t^\alpha - A^\alpha t^\beta) dS_{\alpha\beta} &= \int_S (A^\beta t^\alpha - A^\alpha t^\beta) \frac{1}{2} (\ell_\alpha n_\beta - \ell_\beta n_\alpha) dA = \int_S A^\beta t^\alpha (\ell_\alpha n_\beta - \ell_\beta n_\alpha) dA \\
 &= \frac{1}{8\pi} \int_S (\nabla_\gamma h^{\beta\gamma} - \nabla^\beta h) (\ell^\alpha - \underbrace{\Omega_H \phi^\alpha}_0) (\ell_\alpha n_\beta - \ell_\beta n_\alpha) dA \\
 &\quad \phi \cdot \ell = \phi \cdot n = 0 \\
 &= \frac{1}{8\pi} \int_S \ell^\alpha \nabla^\beta h_{\alpha\beta} dA.
 \end{aligned} \tag{9.91}$$

where we have used $\ell \cdot \ell = 0$, $\ell \cdot n = -1$, and $\ell^\alpha \nabla_\alpha h = 0$ in the last equality.

We need to relate the integrand to $\delta\kappa$, and we compute $\delta\kappa$ as follows. From the definition of κ , we have

$$\delta\kappa = -\delta(\ell^\beta n^\alpha \nabla_\beta \ell_\alpha) = -\delta\ell^\beta n^\alpha \nabla_\beta \ell_\alpha - \delta n^\alpha \ell^\beta \nabla_\beta \ell_\alpha - \ell^\beta n^\alpha \delta(\nabla_\beta \ell_\alpha). \tag{9.92}$$

Now on the horizon,

$$\delta\ell^\alpha = \delta k^\alpha = \delta\Omega_H \phi^\alpha, \tag{9.93}$$

because $\delta t^\alpha = 0 = \delta\phi^\alpha$. In the third term, we can replace ℓ_α by its value k_α on the horizon, because the only derivative is along the horizon. Then Killing's equation gives

$$\delta(\nabla_\beta \ell_\alpha) = \delta(\nabla_{[\beta} k_{\alpha]}) = \nabla_{[\beta} \delta k_{\alpha]},$$

$$\delta\kappa = -n^\alpha \delta\Omega_H \phi^\beta \nabla_\beta \ell_\alpha - \delta n^\alpha \ell^\beta \nabla_\beta \ell_\alpha - \ell^\beta n^\alpha \nabla_{[\beta} \delta k_{\alpha]}. \tag{9.94}$$

On to the second term on the right: Because the horizon is unchanged in our gauge, $\ell_\alpha(\lambda)$ is parallel to the null normal $\nabla_\alpha u$ to H , implying $\delta\ell_\alpha$ is parallel to $\nabla_\alpha u$: $\delta\ell_\alpha = f\ell_\alpha$, some function f on H , and we can write

$$\begin{aligned}
 -\delta n^\alpha \ell^\beta \nabla_\beta \ell_\alpha &= -\delta n^\alpha \kappa \ell_\alpha = \kappa n^\alpha \delta\ell_\alpha = -\kappa f \\
 &= f n^\alpha \ell_\beta \nabla^\beta \ell_\alpha = n^\alpha \ell_\beta \nabla_\beta (f\ell_\alpha), \quad \text{using } \ell^\beta \nabla_b f = k^\beta \nabla_b f = 0 \text{ on } H, \\
 &= \ell^\beta n^\alpha \nabla_\beta \delta\ell_\alpha = \ell^\beta n^\alpha \nabla_\beta \delta k_\alpha.
 \end{aligned} \tag{9.95}$$

The second and third terms on the right of (9.94) now combine to give

$$\frac{1}{2} (\ell^\alpha n^\beta + \ell^\beta n^\alpha) \nabla_\alpha \delta k_\beta.$$

The shear and divergence, $\sigma_{\alpha\beta}(\lambda)$ and $\theta(\lambda)$ vanish, so $0 = \delta(m^\alpha \bar{m}^\beta \nabla_\alpha k_\beta)$; because the position of the horizon is unchanged, δm^α is a linear combination of ℓ^α , m^α and $\bar{m}^\alpha$, implying $\delta m^\alpha \bar{m}^\beta \nabla_\alpha k_\beta = 0$. Then $m^\alpha \bar{m}^\beta \nabla_\alpha \delta k_\beta = 0$ on H , and we have

$$\begin{aligned}
 \frac{1}{2} (\ell^\alpha n^\beta + \ell^\beta n^\alpha) \nabla_\alpha \delta k_\beta &= -\frac{1}{2} (-\ell^\alpha n^\beta - \ell^\beta n^\alpha + m^\alpha \bar{m}^\beta + m^\beta \bar{m}^\alpha) \nabla_\alpha \delta k_\beta = -\frac{1}{2} \nabla^\alpha \delta k_\alpha \\
 &= -\frac{1}{2} \ell^\alpha \nabla^\beta h_{\alpha\beta}.
 \end{aligned} \tag{9.96}$$

In the last equality, we have used Eq. (9.93), $\delta k_\alpha = \delta(g_{\alpha\beta} k^\beta) = h_{\alpha\beta} k^\beta + \Omega_H \phi_\beta$, and $\nabla^\beta \phi_\beta = 0$. Then

$$\delta\kappa = -\frac{1}{2} \ell^\alpha \nabla^\beta h_{\alpha\beta} - n^\alpha \delta\Omega_H \phi^\beta \nabla_\beta \ell_\alpha = -\frac{1}{2} \ell^\beta \nabla^\alpha h_{\alpha\beta} - \delta\Omega_H n^\alpha \ell^\beta \nabla_\beta \phi_\alpha, \tag{9.97}$$

this time using $\mathcal{L}_\phi \ell_\alpha = 0$.

Using this relation to replacing $\frac{1}{2}\ell^\alpha \nabla^\beta h_{\alpha\beta}$ in Eq. (9.91) gives for the surface integral over the horizon

$$\int_S (A^\beta t^\alpha - A^\alpha t^\beta) dS_{\alpha\beta} = -\frac{1}{4\pi} \int_S \delta\kappa dA - \delta\Omega_H \frac{1}{4\pi} \int_S \nabla^\alpha \phi^\beta \ell_\alpha n_\beta dA = -\frac{1}{4\pi} \delta\kappa A - 2\delta\Omega_H J_H,$$

where, in the last line, we used the antisymmetry of $\nabla^\alpha \phi^\beta$. Thus the variation (9.85) of $\int_\Sigma d^3x R \sqrt{-g}/8\pi$ yields

$$\delta M = -\frac{1}{4\pi} \delta\kappa A - 2\delta\Omega_H J_H + \delta_2 M_{\text{matter}}, \quad (9.98)$$

as claimed.

Adding this to the relation $\delta M = \delta(\kappa A/4\pi + 2\Omega_H J_H) + \delta_1 M_{\text{matter}}$, Eq. (9.80), and dividing by 2, gives the first law,

$$\delta M = \frac{1}{8\pi} \kappa \delta A + \Omega_H \delta J_H + \delta M_{\text{matter}}, \quad (9.58)$$

with

$$\delta M_{\text{matter}} = \frac{1}{2} (\delta_1 M_{\text{matter}} + \delta_2 M_{\text{matter}}).$$

In Eq. (9.87) for $\delta_2 M_{\text{matter}}$, we can write $\sqrt{-g} d^3x = t^\alpha dS_\alpha = k^\alpha dS_\alpha$, because ϕ^α lies in Σ and is therefore orthogonal to dS_α . Using Eq. (9.81) for $\delta_1 M_{\text{matter}}$, we then have

$$\begin{aligned} \delta M_{\text{matter}} &= -\frac{1}{8\pi} \delta \int G_\beta^\alpha k^\beta dS_\alpha + \frac{1}{16\pi} \int G^{\beta\gamma} h_{\beta\gamma} k^\alpha dS_\alpha \\ &= -\delta \int T_\beta^\alpha k^\beta dS_\alpha + \frac{1}{2} \int T^{\beta\gamma} h_{\beta\gamma} k^\alpha dS_\alpha. \end{aligned} \quad (9.99)$$

9.5 Gauss

For an antisymmetric tensor $F^{\alpha\beta}$, $\nabla_\nu F^{\mu\nu} = (-g)^{-1/2} \partial_\nu [(-g)^{1/2} F^{\mu\nu}]$. This is the same identity that gives Gauss's law for a vector A^ν ; the extra Christoffel symbol in the derivative vanishes because of the antisymmetry of F . Consider a $t = \text{constant}$ surface Σ bounded by an $r = \text{constant}$ surface $\partial\Sigma$. (For any smooth 3-dimensional submanifold with the topology of a ball bounded by a topological sphere, just choose coordinates with t constant on the ball and r constant on the boundary.) Write $dS_\nu = \nabla_\nu t \sqrt{-g} d^3x$ and $dS_{\mu\nu} = \sqrt{-g} \nabla_{[\mu} t \nabla_{\nu]} r d^2x$. Then

$$\begin{aligned} \int_\Sigma \nabla_\beta F^{\alpha\beta} dS_\alpha &= \int_\Sigma \partial_\nu (F^{\mu\nu} \sqrt{-g}) \nabla_\mu t d^3x = \int_\Sigma F^{\mu\nu} \nabla_\mu t \nabla_\nu r \sqrt{-g} d^2x \\ &= \int_{\partial\Sigma} F^{tr} \sqrt{-g} d\theta d\phi \\ &= \int_{\partial\Sigma} F^{\alpha\beta} dS_{\alpha\beta}. \end{aligned}$$

If n_α and $\tilde{n}_\alpha$ are two orthogonal unit normals spanning the tangent space to $\partial\Sigma$, $dS_{\alpha\beta} = n_{[\alpha} \tilde{n}_{\beta]} dA$. For $F^{\mu\nu}$ the electromagnetic field, this gives the electric charge:

$$4\pi j^\alpha = \nabla_\beta F^{\alpha\beta} \implies \frac{1}{4\pi} \int_\Sigma j^\alpha dS_\alpha = \frac{1}{4\pi} \int_{\partial\Sigma} F^{tr} r^2 \sin\theta d\theta d\phi = \frac{1}{4\pi} \int E^r r^2 d\Omega = Q.$$

9.6 First Law in a Hamiltonian Framework

9.6.1 The gravitational Hamiltonian

The first law for black-holes is simplest to state (but not simplest to prove) when written in terms of the Hamiltonian for gravity. A summary is given in Wald '92, based on work by Wald and Sudarsky. [84, 96] We'll go over the statement of it first, then verify the claims.

Write the expressions for the Hamiltonian and momentum constraints in the form

$$\mathcal{H} = (^3R - K^{ab}K_{ab} + K^2)\sqrt{\gamma}, \quad \mathcal{H}_a = 2D^b(K_{ab} - \gamma_{ab}K)\sqrt{\gamma}. \quad (9.100)$$

We will show: A Hamiltonian governing the spacetime geometry is

$$H_G = \int d^3x [\alpha \mathcal{H} + \beta^a \mathcal{H}_a]. \quad (9.101)$$

the momentum conjugate to γ_{ab} is

$$\pi^{ab} = -(K^{ab} - \gamma^{ab}K)\sqrt{\gamma}. \quad (9.102)$$

Varying the Hamiltonian gives the equations of motion,

$$\dot{\gamma}_{ab} = \frac{\delta H_G}{\delta \pi^{ab}}, \quad \dot{\pi}^{ab} = -\frac{\delta H_G}{\delta \gamma_{ab}}, \quad (9.103)$$

where $(\dot{})$ is the time derivative associated with $t^\alpha = \alpha n^\alpha + \beta^\alpha$, and where α , β^a , h_{ab} and π^{ab} vanish rapidly enough at infinity that there are no surface terms.⁵

The first law involves the opposite case, with t^α the timelike Killing vector and $\beta^\alpha = \phi^a$ the rotational Killing vector of a stationary, axisymmetric vacuum black hole, and of a perturbation to a nearby stationary, axisymmetric black hole. In this case $\dot{\gamma}_{ab} = 0$ and $\dot{\pi}_{ab} = 0$; the field equation is satisfied, so

$$\frac{\delta H_G}{\delta \pi^{ab}} = 0, \quad \frac{\delta H_G}{\delta \gamma_{ab}} = 0;$$

and the constraint equations are satisfied by the perturbed and unperturbed solutions, so $\delta \mathcal{H} = 0$ and $\delta \mathcal{H}_a = 0$. But the surface terms are nonzero and we are left with

$$\text{surface term at } \infty = \text{surface term at horizon},$$

or

$$-\delta M + \Omega_H \delta J = -\kappa \frac{\delta A}{8\pi}. \quad (9.104)$$

To check all this, we will find the gravitational Hamiltonian, show that the surface term at infinity is $-\delta M + \Omega_H \delta J$, and then use our original derivation of the first law to infer that the surface term at the horizon is $-\kappa \delta A / 8\pi$. We first add to the Lagrangian density ${}^4R\sqrt{|g|}$ a total divergence that replaces the

⁵For asymptotically flat spacetimes, the Hamiltonian H_G is not the true Hamiltonian, because the surface terms are nonzero for generic perturbations that preserve asymptotic flatness. Moreover, the value of Hamiltonian for solutions to the field equation should be the mass (energy) M of the spacetime, but H_G vanishes. The two problems are related: Adding a divergence that gets rid of second derivatives in H_G removes the surface term, and the revised Hamiltonian is equal to M for solutions to the field equation (equivalently, for initial data g_{ab} , π^{ab} satisfying the constraint equations). A similar problem arises for the Einstein Lagrangian $R\sqrt{|g|}$, and a first-order Lagrangian is presented in Sect. 10.2 below.

second time derivatives in ${}^4R$ by a kinetic term quadratic in first time derivatives. From the contracted Gauss-Codazzi equation (1.70) in the initial value notes, we have

$${}^3R = \gamma^{\alpha\beta} {}^3R_{\alpha\beta} = \gamma^{\alpha\beta} \gamma^{\gamma\delta} {}^4R_{\alpha\gamma\beta\delta} + K^{\alpha\beta} K_{\alpha\beta} - K^2 \quad (9.105)$$

which we can rewrite as

$${}^4R - 2\nabla_\alpha (n^\gamma \nabla_\gamma n^\alpha - n^\alpha \nabla_\gamma n^\gamma) = {}^3R + K_{\alpha\beta} K^{\alpha\beta} - K^2. \quad (9.106)$$

That is, the right side differs from ${}^4R$ by a total divergence, so the Lagrangian

$$\mathcal{L} = (K_{ab} K^{ab} - K^2 + {}^3R) \alpha \sqrt{\gamma}. \quad (9.107)$$

gives the field equations for variations that vanish fast enough at infinity.

Because ${}^3R$ involves only spatial derivatives of γ_{ab} , the time derivative $\dot{\gamma}_{ab}$ occurs only in the terms involving the extrinsic curvature K_{ab} , so the momentum conjugate to γ_{ab} with respect to $\mathcal{L}$ is given by

$$\pi^{ab} := \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ab}} = \frac{\partial \mathcal{L}}{\partial K_{ab}} \frac{\partial K_{ab}}{\partial \dot{\gamma}_{ab}} = -(K^{ab} - \gamma^{ab} K) \sqrt{\gamma}. \quad (9.108)$$

where we have used Eq. (7.88), namely

$$K_{ab} = -\frac{1}{2\alpha} (\dot{\gamma}_{ab} - \mathcal{L}_{\beta} \gamma_{ab}). \quad (7.88)$$

In the canonical formulation of general relativity, γ_{ab} , π^{ab} , α and β^a are regarded as independent gravitational field variables, the arguments of the geometrical Hamiltonian

$$H_G = \int d^4x (\pi^{ab} \dot{\gamma}_{ab} - \mathcal{L}). \quad (9.109)$$

To write H_G as a function of the four variables γ_{ab} , π^{ab} , α and β^a , we first invert Eq. (9.108) to write K^{ab} in terms of π^{ab} : The trace of Eq. (9.108) gives $\pi = 2K\sqrt{\gamma}$, where $\pi := \gamma_{ab}\pi^{ab}$. From (9.108) itself we then have

$$K_{ab} = -\frac{1}{\sqrt{\gamma}} \left(\pi_{ab} - \frac{1}{2} \gamma_{ab} \pi \right), \quad (9.110)$$

and the Lagrangian density (9.107) has the form

$$\mathcal{L} = \alpha \left[\sqrt{\gamma} {}^3R + \frac{1}{\sqrt{\gamma}} \left(\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) \right], \quad (9.111)$$

Next use (9.110) and (7.105a) to write $\dot{\gamma}_{ab}$ in terms of π_{ab} :

$$\dot{\gamma}_{ab} = \frac{2}{\sqrt{\gamma}} \alpha \left(\pi_{ab} - \frac{1}{2} \gamma_{ab} \pi \right) + D_a \beta_b + D_b \beta_a. \quad (9.112)$$

The Hamiltonian density now takes the form

$$\pi^{ab} \dot{\gamma}_{ab} - \mathcal{L} = \frac{\alpha}{\sqrt{\gamma}} \left(\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) - \alpha \sqrt{\gamma} {}^3R + 2\pi^{ab} D_a \beta_b \quad (9.113)$$

$$= \alpha \mathcal{H} + \beta^a \mathcal{H}_a + D_a (2\pi^{ab} \beta_b), \quad (9.114)$$

where the scalar density

$$\mathcal{H} := \frac{1}{\sqrt{\gamma}} \left(\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) - \sqrt{\gamma} {}^3R, \quad (9.115)$$

and the vector density

$$\mathcal{H}_a := -2D_b \pi_a^b, \quad (9.116)$$

are the Hamiltonian and momentum constraints, written in terms of π_{ab} .

To relate geometrical variables to mass and angular momentum, we need to multiply the geometrical Hamiltonian H_G by $c^4/(16\pi G) = 1/16\pi$ to get the physical Hamiltonian $H := \frac{1}{16\pi} \int d^3x (\pi^{ab} \dot{\gamma}_{ab} - \mathcal{L})$.

The gravitational action $I = \frac{1}{16\pi} \int d^3x \mathcal{L}$ can now be written in the form

$$I = \frac{1}{16\pi} \int d^4x (\pi^{ab} \dot{\gamma}_{ab} - \alpha \mathcal{H} - \beta^a \mathcal{H}_a), \quad (9.117)$$

and the gravitational Hamiltonian is

$$H = \frac{1}{16\pi} \int d^3x [\alpha \mathcal{H} + \beta^a \mathcal{H}_a]. \quad (9.118)$$

Because β^a and α do not appear in $\mathcal{H}$ or $\mathcal{H}_a$, defined as functions of π^{ab} and γ_{ab} , the variational derivatives $\delta I/\delta\alpha$ and $\delta I/\delta\beta^a$ are the constraint equations, the vacuum constraints

$$\mathcal{H} = 0, \quad \mathcal{H}_a = 0, \quad (9.119)$$

and the dynamical Einstein equations are given by 9.103 for variations vanishing rapidly enough at spatial infinity and at the horizon that there are no surface terms.

9.6.2 Surface terms and the first law

In the first law of black hole thermodynamics, our interest is in the opposite case, in surface terms for variations associated with nearby solutions for which $\dot{\gamma}_{ab} = 0 = \dot{\pi}^{ab}$. As described at the beginning of this section, because solutions to the Einstein field equation satisfy $\mathcal{H} = 0$ and $\mathcal{H}_a = 0$, the Hamiltonian H also vanishes, implying the variation $\delta H = 0$. Now δH has the form

$$\delta H = \int d^3x \left[\frac{\delta H}{\delta \gamma_{ab}} \delta \gamma_{ab} + \frac{\delta H}{\delta \pi^{ab}} \delta \pi^{ab} \right] + \text{surface terms at infinity and at the horizon.} \quad (9.120)$$

Because the solutions we are looking at are time independent, we have

$$\frac{\delta H}{\delta \gamma_{ab}} = 0 = \frac{\delta H}{\delta \pi^{ab}}, \quad (9.121)$$

whence

$$0 = \delta H = \text{surface terms at infinity and at the horizon.} \quad (9.122)$$

We consider an evolution along $k^\alpha = t^\alpha + \Omega_H \phi^\alpha$ with Ω_H fixed. In this case,

$$H = \frac{1}{16\pi} \int d^3x [\alpha \mathcal{H} + \Omega_H \phi^a \mathcal{H}_a],$$

and in

$$\alpha\mathcal{H} = \frac{\alpha}{\sqrt{\gamma}} \left(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2 \right) - \alpha\sqrt{\gamma} {}^3R,$$

the first term (the part quadratic in π^{ab}) has no derivatives of γ_{ab} or π^{ab} , so its variation gives no surface terms. As a result, in

$$\delta(\alpha\mathcal{H}) = \delta \left[\frac{\alpha}{\sqrt{\gamma}} \left(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2 \right) - \alpha\sqrt{\gamma} {}^3R \right],$$

the surface term comes only from $\delta(-\alpha\sqrt{\gamma} {}^3R) = \delta(-\sqrt{|g|} {}^3R)$. From Eq. (10.14), we have

$$\delta(-\sqrt{|g|} {}^3R) = \left[(R^{ab} - \frac{1}{2}g^{ab}R)h_{ab} - D_a A^a \right] \sqrt{|g|}, \quad (9.123)$$

where

$$A^a = D_b h^{ab} - D^a {}^3h, \text{ with } {}^3h = \gamma^{ab}h_{ab}. \quad (9.124)$$

The surface term comes from the divergence $-D_a A^a$. Here, at order $1/r$ only $h_{rr} = 2\delta M/r$ is nonzero. Then ${}^3h = h_{rr} + O(r^{-2})$, $h^{rr} = h_{rr} + O(r^{-2})$, and the first surface term is

$$-\frac{1}{16\pi} \int_{\infty} A^a dS_a = -\frac{1}{16\pi} \int_{\infty} \left[\frac{1}{r^2} \partial_r (r^2 h_{rr}) - \partial^r h_{rr} \right] r^2 d\Omega = -\delta M. \quad (9.125)$$

This is the ADM expression for the mass, involving only the spatial metric; it is named after Arnowitt, Deser, and Misner, who introduced it in their famous 3+1 paper [5].

A second surface term comes from $\Omega_H \delta(\phi^a \mathcal{H}_a)$. In fact, $\int_{\Sigma} \phi^a \mathcal{H}_a$ is itself a sum of surface terms at infinity and at the horizon, each proportional to the angular momentum. We convert $\delta(\phi^a \mathcal{H}_a)$ to the form Eq. (9.29b) of J as follows: With n_{α} and $\hat{r}_{\alpha}$ the unit normals to Σ , we have

$$\nabla^{\alpha} \phi^{\beta} dS_{\alpha\beta} = \nabla^{\alpha} \phi^{\beta} \hat{r}_{[\alpha} n_{\beta]} dS = \nabla^{\alpha} \phi^{\beta} \hat{r}_{\alpha} n_{\beta} dS,$$

using in the last equality the antisymmetry of $\nabla^{\alpha} \phi^{\beta}$. Then $\phi^{\beta} n_{\beta} = 0$ implies

$$\begin{aligned} \nabla^{\alpha} \phi^{\beta} \hat{r}_{\alpha} n_{\beta} &= -\phi^{\beta} \hat{r}^{\alpha} \nabla_{\alpha} n_{\beta} = K_{\alpha\beta} \phi^{\beta} \hat{r}^{\alpha} = -\frac{1}{\sqrt{\gamma}} \pi^{ab} \phi_a \hat{r}_b, \\ 8\pi J &= \int_{\infty} \nabla^{\alpha} \phi^{\beta} dS_{\alpha\beta} = -\int_{\infty} \pi^{ab} \phi_a \hat{r}_b dS = -\int_{\infty} \pi^{ab} \phi_a dS_b. \end{aligned} \quad (9.126)$$

We can now find our second surface term as follows: From

$$\frac{1}{16\pi} \delta \int_{\Sigma} \Omega_H \phi^a \mathcal{H}_a = -2\Omega_H \delta \int_{\Sigma} (\phi_a D_b \pi^{ab}) d^3x = -2\Omega_H \delta \int_{\Sigma} D_b (\pi^{ab} \phi_a) d^3x,$$

the surface term at infinity is

$$\frac{1}{16\pi} \left(-2\Omega_H \delta \int_{\infty} \pi^{ab} \phi_a dS_b \right) = \Omega_H \delta J. \quad (9.127)$$

The sum of the two surface terms at infinity, (9.125) and (9.127), is thus

$$\text{surface term at infinity} = -\delta M + \Omega_H \delta J, \quad (9.128)$$

as claimed. Our proof of the first law implies for the sum of terms at the horizon,

$$\text{surface term at the horizon} = -\kappa \frac{\delta A}{8\pi}. \quad (9.129)$$

A direct calculation of the surface term at the horizon would give an alternative proof of the first law,

$$\delta M = \Omega_H \delta J + \kappa \frac{\delta A}{8\pi}, \quad (9.130)$$

but the calculation, first done by Sudarsky and Wald,[84] involves theorems on black holes whose proofs need too much preparation.

Chapter 10

Action, Noether, Surface term

The first section in this chapter, showing that $R\sqrt{|g|}$ is a Lagrangian density for the Einstein equation, is a natural part of a second semester GR course. The subsequent sections are less standard: They use Noether's theorem to introduce conserved quantities in GR, following Katz[51], Chrusciel[26], and Sorkin[82]. To do so, one uses an action that involves only first derivatives of the metric, or, equivalently an action that differs from $\int R\sqrt{|g|}d^4x$ by a surface term involving the extrinsic curvature K [99]. The revised action is needed because $\int R\sqrt{|g|}d^4x$ is not a true action: Variations of the action for which $\delta g_{\alpha\beta}$ vanishes on an initial and final surface– and is well-behaved asymptotically– do not yield the field equations. Instead, they leave a divergence whose integral is a nonvanishing surface term. These last sections may be helpful to readers using work by Hojman, Kuchar & Teitleboim[49] relating surface terms to conserved quantities, or work by York[99] and later Gibbons & Hawking[42] using the first-order action written as the Einstein-Hilbert action with an extrinsic curvature surface term.

10.1 Variation of the gravitational (Hilbert) action

Varying $\int d^n x R\sqrt{|g|}$ in any dimension.

The Einstein-Hilbert action is

$$I = \int \mathcal{L} d^4x = \frac{1}{16\pi} \int R\sqrt{|g|} d^4x. \quad (10.1)$$

Finding the field equations by varying this action is a quick calculation once one realizes that the variation of the Ricci tensor δR_{ab} is a pure divergence

$$\delta R_{ab} = \nabla_c (B^c_{ab}), \quad (10.2)$$

some tensor B^c_{ab} .

We'll do the easy part first, getting the field equations, and then do the harder part, showing that δR_{ab} is a divergence and so contributes only a divergence $\nabla_a A^a$ to the variation of $\mathcal{L}$. For the first law, the field equations vanish and it is A^a that is the important term.

Here's the quick calculation:

$$\begin{aligned} \delta R &= \delta(R_{ab}g^{ab}) = -R_{ab}h^{ab} + g^{ab}\delta R_{ab} \\ &= -R_{ab}h^{ab} + \nabla_c A^c, \end{aligned} \quad (10.3)$$

where $A^c = B^{cb}$. Eq. (10.3) and

$$\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}h$$

immediately give

$$\frac{1}{\sqrt{|g|}}\delta(R\sqrt{|g|}) = -(R^{ab} - \frac{1}{2}g^{ab}R)h_{ab} + \nabla_a A^a, \quad (10.4)$$

as claimed.

Now we find A^a . We start by varying the Riemann tensor, using the commutator

$$R^a{}_{bcd}v^b = [\nabla_c, \nabla_d]v^a, \quad (10.5)$$

for any vector v^a . For fixed v^a , we need to compute

$$\delta R^a{}_{bcd}v^b = \delta[\nabla_c \nabla_d v^a - \nabla_d \nabla_c v^a]. \quad (10.6)$$

The perturbed covariant derivative operator, $\delta\nabla_a$, acts as a tensor¹ $\delta\Gamma^c{}_{ba}$; on a vector v^a , its action is

$$\delta\nabla_c v^a = \delta\Gamma^a{}_{bc}v^b, \quad (10.7)$$

where

$$\delta\Gamma^a{}_{bc} = \frac{1}{2}(\nabla_b h^a{}_c + \nabla_c h^a{}_b - \nabla^a h_{bc}). \quad (10.8)$$

Eq. (10.8) can be derived from the relation

$$0 = \delta(\nabla_a g_{bc}) = \nabla_a h_{bc} - \delta\Gamma^d{}_{ba}g_{dc} - \delta\Gamma^d{}_{ca}g_{bd}, \quad (10.9)$$

via the linear combination

$$0 = \frac{1}{2}g^{ad}[\delta(\nabla_b g_{cd}) + \delta(\nabla_c g_{bd}) - \delta(\nabla_d g_{bc})]. \quad (10.10)$$

To compute $\delta R_{abc}{}^d$, we write

$$\begin{aligned} \delta(\nabla_c \nabla_d v^a) &= -\delta\Gamma^b{}_{cd}\nabla_b v^a + \delta\Gamma^a{}_{bc}\nabla_d v^b + \nabla_c \delta(\nabla_d v^a) \\ &= -\delta\Gamma^b{}_{cd}\nabla_b v^a + \delta\Gamma^a{}_{bc}\nabla_d v^b + \delta\Gamma^a{}_{bd}\nabla_c v^b \\ &\quad + \nabla_c(\delta\Gamma^a{}_{bd})v^b. \end{aligned} \quad (10.11)$$

Because the first three terms on the right side are, together, symmetric in c and d , Eq. (10.6) implies, for all v^a ,

$$\delta R^a{}_{bcd}v^b = (\nabla_c \delta\Gamma^a{}_{bd} - \nabla_d \delta\Gamma^a{}_{bc})v^b,$$

whence

$$\delta R^a{}_{bcd} = \nabla_c \delta\Gamma^a{}_{bd} - \nabla_d \delta\Gamma^a{}_{bc}. \quad (10.12)$$

The perturbation of R_{ab} follows immediately:

$$\delta R_{ab} = \delta R^c{}_{acb} = \nabla_c \delta\Gamma^c{}_{ab} - \nabla_b \delta\Gamma^c{}_{ca}; \quad (10.13)$$

¹Although the Christoffel symbols $\Gamma^\mu{}_{\nu\lambda}$ do not naturally constitute a tensor, the difference between the Christoffel symbols of two different covariant derivative operators do so. To linear order in the difference, this is implied by the explicit covariance of Eq. (10.8). From p. 100 a tensor field is a map of vector fields that is linear under multiplication by scalar fields, and the difference between two covariant derivative operators, $\tilde{\nabla}_a - \nabla_a$ is linear in this sense: $(\tilde{\nabla}_a - \nabla_a)(f\xi^b) = f(\tilde{\nabla}_a - \nabla_a)\xi^b$. This was *Exercise 49*.

and we have

$$\begin{aligned}\delta R &= \delta(R_{ab}g^{ab}) = -R_{ab}h^{ab} + g^{ab}\delta R_{ab} \\ &= -R_{ab}h^{ab} + \nabla_a A^a,\end{aligned}\tag{10.14}$$

where

$$\begin{aligned}A^a &= (g^{bc}\delta\Gamma^a_{bc} - g^{ac}\delta\Gamma^b_{bc}) = (g^{ac}g^{bd} - g^{ab}g^{cd})\nabla_b h_{cd} \quad \text{or} \\ A^a &= \nabla_b h^{ab} - \nabla^a h.\end{aligned}\tag{10.15}$$

To summarize: The variation (10.4) has the form

$$\frac{1}{\sqrt{|g|}}\delta(R\sqrt{|g|}) = -G^{ab}h_{ab} + \nabla_a \nabla_b (h^{ab} - \nabla^a h).\tag{10.16}$$

The remaining sections of this chapter are related to conserved quantities. A generic spacetime has no Killing vectors and no natural definition of energy-momentum conservation. In an asymptotically flat spacetime, however, one can recover the Poincaré group as a symmetry group at spatial infinity. (See, for example, ADM[6], Geroch, Ashtekar-Hansen[9], Chrusciel[28], Sommers). At null infinity (in the limit as one moves away along outward null geodesics), one can again recover a unique group of asymptotic translations and a corresponding conserved energy-momentum of the spacetime, with the mass of the spacetime decreasing by the energy radiated in gravitational (or electromagnetic) waves between successive null hypersurfaces (Bondi, van der Burg, Metzner[17], Sachs[72], Penrose[63]). But the presence of radiation at null infinity leads to weaker conditions on asymptotic fall-off than at spatial infinity because radiation lies within the future light cone of its source. The weaker asymptotic conditions at null infinity lead to a larger group of asymptotic symmetries (the Bondi-Metzner-Sachs or BMS group), in which one cannot uniquely define angular momentum: The asymptotic symmetry group includes a family of Poincaré subgroups, with no preferred choice. The conserved angular momentum in the following discussion, is then appropriate for spatial infinity or for the null infinity of a stationary spacetime, or for an arbitrary choice of an asymptotic Lorentz subgroup of the symmetry group at null infinity for a generic asymptotically flat spacetime.

10.2 First-order action

The presentation in this and the next section is tied to work by Katz[51], Chrusciel[26], and Sorkin[82]; their papers follow earlier work by Hojman, Kuchar & Teitelboim[49] that associates surface terms with conserved quantities. See also Trautman [97, 93] for careful treatments of Noether's theorem.

If a function $I(\phi)$ on a space of fields, $\{\phi\}$, is an action for the field equation $E(\phi) = 0$, then solutions to $E = 0$ are extrema of I for perturbations of the fields that vanish on an initial and final hypersurface. Perturbations $h_{\alpha\beta} = \delta g_{\alpha\beta}$ from one asymptotically flat solution to another do *not* extremize the Hilbert action because the surface terms at spatial infinity are nonzero. This is related to the presence of second derivatives of the metric in R and to the fact that the metric itself is nonzero at infinity. One obtains a true action for asymptotically flat spacetimes by adding a surface term, $-\int_\infty B^a dS_a$, whose

variation cancels the surface term $\int_{\infty} A^{\alpha} dS_{\alpha}$. Equivalently, one adds to the Lagrange density the term $-\nabla_{\alpha} B^{\alpha} \sqrt{-g}$.

We previously found the Hamiltonian form of the action by adding to the Lagrangian density a total divergence that removed the second time derivatives. Here we are removing all second derivatives. In Sect. (10.4), we show that the surface term is proportional to $\int K dS$, with K the extrinsic curvature of the boundary of the region of integration, and this is the way in which York [99] and Gibbons & Hawking[42] write the first-order action.

Denote by ∂_{α} the covariant derivative operator of the flat metric $\eta_{\alpha\beta}$ (introduced previously in *Exercise 48* and *Exercise 49*) and let ξ^{α} be a Killing vector of $\eta_{\alpha\beta}$. Denote by $C^{\alpha}_{\beta\gamma}$ the tensorial form of the Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$,

$$C^{\alpha}_{\beta\gamma} := \frac{1}{2} g^{\alpha\delta} (\partial_{\beta} g_{\gamma\delta} + \partial_{\gamma} g_{\beta\delta} - \partial_{\delta} g_{\beta\gamma}). \quad (10.17)$$

We cancel the second derivatives in $R\sqrt{-g}$ by subtracting the divergence $\partial_{\alpha} \mathcal{B}^{\alpha}$, where

$$\mathcal{B}^{\alpha} = B^{\alpha} \sqrt{-g}, \text{ with} \quad (10.18a)$$

$$B^{\alpha} := g^{\beta\gamma} C^{\alpha}_{\beta\gamma} - g^{\alpha\beta} C^{\gamma}_{\gamma\beta} \quad (10.18b)$$

$$= (g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\beta} g^{\gamma\delta}) \partial_{\beta} g_{\gamma\delta}. \quad (10.18c)$$

From Eq. (10.15),

$$\delta B^{\alpha} = A^{\alpha}, \text{ at leading order in } r^{-1}. \quad (10.19)$$

Then, in the variation of the action $\int \mathcal{L} d^4x$, with

$$\mathcal{L} := R\sqrt{-g} - \partial_{\alpha} \mathcal{B}^{\alpha}, \quad (10.20)$$

the surface terms at spatial infinity vanish.

We now show that $\mathcal{L}$ is quadratic in Γ ,

$$\mathcal{L} := (C_{\alpha\beta\gamma} C^{\beta\alpha\gamma} - C^{\alpha\beta}_{\beta} C^{\beta}_{\beta\alpha}) \sqrt{-g}. \quad (10.21)$$

This is the $\Gamma\Gamma$ part of $R\sqrt{-g}$, but with the opposite sign.

Here's the calculation: From Eq. (2.49), the usual form of R in terms of Γ is

$$R = g^{\gamma\delta} (\partial_{\alpha} C^{\alpha}_{\gamma\delta} - \partial_{\delta} C^{\alpha}_{\gamma\alpha}) + C^{\alpha\beta}_{\beta} C^{\gamma}_{\gamma\alpha} - C_{\alpha\beta\gamma} C^{\beta\alpha\gamma}. \quad (10.22)$$

Using Eqs. (10.18a),(10.18b), we have

$$\partial_{\alpha} \mathcal{B}^{\alpha} = \partial_{\alpha} (g^{\beta\gamma} \sqrt{-g}) C^{\alpha}_{\beta\gamma} - \partial_{\alpha} (g^{\alpha\beta} \sqrt{-g}) C^{\gamma}_{\gamma\beta} + g^{\gamma\delta} (\partial_{\alpha} C^{\alpha}_{\gamma\delta} - \partial_{\delta} C^{\alpha}_{\gamma\alpha}) \sqrt{-g}.$$

We replace derivatives of the metric by C 's, using $0 = \nabla_{\alpha} g_{\beta\gamma} = \partial_{\alpha} g_{\beta\gamma} - C^{\delta}_{\beta\alpha} g_{\delta\gamma} - C^{\delta}_{\alpha\gamma} g_{\beta\delta}$:

$$\partial_{\alpha} (g^{\beta\gamma} \sqrt{-g}) = (-g^{\beta\sigma} g^{\gamma\tau} \partial_{\alpha} g_{\sigma\tau} + \frac{1}{2} g^{\beta\gamma} g^{\sigma\tau} \partial_{\alpha} g_{\sigma\tau}) \sqrt{-g} = (-C^{\beta}_{\alpha}{}^{\gamma} - C^{\gamma}_{\alpha}{}^{\beta} + g^{\beta\gamma} C^{\delta}_{\alpha\delta}) \sqrt{-g}$$

$$\partial_{\alpha} (g^{\alpha\beta} \sqrt{-g}) = -C^{\beta\alpha}_{\alpha} \sqrt{-g}.$$

Then

$$\begin{aligned} \partial_{\alpha} (g^{\beta\gamma} \sqrt{-g}) C^{\alpha}_{\beta\gamma} &= (-2C_{\alpha\beta\gamma} C^{\beta\alpha\gamma} + C^{\alpha\beta}_{\beta} C^{\gamma}_{\gamma\alpha}) \sqrt{-g} \\ -\partial_{\alpha} (g^{\alpha\beta} \sqrt{-g}) C^{\gamma}_{\gamma\beta} &= C^{\beta\alpha}_{\alpha} C^{\gamma}_{\gamma\beta} \sqrt{-g} \\ \partial_{\alpha} \mathcal{B}^{\alpha} &= [-2C_{\alpha\beta\gamma} C^{\beta\alpha\gamma} + 2C^{\alpha\beta}_{\beta} C^{\gamma}_{\gamma\alpha} + g^{\gamma\delta} (\partial_{\alpha} C^{\alpha}_{\gamma\delta} - \partial_{\delta} C^{\alpha}_{\gamma\alpha})] \sqrt{-g} \\ R\sqrt{-g} - \partial_{\alpha} \mathcal{B}^{\alpha} &= (C_{\alpha\beta\gamma} C^{\beta\alpha\gamma} - C^{\alpha\beta}_{\beta} C^{\gamma}_{\gamma\alpha}) \sqrt{-g}. \quad \square \end{aligned}$$

10.3 Noether

Noether's theorem relates symmetries to conserved currents. It can be stated and quickly derived as follows. Let $\mathcal{L}$ be a Lagrangian density for a field or set of fields Φ^I . (For us, $\Phi^I = g_{\alpha\beta}$.) Varying the action $I = \int_M \mathcal{L} d^4x$ gives the field equations $E_I = 0$ satisfied by ϕ^I ; that means

$$\delta \mathcal{L} = E_I \delta \Phi^I + \nabla_\alpha \mathcal{A}^\alpha(\Phi, \delta \Phi). \quad (10.23)$$

Because $\mathcal{A}^\alpha$ is a density, its divergence is independent of derivative operator and could equally well have been written $\partial_\alpha \mathcal{A}^\alpha$.

The symmetries of Noether's theorem are maps of the space $\{\Phi\}$ of fields on M , maps from $\{\Phi\}$ to $\{\Phi\}$, that change $\mathcal{L}$ by a total divergence whether or not Φ satisfies the field equations. Consider a smooth family $\Psi_\lambda : \{\Phi\} \rightarrow \{\Phi\}$ of such maps. Denote by $\hat{\delta}$ the corresponding “infinitesimal symmetry.” That is, $\hat{\delta}\Phi$ is the derivative of Φ along the family of maps: $\hat{\delta}\Phi = \left. \frac{d}{d\lambda} \Psi_\lambda[\Phi] \right|_{\lambda=0}$. Then $\hat{\delta}\mathcal{L}$ is a total divergence,

$$\hat{\delta}\mathcal{L} = \nabla_\alpha \Theta^\alpha. \quad (10.24)$$

When the field equations are satisfied, taking as $\delta\Phi$ the symmetry $\hat{\delta}\Phi$ gives a conserved current

$$\mathcal{J}^\alpha = \mathcal{A}^\alpha(\Phi, \hat{\delta}\Phi) - \Theta^\alpha : \quad (10.25)$$

$$0 = \delta \mathcal{L} - \delta \mathcal{L} = \delta \mathcal{L} - \hat{\delta} \mathcal{L} = \nabla_\alpha \mathcal{J}^\alpha, \quad (10.26)$$

and the associated conserved quantity is the *Noether charge*, the integral of $\mathcal{J}^\alpha$ over a spacelike hypersurface Σ ,

$$Q = \int_\Sigma \mathcal{J}^\alpha d\sigma_\alpha, \quad (10.27)$$

where $d\sigma_\alpha = \eta_{\alpha\beta\gamma\delta} dS^{\beta\gamma\delta}$, with $\eta_{\alpha\beta\gamma\delta}$ the totally antisymmetric tensor density of weight -1 with $\eta_{0123} = 1$ (for any basis).² For Σ a $t = \text{constant}$ surface, $d\sigma_\alpha = \nabla_\alpha t d^3x$.

Example: A massive scalar field on a spacetime with metric $g_{\alpha\beta}$ has Lagrange density

$$\mathcal{L} = -\frac{1}{2} (\nabla^\alpha \Phi \nabla_\alpha \Phi + m^2 \Phi^2) \sqrt{-g}.$$

A Killing vector ξ^α of the metric $g_{\alpha\beta}$ generates a family of diffeos for which $\delta_\xi \Phi = \mathcal{L}_\xi \Phi$ and

$$\delta \mathcal{L} = \mathcal{L}_\xi \mathcal{L} = \nabla_\alpha (\mathcal{L} \xi^\alpha).$$

The diffeos generated by ξ^α are thus symmetries, and $\Theta^\alpha = \mathcal{L} \xi^\alpha$.

We find $\mathcal{A}^\alpha$ by varying $\mathcal{L}$ for a general $\delta\Phi$, writing

$$\delta \mathcal{L} = -(\nabla^\alpha \Phi \nabla_\alpha \delta\Phi + m^2 \Phi \delta\Phi) \sqrt{-g} = (\nabla_\alpha \nabla^\alpha - m^2) \Phi \delta\Phi \sqrt{-g} - \nabla_\alpha (\nabla^\alpha \Phi \delta\Phi \sqrt{-g}).$$

²Like δ_α^β the density $\eta_{\alpha\beta\gamma\delta}$ is *natural*, invariant under diffeos, and hence with vanishing Lie derivatives: $\mathcal{L}_\xi \eta_{\alpha\beta\gamma\delta} = 0$, for any ξ . It does not need a metric for its definition, but once $g_{\alpha\beta}$ is defined,

$$\eta_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta} / \sqrt{-g}.$$

Then $\mathcal{A}^\alpha = -\nabla^\alpha \Phi \delta \Phi \sqrt{-g}$. For $\delta \Phi = \mathcal{L}_\xi \Phi$, we have

$$\begin{aligned}\mathcal{A}^\alpha &= -\nabla^\alpha \Phi \mathcal{L}_\xi \Phi \sqrt{-g}, & \Theta^\alpha &= \xi^\alpha \mathcal{L} \\ \mathcal{J}^\alpha &= -\nabla^\alpha \Phi \mathcal{L}_\xi \Phi \sqrt{-g} - \Theta^\alpha = \left[-\nabla^\alpha \Phi \mathcal{L}_\xi \Phi + \frac{1}{2} \xi^\alpha (\nabla^\beta \Phi \nabla_\beta \Phi + m^2 \Phi^2) \right] \sqrt{-g}.\end{aligned}$$

The corresponding Noether charge is then

$$Q = \int \mathcal{J}^t d^3x = \int \left[-\nabla^t \Phi \mathcal{L}_\xi \Phi + \frac{1}{2} \xi^t (\nabla^\beta \Phi \nabla_\beta \Phi + m^2 \Phi^2) \right] \sqrt{-g} d^3x. \quad (10.28)$$

If the $t = \text{constant}$ hypersurface is invariant under rotations or spatial translations, $-\xi^\alpha \nabla_\alpha t = \xi^t = 0$, and

$$Q = - \int \partial^t \Phi \mathcal{L}_\xi \Phi \sqrt{-g} d^3x = \int \pi \mathcal{L}_\xi \Phi d^3x, \quad (10.29)$$

with $\pi = -\partial^t \Phi \sqrt{-g}$ the momentum density conjugate to Φ , equal to $\partial_t \Phi \sqrt{-g}$ in Minkowski space. Thus if the metric is invariant under translations along ∂_x or is invariant under rotations generated by ∂_ϕ , the scalar field's linear momentum P_x or angular momentum J will be conserved, and they are given by

$$P_x = \int \pi \partial_x \Phi d^3x, \quad J = \int \pi \partial_\phi \Phi d^3x. \quad (10.30)$$

For a time-independent metric, the field's energy is conserved. Because $\xi^\alpha = t^\alpha$ and $\xi^t = t^\alpha \nabla_\alpha t = 1$, we get $\pi \mathcal{L}_\xi \Phi - \mathcal{L} = \pi \dot{\Phi} - \mathcal{L}$ instead of $\pi \dot{\Phi}$ in the integrand of Eq. (10.29):

$$E = \int (\pi \dot{\Phi} - \mathcal{L}) d^3x. \quad (10.31)$$

In particular, in Minkowski space,

$$E = \int \frac{1}{2} \left[\dot{\Phi}^2 + \nabla \Phi \cdot \nabla \Phi + m^2 \Phi^2 \right] \sqrt{-g} d^3x. \quad (10.32)$$

Gravity

We now turn to the conserved quantities associated with the first-order gravitational Lagrangian (10.20), (10.21),

$$\mathcal{L} = R\sqrt{|g|} + \nabla_\alpha \mathcal{B}^\alpha. \quad (10.33)$$

We will find that, for a Killing vector of the flat metric $\eta_{\alpha\beta}$, the associated conserved quantity can be written as a surface integral at infinity³ of the form

$$Q = 2 \int_\infty (\nabla^{[\alpha} \xi^{\beta]} + B^{[\alpha} \xi^{\beta]}) dS_{\alpha\beta}, \quad (10.34)$$

³Do not be confused by the cancellation of surface integrals at infinity from $\mathcal{A}$ and $\delta \mathcal{B}$ in varying the action and the lack of cancellation in the contributions to Q from $\mathcal{A}$ and $\delta \mathcal{B}$. The surface integral at infinity from varying the action involves $\int_\infty \mathcal{A}$, whereas the contribution from $\mathcal{A}$ to Q comes from $\int_\Sigma \mathcal{A}$. We will see that $\mathcal{A}(\mathcal{L}_\xi g)$ is a divergence $\nabla \cdot \nabla \xi$, allowing us to write $\int_\Sigma \mathcal{A} = \int_\infty \nabla \xi$. Similarly, the contribution from $\mathcal{B}$ to δI involves $\int_\infty \delta \mathcal{B}$, whereas the contribution to Q from $\delta \mathcal{B} = \mathcal{L}_\xi \mathcal{B}$ involves $\int_\Sigma \delta \mathcal{B} = \int_\infty \mathcal{B} \xi$.

or, with $\mathcal{L}$ replaced by the correctly normalized Lagrangian density $\frac{1}{16\pi}\mathcal{L}$,

$$Q = \frac{1}{8\pi} \int_{\infty} (\nabla^{[\alpha}\xi^{\beta]} + B^{[\alpha}\xi^{\beta]}) dS_{\alpha\beta}. \quad (10.35)$$

Now $\mathcal{L}$ is constructed from $g_{\alpha\beta}$ and the derivative operator ∂_{α} of the flat metric $\eta_{\alpha\beta}$, so dragging both $g_{\alpha\beta}$ and $\eta_{\alpha\beta}$ by a diffeo just drags $\mathcal{L}$ by the diffeo. For a Killing vector ξ^{α} of $\eta_{\alpha\beta}$, dragging $g_{\alpha\beta}$ alone by the diffeo associated with ξ^{α} is equivalent to dragging both $g_{\alpha\beta}$ and $\eta_{\alpha\beta}$ by the diffeo, because the diffeo doesn't change $\eta_{\alpha\beta}$. Then the change in $\mathcal{L}$ from a symmetry perturbation $h_{\alpha\beta} = \mathcal{L}_{\xi}g_{\alpha\beta}$ is

$$\hat{\delta}\mathcal{L} = \mathcal{L}_{\xi}\mathcal{L} = \nabla_{\alpha}(\xi^{\alpha}\mathcal{L}), \quad (10.36)$$

so

$$\Theta^{\alpha} = \xi^{\alpha}\mathcal{L}. \quad (10.37)$$

Call $\mathcal{A}^{\alpha}$ the divergence associated with $\delta\mathcal{L}$ for an arbitrary $h_{\alpha\beta}$,

$$\delta\mathcal{L} = -G^{\alpha\beta}h_{\alpha\beta} + \nabla_{\alpha}\mathcal{A}^{\alpha}.$$

Using the form (10.33) of $\mathcal{L}$ and Eq.(10.15), we have

$$\begin{aligned} \delta\mathcal{L} &= \delta(R\sqrt{-g}) - \nabla_{\alpha}\delta\mathcal{B}^{\alpha} = [-G^{\alpha\beta}h_{\alpha\beta} + \nabla_{\alpha}\nabla_{\beta}(h^{\alpha\beta} - \nabla^{\alpha}h)]\sqrt{-g} + \nabla_{\alpha}\delta\mathcal{B}^{\alpha}, \\ \mathcal{A}^{\alpha} &= (\nabla_{\beta}h^{\alpha\beta} - \nabla^{\alpha}h)\sqrt{-g} + \delta\mathcal{B}^{\alpha}. \end{aligned}$$

To find the conserved current, we need to evaluate $\mathcal{A}^{\alpha}$ for $h_{\alpha\beta} = \mathcal{L}_{\xi}g_{\alpha\beta}$. Its first term is

$$[\nabla_{\beta}(\nabla^{\alpha}\xi^{\beta} + \nabla^{\beta}\xi^{\alpha}) - 2\nabla^{\alpha}\nabla_{\beta}\xi^{\beta}]\sqrt{-g} = [\nabla_{\beta}(\nabla^{\beta}\xi^{\alpha} - \nabla^{\alpha}\xi^{\beta}) + 2R^{\alpha}_{\beta}\xi^{\beta}]\sqrt{-g}. \quad (10.38)$$

Second term: From its definition (10.18), $\mathcal{B}^{\alpha}$ is constructed from $\eta_{\alpha\beta}$ and $g_{\alpha\beta}$, so $\delta\mathcal{B}^{\alpha} = \mathcal{L}_{\xi}\mathcal{B}^{\alpha}$. The Lie derivative of a vector density can be written as

$$\mathcal{L}_{\xi}\mathcal{B}^{\alpha} = \nabla_{\beta}(\mathcal{B}^{\alpha}\xi^{\beta} - \mathcal{B}^{\beta}\xi^{\alpha}) + \xi^{\alpha}\nabla_{\beta}\mathcal{B}^{\beta},$$

and replacing $\nabla_{\beta}\mathcal{B}^{\beta}$ in the last term by $\mathcal{L} - R\sqrt{-g}$ gives

$$\delta\mathcal{B}^{\alpha} = \nabla_{\beta}(\mathcal{B}^{\alpha}\xi^{\beta} - \mathcal{B}^{\beta}\xi^{\alpha}) + \xi^{\alpha}(\mathcal{L} - R\sqrt{-g}). \quad (10.39)$$

Then

$$\mathcal{A}^{\alpha} = \nabla_{\beta}[(\nabla^{\beta}\xi^{\alpha} - \nabla^{\alpha}\xi^{\beta})\sqrt{-g} + \mathcal{B}^{\alpha}\xi^{\beta} - \mathcal{B}^{\beta}\xi^{\alpha}] + (\xi^{\alpha}\mathcal{L} + 2\xi^{\beta}G^{\alpha}_{\beta})\sqrt{-g}. \quad (10.40)$$

With $\mathcal{B}^{\alpha}$ written as $B^{\alpha}\sqrt{-g}$, Eqs. (10.37) and (10.40) give for the conserved current $\mathcal{J}^{\alpha} = \Theta^{\alpha} - \mathcal{A}^{\alpha}$ the form

$$\mathcal{J}^{\alpha} = \nabla_{\beta}(\nabla^{\beta}\xi^{\alpha} - \nabla^{\alpha}\xi^{\beta} + B^{\alpha}\xi^{\beta} - B^{\beta}\xi^{\alpha})\sqrt{-g} + 2\xi^{\beta}G^{\alpha}_{\beta}\sqrt{-g}. \quad (10.41)$$

When $G_{\alpha\beta} = 0$, the corresponding conserved quantity is given by

$$\begin{aligned} Q &= \int_{\Sigma} \nabla_{\beta} (\nabla^{\beta}\xi^{\alpha} - \nabla^{\alpha}\xi^{\beta} + B^{\alpha}\xi^{\beta} - B^{\beta}\xi^{\alpha}) dS_{\alpha} \\ &= \int_{\infty} (\nabla^{\alpha}\xi^{\beta} - \nabla^{\beta}\xi^{\alpha} + B^{\alpha}\xi^{\beta} - B^{\beta}\xi^{\alpha}) dS_{\alpha\beta}, \end{aligned} \quad (10.42)$$

or, with the normalized action,

$$Q = \frac{1}{8\pi} \int_{\infty} (\nabla^{\alpha} \xi^{\beta} + B^{[\alpha} \xi^{\beta]}) dS_{\alpha\beta}, \quad (10.43)$$

as claimed, with B^{α} given by Eq. (10.18c). The first term, involving $\nabla^{\beta} \xi^{\alpha}$ gives the Komar expression for a conserved quantity associated with ξ^{α} . It is the only contribution when ξ^{α} is a rotational Killing vector, giving for $\xi = \phi$,

$$Q[\phi] = J. \quad (10.44)$$

For $\xi^{\alpha} = t^{\alpha}$ (i.e., an asymptotically timelike Killing vector with asymptotic norm 1),

$$Q[t] = -M. \quad (10.45)$$

The term involving B^{α} contributes to the mass an amount equal to the mass from the Komar expression. This is why, when one writes the mass in terms of the Komar expression alone, one needs a coefficient $1/4\pi$, twice the factor of $1/8\pi$ in the expression for linear and angular momentum. (The sign difference, $-1/4\pi$ instead of $1/4\pi$, is the usual Lorentz signature difference in sign in $\delta I = -E\delta t + p\delta x = p_{\mu}\delta x^{\mu}$.)

10.4 Surface term and extrinsic curvature K

The surface term that arises from varying the Einstein-Hilbert action involves derivatives of the metric on the boundary $\partial\Omega$ of a 4-dimensional region Ω of integration. As York[99] and later Gibbons & Hawking[42] pointed out, for variations of the metric with

$$h_{\alpha\beta} = 0 \text{ on } \partial\Omega, \quad (10.46)$$

the surface term $\int_{\partial\Omega} A^{\alpha} dS_{\alpha}$ is proportional to the integral of δK . The reason is simple: Because $h_{\alpha\beta}$ vanishes on the boundary, so do its tangential derivatives. All that remains is the normal derivative, and the scalar built from the normal derivative of the metric is K .

More formally, on $\partial\Omega$, $h_{\alpha\beta} = 0$ implies

$$\delta n_{\alpha} = 0, \quad \delta n^{\alpha} = 0, \quad \delta \gamma_{\alpha\beta} = 0, \quad \delta \gamma^{\alpha}_{\beta} = 0, \quad (10.47a)$$

$$\gamma^{\alpha\beta} \nabla_{\beta} h_{\gamma\delta} |_{\partial\Omega} = 0, \quad \gamma^{\alpha\beta} \nabla_{\beta} \delta n_{\gamma} |_{\partial\Omega} = 0 \text{ (tangential derivatives vanish)}. \quad (10.47b)$$

For a spacelike part of $\partial\Omega$, from Eq. (7.60), the trace of the extrinsic curvature is

$$K = -\gamma^{\alpha}_{\beta} \nabla_{\alpha} n^{\beta}.$$

Using Eqs. (10.47), we have

$$\begin{aligned} \delta K|_{\Omega} &= -\gamma^{\alpha\beta} (\delta \nabla_{\alpha}) n_{\beta} = \gamma^{\alpha\beta} \delta \Gamma_{\gamma\alpha\beta} n^{\gamma} \\ &= \gamma^{\alpha\beta} \frac{1}{2} (\nabla_{\alpha} h_{\beta\gamma} + \nabla_{\beta} h_{\alpha\gamma} - \nabla_{\gamma} h_{\alpha\beta}) n^{\gamma} \\ &= -\frac{1}{2} \gamma^{\alpha\beta} n^{\gamma} \nabla_{\gamma} h_{\alpha\beta}, \quad \text{by Eq.(10.47b) (tangential derivatives vanish)}. \end{aligned}$$

The integrand of the surface term has the form

$$\begin{aligned}
 A^\alpha n_\alpha &= n^\alpha g^{\beta\gamma} (\nabla_\beta h_{\alpha\gamma} - \nabla_\alpha h_{\beta\gamma}) = n^\alpha (\gamma^{\beta\gamma} - n^\beta n^\gamma) (\nabla_\beta h_{\alpha\gamma} - \nabla_\alpha h_{\beta\gamma}) \\
 &= n^\alpha \gamma^{\beta\gamma} (\nabla_\beta h_{\alpha\gamma} - \nabla_\alpha h_{\beta\gamma}) \quad (\text{by } \alpha\text{-}\beta \text{ antisymmetry}) \\
 &= -n^\alpha \gamma^{\beta\gamma} \nabla_\alpha h_{\beta\gamma} \quad (\text{tangential derivatives vanish}) \\
 &= 2\delta K.
 \end{aligned}$$

For a timelike part of $\partial\Omega$, with the sign convention,

$$K = \gamma^{\alpha\beta} \frac{1}{2} \mathcal{L}_{\mathbf{n}} \gamma_{\alpha\beta} \quad \partial M \text{ timelike}, \quad (10.48)$$

we have

$$A^\alpha n_\alpha = -2\delta K \quad \partial M \text{ timelike}. \quad (10.49)$$

Then

$$\begin{aligned}
 \delta \int R \sqrt{|g|} d^4x &= - \int G^{\alpha\beta} h_{\alpha\beta} \sqrt{|g|} d^4x + \int_{\partial M} A^\alpha n_\alpha dS \\
 &= - \int G^{\alpha\beta} h_{\alpha\beta} \sqrt{|g|} d^4x + 2 \int_{\partial M} e \delta K dS,
 \end{aligned}$$

where

$$e = \begin{cases} 1, & \partial\Omega \text{ spacelike} \\ -1 & \partial\Omega \text{ timelike} \end{cases}.$$

Thus, by adding to the Einstein-Hilbert action a surface term involving the trace of the extrinsic curvature,

$$\hat{I} := \frac{1}{16\pi} \left[\int_{\Omega} R \sqrt{|g|} d^4x - 2 \int_{\partial\Omega} e K dS \right], \quad (10.50)$$

we obtain an action that enforces the field equations for metric perturbations vanishing on $\partial\Omega$:

$$\delta \hat{I} = - \frac{1}{16\pi} \int_{\Omega} G^{\alpha\beta} h_{\alpha\beta} \sqrt{|g|} d^4x. \quad (10.51)$$

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Appendix A

Integration, Forms and Densities

A.1 Forms and densities

The tensor $\epsilon_{a\dots b}$ and the quantity $\sqrt{|g|}$ that appear in the alternative ways of writing an integral are, respectively, an example of a form and a scalar density. As we will see, there is a duality between forms and densities that underlies a equivalence between Stokes's theorem and Gauss's theorem. We define forms and densities, present the duality that relates them, and go on to the corresponding duality relating the integral theorems.

Forms.

Definition. A p -form $\sigma_{a\dots b}$ is an antisymmetric, covariant tensor with p indices.

In particular, a scalar f is a 0-form, a covariant vector A_a is a 1-form, and an antisymmetric 2-index tensor F_{ab} is a 2-form.

Definition. The exterior derivative $d\sigma$ of a p -form σ is the $p + 1$ form

$$(d\sigma)_{ab\dots c} = (p + 1)\nabla_{[a}\sigma_{b\dots c]}. \quad (\text{A.1})$$

The factor $p + 1$ is the number of independent ways of distributing the $p + 1$ indices between ∇ and σ . The antisymmetry implies that $d\sigma$ is independent of the derivative operator; in any chart it has components

$$(d\sigma)_{ij\dots k} = (p + 1)\partial_{[i}\sigma_{j\dots k]}. \quad (\text{A.2})$$

Antisymmetry and the commutativity of partial derivatives imply for any form σ

$$d^2\sigma = 0. \quad (\text{A.3})$$

Lie derivatives and exterior derivatives commute,

$$\mathcal{L}_{\mathbf{v}}d\sigma = d\mathcal{L}_{\mathbf{v}}\sigma, \quad (\text{A.4})$$

and the two derivatives satisfy the Cartan identity,

$$\mathcal{L}_{\mathbf{v}}\sigma = v \cdot (d\sigma) + d(v \cdot \sigma), \text{ where } (v \cdot \sigma)_{a\dots b} := v^c\sigma_{ca\dots b}. \quad (\text{A.5})$$

Each relation can be proved by induction on p (the number of indices of σ). For an n -form in n dimensions, the second relation can be written as

$$\mathcal{L}_{\mathbf{v}}\sigma_{a\dots b} = \nabla_c(\sigma_{a\dots b}v^c), \quad (\text{A.6})$$

with special case

$$\mathcal{L}_{\mathbf{v}}\epsilon_{a\dots b} = \nabla_c(\epsilon_{a\dots b}v^c) = \epsilon_{a\dots b}\nabla_cv^c. \quad (\text{A.7})$$

In n -dimensions, any nonzero n -form is functionally proportional to any other, because each has only one independent component. In particular, because any n -form σ is given by $\sigma_{a\dots b} = f\epsilon_{a\dots b}$ for some scalar f , the integral of an n -form is well-defined, given in any coordinate system by

$$\int_{\Omega} \sigma_{a\dots b} dS^{a\dots b} = \int \sigma_{1\dots n} dx^1 \cdots dx^n.$$

(Again the integral is well defined because a change of coordinates multiplies the value of a p -form by the Jacobian of the transformation.)

Densities.

A *scalar density* $\mathfrak{f}$ (of weight 1), by definition, transforms under a change of coordinates in the same way as one component of an n -form in n dimensions: $\mathfrak{f} \rightarrow \left| \frac{\partial x}{\partial x'} \right| \mathfrak{f}$. Just as one can write any n -form as a multiple of $\epsilon_{a\dots b}$ (once one is given a metric), one can write any scalar density as a scalar multiple of $\sqrt{|g|}$:

$$\mathfrak{f} = f\sqrt{|g|}, \quad (\text{A.8})$$

for some scalar f (namely $\mathfrak{f}/\sqrt{|g|}$). One can analogously introduce vector and tensor densities by transformation laws that differ from those of vectors and tensors by the Jacobian of the transformation: The change of the components of a vector density under a coordinate transformation is given by $j^i \rightarrow \left| \frac{\partial x}{\partial x'} \right| \frac{\partial x'^i}{\partial x^k} j^k$. Again one can write any vector density in the form $j^a = j^a \sqrt{|g|}$, with j^a a vector field.

The Lie derivative of a scalar or tensor density can be deduced from this fact as follows (a metric-free derivation uses the geometric definition of Lie derivative and the fact that the action of a diffeo on a density differs from its action on a tensor by the inverse Jacobian of the diffeo): We have

$$\mathcal{L}_{\mathbf{v}}\sqrt{|g|} = \frac{\partial\sqrt{|g|}}{\partial g_{ab}} \mathcal{L}_{\mathbf{v}}g_{ab} = \frac{1}{2} \frac{1}{\sqrt{|g|}} \frac{\partial g}{\partial g_{ab}} \mathcal{L}_{\mathbf{v}}g_{ab}.$$

Now the coefficient of the component g_{ij} in the determinant g is the minor Δ^{ij} of g_{ij} , and the inverse metric is given by $g^{ij} = \frac{\Delta^{ij}}{g}$. Thus $\frac{\partial g}{\partial g_{ij}} = \Delta^{ij} = gg^{ij}$, and we have

$$\mathcal{L}_{\mathbf{v}}\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}g^{ab}\mathcal{L}_{\mathbf{v}}g_{ab} = \sqrt{|g|}\nabla_av^a = \nabla_a(\sqrt{|g|}v^a). \quad (\text{A.9})$$

Finally, Eqs. (2.70) and (A.9) imply that the Lie derivative of a general scalar density $\mathfrak{f}$ is

$$\mathcal{L}_{\mathbf{v}}\mathfrak{f} = \nabla_a(\mathfrak{f}v^a). \quad (\text{A.10})$$

Duality. A duality between p -forms $\sigma_{a_1\dots a_p}$ and antisymmetric tensor densities $\mathcal{A}^{a_1\dots a_q}$, with $q = n - p$ indices (n the dimension of the space) is given by

$$\mathcal{A}^{a_1\dots a_q} = \frac{1}{p!}\sqrt{|g|}\epsilon^{a_1\dots a_q b_1\dots b_p}\sigma_{b_1\dots b_p}. \quad (\text{A.11})$$

The inverse relation is $\sigma_{a_1\dots a_p} = \pm \frac{1}{q!}|g|^{-1/2}\epsilon_{a_1\dots a_p b_1\dots b_q}\mathcal{A}^{b_1\dots b_q}$, where the sign is positive for a positive definite metric, negative for a metric with Lorentz signature.

The divergence $\nabla \cdot \mathcal{A}$, of an antisymmetric density,

$$(\nabla \cdot \mathcal{A})^{b\dots c} := \nabla_a \mathcal{A}^{ab\dots c}, \quad (\text{A.12})$$

is dual to the exterior derivative $d\sigma$ of the form σ dual to $\mathcal{A}$, and it is independent of the choice of ∇ , having components

$$\nabla_k \mathcal{A}^{ki\dots j} = \partial_k \mathcal{A}^{ki\dots j}. \quad (\text{A.13})$$

In particular, the equation for the divergence of a vector density $\mathcal{A}^a$,

$$\nabla_a \mathcal{A}^a = \partial_i \mathcal{A}^i, \quad (\text{A.14})$$

is equivalent (once one has a metric) to the familiar relation $\nabla_a A^a = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} A^i)$.

The relations (A.3), (A.4), and (A.5) are dual to the relations

$$\nabla \cdot (\nabla \cdot \mathcal{A}) = 0, \quad (\text{A.15})$$

$$\mathcal{L}_v \nabla \cdot \mathcal{A} = \nabla \cdot \mathcal{L}_v \mathcal{A}, \quad (\text{A.16})$$

$$\mathcal{L}_v \mathcal{A} = \nabla \cdot (v \wedge \mathcal{A}) + v \wedge \nabla \cdot \mathcal{A}, \quad (\text{A.17})$$

where $(v \wedge \mathcal{A})^{ab\dots c} := (q+1)v^{[a}\mathcal{A}^{b\dots c]}$.

On an n -dimensional manifold, integrals are naturally defined for n -forms or for scalar densities, which can be said to be *dual* to their corresponding n -forms. That is, the integrals

$$\int \omega_{a\dots b} dS^{a\dots b} = \int \omega_{1\dots n} dx^1 \dots dx^n \quad \text{and} \quad \int f dx^1 \dots dx^n \quad (\text{A.18})$$

are well defined because under a change of coordinates, each integrand is multiplied by the Jacobian of the transformation. The mathematical literature adopts an index-free notation in which the integral of an n -form ω over an n -dimensional region Ω is written $\int_{\Omega} \omega$.

A.2 Stokes's theorem and Gauss's theorem for antisymmetric tensors

The simplest version of Stokes's theorem is its 2-dimensional form, namely Green's theorem:

$$\int_S (\partial_x A_y - \partial_y A_x) dx dy = \int_c (A_x dx + A_y dy),$$

where c is a curve bounding the 2-surface S . The theorem involves the integral over a 2-surface of the antisymmetric tensor $\nabla_a A_b - \nabla_b A_a$. In three dimensions, the tensor is dual to the curl of $\mathbf{A}$: $(\nabla \times \mathbf{A})^a = \epsilon^{abc} \nabla_b A_c$; and Stokes's generalization of Green's theorem can be written in either the form

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_c \mathbf{A} \cdot d\mathbf{l}$$

or in terms of the antisymmetric tensor $\nabla_a A_b - \nabla_b A_a$

$$\int_S (\nabla_a A_b - \nabla_b A_a) dS^{ab} = \int_c A_a dl^a, \quad (\text{A.19})$$

where, for an antisymmetric tensor F_{ab} , $F_{ab}dS^{ab}$ means $F_{12}dx^1dx^2 + F_{23}dx^2dx^3 + F_{31}dx^3dx^1$. Written in this form, the theorem is already correct in a curved spacetime. The reason is that the antisymmetric derivative $\nabla_a A_b - \nabla_b A_a$ has in curved space the same form it has in flat space: As we have seen, its components in any coordinate system are just $\partial_i A_j - \partial_j A_i$.

As a result, the flat-space proof of Green's theorem and Stokes's theorem, based on the Fundamental Theorem of calculus ($\int_a^b f'(x)dx = f(b) - f(a)$), holds in curved space as well: Let S be a coordinate square in a surface of constant x^3 . Then

$$\int_S (\partial_1 A_2 - \partial_2 A_1) dx^1 dx^2 = \int_c (A_1 dx^1 + A_2 dx^2),$$

with the boundary of the square traversed counterclockwise as seen from above the square.

Gauss's theorem,

$$\int_V \nabla_a A^a d^3x = \int_S A^a dS_a, \quad (\text{A.20})$$

with S a surface bounding the volume V again has a simple generalization to curved space. Although the divergence $\nabla_a A^a$ does not have components $\partial_i A^i$, the divergence of $\mathcal{A}^a := A^a \sqrt{g}$ does: $\nabla_a \mathcal{A}^a = \partial_i A^i$. As we have seen, the flat-space proof of Gauss's theorem follows from an integration over a coordinate cube using the fundamental theorem of calculus for the integral over each coordinate.

A.2.1 Integrals on a submanifold

A metric $g_{\alpha\beta}$ on spacetime induces a 3-metric ${}^3g_{ab}$ on a 3-dimensional hypersurface V , its pullback to V . Given the 3-metric g_{ab} , the 3-form ϵ_{abc} on V is uniquely defined up to sign. When one chooses coordinates $\{x^0, x^1, x^2, x^3\}$ on a spacetime, one ordinarily chooses the tensor $\epsilon_{\alpha\beta\gamma\delta}$ for which the coordinate system has positive orientation, for which $\epsilon_{0123} = \sqrt{|g|}$, not $-\sqrt{|g|}$, and one calls the direction of increasing x^0 the future. When $\epsilon_{0123} > 0$ and when x^0 is constant on V and increasing to the future, we will say the corresponding positive orientation V is the one for which $\epsilon_{123} = \sqrt{{}^3g}$.

In discussing hypersurfaces in spacetime it is convenient to avoid the distinction between tensors on V and tensors on M . In the same way that a vector v^a on V is uniquely identified with a vector v^α on M for which $v^\alpha n_\alpha = 0$, we can identify the 3-metric ${}^3g_{ab}$ and 3-form ϵ_{abc} with tensors ${}^3g_{\alpha\beta}$ and $\epsilon_{\alpha\beta\gamma}$ on M that are orthogonal to n_α and whose spatial indices coincide with those of ${}^3g_{ab}$ and ϵ_{abc} .

$${}^3g_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta, \quad \epsilon_{\alpha\beta\gamma} = n^\delta \epsilon_{\delta\alpha\beta\gamma}, \quad (\text{A.21})$$

where n^α is the future pointing unit normal to V , namely

$$n_\mu = \frac{-\delta_\mu^0}{(-g^{00})^{1/2}}. \quad (\text{A.22})$$

We then have $n^0 = \sqrt{-g^{00}} > 0$, and as we show next, the spatial components of $\epsilon_{\alpha\beta\gamma}$ are $\epsilon_{ijk} = \sqrt{{}^3g}$.

It follows immediately from Eq. (A.21) that ${}^3g_{\alpha\beta}$ satisfies the defining conditions ${}^3g_{\alpha\beta} n^\beta = 0$ and ${}^3g_{ab} u^a v^b = {}^3g_{\alpha\beta} u^\alpha v^\beta = g_{\alpha\beta} u^\alpha v^\beta$, for u^α and v^α vectors on V (equivalently ${}^3g_{\alpha\beta}$ has spatial components g_{ij}).

The spatial components of $\epsilon_{\alpha\beta\gamma}$ are

$$\epsilon_{\mu jk} n^\mu = (-g^{00})^{1/2} \epsilon_{0ijk},$$

or $\epsilon_{123} = \sqrt{gg^{00}}$, implying

$$\epsilon_{123} = \sqrt{^3g}. \quad (\text{A.23})$$

Warning: Although contravariant tensors $T^{\alpha\cdots\beta}$ are tensors on V if and only if their 0th components all vanish, the 0th components of covariant tensors $T_{\alpha\cdots\beta}$ on V will not generally vanish, because in general $g_{0i} \neq 0$. The information about the 3-dimensional objects is carried by the spatial components, and the spatial components of the corresponding 3-dimensional and 4-dimensional tensors coincide.

One writes

$$\int dV = \int_V \epsilon_{\alpha\beta\gamma} dS^{\alpha\beta\gamma}, \quad (\text{A.24})$$

for the volume of V , where one thinks of $dS^{\alpha\beta\gamma}$ as antisymmetric, with nonzero components $\pm \frac{1}{3!} dx^i dx^j dx^k$, with the sign depending on whether i, j, k is an even or odd permutation of 1, 2, 3. In coordinates $\{x^0, x^1, x^2, x^3\}$ with V an $x^0 = \text{constant}$ surface,

$$\int dV = \int \sqrt{^3g} d^3x = \int \sqrt{gg^{00}} dx^1 dx^2 dx^3.$$

For any 3-form,

$$\int \omega_{\alpha\beta\gamma} dS^{\alpha\beta\gamma} = \int \omega_{123} dx^1 dx^2 dx^3.$$

If S is a 2-dimensional surface with unit normals n^α and m^α , and we pick the order of the two normals correctly, its area is given in terms of $\epsilon_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} n^\alpha m^\beta$ by

$$\int_S dS = \int_S \epsilon_{\alpha\beta} dS^{\alpha\beta}. \quad (\text{A.25})$$

In a chart x^0, x^1, x^2, x^3 , with S a surface of constant x^0 and x^1 , we have

$$\epsilon_{23} = \sqrt{^2g} = [g_{22}g_{33} - (g_{23})^2]^{1/2},$$

and $\int dS = \int \sqrt{^2g} dx^2 dx^3$.

Example. The magnetic flux through a surface S of radius R in t, r, θ, ϕ coordinates is

$$\int_S F_{ij} dS^{ij} = \int_S F_{\theta\phi} d\theta d\phi.$$

Finally, the integral of a scalar along a curve c is $f = \int_c f dl$, with dl proper length along l ; and the corresponding line integral of a 1-form along c is

$$\int_c A_\alpha dl^\alpha = \int_c A_\alpha t^\alpha dl,$$

where t^α is the unit tangent to c .

Summary of Notation

$$\begin{aligned}
\int f d^4V &= \int f \epsilon_{\alpha\beta\gamma\delta} dS^{\alpha\beta\gamma\delta} = \int f \sqrt{|g|} d^4x \\
\int \omega &= \int \omega_{\alpha\beta\gamma\delta} dS^{\alpha\beta\gamma\delta} = \int \omega_{0123} d^4x \\
\int f dV &= \int f \epsilon_{\alpha\beta\gamma} dS^{\alpha\beta\gamma} = \int f \sqrt{^3g} d^3x = \int_{x^0 \text{ constant}} f \sqrt{g g^{00}} d^3x \\
\int \omega &= \int \omega_{\alpha\beta\gamma} dS^{\alpha\beta\gamma} = \int_{x^0 \text{ constant}} \omega_{123} dx^1 dx^2 dx^3 \\
\int f dS &= \int f \epsilon_{\alpha\beta} dS^{\alpha\beta} = \int f \sqrt{^2g} d^2x = \int_{x^0, x^1 \text{ constant}} f [g_{22}g_{33} - (g_{23})^2]^{1/2} d^2x \\
\int \omega &= \int \omega_{\alpha\beta} dS^{\alpha\beta} = \int_{x^0, x^1 \text{ constant}} \omega_{23} dx^2 dx^3 \\
\int_c f dl &= \int_{x^0, x^1, x^2 \text{ constant}} f \sqrt{g_{33}} dx^3 \\
\int_c \omega &= \int_c \omega_\alpha dl^\alpha = \int_{x^0, x^1, x^2 \text{ constant}} \omega_3 dx^3.
\end{aligned}$$

Example: The integral form of baryon mass conservation $\nabla_\alpha(\rho u^\alpha) = 0$ is

$$\begin{aligned}
0 &= \int_\Omega \nabla_\alpha(\rho u^\alpha) d^4V = \int_{\partial\Omega} \rho u^\alpha dS_\alpha \\
&= \int_{V_2} \rho u^\alpha dS_\alpha + \int_{V_1} \rho u^\alpha dS_\alpha.
\end{aligned}$$

Here the fluid is taken to have finite spatial extent, and the spacetime region Ω is bounded by the initial and final spacelike hypersurfaces V_1 and V_2 . In a coordinate system for which V_1 and V_2 are surfaces of constant t , with t increasing to the future, we have $dS_\mu = \nabla_\mu t \sqrt{|g|} d^3x = \delta_\mu^t \sqrt{|g|} d^3x$ on V_2 , $dS_\mu = -\delta_\mu^t \sqrt{|g|} d^3x$ on V_1 , and

$$\int_\Omega \nabla_\alpha(\rho u^\alpha) d^4V = \int_{V_2} \rho u^t \sqrt{|g|} d^3x - \int_{V_1} \rho u^t \sqrt{|g|} d^3x. \quad (\text{A.26})$$

If, on a slicing of spacetime one chooses on each hypersurface V a surface element dS_α along $+\nabla_\alpha t$, the conservation law is then

$$M_0 = \int_V \rho u^\alpha dS_\alpha = \text{constant}. \quad (\text{A.27})$$

Note that the fact that one can write the conserved quantity associated with a current j^α in the form,

$$\int_V j^\alpha dS_\alpha = \int_V j^t \sqrt{|g|} d^3x,$$

means that there is no need to introduce n_α and $\sqrt{^3g}$ to evaluate the integral. This fact is *essential* if one is evaluating an integral $\int j^\alpha dS_\alpha$ over a null surface, where there is no unit normal. The flux of energy or of baryons across the horizon of a Schwarzschild black hole, for example, can be computed in Eddington-Finkelstein or Kruskal coordinates: In ingoing Eddington-Finkelstein coordinates v, r, θ, ϕ , the horizon is a surface of constant r , and we have

$$\int j^\alpha dS_\alpha = \int j^r \sqrt{|g|} dv d\theta d\phi.$$

Generalized divergence theorem

The key to Gauss's theorem is the fact that the divergence of a vector density has the form $\partial_i \left(\sqrt{|g|} A^i \right)$ or $\partial_i \mathcal{A}^i$. This is true of any q -index antisymmetric tensor density, $\mathcal{A}^{a \cdots b} = A^{a \cdots b} \sqrt{|g|}$, and an analogous theorem holds. Because the text uses only the case of a two-index antisymmetric tensor, the electromagnetic field $F^{\alpha\beta}$, and because the way one extends the proof will be clear, we will give the generalization in detail for this case. The theorem now relates an integral over an $n - 1$ -dimensional submanifold S of M to an integral over its $n - 2$ -dimensional boundary ∂S :

$$\int_S \nabla_b A^{ab} dS_a = \int_{\partial S} A^{ab} dS_{ab}, \quad (\text{A.28})$$

where the sign of dS_a and the meaning of dS_{ab} are defined as follows.

Let $x^1, \dots, x^n$ be a positively oriented chart on a subset of M for which x^1 is constant on S , and with x^2 constant on ∂S and increasing outward. Choose the sign of dS_a by $dS_a = \epsilon_{ab \cdots c} dS^{b \cdots c}$ or, equivalently, by requiring $dS_a = \nabla_a x^1 \sqrt{|g|} dx^2 \cdots dx^n$. The volume element dS_{ab} is similarly chosen to satisfy, in our oriented chart,

$$dS_{ab} = \nabla_{[a} x^1 \nabla_{b]} x^2 \sqrt{|g|} dx^3 \cdots dx^n. \quad (\text{A.29})$$

Then in the coordinates' domain,

$$\int_S \nabla_b A^{ab} dS_a = \int_S \nabla_b (A^{ab} \nabla_a x^1 \sqrt{|g|}) dx^2 \cdots dx^n. \quad (\text{A.30})$$

But the last integrand is just the divergence of the vector density $\tilde{\mathcal{A}}^b = A^{ab} \nabla_a x^1 \sqrt{|g|}$, and we have already proved Gauss's law for this case:

$$\int_S \nabla_b \tilde{\mathcal{A}}^b \sqrt{|g|} dx^2 \cdots dx^n = \int_{\partial S} \tilde{\mathcal{A}}^b d\tilde{S}_b, \quad (\text{A.31})$$

where, in each chart, $d\tilde{S}_b = \nabla_b x^2 \sqrt{|g|} dx^3 \cdots dx^n$. Finally, $\tilde{\mathcal{A}}^b d\tilde{S}_b = A^{ab} dS_{ab}$, whence Eq. (A.31) is identical to the generalized divergence theorem, Eq. (A.28).

Example (Electric charge). Let V be a ball containing a charge Q . The 4-dimensional form of Gauss's law relates the charge $Q = \int_V j^\alpha dS_\alpha$ in V to the electric flux $\int F^{\alpha\beta} dS_{\alpha\beta}$ through the 2-dimensional surface of V :

$$Q = \int_V j^\alpha dS_\alpha = \frac{1}{4\pi} \int_V \nabla_\beta F^{\alpha\beta} dS_\alpha = \frac{1}{4\pi} \int_{\partial V} F^{\alpha\beta} dS_{\alpha\beta}. \quad (\text{A.32})$$

Pick positively oriented coordinates t, r, θ, ϕ for which V is a $t = \text{constant}$ surface and ∂V an $r = \text{constant}$ surface. Then

$$Q = \frac{1}{4\pi} \int F^{tr} \sqrt{|g|} d\theta d\phi. \quad (\text{A.33})$$

Flat space:

$$F^{tr} = E^r = \frac{Q}{r^2} \implies \int_V j^\alpha dS_\alpha = \frac{1}{4\pi} \int \left(\frac{Q}{r^2} \right) (r^2 \sin \theta) d\theta d\phi = \frac{Q}{4\pi} \int d\Omega = Q.$$

For a q -index antisymmetric tensor $A^{a\cdots bc}$, the generalized divergence theorem takes the form $\int_S \nabla_c A^{a\cdots bc} dS_{a\cdots b} = \int_{\partial S} A^{a\cdots bc} dS_{a\cdots bc}$.

A.2.2 Stokes's theorem

The divergence $\nabla_a A^a$ of a vector on an n -dimensional manifold is dual to the exterior derivative $(d\omega)_{ab\cdots c} = n \nabla_{[a} \omega_{b\cdots c]}$ of the $(n-1)$ -form ω dual to A^a . That is, with

$$\omega_{b\cdots c} := A^a \epsilon_{ab\cdots c}, \quad (\text{A.34})$$

we have

$$\epsilon_{ab\cdots c} \nabla_d A^d = (d\omega)_{ab\cdots c}. \quad (\text{A.35})$$

Because both sides of the equation are n -forms, one need only check one component: $n \nabla_{[1} \omega_{2\cdots n]} = \nabla_1 (A^d \epsilon_{d2\cdots n}) - \nabla_2 (A^d \epsilon_{d13\cdots n}) - \cdots - \nabla_n (A^d \epsilon_{d2\cdots n-11}) = \epsilon_{12\cdots n} \nabla_d A^d$. The corresponding dual of the generalized divergence theorem above is

Stokes's Theorem. Let ω be an $n-1$ -form on an n -dimensional manifold S with boundary ∂S . Then

$$\int_S d\omega = \int_{\partial S} \omega. \quad (\text{A.36a})$$

In index notation,

$$\int_S (d\omega)_{ab\cdots c} dS^{ab\cdots c} = \int_{\partial S} \omega_{b\cdots c} dS^{b\cdots c}. \quad (\text{A.36b})$$

The theorem implicitly assumes an orientation for ∂S obtained from that of S by requiring that, if $x^1, \dots, x^n$ is a positively oriented chart on S with ∂S a surface of constant x^1 and x^1 increasing outward, then $x^2, \dots, x^n$ is a positively oriented chart for ∂S .

Proof. This dual of Gauss's theorem follows quickly from Eq. (2.112), in the form

$$\int_S \nabla_d A^d \epsilon_{ab\cdots c} dS^{ab\cdots c} = \int_{\partial S} A^a dS_a. \quad (\text{A.37})$$

Define A^a by Eq. (A.34), and note that, with the orientation chosen above, $dS_a = \epsilon_{ab\cdots c} dS^{b\cdots c}$. Then

$$\int_S \nabla_d A^d \epsilon_{ab\cdots c} dS^{ab\cdots c} = \int_{\partial S} A^a \epsilon_{ab\cdots c} dS^{b\cdots c}, \quad (\text{A.38})$$

and, from Eqs. (A.34) and (A.35), the result follows:

$$\int_S (d\omega)_{ab\cdots c} dS^{ab\cdots c} = \int_{\partial S} \omega_{b\cdots c} dS^{b\cdots c}. \quad \square \quad (\text{A.39})$$

Example: Stokes's theorem in three dimensions.

Let A_a be a 3-vector. $(dA)_{ab} = \nabla_a A_b - \nabla_b A_a$

$$\int_S (\nabla_a A_b - \nabla_b A_a) dS^{ab} = \int_c A_a dl^a,$$

c the curve bounding S . As noted at the beginning of this section, this is equivalent to the usual form of Stokes's theorem in vector calculus:

$$\int_S \vec{\nabla} \times \vec{A} \cdot d\vec{S} = \int_S (\epsilon^{abc} \nabla_b A_c) \epsilon_{ade} dS^{de} = \int_S (\nabla_a A_b - \nabla_b A_a) dS^{ab} = \int_c \vec{A} \cdot d\vec{l}.$$

A.2.3 Diffeomorphism invariance

The usual invariance of an integral under a coordinate transformation has as its active equivalent the invariance of an integral under a diffeo. Let $\omega_{a\dots b}$ be an n -form on an n -dimensional volume V . With $\psi(V)$ the image of V and $\psi\omega_{a\dots b}$ the dragged n -form, the invariance relation is

$$\int_{\psi(V)} \psi\omega = \int_V \omega. \quad (\text{A.40})$$

As in the case of coordinate transformations, invariance under diffeos follows from the fact that the components of $\omega_{a\dots b}$ change by a Jacobian. Intuitively, the invariance follows from the equivalence of diffeo-related tensors on diffeo-related domains.

We'll now use a corollary for a family of diffeos, ψ_λ , to obtain a relation between conservation of vorticity and conservation of circulation:

Let ξ^a be the generator of the family ψ_λ . Then

$$\frac{d}{d\lambda} \int_{\psi_\lambda(V)} \omega = \int_V \mathcal{L}_\xi \omega. \quad (\text{A.41})$$

The proof is immediate from Eq. (2.102), starting from Eq. (A.40) in the form

$$\int_{\psi_\lambda(V)} \omega = \int_V \psi_{-\lambda} \omega: \quad \frac{d}{d\lambda} \int_{\psi_\lambda(V)} \omega = \int_V \frac{d}{d\lambda} \psi_{-\lambda} \omega = \int_V \mathcal{L}_\xi \omega.$$

The equations of motion for a fluid imply that the flow conserves the vorticity,

$$\omega_{\alpha\beta} := \nabla_\alpha(hu_\beta) - \nabla_\beta(hu_\alpha), \quad (\text{A.42})$$

where $h = \frac{\epsilon+p}{\rho}$, is the fluid's enthalpy (ρ_0 is the baryon mass density). The differential conservation law is

$$\mathcal{L}_u \omega_{\alpha\beta} = 0. \quad (\text{A.43})$$

Then, denoting by ψ_τ the family of diffeos describing the flow of a fluid with 4-velocity u^α , we have

$$\begin{aligned} 0 &= \int_S \mathcal{L}_u \omega_{\alpha\beta} dS^{\alpha\beta} = \int_c \mathcal{L}_u(h u_\alpha) dl^\alpha \\ &= \frac{d}{d\tau} \int_{c_\tau} h u_\alpha dl^\alpha, \end{aligned} \quad (\text{A.44})$$

where Stokes's theorem was used to obtain the second equality and Eq. (A.41) was used in the last equality. That is, the line integral,

$$\int_{c_\tau} h u_\alpha dl^\alpha = \int_{c_\tau} \frac{\epsilon+p}{\rho} u_\alpha dl^\alpha, \quad (\text{A.45})$$

(the *circulation* of the flow along a closed curve) is independent of τ , conserved by the fluid flow. In the Newtonian limit, the conserved circulation is just

$$\int_{c_t} v_a dl^a; \quad (\text{A.46})$$

that is, the line integral of the fluid velocity along a curve embedded in (moving with) the fluid is constant. The corresponding differential law, the Newtonian limit of Eq. (A.43), is

$$(\partial_t + \mathcal{L}_v) \nabla_{[a} v_{b]} = 0. \quad (\text{A.47})$$

A.3 Cartan calculus

This section uses the first three equations in Appendix A.1, defining forms and the exterior derivative but nothing else from that section.

Definition. The *wedge product* $\sigma \wedge \tau$ of a p -form $\sigma_{a\dots b}$ and a q -form $\tau_{c\dots d}$ is the $(p+q)$ -form

$$(\sigma \wedge \tau)_{a\dots bc\dots d} := \frac{(p+q)!}{p!q!} \sigma_{[a\dots b} \tau_{c\dots d]}. \quad (\text{A.48})$$

The factor $\binom{p+q}{q} = \frac{(p+q)!}{p!q!}$ gives a number of terms equal to the number of independent ways of assigning p indices to σ and q indices to τ .

Examples. If σ and τ are 1-forms,

$$(\sigma \wedge \tau)_{ab} = \sigma_a \tau_b - \sigma_b \tau_a; \quad (\text{A.49})$$

equivalently,

$$(\sigma \wedge \tau)(u, v) = \sigma(u)\tau(v) - \sigma(v)\tau(u).$$

If σ is a 1-form and τ is a 2-form,

$$(\sigma \wedge \tau)_{abc} = \sigma_a \tau_{bc} + \sigma_b \tau_{ca} + \sigma_c \tau_{ab}. \quad (\text{A.50})$$

A.3.1 The Riemann Tensor in a General Frame

Cartan's formalism provides an efficient way to calculate the Riemann tensor, picking out a minimal set of connection coefficients to be evaluated. One uses an orthonormal frame, and we will begin by finding the components of the Riemann tensor with respect to an arbitrary frame, e_i . Starting from the definition

$$\frac{1}{2} R^a_{bcd} v^b = \nabla_{[c} \nabla_{d]} v^a, \quad (\text{A.51})$$

we have,

$$\nabla_{[c} \nabla_{d]} e_i^a = \frac{1}{2} R^a_{bcd} e_i^b.$$

Then

$$\begin{aligned} \nabla_k \nabla_l e_j^a &= e_k^c e_l^d \nabla_c \nabla_d e_j^a = e_k^c \nabla_c (e_l^d \nabla_d e_j^a) - (e_k^c \nabla_c e_l^d) \nabla_d e_j^a \\ &= \nabla_k (\Gamma_{jl}^m e_m^a) - \Gamma_{lk}^m e_m^d \nabla_d e_j^a \\ &= e_k \Gamma_{jl}^m e_m^a + \Gamma_{jl}^m \Gamma_{mk}^i e_i^a - \Gamma_{lk}^m \Gamma_{jm}^i e_i^a. \end{aligned}$$

Finally, antisymmetrizing on k and l , we have

$$R^i_{jkl} = ([\nabla_k, \nabla_l] e_j)^i = e_k \Gamma_{jl}^i - e_l \Gamma_{jk}^i + \Gamma_{mk}^i \Gamma_{jl}^m - \Gamma_{ml}^i \Gamma_{jk}^m + 2 \Gamma_{jm}^i \Gamma_{kl}^m \quad (\text{A.52})$$

Here $e_k f$ means the derivative of a scalar f along e_k : That is, $e_k f \equiv e_k^a \nabla_a f$. Eq. (A.52) agrees with the usual expression for the coordinate components when $e_i = \partial_i$, for then $\Gamma_{[kl]}^m = 0$.

A.3.2 Cartan's structure equations

A definition of the exterior derivative of a 1-form that does not make use of a covariant derivative operator is

$$d\sigma(u, v) = u \cdot d[\sigma(v)] - v \cdot d[\sigma(u)] - \sigma([u, v]), \quad (\text{A.53})$$

where df means the gradient of the scalar f ; in the equation, $\sigma(v)$ and $\sigma(u)$ are, of course, scalars. (A similar equation can be written down for the exterior derivative of a p -form.) Here's a check that the two equations for $d\sigma$ are equivalent:

$$\begin{aligned} d\sigma(u, v) &= (\nabla_a \sigma_b - \nabla_b \sigma_a) u^a v^b = u^a \nabla_a (\sigma_b v^b) - v^b \nabla_b (\sigma_a u^a) - u^a (\nabla_a v^b) \sigma_b + v^b (\nabla_b u^a) \sigma_a \\ &= u \cdot d[\sigma(v)] - v \cdot d[\sigma(u)] - \sigma([u, v]). \end{aligned}$$

Calculations repeatedly use the identity, for any form ω and scalar f ,

$$d(f\omega) = df \wedge \omega + f d\omega. \quad (\text{A.54})$$

This follows immediately from the Leibnitz rule and our initial definitions of d and $\wedge$, (A.1) and (A.48):

$$d(f\omega)_{ab\dots c} = (p+1)\nabla_{[a}(f\omega_{b\dots c]}) = (p+1)\nabla_{[a}f \omega_{b\dots c]} + f(p+1)\nabla_{[a}\omega_{b\dots c]} = (df \wedge \omega + f d\omega)_{ab\dots c}$$

In particular, for two scalars f and g , because $d^2g = 0$, the identity is

$$d(fdg) = df \wedge dg. \quad (\text{A.55})$$

The connection coefficients (Christoffel symbols) will be calculated by taking exterior derivatives of the dual basis vectors ω^i . Because $\omega^i(e_j) = \delta_j^i$ and the components δ_j^i are constants, Eq. (A.53) gives

$$(d\omega^i)_{jk} = d\omega^i(e_j, e_k) = -\omega^i([e_j, e_k]) = -c_{jk}^i. \quad (\text{A.56})$$

Then Γ will be obtained below from $d\omega$ using Eq. (2.37), namely

$$\Gamma_{ijk} = \frac{1}{2}(c_{kij} - c_{jki} - c_{ikj}). \quad (\text{A.57})$$

Cartan defines connection 1-forms

$$\omega^i_j = \Gamma^i_{jk} \omega^k, \quad (\text{A.58})$$

or $(\omega^i_j)_k = \Gamma^i_{jk}$. Equivalently,

$$\nabla_a e_j^b = \omega^i_{ja} e_i^b. \quad (\text{A.59})$$

Then for an orthonormal frame, or any frame in which the metric has constant components,

$$\omega_{ij} = -\omega_{ji}. \quad (\text{A.60})$$

Check: $\omega_{ija} = e_{ib} \nabla_a e_j^b = \nabla_a(g_{ij}) - e_j^b \nabla_a e_{ib} = -e_{jb} \nabla_a e_i^b = -\omega_{jia}$.

Cartan's first structure equation relates ω^i_j to $d\omega^i$:

$$d\omega^i = -\omega^i_j \wedge \omega^j. \quad (\text{A.61})$$

Proof: Using Eqs. (A.56), (2.28), and then $(\omega^i)_j = \delta_j^i$, we have

$$(d\omega^i)_{kl} = -c_{kl}^i = \Gamma^i_{kl} - \Gamma^i_{lk} = (\omega^i_j)_l (\omega^j)_k - (\omega^i_j)_k (\omega^j)_l = -(\omega^i_j \wedge \omega^j)_{kl}.$$

Eq. (A.57) is essentially the inverse of this last equation, allowing one to write ω_{ij} in terms of $d\omega^i$. After rearranging the c 's and using Eq. (A.56) to replace c_{ijk} by $-(d\omega^i)_{jk} \equiv -d\omega_{i,jk}$, Eq. (A.57) becomes

$$\omega_{ij,k} = \frac{1}{2}(d\omega_{i,jk} + d\omega_{j,ki} - d\omega_{k,ij}), \quad (\text{A.62})$$

Here $\omega_{ij,k}$ means $(\omega_{ij})_k$. Once one has computed the connection coefficients using (A.61) or (A.62), the Riemann tensor is found from Cartan's second structure equation, namely

$$\mathcal{R}^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j. \quad (\text{A.63})$$

where $\mathcal{R}^i_j$ is a 2-form defined by $\mathcal{R}^i_j := R^i_{jab}$.

Proof of (A.63): We must check that (A.63) agrees with (A.52). Now from Eq. (A.53), the first term on the right of (A.63) is

$$\begin{aligned} d\omega^i_j(e_k, e_l) &= e_k[\omega^i_j(e_l)] - e_l[\omega^i_j(e_k)] - \omega^i_j([e_k, e_l]) \\ &= e_k\Gamma^i_{jl} - e_l\Gamma^i_{jk} - \omega^i_{j,m}c^m_{kl} \\ &= e_k\Gamma^i_{jl} - e_l\Gamma^i_{jk} + 2\Gamma^i_{jm}\Gamma^m_{[kl]}, \end{aligned}$$

where we used Eq. (2.28) in the last line. The second term in (A.63) is

$$\omega^i_m \wedge \omega^m_j(e_k, e_l) = \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk}.$$

Thus (A.63) implies

$$\begin{aligned} R^i_{jkl} &= (d\omega^i_j + \omega^i_m \wedge \omega^m_j)(e_k, e_l) \\ &= e_k\Gamma^i_{jl} - e_l\Gamma^i_{jk} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk} + 2\Gamma^i_{jm}\Gamma^m_{[kl]}, \end{aligned}$$

in agreement with (A.52).

Exercise. Write down the independent components of the Riemann tensor in an orthonormal frame for the 2-sphere with metric

$$ds^2 = a^2(d\theta^2 + \sin^2 \theta d\phi^2) = a^2 d\Omega^2,$$

using Cartan calculus. Find the Ricci scalar.

Exercise with solution. Compute the independent components of the Riemann tensor for a time-independent spherically symmetric (Schwarzschild) metric of the form

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\lambda} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (\text{A.64})$$

with Φ and λ functions of r .

Solution on next page.

Exercise. Repeat the calculation of the first exercise above for the 3-sphere:

$$ds^2 = a^2(d\chi^2 + \sin^2 \chi d\Omega^2)$$

and check that $R_{abcd} = a^{-2}(g_{ac}g_{bd} - g_{ad}g_{bc})$.

Exercise. Find the independent components of the Riemann and Ricci tensor in orthonormal frames for the FRW spacetimes with metric

$$ds^2 = -dt^2 + a(t)^2(d\chi^2 + \sin^2 \chi d\Omega^2)$$

and

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2).$$

Solution to Schwarzschild exercise.

When terms in the solution below are written in gray, it means they vanish.

Spherical symmetry and the fact that the frame on the sphere is orthonormal means that Riemann, Ricci, and Einstein tensor components are invariant under the index replacement $2 \leftrightarrow 3$: $R_{1212} = R_{2323}$.

$$\omega^0 = e^\Phi dt, \quad \omega^1 = e^\lambda dr, \quad \omega^2 = r d\theta, \quad \omega^3 = r \sin \theta d\phi.$$

$$d\omega^0 = \Phi' e^\Phi dr \wedge dt = -\Phi' e^{-\lambda} \omega^0 \wedge \omega^1, \quad d\omega^1 = 0 \quad \text{That is, } d[f(r)dr] = f'(r)dr \wedge dr = 0.$$

$$d\omega^2 = dr \wedge d\theta = \frac{1}{r} e^{-\lambda} \omega^1 \wedge \omega^2$$

$$d\omega^3 = d(r \sin \theta) \wedge d\phi = \sin \theta dr \wedge d\phi + r d(\sin \theta) \wedge d\phi = \frac{e^{-\lambda}}{r} \omega^1 \wedge \omega^3 + \frac{\cot \theta}{r} e^{-\lambda} \omega^2 \wedge \omega^3$$

Notice now that $(d\omega^\lambda)_{\mu\nu}$ is nonzero only if the set of three indices is 001, 212, 313 or 323, with λ the first index. That means Γ is nonzero only for these four sets of indices.

$$\omega_{01} = \Gamma_{010} \omega^0 = \frac{1}{2} (d\omega_{0.10} + d\omega_{1.00} - d\omega_{0.01}) \omega^0 = -\Phi' e^{-\lambda} \omega^0 = -\Phi' e^{\Phi-\lambda} dt \quad (\Gamma_{011} = \Gamma_{012} = \Gamma_{013} = 0).$$

$$\omega_{02} = \omega_{03} = 0 \quad (\text{No nonvanishing } d\omega_{\lambda,\mu\nu} \text{ has indices } 02 \text{ or } 03).$$

$$\omega_{12} = \Gamma_{122} \omega^2 = \frac{1}{2} (d\omega_{1.22} + d\omega_{2.21} - d\omega_{2.12}) \omega^3 = -\frac{1}{r} e^{-\lambda} \omega^2 = -e^{-\lambda} d\theta$$

$$\omega_{13} = -\frac{1}{r} e^{-\lambda} \omega^3 = -e^\lambda \sin \theta d\phi$$

$$\omega_{23} = \Gamma_{233} \omega^3 = \frac{1}{2} (d\omega_{2.33} + d\omega_{3.32} - d\omega_{3.23}) \omega^3 = -\frac{1}{r} \cot \theta \omega^3 = -\cos \theta d\phi.$$

$$\mathcal{R}^0_1 = d\omega^0_1 + \omega^0_\mu \wedge \omega^\mu_1 = d(\Phi' e^{\Phi-\lambda} dt) = -[\Phi'' + \Phi'(\Phi' - \lambda')] e^{-2\lambda} \omega^0 \wedge \omega^1$$

$$\mathcal{R}^0_2 = d\omega^0_2 + \omega^0_\mu \wedge \omega^\mu_2 = \omega^0_1 \wedge \omega^1_2 = (\Phi' e^{-\lambda} \omega^0) \wedge \left(-\frac{1}{r} e^{-\lambda} \omega^2 \right) = -\frac{1}{r} \Phi' e^{-2\lambda} \omega^0 \wedge \omega^2$$

$$\mathcal{R}^0_3 = -\frac{1}{r} \Phi' e^{-2\lambda} \omega^0 \wedge \omega^3 \quad \text{by spherical symmetry}$$

$$\mathcal{R}^1_2 = d\omega^1_2 + \omega^1_\mu \wedge \omega^\mu_2 = d(-e^{-\lambda} d\theta) = \frac{1}{r} \lambda' e^{-2\lambda} \omega^1 \wedge \omega^2$$

$$\mathcal{R}^1_3 = \frac{1}{r} \lambda' e^{-2\lambda} \omega^1 \wedge \omega^3 \quad \text{by spherical symmetry}$$

$$\mathcal{R}^2_3 = d\omega^2_3 + \omega^2_1 \wedge \omega^1_3 = d(-\cos \theta d\phi) + \left(\frac{1}{r} e^{-\lambda} \omega^2 \right) \wedge \left(-\frac{1}{r} e^{-\lambda} \omega^3 \right) = \frac{1}{r^2} (1 - e^{-2\lambda}) \omega^2 \wedge \omega^3.$$

$$R_{0101} = [\Phi'' + \Phi'(\Phi' - \lambda')] e^{-2\lambda} \quad R_{2323} = \frac{1}{r^2} (1 - e^{-2\lambda})$$

$$R_{0202} = R_{0303} = \frac{1}{r} \Phi' e^{-2\lambda} \quad R_{1212} = R_{1313} = \frac{1}{r} \lambda' e^{-2\lambda}.$$