Center of mass, spin supplementary conditions, and the momentum of spinning particles

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Abstract

We discuss the problem of defining the center of mass in general relativity and the so-called spin supplementary condition. The different spin conditions in the literature, their physical significance, and the momentum-velocity relation for each of them are analyzed in depth. The reason for the non-parallelism between the velocity and the momentum, and the concept of "hidden momentum", are dissected. It is argued that the different solutions allowed by the different spin conditions are equally valid descriptions for the motion of a given test body, and their equivalence is shown to dipole order in curved spacetime. These different descriptions are compared in simple examples.

1 Introduction

An old problem in the description of the dynamics of test particles endowed with multipole structure is the fact that, even for a free *pole-dipole* particle (i.e., with a momentum vector P^α, and a spin 2-form $S_{\alpha\beta}$ as its only two relevant moments) in flat spacetime, the equations of motion resulting from the conservation laws $T^{\alpha\beta}_{\ \ ;\beta} = 0$ do not yield a determinate system, since there exist 3 more unknowns than equations. The so-called "spin supplementary condition", $S^{\alpha\beta}u_\beta = 0$, for some unit timelike vector u^α, first arose as a means of closing the system, by killing off 3 components of $S^{\alpha\beta}$. Its physical significance remained however obscure, especially in the earlier treatments that dealt with point particles [1, 2, 3, 4] (see also in this respect [5]). Later treatments, most notably the works by Möller [6, 7], dealing with extended bodies, shed some light on

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the interpretation of the spin condition, as it being a choice of representative point in the body; more precisely, choosing it as the center of mass ("centroid") as measured in the rest frame of an observer of 4-velocity u^α — since in relativity, the center of mass of a spinning body is an observer-dependent point. Different choices have been proposed; the best known ones are the Frenkel-Mathisson-Pirani (FMP) condition [1, 8], which chooses the centroid as measured in a frame comoving with it; the Corinaldesi-Papapetrou (CP) condition [9], which chooses the centroid measured by the observers of zero 3-velocity ($u^i = 0$) in a given coordinate system; and the Tulczyjew-Dixon (TD) condition [10, 11], which chooses the centroid measured in the zero 3-momentum frame ($u^\alpha \propto P^\alpha$). A more recent condition, proposed in [12, 13], dubbed herein the "Ohashi-Kyrian-Semerák (OKS) condition" (which, as we shall see, seems to be favored in many applications), chooses the centroid measured with respect to some u^α parallel-transported along its worldline. The spin condition generally remained, however, a not well understood problem (this is true even today), not being clear, namely, its status as a choice (the discussion is sometimes put in terms of which are the "correct" and the "wrong" conditions for each type of particle, see see introduction of [14] for a review), the differences arising from the different choices, and what it means to consider different solutions corresponding to *the same physical motion*. Also, some aspects of each condition have been poorly understood, especially the FMP condition and its famous helical motions [15]. The rules for transition between spin conditions, and the quantities that are fixed (for different solutions corresponding to the same physical body), were established in [13], where the numerical solutions were compared in the Kerr spacetime, and it was shown that, within the limit of validity of the pole-dipole approximation, the different solutions are contained within a minimal worldtube, formed by all the possible positions of the center of mass, which lies inside the convex hull of the body's worldtube. These rules were further discussed in [16], and used to show that the helical motions are fully consistent solutions, always contained within the minimal worldtube (and to clarify the misunderstanding that led to the contrary claims in the literature).

The non-parallelism between the momentum and the velocity of a multipole particle subject to external fields, and its relation with the spin supplementary condition, is another old problem. A significant step towards its understanding was taken in [17], where a generalized concept of "hidden momentum" (first discovered in the context of classical electrodynamics [18, 19, 20, 21, 22]) was introduced in general relativity, and applied to the study of the TD and CP conditions (the latter designated therein by a different name, the "laboratory frame centroid"). These ideas were further worked out, with emphasis on the FMP condition, in recent works by the authors [16, 23].

In this paper, we discuss in detail the different spin conditions in general relativity, the centroids that they determine, their uniqueness/non-uniqueness, and the momentum-velocity relation arising from each of them. The different solutions given by the different spin conditions corresponding to the same physical motion are compared in simple examples, and their differences dissected. Building on the works in [13] and in

[16] (where the equivalence was shown for free particles in flat spacetime), we prove the equivalence of the solutions to dipole order in curved spacetime; in particular, we clarify the dependence of the spin-curvature force on the spin condition, as being precisely what ensures the equivalence, and the connection of that with the geodesic deviation equation.

1.1 Notation and conventions

1. Signature $-+++$; $\epsilon_{\alpha\beta\sigma\gamma} \equiv \sqrt{-g}[\alpha\beta\gamma\delta]$ is the Levi-Civita tensor, and we follow the orientation $[1230] = 1$ (i.e., in flat spacetime $\epsilon_{1230} = 1$); $\epsilon_{ijk} \equiv \epsilon_{ijk0}$. Riemann tensor: $R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + ...$

2. $(h^u)^{\alpha}{}_{\beta} \equiv \delta^{\alpha}_{\beta} + u^{\alpha}u_{\beta}$ denotes the projector orthogonal to a unit time-like vector u^{α}.

3. *The three basic vectors in the description of an extended body.* P^{α} is the momentum; $U^{\alpha} \equiv dz^{\alpha}/d\tau$ is the tangent vector to the reference worldline $z^{\alpha}(\tau)$; the vector field involved in the *spin condition* $S^{\alpha\beta}u_{\beta} = 0$ is generically denoted by u^{α}.

4. "Centroid", "center of mass", "CM": have all the *same meaning* herein. $x^{\alpha}_{\rm CM}(u) \equiv$ centroid as measured by an observer of 4-velocity u^{α}.

2 Center of mass in relativity and the significance of the spin supplementary condition

In the multipole scheme an extended body is represented by a set of moments of its current density 4-vector j^{α} (the "electromagnetic skeleton") and a set of moments of the energy momentum tensor $T^{\alpha\beta}$, called "inertial" or "gravitational" moments (forming the so-called [8] "gravitational skeleton"), defined with respect to a reference worldline $z^{\alpha}(\tau)$ which is taken to be some representative point of the body, and whose motion aims to represent the "bulk" motion of the body. The natural choice for such point is the body's center of mass (CM); however, in relativity, the CM of a spinning body is observer-dependent. This is illustrated in Fig. 1. In order to establish how the center of mass changes with the observer, we need reasonable definitions of momentum, angular momentum, mass and center of mass. In flat spacetime these are all well defined notions; but it is not so in curved spacetime, as they consist of integrals which amount to adding tensors defined at different (albeit close, if the body is assumed small) points; different generalizations of these notions have been proposed (see e.g. [11, 24, 25]). The discussion herein is aimed to be as general as possible; for that we use the following definitions that hold reasonable (at least to lowest orders) regardless of the particular multipole scheme followed.

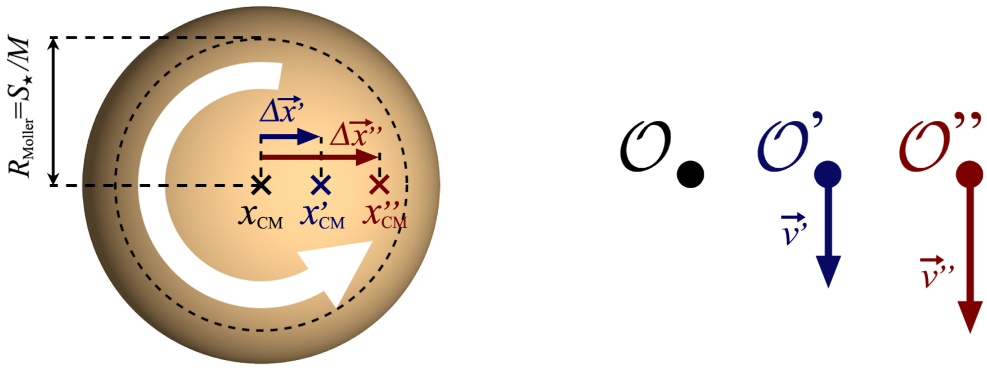

Figure 1: A free spinning spherical body in flat spacetime. Observer $\mathcal{O}$, at rest relative to the axis of rotation, measures the centroid $x^\alpha_{\rm CM}$ to coincide with the sphere's geometrical center (and with the rotation axis). $x^\alpha_{\rm CM} \equiv x_{\rm CM}(P)$ is the centroid as measured in the $P^i = 0$ frame. Observers $\mathcal{O}'$ and $\mathcal{O}''$ moving (relative to $\mathcal{O}$) with velocities $\vec{v}'$, $\vec{v}''$ *opposite* to the rotation of the body, see points on the right side of the body moving faster than those on the left side; hence for these observers the right side of the body is more massive, and the centroid they measure is shifted to the right by $\Delta\vec{x} = \vec{v} \times \vec{S}_\star/M$. The larger the speed v the larger the shift; when v equals the speed of light, the shift takes its maximum value, with the centroid lying in the circle of radius $R_{\rm Moller} = S_\star/M$.

Consider a system of Riemann normal coordinates $\{x^{\hat{\alpha}}\}$ (e.g. [26, 24]) centered at the point z^α of the reference worldline, associated to the orthonormal frame $\mathbf{e}_{\hat{\alpha}}$ at that point, and take it to be momentarily comoving with some observer $\mathcal{O}$ of 4-velocity u^α (*not* necessarily tangent to the curve $z^\alpha(\tau)$); that is, at z^α, $\mathbf{e}_{\hat{0}} = u^\alpha$ and the triad $\mathbf{e}_{\hat{i}}$ spans the instantaneous rest space of $\mathcal{O}$. We define the momentum P^α, angular momentum $S^{\alpha\beta}$, mass $m(u)$ and centroid $x^\alpha_{\rm CM}(u)$ of the particle with respect to $\mathcal{O}$ as the tensors at $z^\alpha(\tau)$ (respectively point) whose components (respectively coordinates) in this chart are

$$P^{\hat{\alpha}} \equiv \int_{\Sigma(z,u)} T^{\hat{\alpha}\hat{0}} d\Sigma \equiv \int_{\Sigma(z,u)} T^{\hat{\alpha}\hat{\beta}} d\Sigma_{\hat{\beta}} \ , \tag{1}$$

$$S^{\hat{\alpha}\hat{\beta}} \equiv 2\int_{\Sigma(z,u)} x^{[\hat{\alpha}} T^{\hat{\beta}]\hat{\gamma}} d\Sigma_{\hat{\gamma}} \ , \tag{2}$$

$$m(u) \equiv -P^\alpha u_\alpha = \int_{\Sigma(z,u)} T^{\hat{0}\hat{\gamma}} d\Sigma_{\hat{\gamma}} \ , \tag{3}$$

$$x^{\hat{\alpha}}_{\rm CM}(u) \equiv \frac{\int_{\Sigma(z,u)} x^{\hat{\alpha}} T^{\hat{0}\hat{\gamma}} d\Sigma_{\hat{\gamma}}}{m(u)} \ . \tag{4}$$

Here $\Sigma(z,u) \equiv \Sigma(z(\tau),u)$ is the spacelike hypersurface generated by all geodesics orthogonal to the timelike vector u^α at the point z^α (in normal coordinates it coincides with the spatial hypersurface $x^{\hat{0}} = 0$), $d\Sigma$ is the 3-volume element on $\Sigma(z,u)$, and $d\Sigma_\gamma \equiv -u_\gamma d\Sigma$. These definitions correspond to the ones given in [24], and have a well defined mathematical meaning, which can be written in the manifestly covariant form (65)-(66) below. They also correspond, to a good approximation, to the ones given in Dixon's schemes [11, 25]. This is discussed in detail in Appendix A. Note that although we used normal coordinates to perform the integrations above, the end results P^α and $S^{\alpha\beta}$ are tensors, which can now be expressed in any frame[1].

The vector

$$(d^u_{\rm G})^\alpha \equiv -S^{\alpha\beta}u_\beta \tag{5}$$

yields the "mass dipole moment" as measured by the observer $\mathcal{O}$ (of 4-velocity u^α), and

$$\Delta x^\alpha = -\frac{S^{\alpha\beta}u_\beta}{m(u)} \tag{6}$$

can be interpreted as the shift, or the "displacement", of the centroid $x^\alpha_{\rm CM}(u)$ relative to the reference worldline $z^\alpha(\tau)$. This is readily seen in the frame $\hat{e}_\alpha$, where $u^{\hat{i}} = 0$ and $S^{\hat{i}\hat{\beta}}u_{\hat{\beta}} = -S^{\hat{i}\hat{0}}$, and so from Eq. (2) we have

$$S^{\hat{i}\hat{0}} = 2\int_{\Sigma(z,u)} x^{[\hat{i}}T^{\hat{0}]\hat{\gamma}}d\Sigma_{\hat{\gamma}} = \int_{\Sigma(z,u)} x^{\hat{i}}T^{\hat{0}\hat{\gamma}}d\Sigma_{\hat{\gamma}} \equiv m(u)x^{\hat{i}}_{\rm CM}(u); \tag{7}$$

note that $x^{\hat{0}} = 0$, since the integration is performed in the geodesic hypersurface $\Sigma(z,u)$ orthogonal to u^α at $z^\alpha(\tau)$. Hence $\Delta x^{\hat{i}} = S^{\hat{i}\hat{0}}/m(u)$ yields the coordinates $x^{\hat{i}}_{\rm CM}(u)$ of the center of mass measured by $\mathcal{O}$, in the normal system $\{x^{\hat{\alpha}}\}$. Since the latter is constructed from geodesics radiating out of z^α, $\Delta\mathbf{x}$ is the vector at z^α tangent to the geodesic connecting z^α and $x^\alpha_{\rm CM}(u)$, and whose length equals that of the geodesic; that is, $x^\alpha_{\rm CM}(u)$ is the image by the geodesic exponential map of $\Delta\mathbf{x}$: $x^\alpha_{\rm CM}(u) = \exp_z(\Delta\mathbf{x})$. In flat spacetime (where vectors are arrows connecting two points), $\Delta\mathbf{x}$ reduces to the displacement vector from z^α to $x^\alpha_{\rm CM}(u)$; in curved spacetime it is still a reasonable notion of center of mass shift, and so (5) is a sensible definition of mass dipole moment. In particular, its vanishing for some observer means that one is choosing z^α as the center of mass $x^\alpha_{\rm CM}(u)$ as measured by that observer. That is, the condition

$$S^{\alpha\beta}u_\beta = 0, \tag{8}$$

implying, in the frame $\hat{e}_\alpha$, $S^{\hat{i}\hat{0}} = 0 \Rightarrow x^{\hat{i}}_{\rm CM}(u) = 0$, states that the reference worldline is the center of mass as measured by the observer $\mathcal{O}(u)$ (or, equivalently, that the mass

[1] One could say the same about the point $x^\alpha_{\rm CM}(u)$, although one must bear in mind when transforming its coordinates to the new frame that it will still be the CM as measured by the specific observer u^α, and *not* the CM as measured in the new frame.

dipole vanishes for $\mathcal{O}(u)$). Eq. (8), for some timelike vector field u^α defined (at least) along $z^\alpha(\tau)$, is known as the "spin supplementary condition", which one needs to impose in order to have a determined system of equations of motion, as we shall see in the next section. As we have just seen, one can generically interpret it *as a choice of center of mass.*

In order to see how the center of mass changes with the observer, let us for simplicity consider the case with no electromagnetic field, $F^{\alpha\beta} = 0$; in this case, as explained in detail in Appendix A.1, under the assumption that the size of the body is small compared with the scale of the curvature, the moments (1)-(2) do not depend on the argument u^α of $\Sigma(z,u)$; that is, they depend on the point along the reference worldline $z^\alpha(\tau)$, but not on the particular *geodesic* hypersurface Σ through it. We may thus regard $P^\alpha(\tau)$ and $S^{\alpha\beta}(\tau)$ as well defined functions on $z^\alpha(\tau)$. We shall also introduce the following relations which will be useful throughout this paper. Let u^α and u'^α be the 4-velocities of two different observers. We can write (e.g. [27])

$$u'^\alpha = \gamma(u,u')(u^\alpha + v^\alpha(u',u)); \quad \gamma(u,u') \equiv -u^\alpha u'_\alpha = \frac{1}{\sqrt{1 - v^\alpha v_\alpha}} , \tag{9}$$

where $v^\alpha(u',u)$ is a vector orthogonal to u^α, whose space components v^i yield the ordinary 3-velocity of the observer u'^α in the frame $u^i = 0$ (i.e., the velocity of the observer u'^α relative to the observer u^α). Choose z^α to be the CM as measured by u^α: $z^\alpha = x^\alpha_{\rm CM}(u)$; that is, choose $S^{\alpha\beta}u_\beta = 0$. In order to obtain the mass dipole measured by u'^α, one just has to contract $S^{\alpha\beta}$ with u'_β: $(d_G^{u'})^\alpha \equiv -S^{\alpha\beta}u'_\beta$; this is because, under the assumptions above, $S^{\alpha\beta}$ does not depend on the normal to the hypersurface $\Sigma(z)$, and thus, in the $u'^i = 0$ frame, we may write $-S^{\alpha\beta}u'_\beta$ in the form (7), only with u' in the place of u. The shift of the centroid $x^\alpha_{\rm CM}(u')$ measured by u'^α relative to $x^\alpha_{\rm CM}(u)$ is thus

$$\Delta x^\alpha = -\frac{S^{\alpha\beta}u'_\beta}{m(u')} = -\gamma(u,u')\frac{S^{\alpha\beta}v_\beta(u',u)}{m(u')} , \tag{10}$$

cf. Eq. (6). Especially interesting is the case $u^\alpha = P^\alpha/M$, where we denoted $M \equiv \sqrt{-P^\alpha P_\alpha}$; this amounts to choosing z^α as the CM as measured in the $P^i = 0$ frame, $z^\alpha = x^\alpha_{\rm CM}(P)$. In this case

$$\Delta x^\alpha = -\frac{S_\star^{\alpha\beta}v_\beta}{M} , \tag{11}$$

where $v^\alpha \equiv v^\alpha(u',P)$ is the velocity of the observer u'^α relative to the $P^i = 0$ frame, and we denoted by $S_\star^{\alpha\beta}$ the angular momentum taken with respect to $z^\alpha = x^\alpha_{\rm CM}(P)$ (note that the tensor $S^{\alpha\beta}$ depends on the choice of z^α, cf. Eq. (2); for the same body, $S^{\alpha\beta}$ is in general different for different z^α's). Let us denote also the corresponding spin vector $S_\star^\alpha$, so that $S_\star^{\alpha\beta} = \epsilon^{\alpha\beta}{}_{\gamma\delta}S_\star^\gamma P^\delta/M$. The space part (both in the $u'^i = 0$ and in the

$P^i = 0$ frames, as Δx^α is orthogonal to both u'^α and P^α) reads

$$\Delta x^i = \frac{(\vec{S}_\star \times \vec{v})^i}{M} \ . \tag{12}$$

Thus the set of all shift vectors corresponding to all possible observers spans a disk of radius $R_{\rm Moller} = S_\star/M$, centered at $x^\alpha_{\rm CM}(P)$ and orthogonal to $S^\alpha_\star$ and P^α, in the tangent space at $x^\alpha_{\rm CM}(P)$. This statement can roughly be rephrased as saying that the set of all possible positions of the center of mass as measured by the different observers is contained (and fills) such disk (in flat spacetime this is an exact statement, originally by Möller [7]). Let us dub such disk the "disk of centroids", and its radius $R_{\rm Moller}$ the *Möller radius*.

In order to illustrate how this works, consider for simplicity the setup in Fig. 1: a free spinning spherical body in flat spacetime. Observer $\mathcal{O}$, at rest relative to the axis of rotation, clearly must (by symmetry) measure the CM to coincide with the body's geometrical center (and with the rotation axis). The rest frame of such an observer corresponds in this case to the $P^i = 0$ frame (this statement will be made obvious in Sec. 3.2.3 by Eq. (23)). Consider now other observers, $\mathcal{O}'$ and $\mathcal{O}''$, moving (relative to $\mathcal{O}$) with velocities $\vec{v}'$ and $\vec{v}''$, opposite to the rotation of the body; for these observers the center of mass is shifted to the right, as they measure the right side of the body to be more massive. The larger the speed v the larger the shift; when v equals the speed of light, the shift takes its maximum value, with the centroid lying in the circle of radius $R_{\rm Moller}$.

In spacetime, the set of all possible centroid worldlines forms a worldtube — the "minimal worldtube" [13], see Fig. 2 — typically very narrow[2], and always contained within the *convex hull* of the body's worldtube (see [24] for its precise definition). This can be shown in different ways. In flat spacetime, it is not difficult to show (see e.g. [28] p. 313), that if the mass density-energy density $\rho(u) = T^{\alpha\beta}u_\beta u_\alpha$ is positive everywhere within the body and with respect to all observers u^α (i.e., if the weak energy condition holds everywhere within the body), then the center of mass with respect to any u^α must be within the body's convex hull. The flat spacetime arguments apply just as well in a local Lorentz frame $\{x^{\hat{\alpha}}\}$ (under the assumption above that the body is small enough so that we can take it to be nearly orthonormal throughout it). In the same framework one can show that $R_{\rm max}$ is the minimum size that a *classical* particle can have in order to have finite spin without containing mass-energy flowing faster than light, that is, without violating the dominant energy condition. The dominant energy condition implies $\rho \geq |\vec{J}|$, where $J^{\hat{i}} \equiv -T^{\hat{\alpha}\hat{i}}u_{\hat{\alpha}}$. Let a be the largest dimension of the body; in the local Lorentz frame centered at $x^\alpha_{\rm CM}(P)$ and such that $P^{\hat{i}} = 0$, we may

[2]For the fastest spinning celestial body known to date, the pulsar PSR J1748-2446ad (rotation frequency 716 Hz, estimated radius $a = 16$ km), whose equatorial velocity is 0.23c, $R_{\rm Moller} \simeq 0.1a$.

write,

$$S_\star = \left|\int \vec{r}\times\vec{J}d^3x\right| \leq \int r|\vec{J}|d^3x \leq \int \rho r d^3x \leq Ma \;\Leftrightarrow\; a \geq \frac{S_\star}{M}\ . \tag{13}$$

3 The momentum-velocity relation

The force and the spin evolution equations for a multipole particle in an external electromagnetic and gravitational field are [11]

$$\frac{DP^\alpha}{d\tau} = qF^{\alpha}_{\ \beta}U^\beta + \frac{1}{2}F^{\mu\nu;\alpha}\mu_{\mu\nu} + F^{\alpha}_{\ \gamma;\beta}U^\gamma d^\beta + F^{\alpha}_{\ \beta}\frac{Dd^\beta}{d\tau}\ ,$$
$$-\frac{1}{2}R^{\alpha}_{\ \beta\mu\nu}S^{\mu\nu}U^\beta + F^\alpha(2^{\mathrm{N}>1}) \tag{14}$$

$$\frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha}U^{\beta]} + \tau^{\alpha\beta} \tag{15}$$

where q, d^α and $\mu_{\alpha\beta}$ are, respectively, the particle's charge, electric dipole vector, and magnetic dipole 2-form (for their precise definitions, see [23]). $F^\alpha(2^{\mathrm{N}>1})$ denotes the force (gravitational and electromagnetic) due to the quadrupole and higher moments, and $\tau^{\alpha\beta}$ is sometimes called the "torque" tensor. $U^\alpha \equiv dz^\alpha/d\tau$ is the tangent to the reference worldline $z^\alpha(\tau)$. These equations form an undetermined system even in the case $DP^\alpha/d\tau = 0$ and $\tau^{\alpha\beta} = 0$ (for there would be 13 unknowns: P^α, 3 independent components of U^α, and 6 independent components of $S^{\alpha\beta}$, for only 10 equations), manifesting the need for a supplementary condition, which amounts to specify the worldline $z^\alpha(\tau)$, relative to which the moments are taken. The condition $S^{\alpha\beta}u_\beta = 0$, for some unit timelike vector field u^α defined along z^α, kills off 3 components of the angular momentum and makes that choice, requiring, as explained in the previous section, $z^\alpha(\tau)$ to be the centroid as measured by an observer of 4-velocity u^α. Contracting (15) with u_β one obtains an expression for the momentum of the particle,

$$P^\alpha = \frac{1}{\gamma(u,U)}\left(m(u)U^\alpha + S^{\alpha\beta}\frac{Du_\beta}{d\tau} + \tau^{\alpha\beta}u_\beta\right)\ , \tag{16}$$

where $\gamma(U,u) \equiv -U^\alpha u_\alpha$, $m(u) \equiv -P^\alpha u_\alpha$, and in the second term we used $S^{\alpha\beta}u_\beta = 0$. Eq. (16) tells us that, in general, P^α is not parallel to the CM 4-velocity U^α; in this section we shall will discuss the reason for that.

The vector P^α can be split in its projections parallel and orthogonal to the CM 4-velocity U^α:

$$P^\alpha = P^\alpha_{\mathrm{kin}} + P^\alpha_{\mathrm{hid}};\quad P^\alpha_{\mathrm{kin}} \equiv mU^\alpha,\ P^\alpha_{\mathrm{hid}} \equiv (h^U)^\alpha_{\ \beta}P^\beta\ , \tag{17}$$

where $m \equiv -P^\alpha U_\alpha$ is the the "proper mass", i.e., the energy of the particle as measured in the CM frame, and

$$(h^U)^\alpha_{\ \beta} \equiv U^\alpha U_\beta + \delta^\alpha_{\ \beta}$$

is the projector orthogonal to U^α. We dub the parallel projection $P^\alpha_{\rm kin} = mU^\alpha$ "kinetic momentum" associated with the motion of the center of mass; it is the most familiar part of P^α, formally similar to the momentum of a monopole particle. The component $P^\alpha_{\rm hid}$ orthogonal to U^α is the so-called "hidden momentum" (e.g. [17]). The reason for the latter denomination is seen taking the perspective of an observer $\mathcal{O}(U)$ comoving with the particle: in the frame of $\mathcal{O}(U)$ (i.e., the $U^i = 0$ frame) the 3-momentum is in general not zero: $\vec{P} = \vec{P}_{\rm hid} \neq 0$; however, by definition, the particle's CM is at rest in that frame; hence this momentum must be somehow hidden in the particle. $P^\alpha_{\rm hid}$ consists of two parts of distinct origin: $P^\alpha_{\rm hid} = P^\alpha_{\rm hidI} + P^\alpha_{{\rm hid}\tau}$,

$$P^\alpha_{\rm hidI} \equiv \frac{1}{\gamma(u,U)}(h^U)^\alpha_{\ \sigma} S^{\sigma\beta}\frac{Du_\beta}{d\tau} \ ; \tag{18}$$

$$P^\alpha_{{\rm hid}\tau} \equiv \frac{1}{\gamma(u,U)}(h^U)^\alpha_{\ \sigma}\tau^{\sigma\beta}u_\beta, \tag{19}$$

which we shall explain. $P^\alpha_{\rm hidI}$ is a term that depends only on the spin supplementary condition, i.e., on the choice of the field u^α relative to which the centroid is computed. In this sense we say it is *gauge*. This type of hidden momentum was first discussed in [17] (dubbed "kinematical" therein). The vector u^α needs only to be defined *along* $z^\alpha(\tau)$; but if one takes it as belonging to some observer congruence in spacetime (one can always do such an extension), and decomposing

$$u_{\alpha;\beta} = -(a^u)_\alpha u_\beta - \epsilon_{\alpha\beta\gamma\delta}\omega^\gamma u^\delta + \theta_{\alpha\beta} \tag{20}$$

where $(a^u)^\alpha \equiv u^\alpha_{\ ;\beta}u^\beta$ is the acceleration *of the observers* u^α, $\omega^\alpha = \frac{1}{2}\epsilon^{\alpha\lambda\sigma\tau}u_\tau u_{[\sigma;\lambda]}$ their vorticity, and $\theta_{\alpha\beta} \equiv (h^u)^\lambda_\alpha (h^u)^\nu_\beta u_{(\lambda;\nu)}$ the shear/expansion, we may write

$$P^\alpha_{\rm hidI} = \frac{1}{\gamma(u,U)}(h^U)^\alpha_{\ \sigma} S^\sigma_{\ \beta}\left(\gamma(u,U)(a^u)^\beta - \epsilon^\beta_{\ \mu\gamma\delta}u^\delta U^\mu \omega^\gamma + \theta^{\beta\gamma}U_\gamma\right) \ . \tag{21}$$

The kinematical quantities in Eq. (20) are connected to "inertial forces", namely $G^\alpha = -(a^u)^\alpha$ and $H^\alpha = \omega^\alpha$ are, respectively, the "gravitoelectric field" and the "Fermi-Walker gravitomagnetic field" as measured by the congruence of observers u^α, see [29, 27]. For this reason we dub $P^\alpha_{\rm hidI}$ "inertial" hidden momentum.

$P_{{\rm hid}\tau}$ is associated to the "torque" tensor $\tau^{\alpha\beta}$ and (in general) consists of two parts: one which is again gauge and arises for certain choices of reference worldline z^α (i.e., of the field u^α) *when* a physical torque acts on the particle, plus another part which is not gauge, and cannot be made to vanish by any center of mass choice. Following [17] we dub the latter "dynamical hidden momentum". To dipole order, this dynamical part consists of a form of mechanical momentum that arises in electromagnetic systems, first discovered in [18], and since discussed in number of papers, e.g. [19, 20, 21], including recent works [22, 17, 30]. To quadrupole and higher orders, there are both electromagnetic and gravitational contributions to $\tau^{\alpha\beta}$, and thus to $P_{{\rm hid}\tau}$.

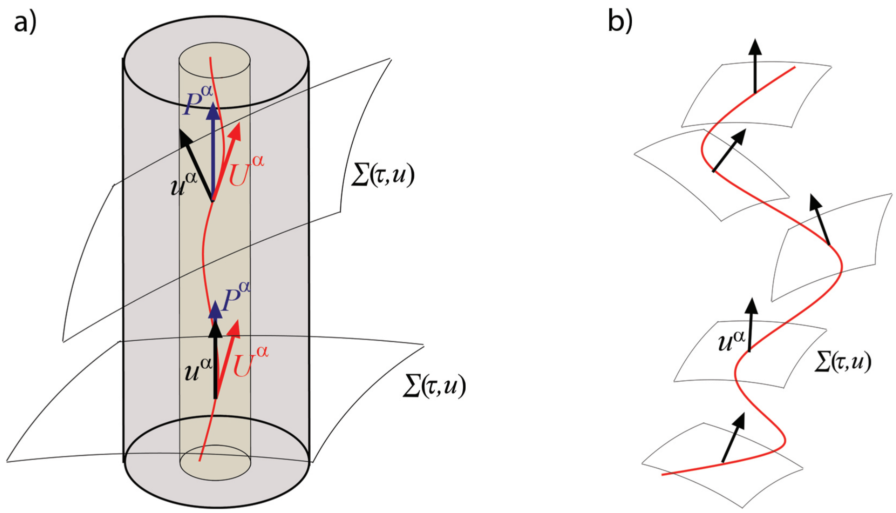

Figure 2: a) the body's worldtube (larger cylinder), the worldtube of centroids (narrow inner cylinder), and the *three* basic vectors involved in the description of the motion: the momentum P^α, the 4-velocity $U^\alpha = dz^\alpha/d\tau$, and the vector field u^α involved in the spin supplementary condition $S^{\alpha\beta}u_\beta = 0$. The vector u^α is orthogonal to the hypersurfaces $\Sigma(\tau, u)$ at z^α, and has the interpretation of the 4-velocity of the observer measuring the centroid. These three vectors are *not* parallel in general. b) A curve with a varying u^α along it; that leads to a varying shift, see Fig. 1, leading to a non-zero velocity of the centroid in the $P^i = 0$ frame, cf. Eq. (23), and possibly to an acceleration without any force involved.

3.1 "Inertial hidden momentum": center of mass shift and the decoupling of U^α from P^α

Eq. (18) tells us that when u^α varies along $z^\alpha(\tau)$ (i.e., $Du^\alpha/d\tau \neq 0$), in general $P^\alpha_{\rm hidI} \neq 0$, thus U^α is not parallel to P^α. This comes as a natural consequence of what we discussed in Sec. 2 about the observer dependence of the center of mass. Recall the situation in Fig. 1, a *free* spinning particle in *flat spacetime*: the centroid measured by observers moving relative to $\mathcal{O}$ are shifted relative to $x_{\rm CM}(\mathcal{O})$. If the velocity of these observers changes along $z^\alpha(\tau)$, e.g., if at an instant τ' we have $u^\alpha(\tau') = u'^\alpha$, and at τ'' $u^\alpha(\tau'') = u''^\alpha$, the shift changes accordingly, giving rise to a non-trivial velocity of the centroid. That is, superfluous centroid motions can be generated just by changing u^α along $z^\alpha(\tau)$. The momentum, however, remains the same, $DP^\alpha/d\tau = 0$ cf. Eq. (14); thus the situation may be cast as the centroid acquiring a non-zero velocity in the $P^i = 0$ frame (which is in this case the rest frame of the observer $\mathcal{O}$). This amounts to saying that U^α gains a component orthogonal to P^α (denote it by $U^\alpha_\perp$); conversely, there is a component of P^α orthogonal to U^α, which is the hidden momentum. Let us see this in detail. Denote by $U_{\|}$ and $U_{\perp}^{\alpha}$, respectively, the components of U^{α} parallel and orthogonal to P^{α},

$$U^{\alpha}=U_{\|}^{\alpha}+U_{\perp}^{\alpha}\ ;\qquad U_{\|}^{\alpha}\equiv\frac{m}{M^{2}}P^{\alpha}\ ;\qquad U_{\perp}^{\alpha}\equiv(h^{P})^{\alpha}{}_{\beta}U^{\beta}\ ,\tag{22}$$

where $m\equiv-U^{\alpha}P_{\alpha}$, and $(h^{P})^{\alpha}{}_{\beta}\equiv P^{\alpha}P_{\beta}/M^{2}+\delta^{\alpha}{}_{\beta}$ denotes the projector in the direction orthogonal to P^{α}. $U_{\perp}^{\alpha}$ is, up to a γ factor, *the 3-velocity of the centroid* in the $P^{i}=0$ frame, cf. Eq. (9) above (substitute therein $u'^{\alpha}=U^{\alpha}$, $u^{\alpha}=P^{\alpha}/M$). In the special case $\tau^{\alpha\beta}=0$, we have from Eq. (16)

$$U_{\perp}^{\alpha}=-\frac{1}{m(u)}(h^{P})^{\alpha}{}_{\sigma}S^{\sigma\beta}\frac{Du_{\beta}}{d\tau}\ ,\tag{23}$$

showing that indeed the variation of u^{α} along $z^{\alpha}(\tau)$ leads to a centroid moving in the zero 3-momentum frame, and to a non-parallelism between U^{α} and P^{α} (it is actually the sole reason for that in the special case $\tau^{\alpha\beta}=0$). *If we further specialize* to the case of a free particle (depicted in Figs. 1-2), $DP^{\alpha}/d\tau=0$, and noting that the centroid shift can be written as $\Delta x^{\alpha}=-(x_{\rm CM}^{\alpha}(P)-x_{\rm CM}^{\alpha}(u))=S^{\alpha\beta}P_{\beta}/M^{2}$, cf. Eq. (10), the shift variation along z^{α} becomes

$$\frac{D\Delta x^{\alpha}}{d\tau}=\frac{P_{\beta}}{M^{2}}\frac{DS^{\alpha\beta}}{d\tau}=U^{\alpha}-\frac{m}{M^{2}}P^{\alpha}=U_{\perp}^{\alpha}\ .\tag{24}$$

In the second equality we used Eq. (15), in the third we used Eqs. (22). That is, the variation of the shift equals the component of U^{α} orthogonal to P^{α}, mathematically formalizing the heuristic arguments in Figs. 1 and 2b). One should note however that, although this reasoning is useful to gain intuition, in the general case ($DP^{\alpha}/d\tau\neq0$) Eq. (24) does not hold, and $U_{\perp}^{\alpha}$ is not just the variation of Δx^{α}; this is because the centroid $x_{\rm CM}^{\alpha}(P)$ is in general no longer at rest in the $P^{i}=0$ frame. (When one employs the TD condition, $u^{\alpha}=P^{\alpha}$, if $DP^{\alpha}/d\tau\neq0$ then the centroid 4-velocity is not in general parallel to P^{α}, cf. Eq. (23) or, explicitly, Eq. (28)). For the general case the argument can be given as follows: *the centroid position depends on the field u^{α} relative to which it is measured, and its velocity on the variation of u^{α}; P^{α}, however, is unaffected by that, which means that in general $P^{\alpha}\nparallel U^{\alpha}$.* This is precisely what Eq. (23) says.

When $U_{\perp}^{\alpha}\neq0$, then (obviously) P^{α} has a component $P_{\rm hid}^{\alpha}$ orthogonal to U^{α}. Noting from (22) that $P^{\alpha}=M^{2}(U^{\alpha}-U_{\perp}^{\alpha})/m$, and from Eqs. (17) that $U^{\alpha}=(P^{\alpha}-P_{\rm hid}^{\alpha})/m$, we obtain the following relations between the hidden momentum and $U_{\perp}^{\alpha}$:

$$U_{\perp}^{\alpha}=-\frac{1}{m}(h^{P})^{\alpha}{}_{\beta}P_{\rm hid}^{\beta};\qquad P_{\rm hid}^{\alpha}=-\frac{M^{2}}{m}(h^{U})^{\alpha}{}_{\beta}U_{\perp}^{\beta}$$

(these are fully general expressions, valid when $\tau^{\alpha\beta}\neq0$).

Differentiating (23) with respect to τ, we see that when $D^{2}u^{\alpha}/d\tau^{2}\neq0$, in general the centroid acceleration $a^{\alpha}=DU^{\alpha}/d\tau$ will be non-zero, i.e., it will accelerate without the action of a force. That can lead to exotic motions; an example of that are the famous Mathisson helical motions, as shown in [16]; the same principle also leads to the bobbings in the "tetherballs" studied in [17] (in this case a force is involved, but it is not parallel to the acceleration), or the ones studied in Sec. 3.4. Of course, such effects can always be made to vanish by a choosing some u^α parallel transported along $z^\alpha(\tau)$; hence one can say that they are a complicated description for the same physics that, in principle, could be described in a simpler manner. In Fig. 2 we illustrate the situation for a free particle in flat spacetime: the worldline $z^\alpha(\tau)$ of a centroid measured by a field of observers u^α that varies along it has, in general, superfluous motions. These are confined to the worldtube of centroids, which is straight tube (always within the convex hull of the body's worldtube, see Sec. 2) parallel to the constant momentum P^α, and whose cross section orthogonal to P^α and $S_\star^\alpha$ is the the disk of centroids illustrated in Fig. 1. Choosing $Du^\alpha/d\tau = 0$ (e.g., inertial frames), the centroid worldlines obtained are straight lines parallel to P^α, yielding the simplest description possible for this problem.

3.2 Center of mass and momentum-velocity relation of the different spin conditions

In this section we shall consider, for simplicity, the case $\tau^{\alpha\beta} = 0$, so that the only hidden momentum present is the inertial hidden momentum P^α_{hidI}. Although all forms of hidden momentum have some sort of dependence on the spin condition, by the circumstance that $U^\alpha \equiv dz^\alpha/d\tau$ depends on the reference worldline $z^\alpha(\tau)$ chosen, P^α_{hidI} is the part that arises *solely* from it. Note that $\tau^{\alpha\beta} = 0$ corresponds for instance to the case of pole-dipole particles in purely gravitational systems.

3.2.1 The Corinaldesi-Papapetrou (CP) condition

This spin condition was introduced in [9] for the Schwarzschild spacetime, where it was cast, in Schwarzschild coordinates, as $S^{i0} = 0$. One can write it covariantly as $S^\alpha{}_\beta u^\beta_{\text{lab}} = 0$, with u^α_{lab} corresponding to observers that have zero 3-velocity in such coordinates, $u^i_{\text{lab}} = 0$. These are the so-called *"static observers",* whose 4-velocity is parallel to the time Killing vector: $u^\alpha_{\text{lab}} = u^\alpha_{\text{static}} \propto \partial/\partial t$. Hence, this condition chooses as reference worldline the centroid measured by the static observers. It can be generalized taking the static observers of other stationary spacetimes, or, as done in [13], to arbitrary metrics taking the congruence of observers with zero 3-velocity in the coordinate system chosen (let us dub it the "laboratory" frame). This effectively amounts to considering an arbitrary congruence of observers, which will be the problem discussed below: take a matter distribution described by the energy-momentum tensor $T^{\alpha\beta}(x)$, and a congruence of observers u^α_{lab} one may arbitrarily fix; then find the worldlines z^α obeying the condition $S^\alpha{}_\beta(z)u^\beta_{\text{lab}}(z) = 0$ — which demands z^α to be the center of mass as measured by the observer $u^\alpha_{\text{lab}}(z)$ *located at that precise point.* At first sight, it does

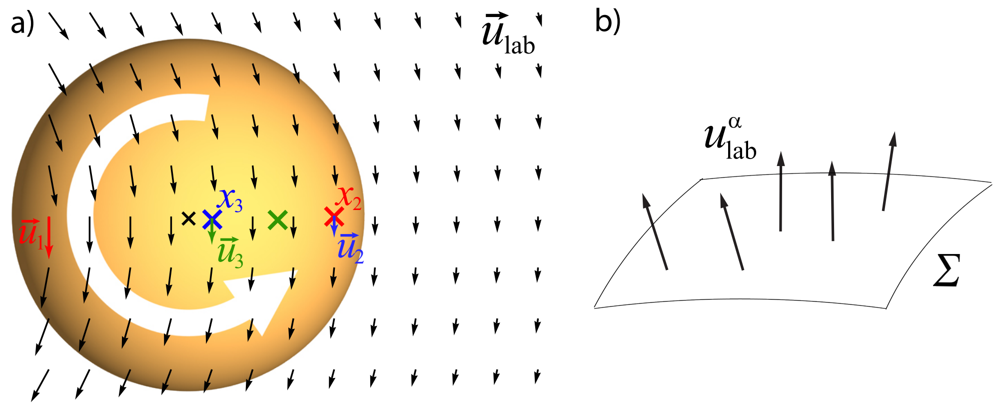

Figure 3: a) Centroid as measured by different observers of the congruence $u^{\alpha}_{\rm lab}$. Colors specify an observer and the corresponding centroid. Observer $u^{\alpha}_{\rm lab}(x_1) \equiv u^{\alpha}_1$ measures the centroid to be at $x^{\alpha}_2 \equiv x^{\alpha}_{\rm CM}(u_1)$. The observer $u^{\alpha}_{\rm lab}(x_2) \equiv u^{\alpha}_2$ at x^{α}_2 is a different one, thus its centroid will in general be at a different point x^{α}_3. Observer $u^{\alpha}_{\rm lab}(x_3) \equiv u^{\alpha}_3$ at x^{α}_3 measures the centroid to be at yet another different point, and so on. b) For the observers $u^{\alpha}_{\rm lab}$ to agree on the centroid position, they must be orthogonal to the *same* geodesic hypersurface Σ. In this case the condition $S^{\alpha}{}_{\beta}u^{\beta}_{\rm lab} = 0$ fixes an *unique* worldline.

not even seem obvious that such solutions exist. For when one considers an observer $u^{\alpha}_{\rm lab}(x_1)$ at a given point x^{α}_1, the centroid with respect to $u^{\alpha}_{\rm lab}(x_1)$ will be at some point x^{α}_2, in general not coinciding with x^{α}_1; and then at the site x^{α}_2, the observer $u^{\alpha}_{\rm lab}(x_2)$ that lies there is a different one, and measures its centroid to be in yet another different point x^{α}_3, and so on. This is illustrated in Fig. 3a).

We shall now show that the solution indeed always exists, but in general it is not unique. Consider the vector field (the mass dipole with respect to the observer $u^{\beta}_{\rm lab}(z)$)

$$d^{\alpha}_{\rm G}(z) = -S^{\alpha}{}_{\beta}(z, u_{\rm lab})u^{\beta}_{\rm lab}(z)\ ,$$

which is a function of z^{α}, where $S^{\alpha}{}_{\beta}(z, u_{\rm lab})$ is the angular momentum taken about z^{α} and in the geodesic hypersurface orthogonal to $u^{\alpha}_{\rm lab}$ at z^{α}. Consider moreover the intersection of the convex hull of the body's worldtube W with some arbitrary spacelike hypersurface Σ, see Fig. 4; and let $\vec{d}_{\rm G}(z)$ be the projection of $d^{\alpha}_{\rm G}(z)$ on Σ. At the boundary of the region $W \cap \Sigma$ it is clear from the definition of $S^{\alpha\beta}(z, u)$ in Eq. (2) that $\vec{d}_{\rm G}(z)$ points inwards (since $T_{\alpha\beta}u^{\alpha}_{\rm lab}u^{\beta}_{\rm lab} > 0$). Given that $d^{\alpha}_{\rm G}(z)$ is a *continuous* vector field (since $u^{\beta}_{\rm lab}$ is an observer *congruence*), the Brouwer fixed point theorem implies that the flow of $\vec{d}_{\rm G}$ must have a fixed point; i.e., $\vec{d}_{\rm G} = 0$ at at least one point within $W \cap \Sigma$. Since $d^{\alpha}_{\rm G}(z)$ is a space-like vector, this effectively means that $d^{\alpha}_{\rm G}(z) = 0$ at that point.

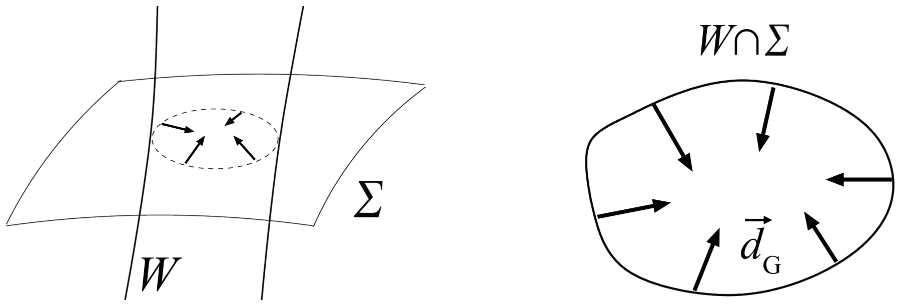

Figure 4: The vector field $d^\alpha_G(z)$ (i.e., the mass dipole as measured by the observers $u^\alpha_{\rm rest}$ at z^α), at the boundary of the region formed by the intersection of the convex hull W of the body's worldtube with some space-like hypersurface Σ, always points inwards. Since the field $d^\alpha_G(z)$ is spacelike and continuous, the Brouwer fixed point theorem ensures that $d^\alpha_G(z) = 0$ at *at least one* point $z^\alpha \in W \cap \Sigma$. In other words, there is at least one point which is the center of mass as measured by the observer $u^\alpha_{\rm lab}$ *at that point.*

The argument above is analogous to the one followed by Madore [24] to produce a similar proof for the vector field $S^\alpha{}_\beta(z,P)P^\beta(z)$.

Hence, at least one worldline z^α will exist such that $S^\alpha{}_\beta(z)u^\beta_{\rm lab}(z) = 0$; but in general it is not unique. The analysis in Sec. 3.2.3 below provides an example. Take, in flat spacetime, the congruence $u^\alpha_{\rm lab}$ to be observers rotating rigidly with the angular velocity of Mathisson's helical motions, $\Omega = M/S_\star$ (i.e., take the laboratory frame to be the observers at rest in the coordinates associated to a frame rotating with angular velocity Ω), *opposite* to the sense of rotation of the body, and around the centroid measured in the $P^i = 0$ frame, $x^\alpha_{\rm CM}(P)$. In this case, every point z^α within the worldtube of centroids is a center of mass with respect to the observers at rest in this frame, i.e., every such point is a solution of $S^\alpha{}_\beta(z)u^\beta_{\rm lab}(z) = 0$.

In some cases the solution is unique; it is clearly so when the observers of the congruence agree on the centroid position (note however that this is a sufficient, but *not* necessary condition for uniqueness). The moments (thus the centroid, Eq. (4)) are defined integrating on geodesic hypersurfaces $\Sigma(z,u)$ orthogonal to $u^\alpha(z)$ at some z^α; in order for different observers to agree on the centroid position, they should be orthogonal to the same geodesic hypersurface, i.e., $u^\alpha_{\rm lab}(x)$ must be parallel[3] along Σ, cf. Fig. 3b). That amounts to saying that $\nabla_{\mathbf{X}} u^\alpha_{\rm lab} = 0$ for any spatial vector X^α tangent to $\Sigma(z, u_{\rm lab})$. From the general decomposition (20), we have

$$\nabla_{\mathbf{X}} u^\alpha_{\rm lab} = -\epsilon_{\alpha\beta\gamma\delta}X^\beta\omega^\gamma u^\delta + \theta_{\alpha\beta}X^\beta \ , \tag{25}$$

telling us that in order for $u^\alpha_{\rm lab}(x)$ to be parallel along Σ, the congruence cannot have vorticity nor shear/expansion, $\omega^\alpha = \theta_{\alpha\beta} = 0$ (note that if $\omega^\alpha \neq 0$, the congruence is

[3]Such observers are said to be "kinematically comoving" (see [31] Sec. 6.1).

not even hypersurface orthogonal; and if $\omega^\alpha = 0$ but $\theta_{\alpha\beta} \neq 0$, there are hypersurfaces orthogonal to it, but they are not geodesic, so do not coincide with any Σ). That is, it must be a congruence of static observers in a *static* spacetime. In this case, within the regime where the normal coordinates can be taken as nearly rectangular throughout the body[4] (that amounts to taking $\lambda \ll 1$ in Eq. (68), which is reasonable in this context, see Appendix A and Footnote 13), the observers $u^\alpha_{\rm lab}$ will agree on the centroid position. If one starts with an observer $u^\alpha_1 \equiv u^\alpha_{\rm lab}(x_1)$ at a point x^α_1, and computes the centroid it measures from Eq. (4), the worldline $z^\alpha = x^\alpha_{\rm CM}(u_1)$ obtained will therefore obey the CP condition $S^\alpha{}_\beta(z)u^\beta_{\rm lab}(z) = 0$, since repeating the computation in the normal coordinates of the observer $u^\alpha_{\rm lab}(z)$ at z^α yields the same result. An example is when $u^\alpha_{\rm lab}$ are the observers associated to a global inertial frame in flat spacetime. In this case $u^\alpha_{\rm rest}$ not only is the same vector everywhere, as one can set the Lorentz frames of each observer $u^\alpha_{\rm rest}(x)$ in an hyperplane Σ to be the same up to spatial translations; so all observers of this frame will measure the centroid at the same point (i.e. there is a well defined, unique centroid associated to such frame). This is an *exact* statement in this case.

Momentum-velocity relation.— Since $u^\alpha_{\rm lab}$ is a well defined vector field in the region of interest, we may write $Du^\alpha_{\rm lab}/d\tau = u^{\alpha;\beta}_{\rm lab}U_\beta$, and therefore, from Eq. (21), the momentum reads

$$P^\alpha = mU^\alpha + \frac{1}{\gamma}(h^U)^\alpha{}_\sigma S^\sigma{}_\beta\left(\gamma G^\beta - \epsilon^\beta{}_{\mu\gamma\delta}u^\delta_{\rm lab}U^\mu\omega^\gamma - \theta^{\beta\gamma}U_\gamma\right) \; ; \tag{26}$$

here $\gamma \equiv -u^\alpha_{\rm lab}U_\alpha$, $G^\alpha = -\nabla_{\mathbf{u}_{\rm lab}}u^\alpha_{\rm lab}$ is minus the acceleration of the laboratory observers (i.e., the gravitoelectric field), ω^γ is their vorticity (or the Fermi-Walker gravitomagnetic field [29, 27]), and $\theta_{\alpha\beta}$ their shear/expansion tensor. Hence we have a well defined expression for P^α in terms of U^α, $S^{\alpha\beta}$ and the kinematics of the congruence $u^\alpha_{\rm lab}$, telling us that P^α differs from mU^α only if the laboratory observers measure inertial forces (i.e., if they are accelerated, rotating, or shearing/expanding).

3.2.2 The Tulczyjew-Dixon (TD) condition

The condition $S^{\alpha\beta}P_\beta = 0$ amounts to choosing $u^\alpha = P^\alpha/M$, i.e., the centroid is the one as measured in the zero 3-momentum frame. As shown in [33, 34], for a given matter distribution, described by the energy-momentum tensor $T^{\alpha\beta}(x)$, there is only

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[4] To see the reason for this assumption, consider two observers $u^\alpha_1 = u^\alpha_{\rm lab}(x_1)$ and $u^\alpha_2 = u^\alpha_{\rm lab}(x_2)$, orthogonal to the same geodesic hypersurface Σ. Let $\{x^{\hat\alpha}\}$ and $\{x^{\tilde\alpha}\}$, respectively, denote their normal coordinate systems, related by $x^{\tilde\alpha} = \Lambda^{\tilde\alpha}{}_{\hat\beta}(x^{\hat\beta} - x^{\hat\beta}_2)$ (where $\Lambda^{\tilde\alpha}{}_{\hat\beta}$ is a function of $x^{\hat\beta}$). They will agree on the centroid position if $x^{\tilde\alpha}_{\rm CM}(u_2) = \Lambda^{\tilde\alpha}{}_{\hat\beta}(x^{\hat\beta}_{\rm CM}(u_1) - x^{\hat\beta}_2)$. From Eq. (4) we see that it is the case when $\mathbf{d}x^{\hat 0} = \mathbf{d}x^{\tilde 0}$, $x^{\tilde i} = \Lambda^{\tilde i}{}_{\hat j}(x^{\hat j} - x^{\hat j}_2)$, with $\Lambda^{\tilde i}{}_j$ a *constant* matrix. Due to the curvature, however, this cannot be exactly so; choosing $\partial_{\tilde\alpha}|_{x_2} \simeq \partial_{\hat\alpha}|_{x_2}$, we have $\Lambda^{\tilde\alpha}{}_{\hat\beta} = \delta^{\tilde\alpha}{}_{\hat\beta} + \mathcal{O}(\|\mathbf{R}\|\hat x\hat x_2) + \mathcal{O}(\|\mathbf{R}\|\hat x^2_2)$, e.g. Eq. (11.12) of [32]. It follows that, for all observers *within* the body's convex hull, $\|x^{\tilde\alpha}_{\rm CM}(u_2) - x^{\tilde\alpha}_{\rm CM}(u_1)\|/a \lesssim \lambda$, $\lambda = \|\mathbf{R}\|a^2$; hence $x^{\tilde\alpha}_{\rm CM}(u_2) \simeq x^{\tilde\alpha}_{\rm CM}(u_1)$ if $\lambda \ll 1$. This is, as expected, the condition that the metric $g_{\hat\alpha\hat\beta} = \eta_{\hat\alpha\hat\beta} + \mathcal{O}(\|\mathbf{R}\|\hat x^2)$ can be taken as nearly flat throughout the body.

one worldline $z^\alpha(\tau)$ such that $S^{\alpha\beta}P_\beta = 0$ ($S^{\alpha\beta}$ and P^α being both evaluated at z^α, and using the hypersurface $\Sigma(z,P)$ orthogonal to P^α at z^α). In other words, this spin condition *specifies an unique worldline*. It is the central worldline of the worldtube of centroids, as seen from Eq. (12). From (16)-(17), we have the expressions for the momentum

$$P^\alpha = \frac{1}{m}\left(M^2 U^\alpha + S^{\alpha\beta}\frac{DP_\beta}{d\tau}\right) = mU^\alpha + \frac{1}{m}(h^U)^\alpha_{\ \sigma} S^{\sigma\beta}\frac{DP_\beta}{d\tau} \ . \tag{27}$$

Here $DP^\alpha/d\tau$ is the force, given by Eq. (14); in the absence of electromagnetic field, and to pole-dipole order, this expression can be manipulated into the well known expression (e.g. [14])

$$U^\alpha = \frac{m}{M^2}\left(P^\alpha + \frac{2S^{\alpha\nu}R_{\nu\tau\kappa\lambda}S^{\kappa\lambda}P^\tau}{4M^2 + R_{\alpha\beta\gamma\delta}S^{\alpha\beta}S^{\gamma\delta}}\right) \ , \tag{28}$$

determining U^α *uniquely* in terms[5] of P^α, $S^{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$. A more general expression for the case when $F_{\alpha\beta} \neq 0$, and to arbitrary multipole order, is given in Eq. (35) of [17].

3.2.3 The Frenkel-Mathisson-Pirani (FMP) condition. Helical motions.

The condition $S^{\alpha\beta}U_\beta = 0$, i.e., $u^\alpha = U^\alpha$, states that the centroid is measured in its own rest frame; in other words, it chooses the center of mass as measured by an observer comoving with it. This condition does not yield an unique worldline though: it is infinitely degenerate. For a given matter distribution, described by $T^{\alpha\beta}(x)$, there are infinitely many worldlines z^α through it such that $S^{\alpha\beta}U_\beta = 0$. Indeed, any point within the disk of centroids can be a solution (i.e., can be a center of mass as measured in its proper frame) provided that it moves with the appropriate velocity. In order to see that, we start by an heuristic argument, originally due to Möller [7]: consider, in Fig. 1, a point in circular motion *opposite* to the rotation of the body, with a radius $R = v'S_\star/M$ such that it passes through the centroid $x^\alpha_{\rm CM}(u')$ measured by the observer $\mathcal{O}'$, and having *therein* the same velocity as $\mathcal{O}'$. Such point instantaneously coincides with $x^\alpha_{\rm CM}(u')$, and at the same time is at rest with respect to $\mathcal{O}'$; it is thus a center of mass computed in its own rest frame, and will be so at every instant as the motion is circular. The angular velocity of such points is constant, $\Omega = v'/R = M/S_\star$ (i.e., does not depend on R). That is, consider a disk of the same size of the disk of centroids, *rigidly* rotating about the centroid $x^\alpha_{\rm CM}(P)$ measured by $\mathcal{O}$; any point of such disk is a centroid computed in its rest frame, and is thus a solution of $S^{\alpha\beta}U_\beta = 0$. This is the origin of the helical motions (in a frame moving with respect to $\mathcal{O}$, the circular motions become helices).

[5]The factor m/M^2 (involving U^α via m) can be determined by the normalization condition $U^\alpha U_\alpha = -1$.

These facts can be explicitly checked from the equations of motion. First we note that, with this spin condition, the momentum becomes, cf. Eq. (16),

$$P^\alpha = mU^\alpha + S^{\alpha\beta}a_\beta = mU^\alpha + \epsilon^{\alpha\beta}{}_{\gamma\delta}a_\beta S^\gamma U^\delta \ , \tag{29}$$

where S^α is the spin vector defined by

$$S^\alpha = \frac{1}{2}\epsilon^\alpha{}_{\beta\mu\nu}S^{\mu\nu}U^\beta; \qquad S^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}S_\mu U_\nu \ . \tag{30}$$

Noting from (29) that $P^\alpha a_\alpha = P^\alpha S_\alpha = 0$, and using $S^{\alpha\beta}U_\beta = 0$, the component of the 4-velocity orthogonal to P^α is, from Eq. (23),

$$U_\perp^\alpha = -\frac{1}{M^2}\epsilon^{\alpha\beta\mu\nu}S_\mu P_\nu \frac{DU_\beta}{d\tau} \tag{31}$$

which in the $P^i = 0$ frame reads

$$\vec{U} + \frac{1}{M}\frac{D\vec{U}}{d\tau} \times \vec{S} = 0 \ . \tag{32}$$

This is a differential equation for the space components $\vec{U}$; as discussed above, $\vec{v} = \vec{U}/\gamma(P,U)$ has the interpretation of 3-velocity of the centroid in the $P^i = 0$ frame. Take now for simplicity the case of a free particle in flat spacetime; in this case, from Eq. (14) we have $DP^\alpha/d\tau = 0$; also, from Eq. (34) below, it follows that $DS^\alpha/d\tau = 0$ (since $S^\alpha a_\alpha = 0$, which can be seen substituting (29) in $DP^\alpha/d\tau = 0$); thus M and $\vec{S}$ in (32) are constants, and the solution for the reference worldline z^α ($\vec{U} = d\vec{z}/d\tau$) is, in rectangular coordinates (taking $\vec{S}$ along $\vec{e}_z$),

$$z^\alpha(\tau) = \left(\gamma\tau, -R\cos\left(\frac{v\gamma}{R}\tau\right), R\sin\left(\frac{v\gamma}{R}\tau\right), 0\right) \tag{33}$$

where $R = v\gamma S/M$, and $\gamma \equiv \gamma(P,U) = -P^\alpha U_\alpha/M = \sqrt{1-v^2}$; v can take any value between 0 and 1. These are circular motions of radius R and frequency $\Omega = M/\gamma S$, centered about the centroid $x^\alpha_{\rm CM}(P)$ measured in the $P^i = 0$ frame. They may not seem at first the same motions we deduced from the heuristic argument above; in particular, the fact γ can be arbitrarily large has led some authors to believe that the radius of these motions, for a given body, can be arbitrary, and for this reason deemed them unphysical [35, 36, 37]. That is not the case; the reason for that is that S is different for all the helical representations corresponding to the same body. Let z^α and z'^α denote two different helical solutions. The scalar $S = \sqrt{S^\alpha S_\alpha} = \sqrt{S^{\alpha\beta}S_{\alpha\beta}}/2$, for a spin tensor obeying $S^{\alpha\beta}U_\beta = 0$, is the magnitude of the angular momentum taken about $z^\alpha = x^\alpha_{\rm CM}(U)$. It should in be different, *for the same matter distribution* $T_{\alpha\beta}(x)$, from $S' = \sqrt{S'^{\alpha\beta}S'_{\alpha\beta}}/2$, since $S'^{\alpha\beta}$, obeying $S'^{\alpha\beta}U'_\beta = 0$, is the angular momentum about a different point, $z'^{\alpha} = x^{\alpha}_{\rm CM}(U')$. It is shown in Sec. IV of [16] that, for all helical motions $S = S_\star/\gamma$, where $S_\star = \sqrt{S_\star^{\alpha\beta} S_{\star\alpha\beta}/2}$ is the magnitude of the angular momentum taken about $x^{\alpha}_{\rm CM}(P)$ (i.e., $S_\star^{\alpha\beta} P_\beta = 0$). So indeed these motions have a *finite* radius and constant frequency, as deduced above:

$$\Omega = \frac{M}{S_\star}; \qquad R = \frac{v S_\star}{M} \ .$$

Hence we see that the famous helical motions are just another exotic effect generated by the variation, along $z^{\alpha}(\tau)$, of the field of observers u^{α} ($= U^{\alpha}$, in this case) with respect to which the centroid is computed; what is special about them is that in this case the non-trivial motion induced on the centroid is such that the latter is always at rest with respect to the observer measuring it. Thus they are *not* unphysical, contrary to some claims in the literature; but they do not contain new physics either, they are just alternative, unnecessarily complicated descriptions for physical motions that can be described through simpler representations: for example, the non-helical solution that this spin condition also allows, which in the case of a free particle in flat spacetime is uniform straight line motion (corresponding to $v = 0$, $R = 0$, in Eq. (33) above).

It is also worth noting that, from a dynamical perspective, the consistency of the helical motions (namely, the fact the centroid accelerates *without any force*) is explained through an interchange between kinetic momentum $P^{\alpha}_{\rm kin} = mU^{\alpha}$ and hidden momentum $P^{\alpha}_{\rm hidI} = S^{\alpha\beta} a_\beta$, which occurs in a way that their variations cancel out at every instant, such that $P^{\alpha} = mU^{\alpha} + P^{\alpha}_{\rm hidI}$ remains constant; see Fig. 3 of [16]. This is exactly the same principle behind the bobbings due to $P^{\alpha}_{\rm hidI}$ discussed in Sec. 3.4 below.

Features of the FMP condition: Fermi-Walker transport and gravito-electromagnetic analogies

If one employs the Frenkel-Mathisson-Pirani condition, the spin vector of a gyroscope (if $\tau^{\alpha\beta} = 0$) is Fermi-Walker transported along the worldlines of *any of the centroids* obeying this condition. This can easily be seen substituting Eq. (30) in (15) to obtain

$$\frac{DS^{\alpha}}{d\tau} = S_\nu a^{\nu} U^{\alpha} \ . \tag{34}$$

This is the most natural description for the spin evolution, where the mathematical definition of a locally non-rotating frame meets the physical one: gyroscopes "oppose" changes in direction of their rotation axes; the axis of torque-free gyroscopes define *physically* the non-rotating frames. On the other hand, Fermi-Walker transport is the *mathematical* definition of a non-rotating frame $\mathbf{e}_{\hat\alpha}$ adapted to an arbitrarily accelerated observer: $\nabla_{\mathbf{u}} \mathbf{e}_{\hat\beta} = \Omega^{\hat\alpha}_{\ \hat\beta} \mathbf{e}_{\hat\alpha}$, $\Omega^{\alpha\beta} = 2u^{[\alpha} a^{\beta]}$; that is, it admits "rotation" (actually boost) in the time-space plane formed by U^{α} and a^{α}, unavoidable to keep the time axis of the tetrad parallel to the 4-velocity ($\mathbf{U} = \mathbf{e}_{\hat{0}}$), so that the triad $\mathbf{e}_{\hat{i}}$ spans the observer's local rest space; but no additional spatial rotation (i.e., the axes $\mathbf{e}_{\hat{i}}$ orthogonal to a^α are parallel transported).

Another interesting feature of this spin condition is that is gives rise to three *exact* gravito-electromagnetic analogies [23, 29]: i) the spin-curvature force (penultimate term of Eq. (14)) becomes $F^\alpha_{\rm G} = -\mathbb{H}^{\beta\alpha}S_\beta$, where $\mathbb{H}_{\alpha\beta} \equiv \star R_{\alpha\beta\gamma\delta}U^\beta U^\gamma$, analogous to the force on a magnetic dipole (second term of Eq. (14)), $F^\alpha_{\rm EM} = B^{\beta\alpha}\mu_\beta$, where $B_{\alpha\beta} \equiv \star F_{\alpha\mu;\beta}U^\mu$; ii) Eq. (34) becomes, in an orthonormal frame "adapted" to a congruence of observers, $dS^{\hat{i}}/d\tau = (\vec{S} \times \vec{H})^{\hat{i}}/2$, where $\vec{H}$ is the "gravitomagnetic field", analogous to the precession of a magnetic dipole, $D\vec{S}/d\tau = \vec{\mu} \times \vec{B}$ (first term of Eq. (61)); iii) the inertial hidden momentum, cf. Eq. (29), is $P^\alpha_{\rm hidI} = \epsilon^\alpha_{\ \beta\gamma\delta}U^\delta S^\beta G^\gamma$, with $G^\alpha = -a^\alpha$ the "gravitoelectric" field as measured in the centroid frame, formally analogous to the electromagnetic hidden momentum, Eq. (64) below. These analogies (apart from their theoretical interest) provide useful insight to study some problems; they are discussed in detail in [23].

The downside of this condition is the fact that it is not always easy to set up the non-helical solution. It is done through suitable ansatzs in Sec. 3.4 below (at an approximate level), or at an exact level, in *very special* systems, in [23] (therein it is seen to be a good choice, as it takes advantage of the symmetries of the problems to yield the simplest equations). Prescriptions in the case of Schwarzschild and Kerr spacetimes are also proposed in [38, 39]; however no general rule is known.

3.2.4 The Ohashi-Kyrian-Semerák (OKS) spin condition

This condition, introduced in [12], and first discussed in depth in [13], amounts to choosing a vector field u^α parallel transported along $z^\alpha(\tau)$, $Du^\alpha/d\tau = 0$, which causes the inertial hidden momentum $P^\alpha_{\rm hidI}$ and its associated gauge motions to vanish, cf. Eq. (18). In the general case where the torque tensor $\tau^{\alpha\beta}$ is non-zero, as we shall see in Sec. 3.5.1 below, some superfluous motions may still be present though, due to the pure gauge part of the hidden momentum $P^\alpha_{{\rm hid}\tau}$ related to $\tau^{\alpha\beta}$ (in gravitational systems, $P^\alpha_{{\rm hid}\tau}$ is usually less important, as it involves the particle's quadrupole moment). When $\tau^{\alpha\beta} = 0$ (the problem at hand herein), it yields the simplest momentum velocity relation possible, $P^\alpha = mU^\alpha$, and a centroid that accelerates only when there is a force, $ma^\alpha = (h^U)^\alpha_{\ \beta}F^\beta$ (this becomes $ma^\alpha = F^\alpha$ for pole-dipole particles in gravitational fields, since $m = M \equiv \sqrt{-P^\alpha P_\alpha}$ is constant, as readily seen contracting Eqs. (14) or (39) with P^α). Eq. (15) also takes a simple form, yielding a spin tensor $S^{\alpha\beta}$ parallel transported along $z^\alpha(\tau)$, $DS^{\alpha\beta}/d\tau = 0$.

This condition obviously does not specify an unique worldline through the body; it is infinitely degenerate, because there are infinite possible choices of u^α (the only restriction imposed is $Du^\alpha/d\tau = 0$); but another of its advantages [13] is that one does not need[6] to explicitly determine u^α to solve the equations of motion (for pole-dipole particles), only its value at the initial point is needed. These properties together make this condition the most suitable (at least in that case) for numerical implementation.

3.2.5 Uniqueness of the centroid vs determinacy of the equations

There are some *apparent* contradictions in the literature regarding the uniqueness of the worldline specified by the different spin conditions, and what that means in terms of the determinacy of the equations of motion. On the one hand most authors (e.g. [7, 10, 25, 13, 17]) argue, in agreement with the discussion above, that the FMP condition does not uniquely specify a worldline through the body; on the other hand, it has recently been argued [40, 41] that it uniquely specifies the motion, given certain initial conditions. Also, in [10, 13], it is said that the CP condition yields an unique solution, whereas in the analysis above we have seen that, depending on the coordinate system chosen, it may or may not yield an unique center of mass. Our considerations above are based on starting with a test body whose matter distribution is described by an energy-momentum tensor $T^{\alpha\beta}(x)$, and asking the following question: given $T^{\alpha\beta}(x)$, does the condition $S^{\alpha\beta}u_\beta = 0$ yield an unique worldline? As we have seen, from the four conditions studied above, the answer is affirmative, as a general statement, only for the TD condition.

But if one takes the perspective of the the initial value problem for the equations of motion (14)-(15), the impact of the uniqueness/non-uniqueness of the center of mass definition is not straightforward. First of all one should notice that, without further assumptions, the system (14)-(15), supplemented by (8), can be determined only to dipole order and if $F^{\alpha\beta} = 0$ (otherwise one needs evolution laws for $\mu^{\alpha\beta}$, d^α, and the higher order electromagnetic and gravitational moments). In this case, all the conditions yield a well defined solution if sufficient initial conditions are provided; and it is *the type of initial data* needed to determine the equations that depends on the nature of center of mass definition given by each of the conditions.

On general grounds one can say that if the equations of motion can be written as the *explicit* functions (dot denotes ordinary derivative along $\mathbf{U}$)

$$\dot{z}^\alpha(\tau) \equiv U^\alpha = U^\alpha(\mathbf{z}, \mathbf{P}, S^{\mu\nu}); \qquad \dot{P}^\alpha = f^\alpha(\mathbf{z}, \mathbf{P}, S^{\mu\nu}); \qquad \dot{S}^{\alpha\beta} = g^{\alpha\beta}(\mathbf{z}, \mathbf{P}, S^{\mu\nu})$$

then, given the initial values $\{z^\alpha, P^\alpha, S^{\alpha\beta}\}|_{\rm in}$, the system is determined. The first equation is the *explicit* velocity-momentum relation; but the other two also require such a relation, as can be seen by writing explicitly

$$\dot{P}^\alpha = \Gamma^\alpha_{\nu\mu} P^\mu U^\nu - \frac{1}{2} R^\alpha_{\ \beta\gamma\delta} U^\beta S^{\gamma\delta}; \qquad \dot{S}^{\alpha\beta} = 2\Gamma^{[\alpha}_{\nu\mu} S^{\beta]\mu} U^\nu + 2P^{[\alpha}U^{\beta]}.$$

[6] We thank O. Semerák and A. Harte for discussions on these issues.

Thus, in order to have $\dot{P}^\alpha$ and $\dot{S}^{\alpha\beta}$ as explicit functions of $(\mathbf{z}, \mathbf{P}, S^{\mu\nu})$, we need to have an explicit relation $U^\alpha(\mathbf{z}, \mathbf{P}, S^{\mu\nu})$.

In the case of the OKS condition, since one has simply $U^\alpha = P^\alpha/m = P^\alpha/M$, cf. Sec. 3.2.4, the statements above obviously hold, and the solution is determined given $\{z^\alpha, P^\alpha, S^{\alpha\beta}\}|_{\rm in}$, or, equivalently, $\{z^\alpha, S^{\alpha\beta}, U^\alpha, m\}|_{\rm in}$.

The situation is similar for the TD condition (only with a more complicated velocity-momentum relation). Eq. (28) is an explicit relation $U^\alpha(\mathbf{z}, \mathbf{P}, S^{\mu\nu})$; thus, given $\{z^\alpha, P^\alpha, S^{\alpha\beta}\}|_{\rm in}$, the solution is determined. The initial data $\{z^\alpha, S^{\alpha\beta}, U^\alpha, m\}|_{\rm in}$ is equally sufficient because one can extract $P^\alpha|_{\rm in}$ from Eq. (27) (substituting therein $DP^\alpha/d\tau$ by the explicit expression $-R^\alpha_{\ \beta\gamma\delta}U^\beta S^{\gamma\delta}/2$; an equation for M^2 is obtained by squaring (27)).

The case of the CP condition is also essentially similar. One obtains an explicit relation $U^\alpha(\mathbf{z}, \mathbf{P}, S^{\mu\nu})$ as follows[7]. Substitute decomposition (20) in Eq. (16), with $u^\alpha = u^\alpha_{\rm rest}$, to obtain

$$P^\alpha = \frac{-P_\alpha u^\alpha_{\rm lab}}{\gamma}U^\alpha + \frac{1}{\gamma}S^\alpha_{\ \beta}\left(\gamma G^\beta - \epsilon^\beta_{\ \mu\gamma\delta}u^\delta_{\rm lab}U^\mu\omega^\gamma - \theta^{\beta\gamma}U_\gamma\right) , \tag{35}$$

where $\gamma \equiv -U_\alpha u^\alpha_{\rm lab}$. This is an equation for P^α in terms of U^α, $S^{\alpha\beta}$, and the quantities G^α, ω^α, and $\theta_{\alpha\beta}$ which are given in advance (see Sec. 3.2.1 and the equivalent Eq. (26)). We need now to solve for U^α. Expressing (35) in its components in a frame where $u^i_{\rm lab} = 0$, we obtain

$$A^i_{\ k}v^k = P^i - S^i_{\ j}G^j , \qquad A^i_{\ k} \equiv \left[P^0\delta^i_{\ k} - S^i_{\ j}\left(\epsilon^{jk}_{\ \ l}\omega^\gamma + \theta^{jk}\right)\right] ,$$

where $v^i = U^i/U^0$ is the centroid velocity in the $u^i_{\rm lab} = 0$ frame. This is a system of linear equations for the three components v^k, with solution $v^i = [A^{-1}]^i_{\ k}[P^k - S^k_{\ j}G^j]$. The component U^0 (and subsequently, U^i) is then obtained from the normalization condition $-1 = U^\alpha U_\alpha = -(U^0)^2(1-v^2)$. We thus end up with an explicit relation $U^\alpha(\mathbf{z}, \mathbf{P}, S^{\mu\nu})$, meaning that, given the initial values $\{z^\alpha, S^{\alpha\beta}, P^\alpha\}|_{\rm in}$, the solution is determined, as asserted in [13]. The set $\{z^\alpha, S^{\alpha\beta}, U^\alpha, m\}|_{\rm in}$ is also sufficient, in agreement with the claim in [10], because one immediately obtains $P^\alpha|_{\rm in}$ from (26). Finally, note that this is a distinct problem from the one addressed in Sec. 3.2.1 (where we started just with a matter distribution $T^{\alpha\beta}(x)$ and imposed $S^\alpha_{\ \beta}u^\beta_{\rm lab} = 0$, in which case, as we have seen, the solution always exists but in general is *not* unique). Herein one *assumes* the existence of some $T^{\alpha\beta}(x)$ that is compatible with the initial conditions prescribed (conversely, in the prescription of Sec. 3.2.1 there is no longer freedom to choose an arbitrary initial position z^α).

The case of the FMP condition has some important differences. The momentum velocity relation is (29); the acceleration can be written as (cf. Eq. (24) of [13])

$$a^\alpha(\mathbf{z}, \mathbf{U}, S^{\mu\nu}) = \frac{1}{S^2}\left(\frac{1}{m}F^\mu S_\mu S^\alpha - P_\gamma S^{\alpha\gamma}\right) \tag{36}$$

[7] We thank O. Semerák for his input on this issue.

with S^α defined by (30), $S^2 \equiv \sqrt{S^\alpha S_\alpha}$, and $F^\alpha \equiv DP^\alpha/d\tau$. Substituting in (29) one obtains an explicit relation $P^\alpha(\mathbf{z}, \mathbf{U}, S^{\mu\nu})$. However, one cannot a priori guarantee that such relation can be inverted into a relation $U^\alpha(\mathbf{z}, \mathbf{P}, S^{\mu\nu})$ (such problem, in the general case, has not yet been tackled in the literature, to the authors' knowledge). In the *special case of a free particle in flat spacetime*, we have, from (29),

$$a^\alpha(\mathbf{P}, S^{\mu\nu}) = -\frac{S^{\alpha\beta}P_\beta}{S^2}; \qquad U^\alpha(\mathbf{P}, S^{\mu\nu}) = \frac{1}{m}\left(P^\alpha + \frac{1}{S^2}S^{\alpha\mu}S_{\mu\beta}P^\beta\right). \tag{37}$$

This is an explicit relation $U^\alpha(\mathbf{P}, S^{\mu\nu})$ (m can be determined through the condition $U^\alpha U_\alpha = -1$); therefore, in agreement with the claims in [40, 41], the motion is indeed determined given the initial data $\{z^\alpha, S^{\alpha\beta}, P^\alpha\}|_{\rm in}$ (i.e., this set of data specifies one particular solution of the *degenerate* condition $S^{\alpha\beta}U_\beta = 0$). On the other hand (unlike the situation for the other three spin conditions), the set of data $\{z^\alpha, S^{\alpha\beta}, U^\alpha, m\}|_{\rm in}$ is *not* enough; one needs, additionally, the initial acceleration a^α, in agreement with the claims in e.g. [11, 17]. This is clear from Eqs. (29), (37): the set of initial data $\{S^{\alpha\beta}, P^\alpha\}|_{\rm in}$ *is equivalent* to $\{S^{\alpha\beta}, U^\alpha, m, a^\alpha\}|_{\rm in}$. These features are readily understood in the framework of the discussion in Sec. 3.2.3: as we have seen, the motion of an helical solution $z^\alpha = x^\alpha_{\rm CM}(U)$ is a superposition of a circular motion centered at the centroid measured in the $P^i = 0$ frame, $x^\alpha_{\rm CM}(P)$, of radius $R = \|\Delta\mathbf{x}\|$ and angular velocity $\vec{\omega} = -M\vec{S}_\star/S_\star^2$, combined with a boost of 4-velocity P^α/M. $\Delta x^\alpha = z^\alpha - x^\alpha_{\rm CM}(P)$ is the shift of z^α relative to the center of the helix. Given $z^\alpha_{\rm in}$, $S^{\alpha\beta}$, and P^α, one obtains $x^\alpha_{\rm CM}(P)|_{\rm in} = z^\alpha_{\rm in} - \Delta x^\alpha_{\rm in}$ from the expression $\Delta x^\alpha = S^{\alpha\beta}P_\beta/M^2$, cf. Eq. (10); $S_\star^{\alpha\beta}$ follows using $S^{\alpha\beta} = S_\star^{\alpha\beta} + 2P^{[\alpha}\Delta x^{\beta]}$, and therefore the motion is completely determined. On the other hand, if instead of $P^\alpha|_{\rm in}$ one is given $\{U^\alpha, m\}|_{\rm in}$, one cannot determine Δx^α; that is the reason why one needs the acceleration, as it contains precisely the same information: $a^\alpha = -\Delta x^\alpha M^2/S^2$, cf. Eq. (37a).

3.3 The dependence of the spin-curvature force on the spin condition; equivalence of the spin conditions.

We have seen that the significance of the spin condition $S^{\alpha\beta}u_\beta = 0$ is that of a choice of representative worldline z^α in the body, more precisely requiring such worldline to be, at each event, the center of mass as measured by an observer of 4-velocity u^α. We have thereby implied that the different spin conditions yield different, but *equivalent* descriptions of the motion of a given body, all contained within the worldtube of centroids, which in turn lies within the convex hull of the body's worldtube. That is easy to see for a free particle in flat spacetime, where indeed the different solutions stay close forever and within the straight worldtube depicted in Fig. 2. However, when external non-homogenous fields are present, changing z^α means not only changing the point where the fields (i.e., $F^{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$) are evaluated, but also changing the moments

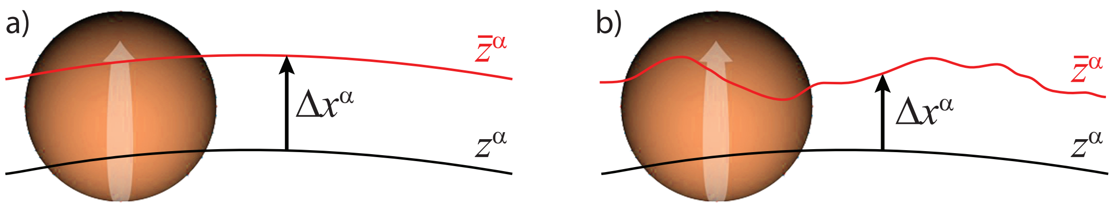

Figure 5: a) Two different centroids of the OKS condition move nearly parallel ($D\Delta x^\alpha/d\tau = 0$). In a curved spacetime, that implies that the force $DP^\alpha/d\tau$ along the two worldlines must be different (e.g., if $z^\alpha(\tau)$ is a geodesic, $\bar{z}^\alpha(\bar{\tau})$ cannot be). b) Centroids $\bar{z}^\alpha(\bar{\tau})$ of other spin conditions accelerate relative to OKS centroids due to the gauge motions induced by the variation of u^α along $\bar{z}^\alpha$ (inertial hidden momentum).

($S^{\alpha\beta}$, $\mu_{\alpha\beta}$, d^α, and the $2^{N>1}$ moments) themselves, on which the forces and torques also depend. These two changes would in principle compensate each other; the larger part of the compensation comes from the lower order terms, and a smaller part (negligible to some extent) from the higher order terms. Hence, in an approximation where only moments up to 2^Nth order are kept, the different worldlines will eventually diverge. However, this does not mean that the spin condition is not a gauge choice after all; in fact, it just marks the limit of validity of the given approximation [13]. The subtlety involved in this compensation is that, except for the case of flat spacetime, it does *not* mean that the force is the same for different choices of z^α.

In order to see this, let us consider first in Newtonian mechanics, the problem of describing an extended body through different reference points; for more details on this problem, we refer to Sec. 3 of [42]. Consider a spherical body in a gravitational field $\vec{G}(\mathbf{x})$. If one takes z^i to be the body's center of mass $z^i = x^i_{\rm CM} \equiv \int \rho x^i d^3x/m$ — an unique point, in Newtonian mechanics — then, with respect to z^i, the body is effectively a monopole, and the only force present is the usual (monopole) gravitational force $\vec{F} = \vec{F}_{\rm g} = m\vec{G}(\mathbf{z})$. Now take a different reference point $\bar{z}^i = z^i + \Delta x^i$ (not a centroid) say, at the boundary of the sphere. The monopole force changes to $\vec{\bar{F}}_{\rm g} = m\vec{G}(\mathbf{z}+\Delta\mathbf{x})$; but, on the other hand, the particle has a mass dipole moment $\vec{d}_{\rm G} = \int \rho\vec{x}d^3x = -m\Delta\vec{x}$ about $\bar{z}^i$ (as well as quadrupole and higher order moments). The dipole force is $\bar{F}^i_{\rm dip} = \nabla_j G^i d^j_{\rm G}$; hence to dipole order, we have the *same net* Newtonian force:

$$\bar{F}^i = \bar{F}^i_{\rm g} + \bar{F}^i_{\rm dip} = mG^i(\mathbf{z}+\Delta\mathbf{x}) - m\nabla_j G^i(\mathbf{z}+\Delta\mathbf{x})\Delta x^j \simeq mG^i(\mathbf{z}) = F^i \ . \tag{38}$$

In General Relativity the situation is different because the lowest order gravitational force is the (dipole order) spin-curvature force

$$F^\alpha = -\frac{1}{2}R^\alpha{}_{\beta\mu\nu}S^{\mu\nu}U^\beta = \star R_{\beta\mu\nu}{}^{\alpha}U^\mu S^\beta u^\nu \ , \tag{39}$$

cf. Eq. (14), which depends explicitly on the spin condition $S^{\alpha\beta}u_\beta = 0$, i.e., on the choice of the centroid z^α. Such dependence is not compensated by a change in the monopole force (which does not exist), nor by the higher order terms (if that was the case, the pole-dipole approximation would not even make sense, for, as we shall see in Sec. 3.4.2, the differences in the force under different spin conditions are of the same order of magnitude as the force itself). Hence, the *net force* $F^\alpha = \nabla_U P^\alpha$ is different for different z^α's, which is natural in a curved spacetime, since the differentiation is along different curves. On the other hand, although the monopole force $\vec{F}_g$ (or $\vec{G}$) has no *physical* existence in the relativistic theory, there is a counterpart to the the *tidal forces* arising from the variation of these fields from point to point, $\nabla_j G_i$, which comes from the curvature tensor (see below). And the crucial point here is that the change in the force F^α when one changes z^α is precisely the one needed to compensate for the tidal forces which "try" to make the worldlines diverge.

This can be formalized as follows. Take two different centroids with worldlines z^α and $\bar{z}^\alpha$, defined by $S^{\alpha\beta}u_\beta = 0$ and $\bar{S}^{\alpha\beta}\bar{u}_\beta = 0$, respectively. $S^{\alpha\beta}$ is the angular momentum about z^α and $\bar{S}^{\alpha\beta}$ the angular momentum about $\bar{z}^\alpha$. Extend (in a region small enough so that they do not intersect) these worldlines to a congruence of curves encompassing both z^α and $\bar{z}^\alpha$; take them to be infinitesimally close, so that one can employ the usual first order deviation equations (Eq. (41) below), and write a connecting vector as $\Delta x^\alpha = \bar{z}^\alpha(\tau) - z^\alpha(\tau)$. Take moreover u^α to be parallel transported along z^α (i.e., it obeys Ohashi-Kyrian-Semerák spin condition), so that $P^\alpha_{\rm hidI} = 0 \Rightarrow P^\alpha = MU^\alpha$, and let the field $\bar{u}^\alpha$ be arbitrary. Noting that P^α can be taken as the same for z^α and $\bar{z}^\alpha$ (see Appendix A.1), it follows from Eqs. (17) that $P^\alpha = \bar{m}\bar{U}^\alpha + \bar{P}^\alpha_{\rm hid}$, where $\bar{m} \equiv -P^\alpha \bar{U}_\alpha$; contracting with P_α to obtain an expression for $\bar{m}/M$, and using Eqs. (22), one obtains

$$\bar{U}^\alpha = \frac{P^\alpha}{M} + \left(\sqrt{1 + \frac{\bar{P}^\alpha_{\rm hid} P_\alpha}{M^2}} - 1\right)\frac{P^\alpha}{M} + \bar{U}^\alpha_\perp \equiv \frac{P^\alpha}{M} + \left[\text{terms in } \bar{U}^\alpha_\perp\right]. \tag{40}$$

$U^\alpha_\perp$ and $P^\alpha_{\rm hid} = P^\alpha_{\rm hidI}$ are gauge and reciprocal quantities; one can write one in terms of the other using Eqs. (23). $\bar{U}^\alpha \neq P^\alpha/M$ only if $\bar{U}^\alpha_\perp \neq 0$ (or equivalently if $P^\alpha_{\rm hid} \neq 0$). From the deviation equation for accelerated worldlines [43], we have

$$\begin{aligned}\frac{D^2\Delta x^\alpha}{d\tau^2} &= -\,\mathbb{E}^{\alpha\beta}\Delta x_\beta + \nabla_{\Delta\mathbf{x}} a^\alpha = -\mathbb{E}^{\alpha\beta}\Delta x_\beta + \left(\nabla_{\bar{U}}\bar{U}^\alpha - \nabla_U U^\alpha\right)\\ &= -\,\mathbb{E}^{\alpha\beta}\Delta x_\beta + \frac{1}{M}\left(\bar{F}^\alpha - F^\alpha\right) + \left[\text{terms in } \bar{U}^\alpha_\perp\right]\ ,\end{aligned} \tag{41}$$

where $\mathbb{E}_{\alpha\beta} \equiv R_{\alpha\mu\beta\nu}U^\mu U^\nu$ is the "gravitoelectric" tidal tensor, which is the relativistic counterpart of the Newtonian tidal tensor $\nabla_j G_i$. In the third equality we used Eq. (40) and the following: M is a conserved quantity for the OKS spin condition ($\nabla_U M = 0$ along z^α if $P^\alpha \parallel U^\alpha$, as readily seen contracting (39) with P^α), so that $F^\alpha = DP^\alpha/d\tau = Ma^\alpha$; and that along $\bar{z}^\alpha$ one has $\nabla_{\bar{U}} M = 0 + \left[\text{terms in } \bar{U}^\alpha_\perp\right]$.

Since Δx^{α} is infinitesimal, we can write (as in flat spacetime),

$$\bar{S}^{\alpha\beta} = S^{\alpha\beta} + 2P^{[\alpha}\Delta x^{\beta]} \tag{42}$$

and therefore the difference between the forces is

$$\begin{aligned} \bar{F}^{\alpha} - F^{\alpha} &= -\frac{1}{2M} R^{\alpha}{}_{\beta\gamma\delta} P^{\beta} \left(\bar{S}^{\gamma\delta} - S^{\gamma\delta} \right) + \left[\text{terms in } \bar{U}^{\alpha}_{\perp} \right] \\ &= M \mathbb{E}^{\alpha\beta} \Delta x_{\beta} + \left[\text{terms in } \bar{U}^{\alpha}_{\perp} \right] \end{aligned} \tag{43}$$

where the terms in $\bar{U}^{\alpha}_{\perp}$ are of order $\mathcal{O}(S^2)$. Substituting in (41), we obtain

$$\frac{D^2 \Delta x^{\alpha}}{d\tau^2} = 0 + \left[\text{terms in } \bar{U}^{\alpha}_{\perp} \right] \ .$$

That is, the worldline deviation of the two solutions reduces to terms involving $U^{\alpha}_{\perp}$ (i.e., P^{α}_{hidI}), that we have seen in Sec. 3.1 to be gauge (arising just from the choice of observers relative to which the centroids are computed). This is illustrated in Fig. 5. In particular, if one takes two different solutions of the OKS condition (so that no superfluous motions come into play[8]) we have simply $D^2\Delta x^{\alpha}/d\tau^2 = 0$, i.e., there is no relative acceleration between the worldlines, which is *guaranteed* by the difference between the forces $F^{\alpha} = \nabla_U P^{\alpha}$ and $\bar{F}^{\alpha} = \nabla_{\bar{U}} P^{\alpha}$.

The situation becomes especially enlightening (and the correspondence with the Newtonian theory closer) in the limit of weak *static* fields and slow motion of Sec. 3.4. In this case the *coordinate* acceleration (for a stationary field) of the centroid z^{α} is

$$m\frac{d^2 z^i}{d\tau^2} = mG^i(\mathbf{z}) + F^i - \frac{DP^i_{\text{hid}}}{d\tau}$$

where $\vec{G}$ is the Newtonian field (more precisely, a fictitious, or "inertial" field, that mimics Newton's $\vec{G}$ in the coordinate acceleration. Is is also known as the "gravitoelectric" field, e.g. [29]). The coordinate acceleration of the centroid $\bar{z}^{\alpha}$ is

$$m\frac{d^2 \bar{z}^i}{d\bar{\tau}^2} = mG^i(\bar{\mathbf{z}}) + \bar{F}^i - \frac{D\bar{P}^i_{\text{hid}}}{d\bar{\tau}} = mG^i(\bar{\mathbf{z}}) + F^i + m\mathbb{E}^{ij}\Delta x_j - \frac{D\bar{P}^i_{\text{hid}}}{d\bar{\tau}};$$

in the second equality we used (43) neglecting the $U^{\alpha}_{\perp}$ terms therein (as they are of order $\mathcal{O}(S^2)$). To first order in $\Delta\mathbf{x}$, $G^i(\bar{\mathbf{z}}) \simeq G^i(\mathbf{z}) + \nabla^j G^i \Delta x_j$; and since, for a stationary field, to linear order (see [29]), $\mathbb{E}_{ij} = -\nabla_j G_i$, then $mG^i(\bar{\mathbf{z}}) + \bar{F}^i = mG^i(\mathbf{z}) + F^i$, i.e., the sum of the spin curvature and the Newtonian forces is the same for both worldlines,

[8] Only when $\nabla_{\mathbf{U}} u^{\alpha} = \nabla_{\bar{\mathbf{U}}} \bar{u}^{\alpha} = 0$ should one expect two different centroids to of the same body to move parallel, even in flat spacetime, as explained in Sec. 3.1 (see also Fig. 2b)). Otherwise (i.e., when $P^{\alpha}_{\text{hidI}} \neq 0$) they can have an arbitrary relative motion, cf. Fig. 5.

the change in one compensating for the other, just like the case with the monopole and dipole forces in the Newtonian problem above, cf. Eq. (38). We have thus

$$m\frac{d^2\bar{z}^i}{d\bar{\tau}^2} = mG^i(\mathbf{z}) + F^i - \frac{D\bar{P}^i_{\rm hid}}{d\bar{\tau}} = m\frac{d^2 z^i}{d\tau^2} + \frac{DP^i_{\rm hid}}{d\tau} - \frac{D\bar{P}^i_{\rm hid}}{d\bar{\tau}} \ . \tag{44}$$

Hence, barring hidden momentum terms, the coordinate acceleration for $\bar{z}^\alpha(\bar{\tau})$ is the same as for $z^\alpha(\tau)$. This means, in particular, that the different solutions of the OKS condition are trajectories that run parallel (as both the coordinate acceleration and the velocity are the same for all of them).

3.4 Comparison of the spin conditions in simple examples

In this section we consider the two simple setups illustrated in Fig. 6 — a spinning charged body (with $\vec{\mu} = 0$, and whose only non-vanishing electromagnetic moment is q, so that $P^\alpha_{{\rm hid}\tau} = 0$) orbiting a Coulomb charge in flat spacetime, and a spinning body orbiting a Schwarzschild black hole, both particles having spin $\vec{S}$ lying in the orbital plane — and compare the description of the motion given by the different spin conditions. Such comparison will be done ensuring that one is dealing with the worldlines of different centroids corresponding to the same physical body (i.e., the same matter distribution $T^{\alpha\beta}(x)$). We will be using the weak field slow motion approximation, for two reasons: first, because it is sufficient to illustrate the effects of interest; second, and more importantly, to make clear that the choice of spin condition (and the resulting hidden momentum) impacts the equations of motion at leading order, and thus these effects must be taken into account in any linearized theory or Post-Newtonian approximation.

3.4.1 Electromagnetic system

The Corinaldesi-Papapetrou (CP) condition, which sets the reference worldline z^α as being center of mass $x^\alpha_{\rm CM}(u_{\rm rest})$ measured in the "laboratory" frame (chosen as the congruence of static observers $u^\alpha_{\rm rest}$, at rest with respect to the source), coincides in this case with one of the solutions of the Ohashi-Kyrian-Semerák (OKS) condition, because such frame is inertial, and therefore $\nabla_{\mathbf{U}} u^\alpha_{\rm rest} = 0 \Rightarrow P^\alpha_{\rm hidI} = 0$, cf. Eq. (18). The momentum is thus parallel to the 4-velocity $P^\alpha = mU^\alpha$, and the equation of motion for the centroid reduces to

$$ma^\alpha = F^\alpha = qF^{\alpha\beta}U_\beta \ ; \qquad \vec{F} = q\vec{E}(U) = qQ\frac{\vec{r}}{r^3} + \mathcal{O}(v^2) \ , \tag{45}$$

whose well known solution for a Coulomb field is an ellipse. In particular, a particle with an initial velocity in the $x_O y$ plane, and equaling that of a circular orbit, will follow a circular orbit in that plane (regardless of its spin); and a particle with initial radial velocity will move radially, cf. Fig. 6a). To compare with the description given by other

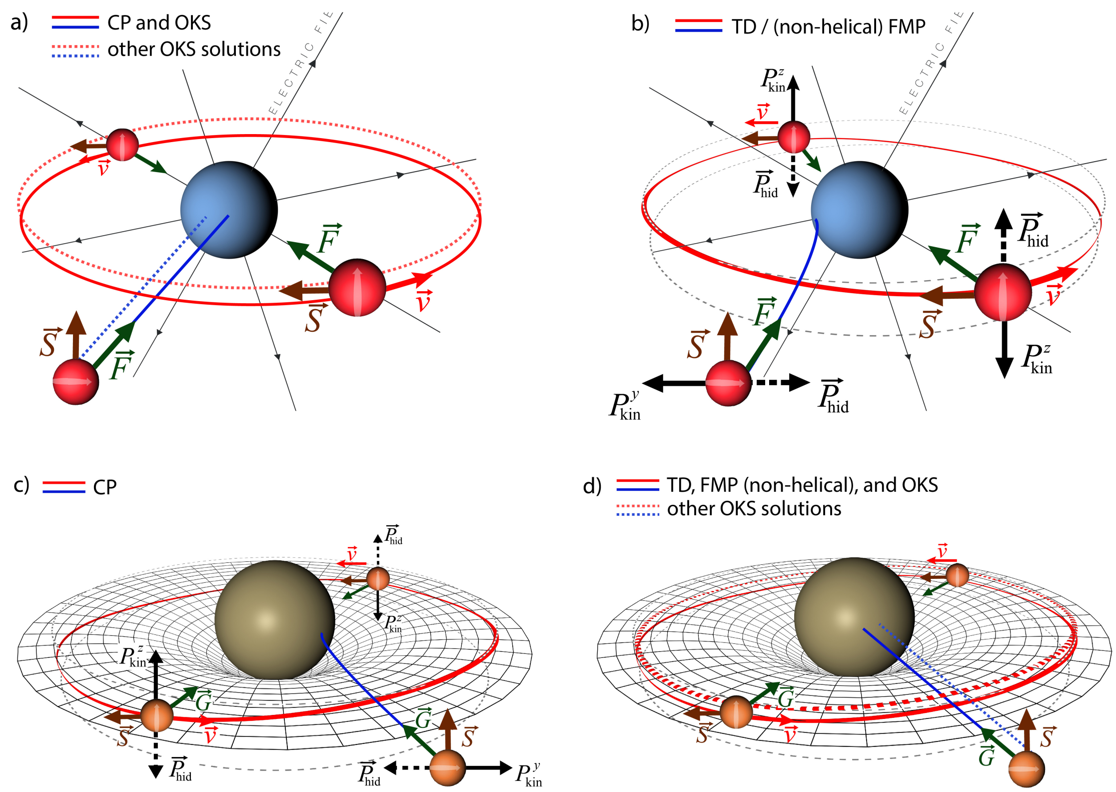

Figure 6: Comparison of different spin conditions ($S^{\alpha\beta}u_\beta = 0$) in two analogous physical systems: a)-b) A spinning charged particle (but with $\vec{\mu} = 0$) orbiting a Coulomb charge in *flat spacetime*; c)-d) a spinning particle in the Schwarzschild spacetime. The CP condition, $u^\alpha = u^\alpha_{\rm lab}$, chooses the centroid as measured by the observers at rest in the background (the "laboratory" frame); the FMP condition, $u^\alpha = U^\alpha$, and the TD condition, $u^\alpha = P^\alpha/M$, choose the centroid as measured in the frame comoving, or nearly comoving (respectively) with it. We consider only the *non-helical* FMP solution. In the electromagnetic system, the CP condition yields $Du^\alpha/d\tau = 0 \Rightarrow P^\alpha = mU^\alpha$ (since the laboratory frame is inertial), there is no hidden momentum nor exotic motions; trajectories are ellipses and, for particles with initial radial velocity, straight lines, cf. Fig. 6a). For the TD/FMP conditions, Fig. 6a), $Du^\alpha/d\tau \neq 0$ since a force $\vec{F}$ acts on the particle, leading to a hidden momentum $\vec{P}_{\rm hid} \simeq -\vec{S}\times\vec{F}/m$ that modifies the trajectories. A bobbing is added to the elliptical trajectories due to the oscillation of $\vec{P}_{\rm hid} = P^z\vec{e}_z$ along the orbit; and instead of a radial motion, the centroid deflects. In the gravitational system the situation is *reversed*: $Du^\alpha/d\tau \approx 0 \Rightarrow P^\alpha_{\rm hid} \approx 0$ for TD/FMP conditions, and it is for the CP condition (since the laboratory observers are accelerated) that there is a hidden momentum $\vec{P}_{\rm hid} \simeq \vec{S}\times\vec{G} \neq 0$. TD/FMP centroids with initial radial velocity move radially, whereas the corresponding CP centroid deflects. $\vec{P}_{\rm hid}$ also induces a bobbing in nearly elliptical orbits (adding to the existing bobbing caused by the spin-curvature force, which is *not* gauge, but has the same form up to a factor of three). In both systems $P^\alpha_{\rm hid}$ and its induced motions are gauged away by the Ohashi-Kyrian-Semerák (OKS) condition; different OKS centroids move nearly parallel to each other.

centroids (corresponding to other OKS solutions, or to other spin conditions), we note that i) these have worldlines $\bar{z}^\alpha(\bar{\tau})$ related to $z^\alpha(\tau)$ by (see Eq. (10))

$$\bar{z}^\alpha = z^\alpha + \Delta x^\alpha \ ; \qquad \Delta\vec{x} \simeq \frac{\vec{S}\times\vec{v}}{m} \ , \tag{46}$$

where $\vec{v}$ is the particle's velocity with respect to the laboratory observers (i.e., $U^\alpha = \gamma(U, u_{\rm lab})(u^\alpha_{\rm lab} + v^\alpha)$, cf. Eq. (9)); ii) the particle's momentum P^α is, to a good approximation, the same for all spin conditions, cf. Appendix A.1.1; and iii) regardless of the reference worldline chosen, to the accuracy at hand, the net force on the body is the same, $\vec{F} = \vec{\bar{F}}$ (unlike one might expect, since the fields are evaluated at different points). Point iii) is explained by arguments analogous to the ones given in Sec. 3.3 for the Newtonian problem: when one changes z^α the particle's moments change as well; if the particle was a monopole with respect to z^α, then about $\bar{z}^\alpha$ it will have an electric dipole moment $\vec{d} = -q\Delta\vec{x}$, as well as higher order moments. Whereas $\vec{F}$ is just the Coulomb force $\vec{F} = q\vec{E}(\mathbf{z})$, to dipole order $\vec{\bar{F}} = q\vec{E}(\bar{\mathbf{z}}) + \bar{F}^i_{\rm dip}$, where $\bar{F}^i_{\rm dip} = \nabla_j E^i d^j + \mathcal{O}(v^2)$ is the force due to the electric dipole, cf. third term of Eq. (14). Hence

$$\bar{F}^i = qE^i(\bar{\mathbf{z}}) + \bar{F}^i_{\rm dip} = qE^i(\mathbf{z}+\Delta\mathbf{x}) - q\nabla_j E^i(\mathbf{z}+\Delta\mathbf{x})\Delta x^j \simeq qE^i(\mathbf{z}) = F^i \ . \tag{47}$$

Therefore, any difference in the acceleration of the two centroids is due solely to the hidden momentum

$$m\vec{\bar{a}} = \vec{\bar{F}} - \frac{D\vec{\bar{P}}_{\rm hid}}{d\tau} = \vec{F} - \frac{D\vec{\bar{P}}_{\rm hid}}{d\tau} = m\vec{a} - \frac{D\vec{\bar{P}}_{\rm hid}}{d\tau} \tag{48}$$

This tells us that different solutions of the OKS condition, for which $\bar{P}^\alpha_{\rm hidI} = 0$ (corresponding to the centroids as measured by observers moving with constant velocity with respect to $u^\alpha_{\rm rest}$), yield different worldlines all (nearly) parallel to the CP condition, since they have the same acceleration, and the same 4-velocity $U^\alpha \parallel P^\alpha$. In particular, other OKS centroids $\bar{z}^\alpha$ corresponding the same physical motion whose description through z^α is radial motion, are non-radial straight lines parallel to z^α; and the $\bar{z}^\alpha$'s for which z^α is a circular motion are (non-concentric, in general non-complanar) circles, obtained from the latter by a constant spatial displacement $\Delta\vec{x}$; see Fig. 6a).

The situation is different if one chooses the Frenkel-Mathisson-Pirani (FMP), $u^\alpha = \bar{U}^\alpha$, or the Tulczyjew-Dixon (TD) condition, $u^\alpha = P^\alpha/M$, which pick as representative point the centroid as measured in the frame comoving, or nearly comoving, respectively, with it. Since a force acts on the particle (implying $DP^\alpha/d\tau \neq 0$ and $\bar{a}^\alpha \neq 0$), it follows that $Du^\alpha/d\tau \neq 0$ and $P^\alpha_{\rm hidI} \neq 0$, and therefore P^α is not is not parallel to $\bar{U}^\alpha$, cf. Eqs. (27) and (29). From Eq. (27) (and noting from (28) that $M = m + \mathcal{O}(S^2)$), we have for the TD condition

$$P^\alpha = m\bar{U}^\alpha + \frac{S^{\alpha\beta}}{m}F_\beta + \mathcal{O}(S^2) = m\bar{U}^\alpha + \frac{1}{m}\epsilon^{\alpha\beta}{}_{\gamma\delta}F_\beta S^\gamma \bar{U}^\delta + \mathcal{O}(S^2) \ . \tag{49}$$

This corresponds also to the momentum of the *non-helical* FMP solution. In order to see that, first take (49) as an ansatz, and observe it obeys, to the accuracy at hand, the FMP equations of motion. Namely, substituting (49) in the explicit equation for the acceleration[9] (36), one gets

$$\bar{a}^{\alpha}=\frac{1}{S^2}\left(\frac{1}{m}F^{\mu}S_{\mu}S^{\alpha}-\frac{S_{\gamma}^{\ \beta}}{m}F_{\beta}S^{\alpha\gamma}\right)+\mathcal{O}(S)=\frac{F^{\alpha}}{m}+\mathcal{O}(S) \tag{50}$$

(in the second equality we noted that since $S_{\gamma}^{\ \beta}S^{\alpha\gamma}=S^{\alpha}S^{\beta}-(h^{U})^{\alpha}_{\ \mu}S^2$ and $F^{\alpha}U_{\alpha}=0$, it follows that $-S_{\gamma}^{\ \beta}S^{\alpha\gamma}F_{\beta}/S^2=F_{\perp}^{\alpha}\equiv$ projection of F^{α} orthogonal to S^{α}); then, substituting (50) in (29), leads consistently to (49). Eq. (50) states that the acceleration comes, in first approximation (i.e., to zeroth order in S), from the force F^{α}, which is what one expects for a non-helical solution (and the rationale for taking (49) as an ansatz for (29)). Note from Eq. (33) that, for a free particle in flat spacetime, the acceleration of the helical motions is $a=\gamma^2 vM/S=\mathcal{O}(S^{-1})\neq\mathcal{O}(S)$. The helices are effectively precluded from moment we impose $P^{\alpha}\simeq m\bar{U}^{\alpha}+S^{\alpha\beta}F_{\beta}/m$.

We can therefore write for the inertial hidden momentum of both the TD or (non-helical) FMP conditions

$$P_{\rm hidI}\simeq\frac{1}{m}\epsilon^{\alpha\beta}_{\ \ \gamma\delta}F_{\beta}S^{\gamma}\bar{U}^{\delta}\ . \tag{51}$$

This hidden momentum leads to exotic motions of the centroid. From Eq. (48),

$$\begin{aligned} m\vec{\bar{a}}=&m\vec{a}-\frac{D\vec{\bar{P}}_{\rm hid}}{d\tau}=m\vec{a}+\frac{1}{m}\vec{S}\times\vec{F}_{,l}\bar{v}^{l}+\mathcal{O}(v^2)\\ =&m\vec{a}+\frac{qQ}{m}\frac{1}{r^3}\left(\vec{\bar{v}}\times\vec{S}-3\frac{(\vec{\bar{v}}\cdot\vec{r})\vec{r}\times\vec{S}}{r^2}\right)\ . \end{aligned} \tag{52}$$

Here $\vec{r}=x\vec{e}_x+y\vec{e}_y$, since $\vec{F}(=\vec{\bar{F}})$ is the Coulomb force in the x_Oy plane.

Nearly circular motion. — Let us start by the motion above whose description through z^{α} (i.e. through the laboratory frame centroid, given by the CP/OKS conditions) was a circular motion in the x_Oy plane. Since we assume that $\vec{S}$ lies on the x_Oy plane, it follows that $\vec{P}_{\rm hidI}=P_{\rm hidI}\vec{e}_{\rm z}$,

$$P^{x}=m\bar{v}^{x}=mv^{x};\qquad P^{y}=m\bar{v}^{y}=mv^{y};\qquad P^{z}=m\bar{v}^{z}+P_{\rm hidI}$$

(where we noted that since P^{α} is the same regardless of the spin condition, the components of the centroid velocity in the x_Oy plane are the same as for the CP condition:

[9] In [13], where Eq. (36) was originally derived, F^{α} was taken to be the spin-curvature force (39); it is however easy to check, following the derivation therein, that it holds for an arbitrary force, as long as Eq. (29) holds.

$\bar{v}^x = v^x$, $\bar{v}^y = v^y$). Therefore, $\vec{\bar{v}} \cdot \vec{r} = 0$, $m\bar{a}^x = ma^x = F^x$, $m\bar{a}^y = ma^y = F^y$, and

$$m\bar{a}^z = \frac{qQ}{m}\frac{1}{r^3}(\vec{v} \times \vec{S})^z \ , \tag{53}$$

cf. Eq. (52). Thus, the projection of the motion in the x_Oy plane is circular, identical to z^α; and since $\vec{S}$ is constant to this accuracy[10], $\bar{a}^z$ oscillates between positive and negative values along the orbit, leading to a bobbing motion, depicted in Fig. 6b). This bobbing can easily be understood as follows. The particle's total momentum along z is constant (since there is no force along z, $DP^z/d\tau = 0$), and equal to zero, as one can see (since P^α is the same) from the results above of the CP/OKS conditions. On the other hand, from Eq. (51), there is a hidden momentum along z, $P^z_{\rm hid} = (\vec{S} \times \vec{F})^z/m$, oscillating along the orbit (from $P^z_{\min} = -SF/m$ to $P^z_{\max} = +SF/m$); this means that the centroid bobs up and down in order for the kinetic momentum $m\bar{v}^z$ to cancel out $P^z_{\rm hid}$, keeping $P^z = 0$. Without loss of generality, we may take $\vec{S} = S\vec{e}_x$, $v^y = v\cos\omega\tau$, and thus $(\vec{v} \times \vec{S})^z = -vS\cos\omega\tau$; integrating $\ddot{\bar{z}} = \bar{a}^z$, Eq. (53), and noticing that $m\omega^2 r = Qq/r^2$ (as the motion is circular in the x_Oy plane), we obtain $z = (vS/m)\cos\omega\tau$, describing oscillations of half amplitude vS/m.

Nearly radial motion. — As we have seen above, for the CP or the OKS conditions, it follows from Eqs. (45) that a particle with radial initial velocity will move radially, regardless of its spin. Take the case that the particle is dropped from rest at some point on the x axis; it will move in straight line along x towards the source, and we thus have that $P^y = P^z = 0$, $P^x = P$. Take $\vec{S}$ to be along z. For the FMP or the TD conditions the situation is different; there is a hidden momentum, given by Eq. (51),

$$\vec{P}_{\rm hid} = -\frac{1}{m}\vec{S} \times \vec{F} = \frac{1}{m}SF\vec{e}_y$$

(increasing as the particle approaches the source, since F increases), which causes the centroid to deflect in the negative y direction, in order to keep $P^y = P^y_{\rm hid} + m\bar{U}^y = 0$. This is depicted in Fig. 6b).

3.4.2 Gravitational system

In the gravitational case, the situation is reversed in comparison to the electromagnetic system: now it is the "laboratory frame" (i.e., the observers at rest in the background) which is accelerated, therefore $\nabla_{\mathbf{U}} u^\alpha_{\rm rest} \neq 0$ and $P^\alpha \nparallel U^\alpha$ when the centroid is computed in such frame, cf. Eq. (26); and it is when the centroid is computed in the comoving (FMP condition) or nearly comoving frame (TD condition) that we have $P \simeq mU^\alpha$, since the only force present is the spin curvature force (39), which yields a $\mathcal{O}(S^2)$

[10]Since $\tau^{\alpha\beta} = 0$, S^α is Fermi-Walker transported for the MP condition, Eq. (30), and approximately so for the TD condition (cf. Eq. (7.11) of [25]); since Eq. (53) is of first order in v, $\vec{S}$ can be taken constant therein.

contribution to the hidden momentum, cf. Eq. (49). This is what we are now going to see in detail.

As we have seen in Sec. 3.3, the force (39) depends explicitly on the spin condition. For the FMP and the TD conditions, it can be written to lowest order as [23]

$$\bar{F}^i = -2\epsilon_{jkl}v^k G^{l,i} S^j + \epsilon^i{}_{lj} G^l{}_{,k} v^k S^j \simeq m\bar{a}^i \ , \tag{54}$$

where $\vec{G} = -m_S \vec{r}/r^3$ is the Newtonian (or gravitoelectric) field evaluated at $\bar{z}^\alpha$, m_S is the mass of the Schwarzschild black hole, and in the second equality we used $\vec{P} = m\vec{\bar{U}} + \mathcal{O}(S^2) \Rightarrow m\vec{\bar{a}} \simeq \vec{\bar{F}}$, as follows from (49). Explicitly:

$$m\vec{\bar{a}} \simeq \vec{\bar{F}} = -\frac{3m_S}{r^3}\left[\vec{v}\times\vec{S} + \frac{2\vec{r}[(\vec{v}\times\vec{r})\cdot\vec{S}]}{r^2} + \frac{(\vec{v}\cdot\vec{r})\vec{S}\times\vec{r}}{r^2}\right] \ . \tag{55}$$

Notice the first term, formally analogous to the first term of (52), which caused the bobbing in the electromagnetic system; but note as well that despite the similarity, they have very different origins: the latter comes from the inertial hidden momentum, whereas the former comes from the spin-curvature force.

The coordinate acceleration is given by the sum of $m\vec{\bar{a}}$ with the (radial) Newtonian "force" $m\vec{G}$,

$$m\frac{d^2\bar{z}^i}{d\bar{\tau}^2} = mG^i(\bar{\mathbf{z}}) + \bar{F}^i \ .$$

For the CP condition the situation is different, because this is now the case where the field $u^\alpha = u^\alpha_{\rm lab}$ (relative to which the centroid is computed) is not parallel transported along $z^\alpha(\tau)$, $\nabla_{\mathbf{U}} u^\alpha_{\rm lab} \neq 0$; therefore there is hidden momentum (cf. Eqs. (18) and (26)):

$$P^\alpha_{\rm hidI} = -(h^U)^\alpha{}_\sigma S^\sigma{}_\beta G^\beta = -(h^U)^\alpha{}_\sigma \epsilon^\sigma{}_{\beta\gamma\delta} G^\beta S^\gamma u^\delta_{\rm lab} \quad \Rightarrow \quad \vec{P}_{\rm hidI} \simeq \vec{S}\times\vec{G} \ . \tag{56}$$

The spin-curvature force takes also a different form with this condition,

$$F^i = -\epsilon_{jkl}v^k G^{l,i} S^j + \epsilon^i{}_{lj} G^l{}_{,k} v^k S^j \tag{57}$$

(notice the relative factor of 2 comparing the first terms of (54) and (57)). The latter difference however is compensated by the difference between $\vec{G}(\mathbf{z})$ and $\vec{G}(\bar{\mathbf{z}})$, as explained in Sec. 3.3: $G^i(z) \simeq G^i(\bar{z}) - G^{i,j}\Delta x_j$, with $\Delta\vec{x} = \vec{S}\times\vec{v}/m$, cf. Eqs. (46); that is, $G^i(z) \simeq G^i(\bar{z}) - \epsilon_{jkl}S^j v^k G^{l,i}/m$, and therefore $mG^i(\mathbf{z}) + F^i = mG^i(\bar{\mathbf{z}}) + \bar{F}^i$. The coordinate acceleration is thus given by

$$m\frac{d^2 z^i}{d\tau^2} = mG^i(\mathbf{z}) + F^i - \frac{DP^i_{\rm hid}}{d\tau} = m\frac{d^2\bar{z}^i}{d\bar{\tau}^2} - \frac{DP^i_{\rm hid}}{d\tau} \ . \tag{58}$$

That is, the coordinate acceleration of the CP worldline $z^\alpha(\tau)$ differs from that of the worldline $\bar{z}^\alpha(\bar{\tau})$ of the TD/FMP conditions only by the hidden momentum term involved in the former. From (56) we have

$$\frac{DP^i_{\rm hid}}{d\tau} \simeq P^\alpha_{{\rm hid},i}v^i = \epsilon^i{}_{jl}S^jG^l{}_{,k}v^k = \frac{m_{\rm S}}{r^3}\left(\vec{v}\times\vec{S} + 3\frac{(\vec{v}\cdot\vec{r})\vec{S}\times\vec{r}}{r^2}\right) \tag{59}$$

where we used the fact that, to this accuracy, $D\vec{S}/d\tau \approx 0$.

Nearly circular motion. — As in the electromagnetic case, we assume $\vec{S} \in x_O y$, and so, for a nearly circular orbit, $(\vec{v}\times\vec{r})\cdot\vec{S} \simeq 0$, $\vec{v}\cdot\vec{r} \simeq 0$; therefore, the second term of (55) and the last term of (55) and (59) vanish. We have thus for the FMP and TD conditions

$$m\frac{d^2\bar{z}}{d\bar{\tau}^2} = mG^z(\bar{\mathbf{z}}) - \frac{3m_{\rm S}}{r^3}(\vec{v}\times\vec{S})^z \ ,$$

and for the CP condition

$$m\frac{d^2z}{d\tau^2} = m\frac{d^2\bar{z}}{d\bar{\tau}^2} - \frac{m_{\rm S}}{r^3}(\vec{v}\times\vec{S})^z \ .$$

Both coordinate accelerations oscillate along the orbit, due to the terms $\vec{v}\times\vec{S}$ (since $\vec{S}$ is approximately constant), leading to a bobbing motion. Hence, by contrast with the electromagnetic system, in this case a bobbing is present regardless of the spin condition (or the presence of hidden momentum); it is just larger for the CP condition, because the contribution for the bobbing from the hidden momentum adds to the bobbing caused by the spin-curvature force (they have the same form, only different factors).

Nearly radial motion. — For a particle in radial motion in Schwarzschild spacetime, the spin-curvature force under the FMP/TD conditions is *exactly* zero, $\bar{F}^\alpha = DP^\alpha/d\bar{\tau} = 0$, as shown in [23] (in the weak field and slow motion regime, one can check that from Eq. (55) above, by noting that the second term is zero, and the first and third terms cancel out when $\vec{r} \parallel \vec{v}$). The hidden momentum is also exactly zero for the TD and the non-helical FMP solutions, so $P^\alpha = m\bar{U}^\alpha$, and thus $D\bar{U}^\alpha/d\bar{\tau} = 0$. When dropped from rest, the particle moves along a geodesic towards the source. Take the motion to be along the x axis, so that $P^y = P^z = 0$, $P^x = P$, and take $\vec{S}$ along z.

For the worldline $z^\alpha(\tau)$ given by the CP condition, there will be a non-vanishing spin-curvature force, cf. Eq. (43); which, as shown above and in Sec. 3.3, just compensates for the difference in the Newtonian field $\vec{G}$ on the two worldlines, so that the coordinate acceleration differs only due to the hidden momentum terms, cf. Eqs. (44) and (58). Since the momentum P^α is the same regardless of the spin condition, the hidden momentum (56) that arises with this spin condition causes the centroid z^α to deflect in the y direction as it approaches the source, in order to keep

$$P^y = mU^y + (\vec{S}\times\vec{G})^y = 0 \ ,$$

just like the situation in the electromagnetic system for the FMP and the TD condition. This is depicted in Fig. 6d). Hence, the situation is *opposite* to the electromagnetic analogue: for the FMP and TD conditions we have no hidden momentum, and $\bar{z}^\alpha$ has straight line radial motion; for the CP condition there is hidden momentum and a centroid that deflects from radial motion.

Finally, if one takes solutions $z'^\alpha(\tau')$ of the OKS condition, other than the one that (to this accuracy) coincides with the centroid z^α of the FMP and TD conditions, we have from (44) $d^2z'^i/d\tau^2 = d^2z'^i/d\tau^2$, and thus $z'^\alpha(\tau')$ are curves that run approximately parallel to the trajectories of the TD/FMP (non-helical) conditions.

3.5 Hidden momentum arising from the "torque" tensor $\tau^{\alpha\beta}$

In this section we briefly discuss the hidden momentum (19) that is related to the torque tensor $\tau^{\alpha\beta}$. It is useful to split

$$\tau^{\alpha\beta} = \tau^{\alpha\beta}_{\rm DEM} + \tau^{\alpha\beta}_{\rm QEM} + \tau^{\alpha\beta}_{\rm QG} + ... \tag{60}$$

where [11]

$$\tau^{\alpha\beta}_{\rm DEM} = 2\mu^{\theta[\beta}F^{\alpha]}{}_{\theta} + 2d^{[\alpha}F^{\beta]}{}_{\gamma}U^\gamma \tag{61}$$

is the electromagnetic dipole torque, $\tau^{\alpha\beta}_{\rm QEM}$ and $\tau^{\alpha\beta}_{\rm QG}$ are, respectively, the quadrupole electromagnetic and gravitational torques (the lowest order torque in the gravitational case), see [23] for the explicit expressions. All these torques (plus the higher order ones) will contribute to the momentum via Eqs. (17)-(19). A hidden momentum $P_{{\rm hid}\tau}$ is originated whenever $\tau^{\alpha\beta}$ has a component along the vector field u^α, cf. Eq. (19), and it may be cast into two parts: a part which is is pure gauge like the inertial hidden momentum $P^\alpha_{\rm hidI}$ (comes from the choice of the reference worldline $z^\alpha(\tau)$; may be made to vanish by suitable choices), and another part, which arises in some physical systems, that is not gauge. Let us discuss these two subtypes of hidden momentum separately.

3.5.1 The pure gauge hidden momentum that arises from $\tau_{\alpha\beta}$

This contribution is easier to understand if we think about a simple example. Consider a spinning particle in flat spacetime as depicted in Fig. 1, with no forces ($DP^\alpha/d\tau = 0$), but now under a torque. Consider moreover $\tau^{\alpha\beta}$ to be spatial and orthogonal[11] to P^α: $\tau^{\alpha\beta}P_\beta = 0$. In this case, just like for a torque-free particle, the centroid $x^\alpha_{\rm CM}(P)$ ("$x_{\rm CM}$" in Fig. 1), given by the condition $S^{\alpha\beta}P_\beta = 0$, is *at rest* in the $P^i = 0$ frame (note that $P^\alpha \parallel U^\alpha$ for the reference worldline $z^\alpha = x^\alpha_{\rm CM}(P)$, as follows from Eq. (16) with $u^\alpha = P^\alpha/M$). Since $x^\alpha_{\rm CM}(P)$ is unaffected by the torque, it remains at rest at the body's geometrical center, regardless of the fact that the spin of the particle is

[11] e.g., the torque on an electric dipole in an uniform electromagnetic field, when z^α is the common centroid given by the TD or the (non-helical) FMP condition.

varying. Now consider another *inertial* observer (4-velocity $\bar{u}^\alpha$) moving with respect to the $P^i = 0$ frame with *constant* velocity $\vec{v}$ (so that $D\bar{u}^\alpha/d\tau = 0$, ensuring that no inertial hidden momentum comes into play); not only the centroid $x^\alpha_{\rm CM}(\bar{u})$ it measures is shifted to the right relative to $x^\alpha_{\rm CM}(P)$, as depicted in Fig. 1, as *the shift* (10)-(12) *also varies*, since $S^\alpha_\star$ varies due to the torque:

$$\frac{D\Delta x^\alpha}{d\tau} = -\frac{1}{m(\bar{u})}\frac{DS^{\alpha\beta}_\star}{d\tau}\bar{u}_\beta = -\frac{1}{m(\bar{u})}\tau^{\alpha\beta}\bar{u}_\beta \tag{62}$$

(i.e., the body's rotation velocity varies, causing Δx^α to vary). This means that the centroid $\bar{z}^\alpha = x^\alpha_{\rm CM}(\bar{u})$ will be moving in the $P^i = 0$ frame; i.e., its 4-velocity $\bar{U}^\alpha = d\bar{z}^\alpha/d\bar{\tau}$ will have a component orthogonal to P^α, which reads (in this special case that $DP^\alpha/d\tau = 0$, so that Eqs. (24) hold) $\bar{U}^\alpha_\perp = D\Delta x^\alpha/d\bar{\tau}$.

In the general case when there are forces acting on the particle, however, as already mentioned in Sec. 3.2, one should not think of $\bar{U}^\alpha_\perp$ as the velocity of the centroid $\bar{z}^\alpha$ relative to $z^\alpha = x^\alpha_{\rm CM}(P)$, because the latter is *not* at rest in the $P^i = 0$ frame. The general argument should be given instead as: *the position of the centroid* $\bar{z}^\alpha = x^\alpha_{\rm CM}(\bar{u})$ *as measured by a given observer* $\bar{u}^\alpha$ *depends on the body's angular momentum*; *when the latter varies due to the action of a torque,* $x^\alpha_{\rm CM}(\bar{u})$ *moves accordingly;* P^α*, however, is unaffected, leading to* $P^\alpha \nparallel U^\alpha$. The general (with $D\bar{u}^\alpha/d\tau = 0$) expression for $\bar{U}^\alpha_\perp$ formalizing this statement follows from Eqs. (16) and (22)[12]:

$$\bar{U}^\alpha_\perp = -\frac{1}{m(\bar{u})}(h^P)^\alpha_{\ \sigma}\bar{\tau}^{\sigma\beta}\bar{u}_\beta\ . \tag{63}$$

Finally, if $U^\alpha_\perp \neq 0$, then $P^\alpha_{\rm hid} \neq 0$ — i.e., when the centroid moves in the $P^i = 0$ frame, the momentum P^i is not zero in the centroid frame (the $\bar{U}^i = 0$ frame); thus there is hidden momentum, the two effects being reciprocal (and mere consequences of the fact that $P^\alpha \nparallel \bar{U}^\alpha$), cf. Eqs. (23).

3.5.2 "Dynamical" hidden momentum

In general, the momentum of a multipole particle subject to electromagnetic and gravitational fields is not parallel to its 4-velocity regardless of the spin condition; that happens when $\tau^{\alpha\beta}$ is not a spatial tensor (i.e., when $\tau^{\alpha\beta}u_\beta \neq 0$ for all unit timelike vectors u^β), and is related to a type of hidden momentum which occurs in some physical systems and is *not* gauge. Following [17], we dub this part of $P^\alpha_{{\rm hid}\tau}$ the "dynamical" hidden momentum. To dipole order, it arises in magnetic dipoles; let us then consider the case when $\tau^{\alpha\beta} = 2\mu^{\theta[\beta}F^{\alpha]}_{\ \ \theta}$ in Eqs. (60)-(61). Take the magnetic dipole moment

[12] To obtain (62) from (63) in the special case above, one uses $d\tau = \gamma(\bar{U},P)d\bar{\tau}$ to write $\bar{U}^\alpha_\perp = \gamma(\bar{U},P)D\Delta x^\alpha/d\tau$, computes $D\bar{S}^{\alpha\beta}/d\bar{\tau}$ from (42) using (15) to obtain $\bar{\tau}^{\alpha\beta} = \gamma(\bar{U},P)\tau^{\alpha\beta}$, and finally uses the assumption above $\tau^{\alpha\beta}P_\beta = 0 \Rightarrow (h^P)^\alpha_{\ \sigma}\bar{\tau}^{\sigma\beta} = \bar{\tau}^{\alpha\beta}$.

to be proportional to the spin, $\mu^{\alpha\beta} = \sigma S^{\alpha\beta}$; we have from (19), for an arbitrary spin condition $S^{\alpha\beta}u_\beta = 0$,

$$P^\alpha_{\text{hid}\tau} = -\frac{1}{\gamma(u,U)}(h^U)^\alpha_{\ \sigma}\mu^{\sigma\beta}(E^u)_\beta \equiv P^\alpha_{\text{hidEM}} \ ,$$

where $(E^u)^\alpha = F^\alpha_{\ \beta}u^\beta$. If $(E^u)^\alpha \nparallel \mu^\alpha$, P^α_{hidEM} can never be zero, because then $\mu^{\beta\sigma}(E^u)_\beta \neq 0$ and is a space-like vector, thus cannot be parallel to any U^α. For the FMP condition ($u^\alpha = U^\alpha$), P^α_{hidEM} takes the suggestive form

$$P^\alpha_{\text{hidEM}} = -\mu^{\alpha\beta}E_\beta = \epsilon^\alpha_{\ \beta\gamma\delta}\mu^\beta E^\gamma U^\delta \ , \tag{64}$$

where $E^\alpha = F^\alpha_{\ \beta}U^\beta$ is the electric field *as measured in the centroid frame.* In such frame, and in vector notation, $\vec{P}_{\text{hidEM}} = \vec{\mu}\times\vec{E}$, which the most usual form in the literature for the hidden momentum that a magnetic dipole acquires under an external electromagnetic field (e.g. [19, 20, 21, 22]). It equals *minus* the electromagnetic field momentum $\vec{P}_\times$ generated by a magnetic dipole when placed in the external electromagnetic field, which, in the particle's frame, reads (see e.g. [19, 22, 23]) $\vec{P}_\times = \int \vec{E}\times\vec{B}_{\text{dipole}} = -\vec{\mu}\times\vec{E}$. It should however be noted that $\vec{P}_{\text{hidEM}}$ is *not* field momentum; it is *purely mechanical in nature*, which can be understood through simple models, see e.g. [19, 22] (in particular Fig. 9 of [22]). Such momentum plays an important role in the conservation laws. Consider, for example, a magnetic dipole *at rest* in an external electric field; since no force is exerted on the particle, the setup is *stationary*; it follows from the conservation equations $(T_{\text{tot}})^{\alpha\beta}_{\ \ ;\beta} = 0$ that the total spatial momentum $\vec{P}_{\text{tot}} \equiv \vec{P}_{\text{matter}} + \vec{P}_{\text{EM}}$ (i.e., the matter momentum plus the field momentum) must vanish. The momentum of the electromagnetic field, $\vec{P}_{\text{EM}} = \vec{P}_\times$, however, is not zero; it is the momentum $\vec{P}_{\text{matter}} = \vec{P}_{\text{hidEM}} = -\vec{P}_{\text{EM}}$, hidden in the dipole, that cancels out $\vec{P}_{\text{EM}}$, ensuring $\vec{P}_{\text{tot}} = 0$, as required by the conservation laws.

P^α_{hidEM} also leads to exotic motions, quite analogous to the ones coming from the inertial hidden momentum studied in Sec. 3.4, as one would expect from the formal analogy between (64) and the the inertial hidden momentum under this spin condition, $P_{\text{hidI}} = -\epsilon^\alpha_{\ \beta\gamma\delta}S^\beta a^\gamma U^\delta$, cf. Eq. (29). Indeed, if in the application in Fig. a)-b) we considered particles with dipole moment $\mu^\alpha = \sigma S^\alpha \neq 0$, there would be a bobbing (in addition to the one caused by P^α_{hidI}) for a particle orbiting the source, and, in the case of a particle in *initially* radial motion, there would be a sideways dipole force on it, but due to P^α_{hidEM} the particle's sideways acceleration would actually be *opposite* to the force. This effect is discussed in detail in [23]. However, a crucial difference exists between these effects and the effects discussed in the previous sections: the hidden momentum in Eq. (64) is *not* gauge, nor the motions generated by it are (in general) made to vanish by any choice of center of mass.

4 Conclusion

In this paper we have discussed and compared in detail the different spin supplementary conditions in the literature, with special attention being given to the lesser-studied (but potentially useful) Corinaldesi-Papapetrou (CP) and Ohashi-Kyrian-Semerák (OKS) spin conditions. One of the main points is that the different solutions allowed by the different spin conditions are equivalent descriptions of the motion of a given body. We have shown this equivalence to pole-dipole order, explaining the change of the spin-curvature force under the different conditions — which is seen to be precisely what ensures the consistency of the different solutions, as it has the magnitude needed to prevent the worldlines from deviating due to tidal effects of a curved spacetime. This builds up on the work in [16] (dealing with free particles in flat spacetime) and backs the claims in [13] about the equivalence of all spin conditions in a curved spacetime.

We clarified the origin of the non-parallelism between U^α and P^α, which can be cast as the particle possessing a "hidden momentum", a concept introduced in General Relativity in [17], and further developed herein. It consists of two main parts: an "inertial" part $P^\alpha_{\rm hidI}$ that arises solely from the spin condition (i.e., from the choice of the observers relative to which the center of mass is measured), which we therefore cast as gauge, and another term $P^\alpha_{{\rm hid}\tau}$ arising from the torque tensor $\tau_{\alpha\beta}$, which generically subdivides into a part that again is gauge (arising from the motion of the centroid measured by *some* observers that is induced when $S^{\alpha\beta}$ varies due to $\tau_{\alpha\beta}$), and a "dynamical" part, which is not gauge. The latter, to dipole order, consists of a form of hidden momentum that arises in electromagnetic systems, and was previously known from treatments in classical electrodynamics.

The differences between the various spin conditions were discussed and illustrated with suitable examples; in particular the reciprocity (first noted in [17]) that exists when one compares spinning particles under an electromagnetic field in flat spacetime to spinning particles in a gravitational field: in the first case, when one picks the centroid as measured in the Laboratory frame (corresponding to the CP/OKS conditions), there is no inertial hidden momentum, $P^\alpha_{\rm hidI} = 0$, and thus (if $P^\alpha_{{\rm hid}\tau} = 0$) the momentum velocity relation is simply $P^\alpha = mU^\alpha$; and when one computes the center of mass in the comoving frame (FMP/TD conditions), P^α is no longer parallel to U^α, leading to exotic motions (like bobbings). In the gravitational case the situation is reversed: when one chooses the TD or the (non-helical) FMP conditions, P^α is approximately parallel to U^α; and it is when one chooses the Laboratory centroid (CP condition) that hidden momentum arises.

All the spin conditions studied present interesting features. The CP condition yields a natural description, as it amounts to compute the centroid in the same frame where the motion is observed (the "Laboratory" frame, which is given in advance); it leads however to considerable superfluous motions in gravitational systems. The TD condition defines always an unique center of mass, which is the central worldline of the worldtube of centroids (can thus be thought of as describing the "bulk" motion of such worldtube). The FMP condition yields the most natural transport law for the spin vector, and also gives rise to exact gravito-electromagnetic analogies (see [23]); however it is not always easy to single out the non-helical solution from the (infinite) helical solutions allowed by this condition (the latter should be avoided, as they are but unnecessarily complicated descriptions of the motion, as discussed in Sec. 3.2.3), and no general prescription for that is known. As for the OKS condition, it always gauges away the inertial hidden momentum and its induced motions, ensuring the simplest equations for the centroid motion; in the absence of torques, one has $F^\alpha = ma^\alpha$, i.e., these are Newtonian-like (or "dynamical") centroids, which accelerate only if there is a force.

It is however crucial to notice that in spite of the equivalence of the descriptions, and the fact that trajectories of the different spin conditions are contained within the (convex hull of the) body's worldtube, their differences, and the superfluous motions induced by some of them are not negligible (even in weak field, slow motion approximations), and should not be overlooked. As it is also important to distinguish these motions from the physical effects. For, as we have exemplified in Sec. 3.4, the pure gauge contribution to the centroid acceleration with the CP condition is of the same order of magnitude as the one from the spin-curvature force itself; and it can actually be much larger, as is the case of the acceleration of the outer helical solutions of the FMP condition, which can be made arbitrarily large.

A Momentum and angular momentum in curved spacetime

In rectangular coordinates in flat spacetime, the momenta P^α and $S^{\alpha\beta}$ of an extended body, as measured by some observer of 4-velocity u^α, are well defined by the integrals

$$P^\alpha = \int_{\Sigma(z,u)} T^{\alpha\beta} d\Sigma_\beta; \qquad S^{\alpha\beta} = 2\int_{\Sigma(z,u)} r^{[\alpha} T^{\beta]\gamma} d\Sigma_\gamma,$$

where $\Sigma(z,u)$ is the hyperplane orthogonal to u^α (the rest space of u^α), and $r^\alpha = x^\alpha - z^\alpha$ is the vector connecting the reference worldline z^α to the point of integration x^α. In curved spacetime the situation is different, as these integrals amount to summing tensors defined at different points; different generalizations of the flat spacetime notions have been proposed in the literature (e.g. [11, 24, 25]), none of them seeming a priori more natural than the others. Herein we discuss the mathematical meaning of the definitions used in this work, and how they relate to the schemes by Dixon [11, 25].

All schemes agree on generalizing $\Sigma(z,u)$ by the *geodesic* hypersurface orthogonal to u^α, and on replacing r^α by the vector $\mathbf{X} \in \mathcal{T}_z$ tangent to the geodesic connecting z^α and x^α, and whose length equals that of the geodesic. That is, $\mathbf{X} = \Phi(x)$, where $\Phi \equiv \exp_z^{-1}$ is the *inverse* exponential map, mapping points in the spacetime manifold to vectors in the tangent space $\mathcal{T}_z$, $\Phi : \mathcal{M} \to \mathcal{T}_z$. Where the schemes differ is in the way the vector $\mathcal{A}^\alpha \equiv T^{\alpha\beta} d\Sigma_\beta$ is integrated. We adhere to the scheme proposed in [24]: using the natural map for tensors induced by $\exp_z$ to pull back the energy-momentum tensor and the volume element to $\mathcal{T}_z$, and integrate therein, which is then a well defined tensor operation. Let $\mathbf{\Omega}^{\hat{\alpha}}$ denote an orthonormal co-frame on $\mathcal{T}_z$; the moments can then be written in the manifestly covariant form

$$\mathbf{P}(\mathbf{\Omega}^{\hat{\alpha}}) = \int_{\Sigma(z,u)} \mathbf{T}(\Phi^* \mathbf{\Omega}^{\hat{\alpha}}, d\mathbf{\Sigma}) \ ; \tag{65}$$

$$\mathbf{S}(\mathbf{\Omega}^{\hat{\alpha}}, \mathbf{\Omega}^{\hat{\beta}}) = 2 \int_{\Sigma(z,u)} \mathbf{X}(\mathbf{\Omega}^{[\hat{\alpha}}) \mathbf{T}(\Phi^* \mathbf{\Omega}^{\hat{\beta}]}, d\mathbf{\Sigma}) \ . \tag{66}$$

Note that since $\mathbf{T}(\Phi^* \mathbf{\Omega}^{\hat{\alpha}}, d\mathbf{\Sigma}) = (\exp_z^* \mathbf{T})(\mathbf{\Omega}^{\hat{\alpha}}, \exp_z^* d\mathbf{\Sigma})$, one is indeed pulling back the integrands from $\mathcal{M}$ to $\mathcal{T}_z$. Note also that Eqs. (65)-(66) are *equivalent* to (1)-(2), i.e., they just amount to perform the integration in a system of Riemann normal coordinates $\{x^{\hat{\alpha}}\}$ centered at z^α (the coordinates naturally adapted to the exponential map). This is because such system is constructed from geodesics radiating out of z^α; thus the components of $\mathbf{X}$, in global Lorentz coordinates in $\mathcal{T}_z$, are equal to the coordinates $x^{\hat{\alpha}}$ on $\mathcal{M}$; also the basis 1-forms of such system are the pullbacks of $\mathbf{\Omega}^{\hat{\alpha}}$ to $\mathcal{M}$, $dx^{\hat{\alpha}} = \Phi^* \mathbf{\Omega}^{\hat{\alpha}}$; and, taking it comoving with u^α (i.e., $u^\alpha = \mathbf{e}_{\hat{0}}$), $\Sigma(z,u)$ coincides with the spatial hypersurface $x^{\hat{0}} = 0$.

Let us now compare these definitions with other schemes in the literature. In [11], P^α and $S^{\alpha\beta}$ are defined as

$$P^\kappa_{\text{Dix}} = \int_{\Sigma(z,u)} \bar{g}_\alpha^{\ \kappa} T^{\alpha\beta} d\Sigma_\beta; \qquad S^{\kappa\lambda}_{\text{Dix}} = -2 \int_{\Sigma(z,u)} \sigma^{[\kappa} \bar{g}_\alpha^{\ \lambda]} T^{\alpha\beta} d\Sigma_\beta, \tag{67}$$

where $\sigma^\kappa(x,z) = -(\Phi(x))^\kappa = -X^\kappa$, cf. [44]. These definitions thus differ from (65)-(66) only in the way the vector $\mathcal{A}^\alpha \equiv T^{\alpha\beta} d\Sigma_\beta$ is integrated: $\bar{g}_\alpha^{\ \kappa}$ is a bitensor which parallel transports $\mathcal{A}^\alpha$ at x^α to z^κ along the geodesic connecting the two points, so that the integral is performed over vectors $\mathcal{A}^\kappa|_z = \bar{g}_\alpha^{\ \kappa} \mathcal{A}^\alpha|_x$ defined at z^κ (in [25, 44] different propagators, $K_\alpha^{\ \kappa}$, $H_\alpha^{\ \kappa}$ in the notation therein, are employed; the two schemes are not equivalent though, as noted in [25]). Writing $\bar{g}_{\hat{\beta}}^{\ \hat{\alpha}} \mathcal{A}^{\hat{\beta}}|_x = \mathcal{A}^{\hat{\alpha}}|_x + \Delta\mathcal{A}^{\hat{\alpha}}$, with $\Delta\mathcal{A}^{\hat{\alpha}} = -\int_x^z \Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}}(x') \mathcal{A}^{\hat{\beta}} dx'^{\hat{\gamma}}$, expanding the integrand in Taylor series around z^α, and noting that, in the normal coordinates $\{x^{\hat{\alpha}}\}$ (see e.g. [26]), we have $\Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}}(z) = 0$ and $\|\Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma},\hat{\delta}}(z)\| \sim \|\mathbf{R}\|$, where $\|\mathbf{R}\| \equiv \sqrt{R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}}$ denotes the magnitude of the curvature, we have $\Delta\mathcal{A}^\alpha = \mathcal{O}(\|\boldsymbol{\mathcal{A}}\| \|\mathbf{R}\| x^2)$. Therefore $P^{\hat{\alpha}}_{\text{Dix}} = P^{\hat{\alpha}} + \mathcal{O}(\lambda \|\mathbf{P}\|)$, where

$$\lambda = \|\mathbf{R}\| a^2 \ , \tag{68}$$

and a is the largest dimension of the body. Thus, when $\lambda \ll 1$, i.e., when the curvature is not too strong compared to the scale of the size of the body[13], $P^{\hat{\alpha}}_{\rm Dix} \simeq P^{\hat{\alpha}}$. The two schemes are actually indistinguishable in a pole-dipole approximation, where only terms to linear order in x are kept in the integrals defining the moments; the resulting equations of motion are the same (compare Eqs. (43), (49) of [24] with Eqs. (6.31)-(6.32) of [11], or Eqs. (7.1)-(7.2) of [25]), both schemes leading to the well known Mathisson-Papapetrou equations (the latter derived using less sophisticated formalisms). These conclusions are natural, for the metric in Riemann normal coordinates is (e.g. [26]) of the form $g_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} + \mathcal{O}(\|\mathbf{R}\|x^2)$; hence the assumption $\lambda \ll 1$ amounts to say that, *for the computation* of P^α and $S^{\alpha\beta}$, one may, to a good approximation, take the spacetime as nearly flat throughout the body.

A.1 The dependence of the particle's momenta on Σ

The momenta (1)-(2) depend, in general, on the spacelike hypersurface $\Sigma(z,u) \equiv \Sigma(z(\tau),u)$ on which the integration is performed, see e.g. [11, 24, 25]. This is so even in flat spacetime; when forces and torques act on the body, it is clear that $P^\alpha(z,u)$, $S^{\alpha\beta}(z,u)$ depend on $z^\alpha(\tau)$, and also on the argument u^α of Σ. Curvature brings additional complications, as u^α is no longer a "free vector", and Σ itself is in principle point dependent. Herein we shall show that, in the absence of electromagnetic field ($F^{\alpha\beta} = 0$), and under the assumption $\lambda \ll 1$ made above, for hypersurfaces $\Sigma(z,u)$ *through a point* z^α *within the body*'s convex hull, the u^α dependence of the momentum and angular momentum is negligible.

Denote by $\boldsymbol{\xi} = dx^{\hat{\alpha}}$ a particular basis 1-form of the Riemann normal coordinate system $\{x^{\hat{\alpha}}\}$; $P^{\boldsymbol{\xi}} \equiv P^\alpha \xi_\alpha$ is thus the $\boldsymbol{\xi}$ component of P^α. From definition (1), and since ξ_α has constant components, we may write the $\boldsymbol{\xi}$ component of the momentum as the integral of a vector $A^\alpha \equiv T^{\alpha\beta}\xi_\beta$ on a 3-surface,

$$P^{\boldsymbol{\xi}}(z,u) = \xi_{\hat{\alpha}} \int_{\Sigma(z,u)} T^{\hat{\alpha}\hat{\beta}} d\Sigma_{\hat{\beta}} = \int_{\Sigma(z,u)} T^{\hat{\alpha}\hat{\beta}} \xi_{\hat{\alpha}} d\Sigma_{\hat{\beta}} = \int_{\Sigma(z,u)} A^\beta d\Sigma_\beta \ .$$

Take $u^\alpha = P^\alpha/M$, and consider another vector u'^α at the same point z^α; the $\boldsymbol{\xi}$ component of the difference between the momenta computed in the hypersurfaces $\Sigma(z,u')$ and $\Sigma(z,u)$, $\Delta P^{\boldsymbol{\xi}} \equiv P^{\boldsymbol{\xi}}(z,u') - P^{\boldsymbol{\xi}}(z,u)$ is, from an application of the Gauss theorem (see Fig. 7),

$$\Delta P^{\boldsymbol{\xi}} = \int_{\Sigma(z,u)} A^\beta d\Sigma_\beta - \int_{\Sigma(z,u')} A^\beta d\Sigma_\beta = \int_{V_{Left}} A^\beta_{\ ;\beta} dV - \int_{V_{Right}} A^\beta_{\ ;\beta} dV \ .$$

[13] For example, in the case of the Schwarzschild spacetime, $\|\mathbf{R}\| \sim m_{\rm S}/r^3$, $\lambda = (m_{\rm S}/r)(a^2/r^2)$; since $m_{\rm S}/r < 1$ for any point outside the horizon, $\lambda \ll 1$ is guaranteed just by taking the size of the body much smaller than its the distance to the source, $r^2 \gg a^2$.

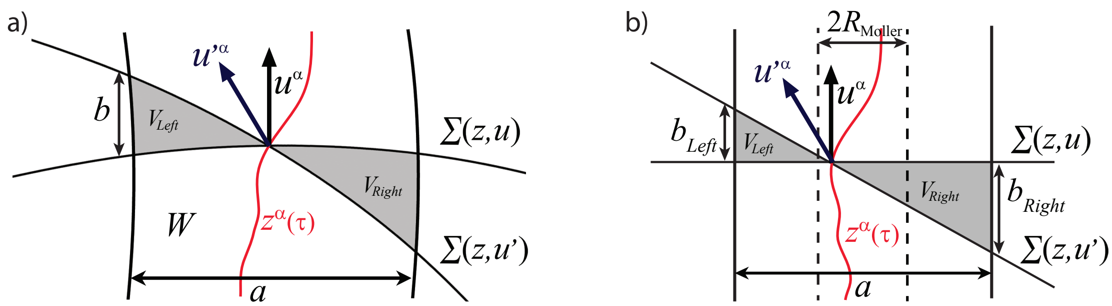

Figure 7: Shadowed regions V_{Left} and V_{Right} are the 4-volumes delimited by the hypersurfaces $\Sigma(z,u')$, $\Sigma(z,u)$, and the boundary of the body's worldtube, of convex hull W. u^α is chosen parallel to P^α. a) curved spacetime; b) flat spacetime.

Here V_{Left} and V_{Right} denote the shadowed regions of Fig. 7 (where $A^\alpha \neq 0$), i.e., the "left" and "right" 4-volumes delimited by $\Sigma(z,u')$, $\Sigma(z,u)$ and the boundary of the body's worldtube. Now, using the conservation law $T^{\alpha\beta}_{\ \ ;\beta} = 0$, one notes that

$$A^\beta_{\ ;\beta} = T^{\alpha\beta}_{\ \ ;\beta}\xi_\alpha + T^{\alpha\beta}\xi_{\alpha;\beta} = T^{\alpha\beta}\xi_{\alpha;\beta}\ ;$$

thus

$$\Delta P^{\boldsymbol{\xi}} = \int_{V_{Left}} T^{\alpha\beta}\xi_{\alpha;\beta}dV - \int_{V_{Right}} T^{\alpha\beta}\xi_{\alpha;\beta}dV\ .$$

Since $\boldsymbol{\xi}$ is a basis 1-form, $\xi_{\hat{\alpha},\hat{\beta}} = 0$, and

$$\xi_{\hat{\alpha};\hat{\beta}} = -\Gamma^{\hat{\gamma}}_{\hat{\alpha}\hat{\beta}}\xi_{\hat{\gamma}} = \mathcal{O}(\|\mathbf{R}\|x)\ ;$$

therefore

$$|\Delta P^{\boldsymbol{\xi}}| \lesssim \|\mathbf{R}\| \int_V T^{\hat{0}\hat{0}}|x|dV = \|\mathbf{R}\|V\left\langle T^{\hat{0}\hat{0}}|x|\right\rangle\ ,$$

where $V \equiv V_{Left} + V_{Right}$, $\langle\cdot\rangle$ denotes the average on the shadowed region of Fig. 7a, and we noted that $T^{\hat{0}\hat{0}}$ is the largest component of $T^{\alpha\beta}$ and always positive. Since $b < av(u',u)$ (see Fig. 7), with $v^\alpha(u',u)$ defined by Eq. (9), and $v(u',u) < 1$, then $\langle|x|\rangle < a$; also, $V\left\langle T^{\hat{0}\hat{0}}\right\rangle < Mav(u',u)$; hence we get

$$|\Delta P^{\boldsymbol{\xi}}| \lesssim M\lambda v(u',u) = \|\mathbf{P}\|\lambda v(u',u)\ , \tag{69}$$

showing that ΔP^α is negligible compared to P^α under the restriction above on the strength of the gravitational field, $\lambda \ll 1$ (the same under which the different multipole schemes become equivalent, and one can take local Lorentz coordinates as nearly rectangular throughout the extension of the body; see also footnote 13). In the application in Sec. 3.4 — Schwarzschild spacetime, far field limit — we can write

$$|\Delta P^{\boldsymbol{\xi}}| \lesssim \frac{M m_{\rm S}}{r^3} a^2 v(u',u) \simeq \|P_{\rm hidI}\| \frac{a}{R_{\rm Moller}} \frac{a}{r} v(u',u)$$

where $P^\alpha_{\rm hidI}$ is the inertial hidden momentum *of the CP condition*, Eq. (56) ($P^\alpha_{\rm hidI}$ is zero or negligible for the other solutions). Thus ΔP^α is negligible compared to $P^\alpha_{\rm hidI}$ under the condition $\frac{R_{\rm Moller}}{a} \gg \frac{a}{r} v(u',u)$, which is reasonable in a problem where the particle's spin is worth taking into account (e.g., in the problem of nearly circular motion in Sec. 3.4 this amounts to taking $\omega_{\rm body} \gg \omega_{\rm orbit}$, where $\omega_{\rm body}$ and $\omega_{\rm orbit}$ are the body's rotation and orbital angular velocities).

Through an analogous procedure, one can show that the dependence of $S^{\alpha\beta}$ on u^α is negligible in this regime. Let $\mathscr{J}^{\hat\alpha\hat\beta\hat\gamma} \equiv 2x^{[\hat\alpha} T^{\hat\beta]\hat\gamma}$, so that $S^{\hat\alpha\hat\beta} = \int_{\Sigma(z,u)} \mathscr{J}^{\hat\alpha\hat\beta\hat\gamma} d\Sigma_{\hat\gamma}$; and consider the two basis *spatial* 1-forms $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Constructing the vector $\mathscr{J}^\gamma \equiv \mathscr{J}^{\alpha\beta\gamma}\xi_\alpha\eta_\beta$, we can write the $\boldsymbol{\xi}\otimes\boldsymbol{\eta}$ component of $S^{\alpha\beta}$ as $S^{\boldsymbol{\xi}\boldsymbol{\eta}}(z,u) = \int_{\Sigma(z,u)} \mathscr{J}^\beta d\Sigma_\beta$. By the Gauss theorem,

$$\begin{aligned}\Delta S^{\boldsymbol{\xi}\boldsymbol{\eta}} &= \int_{V_{Left}} \mathscr{J}^\beta{}_{;\beta} dV - \int_{V_{Right}} \mathscr{J}^\beta{}_{;\beta} dV \ \sim \ \|\mathbf{R}\| \int_V x^2 |\vec{J}| dV = \|\mathbf{R}\| V \langle |\vec{J}| x^2 \rangle \\ &< \|\mathbf{R}\| a^2 V \langle |\vec{J}| \rangle = \lambda V \langle |\vec{J}| \rangle \lesssim \lambda a^4 v(u',u) \langle |\vec{J}| \rangle \ ,\end{aligned}$$

where $J^{\hat i} = T^{\hat 0 \hat i}$. In the second relation again we used $\Gamma^{\hat\gamma}_{\hat\alpha\hat\beta} = \mathcal{O}(\|\mathbf{R}\| x)$. Since $S = \mathcal{O}(a^4 \langle |\vec{J}| \rangle)$, cf. Eq. (13), we see that indeed when $\lambda \ll 1$, $\|\Delta S^{\boldsymbol{\xi}\boldsymbol{\eta}}\| \ll S$.

A.1.1 The case with electromagnetic field

When $F^{\alpha\beta} \neq 0$, the conservation law is $T^{\alpha\beta}{}_{;\beta} = F^{\alpha\beta} j_\beta$ (denoting by $T^{\alpha\beta}$ *the particle's* energy momentum tensor). Consider for simplicity flat spacetime, and let $\boldsymbol{\xi}$ be a basis 1-form of a *global* Lorentz system; then $T^{\alpha\beta}{}_{;\beta}\xi_\alpha = (T^{\alpha\beta}\xi_\alpha)_{;\beta} = F^{\alpha\beta} j_\beta \xi_\alpha \equiv A^\beta{}_{;\beta}$. Note that $f^\alpha \equiv F^{\alpha\beta} j_\beta$ is the Lorentz force density. It follows (see Fig. 7b))

$$\Delta P^{\boldsymbol{\xi}} = \int_{V_{Left}} A^\beta{}_{;\beta} dV - \int_{V_{Right}} A^\beta{}_{;\beta} dV = \xi_\alpha \left(V_{Left} \langle f^\alpha \rangle_{Left} - V_{Right} \langle f^\alpha \rangle_{Right} \right) \ .$$

We have $V_{Left} = V^{(3)}_{Left} b_{Left}/2$, $V_{Right} = V^{(3)}_{Right} b_{Right}/2$ (where $V^{(3)}$ denote 3-volumes orthogonal to u^α). Herein we allow z^α to be any point within the worldtube of centroids; it follows that

$$\begin{aligned} b_{Left} &\geq v(u,u') \left(\frac{a}{2} - R_{\rm Moller} \right); \qquad & b_{Right} &\leq v(u,u') \left(\frac{a}{2} + R_{\rm Moller} \right) \\ V^{(3)}_{Left} &\sim a^2 \left(\frac{a}{2} - R_{\rm Moller} \right); \qquad & V^{(3)}_{Right} &\sim a^2 \left(\frac{a}{2} + R_{\rm Moller} \right) \ . \end{aligned}$$

Let $\langle f^\alpha \rangle_{Left} = \langle f^\alpha \rangle_{Right} + \Delta f^\alpha$, with $\|\Delta f^\alpha\| \lesssim \|\nabla_\beta f^\alpha\| a$; we obtain

$$|\Delta P^{\boldsymbol{\xi}}| \lesssim \|F_{\rm L}\| R_{\rm Moller} v(u', u) + \|\nabla_j F_{\rm L}^\alpha\| v(u', u) a^2 \ . \quad (70)$$

Hence $\Delta P^{\boldsymbol{\xi}}$ has, as upper bound, the sum of two terms: the impulse of the Lorentz force in the time interval $R_{\rm Moller} v(u', u)$ (as measured in the $u^i = 0$ frame) between the two points where the hyperplane $\Sigma(z, u')$ crosses the worldtube of centroids, plus a term analogous to the gravitational one (69). For the field of a Coulomb charge, discussed in Sec. 3.4.1, they read

$$\|F_{\rm L}\| R_{\rm Moller} v(u', u) = |E_{\rm p}| v(u', u) \frac{R_{\rm Moller}}{r} \sim \|P_{\rm hidI}\| v(u', u)$$
$$\|\nabla_j F_{\rm L}^\alpha\| v(u', u) a^2 = |E_{\rm p}| v(u', u) \frac{a^2}{r^2} \sim \|P_{\rm hidI}\| \frac{a}{R_{\rm Moller}} \frac{a}{r} v(u', u)$$

where $E_{\rm p} = qQ/r$ is the electric potential energy, and $P^\alpha_{\rm hidI}$ is the inertial hidden momentum *of the TD/FMP (non-helical) solutions*, Eq. (51) (for the CP/OKS conditions, $P^\alpha_{\rm hidI} = 0$). Assuming $|E_{\rm p}| < M$, if $R_{\rm Moller}/r \ll 1$ and $a^2/r^2 \ll 1$ (as is the case in the far-field regime), then ΔP^α is negligible compared to P^α: $|\Delta P^{\boldsymbol{\xi}}| \ll M = \|\mathbf{P}\|$. It is also negligible compared to $\|P_{\rm hidI}\|$ under the following conditions: i) $v(u', u) \ll 1$ so that the first term of (70) can be neglected (this is guaranteed by the slow motion in Sec. 3.4); ii) that $\frac{R_{\rm Moller}}{a} \gg \frac{a}{r} v(u', u)$, a condition analogous to the one we obtained gravitational case above, which is reasonable whenever the particle's spin is worth taking into account.

Note that the argument above can equally be used to show that P^α does not depend on the spin condition. Start with the TD centroid: $z^\alpha = x^\alpha_{\rm CM}(u)$, with $u^\alpha = P^\alpha/M$; the centroids $x^\alpha_{\rm CM}(u')$ of other spin conditions are reached by $x^\alpha_{\rm CM}(u') = x^\alpha_{\rm CM}(u) + \Delta x^\alpha$, with $\Delta x^\alpha \in \Sigma(u, z)$, cf. Eq. (10). Since the argument above applies to any spacelike hyperplane $\Sigma(u', z')$ through any arbitrary centroid z'^α on $\Sigma(u, z)$, it effectively means that, to the accuracy at hand, P^α does not depend on the particular centroid chosen.

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