The Weyl-Cartan Gauss-Bonnet gravity

Zahra Haghani[1],[*] Nima Khosravi[2],[†] and Shahab Shahidi[1‡]
[1] *School of Physics, Damghan University, Damghan, 41167-36716, Iran and*
[2] *School of Astronomy, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5531, Tehran, Iran*
(Dated: October 10, 2014)

In this paper, we consider the generalized Gauss-Bonnet action in 4-dimensional Weyl-Cartan space-time. In this space-time, the presence of torsion tensor and Weyl vector implies that the generalised Gauss-Bonnet action will not be a total derivative in four dimension space-time. It will be shown that the higher than two time derivatives can be removed from the action by choosing suitable set of parameters. In the special case where only the trace part of the torsion remains, the model reduces to GR plus two vector fields. One of which is massless and the other is massive. We will show that there exists a region in parameter space where the model is free from tachyon and Ostrogradski instabilities.

PACS numbers: 04.50.Kd, 04.20.Fy

I. INTRODUCTION

In 1918 Weyl proposed a new geometry to unify electromagnetism with Einstein's general relativity [1]. In Riemanian geometry one has a priori condition that the length of a vector should not change during the parallel transportation. In the Weyl geometry, this assumption is dropped and so a parallel transported vector has different length and direction with respect to the original vector. The gravitational theory which is built on the Weyl geometry is known as the Einstein-Weyl gravity [1]. In Einstein-Weyl gravity the connection is no longer metric compatible, so, the covariant derivative of the metric is not zero. Instead one has the relation

$$\tilde{\nabla}_\mu g_{\nu\rho} = Q_{\mu\nu\rho}, \tag{1}$$

where the tensor $Q_{\mu\nu\rho}$ is symmetric with respect to its last two indices. Weyl proposed the special case $Q_{\mu\nu\rho} \propto w_\mu g_{\nu\rho}$ for his theory where w_μ is the Weyl vector. One of the important consequences of this geometry is that the unit vector changes through parallel transportation. Suppose that the length of an arbitrary vector field A^μ is l. During the parallel transportation, the variation of the length of A^μ can be written in terms of the Weyl vector as

$$dl = l w_\mu dx^\mu. \tag{2}$$

For a closed curve, the length of the vector A^μ changes as

$$l \to l - \int_S l W_{\mu\nu} dS^{\mu\nu}, \tag{3}$$

where S is the area of the closed curve, $dS^{\mu\nu}$ is the infinitesimal element of the surface, and

$$W_{\mu\nu} = \partial_\mu w_\nu - \partial_\nu w_\mu, \tag{4}$$

is called the Weyl's length curvature which is the same as the electromagnetic field strength. This implies that one has the freedom to choose the unit length at each point, which is the Weyl gauge freedom [1]. A variety of works have been done in the Weyl geometry including the cosmology [2], relations to scalar-tensor [3] and teleparallel theories [4].

One can also restrict the form of Weyl vector to be a derivative of a scalar as $w_\mu = \partial_\mu \phi$ [5]. In this case the length curvature $W_{\mu\nu}$ vanishes and one can then define a fixed unit length at each point. We note that the unit length varies at different points. The resulting theory is known as the Weyl integrable theory [6].

Another generalization of Einstein gravity can be proposed by assuming existence of an assymetric connection on the space-time manifold. The first attempt for this purpose is due to Eddington in 1921 in order to generalize the Einstein's general relativity to get some insights about microscopic Physics [7]. The major attempt in this way was done by Cartan in 1922 where he defined the torsion tensor as the antisymmetric part of the general connection [8]. The theory based on this assumption is called the Einstein-Cartan theory. Cartan believed that the torsion tensor should be related in some way to the angular momentum of the matter content of the universe. So, the torsion should vanish in the absence of matter [8]. In Einstein-Cartan theory the metric and torsion tensors are considered as independent dynamical variables. The energy-momentum tensor of a massive spin particle is in general asymmetric. So, one can not consider the spin massive particle as a source in Einstein's general relativity. This is the main motivation for the use of Einstein-Cartan theory to consider the gravity theory of a massive spin particle [9]. Many works have been done in the context of torsion theories, including the teleparallel theories [10], and the combinations of Weyl and Cartan space-times [11, 12].

There is another way to generalize the Einstein's general relativity, by adding to the Ricci scalar, some other higher order combinations of the Riemann tensor and its contractions, as in $f(R)$ gravity theories [13, 15]. An-

[*]Electronic address: z.haghani@du.ac.ir
[†]Electronic address: nima@ipm.ir
[‡]Electronic address: s.shahidi@du.ac.ir

other attempt was done by Kretschmann [14] in 1917 by introducing the action of the form

$$S_K = \int d^4x\sqrt{-g}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \tag{5}$$

instead of the Einstein-Hilbert action. The above action has higher than second order time derivatives of the metric in its field equation and hence contains ghost instabilities. It turns out that the unique combination of two Riemann tensors and its contractions which leads to at most second order time derivatives in the field equation is of the form

$$S_{GB} = \int d^4x\sqrt{-g}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2), \tag{6}$$

which is called the Gauss-Bonnet term. In four space-time dimensions, this term can be written as a total derivative, and can be dropped from the field equations [15]. This leads to the conclusion that in 4D, the Riemanian geometry together with the condition of stability has a unique candidate for the gravitational action, which is the Einstein-Hilbert action.

In non-Riemanian geometries such as Weyl and Cartan geometries, the above conclusion is no longer true and the Gauss-Bonnet term will not become total derivative. In [16] the Gauss-Bonnet combination was obtained in the context of Weyl geometry, and it turns out that the remaining term in 4D is the Weyl vector kinetic term

$$S_{GB} \propto \int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}. \tag{7}$$

It is the aim of the present paper to generalize the above argument to the case of Weyl-Cartan space-time. Similar to the Einstein-Weyl space-time, in the Weyl-Cartan space-time the Gauss-Bonnet term will not be a total derivative. The theory is not in general Ostrogradski stable. In order to have a stable theory one should constrain the parameter space of the model as we will do in the next section.

The Weyl-Cartan model has also been considered in the context of Weitzenboch gravity in [11]. The authors have added the kinetic terms for the Weyl vector and the torsion tensor by hand, using the trace of torsion tensor. We will see in this paper that considering the Gauss-Bonnet action can produce automatically all the kinetic terms of [11]. It is worth mentioning that the theory [11] has a potential ghost, noting that in the Weitzenboch gravity the torsion has some relation to the Ricci scalar, and as a result the torsion kinetic term has more than second time derivatives. However, the present paper is free from the aforementioned instability due to the absence of the Weitzenboch condition. One should note that the torsion self-interaction term $\nabla_\mu T\nabla^\mu T$ with $T = T^\mu T_\mu$ and $T_\mu = T^\nu{}_{\mu\nu}$, in [11] can not be produced in the present context, because it is fourth order in the torsion and second order in derivatives. In order to produce such term one should consider the higher order Lovelock terms in the action.

The present theory in general may have some tachyon instabilities but the analysis is very complicated because of the appearance of the torsion tensor. In section II the generalized Gauss-Bonnet action in Weyl-Cartan space-time is introduced and shown that the higher than two time derivatives are removed in the action. In section III we will consider a restricted form for the torsion tensor and obtain the healthy region of the parameter space in which all instabilities are removed.

II. THE MODEL

The Weyl geometry proposal induces a new vector which results in non-metricity of the connection i.e. $\bar\nabla_\mu g_{\alpha\beta} \neq 0$ where $\bar\nabla_\mu$ is the covariant derivative with respect to Weyl connection. Mathematically, the Weyl geometry has a special form of non-metricity i.e.

$$\bar\nabla_\mu g_{\nu\sigma} = 2w_\mu g_{\nu\sigma}, \tag{8}$$

where w_μ is the Weyl vector. So the Weyl connection can be obtained as

$$\bar\Gamma^\lambda{}_{\mu\nu} = \left\{ {}_{\mu}{}^{\lambda}{}_{\nu} \right\} + Q^\lambda_{\mu\nu}, \tag{9}$$

where

$$Q^\lambda_{\mu\nu} = g_{\mu\nu}w^\lambda - \delta^\lambda_\mu w_\nu - \delta^\lambda_\nu w_\mu \tag{10}$$

and $\left\{ {}_{\mu}{}^{\lambda}{}_{\nu} \right\}$ is the Christoffel symbol. In addition one may generalize the above connection by adding the effects of the torsion into it as

$$\Gamma^\lambda{}_{\mu\nu} = \left\{ {}_{\mu}{}^{\lambda}{}_{\nu} \right\} + Q^\lambda_{\mu\nu} + C^\lambda_{\mu\nu}. \tag{11}$$

Note that the third term is named contorsion tensor defined as

$$C^\lambda{}_{\mu\nu} = T^\lambda{}_{\mu\nu} - g^{\lambda\beta}g_{\sigma\mu}T^\sigma{}_{\beta\nu} - g^{\lambda\beta}g_{\sigma\nu}T^\sigma{}_{\beta\mu}, \tag{12}$$

where we have defined the torsion tensor $T^\lambda{}_{\mu\nu}$ as

$$T^\lambda{}_{\mu\nu} = \frac{1}{2}\left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right). \tag{13}$$

It is easy to show that the additional torsion does not affect the non-metricity relation i.e. the relation (8) is still valid. By using the metric one can build $C_{\lambda\mu\nu} = g_{\lambda\sigma}C^\sigma{}_{\mu\nu}$ which is antisymmetric with respect to its two first indices by having in mind that the torsion tensor is antisymmetric with respect to its down indices in $T^\sigma{}_{\mu\nu}$.

We define the curvature tensor as

$$K^\lambda{}_{\mu\nu\sigma} = \partial_\nu\Gamma^\lambda{}_{\mu\sigma} - \partial_\sigma\Gamma^\lambda{}_{\mu\nu} + \Gamma^\alpha{}_{\mu\sigma}\Gamma^\lambda{}_{\alpha\nu} - \Gamma^\alpha{}_{\mu\nu}\Gamma^\lambda{}_{\alpha\sigma}. \tag{14}$$

One can decompose the curvature tensor into four parts as

$$K^\lambda{}_{\mu\nu\sigma} = R^\lambda{}_{\mu\nu\sigma} + C^\lambda{}_{\mu\nu\sigma} + Q^\lambda{}_{\mu\nu\sigma} + I^\lambda{}_{\mu\nu\sigma}, \tag{15}$$

where the first term in the right hand side of the above relation is the Riemann curvature tensor defined by the Christoffel symbol and we have defined

$$C^\lambda{}_{\mu\nu\sigma} = \nabla_\nu C^\lambda{}_{\mu\sigma} - \nabla_\sigma C^\lambda{}_{\mu\nu} + C^\alpha{}_{\mu\sigma} C^\lambda{}_{\alpha\nu} - C^\alpha{}_{\mu\nu} C^\lambda{}_{\alpha\sigma}, \tag{16}$$

$$Q^\lambda{}_{\mu\nu\sigma} = \nabla_\nu Q^\lambda{}_{\mu\sigma} - \nabla_\sigma Q^\lambda{}_{\mu\nu} + Q^\alpha{}_{\mu\sigma} Q^\lambda{}_{\alpha\nu} - Q^\alpha{}_{\mu\nu} Q^\lambda{}_{\alpha\sigma}, \tag{17}$$

$$I^\lambda{}_{\mu\nu\sigma} = C^\alpha{}_{\mu\sigma} Q^\lambda{}_{\alpha\nu} + Q^\alpha{}_{\mu\sigma} C^\lambda{}_{\alpha\nu} - C^\alpha{}_{\mu\nu} Q^\lambda{}_{\alpha\sigma} - Q^\alpha{}_{\mu\nu} C^\lambda{}_{\alpha\sigma}, \tag{18}$$

where ∇_μ is covariant derivative with respect to the Christoffel symbol and $I^\lambda{}_{\mu\nu\sigma}$ represents interaction between non-metricity and torsion parts. It is possible to rewrite the purely non-metricity part (17) as

$$\begin{aligned}\frac{1}{2} Q^\lambda{}_{\mu\nu\sigma} =& - \delta^\lambda_\mu \nabla_{[\nu} w_{\sigma]} - \delta^\lambda_{[\sigma} \nabla_{\nu]} w_\mu - g_{\mu[\nu} \nabla_{\sigma]} w^\lambda \\ &+ \delta^\lambda_{[\nu} w_{\sigma]} w_\mu + g_{\mu[\nu} \delta^\lambda_{\sigma]} w^2 + g_{\mu[\sigma} w_{\nu]} w^\lambda \end{aligned} \tag{19}$$

and the interaction part (18) as

$$\begin{aligned}\frac{1}{2} I^\lambda{}_{\mu\nu\sigma} =& -w^\alpha C_{\alpha\mu[\sigma} \delta^\lambda_{\nu]} - w^\alpha C^\lambda{}_{\alpha[\sigma} g_{\nu]\mu} \\ &- w^\lambda C_{\mu[\nu\sigma]} - w_\mu C^\lambda{}_{[\sigma\nu]}. \end{aligned} \tag{20}$$

In order to construct a higher order gravity models e.g. Gauss-Bonnet action, one should multiply the curvature tensor to itself. There are seven different ways to do this

$$\begin{aligned}& K_{\lambda\mu\nu\sigma} K^{\lambda\mu\nu\sigma}, K_{\lambda\mu\nu\sigma} K^{\mu\lambda\nu\sigma}, K_{\lambda\mu\nu\sigma} K^{\nu\sigma\lambda\mu}, \\ & K_{\lambda\mu\nu\sigma} K^{\lambda\nu\mu\sigma}, K_{\lambda\mu\nu\sigma} K^{\nu\mu\lambda\sigma}, K_{\lambda\mu\nu\sigma} K^{\mu\sigma\lambda\nu}, \\ & K_{\lambda\mu\nu\sigma} K^{\sigma\lambda\mu\nu}. \end{aligned} \tag{21}$$

One should note that if the Weyl vector and torsion tensor be zero, only the first three of the above terms reduce to the combination similar to Gauss-Bonnet term, which we will only use them in the following. As is well-known, the Riemann tensor has only one independent contraction.

The Weyl part of the above curvature tensor has two independent contractions

$$\begin{aligned} Q^\lambda{}_{\lambda\mu\nu} &= -4 W_{\mu\nu}, \\ Q^\lambda{}_{\mu\lambda\nu} &= -\nabla_\mu w_\nu + 3 \nabla_\nu w_\mu + g_{\mu\nu} \nabla_\lambda w^\lambda \\ &\quad + 2 w_\mu w_\nu - 2 g_{\mu\nu} w^2, \end{aligned} \tag{22}$$

where we have defined

$$W_{\mu\nu} = \nabla_\mu w_\nu - \nabla_\nu w_\mu. \tag{23}$$

The contortion part of the curvature tensor has only one independent contraction

$$\begin{aligned} C^\lambda{}_{\lambda\mu\nu} &= 0, \\ C^\lambda{}_{\mu\lambda\nu} &= \nabla_\lambda C^\lambda{}_{\mu\nu} + \nabla_\nu C_\mu, \end{aligned} \tag{24}$$

where we have defined $C^\mu = C^{\mu\nu}{}_\nu$.

The interaction part has also one independent contraction which can be written as

$$\begin{aligned} I^\lambda{}_{\lambda\mu\nu} &= 0, \\ I^\lambda{}_{\mu\lambda\nu} &= -w^\alpha (C_{\alpha\mu\nu} + C_{\nu\mu\alpha}). \end{aligned} \tag{25}$$

For the Riemann curvature tensor, we have $R^\lambda{}_{\lambda\mu\nu} = 0$ and $R^\lambda{}_{\mu\lambda\nu} = R_{\mu\nu}$ where $R_{\mu\nu}$ is the standard Ricci tensor. For the contracted curvature tensor, the two independent contractions are

$$K_{\mu\nu} \equiv K^\lambda{}_{\lambda\mu\nu}, \quad \mathcal{K}_{\mu\nu} \equiv \mathcal{K}^\lambda{}_{\mu\lambda\nu}.$$

There are four independent combinations of them as follows

$$K_{\mu\nu} K^{\mu\nu},\ K_{\mu\nu} \mathcal{K}^{\mu\nu},\ \mathcal{K}_{\mu\nu} \mathcal{K}^{\mu\nu},\ \mathcal{K}_{\mu\nu} \mathcal{K}^{\nu\mu}. \tag{26}$$

All the above terms are proportional to $R^{\mu\nu} R_{\mu\nu}$ in the case of zero torsion tensor and Weyl vector.

There is only one independent curvature scalar of the tensor $K^\lambda{}_{\mu\nu\sigma}$ which can be defined by contracting the tensor $\mathcal{K}_{\mu\nu}$ with the metric

$$\begin{aligned} K = R &+ 6 \nabla_\mu w^\mu - 6 w^2 + 2 \nabla_\lambda C^\lambda - C^\alpha C_\alpha \\ &+ C_{\alpha\mu\lambda} C^{\alpha\lambda\mu} - 4 w^\alpha C_\alpha. \end{aligned} \tag{27}$$

Let us propose the following action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} K + S_G, \tag{28}$$

where S_G is the Gauss-Bonnet action defined as

$$\begin{aligned} S_G =& \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \Big[\alpha_1 K^{\alpha\beta\gamma\delta} K_{\alpha\beta\gamma\delta} + \alpha_2 K^{\alpha\beta\gamma\delta} K_{\gamma\delta\alpha\beta} \\ &- \alpha_3 K^{\alpha\beta\gamma\delta} K_{\beta\alpha\gamma\delta} - 4\beta_1 \mathcal{K}_{\beta\gamma} \mathcal{K}^{\beta\gamma} - 4\beta_2 \mathcal{K}_{\beta\gamma} \mathcal{K}^{\gamma\beta} \\ &- 4\beta_3 K_{\alpha\beta} K^{\alpha\beta} - 4\beta_4 K_{\alpha\beta} \mathcal{K}^{\alpha\beta} + K^2 \Big]. \end{aligned} \tag{29}$$

To get the standard Gauss-Bonnet terms in the absence of torsion and non-metricity we need to impose the following constraints on the coefficients

$$\alpha_1 + \alpha_2 + \alpha_3 = 1, \tag{30a}$$

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 1. \tag{30b}$$

We should note that the above action is the most general action for the second order higher gravity in the Weyl-Cartan theory which reduces to the standard Gauss-Bonnet action in the limit of zero Weyl and torsion.

In general the above action has some terms with higher than second time derivatives. These terms can potentially produce some instabilities which are known as the Ostrogradski ghosts. These potentially dangerous terms can be collected as

$$
\begin{aligned}
S_G \supset 4\int d^4x\sqrt{-g}\Big\{&\Big(-2R_{\mu\nu}\nabla^\mu C^\nu + R\nabla^\mu C_\mu - 2R_{\mu\nu}\nabla^\mu w^\nu + R\nabla^\mu w_\mu - 2R_{\mu\nu}\nabla^\alpha C_\alpha{}^{\mu\nu} + R_{\alpha\delta\beta\gamma}\nabla^\alpha C^{\beta\gamma\delta}\Big)\\
&+2(\beta_3+\beta_4)\Big[R_{\mu\nu}\nabla^\mu C^\nu + 2R_{\mu\nu}\nabla^\mu w^\nu + R\nabla^\mu w_\mu + R_{\mu\nu}\nabla^\alpha C_\alpha{}^{\mu\nu}\Big]\Big\}\\
=8\int d^4x\sqrt{-g}&(\beta_3+\beta_4)\Big[R_{\mu\nu}\nabla^\mu C^\nu + R_{\mu\nu}\nabla^\alpha C_\alpha{}^{\mu\nu}\Big],
\end{aligned}
\tag{31}
$$

where we have dropped total derivatives in the second equality. In the above we have used integration by parts and contracted second Bianchi identity. Note that we have eliminated α_1 and β_1 using the relations (30). As one can see, demanding absence of Ostrogradski ghosts imposes a constraint on the coefficients as

$$\beta_3 + \beta_4 = 0. \tag{32}$$

It means the five-dimensional space for coefficients in S_G reduces to a four-dimensional space by demanding absence of instabilities.

In order to write the action S_G in detail, we decompose the action into three parts. The terms which involve only the Weyl vector can be collected as

$$S_W = \rho\int d^4x\sqrt{-g}W_{\mu\nu}W^{\mu\nu}, \tag{33}$$

with

$$\rho = -4(3 + 2\alpha_2 + 2\alpha_3 - 8\beta_4 - 8\beta_2),$$

and we have dropped the total derivative terms. The terms which involves the contortion tensor can be written as

$$
\begin{aligned}
S_C = \int d^4x\sqrt{-g}\Big[&-4\,R^{\alpha\beta\gamma\delta}C_\alpha{}^\nu{}_\gamma C_{\beta\nu\delta} - 8\,R^{\alpha\beta}C^\gamma C_{\alpha\gamma\beta} + 8\,R^{\alpha\beta}C_\alpha{}^{\gamma\delta}C_{\gamma\delta\beta} + 2\,RC^{\alpha\beta\gamma}C_{\alpha\gamma\beta} + 4\,G^{\alpha\beta}C_\alpha C_\beta\\
&+(2-2\,\alpha_2)\Big(\nabla^\alpha C^{\beta\gamma\delta}\,\nabla_\alpha C_{\beta\gamma\delta} - \nabla^\alpha C^{\beta\gamma\delta}\,\nabla_\delta C_{\beta\gamma\alpha} - 4\nabla^\alpha C^{\beta\gamma\delta}\,C_\beta{}^\nu{}_\alpha C_{\gamma\nu\delta} + C^{\alpha\beta\gamma}C_\alpha{}^\delta{}_\gamma C_\beta{}^{\nu\mu}C_{\delta\nu\mu}\\
&-C^{\alpha\beta\gamma}C_\alpha{}^{\delta\nu}C_\beta{}^\mu{}_\nu C_{\delta\mu\gamma}\Big) + 4\,\alpha_2\nabla^\alpha C^{\beta\gamma\delta}\,\nabla_\beta C_{\alpha\delta\gamma} - 8\,\alpha_2\nabla^\alpha C^{\beta\gamma\delta}\,C_\alpha{}^\nu{}_\beta C_{\delta\nu\gamma} + 2\,\alpha_2 C^{\alpha\beta\gamma}C_\alpha{}^{\delta\nu}C_\gamma{}^\mu{}_\beta C_{\nu\mu\delta}\\
&-2\,\alpha_2 C^{\alpha\beta\gamma}C_\alpha{}^{\delta\nu}C_\gamma{}^\mu{}_\delta C_{\nu\mu\beta} + (4-4\,\beta_2)\Big(\nabla^\alpha C_\alpha{}^{\beta\gamma}\,\nabla^\delta C_{\beta\delta\gamma} + 2\nabla^\alpha C^\beta\,\nabla^\gamma C_{\beta\gamma\alpha} - 2\nabla^\alpha C_\alpha{}^{\beta\gamma}\,C^\delta C_{\beta\delta\gamma}\\
&+2\nabla^\alpha C_\alpha{}^{\beta\gamma}\,C_\beta{}^{\delta\nu}C_{\delta\nu\gamma} - 2\nabla^\alpha C^\beta\,C^\gamma C_{\beta\gamma\alpha} + 2\nabla^\alpha C^\beta\,C_\beta{}^{\gamma\delta}C_{\gamma\delta\alpha} - C^\alpha C^\beta C_\alpha{}^{\gamma\delta}C_{\beta\gamma\delta} - 2C^\alpha C_\alpha{}^{\beta\gamma}C_\beta{}^{\delta\nu}C_{\delta\nu\gamma}\\
&+C^{\alpha\beta\gamma}C_{\alpha\gamma}{}^\delta C_\beta{}^{\nu\mu}C_{\nu\mu\delta} - \frac{1}{2}C^{\alpha\beta}C_{\alpha\beta}\Big) + 4\,\beta_2\nabla^\alpha C_\alpha{}^{\beta\gamma}\,\nabla^\delta C_{\gamma\delta\beta} + 8\,\beta_2\nabla^\alpha C^\beta\,\nabla^\gamma C_{\alpha\gamma\beta} - 8\,\beta_2\nabla^\alpha C^\beta\,C^\gamma C_{\alpha\gamma\beta}\\
&-8\,\beta_2\nabla^\alpha C_\alpha{}^{\beta\gamma}\,C^\delta C_{\gamma\delta\beta} + 8\,\beta_2\nabla^\alpha C_\alpha{}^{\beta\gamma}\,C_\gamma{}^{\delta\nu}C_{\delta\nu\beta} + 8\,\beta_2\nabla^\alpha C^\beta\,C_\alpha{}^{\gamma\delta}C_{\gamma\delta\beta} - 4\,\beta_2 C^\alpha C^\beta C_\alpha{}^{\gamma\delta}C_{\beta\delta\gamma}\\
&-8\,\beta_2 C^\alpha C_\alpha{}^{\beta\gamma}C_\gamma{}^{\delta\nu}C_{\delta\nu\beta} + 4\,\beta_2 C^{\alpha\beta\gamma}C_{\alpha\gamma}{}^\delta C_\delta{}^{\nu\mu}C_{\nu\mu\beta} - 4\,C^2\nabla^\alpha C_\alpha + 4\,\nabla^\alpha C_\alpha\,C^{\beta\gamma\delta}C_{\beta\delta\gamma} + C^4 - 2\,C^2C^{\beta\gamma\delta}C_{\beta\delta\gamma}\\
&+C^{\alpha\beta\gamma}C_{\alpha\gamma\beta}C^{\delta\nu\mu}C_{\delta\mu\nu}\Big],
\end{aligned}
\tag{34}
$$

where we have defined $C^2 = C_\mu C^\mu$ and

$$C_{\mu\nu} = \nabla_\mu C_\nu - \nabla_\nu C_\mu. \tag{35}$$

One should note that the tensor $C_{\mu\nu}$ is proportional to the tensor $T_{\mu\nu}$ constructed similarly with the torsion tensor. One can see that the term $T_{\mu\nu}T^{\mu\nu}$ is produced naturally in this model, which is also the kinetic term assumed in [11].

The remaining terms of the action contain a variety of possible interactions between the Weyl vector and the contortion tensor we can be simplified as

$$
\begin{aligned}
S_I = \int d^4x\sqrt{-g}\Big[& -8\,R^{\alpha\beta\gamma\delta}C_{\alpha\gamma\beta}w_\delta + (8\,\beta_2 - 8\,\alpha_2)\big(\nabla^\alpha C_\alpha{}^{\beta\gamma}\,C_\beta{}^\delta{}_\gamma w_\delta - C^{\alpha\beta}\,C_\alpha{}^\gamma{}_\beta w_\gamma - C^\alpha C_\alpha{}^{\beta\gamma}C_\beta{}^\delta{}_\gamma w_\delta \\
& - \nabla^\alpha C_\alpha{}^{\beta\gamma}\,C_\gamma{}^\delta{}_\beta w_\delta + C^\alpha C_\alpha{}^{\beta\gamma}C_\gamma{}^\delta{}_\beta w_\delta - C^{\alpha\beta\gamma}C_{\alpha\gamma}{}^\delta C_\delta{}^\mu{}_\beta w_\mu\big) + (4\,-4\alpha_2)\big(2\nabla^\alpha C^{\beta\gamma\delta}\,C_{\alpha\beta\delta}w_\gamma \\
& + 2\nabla^\alpha C^{\beta\gamma\delta}\,C_{\beta\delta\alpha}w_\gamma + 2C^{\alpha\beta\gamma}C_\alpha{}^\delta{}_\gamma C_\beta{}^\mu{}_\delta w_\mu + w^2C^{\alpha\beta\gamma}C_{\alpha\gamma\beta} - w^2C^{\alpha\beta\gamma}C_{\alpha\beta\gamma} - C^{\beta\gamma\delta}\,C_{\beta\delta\gamma}\nabla^\alpha w_\alpha\big) \\
& + (16\,\beta_2 - 8\,\beta_4 - 4\,\alpha_2 - 8)\big(C^{\alpha\beta}W_{\alpha\beta} - 2\nabla^\alpha C_\alpha{}^{\beta\gamma}\,W_{\beta\gamma} + 2W^{\alpha\beta}C^\gamma C_{\beta\gamma\alpha} - 2W^{\alpha\beta}C_\alpha{}^\gamma{}_\beta w_\gamma + 2W^{\alpha\beta}C_\alpha w_\beta\big) \\
& + (4 - 4\,\beta_2)\big(C^\alpha C^\beta w_\alpha w_\beta - w^2C^2 + w^2C^{\alpha\beta\gamma}C_\gamma{}^\delta{}_\alpha - 2\nabla^\alpha C_\alpha{}^{\beta\gamma}\,C_\gamma w_\beta - 2\nabla^\alpha C^\beta\,C_\alpha w_\beta \\
& + 2C^\alpha C_\alpha{}^{\beta\gamma}C_{\beta\gamma}{}^\delta w_\delta + 2C^\alpha C^\beta C_\alpha{}^\gamma{}_\beta w_\gamma + 2w_\alpha C_\beta\nabla^\alpha C^\beta + 2C^\alpha C^{\beta\gamma}{}_\alpha C_\beta{}^\delta{}_\gamma w_\delta\big) - 8\,\nabla^\alpha C_\alpha{}^{\beta\gamma}\,\nabla_\beta w_\gamma \\
& + 8\,\nabla^\alpha w_\gamma C_\alpha{}^{\gamma\beta}\,w_\beta + (8 - 16\,\alpha_2 + 8\,\beta_2)C^{\alpha\beta\gamma}C_{\alpha\gamma}{}^\delta C_\beta{}^\mu{}_\delta w_\mu + 8\,\nabla^\alpha w^\beta\,C_\beta{}^{\gamma\delta}C_{\gamma\delta\alpha} - 8\,\nabla^\alpha w^\beta\,C^\gamma C_{\alpha\gamma\beta} \\
& + (16 + 8\,\alpha_2 - 24\,\beta_2)C^\alpha C_\alpha{}^{\beta\gamma}w_\beta w_\gamma + (-8 - 8\,\alpha_2 + 16\,\beta_2)C^{\alpha\beta\gamma}C_{\alpha\gamma}{}^\delta w_\beta w_\delta + 8\,\alpha_2\nabla^\alpha C^{\beta\gamma\delta}\,C_{\alpha\beta\gamma}w_\delta \\
& + (-8 + 4\,\alpha_2 + 4\,\beta_2)C^{\alpha\beta\gamma}C_\alpha{}^\delta{}_\gamma w_\beta w_\delta + (12 + 8\,\alpha_2 + 16\,\beta_4 - 32\,\beta_2)W^{\alpha\beta}C_{\alpha\beta}{}^\gamma w_\gamma - 8\,\beta_2\nabla^\alpha C_\alpha{}^{\beta\gamma}\,C_\beta w_\gamma \\
& + (-8\,\alpha_2 - 8 + 16\,\beta_2)C^{\alpha\beta\gamma}C_{\alpha\gamma}{}^\delta C_{\beta\delta}{}^\mu w_\mu + (-8 + 16\,\beta_2)\nabla^\alpha C_\alpha{}^{\beta\gamma}\,C_{\beta\gamma}{}^\delta w_\delta + (8 - 16\,\beta_2)\nabla^\alpha C^\beta\,C_{\alpha\beta}{}^\gamma w_\gamma \\
& + (-8\,\alpha_2 - 8 - 16\,\beta_4 + 32\,\beta_2)W^{\alpha\beta}C_\alpha{}^{\gamma\delta}C_{\gamma\delta\beta} + (-4\,\alpha_2 - 4 + 8\,\beta_2)C^{\alpha\beta\gamma}C_{\alpha\beta}{}^\delta w_\gamma w_\delta\Big].
\end{aligned}
\tag{36}
$$

It is worth mentioning that the term $\nabla_\alpha C^{\alpha\beta\gamma}W_{\beta\gamma}$ contains an interaction term between Weyl vector and torsion tensor which was assumed in [12].

Finally, the full action of the theory can be written as

$$
S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[R - 6w^2 - C^2 + C_{\alpha\mu\lambda}C^{\alpha\lambda\mu} - 4w^\alpha C_\alpha\Big] + S_W + S_C + S_I. \tag{37}
$$

The torsion tensor can be decomposed irreducibly into

$$
T_{\mu\nu\rho} = \frac{2}{3}(t_{\mu\nu\rho} - t_{\mu\rho\nu}) + \frac{1}{3}(Q_\nu g_{\mu\rho} - Q_\rho g_{\mu\nu}) + \epsilon_{\mu\nu\rho\sigma}S^\sigma, \tag{38}
$$

where Q_μ and S^μ are two vector fields. The vector Q_μ is actually the trace of torsion over its first and third indices. The tensor $t_{\mu\nu\rho}$ is symmetric with respect to μ and ν and has the following properties

$$
t_{\mu\nu\rho} + t_{\nu\rho\mu} + t_{\rho\mu\nu} = 0, \quad g_{\mu\nu}t^{\mu\nu\rho} = 0 = g_{\mu\rho}t^{\mu\nu\rho}. \tag{39}
$$

One can decompose the contortion tensor according to the above relation as

$$
C_{\rho\mu\nu} = \frac{4}{3}(t_{\mu\nu\rho} - t_{\rho\nu\mu}) + \frac{2}{3}(Q_\mu g_{\nu\rho} - Q_\rho g_{\mu\nu}) + \epsilon_{\rho\mu\nu\sigma}S^\sigma, \tag{40}
$$

III. SPECIAL CASE FOR CONTORSTION TENSOR

Let us assume that the contortion tensor has the followin simple form

$$
C_{\rho\mu\nu} = \hat{Q}_\mu g_{\nu\rho} - \hat{Q}_\rho g_{\mu\nu}, \tag{41}
$$

where we have assumed that $t_{\mu\nu\rho} = 0$ and $S^\sigma = 0$, and we define the vector $\hat{Q}_\mu = \frac{2}{3}Q_\mu$.

The action can then be expanded as

$$
\begin{aligned}
S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[& R - 6w^2 - 6\hat{Q}^2 + 12w^\alpha\hat{Q}_\alpha \\
& - 4(1 + \alpha_2 - 2)\hat{Q}_{\mu\nu}\hat{Q}^{\mu\nu} \\
& + 8(2 + \alpha_2 + 2\beta_4 - 4\beta_2)\hat{Q}_{\mu\nu}W^{\mu\nu} \\
& - 4(3 + 2\alpha_2 + 2\alpha_3 - 8\beta_2 - 8\beta_4)W_{\mu\nu}W^{\mu\nu}\Big].
\end{aligned}
\tag{42}
$$

In general the above action may have some ghost and or tachyon instabilities. In order to examine this issue, we first diagonalise the kinetic and potential terms for $\hat{Q}_\mu$ and w_μ with the result

$$
S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[R - \frac{1}{4}X_{\mu\nu}X^{\mu\nu} - \frac{1}{4}Y_{\mu\nu}Y^{\mu\nu} - \frac{1}{2}m^2X_\mu X^\mu\Big], \tag{43}
$$

where $X_{\mu\nu}$ and $Y_{\mu\nu}$ are strength tensors respectively according to vectors X_μ and Y_μ which will be defined below. As one can see from the above action, the theory contains one massless and one massive vector field with mass

$$
m^2 = \frac{3(A + 2B + C)}{B^2 - AC}, \tag{44}
$$

where we have defined

$$
A = -4 - 4\alpha_2 + 8\beta_2, \tag{45a}
$$

$$
B = 8 + 8\beta_4 + 4\alpha_2 - 16\beta_2, \tag{45b}
$$

$$
C = -12 + 32\beta_4 - 8\alpha_2 + 32\beta_2 - 8\alpha_3. \tag{45c}
$$

The new fileds can be related to the original fields $\hat{Q}_\mu$ and w_μ as

$$X_\mu = \frac{2}{\sqrt{2\beta^2-1}}\Big[(\alpha\beta\lambda_+ - \lambda_-\sqrt{(1-\alpha^2)(\beta^2-1)})\hat{Q}_\mu + (\beta\lambda_+\sqrt{1-\alpha^2} + \alpha\lambda_-\sqrt{\beta^2-1})w_\mu\Big], \tag{46}$$

$$Y_\mu = -\frac{2}{\sqrt{2\beta^2-1}}\Big[(\beta\lambda_-\sqrt{1-\alpha^2} + \alpha\lambda_+\sqrt{\beta^2-1})\hat{Q}_\mu - (\alpha\beta\lambda_- - \lambda_+\sqrt{(1-\alpha^2)(\beta^2-1)})w_\mu\Big], \tag{47}$$

where we have defined

$$\alpha = \left[\frac{1}{2}\left(1+\frac{A-C}{\sqrt{4B^2+(A-C)^2}}\right)\right]^{\frac{1}{2}}, \tag{48}$$

$$\beta = -\left[\frac{1}{2}\left(\frac{(A+2B+C)\sqrt{4B^2+(A-C)^2}}{4B^2+(A-C)^2+2B(A+C)}+1\right)\right]^{\frac{1}{2}}, \tag{49}$$

and

$$\lambda_\pm^2 = -\frac{1}{2}\left(A+C\pm\sqrt{4B^2+(A-C)^2}\right). \tag{50}$$

In order to have a ghost and tachyon free theory we should have $m^2 > 0$ and the new fields (46) and (47) should be meaningful. We thus conclude that the parameters $\alpha_2, \alpha_3, \beta_2$ and β_4 should satisfy the relations

$$m^2 > 0, \qquad \lambda_\pm^2 > 0, \tag{51}$$

together with reality of square roots. In general, the above conditions can not be solved analytically in full four dimensional parameter space. In the following, we will concentrate our attention to some special cases.

A. Case I: $\beta_2 = 0 = \alpha_3$ and $\alpha_2 = 1$

In this case the the action S_G reduces to

$$S_G = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[K^{\alpha\beta\gamma\delta}K_{\gamma\delta\alpha\beta} - 4\mathcal{K}_{\beta\gamma}\mathcal{K}^{\beta\gamma} + 4\beta_4(K_{\alpha\beta}K^{\alpha\beta} - K_{\alpha\beta}\mathcal{K}^{\alpha\beta}) + K^2\Big], \tag{52}$$

and the constraints (51) satisfy if

$$-\frac{3}{2} < \beta_4 < \frac{1}{2}\left(5\sqrt{2}-7\right). \tag{53}$$

B. Case II: $\beta_2 = 0 = \alpha_3$ and $\beta_4 = 0$

In this case the the action S_G reduces to

$$S_G = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[(1-\alpha_2)K^{\alpha\beta\gamma\delta}K_{\alpha\beta\gamma\delta} + \alpha_2 K^{\alpha\beta\gamma\delta}K_{\gamma\delta\alpha\beta} - 4\mathcal{K}_{\beta\gamma}\mathcal{K}^{\beta\gamma} + K^2\Big], \tag{54}$$

and the constraints (51) satisfy if

$$\frac{2}{1+\sqrt{5}} < \alpha_2 < 2. \tag{55}$$

C. Case III: $\beta_2 = 0 = \alpha_3$

In this case the the action S_G reduces to

$$S_G = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[(1-\alpha_2)K^{\alpha\beta\gamma\delta}K_{\alpha\beta\gamma\delta} + \alpha_2 K^{\alpha\beta\gamma\delta}K_{\gamma\delta\alpha\beta} - 4\times\mathcal{K}_{\beta\gamma}\mathcal{K}^{\beta\gamma} + 4\beta_4(K_{\alpha\beta}K^{\alpha\beta} - K_{\alpha\beta}\mathcal{K}^{\alpha\beta}) + K^2\Big], \tag{56}$$

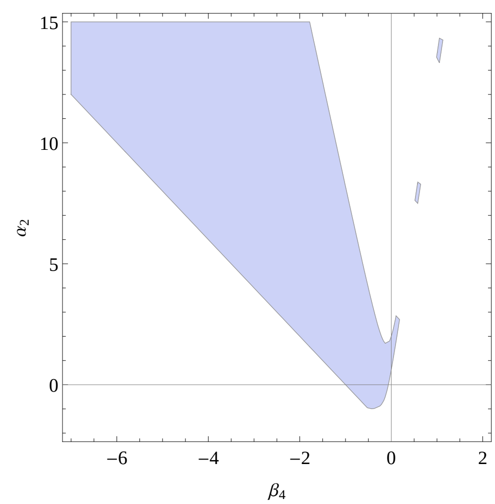

FIG1: Allowed range of β_4 and α_2 for the action (56) to become ghost and tachyon free theory.

In the figure we have plotted the allowed region of parameter space (α_2,β_4) in order to have a ghost and tachyon free bi-vector theory.

IV. CONCLUSION

In this paper, we have introduced a ghost and tachyon free modified theory of gravity by generalizing the geometry to be the Weyl-Cartan space-time. Using the standard Einstein-Hilbert term for this geometry, the action reduces to the Ricci scalar, plus possible mass terms for Weyl vector and the torsion tensor. In this case no kinetic terms for these two new fields can be produced. In order to make the Weyl vector and the torsion tensor dynamical, one can add some kinetic terms by hand, which was done in [11].

In this paper, in order to produce kinetic terms for the Weyl and torsion fields, we have generalized the action to be of Gauss-Bonnet type. In 4D Riemannian geometry the Gauss-Bonnet term becomes a total derivative and dropped from the action. However in the Weyl-Cartan geometry, this term produces a bunch of interaction and kinetic terms for Weyl and torsion fields. In the Weyl-Cartan geometry the curvature tensor has less symmetries than the Riemann tensor. So one can write more than three quadratic terms according to the curvature. In general, the resulting action does not reduce to the standard Gauss-Bonnet action and may have some higher derivative instabilities. Removing the above difficulties, one can obtain a 4-parameter family of theories, which has no Ostrogradski instability.

For further considerations, we have studied a special case of the theory where only the trace part of the torsion tensor is non-zero. In this case, the theory is reduced to general relativity plus one massive and one massless vector fields. The absence of ghost and tachyon instabilities will reduce the parameter space of the theory which we have obtained them for some special cases. One should note that in the full theory one can not specify the viable values of the parameter space. In this paper, we have proved that such region exists.

---

[1] H. Weyl, Sitzungsber. Preuss. Akad. Wiss. 465 (1918) 14.

[2] M. Israelit, Gen. Rel. Gravit. 43 (2011) 751; M. Israelit, arXiv:1212.2208 [gr-qc];

[3] C. Romero, J. B. Fonseca-Neto and M. L. Pucheu, arXiv:1201.1469 [gr-qc].

[4] J. B. Formiga, J. B. Fonseca-Neto and C. Romero, Phys. Rev. D 87 (2013) 067702; J. B. Formiga, Phys. Rev. D. 88 (2013) 068501; J. B. Formiga, arXiv:1401.7500 [gr-qc].

[5] M. Novello and H. Heintzmann, Phys. Lett. A 98 (1983) 10.

[6] F. P. Poulis and J. M. Salim, arXiv:1305.6830 [gr-qc]; J. Miritzis, arXiv:1301.5696 [gr-qc]; Y.Liu, K. Yang and Y. Zhong, JHEP 10 (2010) 069.

[7] A. S. Eddington, Proc. R. Soc. Lond. A 99, (1921) 104; A. S. Eddington, The Mathematical Theory of Relativity, 2nd ed. 1924, Cambridge University, Cambridge.

[8] E. Cartan, C. R. Acad. Sci. (Paris) 174, (1922) 593; E. Cartan, Ann. Ec. Norm. Sup. 40, (1923) 325; E. Cartan, Ann. Ec. Norm. Sup. 41, (1924) 1; E. Cartan, Ann. Ec. Norm. Sup. 42, (1925) 17.

[9] H. Weyl, Phys. Rev. 77 (1950) 699; E. C. G. Stueckelberg, Phys. Rev. 73 (1948) 808; C. Pellegrini and J. Plebanski, K. Dan. Vidensk. Selsk. Mat. Fys. skr. 2 (1963) 4.

[10] R. Ferraro, AIP Conf. Proc. 1471 (2012) 103; R. Aldrovandi and J. G. Pereira, "Teleparallel Gravity", Fundamental Theories of Physics, Vol. 173, Springer;

[11] Z. Haghani, T. Harko, H. R. Sepangi and S. Shahidi, JCAP 10 (2012) 061; Z. Haghani, T. Harko, H. R. Sepangi and S. Shahidi, Phys. Rev. D 88, 044024 (2013).

[12] M. Israelit, Found.Phys. 28 (1998) 205; M. Israelit, arXiv:0712.3998.

[13] H. A. Buchdahl, Mon. Not. Roy. Astro. Soc. 150 (1970) 1; A. A. Starobinski, Phys. Lett. B. 91 (1980) 99; V. Faraoni, arXiv:0810.2602 [gr-qc]; T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82 (2010) 451.

[14] E. Kretschmann, Ann. Phys. (Leipzig), 53 (1917) 575.

[15] A. De Felice and S. Tsujikawa, Living Rev. Rel. 13 (2010) 3.

[16] J. B. Jimenez and T. S. Koivisto, arXiv:1402.1846 [gr-qc].