Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations in ultra-relativistic regime and gravimagnetic moment.

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MPTD-equations in the Lagrangian formulation correspond to the minimal interaction of spin with gravity. Due to the interaction, in the Lagrangian equations instead of the original metric g emerges spin-dependent effective metric $G = g + h(S)$. So we need to decide, which of them the MPTD-particle sees as the space-time metric. We show that MPTD-equations, if considered with respect to original metric, have no physically admissible solutions: acceleration of the particle grows up to infinity as its speed approximates to the speed of light. If considered with respect to G, the theory is consistent. But the metric now depends on spin, so there is no unique space-time manifold for the Universe of spinning particles: each particle probes his own three-dimensional geometry. This can be improved by adding a non-minimal interaction, and gives the modified MPTD-equations with reasonable behavior within the original metric.

Equations of motion of a rotating body in a curved background formulated usually in the multipole approach to description of the body [1–8]. We consider MPTD-equations [31] , which describe the motion in pole-dipole approximation, in the form studied by Dixon (for the relation of the Dixon equations with those of Papapetrou and Tulczyjew see p. 335 in [4] as well as the recent works [7, 8]):

$$\nabla P^\mu = -\frac{1}{4}R^\mu{}_{\nu\alpha\beta}S^{\alpha\beta}\dot{x}^\nu \equiv -\frac{1}{4}\theta^\mu{}_\nu\dot{x}^\nu, \\ \nabla S^{\mu\nu} = 2P^{[\mu}\dot{x}^{\nu]}, \qquad S^{\mu\nu}P_\nu = 0. \tag{1}$$

They are widely used now in computations of spin effects in compact binaries and rotating black holes [9–16], so our results may be relevant in this framework. In the multipole approach, $x^\mu(\tau)$ is called representative point of the body, antisymmetric spin-tensor $S^{\mu\nu}(\tau)$ is associated with inner angular momentum, vector $P^\mu(\tau)$ is called momentum.

In the present work we discuss behavior of MPTD-particle in ultra-relativistic limit, when speed of the particle approximates to the speed of light. Since we are interested in the influence of spin on the trajectory of a particle, we eliminate the momenta from MPTD-equations, thus obtaining a second-order equation for the representative point $x^\mu(\tau)$. To achieve this, we compute derivative of the spin supplementary condition, $\nabla(S^{\mu\nu}P_\nu) = 0$, and take into account that P^2 and $S^{\mu\nu}S_{\mu\nu}$ turn out to be constants of motion of the equations (1), say $\sqrt{-P^2} = k$ and $S^2 = \beta$. Then the derivative reads

$$P^\mu = \frac{k}{\sqrt{-\dot{x}G\dot{x}}}(\tilde{T}\dot{x})^\mu, \qquad \tilde{T}^\mu{}_\nu = \delta^\mu{}_\nu - \frac{1}{8k^2}(S\theta)^\mu{}_\nu, \tag{2}$$

where appeared the matrix G constructed from the "tetrad field" $\tilde{T}$ as follows:

$$G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}(S) \equiv g_{\alpha\beta}\tilde{T}^\alpha{}_\mu\tilde{T}^\beta{}_\nu. \tag{3}$$

Since this is composed from the original metric $g_{\mu\nu}$ plus (spin and field-dependent) contribution $h_{\mu\nu}$, we call G the effective metric produced along the world-line by interaction of spin with gravity.

Substitution of the expression (2) into (1) yields equations without P^μ, modulo the constant of motion k. Before we begin the analysis of the resulting equations, we point out how they can be obtained from variational problem for vector model of spin, see [17] for details.

Consider the relativistic spinning particle described by position $x^\mu(\tau)$ and by vector $\omega^\mu(\tau)$ attached to the point x^μ. The spin-tensor in our model is a composite quantity constructed from ω^μ and its conjugated momentum $\pi^\mu = \frac{\partial L}{\partial\dot{\omega}_\mu}$ as follows:

$$S^{\mu\nu} = 2(\omega^\mu\pi^\nu - \omega^\nu\pi^\mu) = (S^{i0} = D^i,\ S_{ij} = 2\epsilon_{ijk}S_k). \tag{4}$$

Here S_i is three-dimensional spin-vector and D_i is dipole electric moment [18]. The spinning particle in flat space is described by the Lagrangian action [19]

$$S = -\frac{1}{\sqrt{2}}\int d\tau\sqrt{m^2c^2 - \frac{\alpha}{\omega^2}} \\ \times\sqrt{-\dot{x}N\dot{x} - \dot{\omega}N\dot{\omega} + \sqrt{[\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega}]^2 - 4(\dot{x}N\dot{\omega})^2}}. \tag{5}$$

The matrix $N_{\mu\nu} = \eta_{\mu\nu} - \frac{\omega_\mu\omega_\nu}{\omega^2}$ is the projector on the plane orthogonal to ω^ν: $N_{\mu\nu}\omega^\nu = 0$. The double square-root structure in the expression (5) seem to be typical for the vector models of spin [20, 21]. In the spinless limit, $\alpha = 0$ and $\omega^\mu = 0$, the expression (5) reduces to the standard Lagrangian of relativistic particle, $-mc\sqrt{-\dot{x}^\mu\dot{x}_\mu}$. Let us shortly enumerate some properties of the spinning particle. The Lagrangian depends on one free parameter α which determines the value of spin. The value $\alpha = \frac{3\hbar^2}{4}$

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corresponds to a spin one-half particle. L is invariant under reparametrizations as well as under local spin-plane symmetry [22], the latter acts on ω and π but leaves S invariant. So only S is an observable quantity. Canonical quantization of the model yields the Dirac equation [23]. At last, the model admits interaction with an arbitrary electromagnetic [24] and gravitational [25] fields.

The minimal interaction with gravity is achieved by covariantization of the formulation (5), that is we replace $\eta_{\mu\nu} \to g_{\mu\nu}$, and usual derivative by the covariant one, $\dot\omega^\mu \to \nabla\omega^\mu = \frac{d\omega^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}\dot x^\alpha\omega^\beta$.

For the general-covariant phase-space quantities $\dot x^\mu$, $P_\mu = \frac{\partial L}{\partial \dot x^\mu} - \Gamma^\beta_{\alpha\mu}\omega^\alpha\pi_\beta$ and $S^{\mu\nu}$, the variational problem (5) yields the dynamical equations (here G is given by (3) with $\tilde T^\mu{}_\nu = \delta^\mu{}_\nu - \frac{1}{8m^2c^2}(S\theta)^\mu{}_\nu$)

$$P^\mu = \frac{mc}{\sqrt{-\dot xG\dot x}}(\tilde T\dot x)^\mu, \qquad \nabla P^\mu = -\frac{1}{4}\theta^\mu{}_\nu\dot x^\nu\,, \\ \nabla S^{\mu\nu} = 2P^{[\mu}\dot x^{\nu]}\,, \qquad (6)$$

as well as the constraints

$$S^{\mu\nu}P_\nu = 0, \quad P^2 + (mc)^2 = 0, \quad S^2 = 8\alpha. \qquad (7)$$

For the latter use, we also present the first-order (Hamiltonian) action of the theory

$$\int d\tau\; p_\mu\dot x^\mu + \pi_\mu\dot\omega^\mu - \left[\frac{\lambda_1}{2}\left(P^2 + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2}\right)\right. \\ \left. +\lambda_2(\omega\pi) + \lambda_3(P\omega) + \lambda_4(P\pi)\right]. \qquad (8)$$

Comparing (6) and (7) with MPTD-equations (1) and (2), we conclude that all the trajectories of MPTD-equations with given integration constants k and β are described by our spinning particle with mass $m = \frac{k}{c}$ and spin $\alpha = \frac{\beta}{8}$. By the way, we demonstrated that MPTD-equations correspond to the minimal interaction of the spinning particle with gravity.

Lagrangian form of the equations reads

$$\nabla\left[\frac{\tilde T^\mu{}_\nu\dot x^\nu}{\sqrt{-\dot xG\dot x}}\right] = -\frac{1}{4mc}\theta^\mu{}_\nu\dot x^\nu\,, \qquad (9)$$

$$\nabla S^{\mu\nu} = \frac{1}{4mc\sqrt{-\dot xG\dot x}}\dot x^{[\mu}(S\theta\dot x)^{\nu]}\,, \qquad (10)$$

$$S^\mu{}_\nu\dot x^\nu - \frac{1}{8(mc)^2}(S\theta\dot x)^\mu = 0\,. \qquad (11)$$

All the subsequent discussion will be around the factor $\dot xG\dot x$, where appeared the effective metric $G_{\mu\nu}$. The equation for trajectory (9) became singular for the particle's velocity which annihilates this factor, $\dot xG\dot x = 0$. Performing technical computations, we include all the factors into the expression for reparametrization-invariant derivative $D \equiv \frac{1}{\sqrt{-\dot xG\dot x}}\frac{d}{d\tau}$, representing (9) in the form

$$DDx^\mu = f^\mu(Dx, S, R)\,. \qquad (12)$$

The singularity determines behavior of the particle in ultra-relativistic limit. To clarify this point, two comments are in order.

1. Consider the standard equations of spinless particle interacting with electromagnetic field in the physical-time parametrization $x^\mu(t) = (ct, \mathbf{x}(t))$, $\left(\frac{\dot x^\mu}{\sqrt{c^2-\mathbf{v}^2}}\right)^{\cdot} = \frac{e}{mc^2}F^\mu{}_\nu\dot x^\nu$, then the factor is just $c^2 - \mathbf{v}^2$. Rewriting the equations in the form of second law of Newton we find an acceleration. For the case, the longitudinal acceleration reads $a_{||} = \mathbf{va} = \frac{e(c^2-\mathbf{v}^2)^{\frac{3}{2}}}{mc^3}(\mathbf{Ev})$, that is the factor, elevated in some degree, appears on the right hand side of the equation, and thus determines the value of velocity at which the longitudinal acceleration vanishes, $a_{||} \xrightarrow{v\to c} 0$. In resume, for the present case the singularity implies that during its evolution in external background, the particle can not exceed the speed of light c.

2. In a curved space we need to be more careful since the three-dimensional geometry should respect the coordinate independence of the speed of light. The notions for time interval, distance and velocity can be done according the known procedure [26]. For the events x^μ and $x^\mu + dx^\mu$ in curved space $g_{\mu\nu}$, the three-dimensional quantities are

$$dt = -\frac{g_{0\mu}dx^\mu}{c\sqrt{-g_{00}}}, \\ dl^2 = \left(g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}\right)dx^idx^j \equiv \gamma_{ij}(x^0, \mathbf{x})dx^idx^j. \qquad (13)$$

Then three-velocity vector $\mathbf{v}$ is $v^i = \left(\frac{dt}{dx^0}\right)^{-1}\frac{dx^i}{dx^0}$ or, symbolically, $v^i = \frac{dx^i}{dt}$. With these definitions, the four-interval acquires the form similar to special relativity: $g_{\mu\nu}dx^\mu dx^\nu = -dt^2\left(c^2 - \mathbf{v}\gamma\mathbf{v}\right)$, and a particle (photon) with propagation law $\dot xg\dot x = 0$ has the speed equal to c.

To define an acceleration of a particle in the three-dimensional geometry, we need the notion of a constant vector field (or, equivalently, the parallel-transport equation). The three-dimensional vector field with components v^i along a curve $x^i(x^0)$ is called constant, if this obeys [27] $\nabla_0 v^i + \frac{1}{2}(\mathbf{v}\partial_0\gamma\gamma^{-1})^i = 0$. Here the covariant derivative is defined with help of three-dimensional metric $\gamma_{ij}(x^0, \mathbf{x})$, with x^0 considered as a parameter. This definition guarantees that scalar product of two constant fields does not depend on the point where it was computed, $\frac{d}{dx^0}(\mathbf{v}\gamma\mathbf{w}) = 0$. The deviation from a constant velocity is an acceleration

$$a^i = \left(\frac{dt}{dx^0}\right)^{-1}\left[\nabla_0 v^i + \frac{1}{2}(\mathbf{v}\partial_0\gamma\gamma^{-1})^i\right]. \qquad (14)$$

The extra-term appeared in this equation play an essential role [27] to provide that for the geodesic motion we have: $a_{||} \xrightarrow{v\to c} 0$. For the static metric, $\partial_0\gamma = 0$, our definition reduces to that of Landau-Lifshitz, see page 251 in [26].

Let us return to the Lagrangian form (9)-(11) of MPTD-equations. The singular factor contains the effective metric $G = g + h$ where g is the original metric. So we need to decide, which one of them the particle sees as the space-time metric?

Let us use g to define the three-dimensional geometry (13) and (14). Then even in static field (and with $g_{0i} = 0$) we obtain rather surprising result that the longitudinal acceleration grows up to infinity as the particle's speed approximates to the speed of light (the dots state for irrelevant for the present discussion non singular terms)

$$\mathbf{v}\gamma\mathbf{a} = \frac{1}{\sqrt{c^2 - \mathbf{v}^2}} \frac{v_\alpha S^{\alpha\sigma} R_{\sigma\nu\rho\lambda} v^\nu v^\rho (S\theta v)^\lambda}{16(mc)^3} + \ldots \stackrel{v\to c}{\longrightarrow} \infty \,.$$

So MPTD-equations, if considered with respect to original metric, have no physically admissible solutions. As $(\mathbf{v}\gamma\mathbf{a}) \sim \frac{1}{m^3}$, this effect could be more appreciable for neutrino.

Let us use G to define the three-dimensional geometry (13) and (14). In this case the expression for longitudinal acceleration as a function of the force (12) can be obtained in compact form for an arbitrary original metric

$$\mathbf{v}\gamma\mathbf{a} = \frac{c^2 - \mathbf{v}^2}{c^2} \Big[(c^2 - \mathbf{v}^2)(\mathbf{v}\gamma\mathbf{f}) + (\mathbf{v}\gamma)_j \Gamma^j{}_{ab} v^a v^b + \frac{1}{2}\Big(\frac{dt}{dx^0}\Big)^{-1} (\mathbf{v}\partial_0 \gamma \mathbf{v}) \Big] . \qquad (15)$$

For MPTD-particle $\mathbf{f} \sim (c^2 - \mathbf{v}^2)^{-\frac{3}{2}}$, so $(\mathbf{v}\gamma\mathbf{a}) \stackrel{v\to c}{\longrightarrow} 0$, and the theory is consistent with respect to the effective metric G. Since G is spin and field dependent quantity, we conclude that in this picture there is no unique space-time manifold for the Universe of spinning particles: each particle will probe his own three-dimensional geometry.

Can we modify the MPTD-equations to obtain a theory with reasonable behavior with respect to original metric $g_{\mu\nu}$? The inspection of the computation traced above shows that the nonphysical behavior originates from the fact that r. h. s. of the equation for precession of spin (the second equation in (1)) too singularly, $DS \sim (c^2 - \mathbf{v}^2)^{-1}$. Let us improve this behavior. As we have seen above, MPTD-equations result from minimal interaction of spinning particle with gravitational field. We add a nonminimal interaction in such a way that equation for precession of spin, $DS \sim (c^2 - \mathbf{v}^2)^{-1}$, is replaced by $DS \sim (c^2 - \mathbf{v}^2)^0$. This improves the bad behavior of MPTD equations. To achieve this, we add the nonminimal interaction $\frac{\lambda_1}{2}\frac{\kappa}{16}(\theta S) \equiv \frac{\lambda_1}{2}\kappa R_{\alpha\beta\mu\nu}\omega^\alpha\pi^\beta\omega^\mu\pi^\nu$ into the Hamiltonian action (8). By analogy with the magnetic moment, the interaction constant κ is called gravimagnetic moment [28]. The new interaction turns out to be consistent with all the constraints of the model for any value of κ (approximate equations with nonvanishing gravimagnetic moment were discussed in [28–30]). For the particular value $\kappa = 1$, the effective metric G turn into the initial metric g. The Lagrangian equations read

$$\nabla \left[\frac{\bar{m}\dot{x}^\mu}{\sqrt{-\dot{x}g\dot{x}}} \right] = -\frac{1}{4\bar{m}c}\theta^\mu{}_\nu \dot{x}^\nu + \ldots \,, \qquad (16)$$

$$\nabla S^{\mu\nu} = \frac{\sqrt{-\dot{x}g\dot{x}}}{4\bar{m}c}\theta^{[\mu}{}_\sigma S^{\nu]\sigma} + \ldots \,, \qquad (17)$$

$$S^{\mu\nu}\dot{x}_\nu + \ldots = 0 \,, \qquad (18)$$

where appeared the radiation mass $\bar{m} = m^2 + \frac{(\theta S)}{16c^2} + \ldots$, and the dots state for irrelevant for the present discussion contributions due to non homogeneity of a curvature, $O(\nabla R)$. These equations can be compared with (9)-(11). Even in homogeneous field we have modified dynamics for both x and S. In the modified theory

1. Time interval and distance are unambiguously defined within the original space-time metric $g_{\mu\nu}$.
2. Longitudinal acceleration vanishes as $v \to c$.

That is, contrary to MPTD-equations, the modified theory is consistent with respect to the original metric $g_{\mu\nu}$. Hence the modified equations could be more promising for description of the rotating objects in astrophysics.

In conclusion, we note that MPTD-equations follow from particular form assumed for the multipole representation of a rotating body [6]. It would be interesting to find a set of multipoles which yields the modified equations (16) and (17). Also, it would be interesting to find the Lagrangian form of the variational problem (8) with the nonminimal interaction introduced above.

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[31] Contrary to Dixon, we do not assume the proper-time parametrization, that is we do not add the equation $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -c^2$ to the system (1). Our variables are taken in arbitrary parametrization τ, then $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. Covariant derivative is $\nabla P^\mu = \frac{dP^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha P^\beta$ and curvature is $R^\sigma{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\sigma{}_{\lambda\nu} - \partial_\nu\Gamma^\sigma{}_{\lambda\mu} + \Gamma^\sigma{}_{\beta\mu}\Gamma^\beta{}_{\lambda\nu} - \Gamma^\sigma{}_{\beta\nu}\Gamma^\beta{}_{\lambda\mu}$. The square brackets mean antisymmetrization, $\omega^{[\mu}\pi^{\nu]} = \omega^\mu\pi^\nu - \omega^\nu\pi^\mu$. We use the condensed notation $\dot{x}^\mu G_{\mu\nu}\dot{x}^\nu = \dot{x}G\dot{x}$, $N^\mu{}_\nu\dot{x}^\nu = (N\dot{x})^\mu$, $\omega^2 = g_{\mu\nu}\omega^\mu\omega^\nu$, $\mu, \nu = 0, 1, 2, 3$, $v^i\gamma_{ij}a^j = \mathbf{v}\gamma\mathbf{a}$, $i, j = 1, 2, 3$, and so on. Notation for the scalar functions constructed from second-rank tensors are $\theta S = \theta^{\mu\nu}S_{\mu\nu}$, $S^2 = S^{\mu\nu}S_{\mu\nu}$.