Kinetization of scalar field by torsion

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We show that a scalar field without a kinetic term in the Lagrangian density, coupled to a covariant divergence of the torsion vector in the Einstein-Cartan theory of gravity, becomes kinetic in its general-relativistic equivalent formulation. Dynamical scalar fields may therefore be emergent.

The Einstein-Cartan theory of gravity [1, 2] naturally extends general relativity by including the spin angular momentum of matter. The spin-spin interaction arising in this theory [3, 4] may also remove divergent integrals in quantum field theory by providing fermions with spatial extension [5] and avoid the formation of singularities from fermionic matter in black holes and in cosmology [6]. In this theory, the Lagrangian density for the gravitational field is proportional to the Ricci scalar $R = R^i_{i}$ [1]:

$$\mathfrak{L} = -\frac{1}{2\kappa}R\sqrt{-g} + \mathfrak{L}_\mathrm{m}, \tag{1}$$

where $\mathfrak{L}$ is the total Lagrangian density, $R_{ik} = R^j_{ijk}$ is the Ricci tensor, $R^i_{mjk} = \Gamma^{\,i}_{m\,k,j} - \Gamma^{\,i}_{m\,j,k} + \Gamma^{\,i}_{l\,j}\Gamma^{\,l}_{m\,k} - \Gamma^{\,i}_{l\,k}\Gamma^{\,l}_{m\,j}$ is the curvature tensor, $\Gamma^{\,i}_{j\,k}$ is the affine connection, the comma denotes a partial derivative with respect to the coordinates, g is the determinant of the metric tensor g_{ik}, $\mathfrak{L}_\mathrm{m}$ is the Lagrangian density for matter, and $\kappa = 8\pi G/c^4$ is Einstein's gravitational constant. The metricity condition $g_{ij;k} = 0$, where the semicolon denotes a covariant derivative with respect to the affine connection, gives the affine connection $\Gamma^{\,k}_{i\,j} = \{^{\,k}_{i\,j}\} + C^k_{ij}$, where $\{^{\,k}_{i\,j}\} = (1/2)g^{km}(g_{mi,j} + g_{mj,i} - g_{ij,m})$ are the Christoffel symbols,

$$C^i_{jk} = S^i_{jk} + 2S_{(jk)}^{i} \tag{2}$$

is the contortion tensor, $S^i_{jk} = \Gamma^{\,\,i}_{[j\,k]}$ is the torsion tensor, ( ) denotes symmetrization, and [ ] denotes antisymmetrization. The indices can be lowered with the metric tensor and raised with the contravariant metric tensor g^{ik}, as in general relativity [7]. The curvature tensor can be decomposed as $R^i_{klm} = P^i_{klm} + C^i_{km:l} - C^i_{kl:m} + C^j_{km}C^i_{jl} - C^j_{kl}C^i_{jm}$, where P^i_{klm} is the Riemann tensor (the curvature tensor constructed from the Christoffel symbols instead of the affine connection) and the colon denotes a covariant derivative with respect to the Christoffel symbols. We use the notation of [4]. The Ricci scalar can be decomposed as

$$R = P - 4S^i_{:i} - 4S^i S_i - C^{ijk}C_{kij}, \tag{3}$$

where $P = P^j_{ijk}g^{ik}$ is the Riemann scalar and

$$S_i = S^k_{ik} \tag{4}$$

is the torsion vector.

We consider the following Lagrangian density for matter:

$$\mathfrak{L}_\mathrm{m} = \alpha S^i \phi_{,i}\sqrt{-g}, \tag{5}$$

where ϕ is a scalar field and α is a constant. Varying the Lagrangian density with respect to the torsion tensor and equaling this variation to zero gives the Cartan field equations:

$$S^j_{ik} - S_i\delta^j_k + S_k\delta^j_i = -\frac{\kappa}{2}s_{ik}^{j}, \tag{6}$$

where

$$s_i^{jk} = \frac{2}{\sqrt{-g}}\frac{\delta\mathfrak{L}_\mathrm{m}}{\delta C^i_{jk}} \tag{7}$$

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is the spin tensor of matter. The inverse relation is

$$S^i{}_{jk} = -\frac{\kappa}{2}(s_{jk}{}^i + \delta^i_{[j} s_{k]l}{}^l) \tag{8}$$

and its contraction gives the torsion vector:

$$S_i = \frac{1}{4}\kappa s_{ik}{}^k. \tag{9}$$

For the Lagrangian density (5), the spin tensor is

$$s_{ij}{}^k = \frac{\alpha}{2}(\delta^k_i \phi_{,j} - \delta^k_j \phi_{,i}). \tag{10}$$

Its contraction is thus

$$s_{ik}{}^k = -\frac{3\alpha}{2}\phi_{,i}. \tag{11}$$

Substituting the spin tensor and its contraction to the field equations (8) gives the torsion tensor:

$$S^i{}_{jk} = \frac{\alpha\kappa}{8}(\delta^i_j \phi_{,k} - \delta^i_k \phi_{,j}), \tag{12}$$

and the torsion vector:

$$S_i = -\frac{3\alpha\kappa}{8}\phi_{,i}. \tag{13}$$

The contortion tensor (2) is thus

$$C_{ijk} = \frac{\alpha\kappa}{4}(\phi_{,i} g_{jk} - \phi_{,j} g_{ik}). \tag{14}$$

Substituting the torsion vector and contortion tensor to (1), using (3) and (5), and omitting a covariant divergence which does not contribute to the field equations, gives

$$\mathfrak{L} = -\frac{1}{2\kappa}P\sqrt{-g} - \frac{3\alpha^2\kappa}{32}\phi_{,i}\phi_{,k}g^{ik}\sqrt{-g}. \tag{15}$$

The second term on the right-hand side of this equation is the Lagrangian density for matter in the general-relativistic equivalent formulation of the Einstein-Cartan theory. To obtain the Einstein field equations, the Lagrangian density (15) must be varied with respect to the metric tensor and such a variation must be equaled to zero.

The second term on the right-hand side of (15) has a form of a negative kinetic term for the scalar field ϕ. Such a phantom scalar field could be a source of dark energy [8]. Furthermore, the term $\alpha S^i \phi_{,i}\sqrt{-g}$ in the Lagrangian density (5) is dynamically equivalent (differs by a covariant divergence) to a term in which the field ϕ is coupled to the covariant divergence of the torsion vector and has a nonkinetic form:

$$\alpha S^i{}_{:i}\phi\sqrt{-g} = \alpha(S^i{}_{;i} - 2S^i S_i)\phi\sqrt{-g}. \tag{16}$$

Therefore, a kinetic scalar field can be generated from a nonkinetic scalar field by torsion. This result suggests that dynamical scalar fields may be emergent.

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