Beyond Einstein-Cartan gravity: Quadratic torsion and curvature invariants with even and odd parity including all boundary terms

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Abstract. Recently, gravitational gauge theories with torsion have been discussed by an increasing number of authors from a classical as well as from a quantum field theoretical point of view. The Einstein-Cartan(-Sciama-Kibble) Lagrangian has been enriched by the parity odd pseudoscalar curvature (Hojman, Mukku, and Sayed) and by torsion square and curvature square pieces, likewise of even and odd parity. (i) We show that the inverse of the so-called Barbero-Immirzi parameter multiplying the pseudoscalar curvature, because of the topological Nieh-Yan form, can only be appropriately discussed if torsion square pieces are included. (ii) The quadratic gauge Lagrangian with both parities, proposed by Obukhov et al. and Baekler et al., emerges also in the framework of Diakonov et al. (2011). We establish the exact relations between both approaches by applying the topological Euler and Pontryagin forms in a Riemann-Cartan space expressed for the first time in terms of irreducible pieces of the curvature tensor. (iii) Only in a Riemann-Cartan spacetime, that is, in a spacetime with torsion, parity violating terms can be brought into the gravitational Lagrangian in a straightforward and natural way. Accordingly, Riemann-Cartan spacetime is a natural habitat for chiral fermionic matter fields.

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1. Einstein-Cartan theory and weak gravity

In gauge-theoretical approaches to gravity (see [1, 2, 3, 4]), we have the orthonormal coframe 1-form ϑ^α as the translational potential and the connection 1-form $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ as the Lorentz potential. The corresponding field strength are the torsion 2-form T^α and the curvature 2-form $R^{\alpha\beta} = -R^{\beta\alpha}$. The first order gravitational theory in this framework is called the Poincaré gauge theory of gravity (PG).

The simplest model within PG is the Einstein-Cartan theory of gravity (EC), see [5], with the twisted gauge Lagrangian (κ = gravitational and Λ_0 = cosmological constant)‡

$$V_{\mathrm{EC}} := \frac{1}{2\kappa}\left(\eta_{\alpha\beta}\wedge R^{\alpha\beta} - 2\Lambda_0\eta\right) \quad \text{and with} \quad L_{\mathrm{tot}} = V_{\mathrm{EC}} + L(\psi, D\psi)\,, \tag{1}$$

where L is the matter Lagrangian depending on the minimally coupled fermionic/bosonic matter fields $\psi(x)$. This is a viable gravitational theory that deviates from general relativity at extremely high matter densities $\rho \gtrsim \rho_{\mathrm{crit}}$, with $\rho_{\mathrm{crit}} \approx m/\left(\lambda_{\mathrm{Compton}}\ell^2_{\mathrm{Planck}}\right)$ and m is the mass of the field, see also [7]. At the same time it is clear that GR can alternatively be reformulated as a teleparallelism theory with torsion square pieces in the Lagrangian. If we call the Newton-Einstein type of gravity "weak" gravity, then its general quadratic gauge Lagrangian reads (a_0 and a_1, a_2, a_3 are constants):

$$V^{+}_{\mathrm{weak}} = \frac{1}{2\kappa}\left(-a_0 R^{\alpha\beta}\wedge\eta_{\alpha\beta} - 2\Lambda_0\eta + T^\alpha\wedge\sum_{I=1}^{3} a_I\,{}^{\star(I)}T_\alpha\right). \tag{2}$$

Here ${}^{(I)}T_\alpha$ denotes the irreducible pieces of the torsion, with ${}^{(2)}T_\alpha := \vartheta_\alpha\wedge(e_\beta\rfloor T^\beta)/3$ (`vector`, 4 independent components), ${}^{(3)}T_\alpha := e_\alpha\rfloor(T^\beta\wedge\vartheta_\beta)/3$ (`axitor`, 4), and ${}^{(1)}T_\alpha := T_\alpha - {}^{(2)}T_\alpha - {}^{(3)}T_\alpha$ (`tentor`, 16). For the special cases $R^{\alpha\beta} = 0$, enforced by a corresponding Lagrange multiplier term in (2), we recover the teleparallel equivalent of GR, provided local Lorentz invariance of the gravitational Lagrangian is implemented, see [8, 9, 10, 11], and alternatively, for $T^\alpha = 0$, we find GR directly. Thus, GR is hidden in (2) in two totally different ways, a fact often overlooked.

To link up with the experience of GR, we recall that the Riemann-Cartan curvature 2-form $R^{\alpha\beta}$ can be decomposed into the (torsionfree) Riemann curvature $\widetilde{R}^{\alpha\beta}$ and in torsion dependent terms. For the curvature scalar this formula reads (see [6, 12])§

$$-R^{\alpha\beta}\wedge\eta_{\alpha\beta} = -\widetilde{R}^{\alpha\beta}\wedge\eta_{\alpha\beta} - T^\alpha\wedge{}^\star(-{}^{(1)}T_\alpha + 2\,{}^{(2)}T_\alpha + \frac{1}{2}{}^{(3)}T_\alpha) + 2d(\vartheta^\alpha\wedge{}^\star T_\alpha)\,. \tag{3}$$

If we substituted (3) into (2), then apart from a boundary term, see below, the Riemann curvature would emerge and the a_I in the quadratic torsion would get redefined. However, we don't apply this procedure to (2), since we don't want to leave the formalism of first order field theory.

‡ We follow the conventions of [6]. We have the coframe 1-form $\vartheta^\alpha = e_i{}^\alpha dx^i$ and the frame vectors $e_\beta = e^j{}_\beta\partial_j$, with $e_\beta\rfloor\vartheta^\alpha = \delta^\alpha_\beta$. Greek indices are raised and lowered by means of the Minkowski metric $g_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$. The volume 4-form is denoted by η, and $\eta_\alpha = {}^\star\vartheta_\alpha$, $\eta_{\alpha\beta} = {}^\star\vartheta_{\alpha\beta}$, $\eta_{\alpha\beta\gamma} = {}^\star\vartheta_{\alpha\beta\gamma}$, $\eta_{\alpha\beta\gamma\delta} = {}^\star\vartheta_{\alpha\beta\gamma\delta}$, where * is the Hodge star operator and $\vartheta^{\alpha\beta} := \vartheta^\alpha\wedge\vartheta^\beta$, etc.

§ The second minus sign on the right-hand-side of this equation is corrected. In [6], Eq.(5.9.18) was a sign error.

We marked the Lagrangian (2) with a plus sign + for being, as a twisted 4-form, parity even. However, already in 1980, Hojman, Mukku, and Sayed (HMS) [13] and Nelson [14] added the parity odd‖ pseudoscalar curvature piece $R_{\alpha\beta}\wedge\vartheta^{\alpha\beta}$ to the EC Lagrangian, see also [16, 17, 18, 19, 20]. More recently, in the context of the Ashtekar formalism [21], see Kiefer [22], and in loop quantum gravity, see Rovelli [23], this has become popular, see [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37], for related cosmological models see also [38, 39, 40, 41, 42, 43]. Including additionally odd torsion square pieces, we have (b_0, σ_1, σ_2 are constants, ${}^{(3)}R_{\alpha\beta}$ is the irreducible pseudoscalar curvature 2-form)

$$V^{-}_{\text{weak}} = -\frac{b_0}{2\kappa}\,{}^{(3)}R_{\alpha\beta}\wedge\vartheta^{\alpha\beta} + \frac{1}{\kappa}\left(\sigma_1{}^{(1)}T^{\alpha}\wedge{}^{(1)}T_{\alpha} + \sigma_2{}^{(2)}T^{\alpha}\wedge{}^{(3)}T_{\alpha}\right)\,. \tag{4}$$

The inverse of b_0 is sometimes called the Barbero-Immirzi parameter [44, 45, 28]. The total gauge Lagrangian would then read $V^{+}_{\text{weak}} + V^{-}_{\text{weak}}$. However, we should be aware that for weak gravity there exists a boundary term, the untwisted parity odd *Nieh-Yan* 4-form [16],

$$\begin{aligned} B^{-}_{TT} = dC^{-}_{TT} &= \frac{1}{2}\left(T^{\alpha}\wedge T_{\alpha} + R_{\alpha\beta}\wedge\vartheta^{\alpha\beta}\right) = \frac{1}{2}(T^{\alpha}\wedge T_{\alpha} - {}^{\star}X) \\ &= \frac{1}{2}\left({}^{(1)}T^{\alpha}\wedge{}^{(1)}T_{\alpha} + 2\,{}^{(2)}T^{\alpha}\wedge{}^{(3)}T_{\alpha} + {}^{(3)}R_{\alpha\beta}\wedge\vartheta^{\alpha\beta}\right)\,, \end{aligned} \tag{5}$$

with $C^{-}_{TT} := \frac{1}{2}\vartheta^{\alpha}\wedge T_{\alpha}$ and X as the curvature pseudoscalar, $X = \eta_{\alpha\beta\gamma\delta}R^{[\alpha\beta\gamma\delta]}/4!$. We add this form with a suitable constant f_1 to our weak gravity Lagrangian:

$$V_{\text{weak}} = V_{\text{weak}}(a_0; b_0; a_1, a_2, a_3; \sigma_1, \sigma_2; f_1) := V^{+}_{\text{weak}} + V^{-}_{\text{weak}} + \frac{f_1}{\kappa}B^{-}_{TT}\,. \tag{6}$$

It depends on the gravitational constant κ and the cosmological constant λ_0 and, furthermore, on the 8 constants specified in (6). By a suitable choice of f_1, we can compensate either the HMS-term [13] (that is, $b_0 = 0$) or one tensor square term of the torsion (that is, either $\sigma_1 = 0$ or $\sigma_2 = 0$). However, since ${}^{(1)}T^{\alpha}$ depends on 16 independent components, the pseudoscalar curvature only on 1 component, it seems to simplify the Lagrangian to a greater extent, if we kick out the term with ${}^{(1)}T^{\alpha}$. Thus, for the weak gravity Lagrangian we are left with **6** unspecified constants $(a_0, b_0; a_1, a_2, a_3; \sigma_2)$¶.

Looking back at Eq. (3), it could appear that we forgot the boundary term $B^{+}_{TT} = \frac{1}{2}(d\vartheta^{\alpha}\wedge{}^{\star}T_{\alpha})$ and that we could add it as $\frac{f_0}{\kappa}B^{+}_{TT}$ to the Lagrangian (6), see the procedure of Mielke [47, 48]. However, if we compare the Nieh-Yan and the "teleparallel" formulas

$$d\left(\vartheta^{\alpha}\wedge T_{\alpha}\right) = T^{\alpha}\wedge T_{\alpha} - \vartheta^{\alpha}\wedge DT_{\alpha} \quad\text{and}\quad \left(\vartheta^{\alpha}\wedge{}^{\star}T_{\alpha}\right) = T^{\alpha}\wedge{}^{\star}T_{\alpha} - \vartheta^{\alpha}\wedge D\,{}^{\star}T_{\alpha}, \tag{7}$$

respectively, then we recognize that in the former equation DT_{α} can be eliminated via the first Bianchi identity $DT_{\alpha} = R_{\beta\alpha}\wedge\vartheta^{\beta}$, whereas in the latter equation such a trick

‖ Parity even and *odd* torsion square terms were introduced by Purcell [15] even 2 years ealier.

¶ Diakonov et al. [46] found an *equivalent* result, but they eliminated the curvature pseudoscalar. Thus, they are left with the 6 unspecified constants $(a_0; a_1, a_2, a_3; \sigma_1, \sigma_2)$, that is, with the curvature scalar plus the 5 torsion-square pieces.

is impossible. Therefore, we would trade in for torsion square pieces derivatives of the torsion and would mess up the first order character of our Lagrangian.+

If one starts with the EC-Lagrangian and adds *only* an HMS-term, as numerous people do, then, because of the Nieh-Yan form, two torsion square terms of odd parity are induced. Consequently, the other torsion square terms come immediately into focus and —this is one message of our letter—there doesn't seem to exist a sufficient reason to exclude the other torsion square terms. In other words, the whole weak gravity Lagrangian (6) should come under scrutiny.

From a totally different point of view, from observational cosmology and from quantum chromodynamics, there are indications that we may live in a parity violating Universe, see the review by Urban & Zhitnitsky [49]. All the more investigations in a parity odd PG model seem desirable.

2. Quadratic Poincaré gauge theory and strong gravity

If one wants the Lorentz connection as a propagating field, then one has to allow for "strong" gravity of the Yang-Mills type by adding quadratic curvature 4-forms to the weak Lagrangian. The curvature $R^{\alpha\beta}$ of a Riemann-Cartan space has six irreducible pieces: $R^{\alpha\beta} = \sum_{I=1}^{6} {}^{(I)}R^{\alpha\beta}$. We write symbolically, using the self-explanatory computer names for the irreducible terms: `curv` (36 indep. comp.) = `weyl` (10) + `paircom` (9) + `pscalar` (1) + `ricsymf` (9) + `ricanti` (6) + `scalar` (1), see [6] for the exact definitions. In a Riemann space only ${}^{(1)}R^{\alpha\beta}$, ${}^{(4)}R^{\alpha\beta}$, and ${}^{(6)}R^{\alpha\beta}$ are left over. Hence the most general parity even quadratic Lagrangian, with a new dimensionless coupling constant ϱ, reads

$$V^{+}_{\text{strong}} = -\frac{1}{2\varrho} R^{\alpha\beta} \wedge \sum_{I=1}^{6} w_I \, {}^{\star(I)}R_{\alpha\beta} \, . \tag{9}$$

Alerted by the corresponding case in weak gravity, we now search for parity odd terms. They were found to be (see [50, 51]) as

$$\begin{aligned} V^{-}_{\text{strong}} = & -\frac{1}{2\varrho}\left(\mu_1 {}^{(1)}R^{\alpha\beta} \wedge {}^{(1)}R_{\alpha\beta} + \mu_2 {}^{(2)}R^{\alpha\beta} \wedge {}^{(4)}R_{\alpha\beta} \right. \\ & \left. + \mu_3 {}^{(3)}R^{\alpha\beta} \wedge {}^{(6)}R_{\alpha\beta} + \mu_4 {}^{(5)}R^{\alpha\beta} \wedge {}^{(5)}R_{\alpha\beta}\right) , \end{aligned} \tag{10}$$

which are the only quadratic curvature square invariants of odd character in a 4D Riemann-Cartan space. Note that in a Riemann space, that is, when torsion vanishes, only the first piece built up from the Weyl curvature ${}^{(1)}R^{\alpha\beta}$ is left over.

Taking a lesson from the above, we can now search for boundary terms. As in any Yang-Mills theory, we can find an untwisted *Pontryagin* 4-form B^{-}_{RR}. But in gravity the (anholonomic) Lorentz indices of the curvature can be contracted with the help of the

+ We can combine the two equations in (7). This yields yields

$$d\left(\vartheta^{\alpha} \wedge \xi^{\pm}_{\alpha}\right) = T^{\alpha} \wedge \xi^{\pm}_{\alpha} - \vartheta^{\alpha} \wedge D\xi^{\pm}_{\alpha} \qquad \text{with} \qquad \xi^{\pm}_{\alpha} := T_{\alpha} \pm {}^{\star}T_{\alpha} \, . \tag{8}$$

Mielke [47, 48] built a similar linear combination but with the imaginary unit in front of ${}^{\star}T_{\alpha}$, but neither his nor our version seems to lead to firm conclusions so far.

totally antisymmetric Levi-Civita tensor $\eta^{\alpha\beta\gamma\delta}$. Accordingly, we introduce the so-called Lie-dual of the curvature with the Lie star operator ${}^{(*)}$ as

$$R^{(*)\,\alpha\beta} := \frac{1}{2} R_{\mu\nu}\eta^{\mu\nu\alpha\beta} \,. \tag{11}$$

Like $\eta^{\mu\nu\alpha\beta}$, the Lie star ${}^{(*)}$ is twisted. It gives rise to the twisted *Euler* 4-form $B^{+}_{RR^{(*)}}$. Following [6], we have then the following two boundary terms:

$$B^{-}_{RR} = dC^{-}_{RR} = \frac{1}{2} R_{\alpha\beta} \wedge R^{\alpha\beta} \,, \quad B^{+}_{RR^{(*)}} = dC^{+}_{RR^{(*)}} = \frac{1}{2} R_{\alpha\beta} \wedge R^{(*)\alpha\beta} \,. \tag{12}$$

Thus, the strong part of the gauge Lagrangian turns out to be

$$V_{\text{strong}} := V^{+}_{\text{strong}} + V^{-}_{\text{strong}} + \frac{f_2}{\varrho} B^{-}_{RR} + \frac{f_3}{\varrho} B^{+}_{RR^{(*)}} \,. \tag{13}$$

The total quadratic gauge Lagrangian including boundary terms is then

$$V_{\text{gauge}} = V^{+}_{\text{weak}} + V^{-}_{\text{weak}} + V^{+}_{\text{strong}} + V^{-}_{\text{strong}} + \frac{f_1}{\kappa} B^{-}_{TT} + \frac{f_2}{\varrho} B^{-}_{RR} + \frac{f_3}{\varrho} B^{+}_{RR^{(*)}} . \tag{14}$$

For *vanishing torsion*, V_{weak} reduces to V_{GR} with cosmological constant and V_{strong} has only $(w_1, w_4, w_6; \mu_1; f_2, f_3) \neq 0$. By a suitable choice of f_2, f_3, only the two terms with w_4, w_6 survive, that is, those with the tracefree Ricci tensor and the curvature scalar.

3. The role of the Lie-dual of the curvature

Before we continue the investigation of (13), we will derive some rules for manipulating curvature square terms containing a Lie star. Using heavily the computer-algebra system `Reduce` with the `Excalc` package, compare [52, 53, 54, 55], we were able to convert completely the Lie star ${}^{(*)}$ into the Hodge star * according to the following rules: The expression ${}^{(I)}R^{\mu\nu} \wedge {}^{(J)}R^{(*)}_{\mu\nu}$ is diagonal, that is, $\propto \delta^{IJ}$; only the diagonal pieces do not vanish, namely

$${}^{(I)}R^{\mu\nu} \wedge {}^{(I)}R^{(*)}_{\mu\nu} = \pm\, {}^{(I)}R^{\mu\nu} \wedge {}^{\star(I)}R_{\mu\nu} \,, \tag{15}$$

with $+$ for $I = 1, 3, 5, 6$ and with $-$ for $I = 2, 4$. Note that on the left-hand-side of this equation we have the Lie star ${}^{(*)}$, on the right-hand-side, however, the Hodge star *. This implies the relation, derived here for the first time explicitly*,

$$\begin{aligned} R^{\alpha\beta} \wedge R^{(*)}_{\alpha\beta} = {}& {}^{(1)}R^{\alpha\beta} \wedge {}^{\star(1)}R_{\alpha\beta} - {}^{(2)}R^{\alpha\beta} \wedge {}^{\star(2)}R_{\alpha\beta} + {}^{(3)}R^{\alpha\beta} \wedge {}^{\star(3)}R_{\alpha\beta} \\ & - {}^{(4)}R^{\alpha\beta} \wedge {}^{\star(4)}R_{\alpha\beta} + {}^{(5)}R^{\alpha\beta} \wedge {}^{\star(5)}R_{\alpha\beta} + {}^{(6)}R^{\alpha\beta} \wedge {}^{\star(6)}R_{\alpha\beta} \,. \end{aligned} \tag{16}$$

In particular, this shows that the Lie star is superfluous in forming a quadratic Lagrangian, the Hodge star is sufficient.

Comparison with (12) allows us to rewrite the *Euler* 4-form with the help of the Hodge star as

$$B^{+}_{RR^{(*)}} = \frac{1}{2} \left(R^{\alpha\beta} \wedge {}^{\star}R_{\alpha\beta} - 2\, {}^{(4)}R^{\alpha\beta} \wedge {}^{\star(4)}R_{\alpha\beta} - 2\, {}^{(2)}R^{\alpha\beta} \wedge {}^{\star(2)}R_{\alpha\beta} \right) . \tag{17}$$

* By using some simple algebra, Eq.(16) can alternatively be derived in a straightforward way from Eqs.(10.17) to (10.22) of Obukhov [4].

The *Pontryagin* 4-form, also defined in (12), after some algebra, can be expressed in terms of the irreducible pieces of the curvature as (also this relation is new)

$$\begin{aligned} B^{-}_{RR} \;\; &= \frac{1}{2}\big({}^{(1)}R_{\alpha\beta}\wedge {}^{(1)}R^{\alpha\beta} + \; {}^{(5)}R_{\alpha\beta}\wedge {}^{(5)}R^{\alpha\beta} \\ &\quad +2\,{}^{(3)}R_{\alpha\beta}\wedge {}^{(6)}R^{\alpha\beta} + 2\,{}^{(2)}R_{\alpha\beta}\wedge {}^{(4)}R^{\alpha\beta}\big)\;. \end{aligned} \tag{18}$$

4. Comparison with Diakonov et al. [46]

Recently, in the framework of perturbative quantum field theory, new results were reported [46] on including torsion in a gravitational gauge theory for describing fermionic matter, for a review of some earlier results, see [56]. In this context, Diakonov et al. investigated gravitational gauge Lagrangians containing quadratic terms in the gauge fields of even and of odd parity. That is, contributions of the weak and the strong gravity sector (Lorentz gauge bosons) were considered. Generally, those 4-fermion interaction terms will give additional contributions to the energy-momentum current of matter. This aspects might be relevant in the context of quantum cosmological models, see [39, 42, 43]. Other groups address only weak gravity, even though they include Euler and Pontryagin terms, which refer to strong gravity, see Benedetti et al. [57].

In the following, we would like to compare the approach given in [46] with the results we already gave in [51]♯.

4.1. Torsion square invariants

Let us compare our torsion-square invariants in (6), see also [51], with those in [46], Eq.(53). If we add a plus sign for parity even and a minus sign for parity odd terms, the invariants of Diakonov et al., multiplied by the volume form η, read:

$$K^{+}_{1} = 2\,{}^{(1)}T^{\alpha}\wedge {}^{\star(1)}T_{\alpha} - 2\,{}^{(2)}T^{\alpha}\wedge {}^{\star(2)}T_{\alpha} + 2\,{}^{(3)}T^{\alpha}\wedge {}^{\star(3)}T_{\alpha}\,, \tag{19}$$

$$K^{+}_{2} = 3\,{}^{(2)}T^{\alpha}\wedge {}^{\star(2)}T_{\alpha}\,, \tag{20}$$

$$K^{+}_{3} = \; {}^{(1)}T^{\alpha}\wedge {}^{\star(1)}T_{\alpha} + \; {}^{(2)}T^{\alpha}\wedge {}^{\star(2)}T_{\alpha} - 2\,{}^{(3)}T^{\alpha}\wedge {}^{\star(3)}T_{\alpha}\,, \tag{21}$$

$$K^{-}_{4} = \; -2\,{}^{(1)}T^{\alpha}\wedge \; {}^{(1)}T_{\alpha} + 4\,{}^{(2)}T^{\alpha}\wedge \; {}^{(3)}T_{\alpha}\,, \tag{22}$$

$$K^{-}_{5} = \; {}^{(1)}T^{\alpha}\wedge \; {}^{(1)}T_{\alpha} + 4\,{}^{(2)}T^{\alpha}\wedge \; {}^{(3)}T_{\alpha}\,, \tag{23}$$

and the inverse relations are

$${}^{(1)}T^{\alpha}\wedge {}^{\star(1)}T_{\alpha} = \frac{1}{9}(3K^{+}_{1} + K^{+}_{2} + 3K^{+}_{3})\,, \tag{24}$$

$${}^{(2)}T^{\alpha}\wedge {}^{\star(2)}T_{\alpha} = \frac{1}{3}K^{+}_{2}\,, \tag{25}$$

♯ Following essentially Schouten [58], our conventions are [6]: $\vartheta^{\alpha} = e_i{}^{\alpha}dx^i$, $\Gamma^{\alpha\beta} = \Gamma_i{}^{\alpha\beta}dx^i$, $T^{\alpha} := D\vartheta^{\alpha} = \frac{1}{2}\,T_{ij}{}^{\alpha}dx^i\wedge dx^j$, $T_{ij}{}^{\alpha} = 2(\partial_{[i}\vartheta_{j]}{}^{\alpha} + \Gamma_{[ij]}{}^{\alpha})$, $R^{\alpha\beta} := d\Gamma^{\alpha\beta} + \Gamma^{\alpha\gamma}\wedge\Gamma^{\beta}{}_{\gamma} = \frac{1}{2}\,R_{ij}{}^{\alpha\beta}dx^i\wedge dx^j$, $R_{ij}{}^{\alpha\beta} = 2(\partial_{[i}\Gamma_{j]}{}^{\alpha\beta} + \Gamma_{[i}{}^{\alpha\gamma}\Gamma_{j]\,\cdot\,\gamma}{}^{\beta})$, $\mathrm{Ric}_{\alpha} := e_{\beta}\rfloor R_{\alpha}{}^{\beta} = \mathrm{Ric}_{\beta\alpha}\,\vartheta^{\beta}$, with $\mathrm{Ric}_{\alpha\beta} = R_{\gamma\alpha\beta}{}^{\gamma}$, $R := \mathrm{Ric}_{\alpha}{}^{\alpha} = R_{\beta\alpha}{}^{\alpha\beta}$. Diakonov et al. [46] use the conventions of Landau-Lifschitz [59]. Accordingly, there are the following correspondence rules: $R^{\kappa}{}_{\lambda\mu\nu}|_{\text{Diak et al.}} = R_{\mu\nu\lambda}{}^{\kappa}|_{\text{here}}$, $\mathrm{Ric}_{\lambda\mu}|_{\text{Diak et al.}} := R_{\lambda}{}^{\kappa}{}_{\mu\kappa}|_{\text{Diak et al.}} = R_{\mu\kappa}{}^{\kappa}{}_{\lambda}|_{\text{here}} = R_{\kappa\mu\lambda}{}^{\kappa}|_{\text{here}} = \mathrm{Ric}_{\mu\lambda}|_{\text{here}}$.

$$
\begin{aligned}
{}^{(3)}T^{\alpha} \wedge {}^{\star(3)}T_{\alpha} &= \frac{1}{12}(3K_1^+ + 4K_2^+ - 6K_3^+)\,, &(26)\\
{}^{(1)}T^{\alpha} \wedge {}^{(1)}T_{\alpha} &= -\frac{1}{3}(K_4^- - K_5^-)\,, &(27)\\
{}^{(2)}T^{\alpha} \wedge {}^{(3)}T_{\alpha} &= \frac{1}{12}(K_4^- + 2K_5^-)\,. &(28)
\end{aligned}
$$

These 5 invariants agree with those given in [51], Eqs.(30) and (55).

4.2. Curvature square invariants

Diakonov et al. [46], Eq. (59), find 6 even and 4 odd independent quadratic invariants G_I. As we did with the torsion invariants, we translate their component representations into the language of exterior differential forms used by us. We find, after some messy computer checked algebra, the following curvature invariants (multiplied by η):

$$
G_1^+ = R^2\eta = 12\,{}^{(6)}R_{\alpha\beta} \wedge {}^{\star(6)}R^{\alpha\beta}\,, \tag{29}
$$

$$
G_2^+ = R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}\eta = 2\,R_{\alpha\beta} \wedge {}^{\star}R^{\alpha\beta} = 2\sum_{I=1}^{6} {}^{(I)}R^{\alpha\beta} \wedge {}^{\star(I)}R_{\alpha\beta}\,, \tag{30}
$$

$$
\begin{aligned}
G_3^+ &= R_{\mu\nu\rho\lambda}R^{\lambda\rho\nu\mu}\eta\\
&= 2\left({}^{(1)}R^{\alpha\beta} \wedge {}^{\star(1)}R_{\alpha\beta} - {}^{(2)}R^{\alpha\beta} \wedge {}^{\star(2)}R_{\alpha\beta} + {}^{(3)}R^{\alpha\beta} \wedge {}^{\star(3)}R_{\alpha\beta}\right.\\
&\quad\left. + {}^{(4)}R^{\alpha\beta} \wedge {}^{\star(4)}R_{\alpha\beta} - {}^{(5)}R^{\alpha\beta} \wedge {}^{\star(5)}R_{\alpha\beta} + {}^{(6)}R^{\alpha\beta} \wedge {}^{\star(6)}R_{\alpha\beta}\right)\,,
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
G_4^+ &= \left(R^2 - 4\mathrm{Ric}_{\mu\lambda}\mathrm{Ric}^{\lambda\mu} + R_{\mu\nu\rho\lambda}R^{\lambda\rho\nu\mu}\right)\eta\\
&= 2\left({}^{(1)}R^{\alpha\beta} \wedge {}^{\star(1)}R_{\alpha\beta} - {}^{(2)}R^{\alpha\beta} \wedge {}^{\star(2)}R_{\alpha\beta} + {}^{(3)}R^{\alpha\beta} \wedge {}^{\star(3)}R_{\alpha\beta}\right.\\
&\quad\left. - {}^{(4)}R^{\alpha\beta} \wedge {}^{\star(4)}R_{\alpha\beta} + {}^{(5)}R^{\alpha\beta} \wedge {}^{\star(5)}R_{\alpha\beta} + {}^{(6)}R^{\alpha\beta} \wedge {}^{\star(6)}R_{\alpha\beta}\right)\,,
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
G_5^+ &= \left(R^2 - 4\mathrm{Ric}_{\mu\lambda}\mathrm{Ric}^{\mu\lambda} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}\right)\eta\\
&= 2\left({}^{(1)}R^{\alpha\beta} \wedge {}^{\star(1)}R_{\alpha\beta} + {}^{(2)}R^{\alpha\beta} \wedge {}^{\star(2)}R_{\alpha\beta} + {}^{(3)}R^{\alpha\beta} \wedge {}^{\star(3)}R_{\alpha\beta}\right.\\
&\quad\left. - {}^{(4)}R^{\alpha\beta} \wedge {}^{\star(4)}R_{\alpha\beta} - {}^{(5)}R^{\alpha\beta} \wedge {}^{\star(5)}R_{\alpha\beta} + {}^{(6)}R^{\alpha\beta} \wedge {}^{\star(6)}R_{\alpha\beta}\right)\,,
\end{aligned} \tag{33}
$$

$$
G_6^+ = \left(\eta^{\lambda\rho\mu\nu}R_{\mu\nu\rho\lambda}\right)^2\eta = -48\,{}^{(3)}R_{\alpha\beta} \wedge {}^{\star(3)}R^{\alpha\beta}\,, \tag{34}
$$

$$
G_7^- = R\eta^{\lambda\rho\mu\nu}R_{\mu\nu\rho\lambda}\eta = -24\,{}^{(3)}R_{\alpha\beta} \wedge {}^{(6)}R^{\alpha\beta}\,, \tag{35}
$$

$$
\begin{aligned}
G_8^- &= \eta^{\mu\nu\alpha\beta}R_{\mu\nu\rho\lambda}R_{\alpha\beta}{}^{\rho\lambda}\eta\\
&= -4\,{}^{(1)}R_{\alpha\beta} \wedge {}^{(1)}R^{\alpha\beta} - 4\,{}^{(5)}R_{\alpha\beta} \wedge {}^{(5)}R^{\alpha\beta}\\
&\quad - 8\,{}^{(3)}R_{\alpha\beta} \wedge {}^{(6)}R^{\alpha\beta} - 8\,{}^{(2)}R_{\alpha\beta} \wedge {}^{(4)}R^{\alpha\beta}\,,
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
G_9^- &= \eta^{\lambda\rho\gamma\delta}R_{\mu\nu\rho\lambda}R^{\mu\nu}{}_{\delta\gamma}\eta\\
&= -4\,{}^{(1)}R_{\alpha\beta} \wedge {}^{(1)}R^{\alpha\beta} - 4\,{}^{(5)}R_{\alpha\beta} \wedge {}^{(5)}R^{\alpha\beta}\\
&\quad - 8\,{}^{(3)}R_{\alpha\beta} \wedge {}^{(6)}R^{\alpha\beta} + 8\,{}^{(2)}R_{\alpha\beta} \wedge {}^{(4)}R^{\alpha\beta}\,,
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
G_{10}^- &= \eta^{\lambda\rho\alpha\beta}R_{\mu\nu\rho\lambda}R_{\alpha\beta}{}^{\nu\mu}\eta\\
&= -4\,{}^{(1)}R_{\alpha\beta} \wedge {}^{(1)}R^{\alpha\beta} + 4\,{}^{(5)}R_{\alpha\beta} \wedge {}^{(5)}R^{\alpha\beta} - 8\,{}^{(3)}R_{\alpha\beta} \wedge {}^{(6)}R^{\alpha\beta}\,.
\end{aligned} \tag{38}
$$

The inverse relations are convenient for a detailed comparison. They turn out to be

$$
{}^{(1)}R^{\alpha\beta} \wedge {}^{\star(1)}R_{\alpha\beta} = -\frac{1}{12}G_1^+ + \frac{1}{8}\left(G_2^+ + G_3^+ + G_4^+ + G_5^+\right) + \frac{1}{48}G_6^+\,, \tag{39}
$$

$$^{(2)}R^{\alpha\beta}\wedge{}^{\star(2)}R_{\alpha\beta}=\frac{1}{8}\left(G_2^+-G_3^+-G_4^++G_5^+\right)\,, \tag{40}$$

$$^{(3)}R^{\alpha\beta}\wedge{}^{\star(3)}R_{\alpha\beta}=-\frac{1}{48}G_6^+\,, \tag{41}$$

$$^{(4)}R^{\alpha\beta}\wedge{}^{\star(4)}R_{\alpha\beta}=\frac{1}{8}\left(G_2^++G_3^+-G_4^+-G_5^+\right)\,, \tag{42}$$

$$^{(5)}R^{\alpha\beta}\wedge{}^{\star(5)}R_{\alpha\beta}=\frac{1}{8}\left(G_2^+-G_3^++G_4^+-G_5^+\right)\,, \tag{43}$$

$$^{(6)}R^{\alpha\beta}\wedge{}^{\star(6)}R_{\alpha\beta}=\frac{1}{12}G_1^+\,, \tag{44}$$

and

$$^{(1)}R^{\alpha\beta}\wedge{}^{(1)}R_{\alpha\beta}=-\frac{1}{16}\left(G_8^-+G_9^-+2G_{10}^-\right)+\frac{1}{12}G_7^-\,, \tag{45}$$

$$^{(2)}R^{\alpha\beta}\wedge{}^{(4)}R_{\alpha\beta}=-\frac{1}{16}\left(G_8^--G_9^-\right)\,, \tag{46}$$

$$^{(3)}R^{\alpha\beta}\wedge{}^{(6)}R_{\alpha\beta}=-\frac{1}{24}G_7^-\,, \tag{47}$$

$$^{(5)}R^{\alpha\beta}\wedge{}^{(5)}R_{\alpha\beta}=-\frac{1}{16}\left(G_8^-+G_9^--2G_{10}^-\right)\,. \tag{48}$$

It is now straightforward to express the Euler 4-form (17) and the Pontryagin 4-form (18) in terms of the G_I's. We find

$$B^+_{RR^{(*)}}=\frac{1}{4}G_4^+\qquad\text{and}\qquad B^-_{RR}=-\frac{1}{8}G_8^-\,, \tag{49}$$

respectively. This is what Diakonov et al. stressed: that their invariants G_4^+ and G_8^- are boundary terms. These two boundary terms can also be found in our earlier work, see [51], Eqs.(33) and (50).

Hence the results of Diakonov et al. [46] with respect to the quadratic invariants of torsion and curvature coincide with those of [51]. This is also manifest in the Riemannian subcase, that is, for *vanishing torsion* $T^\alpha=0$. Then,

$$^{(2)}R_{\alpha\beta}={}^{(3)}R_{\alpha\beta}={}^{(5)}R_{\alpha\beta}=0\,, \tag{50}$$

or, in terms of the G_I's,

$$G_2^+=G_3^+\,,\quad G_4^+=G_5^+\,,\quad G_6^+=G_7^+=0\,,\quad G_8^+=G_9^+=G_{10}^+\,, \tag{51}$$

which can be read off directly from the Eqs.(29) to (38). Under the condition of vanishing torsion the boundary terms read

$$B^+_{RR^{(*)}}|_{T^\alpha=0}=\frac{1}{2}\left({}^{(1)}R^{\alpha\beta}\wedge{}^{\star(1)}R_{\alpha\beta}-{}^{(4)}R^{\alpha\beta}\wedge{}^{\star(4)}R_{\alpha\beta}+{}^{(6)}R^{\alpha\beta}\wedge{}^{\star(6)}R_{\alpha\beta}\right)\,, \tag{52}$$

$$B^-_{RR}|_{T^\alpha=0}\;=\frac{1}{2}\,{}^{(1)}R^{\alpha\beta}\wedge{}^{(1)}R_{\alpha\beta}\,. \tag{53}$$

5. The number of independent terms in the most general quadratic PG Lagrangian

For the strong gravitational Lagrangian $V_{\rm strong}$ in (13), we can enter in a similar discussion as for $V_{\rm weak}$ in (6): Besides the strong gravitational coupling constant ϱ, we have the 12 constants $(w_1, w_2, ..., w_6; \mu_1, \mu_2, \mu_3, \mu_4; f_2, f_3)$. By a suitable choice of f_2 and f_3, we can compensate the terms containing the Weyl 2-form ${}^{(1)}R^{\alpha\beta}$ with 10 independent components, as can be seen from (17) and (18). Consequently, we are left with the **8** constants $(w_2, ..., w_6; \mu_2, \mu_3, \mu_4)$. Diakonov et al. [46] found the 10 invariants G_I, two of which, namely G_4^+ and G_8^- are boundary terms. Hence they also arrive at 8 independent invariants. Accordingly, also for strong gravity our results match those of Diakonov et al.

Our final gravitational Lagrangian is then††

$$\begin{aligned} V = & \frac{1}{2\kappa}\Big[\,(\,a_0 R - 2\Lambda_0 + b_0 X\,)\,\eta \\ & \qquad + \frac{a_2}{3}\mathcal{V}\wedge{}^\star\mathcal{V} - \frac{a_3}{3}\mathcal{A}\wedge{}^\star\!\mathcal{A} - \frac{2\sigma_2}{3}\mathcal{V}\wedge{}^\star\!\mathcal{A} + a_1{}^{(1)}T^\alpha\wedge{}^{\star(1)}T_\alpha\Big] \\ & - \frac{1}{2\varrho}\Big[\big(\frac{w_6}{12}R^2 - \frac{w_3}{12}X^2 + \frac{\mu_3}{12}RX\big)\,\eta + w_4{}^{(4)}R^{\alpha\beta}\wedge{}^{\star(4)}R_{\alpha\beta} \\ & \qquad + {}^{(2)}R^{\alpha\beta}\wedge\big(w_2{}^{\star(2)}R_{\alpha\beta} + \mu_2{}^{(4)}R_{\alpha\beta}\big) + {}^{(5)}R^{\alpha\beta}\wedge\big(w_5{}^{\star(5)}R_{\alpha\beta} + \mu_4{}^{(5)}R_{\alpha\beta}\big)\Big]\,. \end{aligned} \tag{54}$$

The first two lines represent weak gravity, the last two lines strong gravity. The parity odd pieces are those with the constants $b_0, \sigma_2; \mu_2, \mu_3, \mu_4$. In a Riemann space (where $X = 0$), only two terms of the first line and likewise two terms in the third line survive. All these 4 terms are parity even, that is, only torsion brings in parity odd pieces into the gravitational Lagrangian.

Yo and Nester [60, 61, 62] found that only a small subclass of the Lagrangians (54) is consistent from a Hamiltonian point of view. They, together with Shie, presented such a Lagrangian [63] and found an accelerating cosmological Friedman type model with propagating connection. Shortly afterwards, Nester and his group, see Chen et al. [64], generalized this model and introduced a consistent Lagrangian containing the parity odd pieces $\mathcal{A}$ and X. Since these terms occur quadratically, their Lagrangian was still parity even:

$$\begin{aligned} V_{\text{Chen et al.}} = & \frac{1}{2\kappa}\,(a_0 R - 2\Lambda_0)\,\eta + \frac{1}{6\kappa}\,(a_2\mathcal{V}\wedge{}^\star\mathcal{V} - a_3\mathcal{A}\wedge{}^\star\!\mathcal{A}) \\ & - \frac{1}{24\varrho}\left(w_6 R^2 - w_3 X^2\right)\eta\,. \end{aligned} \tag{55}$$

The next step was done by Baekler et al. [51], see also [65]. They investigated a Lagrangian with three additional pieces with odd parity (55), namely those carrying

††If we introduce the notations R and X for the curvature scalar and the curvature pseudoscalar, then we find ${}^{(6)}R_{\alpha\beta} = -R\,\vartheta_{\alpha\beta}/12$ and ${}^{(3)}R_{\alpha\beta} = -X\eta_{\alpha\beta}/12$, respectively; moreover, for the torsion we can define the 1-forms of $\mathcal{A}$ and $\mathcal{V}$ for the axial vector and the vector torsion ${}^{(3)}T^\alpha = {}^\star(\mathcal{A}\wedge\vartheta^\alpha)/3$ and ${}^{(2)}T^\alpha = -(\mathcal{V}\wedge\vartheta^\alpha)/3$, respectively.

the constants b_0, σ_2, μ_3:

$$V_{\rm BHN} = \frac{1}{2\kappa}(a_0 R - 2\Lambda_0 + b_0 X)\eta + \frac{1}{6\kappa}\left(a_2 \mathcal{V}\wedge{}^\star\mathcal{V} - a_3\mathcal{A}\wedge{}^\star\mathcal{A} - 2\sigma_2\mathcal{V}\wedge{}^\star\mathcal{A}\right) - \frac{1}{24\varrho}\left(w_6 R^2 - w_3 X^2 + \mu_3 R X\right)\eta\,. \tag{56}$$

Now one should analyze the particle content of the lagrangian (54), that is, to find out which modes are propagating decently. This has been done in [51] by the simple method of the *diagonalization of the Lagrangian*. The results turned out to be in agreement with those of the Hamiltonian approach.

It is manifest already by now, looking beyond the Einstein-Cartan theory including parity odd Lagrangians is a field with bright prospects.

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Beyond Einstein-Cartan gravity: Quadratic torsion and curvature invariants with even and odd parity including all boundary terms

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Abstract. Recently, gravitational gauge theories with torsion have been discussed by an increasing number of authors from a classical as well as from a quantum field theoretical point of view. The Einstein-Cartan(-Sciama-Kibble) Lagrangian has been enriched by the parity odd pseudoscalar curvature (Hojman, Mukku, and Sayed) and by torsion square and curvature square pieces, likewise of even and odd parity. (i) We show that the inverse of the so-called Barbero-Immirzi parameter multiplying the pseudoscalar curvature, because of the topological Nieh-Yan form, can be appropriately discussed if torsion square pieces are included. (ii) The quadratic gauge Lagrangian with both parities, proposed by Obukhov et al. and Baekler et al., emerges also in the framework of Diakonov et al. (2011). We establish the exact relations between both approaches by applying the topological Euler and Pontryagin forms in a Riemann-Cartan space expressed for the first time in terms of irreducible pieces of the curvature tensor. (iii) In a Riemann-Cartan spacetime, that is, in a spacetime with torsion, parity violating terms can be brought into the gravitational Lagrangian in a straightforward and natural way. Accordingly, Riemann-Cartan spacetime is a natural habitat for chiral fermionic matter fields.

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1. Einstein-Cartan theory and weak gravity

In gauge-theoretical approaches to gravity (see [1, 2, 3, 4]), we have the orthonormal coframe 1-form ϑ^α as the translational potential and the connection 1-form $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ as the Lorentz potential. The corresponding field strengths are the torsion 2-form T^α and the curvature 2-form $R^{\alpha\beta} = -R^{\beta\alpha}$. The first order gravitational theory in this framework is called the Poincaré gauge theory of gravity (PG).

The simplest model within PG is the Einstein-Cartan theory of gravity (EC), see [5], with the twisted gauge Lagrangian (κ = gravitational and Λ_0 = cosmological constant‡

$$V_{\rm EC} := \frac{1}{2\kappa}\left(\eta_{\alpha\beta}\wedge R^{\alpha\beta} - 2\Lambda_0\eta\right) \quad \text{and with} \quad L_{\rm tot} = V_{\rm EC} + L(\psi, D\psi)\,, \tag{1}$$

where L is the matter Lagrangian depending on the minimally coupled fermionic/bosonic matter fields $\psi(x)$. This is a viable gravitational theory that deviates from general relativity at extremely high matter densities $\rho \gtrsim \rho_{\rm crit}$, with $\rho_{\rm crit} \approx m/\left(\lambda_{\rm Compton}\ell^2_{\rm Planck}\right)$ and m is the mass of the field, see also [10]. At the same time it is clear that GR can alternatively be reformulated as a teleparallelism theory with torsion square pieces in the Lagrangian. If we call the Newton-Einstein type of gravity "weak" gravity, then its general quadratic gauge Lagrangian reads (a_0 and a_1, a_2, a_3 are constants):

$$V^+_{\rm weak} = \frac{1}{2\kappa}\left(-a_0 R^{\alpha\beta}\wedge\eta_{\alpha\beta} - 2\Lambda_0\eta + T^\alpha\wedge\sum_{I=1}^{3} a_I{}^{\star(I)}T_\alpha\right)\,. \tag{2}$$

Here ${}^{(I)}T_\alpha$ denotes the irreducible pieces of the torsion, with ${}^{(2)}T_\alpha := \vartheta_\alpha\wedge(e_\beta\rfloor T^\beta)/3$ (`vector`, 4 independent components), ${}^{(3)}T_\alpha := e_\alpha\rfloor(T^\beta\wedge\vartheta_\beta)/3$ (`axitor`, 4), and ${}^{(1)}T_\alpha := T_\alpha - {}^{(2)}T_\alpha - {}^{(3)}T_\alpha$ (`tentor`, 16). For the special cases $R^{\alpha\beta} = 0$, enforced by a corresponding Lagrange multiplier term in (2), we recover the teleparallel equivalent of GR, provided local Lorentz invariance of the gravitational Lagrangian is implemented, see [11, 12, 13, 14, 15, 16, 17], and alternatively, for $T^\alpha = 0$, we find GR directly. Thus, GR is hidden in (2) in two totally different ways, a fact often overlooked.

To link up with the experience of GR, we recall that the Riemann-Cartan curvature 2-form $R^{\alpha\beta}$ can be decomposed into the (torsionfree) Riemann curvature $\widetilde{R}^{\alpha\beta}$ and in torsion dependent terms. For the curvature scalar this formula reads (see [7, 12])§

$$-R^{\alpha\beta}\wedge\eta_{\alpha\beta} = -\widetilde{R}^{\alpha\beta}\wedge\eta_{\alpha\beta} - T^\alpha\wedge{}^\star(-{}^{(1)}T_\alpha + 2{}^{(2)}T_\alpha + \frac{1}{2}{}^{(3)}T_\alpha) + 2d(\vartheta^\alpha\wedge{}^\star T_\alpha)\,. \tag{3}$$

‡ Following essentially Schouten [6], our conventions are [7]: We have the coframe 1-form $\vartheta^\alpha = e_i{}^\alpha dx^i$ and the frame vectors $e_\beta = e^j{}_\beta\partial_j$, with $e_\beta\rfloor\vartheta^\alpha = \delta^\alpha_\beta$. The connection 1-form is $\Gamma^{\alpha\beta} = \Gamma_i{}^{\alpha\beta}dx^i$. Greek indices are raised and lowered by means of the Minkowski metric $g_{\alpha\beta} = \text{diag}(-1,1,1,1)$. The volume 4-form is denoted by η, and $\eta_\alpha = {}^\star\vartheta_\alpha$, $\eta_{\alpha\beta} = {}^\star\vartheta_{\alpha\beta}$, $\eta_{\alpha\beta\gamma} = {}^\star\vartheta_{\alpha\beta\gamma}$, $\eta_{\alpha\beta\gamma\delta} = {}^\star\vartheta_{\alpha\beta\gamma\delta}$, where * is the Hodge star operator and $\vartheta^{\alpha\beta} := \vartheta^\alpha\wedge\vartheta^\beta$, etc. Furthermore, $T^\alpha := D\vartheta^\alpha = \frac{1}{2}T_{ij}{}^\alpha dx^i\wedge dx^j$, $T_{ij}{}^\alpha = 2(\partial_{[i}\vartheta_{j]}{}^\alpha + \Gamma_{[ij]}{}^\alpha)$, $R^{\alpha\beta} := d\Gamma^{\alpha\beta} + \Gamma^{\alpha\gamma}\wedge\Gamma^\beta{}_\gamma = \frac{1}{2}R_{ij}{}^{\alpha\beta}dx^i\wedge dx^j$, $R_{ij}{}^{\alpha\beta} = 2(\partial_{[i}\Gamma_{j]}{}^{\alpha\beta} + \Gamma_{[i}{}^{\alpha\gamma}\Gamma_{j]\cdot\gamma}{}^{\beta})$, $\text{Ric}_\alpha := e_\beta\rfloor R_\alpha{}^\beta = \text{Ric}_{\beta\alpha}\,\vartheta^\beta$, with $\text{Ric}_{\alpha\beta} = R_{\gamma\alpha\beta}{}^\gamma$, $R := \text{Ric}_\alpha{}^\alpha = R_{\beta\alpha}{}^{\alpha\beta}$. Diakonov et al. [8] use the conventions of Landau-Lifschitz [9]. Accordingly, there are the following correspondence rules: $R^\kappa{}_{\lambda\mu\nu}|_{\text{Diak et al.}} = R_{\mu\nu\lambda}{}^\kappa|_{\text{here}}$, $\text{Ric}_{\lambda\mu}|_{\text{Diak et al.}} := R_\lambda{}^\kappa{}_{\mu\kappa}|_{\text{Diak et al.}} = R_{\mu\kappa}{}^\kappa{}_\lambda|_{\text{here}} = R_{\kappa\mu\lambda}{}^\kappa|_{\text{here}} = \text{Ric}_{\mu\lambda}|_{\text{here}}$.

§ The second minus sign on the right-hand-side of this equation is corrected. In [7], Eq.(5.9.18) was a sign error.

If we substituted (3) into (2), then apart from a boundary term, see below, the Riemann curvature would emerge and the a_I in the quadratic torsion would get redefined. However, we don't apply this procedure to (2), since we don't want to leave the formalism of first order field theory.

We marked the Lagrangian (2) with a plus sign + for being, as a twisted 4-form, parity even. However, already in 1980, Hojman, Mukku, and Sayed (HMS) [18] and Nelson [19] added the parity odd‖ pseudoscalar curvature piece $R_{\alpha\beta}\wedge\vartheta^{\alpha\beta}$ to the EC Lagrangian, see also [21, 22, 23, 24, 25]. More recently, in the context of the Ashtekar formalism [26], see Kiefer [27], and in loop quantum gravity, see Rovelli [28, 29], this has become popular, see [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50], for related cosmological models see also [51, 52, 53, 54, 55, 56]. Including additionally odd torsion square pieces, we have (b_0, σ_1, σ_2 are constants, ${}^{(3)}R_{\alpha\beta}$ is the irreducible pseudoscalar curvature 2-form)

$$V^-_{\rm weak} = -\frac{b_0}{2\kappa}\,{}^{(3)}R_{\alpha\beta}\wedge\vartheta^{\alpha\beta} + \frac{1}{\kappa}\left(\sigma_1{}^{(1)}T^\alpha\wedge{}^{(1)}T_\alpha + \sigma_2{}^{(2)}T^\alpha\wedge{}^{(3)}T_\alpha\right)\,. \qquad (4)$$

The inverse of b_0 is sometimes called the Barbero-Immirzi parameter [57, 58, 35]. The total gauge Lagrangian would then read $V^+_{\rm weak}+V^-_{\rm weak}$. However, we should be aware that for weak gravity there exists a boundary term, the untwisted parity odd *Nieh-Yan* 4-form¶ [21],

$$B^-_{TT} = dC^-_{TT} = \frac{1}{2}\left(T^\alpha\wedge T_\alpha + R_{\alpha\beta}\wedge\vartheta^{\alpha\beta}\right) = \frac{1}{2}(T^\alpha\wedge T_\alpha - {}^\star X)$$
$$= \frac{1}{2}\left({}^{(1)}T^\alpha\wedge{}^{(1)}T_\alpha + 2\,{}^{(2)}T^\alpha\wedge{}^{(3)}T_\alpha + {}^{(3)}R_{\alpha\beta}\wedge\vartheta^{\alpha\beta}\right)\,, \qquad (6)$$

with $C^-_{TT} := \dfrac{1}{2}\vartheta^\alpha\wedge T_\alpha$ and X as the curvature pseudoscalar, $X=\eta_{\alpha\beta\gamma\delta}R^{[\alpha\beta\gamma\delta]}/4!$. We add this form with a suitable constant f_1 to our weak gravity Lagrangian:

$$V_{\rm weak} = V_{\rm weak}(a_0; b_0; a_1, a_2, a_3; \sigma_1, \sigma_2; f_1) := V^+_{\rm weak} + V^-_{\rm weak} + \frac{f_1}{\kappa}B^-_{TT}\,. \qquad (7)$$

It depends on the gravitational constant κ and the cosmological constant λ_0 and, furthermore, on the 8 constants specified in (7). By a suitable choice of f_1, we can compensate either the HMS-term [18] (that is, $b_0 = 0$) or one tensor square term of the torsion (that is, either $\sigma_1 = 0$ or $\sigma_2 = 0$). However, since ${}^{(1)}T^\alpha$ depends on 16 independent components, the pseudoscalar curvature only on 1 component, it seems to simplify the Lagrangian to a greater extent, if we kick out the term with ${}^{(1)}T^\alpha$. Thus, for the weak gravity Lagrangian we are left with **6** unspecified constants $(a_0, b_0; a_1, a_2, a_3; \sigma_2)^+$.

‖ Parity even and *odd* torsion square terms were introduced by Purcell [20] even 2 years ealier.

¶ In a metric-affine spacetime [7] with the distortion 1-form $N_\alpha{}^\beta := \Gamma_a{}^\beta - \widetilde{\Gamma}_a{}^\beta$, we can bring the Nieh-Yan identity in a very compact form, see [59]:

$$d\,(T^\alpha\wedge\vartheta_\alpha) = {}^{(3)}R_\alpha{}^\beta\wedge\vartheta^\alpha\wedge\vartheta_\beta - T^\alpha\wedge N_\alpha{}^\beta\wedge\vartheta_\beta\,. \qquad (5)$$

+ Diakonov et al. [8] found an *equivalent* result, but they eliminated the curvature pseudoscalar. Thus, they are left with the 6 unspecified constants $(a_0; a_1, a_2, a_3; \sigma_1, \sigma_2)$, that is, with the curvature scalar plus the 5 torsion-square pieces.

Looking back at Eq. (3), it could appear that we forgot the boundary term $B^{+}_{TT} = \frac{1}{2} d(\vartheta^\alpha \wedge {}^\star T_\alpha)$ and that we could add it as $\frac{f_0}{\kappa} B^{+}_{TT}$ to the Lagrangian (7), see the procedure of Mielke [60, 61]. However, if we compare the Nieh-Yan and the "teleparallel" formulas

$$d\left(\vartheta^\alpha \wedge T_\alpha\right) = T^\alpha \wedge T_\alpha - \vartheta^\alpha \wedge DT_\alpha \quad \text{and} \quad d(\vartheta^\alpha \wedge {}^\star T_\alpha) = T^\alpha \wedge {}^\star T_\alpha - \vartheta^\alpha \wedge D\,{}^\star T_\alpha, \quad (8)$$

respectively, then we recognize that in the former equation DT_α can be eliminated via the first Bianchi identity $DT_\alpha = R_{\beta\alpha} \wedge \vartheta^\beta$, whereas in the latter equation such a trick is impossible. Therefore, we would trade in for torsion square pieces derivatives of the torsion and would mess up the first order character of our Lagrangian.*

If one starts with the EC-Lagrangian and adds *only* an HMS-term, as numerous people do, then, because of the Nieh-Yan form, two torsion square terms of odd parity are induced. Hence, the Lagrangian can be reformulated as the EC-Lagrangian with specific additional torsion square pieces:

$$V_{\rm EC} + \frac{b_0}{2\kappa}\, {}^{(3)}R_{\alpha\beta} \wedge \vartheta^{\alpha\beta} = V_{\rm EC} - \frac{b_0}{2\kappa}\left({}^{(1)}T^\alpha \wedge {}^{(1)}T_\alpha + 2\, {}^{(2)}T^\alpha \wedge {}^{(3)}T_\alpha\right) + d(\ldots)\,. (10)$$

Then the question can hardly be circumvented, why one should choose only these specific torsion square pieces with very specific constants and why the other torsion square pieces should be forbidden, that is, the torsion square pieces come into focus. Moreover, it is known that GR can be reformulated as a teleparallelism theory with torsion square pieces in the Lagrangian [11, 12, 13, 14, 15, 16, 17]. In other words, the addition of the HMS-term opens the door wide for torsion square Lagrangians.

Classically, it is consistent to consider only the two additional specific terms in (10). However, it is not particularly plausible. If torsion is introduced as a new concept, why should one then introduce it in the highly constrained form of (10)? In loop quantum gravity [29], which is thought of as a fundamental theory of gravity, the truncated Lagrangian (10) is taken as a classical starting point, see [29], Eq. (34), with the argument that also in QCD a similar parity odd piece is used. However, then in the Lagrangian the internal color group $SU(3)$ with its potential A is put in analogy to the local Poincaré group $\mathrm{R}^4 \rtimes SO(1,3)$ with its translation potential ϑ^α and Lorentz potential $\Gamma^{\alpha\beta}$. Apart from the fact that QCD is quadratic in the field strength and (10) is only linear in the curvature, this argument is less than convincing to us.

If gravity is seen in a quantum field theoretical context, see Diakonov et al. [8], for instance, then, as Date et al. [40] have pointed out, the Lagrangian (10) is insufficient anyway: "In a complete theory of gravity, besides the Nieh-Yan topological term, we need to include two other topological terms, the Pontryagin density and the Euler density. This introduces two additional topological parameters associated with such topological terms, besides the parameter η [our b_0] we have discussed here.

* We can combine the two equations in (8). This yields yields

$$d\left(\vartheta^\alpha \wedge \xi^{\pm}_\alpha\right) = T^\alpha \wedge \xi^{\pm}_\alpha - \vartheta^\alpha \wedge D\xi^{\pm}_\alpha \qquad \text{with} \qquad \xi^{\pm}_\alpha := T_\alpha \pm {}^\star T_\alpha\,. \quad (9)$$

Mielke [60, 61] built a similar linear combination but with the imaginary unit in front of ${}^\star T_\alpha$, but neither his nor our version seems to lead to firm conclusions up to now.

Any quantum theory of gravity should have all these three CP-violating topological couplings." Actually, the Euler 4-form is CP-even, see equation (19) or [49].

From a totally different point of view, from observational cosmology and from quantum chromodynamics, there are indications that we may live in a parity violating Universe, see the review by Urban & Zhitnitsky [62]. All the more investigations in a parity odd PG model seem desirable.

2. Quadratic Poincaré gauge theory and strong gravity

If one wants the Lorentz connection as a propagating field, then one has to allow for "strong" gravity of the Yang-Mills type by adding quadratic curvature 4-forms to the weak Lagrangian. The curvature $R^{\alpha\beta}$ of a Riemann-Cartan space has six irreducible pieces: $R^{\alpha\beta} = \sum_{I=1}^{6} {}^{(I)}R^{\alpha\beta}$. We write symbolically, using the self-explanatory computer names for the irreducible terms: `curv` (36 indep. comp.) = `weyl` (10) + `paircom` (9) + `pscalar` (1) + `ricsymf` (9) + `ricanti` (6) + `scalar` (1), see [7] for the exact definitions. In a Riemann space only ${}^{(1)}R^{\alpha\beta}$, ${}^{(4)}R^{\alpha\beta}$, and ${}^{(6)}R^{\alpha\beta}$ are left over. Hence the most general parity even quadratic Lagrangian, with a new dimensionless coupling constant ϱ, reads

$$V^{+}_{\text{strong}} = -\frac{1}{2\varrho} R^{\alpha\beta} \wedge \sum_{I=1}^{6} w_I \, {}^{\star(I)}R_{\alpha\beta} \,. \tag{11}$$

Alerted by the corresponding case in weak gravity, we now search for parity odd terms. They were found to be (see [63, 64]) as

$$\begin{aligned} V^{-}_{\text{strong}} = & -\frac{1}{2\varrho} \left(\mu_1 {}^{(1)}R^{\alpha\beta} \wedge {}^{(1)}R_{\alpha\beta} + \mu_2 {}^{(2)}R^{\alpha\beta} \wedge {}^{(4)}R_{\alpha\beta} \right. \\ & \left. + \mu_3 {}^{(3)}R^{\alpha\beta} \wedge {}^{(6)}R_{\alpha\beta} + \mu_4 {}^{(5)}R^{\alpha\beta} \wedge {}^{(5)}R_{\alpha\beta} \right) , \end{aligned} \tag{12}$$

which are the only quadratic curvature square invariants of odd character in a 4D Riemann-Cartan space. Note that in a Riemann space, that is, when torsion vanishes, only the first piece built up from the Weyl curvature ${}^{(1)}R^{\alpha\beta}$ is left over.

Taking a lesson from the above, we can now search for boundary terms. As in any Yang-Mills theory, we can find an untwisted *Pontryagin* 4-form B^{-}_{RR}. But in gravity the (anholonomic) Lorentz indices of the curvature can be contracted with the help of the totally antisymmetric Levi-Civita tensor $\eta^{\alpha\beta\gamma\delta}$. Accordingly, we introduce the so-called Lie-dual of the curvature with the Lie star operator ${}^{(*)}$ as

$$R^{(*)\,\alpha\beta} := \frac{1}{2} R_{\mu\nu} \eta^{\mu\nu\alpha\beta} \,. \tag{13}$$

Like $\eta^{\mu\nu\alpha\beta}$, the Lie star ${}^{(*)}$ is twisted. It gives rise to the twisted *Euler* 4-form $B^{+}_{RR^{(*)}}$. Following [7], we have then the following two boundary terms:

$$B^{-}_{RR} = dC^{-}_{RR} = \frac{1}{2} R_{\alpha\beta} \wedge R^{\alpha\beta} \,, \quad B^{+}_{RR^{(*)}} = dC^{+}_{RR^{(*)}} = \frac{1}{2} R_{\alpha\beta} \wedge R^{(*)\alpha\beta} \,. \tag{14}$$

Thus, the strong part of the gauge Lagrangian turns out to be

$$V_{\text{strong}} := V^{+}_{\text{strong}} + V^{-}_{\text{strong}} + \frac{f_2}{\varrho} B^{-}_{RR} + \frac{f_3}{\varrho} B^{+}_{RR^{(*)}} \,. \tag{15}$$

The total quadratic gauge Lagrangian including boundary terms is then

$$V_{\rm gauge} = V^{+}_{\rm weak} + V^{-}_{\rm weak} + V^{+}_{\rm strong} + V^{-}_{\rm strong} + \frac{f_1}{\kappa} B^{-}_{TT} + \frac{f_2}{\varrho} B^{-}_{RR} + \frac{f_3}{\varrho} B^{+}_{RR^{(*)}} . \tag{16}$$

For *vanishing torsion,* $V_{\rm weak}$ reduces to $V_{\rm GR}$ with cosmological constant and $V_{\rm strong}$ has only $(w_1, w_4, w_6; \mu_1; f_2, f_3) \neq 0$. By a suitable choice of f_2, f_3, only the two terms with w_4, w_6 survive, that is, those with the tracefree Ricci tensor and the curvature scalar.

3. The role of the Lie-dual of the curvature

Before we continue the investigation of (15), we will derive some rules for manipulating curvature square terms containing a Lie star. Using heavily the computer-algebra system `Reduce` with the `Excalc` package, compare [65, 66, 67, 68], we were able to convert completely the Lie star $^{(*)}$ into the Hodge star * according to the following rules: The expression ${}^{(I)}R^{\mu\nu} \wedge {}^{(J)} R^{(*)}_{\mu\nu}$ is diagonal, that is, $\propto \delta^{IJ}$; only the diagonal pieces do not vanish, namely

$${}^{(I)}R^{\mu\nu} \wedge {}^{(I)}R^{(*)}_{\mu\nu} = \pm\, {}^{(I)}R^{\mu\nu} \wedge {}^{\star(I)}R_{\mu\nu} \, , \tag{17}$$

with $+$ for $I = 1, 3, 5, 6$ and with $-$ for $I = 2, 4$. Note that on the left-hand-side of this equation we have the Lie star $^{(*)}$, on the right-hand-side, however, the Hodge star *. This implies the relation, derived here for the first time explicitly♯,

$$\begin{aligned} R^{\alpha\beta} \wedge R^{(*)}_{\alpha\beta} = {} & {}^{(1)}R^{\alpha\beta} \wedge {}^{\star(1)}R_{\alpha\beta} - {}^{(2)}R^{\alpha\beta} \wedge {}^{\star(2)}R_{\alpha\beta} + {}^{(3)}R^{\alpha\beta} \wedge {}^{\star(3)}R_{\alpha\beta} \\ & - {}^{(4)}R^{\alpha\beta} \wedge {}^{\star(4)}R_{\alpha\beta} + {}^{(5)}R^{\alpha\beta} \wedge {}^{\star(5)}R_{\alpha\beta} + {}^{(6)}R^{\alpha\beta} \wedge {}^{\star(6)}R_{\alpha\beta} \, . \end{aligned} \tag{18}$$

In particular, this shows that the Lie star is superfluous in forming a quadratic Lagrangian, the Hodge star is sufficient.

Comparison with (14) allows us to rewrite the *Euler* 4-form with the help of the Hodge star as

$$B^{+}_{RR^{(*)}} = \frac{1}{2} \left(R^{\alpha\beta} \wedge {}^{\star}R_{\alpha\beta} - 2\, {}^{(4)}R^{\alpha\beta} \wedge {}^{\star(4)}R_{\alpha\beta} - 2\, {}^{(2)}R^{\alpha\beta} \wedge {}^{\star(2)}R_{\alpha\beta} \right) . \tag{19}$$

The *Pontryagin* 4-form, also defined in (14), after some algebra, can be expressed in terms of the irreducible pieces of the curvature as (also this relation is new)

$$\begin{aligned} B^{-}_{RR} = {} & \frac{1}{2} \big({}^{(1)}R_{\alpha\beta} \wedge {}^{(1)}R^{\alpha\beta} + {}^{(5)}R_{\alpha\beta} \wedge {}^{(5)}R^{\alpha\beta} \\ & + 2\, {}^{(3)}R_{\alpha\beta} \wedge {}^{(6)}R^{\alpha\beta} + 2\, {}^{(2)}R_{\alpha\beta} \wedge {}^{(4)}R^{\alpha\beta} \big) \, . \end{aligned} \tag{20}$$

4. Comparison with Diakonov et al. [8]

Recently, in the framework of perturbative quantum field theory, new results were reported [8] on including torsion in a gravitational gauge theory for describing fermionic matter, for a review of some earlier results, see [69]. In this context, Diakonov et al.

♯ By using some simple algebra, Eq.(18) can alternatively be derived in a straightforward way from Eqs.(10.17) to (10.22) of Obukhov [4].

investigated gravitational gauge Lagrangians containing quadratic terms in the gauge fields of even and of odd parity. That is, contributions of the weak and the strong gravity sector (Lorentz gauge bosons) were considered. Generally, those 4-fermion interaction terms will give additional contributions to the energy-momentum current of matter. This aspects might be relevant in the context of quantum cosmological models, see [52, 55, 56]. Other groups address only weak gravity, even though they include Euler and Pontryagin terms, which refer to strong gravity, see Benedetti et al. [70].

In the following, we would like to compare the approach given in [8] with the results we already gave in [64].

4.1. Torsion square invariants

Let us compare our torsion-square invariants in (7), see also [64], with those in [8], Eq.(53). If we add a plus sign for parity even and a minus sign for parity odd terms, the invariants of Diakonov et al., multiplied by the volume form η, read:

$$K_1^+ = 2\,{}^{(1)}T^\alpha \wedge {}^{\star(1)}T_\alpha - 2\,{}^{(2)}T^\alpha \wedge {}^{\star(2)}T_\alpha + 2\,{}^{(3)}T^\alpha \wedge {}^{\star(3)}T_\alpha\,, \tag{21}$$

$$K_2^+ = 3\,{}^{(2)}T^\alpha \wedge {}^{\star(2)}T_\alpha\,, \tag{22}$$

$$K_3^+ = {}^{(1)}T^\alpha \wedge {}^{\star(1)}T_\alpha + {}^{(2)}T^\alpha \wedge {}^{\star(2)}T_\alpha - 2\,{}^{(3)}T^\alpha \wedge {}^{\star(3)}T_\alpha\,, \tag{23}$$

$$K_4^- = -\,2\,{}^{(1)}T^\alpha \wedge {}^{(1)}T_\alpha + 4\,{}^{(2)}T^\alpha \wedge {}^{(3)}T_\alpha\,, \tag{24}$$

$$K_5^- = {}^{(1)}T^\alpha \wedge {}^{(1)}T_\alpha + 4\,{}^{(2)}T^\alpha \wedge {}^{(3)}T_\alpha\,, \tag{25}$$

and the inverse relations are

$${}^{(1)}T^\alpha \wedge {}^{\star(1)}T_\alpha = \frac{1}{9}(3K_1^+ + K_2^+ + 3K_3^+)\,, \tag{26}$$

$${}^{(2)}T^\alpha \wedge {}^{\star(2)}T_\alpha = \frac{1}{3}K_2^+\,, \tag{27}$$

$${}^{(3)}T^\alpha \wedge {}^{\star(3)}T_\alpha = \frac{1}{12}(3K_1^+ + 4K_2^+ - 6K_3^+)\,, \tag{28}$$

$${}^{(1)}T^\alpha \wedge {}^{(1)}T_\alpha = -\frac{1}{3}(K_4^- - K_5^-)\,, \tag{29}$$

$${}^{(2)}T^\alpha \wedge {}^{(3)}T_\alpha = \frac{1}{12}(K_4^- + 2K_5^-)\,. \tag{30}$$

These 5 invariants agree with those given in [64], Eqs.(30) and (55).

4.2. Curvature square invariants

Diakonov et al. [8], Eq. (59), find 6 even and 4 odd independent quadratic invariants G_I. As we did with the torsion invariants, we translate their component representations into the language of exterior differential forms used by us. We find, after some messy computer checked algebra, the following curvature invariants (multiplied by η):

$$G_1^+ = R^2\eta = 12\,{}^{(6)}R_{\alpha\beta} \wedge {}^{\star(6)}R^{\alpha\beta}\,, \tag{31}$$

$$G_2^+ = R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}\eta = 2\,R_{\alpha\beta} \wedge {}^\star R^{\alpha\beta} = 2\sum_{I=1}^{6} {}^{(I)}R^{\alpha\beta} \wedge {}^{\star(I)}R_{\alpha\beta}\,, \tag{32}$$

$$
\begin{aligned}
G_3^+ &= R_{\mu\nu\rho\lambda}R^{\lambda\rho\nu\mu}\eta \\
&= 2\left({}^{(1)}R^{\alpha\beta}\wedge{}^{\star(1)}R_{\alpha\beta} - {}^{(2)}R^{\alpha\beta}\wedge{}^{\star(2)}R_{\alpha\beta} + {}^{(3)}R^{\alpha\beta}\wedge{}^{\star(3)}R_{\alpha\beta}\right. \\
&\quad \left. +{}^{(4)}R^{\alpha\beta}\wedge{}^{\star(4)}R_{\alpha\beta} - {}^{(5)}R^{\alpha\beta}\wedge{}^{\star(5)}R_{\alpha\beta} + {}^{(6)}R^{\alpha\beta}\wedge{}^{\star(6)}R_{\alpha\beta}\right)\,, \qquad (33)\\
G_4^+ &= \left(R^2 - 4\mathrm{Ric}_{\mu\lambda}\mathrm{Ric}^{\lambda\mu} + R_{\mu\nu\rho\lambda}R^{\lambda\rho\nu\mu}\right)\eta \\
&= 2\left({}^{(1)}R^{\alpha\beta}\wedge{}^{\star(1)}R_{\alpha\beta} - {}^{(2)}R^{\alpha\beta}\wedge{}^{\star(2)}R_{\alpha\beta} + {}^{(3)}R^{\alpha\beta}\wedge{}^{\star(3)}R_{\alpha\beta}\right. \\
&\quad \left. -{}^{(4)}R^{\alpha\beta}\wedge{}^{\star(4)}R_{\alpha\beta} + {}^{(5)}R^{\alpha\beta}\wedge{}^{\star(5)}R_{\alpha\beta} + {}^{(6)}R^{\alpha\beta}\wedge{}^{\star(6)}R_{\alpha\beta}\right)\,, \qquad (34)\\
G_5^+ &= \left(R^2 - 4\mathrm{Ric}_{\mu\lambda}\mathrm{Ric}^{\mu\lambda} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}\right)\eta \\
&= 2\left({}^{(1)}R^{\alpha\beta}\wedge{}^{\star(1)}R_{\alpha\beta} + {}^{(2)}R^{\alpha\beta}\wedge{}^{\star(2)}R_{\alpha\beta} + {}^{(3)}R^{\alpha\beta}\wedge{}^{\star(3)}R_{\alpha\beta}\right. \\
&\quad \left. -{}^{(4)}R^{\alpha\beta}\wedge{}^{\star(4)}R_{\alpha\beta} - {}^{(5)}R^{\alpha\beta}\wedge{}^{\star(5)}R_{\alpha\beta} + {}^{(6)}R^{\alpha\beta}\wedge{}^{\star(6)}R_{\alpha\beta}\right)\,, \qquad (35)\\
G_6^+ &= \left(\eta^{\lambda\rho\mu\nu}R_{\mu\nu\rho\lambda}\right)^2\eta = -48\,{}^{(3)}R_{\alpha\beta}\wedge{}^{\star(3)}R^{\alpha\beta}\,, \qquad (36)\\
G_7^- &= R\eta^{\lambda\rho\mu\nu}R_{\mu\nu\rho\lambda}\eta = -24\,{}^{(3)}R_{\alpha\beta}\wedge{}^{(6)}R^{\alpha\beta}\,, \qquad (37)\\
G_8^- &= \eta^{\mu\nu\alpha\beta}R_{\mu\nu\rho\lambda}R_{\alpha\beta}{}^{\rho\lambda}\eta \\
&= -4\,{}^{(1)}R_{\alpha\beta}\wedge{}^{(1)}R^{\alpha\beta} - 4\,{}^{(5)}R_{\alpha\beta}\wedge{}^{(5)}R^{\alpha\beta} \\
&\quad - 8\,{}^{(3)}R_{\alpha\beta}\wedge{}^{(6)}R^{\alpha\beta} - 8\,{}^{(2)}R_{\alpha\beta}\wedge{}^{(4)}R^{\alpha\beta}\,, \qquad (38)\\
G_9^- &= \eta^{\lambda\rho\gamma\delta}R_{\mu\nu\rho\lambda}R^{\mu\nu}{}_{\delta\gamma}\eta \\
&= -4\,{}^{(1)}R_{\alpha\beta}\wedge{}^{(1)}R^{\alpha\beta} - 4\,{}^{(5)}R_{\alpha\beta}\wedge{}^{(5)}R^{\alpha\beta} \\
&\quad - 8\,{}^{(3)}R_{\alpha\beta}\wedge{}^{(6)}R^{\alpha\beta} + 8\,{}^{(2)}R_{\alpha\beta}\wedge{}^{(4)}R^{\alpha\beta}\,, \qquad (39)\\
G_{10}^- &= \eta^{\lambda\rho\alpha\beta}R_{\mu\nu\rho\lambda}R_{\alpha\beta}{}^{\nu\mu}\eta \\
&= -4\,{}^{(1)}R_{\alpha\beta}\wedge{}^{(1)}R^{\alpha\beta} + 4\,{}^{(5)}R_{\alpha\beta}\wedge{}^{(5)}R^{\alpha\beta} - 8\,{}^{(3)}R_{\alpha\beta}\wedge{}^{(6)}R^{\alpha\beta}\,. \qquad (40)
\end{aligned}
$$

The inverse relations are convenient for a detailed comparison. They turn out to be

$$ {}^{(1)}R^{\alpha\beta}\wedge{}^{\star(1)}R_{\alpha\beta} = -\frac{1}{12}G_1^+ + \frac{1}{8}\left(G_2^+ + G_3^+ + G_4^+ + G_5^+\right) + \frac{1}{48}G_6^+\,, \qquad (41) $$

$$ {}^{(2)}R^{\alpha\beta}\wedge{}^{\star(2)}R_{\alpha\beta} = \frac{1}{8}\left(G_2^+ - G_3^+ - G_4^+ + G_5^+\right)\,, \qquad (42) $$

$$ {}^{(3)}R^{\alpha\beta}\wedge{}^{\star(3)}R_{\alpha\beta} = -\frac{1}{48}G_6^+\,, \qquad (43) $$

$$ {}^{(4)}R^{\alpha\beta}\wedge{}^{\star(4)}R_{\alpha\beta} = \frac{1}{8}\left(G_2^+ + G_3^+ - G_4^+ - G_5^+\right)\,, \qquad (44) $$

$$ {}^{(5)}R^{\alpha\beta}\wedge{}^{\star(5)}R_{\alpha\beta} = \frac{1}{8}\left(G_2^+ - G_3^+ + G_4^+ - G_5^+\right)\,, \qquad (45) $$

$$ {}^{(6)}R^{\alpha\beta}\wedge{}^{\star(6)}R_{\alpha\beta} = \frac{1}{12}G_1^+\,, \qquad (46) $$

and

$$ {}^{(1)}R^{\alpha\beta}\wedge{}^{(1)}R_{\alpha\beta} = -\frac{1}{16}\left(G_8^- + G_9^- + 2G_{10}^-\right) + \frac{1}{12}G_7^-\,, \qquad (47) $$

$$ {}^{(2)}R^{\alpha\beta}\wedge{}^{(4)}R_{\alpha\beta} = -\frac{1}{16}\left(G_8^- - G_9^-\right)\,, \qquad (48) $$

$$ {}^{(3)}R^{\alpha\beta}\wedge{}^{(6)}R_{\alpha\beta} = -\frac{1}{24}G_7^-\,, \qquad (49) $$

$$ {}^{(5)}R^{\alpha\beta} \wedge {}^{(5)}R_{\alpha\beta} = -\frac{1}{16}\left(G_8^- + G_9^- - 2G_{10}^-\right) . \tag{50} $$

It is now straightforward to express the Euler 4-form (19) and the Pontryagin 4-form (20) in terms of the G_I's. We find

$$ B^+_{RR^{(*)}} = \frac{1}{4}G_4^+ \qquad \text{and} \qquad B^-_{RR} = -\frac{1}{8}G_8^- , \tag{51} $$

respectively. This is what Diakonov et al. stressed: that their invariants G_4^+ and G_8^- are boundary terms. These two boundary terms can also be found in our earlier work, see [64], Eqs.(33) and (50).

Hence the results of Diakonov et al. [8] with respect to the quadratic invariants of torsion and curvature coincide with those of [64]. This is also manifest in the Riemannian subcase, that is, for *vanishing torsion* $T^\alpha = 0$. Then,

$$ {}^{(2)}R_{\alpha\beta} = {}^{(3)}R_{\alpha\beta} = {}^{(5)}R_{\alpha\beta} = 0 , \tag{52} $$

or, in terms of the G_I's,

$$ G_2^+ = G_3^+ , \quad G_4^+ = G_5^+ , \quad G_6^+ = G_7^+ = 0 , \quad G_8^+ = G_9^+ = G_{10}^+ , \tag{53} $$

which can be read off directly from the Eqs.(31) to (40). Under the condition of vanishing torsion the boundary terms read

$$ B^+_{RR^{(*)}}|_{T^\alpha=0} = \frac{1}{2}\left({}^{(1)}R^{\alpha\beta} \wedge {}^{\star(1)}R_{\alpha\beta} - {}^{(4)}R^{\alpha\beta} \wedge {}^{\star(4)}R_{\alpha\beta} + {}^{(6)}R^{\alpha\beta} \wedge {}^{\star(6)}R_{\alpha\beta}\right) , \tag{54} $$

$$ B^-_{RR}|_{T^\alpha=0} = \frac{1}{2}\, {}^{(1)}R^{\alpha\beta} \wedge {}^{(1)}R_{\alpha\beta} . \tag{55} $$

5. The number of independent terms in the most general quadratic PG Lagrangian

For the strong gravitational Lagrangian V_{strong} in (15), we can enter in a similar discussion as for V_{weak} in (7): Besides the strong gravitational coupling constant ϱ, we have the 12 constants $(w_1, w_2, ..., w_6; \mu_1, \mu_2, \mu_3, \mu_4; f_2, f_3)$. By a suitable choice of f_2 and f_3, we can compensate the terms containing the Weyl 2-form ${}^{(1)}R^{\alpha\beta}$ with 10 independent components, as can be seen from (19) and (20). Consequently, we are left with the **8** constants $(w_2, ..., w_6; \mu_2, \mu_3, \mu_4)$. Diakonov et al. [8] found the 10 invariants G_I, two of which, namely G_4^+ and G_8^- are boundary terms. Hence they also arrive at 8 independent invariants. Accordingly, also for strong gravity our results match those of Diakonov et al.

Our final gravitational Lagrangian is then††

$$ V = \frac{1}{2\kappa}[\,(\,a_0 R - 2\Lambda_0 + b_0 X\,)\,\eta $$

††If we introduce the notations R and X for the curvature scalar and the curvature pseudoscalar, then we find ${}^{(6)}R_{\alpha\beta} = -R\,\vartheta_{\alpha\beta}/12$ and ${}^{(3)}R_{\alpha\beta} = -X\eta_{\alpha\beta}/12$, respectively; moreover, for the torsion we can define the 1-forms of $\mathcal{A}$ and $\mathcal{V}$ for the axial vector and the vector torsion ${}^{(3)}T^\alpha = {}^\star(\mathcal{A} \wedge \vartheta^\alpha)/3$ and ${}^{(2)}T^\alpha = -(\mathcal{V} \wedge \vartheta^\alpha)/3$, respectively.

$$
+\frac{a_2}{3}\mathcal{V}\wedge{}^\star\mathcal{V}-\frac{a_3}{3}\mathcal{A}\wedge{}^\star\mathcal{A}-\frac{2\sigma_2}{3}\mathcal{V}\wedge{}^\star\mathcal{A}+a_1{}^{(1)}T^\alpha\wedge{}^{\star(1)}T_\alpha\Big]
$$
$$
-\frac{1}{2\varrho}\Big[(\frac{w_6}{12}R^2-\frac{w_3}{12}X^2+\frac{\mu_3}{12}RX)\,\eta+w_4{}^{(4)}R^{\alpha\beta}\wedge{}^{\star(4)}R_{\alpha\beta}
$$
$$
+{}^{(2)}R^{\alpha\beta}\wedge(w_2{}^{\star(2)}R_{\alpha\beta}+\mu_2{}^{(4)}R_{\alpha\beta})+{}^{(5)}R^{\alpha\beta}\wedge(w_5{}^{\star(5)}R_{\alpha\beta}+\mu_4{}^{(5)}R_{\alpha\beta})\Big]\,. \qquad (56)
$$

The first two lines represent weak gravity, the last two lines strong gravity. The parity odd pieces are those with the constants $b_0, \sigma_2; \mu_2, \mu_3, \mu_4$. In a Riemann space (where $X = 0$), only two terms of the first line and likewise two terms in the third line survive. All these 4 terms are parity even, that is, only torsion brings in parity odd pieces into the gravitational Lagrangian.

Yo and Nester [71, 72, 73] found that only a small subclass of the Lagrangians (56) is consistent from a Hamiltonian point of view. They, together with Shie, presented such a Lagrangian [74] and found an accelerating cosmological Friedman type model with propagating connection. Shortly afterwards, Nester and his group, see Chen et al. [75], generalized this model and introduced a consistent Lagrangian containing the parity odd pieces $\mathcal{A}$ and X. Since these terms occur quadratically, their Lagrangian was still parity even:

$$
V_{\text{Chen et al.}} = \frac{1}{2\kappa}(a_0R-2\Lambda_0)\,\eta+\frac{1}{6\kappa}(a_2\mathcal{V}\wedge{}^\star\mathcal{V}-a_3\mathcal{A}\wedge{}^\star\mathcal{A})
$$
$$
-\frac{1}{24\varrho}\left(w_6R^2-w_3X^2\right)\eta\,. \qquad (57)
$$

The next step was done by Baekler et al. [64], see also [76, 77]. They investigated a Lagrangian with three additional pieces with odd parity (57), namely those carrying the constants b_0, σ_2, μ_3:

$$
V_{\text{BHN}} = \frac{1}{2\kappa}(a_0R-2\Lambda_0+b_0X)\eta+\frac{1}{6\kappa}(a_2\mathcal{V}\wedge{}^\star\mathcal{V}-a_3\mathcal{A}\wedge{}^\star\mathcal{A}-2\sigma_2\mathcal{V}\wedge{}^\star\mathcal{A})
$$
$$
-\frac{1}{24\varrho}\left(w_6R^2-w_3X^2+\mu_3RX\right)\eta\,. \qquad (58)
$$

Now one should analyze the particle content of the lagrangian (56), that is, to find out which modes are propagating decently. This has been done in [64] by the simple method of the *diagonalization of the Lagrangian.* The results turned out to be in agreement with those of the Hamiltonian approach.

It is manifest already by now, looking beyond the Einstein-Cartan theory including parity odd Lagrangians is a field with bright prospects.

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