Energy-momentum Tensor:Noether vs Hilbert

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We revisit the old problem of the energy-momentum tensor in general relativistic field theories. On the basis of the general covariance we derive a simple equation for the Hilbert and Noether energy-momentum tensors for the scalar and electromagnetic field theories. We see that the two definitions of energy-momentum tensors coincide and identify the Noether current if the space-time has the Killing vector. Relation to the Wald entropy is also briefly discussed.

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I. INTRODUCTION

The issue of the energy-momentum tensor (EMT) in classical field theory has a long history since 1940's as we can see in the major textbooks on general relativity [1]. For more detailed historical accounts, see for example a thesis [2]. Roughly there are two definitions of EMT. The one is the Hilbert EMT defined by $T^{\mu\nu}_{(H)} = \frac{2}{\sqrt{-g}}\frac{\partial L_M}{\partial g_{\mu\nu}}$ where L_M is the Lagrangian density for matter and $g_{\mu\nu}$ is the metric tensor. The $T^{\mu\nu}_{(H)}$ is the source of gravitational field through the Einstein equation

$$R^{\mu\nu} - \frac{R}{2}g^{\mu\nu} = 8\pi G T^{\mu\nu}_{(H)}. \tag{1}$$

On the other hand, the EMT represents the energy density and the momentum flow of matter $j^\mu = T^{\mu}_{(N)\alpha}\xi^\alpha$ which conserve if the spacetime has a symmetry for the infinitesimal transformation of the Lagrangian density generated by the infinitesimal vector ξ^μ called the Killing vector, as dictated by the Noether theorem on the relation between symmetry and conservation law. Through the form of Noether current j^μ, we can read out the Noether EMT $T^{\mu\nu}_{(N)}$.

One can naturally ask whether the two definitions of the energy-momentum tensor coincide, i.e.,$T^{\mu\nu}_{(N)} \stackrel{?}{=} T^{\mu\nu}_{(H)}$. One might quickly check the case of scalar field theory with the Lagrangian density $L_M = -\frac{1}{2}\sqrt{-g}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi$ to find that $T^{\mu\nu}_{(H)} = \frac{2}{\sqrt{-g}}\frac{\partial L_M}{\partial g_{\mu\nu}} = g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\,\partial_\beta\phi + g^{\mu\nu}L_M = -\frac{\partial L_M}{\partial\partial_\mu\phi}g^{\nu\beta}\partial_\beta\phi + g^{\mu\nu}L_M = T^{\mu\nu}_{(N)}$ and hastily assume that the equality $T^{\mu\nu}_{(N)} = T^{\mu\nu}_{(H)}$ holds in general. However, this is not always the case.

Historically this issue was first raised in the context of special relativity. Belinfante and Rosenfeld proposed a procedure to "correct" Noether's energy-momentum tensor $T^{\mu\nu}_{(N)}$ so that it becomes a symmetric tensor [3].

More precisely, consider the matter Lagrangian L_M for the electromagnetic field A_μ given by

$$L_M = -\frac{1}{4}\eta^{\mu\nu}\eta^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}, \tag{2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3}$$

where $\eta^{\mu\nu}$ is the Minkowski metric tensor. The energy-momentum tensor of Noether is given by

$$T_{(N)}{}^{\mu}{}_{\nu} = -\frac{\partial L_M}{\partial\partial_\mu A_\alpha}\partial_\nu A_\alpha + \delta^\mu{}_\nu L_M = F^{\mu\alpha}\partial_\nu A_\alpha + \delta^\mu{}_\nu L_M, \tag{4}$$

which is, with the index ν raised, neither symmetric nor gauge invariant. The Belinfante-Rosenfeld procedure is to add an extra term,

$$T_{(N)}{}^{\mu}{}_{\nu} + \partial_\rho\Psi^{\mu\rho}{}_{\nu} = F^{\mu\alpha}(\partial_\nu A_\alpha - \partial_\alpha A_\nu) + \delta^\mu{}_\nu L_M = F^{\mu\alpha}F_{\nu\alpha} + \delta^\mu{}_\nu L_M, \tag{5}$$

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where $\Psi^{\mu\rho}{}_{\nu} = -F^{\mu\rho}A_{\nu}$ is a tensor which is antisymmetric with respect to μ and ρ and the vacuum Maxwell equation $\partial_\rho F^{\mu\rho} = 0$ is used in the first equality. However, the Belinfante-Rosenfeld procedure cannot be generalized to an arbitrary curved spacetime, since $D_\mu D_\rho \Psi^{\mu\rho}{}_{\nu} \neq 0$ in general.

The total Lagrangian density L_T consists of the gravity part L_G and the matter part L_M,

$$L_T = L_G + L_M. \tag{6}$$

We assume that $L_G = L_G(g_{\mu\nu}, R^{\mu}{}_{\nu\rho\sigma})$ and the scalar matter $L_M = L_M(g^{\mu\nu}, \phi, D_\mu\phi)$ or the matter composed of the electromagnetic field $L_M(g^{\mu\nu}, F_{\mu\nu})$, where $R^{\mu}{}_{\nu\rho\sigma}$ is the Riemann tensor and $F_{\mu\nu}$ is the electromagnetic field strength. The gravitational and matter parts L_G and L_M are separately diffeomorphism covariant. In what follows we mainly deal with the matter part of the Lagrangian density and discuss the total Lagrangian density in Sect 4 to find that the present problem is related to the Wald entropy [5].

In the present work, we will show that the general coordinate covariance of L_M for the matter Lagrangian density implies that

$$T^{\mu\nu}_{(H)} D_\mu \xi_\nu = D_\mu (T^{\mu\nu}_{(N)} \xi_\nu) \tag{7}$$

for an arbitrary infinitesimal coordinate change $x^\mu \to x'^\mu = x^\mu - \xi^\mu(x)$.

The equation (7) rewritten in the form

$$(T^{\mu\nu}_{(H)} - T^{\mu\nu}_{(N)}) D_\mu \xi_\nu = D_\mu T^{\mu\nu}_{(N)} \cdot \xi_\nu \tag{8}$$

should hold for an arbitrary vector ξ^μ and its derivative $D_\mu \xi^\nu$, which are linearly independent. Thefore, the two definitions of EMT coincide

$$T^{\mu\nu}_{(H)} = T^{\mu\nu}_{(N)} \tag{9}$$

and that the EMT conserves

$$D_\mu T^{\mu\nu}_{(N)} = 0. \tag{10}$$

Especially if the spacetime has a symmetry, the vector $\xi^\mu(x)$ satisfies the Killing equation,$D_\mu\xi_\nu + D_\nu\xi_\mu = 0$. From the right hand side of equation (7) we see that the current

$$j^\mu = T^{\mu\nu}_{(N)} \xi_\nu \tag{11}$$

conserves

$$D_\mu j^\mu = 0 \tag{12}$$

for the Killing vector ξ_ν. Note that we do not need an explicit form of the Lagrangian.

At this stage we clearly see why the two definitions of EMT are related. The combination of the infinitesimal coordinate change through the metric tensor and matter fields ensures the general covariance of the Lagrangian.

In Sect 2 we shall prove (7) on the basis of the general covariance for a scalar field theory. Sect 3 is for the electromagnetic field. In Sect 4, we point out that our covariance argument leads to the Wald entropy if we include the gravity part L_G. The final section Sect 5 is devoted to summary and discussions.

II. SCALAR FIELD

In this section we present a derivation of (7) in the scalar field theory as a prototype.

Consider a Lagrangian density $L_M(g^{\mu\nu}, \phi, D_\mu\phi)$ for a scalar field $\phi(x)$ in a curved spacetime endowed with a metric tensor $g^{\mu\nu}$. A simple example is

$$L_M(g^{\mu\nu}, \phi, D_\mu\phi) = \sqrt{-g}\big[-\frac{1}{2} g^{\mu\nu} D_\mu\phi\, D_\nu\phi - \frac{m^2}{2}\phi^2 \big] \tag{13}$$

though we do not use this explicit expression for the Lagrangian $L_M(g^{\mu\nu}, \phi, D_\mu\phi)$ in the proof of (7).

Consider an infinitesimal coordinate transformation $\delta : x^\mu \to x'^\mu = x^\mu - \xi^\mu(x)$ and the corresponding Lie variation,

$$\begin{aligned}
&\delta\phi(x) = \phi'(x) - \phi(x) = \xi^\alpha D_\alpha \phi(x), \\
&\delta D_\mu\phi(x) = (D_\mu\phi)'(x) - \phi(x) = D_\mu\xi^\alpha \cdot D_\alpha\phi(x) + \xi^\alpha D_\alpha D_\mu \phi(x) = D_\mu(\xi^\alpha D_\alpha\phi(x)), \\
&\delta g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = D_\mu\xi_\nu + D_\nu\xi_\mu.
\end{aligned} \tag{14}$$

The Lagrangian density as a whole is a scalar density so that it should transform as

$$\delta L_M = D_\mu(L_M \xi^\mu), \tag{15}$$

which has an alternative expression

$$\delta L_M = \frac{\partial L_M}{\partial g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\partial L_M}{\partial \phi}\delta\phi + \frac{\partial L_M}{\partial D_\mu \phi}\delta D_\mu \phi, \tag{16}$$

where the Lie variations $\delta g^{\mu\nu}, \delta\phi, \delta D_\mu\phi(x)$ are given in (14). Then the right hand side is

$$\begin{aligned}
\delta L_M \quad &= +\frac{\partial L_M}{\partial g_{\mu\nu}}(D_\mu \xi_\nu + D_\nu \xi_\mu) + \frac{\partial L_M}{\partial \phi}\xi^\alpha D_\alpha \phi + \frac{\partial L_M}{\partial D_\mu \phi} D_\mu(\xi^\alpha D_\alpha \phi) \\
&= +\sqrt{-g} T_{(H)}{}^{\mu\nu} D_\mu \xi_\nu + \big[\frac{\partial L_M}{\partial \phi} - D_\mu \frac{\partial L_M}{\partial D_\mu \phi}\big]\xi^\alpha D_\alpha \phi + D_\mu \big[\frac{\partial L_M}{\partial D_\mu \phi}\xi^\alpha D_\alpha \phi\big] \qquad (17) \\
&= +\sqrt{-g} T_{(H)}{}^{\mu\nu} D_\mu \xi_\nu + D_\mu \big[\frac{\partial L_M}{\partial D_\mu \phi}\xi^\alpha D_\alpha \phi\big]. \\
&= \sqrt{-g}\Big[T_{(H)}{}^{\mu\nu} D_\mu \xi_\nu - D_\mu\big[T_{(N)}{}^\mu{}_\nu \xi^\nu\big]\Big] + D_\mu\big[L_M \xi^\mu\big] \ , \qquad (18)
\end{aligned}$$

where $\sqrt{-g}T_{(N)}{}^{\mu\nu} = -\frac{\partial L_M}{\partial D_\mu \phi} D^\nu \phi + g^{\mu\nu} L_M$. The second term of (17) vanishes due to the Euler-Lagrange equation $\frac{\partial L_M}{\partial \phi} - D_\mu \frac{\partial L_M}{\partial D_\mu \phi} = \frac{\partial L_M}{\partial \phi} - \partial_\mu \frac{\partial L_M}{\partial \partial_\mu \phi} = 0$ for Lagrangian density L_M. And the third term of (18) on the right hand side cancels with (15). Therefore, we obtain

$$T^{\mu\nu}_{(H)} D_\mu \xi_\nu = D_\mu(T^{\mu\nu}_{(N)} \xi_\nu). \tag{19}$$

As explained in the introduction, this means the equality of the Hilbert and Noether EMT and the conservation law of Noether energy momentum tensor which holds on-shell as indicated by the use of the Euler-Lagrange equation in the proof. For the particular Lagrangian (13) we have a familiar form of the energy-momentum tensor using $\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} = -\frac{1}{2}(g^{\alpha\mu}g^{\beta\nu} + g^{\beta\mu}g^{\alpha\nu})$ as,

$$T_{(H)}{}_{\mu\nu} = T_{(N)}{}_{\mu\nu} = D_\mu \phi \ D_\nu \phi + \frac{1}{\sqrt{-g}} L_M g_{\mu\nu}. \tag{20}$$

We can understand the equation (7) as the cancellation of the effect of the variation through the metric tensor and that of the fields to preserve the general coordinate covariance of the Lagrangian density.

III. ELECTROMAGNETIC FIELD

As explained in the introduction the issue started with the EMT for the electromagnetic field. We are going to show the key equation (7) assuming that the Lagrangian $L(g^{\mu\nu}, D_\mu A_\nu)$ is given as a function of the field strength $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$ and the metric tensor $g^{\mu\nu}$. Explicitly,

$$L_M = -\frac{\sqrt{-g}}{4} g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}. \tag{21}$$

We follow the same procedure to the case of scalar field especially for the matter Lagrangian density L_M, only composed of metric and antisymmetric tensor. Starting from Lie variation of L_M, we can extend the Lie variation to covariant expression in the second line, due to antisymmetry of $\frac{\partial L_M}{\partial \partial_\mu A_\nu}$ as,

$$\begin{aligned}
\delta L_M \ &= \ \frac{\partial L_M}{\partial g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\partial L_M}{\partial \partial_\mu A_\nu} \delta \partial_\mu A_\nu \ , \\
&= \ \frac{\partial L_M}{\partial g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\partial L_M}{\partial D_\mu A_\nu} \delta D_\mu A_\nu \ , \qquad (22)
\end{aligned}$$

where $\delta g_{\mu\nu}(x) = D_\mu \xi_\nu + D_\nu \xi_\mu$ and $\delta D_\mu A_\nu = D_\mu \xi^\alpha \cdot D_\alpha A_\nu + D_\nu \xi^\alpha \cdot D_\mu A_\alpha + \xi^\alpha D_\alpha D_\mu A_\nu$.

Noting that $\frac{\partial L_M}{\partial D_\mu A_\nu} = 2\frac{\partial L_M}{\partial F_{\mu\nu}} = -2\frac{\partial L_M}{\partial F_{\nu\mu}} = -\frac{\partial L_M}{\partial D_\nu A_\mu}$ is antisymmetric with respect to the indices μ and ν, we see that

$$\begin{aligned}\frac{\partial L_M}{\partial D_\mu A_\nu}\delta D_\mu A_\nu &= \frac{\partial L_M}{\partial D_\mu A_\nu}\big[D_\mu\xi^\alpha\cdot D_\alpha A_\nu + D_\nu\xi^\alpha\cdot D_\mu A_\alpha + \xi^\alpha D_\alpha D_\mu A_\nu\big]\\ &= \frac{\partial L_M}{\partial D_\mu A_\nu}\big[D_\mu\xi^\alpha\cdot D_\alpha A_\nu - D_\mu\xi^\alpha\cdot D_\nu A_\alpha + \xi^\alpha D_\alpha D_\mu A_\nu\big].\end{aligned}$$

A bit of manipulation using antisymmetry of contracted indices μ,ν shows that the second factor on the right hand side becomes

$$\begin{aligned}&D_\mu\xi^\alpha\cdot D_\alpha A_\nu - D_\mu\xi^\alpha\cdot D_\nu A_\alpha + \xi^\alpha D_\alpha D_\mu A_\nu\\ &= D_\mu\xi^\alpha\big[D_\alpha A_\nu - D_\nu A_\alpha\big] + \xi^\alpha\big[D_\alpha, D_\mu\big]A_\nu + \xi^\alpha D_\mu D_\alpha A_\nu\\ &= D_\mu\xi^\alpha\big[D_\alpha A_\nu - D_\nu A_\alpha\big] + \xi^\alpha\big[D_\alpha, D_\mu\big]A_\nu + \xi^\alpha D_\mu\big[D_\alpha A_\nu - D_\nu A_\alpha\big] + \xi^\alpha D_\mu D_\nu A_\alpha\ .\end{aligned}$$

Under the contracted antisymmetric indices μ,ν, the last terms $\xi^\alpha D_\mu D_\nu A_\alpha$ can be made antisymmetric. Then the second and the last terms yield Riemann tensor $R^\beta_{\ \nu\alpha\mu}$, satisfying $R^\beta_{\ \nu\alpha\mu} = -R^\beta_{\ \nu\mu\alpha} = R^\beta_{\ \mu\nu\alpha}$ in the understanding that the indices μ and ν are antisymmetrized. Therefore, we see that

$$\begin{aligned}&\frac{\partial L_M}{\partial D_\mu A_\nu}\delta D_\mu A_\nu\\ &= \frac{\partial L_M}{\partial D_\mu A_\nu}\Big[D_\mu\big[\xi^\alpha(D_\alpha A_\nu - D_\nu A_\alpha)\big] + \xi^\alpha\big[D_\alpha, D_\mu\big]A_\nu + \frac{1}{2}\xi^\alpha\big[D_\mu, D_\nu\big]A_\alpha\Big]\\ &= \frac{\partial L_M}{\partial D_\mu A_\nu}\Big[D_\mu\big[\xi^\alpha(D_\alpha A_\nu - D_\nu A_\alpha)\big] - \xi^\alpha\big[R^\beta_{\ \nu\alpha\mu}A_\beta + \frac{1}{2}R^\beta_{\ \alpha\mu\nu}A_\beta\big]\Big]\\ &= \frac{\partial L_M}{\partial D_\mu A_\nu}\Big[D_\mu(\xi^\alpha(D_\alpha A_\nu - D_\nu A_\alpha)) - \frac{1}{2}\xi^\alpha(R^\beta_{\ \nu\alpha\mu} + R^\beta_{\ \mu\nu\alpha} + R^\beta_{\ \alpha\mu\nu})A_\beta\Big]\\ &= \frac{\partial L_M}{\partial D_\mu A_\nu}D_\mu\big[\xi^\alpha(D_\alpha A_\nu - D_\nu A_\alpha)\big].\end{aligned}$$

The last term in the third line vanishes because of the cyclic identity of the Riemann tensor $R^\beta_{\ \nu\alpha\mu} + R^\beta_{\ \mu\nu\alpha} + R^\beta_{\ \alpha\mu\nu} = 0$. To go back to the equation for the covariance, using $\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} = -\frac{1}{2}(g^{\alpha\mu}g^{\beta\nu} + g^{\beta\mu}g^{\alpha\nu})$, we obtain

$$\begin{aligned}D_\mu(L\delta^\mu{}_\alpha\xi^\alpha) &= \delta L = \frac{\partial L_M}{\partial g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\partial L_M}{\partial D_\mu A_\nu}\delta D_\mu A_\nu = +\sqrt{-g}T_{(H)}{}^{\mu\nu}D_\mu\xi_\nu + \frac{\partial L_M}{\partial D_\mu A_\nu}D_\mu\big[\xi^\alpha(D_\alpha A_\nu - D_\nu A_\alpha)\big]\\ &= +\sqrt{-g}T_{(H)}{}^{\mu\nu}D_\mu\xi_\nu + D_\mu\big[\frac{\partial L_M}{\partial D_\mu A_\nu}\xi^\alpha(D_\alpha A_\nu - D_\nu A_\alpha)\big] - \left(D_\mu\frac{\partial L_M}{\partial D_\mu A_\nu}\right)\xi^\alpha(D_\alpha A_\nu - D_\nu A_\alpha).\end{aligned}$$

The last term vanishes due to the field equation $D_\mu\frac{\partial L_M}{\partial D_\mu A_\nu} = 0$. Finally we obtain

$$T^{\mu\nu}_{(H)}D_\mu\xi_\nu = D_\mu(T^{\mu\nu}_{(N)}\xi_\nu), \tag{23}$$

where the Noether EMT is given by

$$T^\mu_{(N)\alpha} = \frac{1}{\sqrt{-g}}\Big[-\frac{\partial L_M}{\partial D_\mu A_\nu}(D_\alpha A_\nu - D_\nu A_\alpha) + L_M\delta^\mu{}_\alpha\Big]. \tag{24}$$

The equation (5) is recovered without via Belinfante-Rosenfeld procedure.

IV. WALD ENTROPY

As shown by Wald [5], the gravity part of the Lagrangian density L_G gives the entropy part of the black hole analog of the first law of thermodynamics on the basis of general coordinate covariance. In this section we would like to recapitulate the derivation of the Wald entropy from the angle of the previous section. For simplicity, we consider the Einstein-Hilbert Lagrangian density :$L_G(g_{\mu\nu}, R^\mu{}_{\nu\rho\sigma}) = \frac{\sqrt{-g}R}{16\pi G}$ as well as scalar or electro-magnetic fields matter

Lagrangian density : $L_M(g^{\mu\nu},\phi,D_\mu\phi)$ or $L_M(g^{\mu\nu},D_\mu A_\nu)$, respectively. Note that the Noether enrgy-momentum tensor is respectively given by $\overset{(N)}{T_M}{}^{\mu}{}_{\nu}\,(g^{\mu\nu},\phi,D_\mu\phi)=-\frac{\partial L_M}{\partial D_\mu\phi}D_\nu\phi+\delta^\mu{}_\nu L_M$ and $\overset{(N)}{T_M}{}^{\mu}{}_{\nu}\,(g^{\mu\nu},D_\mu A_\nu)=-\frac{\partial L_M}{\partial D_\mu A_\alpha}(D_\nu A_\alpha-D_\alpha A_\nu)+\delta^\mu{}_\nu L_M$.

For the either scalar or electromagnetic matter field, the Lie variation δ of the total Lagrangian density $L_T=L_G+L_M$, the equation (6) induced by the infinitesimal vector ξ^μ yields

$$\begin{aligned}D_\mu(L_G\xi^\mu)+D_\mu(L_M\xi^\mu)=\delta L_T &= \frac{\partial L_G}{\partial R^\mu{}_{\nu\rho\sigma}}\delta R^\mu{}_{\nu\rho\sigma}+\frac{\partial(L_G+L_M)}{\partial g_{\mu\nu}}\delta g_{\mu\nu}+D_\mu\big((-\overset{(N)}{T_M}{}^{\mu}{}_{\nu}+L_M\delta^\mu{}_\nu)\xi^\mu\big)\\ &= \frac{\partial L_G}{\partial R^\mu{}_{\nu\rho\sigma}}\delta R^\mu{}_{\nu\rho\sigma}+\frac{\partial(L_G+L_M)}{\partial g_{\mu\nu}}\delta g_{\mu\nu}-D_\mu(\overset{(N)}{T_M}{}^{\mu}{}_{\nu}\xi^\nu)+D_\mu(L_M\xi^\mu),\end{aligned}$$

where we have used the previous result on the matter part Lie variation. The second term vanishes due to the Einstein equation so that we obtain

$$D_\mu(L_G\xi^\mu)+D_\mu(L_M\xi^\mu)=\frac{\partial L_G}{\partial R^\mu{}_{\nu\rho\sigma}}\delta R^\mu{}_{\nu\rho\sigma}-D_\mu(\overset{(N)}{T_M}{}^{\mu}{}_{\nu}\,\xi^\nu)+D_\mu(L_M\xi^\mu). \tag{25}$$

Combining the Lie variation [1] for Riemann tensor

$$\delta R^\mu{}_{\nu\rho\sigma}=D_\rho\delta\Gamma^\mu_{\nu\sigma}-D_\sigma\delta\Gamma^\mu_{\nu\rho}, \tag{26}$$
$$\delta\Gamma^\mu_{\nu\sigma}=\frac{1}{2}g^{\mu\beta}(D_\nu\delta g_{\beta\sigma}+D_\sigma\delta g_{\beta\nu}-D_\beta\delta g_{\nu\sigma}), \tag{27}$$
$$\delta g_{\beta\sigma}=D_\beta\xi_\sigma+D_\sigma\xi_\beta, \tag{28}$$

and

$$\frac{\partial L_G}{\partial R^\mu{}_{\nu\rho\sigma}}=\frac{\sqrt{-g}}{16\pi G}\frac{1}{2}\big[\delta^\rho{}_\mu g^{\nu\sigma}-\delta^\sigma{}_\mu g^{\nu\rho}\big], \tag{29}$$

we see that

$$\begin{aligned}\frac{\partial L_G}{\partial R^\mu{}_{\nu\rho\sigma}}\delta R^\mu{}_{\nu\rho\sigma} &= \frac{\sqrt{-g}}{16\pi G}D_\mu\big[D_\nu D^\nu\xi^\mu+D_\nu D^\mu\xi^\nu-2D^\mu D_\nu\xi^\nu\big]\\ &= \frac{\sqrt{-g}}{16\pi G}D_\mu\big[-D_\nu(D^\mu\xi^\nu-D^\nu\xi^\mu)+2(D_\nu D^\mu-D^\mu D^\nu)\xi_\nu\big]\\ &= \frac{\sqrt{-g}}{16\pi G}D_\mu\big[-D_\nu(D^\mu\xi^\nu-D^\nu\xi^\mu)+2R^\mu{}_\lambda\xi^\lambda\big]\end{aligned} \tag{30}$$

From equation (25),we obtain

$$\frac{\sqrt{-g}}{16\pi G}D_\mu(R\xi^\mu)=\frac{\sqrt{-g}}{16\pi G}D_\mu\big[-D_\nu(D^\mu\xi^\nu-D^\nu\xi^\mu)+2R^\mu{}_\lambda\xi^\lambda\big]-D_\mu(\overset{(N)}{T_M}{}^{\mu}{}_{\nu}\,\xi^\nu), \tag{31}$$

Using the Einstein equation $R^\mu{}_\lambda-\frac{1}{2}\delta^\mu{}_\lambda R=8\pi G\,\overset{(H)}{T_M}{}^{\mu}{}_{\nu}$, we obtain the conservation law for the Noether current ${j_G}^\mu$ including gravitational entropy, as

$$D_\mu\big[{j_G}^\mu\big]=0, \tag{32}$$
$${j_G}^\mu:=-\frac{\sqrt{-g}}{16\pi G}D_\nu(D^\mu\xi^\nu-D^\nu\xi^\mu)+(\overset{(H)}{T_M}{}^{\mu}{}_{\nu}-\overset{(N)}{T_M}{}^{\mu}{}_{\nu})\xi^\nu, \tag{33}$$

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[1] Although the results of Lie variation $\delta R^\mu{}_{\nu\rho\sigma}=R'^\mu{}_{\nu\rho\sigma}(x)-R^\mu{}_{\nu\rho\sigma}(x)$ looks different from the results of Eq.(26), we can show that Eq.(26) and the Lie variation $\delta R^\mu{}_{\nu\rho\sigma}$ are equivalent by using two kinds of Bianchi identities multiplied by $D^\beta\xi_\nu$ or ξ^β, such as $\big(R^\mu{}_{\beta\rho\sigma}+R^\mu{}_{\rho\sigma\beta}+R^\mu{}_{\sigma\beta\rho}\big)D^\beta\xi_\nu=0$ and $g^{\mu\gamma}\big(D_\nu R_{\gamma\beta\rho\sigma}+D_\gamma R_{\beta\nu\rho\sigma}\big)\xi^\beta=-g^{\mu\gamma}\xi^\beta\big(D_\beta R_{\nu\gamma\rho\sigma}\big)$. More conviniently we may start with (26) to obtain the Lie variation result $\delta R^\mu{}_{\nu\rho\sigma}=-R^\beta{}_{\nu\rho\sigma}D_\beta\xi^\mu+R^\mu{}_{\beta\rho\sigma}D_\nu\xi^\beta+R^\mu{}_{\nu\beta\sigma}D_\sigma\xi^\beta+R^\mu{}_{\nu\rho\beta}D_\sigma\xi^\beta+\xi^\beta D_\beta R^\mu{}_{\nu\rho\sigma}$.

The second term in the Noether current (33) is the ordinary energy flow which are cancelled for the well-known matter (scalar field or electromagnetic field), while the first term represents the heat flow proportional to the entropy, which will show up in the case that the Killing horizon exists after the integration over spacetime.

If we want to see the correspondence to the equation (7), we may rewrite equation (32) as

$$\frac{\sqrt{-g}}{16\pi G}D_\mu D_\nu(D^\mu\xi^\nu - D^\nu\xi^\mu) + \overset{(H)}{T_M}{}^{\mu\nu} D_\mu\xi_\nu - D_\mu(\overset{(N)}{T_M}{}^{\mu}{}_{\nu} D_\mu\xi^\nu) = 0 \tag{34}$$

where $D_\mu \overset{(H)}{T_M}{}^{\mu\nu} = 0$ is used due to the Einstein equation and the Bianchi identity $D_\mu(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) = 0$. In case that the first term in equation (34) is not singular, the term is vanishing, which is reduced to the Hilbert and Noether matter EMT relation (7).

V. SUMMARY AND DISCUSSION

For scalar and electromagnetic field theories, we have shown the key equation (7) for the Hilbert EMT $T^{\mu\nu}_{(H)}$ and the Noether EMT $T^{\mu\nu}_{(N)}$ obtained by an infinitesimal coordinate transformation. Namely, the equation (7) should hold for any infinitesimal vector ξ^μ and independent tensors $D_\mu\xi^\nu$. Therefore, the respective coefficient has to vanish. We can conclude that the two definitions of EMT coincide and the EMT conserves as,

$$T^{\mu\nu}_{(H)} = T^{\mu\nu}_{(N)}, \tag{35}$$

$$D_\mu T^{\mu\nu}_{(N)} = 0. \tag{36}$$

As a byproduct, we see that the Noether EMT $T^{\mu\nu}_{(N)}$ is symmetric since the Hilbert EMT $T^{\mu\nu}_{(H)}$ is by definition. In the case that the spacetime has a symmetry, the corresponding Noether current is given by

$$j^\mu = T^{\mu\nu}_{(N)}\xi_\nu, \tag{37}$$

for a Killing vector ξ_ν.

Actually the precise form of the Noether EMT $T^{\mu\nu}_{(N)}$ is defined by equation (7). We do not claim that the key equation (7) always hold for general fields other than scalar and electromagnetic fields. We have restricted our attention to the case that the Lagrangian contains only the first derivative of fields not the higher derivatives as discussed in the paper [4]. However, it would be very interesting if (7) does not hold, because there are matter energy that acts only as a source of gravity. In other words, the difference between the Noether EMT and the Hilbert EMT $\Delta T^{\nu\nu} = \overset{(N)}{T}{}^{\mu\nu} - \overset{(H)}{T}{}^{\mu\nu}$ would not contribute to gravity.

We would like to see the Belinfante-Rosenfeld procedure in the light of the key equation,

$$T^{\mu\nu}_{(H)}D_\mu\xi_\nu = D_\mu\big[T^{\mu\nu}_{(N)}\xi_\nu + D_\alpha(\Psi^{\mu\alpha\nu}\xi_\nu)\big], \tag{38}$$

by adding $0 = D_\mu D_\alpha(\Psi^{\mu\alpha\nu}\xi_\nu)$ in (7), where $\Psi^{\mu\alpha\nu}$ is antisymmetric with respect to only two former indices, μ, α. The right hand side becomes

$$\begin{aligned} &D_\mu\big[T^{\mu\nu}_{(N)}\xi_\nu + D_\alpha\Psi^{\mu\alpha\nu}\cdot\xi_\nu + \Psi^{\mu\alpha\nu}D_\alpha\xi_\nu)\big] = D_\mu\big[(T^{\mu\nu}_{(N)} + D_\alpha\Psi^{\mu\alpha\nu})\xi_\nu + \Psi^{\mu\alpha\nu}D_\alpha\xi_\nu)\big] \\ &= D_\mu(\tilde{T}^{\mu\nu}_{(N)}\xi_\nu) + D_\mu\Psi^{\mu\alpha\nu}\cdot D_\alpha\xi_\nu + \Psi^{\mu\alpha\nu}D_\mu D_\alpha\xi_\nu\big] = D_\mu(\tilde{T}^{\mu\nu}_{(N)}\xi_\nu) - D_\alpha\Psi^{\mu\alpha\nu}\cdot D_\mu\xi_\nu + \Psi^{\mu\alpha\nu}D_\mu D_\alpha\xi_\nu \ , \end{aligned}$$

where

$$\tilde{T}^{\mu\nu}_{(N)} = T^{\mu\nu}_{(N)} + D_\alpha\Psi^{\mu\alpha\nu} \tag{39}$$

Therefore we obtain

$$T^{\mu\nu}_{(H)}D_\mu\xi_\nu = D_\mu(\tilde{T}^{\mu\nu}_{(N)}\xi_\nu) - D_\alpha\Psi^{\mu\alpha\nu}\cdot D_\mu\xi_\nu + \Psi^{\mu\alpha\nu}D_\mu D_\alpha\xi_\nu \ . \tag{40}$$

Or via

$$T^{\mu\nu}_{(H)}D_\mu\xi_\nu + D_\alpha\Psi^{\mu\alpha\nu}\cdot D_\mu\xi_\nu = D_\mu(\tilde{T}^{\mu\nu}_{(N)}\xi_\nu) + \Psi^{\mu\alpha\nu}D_\mu D_\alpha\xi_\nu \ , \tag{41}$$

we arrive at

$$\tilde{T}^{\mu\nu}_{(H)} D_\mu \xi_\nu = D_\mu(\tilde{T}^{\mu\nu}_{(N)} \xi_\nu) - \frac{1}{2} \Psi^{\mu\alpha\nu} R^\beta{}_{\nu\mu\alpha} \xi_\beta, \tag{42}$$

where

$$\tilde{T}^{\mu\nu}_{(H)} = T_{(H)}{}^{\mu\nu} + D_\alpha \Psi^{\mu\alpha\nu} \tag{43}$$

and $2\Psi^{\mu\alpha\nu} D_\mu D_\alpha \xi_\nu = \Psi^{\mu\alpha\nu}(D_\mu D_\alpha - D_\alpha D_\mu)\xi_\nu = \Psi^{\mu\alpha\nu} R_{\mu\alpha\nu}{}^\beta \xi_\beta = \Psi^{\mu\alpha\nu} R_\nu{}^\beta{}_{\mu\alpha} \xi_\beta = -\Psi^{\mu\alpha\nu} R^\beta{}_{\nu\mu\alpha} \xi_\beta$. These relations (39) and (43) seems to correspond to the Belinfante-Rosenfeld procedure in curved spacetime. As in (42), the relation between two EMT should be rather based on Noether current that is, expressed as the relation between the divergence of the new Noether current $D_\mu\big[\tilde{T}^{\mu\nu}_{(N)} \xi_\nu\big]$, the new Hilbert EMT $\tilde{T}^{\mu\nu}_{(H)}$, the infinitesimal vector ξ^μ and tensor $D_\mu \xi^\nu$.

The argument in the introduction leads to

$$\tilde{T}^{\mu\nu}_{(N)} = \tilde{T}^{\mu\nu}_{(H)} \;\Rightarrow\; T^{\mu\nu}_{(N)} = T^{\mu\nu}_{(H)}$$

$$D_\mu \tilde{T}^{\mu\nu}_{(N)} - \frac{1}{2} \Psi^{\mu\alpha\beta} R^\nu{}_{\beta\mu\alpha} = 0 \;\Rightarrow\; D_\mu T^{\mu\nu}_{(N)} = 0$$

to see that this is the same as (36) to confirm that the additional term ends up with nothing.

From (38) the extended Noether current then becomes

$$j^\mu = T^{\mu\nu}_{(N)} \xi_\nu + D_\alpha(\Psi^{\mu\alpha\nu} \xi_\nu), \tag{44}$$

for the Killing vector ξ_ν. The extra term $D_\alpha(\Psi^{\mu\alpha\nu}\xi_\nu)$ is a kind of topological current which is conserved without the help of equation of motion i.e., "off-shell". We do not know the physical meaning of the second term.

It is straightforward to generalize the matter Lagrangian L_M to arbitrary number of matter fields by just adding them. We do not see any problem in the extension to higher spacetime dimensions and to the modified theories of gravity.

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