The Vierbein Formalism
and Energy-Momentum Tensor of Spinors

Ying-Qiu Gu*

School of Mathematical Science, Fudan University, Shanghai 200433, China

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To study the coupling system of space-time and Fermions, we need the explicit form of the energy-momentum tensor of spinors. The energy-momentum tensor is closely related to the tetrad frames which cannot be uniquely determined by the metric. This flexibility increases difficulties to derive the exact expression and easily leads to ambiguous results. In this paper, we give a detailed derivation for the energy-momentum tensor of Weyl and Dirac spinors. From the results we find that, besides the usual kinetic energy momentum term, there are three kinds of other additional terms. One is the nonlinear self-interactive potential, which acts like negative pressure. The other reflects the interaction of momentum p^μ with tetrad. The third is the spin-gravity coupling term which is a higher order infinitesimal in weak field, but may be important in a neutron star. This term is also closely related with magnetic field of a celestial body. These results are based on the decomposition of usual spin connection into geometrical part and dynamical part, which not only makes calculation simpler, but also highlights their different physical meanings. In addition, we get a new tensor $S^{\mu\nu}_{ab}$ in calculation of tetrad formalism, which plays an important role in the interaction of spinor with gravity.

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*Electronic address: yqgu@fudan.edu.cn

I. INTRODUCTION

The spinor field can give explanation for the accelerating expansion of the universe, and may be a possible candidate of dark matter. It is studied by some researchers in recently years[1–6]. In these works, the space-time is usually Friedmann-Lemaitre-Robertson-Walker type with diagonal metric. The energy-momentum tensor(**EMT**) $T_{\mu\nu}$ of spinors is simple and can be directly derived from Lagrangian of the spinor field in this case. There are some approaches to the general expression of EMT of spinors in curved space-time[6–9]. But the formalisms are usually quite complicated for practical calculation and different from each other. In [7, 8], according to the Pauli's theorem

$$\delta\widetilde{\gamma}^{\alpha} = \frac{1}{2}\widetilde{\gamma}_{\beta}\delta g^{\alpha\beta} + [\widetilde{\gamma}^{\alpha}, M], \tag{1.1}$$

where M is a traceless matrix related to the frame transformation, the EMT for Dirac spinor ϕ was derived as follows,

$$T^{\mu\nu} = \frac{1}{2}\Re\langle\phi^{\dagger}\ (\widetilde{\gamma}^{\mu} i\nabla^{\nu} + \widetilde{\gamma}^{\nu} i\nabla^{\mu})\phi\rangle, \tag{1.2}$$

where $\phi^{\dagger}\ = \phi^{+}\gamma$ is the Dirac conjugation, ∇^{μ} is the usual covariant derivatives for spinor. A detailed calculation for variation of action was performed in [6], and the results were a little different from (1.1) and (1.2).

The following calculation shows that, M is still related with $\delta g^{\mu\nu}$, and provides nonzero contribution to $T^{\mu\nu}$ in general cases. Besides, the covariant derivatives operator $i\nabla_{\mu}$ is not parallel to the classical momentum p_{μ} and the expression (1.2) is complicated for practical calculation and some important effects are covered by the compact form. The exact EMT of spinor was actually not obtained before.

The derivation of $T_{\mu\nu}$ is quite difficult due to non-uniqueness representation and complicated formalism of vierbein or tetrad frames. In this paper, we give a systematical and detailed calculation for EMT of Weyl and Dirac spinors. We clearly establish the relations between tetrad and metric at first, and then solve the Euler derivatives with respect to $g_{\mu\nu}$ to get explicit and rigorous $T_{\mu\nu}$.

From the results we find some new and interesting conclusions. Besides the usual kinetic energy momentum term, we find three kinds of other additional terms in EMT of bispinor. One is the self interactive potential, which acts like negative pressure. The other reflects the interaction of momentum p^μ with tetrad, which vanishes in classical approximation. The third is the spin-gravity coupling term $\Omega_\alpha s^\alpha$, which is a higher order infinitesimal in weak field, but becomes important in a neutron star. This term is the eye of a particle with location and navigation functions, and is closely related with magnetic field of a celestial body. All these results are based on the decomposition of usual spin connection Γ_μ into geometrical part Υ_μ and dynamical part Ω_μ, which not only makes calculation simpler, but also highlights their different physical meanings. In addition, in the calculation of tetrad formalism we find a new tensor $S_{ab}^{\mu\nu}$ which plays an important role in the interaction of spinor with gravity and appears in many places.

The materials are organized as follows: In the next section, we specify all notations, conventions and relevant equations used in the discussion. In the third section, we provide the technical foundations for the following derivation of EMT. We derive the exact EMT of spinor and its classical approximation in section IV, and then we give some simple discussions and illustrations in the last section.

II. NOTATIONS AND EQUATIONS

Denote the metric and coordinates of the Minkowski space-time respectively by

$$\eta_{ab} = \mathrm{diag}(-1, -1, -1,\ 1), \quad \delta X^\mu = (\delta X, \delta Y, \delta Z, c\delta T). \tag{2.1}$$

The Pauli matrices are expressed by

$$\sigma^\mu = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \tag{2.2}$$

$$\widetilde{\sigma}^\mu = (-\vec{\sigma},\ \sigma^0), \qquad \vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3). \tag{2.3}$$

where $a, \mu \in 0, 1, 2, 3$. In this paper, we use the Greek characters stand for indices of curvilinear coordinates and Latin characters for indices of local Minkowski coordinates. The element of the space-time can be expressed by

$$d\mathbf{x} = \widetilde{\gamma}_\mu dx^\mu = \gamma_a \delta X^a, \tag{2.4}$$

where γ_a and $\widetilde{\gamma}^\mu$ are tetrad expressed by Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \widetilde{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \quad \widetilde{\gamma}^\mu = h^\mu{}_a \gamma^a, \quad \widetilde{\gamma}_\mu = l_\mu{}^a \gamma_a, \tag{2.5}$$

which satisfies the $C\ell(1,3)$ Clifford algebra[10–14]

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=\eta_{\mu\nu},\qquad \widetilde{\gamma}_\mu\widetilde{\gamma}_\nu+\widetilde{\gamma}_\nu\widetilde{\gamma}_\mu=g_{\mu\nu}. \tag{2.6}$$

In Minkowski space-time, we have Dirac equation

$$\gamma^\mu i\partial_\mu\phi=m\phi, \tag{2.7}$$

or in chiral form[10]

$$\begin{cases}\sigma^\mu i\partial_\mu\psi=m\widetilde{\psi},\\ \widetilde{\sigma}^\mu i\partial_\mu\widetilde{\psi}=m\psi,\end{cases}\qquad \phi=\begin{pmatrix}\psi\\ \widetilde{\psi}\end{pmatrix}, \tag{2.8}$$

where $\psi,\ \widetilde{\psi}$ are two Weyl spinors.

Denote the Pauli matrices in curved space-time by

$$\begin{cases}\varrho^\mu=h^\mu{}_a\sigma^a,\quad \varrho_\mu=l_\mu{}^a\sigma_a,\\ \widetilde{\varrho}^\mu=h^\mu{}_a\widetilde{\sigma}^a,\quad \widetilde{\varrho}_\mu=l_\mu{}^a\widetilde{\sigma}_a,\end{cases}\qquad \widetilde{\gamma}^\mu=\begin{pmatrix}0&\widetilde{\varrho}^\mu\\ \varrho^\mu&0\end{pmatrix}, \tag{2.9}$$

then we have Clifford algebra as follows

$$\varrho^\mu\widetilde{\varrho}^\nu+\varrho^\nu\widetilde{\varrho}^\mu=\widetilde{\varrho}^\mu\varrho^\nu+\widetilde{\varrho}^\nu\varrho^\mu=2g^{\mu\nu}. \tag{2.10}$$

The Weyl spinor equation (2.8) in curved space-time becomes

$$\begin{cases}\varrho^\mu i(\partial_\mu+\Gamma_\mu)\psi=m\widetilde{\psi},\\ \widetilde{\varrho}^\mu i(\partial_\mu+\widetilde{\Gamma}_\mu)\widetilde{\psi}=m\psi,\end{cases} \tag{2.11}$$

where Γ_μ and $\widetilde{\Gamma}_\mu$ are the spinor affine connections[1, 2, 4–6, 8, 13],

$$\Gamma_\mu=\frac{1}{4}\widetilde{\varrho}_\nu\varrho^\nu_{;\mu},\qquad \widetilde{\Gamma}_\mu=\frac{1}{4}\varrho_\nu\widetilde{\varrho}^\nu_{;\mu}, \tag{2.12}$$

in which $\varrho^\mu_{;\nu}=\partial_\nu\varrho^\mu+\Gamma^\mu_{\alpha\nu}\varrho^\alpha$.

In order to disclose the physical meanings of connection, the total covariant derivatives of spinor can be represented in the following form[15],

$$\varrho^\mu i(\partial_\mu+\Gamma_\mu)=\varrho^\mu\left[i(\partial_\mu+\Upsilon_\mu)+\Omega_\mu\right], \tag{2.13}$$

$$\widetilde{\varrho}^\mu i(\partial_\mu+\widetilde{\Gamma}_\mu)=\widetilde{\varrho}^\mu\left[i(\partial_\mu+\Upsilon_\mu)-\Omega_\mu\right]. \tag{2.14}$$

In which, Υ_μ is related with the grade-1 Clifford algebra, which has only geometrical effect,

$$\Upsilon_\mu\equiv\frac{1}{2}(l^a_\mu\partial_\nu h^\nu_a+\partial_\mu\ln\sqrt{g})=\frac{1}{2}h^\nu_a(\partial_\mu l^a_\nu-\partial_\nu l^a_\mu). \tag{2.15}$$

But Ω_μ is related with grade-3 Clifford algebra[16]. Similarly, for Dirac bispinor in curved space-time we have

$$\nabla_\mu \phi = (\partial_\mu + \hat{\Gamma}_\mu)\phi, \qquad \hat{\Gamma}_\mu = \frac{1}{4}\widetilde{\gamma}_\nu \widetilde{\gamma}^\nu_{;\mu}. \tag{2.16}$$

More generally, the Lagrangian corresponding to (2.11) with electromagnetic interaction and nonlinear self-potential F is given by

$$\begin{aligned}\mathcal{L}_m &= \Re\langle \phi^+ \widetilde{\alpha}^\mu \hat{p}_\mu \phi \rangle + \Omega_\mu \phi^+ \hat{s}^\mu \phi - m\phi^+ \gamma_0 \phi + F, \qquad (2.17)\\ &= \Re\langle \psi^+ \varrho^\mu \hat{p}_\mu \psi + \widetilde{\psi}^+ \widetilde{\varrho}^\mu \hat{p}_\mu \widetilde{\psi}\rangle + \psi^+ \Omega \psi + \widetilde{\psi}^+ \widetilde{\Omega} \widetilde{\psi} - m(\widetilde{\psi}^+ \psi + \psi^+ \widetilde{\psi}) + F, \qquad (2.18)\end{aligned}$$

where $\Re\langle\rangle$ means taking real part, $\widetilde{w} > 0$ is the nonlinear coupling coefficient,

$$F = \frac{1}{2}\widetilde{w}\check{\gamma}^2, \qquad \check{\gamma} = \phi^+ \gamma_0 \phi. \tag{2.19}$$

$(\hat{p}_\mu, \widetilde{\alpha}^\mu, \hat{s}^\mu)$ are respectively momentum, current and spin operators defined by

$$\hat{p}_\mu = i(\partial_\mu + \Upsilon_\mu) - eA_\mu, \quad \widetilde{\alpha}^\mu = \mathrm{diag}(\varrho^\mu, \widetilde{\varrho}^\mu), \quad \hat{s}^\mu = \mathrm{diag}(\varrho^\mu, -\widetilde{\varrho}^\mu). \tag{2.20}$$

Ω and $\widetilde{\Omega}$ are two Hermitian matrix defined by

$$\begin{cases} \Omega \equiv \frac{i}{8}[\varrho^\mu \widetilde{\varrho}^\alpha \partial_\mu \varrho_\alpha - (\partial_\mu \varrho_\alpha)\widetilde{\varrho}^\alpha \varrho^\mu] = \Omega_\mu \varrho^\mu = \omega_a \sigma^a, \\ \widetilde{\Omega} \equiv \frac{i}{8}[\widetilde{\varrho}^\mu \varrho^\alpha \partial_\mu \widetilde{\varrho}_\alpha - (\partial_\mu \widetilde{\varrho}_\alpha)\varrho^\alpha \widetilde{\varrho}^\mu] = -\Omega_\mu \widetilde{\varrho}^\mu = -\omega_a \widetilde{\sigma}^a. \end{cases} \tag{2.21}$$

For any diagonal metric, it easy to check $\Omega = \widetilde{\Omega} = 0$. By straightforward calculation we have[15]

$$\begin{cases} \omega_0 = -\frac{1}{4}(\vec{h}^\alpha \times \vec{h}^\beta)\cdot \partial_\alpha \vec{l}_\beta, \\ \vec{\omega} = -\frac{1}{4}\left(\partial_\alpha l_\beta^{\ 0}(\vec{h}^\alpha \times \vec{h}^\beta) - (h^\alpha_{\ 0}\vec{h}^\beta - h^\beta_{\ 0}\vec{h}^\alpha) \times \partial_\alpha \vec{l}_\beta\right), \\ \Omega_\mu = -\frac{1}{4}\left((\vec{h}^\alpha \times \vec{h}^\beta)\cdot (l_\mu^{\ 0}\partial_\alpha \vec{l}_\beta - \vec{l}_\mu \partial_\alpha l_\beta^{\ 0}) + \vec{l}_\mu \cdot [(h^\alpha_{\ 0}\vec{h}^\beta - h^\beta_{\ 0}\vec{h}^\alpha)\times \partial_\alpha \vec{l}_\beta]\right), \end{cases} \tag{2.22}$$

in which $\vec{\omega} = (\omega^1, \omega^2, \omega^3)$. (2.22) defines the dynamical part of the spinor connection.

III. RELATIONS BETWEEN TETRAD AND METRIC

In this section, we give an explicit representation of tetrad formalism. The derivation of EMT is based on this representation. Different from the cases of vector and tensor, in general relativity the dynamical equations for spinor fields depend on the local tetrad frame, which make the representation of the spinor connection and the EMT quite complicated. Assume that $x^\mu=(x,y,z,ct)$ is the coordinates and δX^a is the element vector in the tangent space at fixed point x^μ. The tetrad $\widetilde{\gamma}^\mu$ cannot be uniquely determined by metric, which can be only determined to an arbitrary local Lorentz transformation. Now we derive some important relations used below.

Lemma 1. *For Pauli matrices (2.2) and (2.3), we have relations*

$$\begin{cases} \sigma^a\widetilde{\sigma}^b\sigma^c-\sigma^c\widetilde{\sigma}^b\sigma^a=2i\epsilon^{abcd}\sigma_d, \\ \widetilde{\sigma}^a\sigma^b\widetilde{\sigma}^c-\widetilde{\sigma}^c\sigma^b\widetilde{\sigma}^a=-2i\epsilon^{abcd}\widetilde{\sigma}_d, \end{cases} \tag{3.1}$$

in which ϵ^{abcd} is the permutation function.

Lemma 1 can be easy checked. We have only 4 nonzero cases for each equation.

For metric $g_{\mu\nu}$, not losing generality we assume that, in the neighborhood of x^μ, dx^0 is time-like and (dx^1,dx^2,dx^3) are space-like. This means $g_{00}\ge 0$ and $g_{kk}\le 0 (k\ne 0)$, and the following definitions of J_k are real numbers

$$J_1=\sqrt{-g_{11}},\quad J_2=\sqrt{\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{vmatrix}},\quad J_3=\sqrt{-\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33}\end{vmatrix}},\quad J_0=\sqrt{-\det(g)}. \tag{3.2}$$

Denote

$$u_1=\begin{vmatrix} g_{11} & g_{12} \\ g_{31} & g_{32}\end{vmatrix},\qquad u_2=\begin{vmatrix} g_{11} & g_{12} \\ g_{01} & g_{02}\end{vmatrix},\qquad u_3=\begin{vmatrix} g_{21} & g_{22} \\ g_{31} & g_{32}\end{vmatrix}, \tag{3.3}$$

and

$$v_1=\begin{vmatrix} g_{12} & g_{13} & g_{10} \\ g_{22} & g_{23} & g_{20} \\ g_{32} & g_{33} & g_{30}\end{vmatrix},\quad v_2=\begin{vmatrix} g_{11} & g_{13} & g_{10} \\ g_{21} & g_{23} & g_{20} \\ g_{31} & g_{33} & g_{30}\end{vmatrix},\quad v_3=\begin{vmatrix} g_{11} & g_{12} & g_{10} \\ g_{21} & g_{22} & g_{20} \\ g_{31} & g_{32} & g_{30}\end{vmatrix}, \tag{3.4}$$

then we have the following conclusion.

Theorem 2. *For LU decomposition of matrix $(g_{\mu\nu})$,*

$$(g_{\mu\nu})=L(\eta_{ab})L^+,\quad (g^{\mu\nu})=U(\eta_{ab})U^+,\quad U=L^*=(L^+)^{-1}, \tag{3.5}$$

with positive diagonal elements, we have the following unique solution

$$L=(L_\mu^{\ a})=\begin{pmatrix} -\frac{g_{11}}{J_1} & 0 & 0 & 0 \\ -\frac{g_{21}}{J_1} & \frac{J_2}{J_1} & 0 & 0 \\ -\frac{g_{31}}{J_1} & \frac{u_1}{J_1\,J_2} & \frac{J_3}{J_2} & 0 \\ -\frac{g_{01}}{J_1} & \frac{u_2}{J_1\,J_2} & -\frac{v_3}{J_2\,J_3} & \frac{J_0}{J_3} \end{pmatrix}, \tag{3.6}$$

$$U=(U^\mu_{\ a})=\begin{pmatrix} \frac{1}{J_1} & \frac{g_{21}}{J_1\,J_2} & \frac{u_3}{J_2\,J_3} & \frac{v_1}{J_3\,J_0} \\ 0 & \frac{J_1}{J_2} & -\frac{u_1}{J_2\,J_3} & -\frac{v_2}{J_3\,J_0} \\ 0 & 0 & \frac{J_2}{J_3} & \frac{v_3}{J_3\,J_0} \\ 0 & 0 & 0 & \frac{J_3}{J_0} \end{pmatrix}. \tag{3.7}$$

(3.6) and (3.7) can be checked directly. For any other solutions of (2.5), we have

Theorem 3. *For any solution of tetrad (2.5) in matrix form* $(l_\mu^{\ a})$ *and* $(h^\mu_{\ a})$*, there exists a local Lorentz transformation* $\delta X'^a=\Lambda^a_{\ b}\delta X^b$ *independent of* $g_{\mu\nu}$*, such that*

$$(l_\mu^{\ a})=L\Lambda^+,\qquad (h^\mu_{\ a})=U\Lambda^{-1}, \tag{3.8}$$

where $\Lambda=(\Lambda^a_{\ b})$ *stands for the matrix of Lorentz transformation.*

Theorem 3 can be checked as follows,

$$(g_{\mu\nu})=L(\eta_{ab})L^+=(l_\mu^{\ a})(\eta_{ab})(l_\mu^{\ a})^+\implies L^{-1}(l_\mu^{\ a})(\eta_{ab})(L^{-1}(l_\mu^{\ a}))^+=(\eta_{ab}). \tag{3.9}$$

Then we have a Lorentz transformation matrix $\Lambda=(\Lambda^a_{\ b})$, such that

$$L^{-1}(l_\mu^{\ a})=\Lambda^+\implies (l_\mu^{\ a})=L\Lambda^+,\quad\text{or}\quad l_\mu^{\ a}=L_\mu^{\ b}\Lambda^a_{\ b}. \tag{3.10}$$

Similarly we have $(h^\mu_{\ a})=U\Lambda^{-1}$.

Remark. *The decomposition (3.5) is nothing but a Gram-Schmidt orthogonalization for vectors* $dx^\mu=(dx,dy,dz,dt)$ *in tangent space-time in the order* $dt\to dz\to dy\to dx$*, namely in vector form* $\delta X=dxL$ *or*

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu=\eta_{ab}\delta X^a\delta X^b \\ &= -(L_x^{\ X}dx+L_y^{\ X}dy+L_z^{\ X}dz+L_t^{\ X}dt)^2 \\ &\quad -(L_y^{\ Y}dy+L_z^{\ Y}dz+L_t^{\ Y}dt)^2-(L_z^{\ Z}dz+L_t^{\ Z}dt)^2+(L_t^{\ T}dt)^2. \end{aligned} \tag{3.11}$$

(3.11) is a direct result of (3.6), but (3.11) manifestly shows the geometrical meanings of the tetrad components $L_\mu{}^a$. Obviously, (3.11) is much convenient for practical calculation of tetrad parameters.

Define a spinor coefficient tensor by

$$S_{ab}^{\mu\nu} \equiv \frac{1}{2}(h^\mu{}_a h^\nu{}_b + h^\nu{}_a h^\mu{}_b)\mathrm{sgn}(a-b). \tag{3.12}$$

$S_{ab}^{\mu\nu}$ is symmetrical for Riemann indices (μ,ν) but anti-symmetrical for Minkowski indices (a,b). (3.12) is important for the following calculation. By representation of (3.6), (3.7) and relation (3.8), we can check the following results by straightforward calculation.

Theorem 4. *For any solution of tetrad (2.5), we have*

$$\frac{\partial l_\alpha{}^n}{\partial g_{\mu\nu}} = \frac{1}{4}(\delta_\alpha^\mu h^\nu{}_m + \delta_\alpha^\nu h^\mu{}_m)\eta^{nm} + \frac{1}{2}S_{ab}^{\mu\nu} l_\alpha{}^a \eta^{nb}. \tag{3.13}$$

$$\frac{\partial h^\alpha{}_a}{\partial g_{\mu\nu}} = -\frac{1}{4}(h^\mu{}_a g^{\alpha\nu} + h^\nu{}_a g^{\mu\alpha}) - \frac{1}{2}S_{ab}^{\mu\nu} h^\alpha{}_n \eta^{nb}. \tag{3.14}$$

Or equivalently,

$$\frac{\partial \varrho_\alpha}{\partial g_{\mu\nu}} = \frac{1}{4}(\delta_\alpha^\mu \varrho^\nu + \delta_\alpha^\nu \varrho^\mu) + \frac{1}{2}S_{ab}^{\mu\nu} l_\alpha{}^a \sigma^b. \tag{3.15}$$

$$\frac{\partial \varrho^\alpha}{\partial g_{\mu\nu}} = -\frac{1}{4}(g^{\mu\alpha}\varrho^\nu + g^{\nu\alpha}\varrho^\mu) - \frac{1}{2}S_{ab}^{\mu\nu} h^\alpha{}_n \eta^{nb}\sigma^a. \tag{3.16}$$

In (3.13)-(3.16) we set $\frac{\partial \varrho_\alpha}{\partial g_{\mu\nu}} = \frac{\partial \varrho_\alpha}{\partial g_{\nu\mu}} = \frac{1}{2}\frac{d\varrho_\alpha}{dg_{\mu\nu}}$ $(\mu\ne\nu)$ *to get the tensor form.* $\frac{d}{dg_{\mu\nu}}$ *means the total derivative for* $g_{\mu\nu}$ *and* $g_{\nu\mu}$. *For given vector* A^α, *we have*

$$A^\alpha \frac{\partial \varrho_\alpha}{\partial g_{\mu\nu}} = \frac{1}{4}(A^\mu \varrho^\nu + A^\nu \varrho^\mu) + \frac{1}{2}S_{ab}^{\mu\nu} A^\alpha l_\alpha{}^a \sigma^b. \tag{3.17}$$

$$A_\alpha \frac{\partial \varrho^\alpha}{\partial g_{\mu\nu}} = -\frac{1}{4}(A^\mu \varrho^\nu + A^\nu \varrho^\mu) - \frac{1}{2}S_{ab}^{\mu\nu} A_\alpha h^\alpha{}_n \eta^{nb}\sigma^a. \tag{3.18}$$

Similarly, for Dirac matrices we have

Corollary 5. *For Dirac matrices (2.5), we have*

$$\frac{\partial \widetilde{\gamma}_\alpha}{\partial g_{\mu\nu}} = \frac{1}{4}(\delta_\alpha^\mu \widetilde{\gamma}^\nu + \delta_\alpha^\nu \widetilde{\gamma}^\mu) + \frac{1}{2}S_{ab}^{\mu\nu} l_\alpha{}^a \gamma^b. \tag{3.19}$$

Or equivalently, for any given vector A^μ, *we have*

$$A^\alpha \frac{\partial \widetilde{\gamma}_\alpha}{\partial g_{\mu\nu}} = \frac{1}{4}(A^\mu \widetilde{\gamma}^\nu + A^\nu \widetilde{\gamma}^\mu) + \frac{1}{2}S_{ab}^{\mu\nu} A^\alpha l_\alpha{}^a \gamma^b. \tag{3.20}$$

IV. ENERGY-MOMENTUM TENSOR OF SPINORS

Now we consider the coupling system of spinor and gravity. The Ricci tensor and scalar curvature is defined by

$$R_{\mu\nu}=\partial_\mu\Gamma^\alpha_{\nu\alpha}-\partial_\alpha\Gamma^\alpha_{\mu\nu}+\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}-\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta},\qquad R\equiv g^{\mu\nu}R_{\mu\nu}.$$

The total Lagrangian of the system reads

$$\mathcal{L}=\frac{1}{2\kappa}\mathcal{L}_g+\mathcal{L}_m=-\frac{1}{2\kappa}(R-2\Lambda)+\mathcal{L}_m, \tag{4.1}$$

where $\kappa=8\pi G$, Λ is the cosmological constant, $\mathcal{L}_m$ the Lagrangian of spinors (2.18). Variation of the Lagrangian (4.1) with respect to $g_{\mu\nu}$ we get Einstein's equation

$$R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R+\Lambda g^{\mu\nu}=\kappa T^{\mu\nu}, \tag{4.2}$$

where $T^{\mu\nu}$ is EMT of the spinors, which is defined by

$$T^{\mu\nu}=-2\frac{\delta(\mathcal{L}_m\sqrt{g})}{\sqrt{g}\delta g_{\mu\nu}}=-2\left(\frac{\partial\mathcal{L}_m}{\partial g_{\mu\nu}}-(\partial_\alpha+\Gamma^\gamma_{\alpha\gamma})\frac{\partial\mathcal{L}_m}{\partial(\partial_\alpha g_{\mu\nu})}\right)-g^{\mu\nu}\mathcal{L}_m, \tag{4.3}$$

where $g=|\det(g_{\mu\nu})|$ and $\frac{\delta}{\delta g_{\mu\nu}}$ is the Euler derivatives.

We take ψ as example to derive relations, because we have similar results for $\widetilde{\psi}$. By (2.18) and (3.16) we have

$$\mathcal{E}^{\mu\nu}_p\equiv\frac{\partial}{\partial g_{\mu\nu}}\Re\langle\psi^+\varrho^\alpha\hat{p}_\alpha\psi\rangle=-\frac{1}{4}\Re\langle\psi^+(\varrho^\mu\hat{p}^\nu+\varrho^\nu\hat{p}^\mu+2S^{\mu\nu}_{ab}h^\alpha_{\ n}\eta^{nb}\sigma^a\hat{p}_\alpha)\psi\rangle. \tag{4.4}$$

By (2.21), (3.1) and (3.17) we rewrite Ω as follows,

$$\begin{aligned}\Omega &= \frac{i}{8}\left(\varrho^\mu\widetilde{\varrho}^\alpha\frac{\partial\varrho_\alpha}{\partial g_{\lambda\kappa}}-\frac{\partial\varrho_\alpha}{\partial g_{\lambda\kappa}}\widetilde{\varrho}^\alpha\varrho^\mu\right)\partial_\mu g_{\lambda\kappa},\\ &= \frac{i}{16}\left(\sigma^a\widetilde{\sigma}^b\sigma^c-\sigma^c\widetilde{\sigma}^b\sigma^a\right)h^\mu_{\ a}S^{\lambda\kappa}_{bc}\partial_\mu g_{\lambda\kappa}, && \text{by (3.17)}\\ &= -\frac{1}{8}\epsilon^{abcd}\sigma_d h^\mu_{\ a}S^{\lambda\kappa}_{bc}\partial_\mu g_{\lambda\kappa}=\frac{1}{8}\epsilon^{dabc}\sigma_d h^\mu_{\ a}S^{\lambda\kappa}_{bc}\partial_\mu g_{\lambda\kappa}. && \text{by (3.1)}\end{aligned} \tag{4.5}$$

By (4.5), we get (Ω^α,ω^a) expressed by $\partial_\alpha g_{\mu\nu}$ as follows,

$$\omega^d=\frac{1}{8}\epsilon^{dabc}h^\alpha_{\ a}S^{\mu\nu}_{bc}\partial_\alpha g_{\mu\nu},\qquad \Omega^\alpha=\frac{1}{8}\epsilon^{dabc}l_d^{\ \alpha}h^\beta_{\ a}S^{\mu\nu}_{bc}\partial_\beta g_{\mu\nu}. \tag{4.6}$$

Then we get

$$\frac{\partial(\psi^+\Omega\psi)}{\partial g_{\mu\nu}} = -\frac{1}{8}\epsilon^{abcd}\breve{\sigma}_d\frac{\partial(h^\alpha_{\ a}S^{\lambda\kappa}_{bc})}{\partial g_{\mu\nu}}\partial_\alpha g_{\lambda\kappa},\qquad \breve{\sigma}_d\equiv\psi^+\sigma_d\psi, \tag{4.7}$$

$$(\partial_\alpha+\Gamma^\beta_{\alpha\beta})\frac{\partial(\psi^+\Omega\psi)}{\partial(\partial_\alpha g_{\mu\nu})} = -\frac{1}{8}\epsilon^{abcd}\left(h^\alpha_{\ a}S^{\mu\nu}_{bc}(\partial_\alpha+\Gamma^\beta_{\alpha\beta})\breve{\sigma}_d+\breve{\sigma}_d\frac{\partial(h^\alpha_{\ a}S^{\mu\nu}_{bc})}{\partial g_{\lambda\kappa}}\partial_\alpha g_{\lambda\kappa}\right), \tag{4.8}$$

and then we have the Euler derivative for $\psi^+\Omega\psi$ as

$$\begin{aligned}
\mathcal{E}^{\mu\nu}_\Omega &\equiv \frac{\delta(\psi^+\Omega\psi)}{\delta g_{\mu\nu}} = \frac{\partial(\psi^+\Omega\psi)}{\partial g_{\mu\nu}} - (\partial_\alpha + \Gamma^\beta_{\alpha\beta})\frac{\partial(\psi^+\Omega\psi)}{\partial(\partial_\alpha g_{\mu\nu})}, \\
&= \frac{1}{8}\epsilon^{abcd}\left[h^\alpha{}_a S^{\mu\nu}_{bc}(\partial_\alpha + \Gamma^\beta_{\alpha\beta})\breve{\sigma}_d + \breve{\sigma}_d\left(\frac{\partial(h^\alpha{}_a S^{\mu\nu}_{bc})}{\partial g_{\lambda\kappa}} - \frac{\partial(h^\alpha{}_a S^{\lambda\kappa}_{bc})}{\partial g_{\mu\nu}}\right)\partial_\alpha g_{\lambda\kappa}\right].
\end{aligned} \tag{4.9}$$

In principle, we can substitute (3.14) into (4.9) and then simplify it to get the final expression of $\mathcal{E}^{\mu\nu}_\Omega$. However, the structure of $\mathcal{E}^{\mu\nu}_\Omega$ has been clear, and we need not to do so complicated calculations now, because it is a second order tenser including only linear first order derivatives $\partial_\mu\breve{\sigma}_d$ and $\partial_\mu g_{\alpha\beta}$. The vectors and tensors constructed by metric and its linear first order derivatives are only $(\Upsilon_\mu, \Omega_\mu)$ multiplied by $g_{\mu\nu}$ and $g^{\alpha\beta}$. Substituting $\breve{\sigma}_d = h^\beta{}_d \breve{\varrho}_\beta$ into (4.9), then the simplified expression should take the following covariant form

$$\mathcal{E}^{\mu\nu}_\Omega = \frac{1}{8}\epsilon^{abcd} h^\alpha{}_a h^\beta{}_b S^{\mu\nu}_{cd}\,\breve{\varrho}_{\beta;\alpha} + g^{\mu\nu}(k_1\Omega^\alpha + k_2\Upsilon^\alpha)\breve{\varrho}_\alpha, \tag{4.10}$$

where (k_1, k_2) are 2 constants to be determined.

For diagonal metric we have $\Omega = S^{\mu\mu}_{ab} = \mathcal{E}^{\mu\mu}_\Omega = 0$, and then by (4.10) we get $k_2 = 0$. In Gaussian normal coordinate system $g_{\mu\nu} = (-\bar{g}_{kl}, 1)$, we have Hamiltonian for linear bispinor[15]

$$\mathbf{H} = -\widetilde{\alpha}^k \hat{p}_k + eA_0 + m\gamma_0 - \Omega_\mu \hat{s}^\mu. \tag{4.11}$$

By (4.3) and (4.10), we have

$$T^{00} = -2\mathcal{E}^{00}_\Omega + \cdots = -2k_1\Omega_\mu \check{s}^\mu + \cdots, \tag{4.12}$$

in which the omitted terms are irrelevant with the following comparison. Since T^{00} is energy of the spinors, in contrast (4.12) with (4.11) we get $k_1 = \frac{1}{2}$. So we get

$$\mathcal{E}^{\mu\nu}_\Omega = \frac{1}{8}\epsilon^{abcd} h^\alpha{}_a h^\beta{}_b S^{\mu\nu}_{cd}\,\breve{\varrho}_{\beta;\alpha} + \frac{1}{2}g^{\mu\nu}\Omega^\alpha\breve{\varrho}_\alpha. \tag{4.13}$$

Similarly, for $\widetilde{\psi}$ we have

$$\mathcal{E}^{\mu\nu}_{\widetilde{\Omega}} = -\frac{1}{8}\epsilon^{abcd} h^\alpha{}_a h^\beta{}_b S^{\mu\nu}_{cd}\,\breve{\widetilde{\varrho}}_{\beta;\alpha} - \frac{1}{2}g^{\mu\nu}\Omega^\alpha\breve{\widetilde{\varrho}}_\alpha, \qquad \breve{\widetilde{\varrho}}_\alpha \equiv \widetilde{\psi}^+\widetilde{\varrho}_\alpha\widetilde{\psi}. \tag{4.14}$$

For total spin $\hat{s}^\mu$ we have

$$\mathcal{E}^{\mu\nu}_s = \mathcal{E}^{\mu\nu}_\Omega + \mathcal{E}^{\mu\nu}_{\widetilde{\Omega}} = \frac{1}{8}\epsilon^{abcd} h^\alpha{}_a h^\beta{}_b S^{\mu\nu}_{cd}\,\check{s}_{\beta;\alpha} + \frac{1}{2}g^{\mu\nu}\Omega^\alpha\check{s}_\alpha, \quad \check{s}_\alpha \equiv \phi^+\hat{s}\phi. \tag{4.15}$$

Substituting (4.4) and (4.15) into (4.3), we get the EMT for Weyl spinors

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{2}\Re\langle\psi^+(\varrho^\mu\hat{p}^\nu+\varrho^\nu\hat{p}^\mu)\psi+\widetilde{\psi}^+(\widetilde{\varrho}^\mu\hat{p}^\nu+\widetilde{\varrho}^\nu\hat{p}^\mu)\widetilde{\psi}\rangle-g^{\mu\nu}\mathcal{L}_m \\ &+\Re\langle\psi^+S^{\mu\nu}_{ab}h^\alpha{}_n\eta^{nb}\sigma^a\hat{p}_\alpha\psi+\widetilde{\psi}^+S^{\mu\nu}_{ab}h^\alpha{}_n\eta^{nb}\widetilde{\sigma}^a\hat{p}_\alpha\widetilde{\psi}\rangle \\ &-\frac{1}{4}\epsilon^{abcd}h^\alpha{}_a h^\beta{}_b S^{\mu\nu}_{cd}\check{s}_{\beta;\alpha}-g^{\mu\nu}\Omega^\alpha\check{s}_\alpha. \end{aligned} \tag{4.16}$$

For nonlinear spinor, by Dirac equation we have $\mathcal{L}_m=-F$.

Substituting the results into (4.16), we get the final EMT of bispinor ϕ as follows,

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{2}\Re\langle\phi^+(\widetilde{\alpha}^\mu\hat{p}^\nu+\widetilde{\alpha}^\nu\hat{p}^\mu+2S^{\mu\nu}_{ab}h^\alpha{}_n\eta^{nb}\alpha^a\hat{p}_\alpha)\phi\rangle+g^{\mu\nu}F \\ &-\frac{1}{4}\epsilon^{abcd}h^\alpha{}_a h^\beta{}_b S^{\mu\nu}_{cd}\check{s}_{\beta;\alpha}-g^{\mu\nu}\Omega_\alpha\check{s}^\alpha. \end{aligned} \tag{4.17}$$

The physical meaning of

$$\mathcal{E}^{\mu\nu}\equiv\frac{1}{4}\epsilon^{abcd}h^\alpha{}_a h^\beta{}_b S^{\mu\nu}_{cd}\check{s}_{\beta;\alpha}=\frac{1}{8}\epsilon^{abcd}h^\alpha{}_a h^\beta{}_b S^{\mu\nu}_{cd}(\partial_\alpha\check{s}_\beta-\partial_\beta\check{s}_\alpha) \tag{4.18}$$

is unclear. Its diagonal components vanish, so it should be very tiny term to keep $T^{\mu\nu}_{;\nu}=0$.

Now we consider the classical approximation for (4.17),

$$\phi^+\widetilde{\alpha}^\nu\phi\to u^\nu\sqrt{1-v^2}\delta^3(\vec{x}-\vec{X}),\quad \hat{p}^\mu\phi\to mu^\mu\phi,\quad F\to w\sqrt{1-v^2}\delta^3(\vec{x}-\vec{X}). \tag{4.19}$$

Substituting (4.19) into (4.17) and noticing $S^{\mu\nu}_{ab}=-S^{\mu\nu}_{ba}$, we have

$$\Re\langle\phi^+S^{\mu\nu}_{ab}h^\alpha{}_n\eta^{nb}\alpha^a\hat{p}_\alpha\phi\rangle\to mS^{\mu\nu}_{ab}u^a u^b\sqrt{1-v^2}\delta^3(\vec{x}-\vec{X})=0. \tag{4.20}$$

Omitting the tiny spin-gravity coupling energy, we get the usual EMT for a classical particle with self-interactive potential

$$T^{\mu\nu}\to(mu^\mu u^\nu+wg^{\mu\nu})\sqrt{1-v^2}\delta^3(\vec{x}-\vec{X}). \tag{4.21}$$

$w>0$ acts like negative pressure[3, 17]. By (4.21) and energy-momentum conservation law $T^{\mu\nu}_{;\nu}=0$, we find only if $w=0$ and m is a constant independent of v, the particle moves along geodesic and the principle of equivalence strictly holds[15, 18].

Some previous works usually use one spinor to represent matter field. This may be not the case, because spinor fields only has a very tiny structure. Only to represent one particle by one spinor field, the matter model can be comparable with general relativity, classical mechanics and quantum mechanics[3, 15, 17, 19]. The many body system should be better described by the Lagrangian similar to the following one,

$$\mathcal{L}_m = \sum_n \Re\langle \phi_n^+ \widetilde{\alpha}^\mu \hat{p}_\mu \phi_n \rangle + \Omega_\mu \phi_n^+ \hat{s}^\mu \phi_n - m \phi_n^+ \gamma_0 \phi_n + F_n, \quad F_n = \frac{1}{2} \widetilde{w} \check{\gamma}_n^2. \tag{4.22}$$

The corresponding EMT is given by

$$\begin{aligned} T^{\mu\nu} \; &= \sum_n \left(\frac{1}{2} \Re\langle \phi_n^+ (\widetilde{\alpha}^\mu \hat{p}^\nu + \widetilde{\alpha}^\nu \hat{p}^\mu + 2 S_{ab}^{\mu\nu} h^\alpha{}_n \eta^{nb} \alpha^a \hat{p}_\alpha) \phi_n \rangle + g^{\mu\nu} F_n \right. \\ & \left. \qquad - \frac{1}{8} \epsilon^{abcd} h^\alpha{}_a h^\beta{}_b S_{cd}^{\mu\nu} (\partial_\alpha \check{s}_{n\beta} - \partial_\beta \check{s}_{n\alpha}) - g^{\mu\nu} \Omega_\alpha \check{s}_n^\alpha \right). \end{aligned} \tag{4.23}$$

The classical approximation becomes

$$T^{\mu\nu} \to \sum_n (m_n u_n^\mu u_n^\nu + w_n g^{\mu\nu}) \sqrt{1 - v_n^2}\, \delta^3(\vec{x} - \vec{X}_n), \tag{4.24}$$

which leads to the EMT for average field of spinor fluid as follows

$$T^{\mu\nu} = (\rho + P) U^\mu U^\nu + (W - P) g^{\mu\nu}. \tag{4.25}$$

The self potential becomes the negative pressure W[20].

V. DISCUSSION AND CONCLUSION

In this paper, according to the explicit relations between tetrad and metric, we derived the exact representation of $T_{\mu\nu}$ of spinors. By splitting the spinor connection into geometrical part Υ_μ and dynamical part Ω_μ, we find some new results. In this EMT, besides the usual kinetic energy momentum term, the nonlinear self-interactive potential W acts like negative pressure and may be important in astrophysics. The term $\Re\langle \phi^+ S_{ab}^{\mu\nu} h^\alpha{}_n \eta^{nb} \alpha^a \hat{p}_\alpha \phi \rangle$ reflects the interaction of momentum p^μ with tetrad, which vanishes in classical approximation. The spin-gravity coupling term is described by $\Omega_\alpha s^\alpha$, which may be the origin for magnetic field of a celestial body. This term also appears in the dynamics of a spinor and has location and navigation functions for the particle. The strong magnetic field in a neutron star may be closely related with this term.

The decomposition of spin connection Γ_μ not only makes calculation simpler, but also highlights their different physical meanings. Only in this representation, we can discover and clarify the special effects as discussed above. In the calculation of tetrad formalism we get a new tensor $S_{ab}^{\mu\nu}$ which plays an important role in the interaction of spinor with gravity and appears in many places. But it has not classical correspondences, and vanishes in classical approximation.

The spinor has only a tiny but marvelous structure. It is an indivisible system and unsuitable to describe many body problem. Only by using one spinor to describe one particle, we get a harmonic picture for field theory and classical mechanics.

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