A torsional completion of gravity for Dirac matter fields and its applications to neutrino oscillations

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In this paper, we consider the torsional completion of gravitation for an underlying background filled with Dirac fields, applying it to the problem of neutrino oscillations: we discuss how in this context neutrinos even when massless and left-handed may nevertheless display oscillations.
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I. INTRODUCTION

The problem of neutrino oscillations stems from the fact that known thermonuclear fusion taking place in solar processes predicts the number of neutrinos reaching earth but only a considerably smaller fraction of them is observed in our detectors. Pontecorvo has been the first who theorized that such conversion may occur whenever the leptonic number are not conserved if neutrinos were massive and with non-degenerate masses: the mechanism relies on the fact that mass and flavour eigenstates do not coincide, and each wave-packet of a given flavour eigenstate is the superposition of all the wave-packets of mass eigenstates, so that the wave-packets may have interference, and a phase acquires a shift [1–3]. A first, theoretical problem about this mechanism is that neutrinos with different masses, produced with the same energy, have different speeds, and so there will be a spatial separation between the wave-packets, but on the other hand, in order for phases to shift, that is in order for interference to have place, the different eigenstates must stay in superposition, and thus they can have no spatial separation; a second, phenomenological problem is the fact that if neutrinos were indeed massive then they would also have to have a right-handed chiral projection, but neither mass nor such a right-handed component have ever been observed directly. A way to solve the theoretical as well as part of the phenomenological problem is to find a mechanism in which neutrino oscillations may occur in the zero-mass limit, as in [4]. However, in this case the right-handed component was still present, and a way to solve also this part of the phenomenological problem is to have the left-handed component alone and where the mixing is due to external interactions of specific type, as it is discussed by Mikheyev and Smirnov and by Wolfenstein, respectively in [5] and in [6]. The MSW mechanism is an economic solution, but as one may wonder, what happens when neutrinos propagate in material vacuum?

In the MSW mechanism neutrino oscillations are due to external interactions that are absent in vacuum, but if it were possible to have neutrinos with self-interactions, then even if we define the vacuum as the absence of external interactions nevertheless there can never be a vacuum for self-interacting fields. Instead of asking the previous question we could simply ask, is it possible to derive self-interactions for neutrinos? And the answer is, yes.

The torsional completion of gravity is the theory that we obtain whenever we do not constrain the most general metric-compatible connection to be symmetric in the two lower indices in holonomic basis, yielding a geometry that is endowed with torsion as well as curvature, called Cartan-Riemann geometry; when this theory is applied to Dirac spinors, torsion couples to the spin density in the same way in which curvature couples to the energy density of the Dirac matter field, giving the well known Sciama-Kibble-Einstein theory. Within such a SKE theory the Dirac matter field equation can be written in a way that is formally equivalent to the Dirac matter field equation in presence of Nambu–Jona-Lasinio potentials for every fermionic field. To have a comprehensive review of all these results[1] and a general introduction see [14].

So the torsional completion of gravity for a geometry hosting Dirac matter fields does provide an induced spin-contact interaction for every spinor field: and in turn, this gives the opportunity to have left-handed neutrinos with mixing of the MSW-type. As a matter of fact, in the past there have been efforts to employ the SKE theory with Dirac fields to describe left-handed massless neutrino oscillations [15, 16], but these attempts were affected by a misconception that hindered research in SKED theory for long, namely the problem of the torsion constant.

The torsion constant problem is the belief that the tor-

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[1] From a historical perspective, it was Cartan who first recognized the role of torsion, also called Cartan tensor, and together with the Riemann curvature, they justify the name of Cartan-Riemann geometry [7–10], while Sciama and Kibble were the first who wrote the field equations coupling torsion to the spin density in the same way in which Einstein wrote the field equations coupling curvature to energy, and so we talk about Sciama-Kibble-Einstein theory [11, 12]; Nambu–Jona-Lasinio potentials should be Heisenberg potentials [13], but tradition has stuck and for the sake of simplicity we will adhere to it in this paper.

sional constant must be positive and small because it is to be the Newton constant, but this misunderstanding only occurs when field equations are derived from an action that is the simplest of all; when instead the most general action is considered, it is not difficult to prove that there are a total of four constants, that is the Newton constant for the gravitational field plus three additional constants corresponding to the three irreducible decompositions of the torsion tensor [17, 18]. In those papers we have studied least-order derivative dynamics for the most general geometry, but we have not focused on the Dirac matter field in order to see what happens for the most general coupling of this spinor, which is what we will do here.

In this case, on the one hand, only one of the three irreducible components of torsion is excited by the spin density tensor, but on the other hand, there is an additional term for each flavour of the spinor that can be added to the action: then there will remain only one of the three constants of torsion coming from geometry, but also there will be one additional constant for each of the spinors that contribute to the torsion-spin coupling.

After the most general action is given, we will see what are the consequences for the neutrino oscillations.

II. SKED THEORY

In this paper, we take $(1+3)$-dimensional spacetimes filled with $\frac{1}{2}$-spin spinor fields, and all these fields will be coupled in terms of least-order derivative field equations.

All definitions we will employ hereafter are given in references [19–22], but for the sake of simplicity we will give all that is needed to fix our convention in the following.

In [23, 24] we argued that torsion $Q_{\rho\mu\nu}$ is completely antisymmetric and thus it is considered to be completely antisymmetric in this paper as well: this is not a loss of generality because in any case torsion would have been obtained to be completely antisymmetric as a consequence of the fact that in the spacetime Dirac spinors have a spin density that is completely antisymmetric itself and for the least-order derivative field equations the coupling between torsion and spin density is algebraic.

From the metric tensor we define the completely antisymmetric pseudo-tensor $\varepsilon_{\rho\mu\nu\alpha}$ with which the completely antisymmetric torsion can be written according to the expression $6Q_{\rho\mu\nu}=W^{\alpha}\varepsilon_{\alpha\rho\mu\nu}$ in terms of an axial vector called axial vector torsion: and once again such a procedure is achieved without any loss of generality.

From the most general connection we can define the curvature tensor $G^{\rho}{}_{\eta\alpha\nu}$ while analogously from the Levi-Civita connection it is possible to define the torsionless curvature tensor $R^{\rho}{}_{\eta\alpha\nu}$ and as the most general connection can be decomposed in terms of the Levi-Civita connection plus torsion similarly the curvature tensor can be decomposed in terms of the torsionless curvature tensor supplemented by specific torsional contributions.

The curvature tensor has a single independent contraction chosen as $G^{\rho}{}_{\eta\rho\nu}=G_{\eta\nu}$ with $G_{\eta\nu}g^{\eta\nu}=G$ and also the torsionless curvature tensor has a single independent contraction given by $R^{\rho}{}_{\eta\rho\nu}=R_{\eta\nu}$ with $R_{\eta\nu}g^{\eta\nu}=R$ which are known as Ricci curvature tensor and scalar and torsionless curvature tensor and scalar, respectively.

Finally, by introducing $\boldsymbol{\gamma}_a$ known as Clifford matrices because they verify $2\mathbb{I}\eta_{ij}=\{\boldsymbol{\gamma}_i,\boldsymbol{\gamma}_j\}$ known as Clifford algebra, and $\{\boldsymbol{\gamma}_i,[\boldsymbol{\gamma}_j,\boldsymbol{\gamma}_k]\}=4i\varepsilon_{ijkq}\boldsymbol{\pi}\boldsymbol{\gamma}^q$ as the relationship implicitly giving $\boldsymbol{\pi}$ as the parity-odd matrix[2] used to obtain the two chiral projections of the spinor field itself.

In terms of the spinorial connection we can define the spinorial covariant derivatives $\boldsymbol{D}_\mu$ and from the torsionless spinorial connection it is possible to define the torsionless spinorial covariant derivative $\boldsymbol{\nabla}_\mu$ as usual.

Because curvatures and covariant derivatives can be decomposed in torsionless curvatures and torsionless covariant derivatives plus torsional contributions while spinors can be decomposed in two chiral projections, then we can employ either the full or the decomposed quantities at will when we will write the dynamical action.

As it has been discussed in [25], in the case of two independent spinor fields the most general Lagrangian is

$$L=-\tfrac{1}{128k}W_\alpha W^\alpha-R+i\overline{\psi}_1\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi_1+i\overline{\psi}_2\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi_2+\\+\tfrac{1}{8}a_1\overline{\psi}_1\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi_1W_\mu+\tfrac{1}{8}a_2\overline{\psi}_2\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi_2W_\mu \tag{1}$$

where mass terms have been neglected: by varying with respect to torsion we obtain the torsion-spin coupling

$$W^\mu=8k(a_1\overline{\psi}_1\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi_1+a_2\overline{\psi}_2\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi_2) \tag{2}$$

which can be plugged back into the Lagrangian yielding

$$L=-R+i\overline{\psi}_1\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi_1+i\overline{\psi}_2\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi_2+\\+ka_1a_2\overline{\psi}_2\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi_2\overline{\psi}_1\boldsymbol{\gamma}_\mu\boldsymbol{\pi}\psi_1-\\-\tfrac{1}{2}ka_1^2\overline{\psi}_1\boldsymbol{\gamma}^\mu\psi_1\overline{\psi}_1\boldsymbol{\gamma}_\mu\psi_1-\tfrac{1}{2}ka_2^2\overline{\psi}_2\boldsymbol{\gamma}^\mu\psi_2\overline{\psi}_2\boldsymbol{\gamma}_\mu\psi_2 \tag{3}$$

showing that the two torsion-spinor interaction terms shift the torsion-spinor coupling constants both in terms of the geometric torsion constant k and in terms of the torsion-spinor coupling constants a_1 and a_2 which do not have equal values when taken in general circumstances.

The case of two spinor fields is clearly extended to a generic number n of spinor fields, so that the geometric torsion constant k is accompanied by n torsion-spinor coupling constants a_i resulting into n self-interaction constants given by ka_i^2 plus $\frac{1}{2}n(n-1)$ coupling constants given by ka_ia_j with $i\neq j$ symmetric in the two labels.

From ka_i^2 and ka_ia_j we can see that if the geometric torsion constant k is positive/negative then each spinor field has self-interactions that are repulsive/attractive respectively, with torsion-spin coupling constants a_i whose sign can determine only the repulsiveness/attractiveness

[2] This matrix was historically introduced to study five dimensional extensions and it was indicated as a gamma matrix with an index five, but such a notation has no meaning in four dimensions and therefore we prefer to adopt a matrix with no index.

of the coupling, and the geometric torsion constant determines the absolute scale of the dynamics, whereas each of the torsion-spin coupling constants determine the relative strengths of the self-interactions and couplings.

III. NEUTRINO OSCILLATIONS

In this section we will employ action (3) neglecting the gravitational interaction and assuming that neutrinos are not only massless but also left-handed in general.

For two flavours, the Hamiltonian of interaction is

$$H_{12}=-ka_1a_2\overline{\nu}_2\gamma^\mu\nu_2\overline{\nu}_1\gamma_\mu\nu_1 \tag{4}$$

where the off-diagonal part is clearly manifest.

In reference [16] such an off-diagonal potential has been used to show that left-handed massless neutrinos have torsionally-induced spin-contact interactions giving rise to oscillations; nevertheless, we have not calculated the length because of the non-linearity of such potential.

In [15] however the authors deal with the non-linear potential by taking neutrinos dense enough as to make the torsional background homogeneous: they thus obtain the formula yielding the phase difference in terms of the nearly constant torsion background W_α as

$$\Delta\Phi_{12}=\tfrac{1}{2}W_t\Delta L \tag{5}$$

depending on the length; (5) is general, but as a final comment the authors remark that the phase difference is negligible because torsion is of the order of magnitude of the Planck scale, which is true but it is valid only when in the coupling equation (2) constants k and a_i are chosen equal to the unity in Planck units, and not in general.

In general, the constant k and a_i are not equal to unity and equation (2) yields torsion with an order of magnitude that is not necessarily the Planck scale: as a matter of fact, it is possible to plug (2) into (5) obtaining that the phase difference is given in terms of the background neutrino energy density $N=\psi^\dagger\psi$ according to

$$\Delta\Phi_{12}=4ka_1N\Delta L \tag{6}$$

depending on the length as in the customary analysis.

However in the customary description the phase difference depends on the length but it is given in terms of the difference of the squared masses according to

$$\Delta\Phi_{12}=\frac{\Delta m_{12}^2}{2E}\Delta L \tag{7}$$

inversely proportional to the energy of neutrinos.

Upon comparison between (6) and (7) we see that the phase differences have an analogous dependence on the length but they also have discrepancies that are encoded in the fact that in (6) it grows with the neutrino energy density while in (7) it drops with the energy of neutrinos.

Whenever we have $ka_i\approx 10^{27}$ the two formulas will coincide in the case of the solar neutrino oscillations.

If the three constants a_i have different values with similar magnitude the obtained formulas fit the solar neutrino oscillations for each of the three neutrino families.

In this scenario three flavours of left-handed massless neutrinos may oscillate fitting all observations.

IV. CONCLUSION

In this paper, the most general SKED theory was used to study left-handed massless neutrinos; as we have seen, the torsionally-induced spin-contact interactions determine oscillations, and for three neutrino families there are three coupling constants giving three oscillation lengths.

As it stands torsion may be responsible for oscillations with no need for physics beyond the standard model.

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