FOUNDATIONS QUADRILOGY

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In this work we present the general differential geometry of a background in which the space-time has both torsion and curvature with internal symmetries being described by gauge fields, and that is equipped to couple spinorial matter fields having spin and energy as well as gauge currents: torsion will turn out to be equivalent to an axial-vector massive Proca field and because the spinor can be decomposed in its two chiral projections, torsion can be thought as the mediator that keeps spinors in stable configurations; we will justify this claim by studying some limiting situations. In what will look like a second introduction, we present in historical manner the way in which quantum principles have come to be mathematically implemented in physical theories, with the aim of isolating the main problems of the quantization protocols; we will do this with the goal of presenting possible solutions and improvements that are based on the concept of spin, and therefore describable within the theory that has been introduced in the first part. Further we present some of the most recent open problems in physics, again with the idea of proposing solutions that are based on the interaction between the spin and the torsion tensor. Finally we briefly sketch a discussion about the existence of some exact solutions and a few of their possible consequences.

Contents

I. INTRODUCTION

In fundamental theoretical physics, there are a number of principles that are assumed, and among them, one of the most important is the principle of covariance, stating that the form of physical laws must be independent on the coordinate system used to write them down; covariance is mathematically translated into the instruction that such physical laws have to be written by employing tensors.

On the other hand, because physical laws describe the shape and evolution of fields, differential operators must be used; but because of covariance, all derivatives in the field equations have to be covariant too: and covariant derivatives can be defined in general upon introducing an object called connection. In their most general form, the covariant derivative of, say, a vector, is given by

$$D_\alpha V^\nu = \partial_\alpha V^\nu + V^\sigma \Gamma^\nu_{\sigma\alpha}$$

where the connection $\Gamma^\alpha_{\nu\sigma}$ has three indices: the upper index and the lower index on the left are the indices involved in the shuffling of the components of the vector, whereas the lower index on the right is the index related to the coordinate with respect to which the derivative is calculated eventually. Hence, there appears to be a clear distinction in the roles played by the left and the right of the lower indices, and therefore the connection cannot be taken to have any kind of symmetry property for indices transposition involving the two lower indices at all.

The fact that in the most general case the connection has no specific symmetry implies that the antisymmetric part of the connection is not zero, and because it turns out to be a tensor, then it is called torsion tensor.

The circumstance for which the torsion tensor is not zero does not follow from arguments of generality alone, but also from explicit examples: for instance, torsion does describe some essential properties of Lie groups, as it was discussed by Cartan. Cartan has been the first who pioneered into studying torsion [1–4], and this is the reason why today torsion is also known as Cartan tensor.

When back at the ending of the XIX century Ricci-Curbastro and Levi-Civita developed absolute differential calculus, or tensor calculus, they did it by assuming zero torsion to simplify computations, and the geometry they eventually obtained was entirely based on the existence of a Riemann metric, and so it was called Riemann geometry; nothing of this geometry is spoiled by allowing torsion to take its place in it, the only difference being that now the metric would be accompanied by the torsion, and the final setting is Riemann-Cartan geometry.

In the RC geometry, then, there are two fundamental objects, that is metric and torsion, or equivalently, metric and connection: the metric is employed to measure distances and angles while the connection is used to compute covariant derivatives. Again, there seems to be no relation between metric and topological properties and therefore metric and connection should be independent, a requirement that is implemented by asking that the covariant derivative of the metric vanishes, and when this happens we talk about metric-compatible connection.

There is another reason why this should be the case, and in order to better see it we have to recall that the metric is employed to measures distances and angles but also raise and lower tensorial indices; as a matter of fact, one may also reverse the argument, starting from the introduction of a fundamental tensor used for raising and lowering tensorial indices, and continuing by seeing how the requirement that raising and then lowering the same tensorial index leave the tensor unchanged implies the fundamental tensor be symmetric and not-degenerate, so being a Riemann metric, as we will have the opportunity to discuss in a while. Furthermore, the requirement that the raising and lowering of tensorial indices be possible in any case, and that is even for tensors that are covariant derivatives of some other tensor, does imply that the covariant derivative of the metric is zero, thus leading to the metric-compatibility of the connection itself.

Metric-compatible connections are decomposed into an antisymmetric part, given by the torsion tensor, and a symmetric part, which is given by a combination of torsion tensors plus a symmetric connection entirely written in terms of the metric and called Levi-Civita connection.

The fact that there exists a term built with torsion but nevertheless symmetric implies that there is more than one symmetric connection, and this is the source of some issue: from the fact that, while the most general connection is metric-compatible, some of its symmetric parts are not; to the fact that, among the different symmetric connections, we cannot choose which encodes the gravitational information according to the equivalence principle.

To better appreciate the above arguments, we have to recall that the principle of equivalence states that it is always possible to find a system of reference in which locally the gravitational field can be neglected; this principle has to be taken in parallel with a result know from geometry as Weyl theorem, stating that it is always possible to find a system of coordinates where in a point the symmetric part of the connection vanishes: putting both principles beside each other makes it clear that on the basis of Weyl theorem the principle of equivalence becomes a statement on the interpretation of gravity as what is described by the symmetric part of the connection.

Now, in a theory in which there is more than one symmetric connection the interpretation of gravity as what is contained in the symmetric connection is ambiguous.

Although quite in general, mathematically, the torsion tensor is non-zero, nevertheless there may be reasons for which, physically, it could be zero: as we just discussed, the principle of equivalence may be that reason [5].

Except that this is not the case: the principle of equivalence, by demanding that the unique gravitational field be stored within a single symmetric connection, may simply be implemented by insisting that all symmetric parts of the connection collapse down to a single one, a requirement that is mathematically translated into the fact that metric-compatible connections with completely antisymmetric torsion alone must be considered; incidentally, this requirement would also imply that all of the connections would have to be metric-compatible, and therefore the problem that some symmetric connections be not metric-compatible does not even arise. Or in alternative, one may even avoid to implement the principle of equivalence mathematically allowing for all possible symmetric connection and then declaring that the Levi-Civita symmetric connection be the one describing gravity; granted that this assumption may well look arbitrary, nevertheless one may assume it just the same. The torsion tensor may be completely antisymmetric or general, but whichever it is between the two, the torsion tensor is non-zero.

Once again, the circumstance for which, also in physical space-times, the torsion tensor is not zero does follows from arguments of generality but also from a specific property of the torsion itself: in fact, by writing the Riemann-Cartan geometry in anholonomic bases, the torsion can be seen as the strength of the potential arising from gauging the translation group, much in the same way in which the curvature is the strength of the potential arising from gauging the rotation group, as shown by Sciama and Kibble. What Sciama and Kibble proved was that torsion is not just a tensor that could be added, but a tensor that must be added, beside curvature, in order to have the possibility to completely describe translations, beside rotations, in a full Poincaré gauge theory [6, 7].

When at the beginning of the XX century Einstein developed his theory of gravity, he did it by assuming zero torsion because when torsion vanishes the Ricci tensor is symmetric and therefore it can be consistently coupled to the symmetric energy tensor, realizing the identification between the space-time curvature and its energy content expressed by Einstein field equations, which is the basic spirit of Einstein gravity; but now we know that generally in physics there is also another quantity of interest called spin, and that in its presence the energy is no longer symmetric: so nowadays having a non-symmetric Ricci tensor, beside a Cartan tensor, would allow for a more exhaustive coupling in gravity, where the curvature would still be coupled to the energy but now torsion would be coupled to the spin, realizing the identification between the space-time curvature and its energy content expressed by Einstein field equations and the identification between space-time torsion and its spin content expressed by the Sciama-Kibble field equations, in what is known to be the Einstein–Sciama-Kibble gravity [8].

The ESK theory of gravity is thus the most complete theory describing the dynamics of the space-time, and because torsion is coupled to the spin in the same spirit in which curvature is coupled to energy then it is the theory of space-time in which the coupling to its matter content is achieved most exhaustively; the pivotal point of the situation is therefore brought to the question asking whether there actually exist something possessing both spin and energy as a form of matter, which can profit from the setting of the ESK gravity to a total extent.

As a matter of fact, such a theory not only exists, but also it is very well known and established, and that is the spinor field theory, thoroughly investigated by Dirac.

With so much of insight, it is an odd circumstance that there be still such a controversy about the role of torsion beside that of curvature in gravity, and there may actually be several reasons for it: the single most important one may be that Einstein gravity was first published in the year 1916 when no spin was known and, despite being then insightful to set the torsion tensor to zero, when in the year 1928 Dirac came with a theory of spinors comprising an intrinsic spin, the successes of Einstein theory of gravity were already too good to make anyone wonder about the possibility of modifying it even slightly.

Of course, this is no scientific reason to hinder research, but sociologically it can be easy to understand why one would not light-heartedly go to look beyond something good, especially today that the successes of the Einstein theory of gravitation have become enormous.

The behaviour of being extremely careful in going beyond the known is justified, but this is not the only cause that keeps physicists from doing so and there are other psychological barriers that push against it in a more active manner: one such example is for instance what we can read in Weinberg's book of gravitation [9], in which the author proves that torsion must be zero as a result of the fact that in the manipulations of the equations of motion for test particles he generalizes Newton's law up to what seems to be the most general of its form and in that most general form no torsion is present whatsoever.

Of course Weinberg's results are correct, but expected: a first reason is that these results come from the fact that in presence of torsion auto-parallel (straightest) and geodesics (shortest) trajectories would fail to coincide, but as we have discussed above there may be reasons to consider torsion to be completely antisymmetric and in this case auto-parallel and geodesic trajectories would become identical. Yet another, important reason is that Weinberg, demonstrating that Newton's law in its most general instance contemplates no torsion, has proven that torsion cannot pertain to macroscopic domains, and this is natural since torsion is coupled to spin, which is a microscopic quantity, and as torsion is correspondingly a property of microscopic domains, it follows there is no reason to expect it in macroscopic cases. Would we want to investigate not the macroscopic situations but the microscopic situations, then we should not use Newton's law but Dirac equation: and then, even by following Weinberg's argument, we would be able to find torsion as a natural concept, as it has been explained very clearly by Hehl in [10]. Over time, there have been other reasons to oppose torsion in gravity even when spinors are taken into account, and we do not want to insist on such a discussion, but interested readers may consult a list of these arguments and their fallacies, for instance, in [11].

At the present state of our knowledge, there is not a single argument against the presence of torsion in gravity and when torsion is coupled to the spin in the same spirit in which curvature is coupled to energy the ensuing theory is the one for which the space-time is coupled to its matter content most exhaustively. Thus the necessity of having torsion beside curvature is reduced to the fact that there is spin beside energy in the most general case of matter distributions that we can find, and hence it is further reduced to the existence of the spinor fields, like those that are defined in the case of the Dirac theory.

As such, it may be a loss failing to investigate torsion gravity coupled to spinor fields, and in the following we are going to present and review the most recent findings.

From an entirely different starting point, we will begin a following part by reviewing the historical foundations of quantum physics: clearly, compared to torsion, quantum physics needs much less formal introduction, but still we believe that the historical presentation is not well known by all and that it is instructive to follow the original path in order to best comprehend some conceptual issues, and most importantly, where quantum principles are not necessarily well mathematically implemented. After filtering out all irrelevant details so to isolate the main issues, we will critically analyze them in order to argue in favour of spin as what could constitute the missing element for a complete description of quantum effects and hence for a way to improve their mathematical implementation.

Such a second introduction is not so independent after all, since the solution we want to present is based on the theory of torsion gravity and gauge potentials for spinor matter fields, the one we presented in the first part.

Furthermore, we will continue to employ the presented theory, with its intrinsic spin-torsion axial coupling, for the assessment of six of the known open problems in the standard models of cosmology and particle physics.

Finally, a discussion on exact solutions is done.

ONE: THEORIES

II. GENERAL GEOMETRY OF MATTER FIELDS

In this first part we introduce the physical theory that shall be our reference all throughout this entire work.

We will start with a most general introduction of the kinematic quantities. And we will continue by establishing their dynamical link in terms of the field equations that will be the central point of this presentation.

A. Geometry and its matter content

In this first section, we build the kinematic background by defining all fields in analogous ways based on the use of symmetry principles, although we shall later see that there shall naturally arise discriminations. Indeed we will find that there will be fields describing the environment, and which will be called geometric fields; and there will be fields that will describe what in such an environment can take place, and which will be called material fields.

1. *Geometric fields: tensor and gauge fields*

In this first subsection, we shall define the fundamental geometry that will form the background for the ensuing material theory, and we will present two different aspects of this geometry: one will be the construction of the most general absolute differential calculus, the other will be the construction of the most general abelian gauge theory.

a. *Tensor fields: from the most general coordinate indices to Lorentz indices*

In the following, we shall define from the most general geometric perspective the concept of tensor field; in doing so we will discriminate tensors according to whether their indices will be coordinate indices or Lorentz indices.

As we have mentioned in the introduction, at the beginning, the most fundamental definition we have to give is that of tensor field: given any two systems of coordinates as x and x' related by the most general coordinate transformation law $x'=x'(x)$ then a set of functions of these coordinates written with respect to the first and second system of coordinates as $T(x)$ and $T'(x')$ and such that for a coordinate transformation they are related by

$$T'^{\alpha\dots\sigma}_{\rho\dots\zeta}=\mathrm{sign}\ \det\left(\frac{\partial x'}{\partial x}\right)\frac{\partial x^{\beta}}{\partial x'^{\rho}}\dots\frac{\partial x^{\theta}}{\partial x'^{\zeta}}\frac{\partial x'^{\alpha}}{\partial x^{\nu}}\dots\frac{\partial x'^{\sigma}}{\partial x^{\tau}}T^{\nu\dots\tau}_{\beta\dots\theta} \quad (1)$$

is called tensor or pseudo-tensor, according to whether the sign of the determinant of the transformation is positive or negative respectively: for tensors with at least two upper or two lower indices, we may switch the two indices getting a tensor called transposition of the original tensor in those two indices, which may happen to be equal to the initial tensor up to the sign plus or minus, in which case the tensor is called symmetric or antisymmetric in those two indices respectively; given a tensor with at least one upper and one lower index, we can consider one of the upper and one of the lower indices forcing them to have the same value and performing the sum over all possible values of the indices, called contraction in those indices, and this process can be repeated until we reach a tensor whose contraction is zero, and said to be irreducible.

As it was again mentioned in the introduction, it is necessary to introduce operations among tensors, which have to be given by algebraic as well as differential operations, and they both have to respect the tensorial structure: algebraic operations can be assigned respecting the tensorial structure, since the sum of two tensors with a given indices disposition is a tensor with the same indices disposition and the product of any two tensors is another tensor; but also differential operations must be introduced covariantly although in general the derivative of a tensor is not a tensor, unless an additional structure is also introduced. To construct an operation that is able to generalize the usual derivative up to a derivative that respects covariance, we begin by noticing that because a tensor is a set of fields, in general it will have two types of variations: the first is due to the fact that tensors fields are fields, coordinate dependent, and so a local structure must be present, which is the partial derivative

$$\begin{aligned}{}_{\mathrm{local}}\Delta T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\beta_i}&=T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\beta_i}(x')-T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\beta_i}(x)=\\&=\partial_{\mu}T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\beta_i}(x)\delta x^{\mu}\end{aligned}$$

at the first order infinitesimal; the second is due to the fact that tensors fields are tensors, so a system of components, and thus a re-shuffling of the different components must be allowed, and the most general form in which this can be done while respecting the fact that the differential structure requires the linearity and the Leibniz rule is to employ a form given according to the expression

$$\begin{aligned}&{}_{\mathrm{structure}}\Delta T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i}=T'^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i}-T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i}=\\&=[(\delta\Gamma^{\alpha_1}_{\theta}T^{\theta\ldots\alpha_j}_{\beta_1\ldots\beta_i}+\ldots+\delta\Gamma^{\alpha_j}_{\theta}T^{\alpha_1\ldots\theta}_{\beta_1\ldots\beta_i})-\\&-(\delta\Gamma^{\theta}_{\beta_1}T^{\alpha_1\ldots\alpha_j}_{\theta\ldots\beta_i}+\ldots+\delta\Gamma^{\theta}_{\beta_i}T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\theta})]\end{aligned}$$

always at the first order of infinitesimal. In full we have

$$\begin{aligned}&\Delta T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i}={}_{\mathrm{local}}\Delta T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i}+{}_{\mathrm{structure}}\Delta T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i}=\\&=\partial_\mu T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i}(x)\delta x^\mu+\\&+[(\delta\Gamma^{\alpha_1}_{\theta}T^{\theta\ldots\alpha_j}_{\beta_1\ldots\beta_i}+\ldots+\delta\Gamma^{\alpha_j}_{\theta}T^{\alpha_1\ldots\theta}_{\beta_1\ldots\beta_i})-\\&-(\delta\Gamma^{\theta}_{\beta_1}T^{\alpha_1\ldots\alpha_j}_{\theta\ldots\beta_i}+\ldots+\delta\Gamma^{\theta}_{\beta_i}T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\theta})]\end{aligned}$$

at the first order infinitesimal, so defining $\delta\Gamma^{\alpha}_{\beta}=\Gamma^{\alpha}_{\beta\mu}\delta x^\mu$ and dividing by δx^μ we obtain that

$$\begin{aligned}&D_\mu T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i}=\partial_\mu T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i}+\\&+(\Gamma^{\alpha_1}_{\theta\mu}T^{\theta\ldots\alpha_j}_{\beta_1\ldots\beta_i}+\ldots+\Gamma^{\alpha_j}_{\theta\mu}T^{\alpha_1\ldots\theta}_{\beta_1\ldots\beta_i})-\\&-(\Gamma^{\theta}_{\beta_1\mu}T^{\alpha_1\ldots\alpha_j}_{\theta\ldots\beta_i}+\ldots+\Gamma^{\theta}_{\beta_i\mu}T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\theta})\end{aligned}$$

after taking the limit; this is the most general form of potential covariant derivative. To see that this derivative is indeed covariant we have to require that $\Gamma^{\alpha}_{\beta\mu}$ transforms with a specific non-tensorial transformation law such as to compensate for the non-tensorial transformation law of the partial derivative: for the simplest case of one tensorial index, we have that the derivative is

$$D_\iota V^\alpha=\partial_\iota V^\alpha+V^\beta\Gamma^{\alpha}_{\beta\iota}$$

whose transformation law is given by

$$\begin{aligned}&\frac{\partial x^\beta}{\partial x'^{\beta'}}\frac{\partial x'^{\alpha'}}{\partial x^\alpha}(\partial_\beta V^\alpha+V^\rho\Gamma^{\alpha}_{\rho\beta})=\frac{\partial x^\beta}{\partial x'^{\beta'}}\frac{\partial x'^{\alpha'}}{\partial x^\alpha}D_\beta V^\alpha=\\&=(D_\beta V^\alpha)'=(\partial_\beta V^\alpha+V^\rho\Gamma^{\alpha}_{\rho\beta})'=\partial_{\beta'}V^{\alpha'}+V^{\rho'}\Gamma'^{\alpha'}_{\rho'\beta'}=\\&=\frac{\partial x^\theta}{\partial x'^{\beta'}}\frac{\partial}{\partial x^\theta}\left(\frac{\partial x'^{\alpha'}}{\partial x^\alpha}V^\alpha\right)+\frac{\partial x^\beta}{\partial x'^{\beta'}}\frac{\partial x'^{\rho'}}{\partial x^\rho}V^\rho\Gamma'^{\alpha'}_{\rho'\beta'}=\\&=\frac{\partial x^\theta}{\partial x'^{\beta'}}\frac{\partial x'^{\alpha'}}{\partial x^\alpha}\frac{\partial V^\alpha}{\partial x^\theta}+\frac{\partial x^\theta}{\partial x'^{\beta'}}\frac{\partial}{\partial x^\theta}\frac{\partial x'^{\alpha'}}{\partial x^\alpha}V^\alpha+\frac{\partial x'^{\rho'}}{\partial x^\rho}V^\rho\Gamma'^{\alpha'}_{\rho'\beta'}\end{aligned}$$

in which terms with the derivatives disappear, leaving

$$\frac{\partial x^\beta}{\partial x'^{\beta'}}\frac{\partial x'^{\alpha'}}{\partial x^\alpha}V^\rho\Gamma^{\alpha}_{\rho\beta}=\frac{\partial x^\theta}{\partial x'^{\beta'}}\frac{\partial}{\partial x^\theta}\frac{\partial x'^{\alpha'}}{\partial x^\alpha}V^\alpha+\frac{\partial x'^{\rho'}}{\partial x^\rho}V^\rho\Gamma'^{\alpha'}_{\rho'\beta'}$$

and since this has to hold for any tensor, then

$$\frac{\partial x^\beta}{\partial x'^{\beta'}}\frac{\partial x'^{\alpha'}}{\partial x^\alpha}\Gamma^{\alpha}_{\rho\beta}=\frac{\partial x^\theta}{\partial x'^{\beta'}}\frac{\partial}{\partial x^\theta}\frac{\partial x'^{\alpha'}}{\partial x^\rho}+\frac{\partial x'^{\rho'}}{\partial x^\rho}\Gamma'^{\alpha'}_{\rho'\beta'}$$

which is the non-tensorial transformation law the set of coefficients $\Gamma^{\alpha}_{\rho\beta}$ has to undergo in order to ensure the tensoriality of the whole derivative, in this very specific case of a vectorial field. But the very same non-tensorial transformation law for $\Gamma^{\alpha}_{\rho\beta}$ can be used for all terms of the most general form of derivative for generic tensors, and as a consequence the result we have obtained is completely general. We needed to do the entire derivation to show that this covariant derivative is the most general that is possible at all. When we talk about generality, we mean both for the structure of the derivative and for the set of coefficients $\Gamma^{\alpha}_{\rho\beta}$ which, thus, have no specific symmetry properties in the lower indices: consequently, we have

$$\Gamma^{\alpha}_{\mu\nu}\equiv\tfrac{1}{2}(\Gamma^{\alpha}_{\mu\nu}+\Gamma^{\alpha}_{\nu\mu})+\tfrac{1}{2}(\Gamma^{\alpha}_{\mu\nu}-\Gamma^{\alpha}_{\nu\mu})$$

where the transformation properties of the full object is inherited by the first part, which is symmetric in the two lower indices and it can be indicated as

$$\Lambda^{\alpha}_{\mu\nu}=\tfrac{1}{2}(\Gamma^{\alpha}_{\mu\nu}+\Gamma^{\alpha}_{\nu\mu})$$

while the second part

$$Q^{\alpha}{}_{\mu\nu}=\Gamma^{\alpha}_{\mu\nu}-\Gamma^{\alpha}_{\nu\mu}$$

transforms as a tensor such that $Q^{\alpha}{}_{\mu\nu}=-Q^{\alpha}{}_{\nu\mu}$ and that is antisymmetric in its second pair of indices. With such a definition, we have the decomposition

$$\Gamma^{\alpha}_{\mu\nu}=\Lambda^{\alpha}_{\mu\nu}+\tfrac{1}{2}Q^{\alpha}{}_{\mu\nu}$$

in the most general case. As in the covariant derivatives the connection enters linearly, the splitting in symmetric and antisymmetric parts sums up to a linear combination of the tensor $Q^{\alpha}{}_{\mu\nu}$ plus the terms linear in the symmetric connection, which therefore forms yet another type of covariant derivative that is defined according to

$$\begin{aligned}&\nabla_\mu T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i}=\partial_\mu T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i}+\\&+(\Lambda^{\alpha_1}_{\theta\mu}T^{\theta\ldots\alpha_j}_{\beta_1\ldots\beta_i}+\ldots+\Lambda^{\alpha_j}_{\theta\mu}T^{\alpha_1\ldots\theta}_{\beta_1\ldots\beta_i})-\\&-(\Lambda^{\theta}_{\beta_1\mu}T^{\alpha_1\ldots\alpha_j}_{\theta\ldots\beta_i}+\ldots+\Lambda^{\theta}_{\beta_i\mu}T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\theta})\end{aligned}$$

and in it the fact that the symmetric connection is indeed symmetric allows for particularly simplified expressions in some special cases: for instance taking the symmetric covariant derivative of a tensor with all lower indices gives

$$\nabla_\mu T_{\beta_1\ldots\beta_i}=\partial_\mu T_{\beta_1\ldots\beta_i}-\Lambda^{\theta}_{\beta_1\mu}T_{\theta\ldots\beta_i}-\ldots-\Lambda^{\theta}_{\beta_i\mu}T_{\beta_1\ldots\theta}$$

which is particularly interesting because we see that the symmetric connection saturates always the same index in the upper position; therefore, if we further specialize onto the case in which the tensor is completely antisymmetric we obtain that the correspondingly completely antisymmetrized form of the covariant derivative eventually reads

$$\begin{aligned}&\nabla_{[\mu}T_{\beta\ldots\rho]}=\nabla_\mu T_{\beta\ldots\rho}-\nabla_\beta T_{\mu\ldots\rho}+\ldots-\nabla_\rho T_{\beta\ldots\mu}=\\&=\partial_\mu T_{\beta\ldots\rho}-\Lambda^{\sigma}_{\beta\mu}T_{\sigma\ldots\rho}-\ldots-\Lambda^{\sigma}_{\rho\mu}T_{\beta\ldots\sigma}-\\&-\partial_\beta T_{\mu\ldots\rho}+\Lambda^{\sigma}_{\mu\beta}T_{\sigma\ldots\rho}+\ldots+\Lambda^{\sigma}_{\rho\beta}T_{\mu\ldots\sigma}+\ldots\\&\ldots-\partial_\rho T_{\beta\ldots\mu}+\Lambda^{\sigma}_{\beta\rho}T_{\sigma\ldots\mu}+\ldots+\Lambda^{\sigma}_{\mu\rho}T_{\beta\ldots\sigma}=\\&=\partial_\mu T_{\beta\ldots\rho}-\partial_\beta T_{\mu\ldots\rho}+\ldots-\partial_\rho T_{\beta\ldots\mu}=\partial_{[\mu}T_{\beta\ldots\rho]}\end{aligned}$$

where all symmetric connections cancelled off leaving an expression written only in terms of partial derivatives but that is a completely antisymmetric covariant derivative in the most general case. This is a very peculiar property of tensors having all lower indices and being completely antisymmetric in all of these indices, and there is an entire domain related to this type of tensors and covariant derivatives, in which tensors are known as forms and the covariant derivatives are part of what is known as exterior calculus; nevertheless, we will not discuss it here because we do not want to introduce even further mathematical concepts and after all forms and exterior derivatives are nothing but a specific type of tensors. For our purposes, the most general covariant derivatives are well enough.

So to summarize what we have been doing so far, we have that the set of functions $\Gamma^{\rho}_{\alpha\beta}$ transforming as

$$\Gamma'^{\rho}_{\sigma\tau}=\left(\Gamma^{\alpha}_{\mu\nu}-\frac{\partial x^{\alpha}}{\partial x'^{\kappa}}\frac{\partial^2 x'^{\kappa}}{\partial x^{\nu}\partial x^{\mu}}\right)\frac{\partial x'^{\rho}}{\partial x^{\alpha}}\frac{\partial x^{\mu}}{\partial x'^{\sigma}}\frac{\partial x^{\nu}}{\partial x'^{\tau}} \tag{2}$$

is called connection and it can be decomposed as

$$\Gamma^{\rho}_{\alpha\beta}=\Lambda^{\rho}_{\alpha\beta}+\tfrac{1}{2}Q^{\rho}{}_{\alpha\beta} \tag{3}$$

where $\Lambda^{\rho}_{\alpha\beta}$ is a set of functions transforming according to the law of a connection but which are symmetric in the two lower indices, called symmetric connection, and

$$Q^{\rho}{}_{\alpha\beta}=\Gamma^{\rho}_{\alpha\beta}-\Gamma^{\rho}_{\beta\alpha} \tag{4}$$

which is a tensor antisymmetric in the two lower indices, and called torsion tensor. In terms of the connection we may write the covariant derivatives, but since there are two different connections we have to write two covariant derivatives, starting with the covariant derivative of the most general connection according to expression

$$D_{\mu}T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\beta_i}=\partial_{\mu}T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\beta_i}+\sum_{k=1}^{k=j}\Gamma^{\alpha_k}_{\sigma\mu}T^{\alpha_1\dots\sigma\dots\alpha_j}_{\beta_1\dots\beta_i}-$$
$$-\sum_{k=1}^{k=i}\Gamma^{\sigma}_{\beta_k\mu}T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\sigma\dots\beta_i} \tag{5}$$

decomposing as

$$D_{\mu}T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\beta_i}=\nabla_{\mu}T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\beta_i}+\tfrac{1}{2}\sum_{k=1}^{k=j}Q^{\alpha_k}_{\sigma\mu}T^{\alpha_1\dots\sigma\dots\alpha_j}_{\beta_1\dots\beta_i}-$$
$$-\tfrac{1}{2}\sum_{k=1}^{k=i}Q^{\sigma}_{\beta_k\mu}T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\sigma\dots\beta_i} \tag{6}$$

with spurious terms that are linear in the torsion tensor and the covariant derivative of the symmetric connection

$$\nabla_{\mu}T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\beta_i}=\partial_{\mu}T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\beta_i}+\sum_{k=1}^{k=j}\Lambda^{\alpha_k}_{\sigma\mu}T^{\alpha_1\dots\sigma\dots\alpha_j}_{\beta_1\dots\beta_i}-$$
$$-\sum_{k=1}^{k=i}\Lambda^{\sigma}_{\beta_k\mu}T^{\alpha_1\dots\alpha_j}_{\beta_1\dots\sigma\dots\beta_i} \tag{7}$$

as it is clear: notice that if we apply such last definition to the particular case of tensors with all lower indices and having the property of being completely antisymmetric and further if we take its completely antisymmetric part, we obtain a form in which all occurrences of the symmetric connection disappear leaving only the form

$$\nabla_{[\nu}T_{\alpha\dots\sigma]}=\partial_{[\nu}T_{\alpha\dots\sigma]}\equiv(\partial T)_{\nu\alpha\dots\sigma} \tag{8}$$

which is still a tensor and such that it is completely antisymmetric, called covariant gradient of the tensor field.

We have introduced the concept of tensor, which was characterized by having two types of indices, upper and lower; they reflected the fact that tensors could transform according to two type of transformations, direct and inverse: because these two types of transformation are two different forms of the same transformation law, the two types of indices should be two different arrangements of the same system of components. In particular, there should not be any difference in the content of information for any two different indices dispositions in any tensor.

What this implies is that it should be possible to move indices up and down without losing or adding anything to the information content: this can be done by considering the Kronecker tensor δ^{α}_{ν} and postulating the existence of two tensors $g_{\alpha\nu}$ and $g^{\alpha\nu}$ in general; then we can define the operation of raising and lowering of tensorial indices by considering that $A^{\pi}g_{\pi\nu}$ and $A_{\pi}g^{\pi\nu}$ are tensors that are related to the initial ones but with the index lowered and raised respectively, and so we may define these two tensors as $A_{\pi}g^{\pi\nu}\equiv A^{\nu}$ and $A^{\pi}g_{\pi\nu}\equiv A_{\nu}$ as the same tensors but with the index moved in a different position with respect to the initial one. While it is certainly useful to have the possibility to perform such an operation, we have also to consider that such an operation has a two-fold ambiguity concerning the fact beside the contractions $A_{\pi}g^{\pi\nu}\equiv A^{\nu}$ and $A^{\pi}g_{\pi\nu}\equiv A_{\nu}$ we may have the contractions $A_{\pi}g^{\nu\pi}\equiv A^{\nu}$ and $A^{\pi}g_{\nu\pi}\equiv A_{\nu}$ too; also we may decide to raise the previously lowered index to the initial position or lower the previously raise index to the initial position, so that the above ambiguity becomes four-fold with $A_{\pi}g^{\pi\nu}g_{\sigma\nu}\equiv A_{\sigma}$ and $A_{\pi}g^{\nu\pi}g_{\sigma\nu}\equiv A_{\sigma}$ as well as $A_{\pi}g^{\pi\nu}g_{\nu\sigma}\equiv A_{\sigma}$ and $A_{\pi}g^{\nu\pi}g_{\nu\sigma}\equiv A_{\sigma}$ as equally good possibilities that may be considered: on the other hand, requiring that raising one index up and then lowering that index down give back the initial tensor in all of the four possibilities leads to the following relationships

$$A_{\mu}(g^{\mu\sigma}g_{\sigma\kappa}-\delta^{\mu}_{\kappa})=0 \qquad A_{\mu}(g^{\sigma\mu}g_{\sigma\kappa}-\delta^{\mu}_{\kappa})=0$$
$$A_{\mu}(g^{\mu\sigma}g_{\kappa\sigma}-\delta^{\mu}_{\kappa})=0 \qquad A_{\mu}(g^{\sigma\mu}g_{\kappa\sigma}-\delta^{\mu}_{\kappa})=0$$

for any possible tensor A_{μ}, so that

$$(g^{\mu\sigma}g_{\sigma\kappa}-\delta^{\mu}_{\kappa})=0 \qquad (g^{\sigma\mu}g_{\sigma\kappa}-\delta^{\mu}_{\kappa})=0$$
$$(g^{\mu\sigma}g_{\kappa\sigma}-\delta^{\mu}_{\kappa})=0 \qquad (g^{\sigma\mu}g_{\kappa\sigma}-\delta^{\mu}_{\kappa})=0$$

identically, and taking the differences

$$g^{\mu\sigma}(g_{\sigma\kappa}-g_{\kappa\sigma})=0 \qquad (g^{\sigma\mu}-g^{\mu\sigma})g_{\sigma\kappa}=0$$
$$g^{\sigma\mu}(g_{\sigma\kappa}-g_{\kappa\sigma})=0 \qquad (g^{\sigma\mu}-g^{\mu\sigma})g_{\kappa\sigma}=0$$

we may work out that

$$g_{\alpha\kappa}=g_{\kappa\alpha}$$
$$g^{\alpha\kappa}=g^{\kappa\alpha}$$

together with the condition

$$g^{\sigma\mu}g_{\kappa\sigma}=\delta^{\mu}_{\kappa}$$

meaning that, seen as matrices, they are symmetric and one the inverse of the other, and so in particular they are non-degenerate, as it has been demonstrated in [12]. This implies is that what has been introduced to raise lower or lower upper indices has all the features of a metric and therefore these two tensors can also be identified with the metric of the space-time; we remark that this is exactly the opposite to the normal approach, where the metric is postulated, and then it is realized it can be used to move up and down indices of tensors. The equivalence of these two a priori unrelated operations is something that looks profound. In addition, there are other considerations to do and which involve the metric determinant as we will discuss now: to begin, we define the following quantity

$$\delta^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} = \det \begin{vmatrix} \delta^{i_1}_{j_1} & \delta^{i_2}_{j_1} & \delta^{i_3}_{j_1} & \delta^{i_4}_{j_1} \\ \delta^{i_1}_{j_2} & \delta^{i_2}_{j_2} & \delta^{i_3}_{j_2} & \delta^{i_4}_{j_2} \\ \delta^{i_1}_{j_3} & \delta^{i_2}_{j_3} & \delta^{i_3}_{j_3} & \delta^{i_4}_{j_3} \\ \delta^{i_1}_{j_4} & \delta^{i_2}_{j_4} & \delta^{i_3}_{j_4} & \delta^{i_4}_{j_4} \end{vmatrix}$$

which is a tensor, antisymmetric in each of its pair of tensorial indices; then by indicating the metric determinant according to $\det(g_{\mu\nu}) = g$ with sign $g = -1$ we see that

$$g' = \det \left|\frac{\partial x}{\partial x'}\right|^2 g$$

which is not the transformation law for a tensor. But in addition we may also define the non-tensorial quantity that is given by $\epsilon_{i_1 i_2 i_3 i_4}$ such that it is equal to the unity for an even permutation of (1234) and minus the unity for an odd permutation of (1234) and zero for a sequence that is not a permutation of (1234) at all: as this set of coefficients is completely antisymmetric with a number of indices that is equal to the dimension, we have that it has only one independent component, transforming as

$$\frac{\partial x^{i_1}}{\partial x'^{i'_1}} \frac{\partial x^{i_2}}{\partial x'^{i'_2}} \frac{\partial x^{i_3}}{\partial x'^{i'_3}} \frac{\partial x^{i_4}}{\partial x'^{i'_4}} \epsilon_{i_1 i_2 i_3 i_4} = \epsilon_{i'_1 i'_2 i'_3 i'_4} \alpha$$

for a given α function to be determined, and because the determinant of any generic matrix can always be written in terms of these coefficients according to the expression given by $\det M = \Sigma_{i_j} \epsilon_{i_1 i_2 i_3 i_4} M^{1 i_1} M^{2 i_2} M^{3 i_3} M^{4 i_4}$ then

$$\det \frac{\partial x}{\partial x'} = \frac{\partial x^{i_1}}{\partial x'^1} \frac{\partial x^{i_2}}{\partial x'^2} \frac{\partial x^{i_3}}{\partial x'^3} \frac{\partial x^{i_4}}{\partial x'^4} \epsilon_{i_1 i_2 i_3 i_4} = \epsilon_{1234} \alpha = \alpha$$

furnishing the α function, so that we have

$$\epsilon_{i'_1 i'_2 i'_3 i'_4} = \det \frac{\partial x'}{\partial x} \frac{\partial x^{i_1}}{\partial x'^{i'_1}} \frac{\partial x^{i_2}}{\partial x'^{i'_2}} \frac{\partial x^{i_3}}{\partial x'^{i'_3}} \frac{\partial x^{i_4}}{\partial x'^{i'_4}} \epsilon_{i_1 i_2 i_3 i_4}$$

which is non-tensorial, but its non-tensoriality perfectly matches that of the determinant of the metric. Therefore, we have that they compensate in the combined form

$$(g^{\frac{1}{2}} \epsilon_{\alpha\nu\sigma\tau})' = \text{sign } \det \left|\frac{\partial x'}{\partial x}\right| \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x^\theta}{\partial x'^\sigma} \frac{\partial x^\rho}{\partial x'^\tau} (g^{\frac{1}{2}} \epsilon_{\beta\mu\theta\rho})$$

which is in fact the transformation law that defines a pseudo-tensorial field; notice however that if we were to define the tensor with all lower indices as

$$\varepsilon_{\alpha\nu\sigma\tau} = \epsilon_{\alpha\nu\sigma\tau} |g|^{\frac{1}{2}}$$

the correspondent tensor with all upper indices would be given according to the following expression

$$\varepsilon^{\alpha\nu\sigma\tau} = \epsilon^{\alpha\nu\sigma\tau} |g|^{-\frac{1}{2}}$$

in order for it to be consistently defined. This difference is necessary, as it can be seen from the fact the relationship

$$\varepsilon^{i_1 i_2 i_3 i_4} \varepsilon_{j_1 j_2 j_3 j_4} = -\delta^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4}$$

as it is very easy to check by performing a straightforward substitution and making all the direct calculations.

To summarize, the object δ^β_α that it is unity or zero according to whether the value of its indices is equal or different is called unity tensor, and then we assume the existence of two tensors $g_{\alpha\kappa}$ and $g^{\alpha\kappa}$ symmetric and seen as matrices one the inverse of the other according to

$$g^{\sigma\mu} g_{\kappa\sigma} = \delta^\mu_\kappa \tag{9}$$

called metric tensors: we may define

$$\delta^{i_0 i_1 i_2 i_3}_{j_0 j_1 j_2 j_3} = \det \begin{vmatrix} \delta^{i_0}_{j_0} & \delta^{i_1}_{j_0} & \delta^{i_2}_{j_0} & \delta^{i_3}_{j_0} \\ \delta^{i_0}_{j_1} & \delta^{i_1}_{j_1} & \delta^{i_2}_{j_1} & \delta^{i_3}_{j_1} \\ \delta^{i_0}_{j_2} & \delta^{i_1}_{j_2} & \delta^{i_2}_{j_2} & \delta^{i_3}_{j_2} \\ \delta^{i_0}_{j_3} & \delta^{i_1}_{j_3} & \delta^{i_2}_{j_3} & \delta^{i_3}_{j_3} \end{vmatrix} \tag{10}$$

as a completely antisymmetric unity tensor, and also the quantity $\epsilon_{i_0 i_1 i_2 i_3}$ equal to the unity, minus unity, or zero according to whether $(i_0 i_1 i_2 i_3)$ is an even, odd, or no permutation of (0123) which can be taken, together with the metric determinant $\det(g_{\mu\nu}) = g$ in general, to define

$$\varepsilon^{\alpha\nu\sigma\tau} = \epsilon^{\alpha\nu\sigma\tau} |g|^{-\frac{1}{2}} \tag{11}$$

and also

$$\varepsilon_{\alpha\nu\sigma\tau} = \epsilon_{\alpha\nu\sigma\tau} |g|^{\frac{1}{2}} \tag{12}$$

which are completely antisymmetric and such that

$$\varepsilon^{i_0 i_1 i_2 i_3} \varepsilon_{j_0 j_1 j_2 j_3} = -\delta^{i_0 i_1 i_2 i_3}_{j_0 j_1 j_2 j_3} \tag{13}$$

called completely antisymmetric pseudo-tensors. So if a tensor with at least one index is multiplied by the metric tensor and the index is contracted with one index of the metric tensor, the result is a tensor in which the index has been moved: in particular if a tensor that is completely antisymmetric in k indices is multiplied by the completely antisymmetric pseudo-tensors and the k indices of the tensor are contracted with k indices of the completely antisymmetric pseudo-tensors, the result is a pseudo-tensor antisymmetric in $(4-k)$ of its indices.

An important point we may ask now concerns the fact that if the indices disposition cannot change the information content of a tensor then this must be true for any tensor, in particular if the tensor is the covariant derivative of some other tensor: consequently we must have that $g^{\alpha\beta} D_\mu T^{\nu\ldots\zeta}_{\beta\rho\sigma\ldots\theta} = D_\mu T^{\alpha\nu\ldots\zeta}_{\rho\sigma\ldots\theta}$ which therefore implies

$$D_\mu T^{\alpha\nu\ldots\zeta}_{\rho\sigma\ldots\theta} = D_\mu (g^{\alpha\beta} T^{\nu\ldots\zeta}_{\beta\rho\sigma\ldots\theta}) = D_\mu g^{\alpha\beta} T^{\nu\ldots\zeta}_{\beta\rho\sigma\ldots\theta} + \\ + g^{\alpha\beta} D_\mu T^{\nu\ldots\zeta}_{\beta\rho\sigma\ldots\theta}$$

so that we are left with the equation

$$D_\mu g^{\alpha\beta}T^{\nu\ldots\zeta}_{\beta\rho\sigma\ldots\theta}=0$$

for any tensor, implying $D_\mu g^{\alpha\beta}=0$ as well; this means that the metric tensor is covariantly constant. Conditions of vanishing of the covariant derivative of the metric tensor mean that the irrelevance of the indices disposition must be valid regardless the differential order of the tensor; if we were to follow the common approach defining the metric first, these conditions would mean that the metric structure and the topological structure are to be independent. This is reasonable since if a vector is constant, its norm should be constant too. It is interesting to notice that since we have two types of covariant derivatives and because the present arguments hold regardless the specific covariant derivative then we have to assume that both covariant derivatives of the metric tensor vanish as $D_\mu g_{\alpha\beta}=\nabla_\mu g_{\alpha\beta}=0$ in general: in particular we have that $D_\theta\varepsilon_{\alpha\beta\mu\nu}=\nabla_\theta\varepsilon_{\alpha\beta\mu\nu}=0$ hold as well. When we expand the connection-metric compatibility condition as

$$\partial_\rho g_{\alpha\beta}-g_{\alpha\mu}\Gamma^\mu_{\beta\rho}-g_{\mu\beta}\Gamma^\mu_{\alpha\rho}=0$$

we may take the three different indices permutations combined together with the definition of torsion to get

$$\Gamma^\rho_{\alpha\beta}=\tfrac{1}{2}Q^\rho{}_{\alpha\beta}+\tfrac{1}{2}(Q_{\alpha\beta}{}^\rho+Q_{\beta\alpha}{}^\rho)+$$
$$+\tfrac{1}{2}g^{\rho\mu}\left(\partial_\beta g_{\alpha\mu}+\partial_\alpha g_{\mu\beta}-\partial_\mu g_{\alpha\beta}\right)$$

in which $Q_{\rho\alpha\sigma}$ is the torsion tensor antisymmetric in the two lower indices, while $(Q_{\alpha\beta\rho}+Q_{\beta\alpha\rho})$ is a tensor symmetric in those indices whereas the remaining coefficients written in terms of the partial derivatives of the metric tensor transform as a connection and they are symmetric in those very indices: this expression shows that the most general connection can be decomposed in terms of the torsion plus a symmetric connection, as we already knew from expression (3), but in addition it tells us the explicit form of $\Lambda^\rho_{\alpha\beta}$ as given by a symmetric combination of two torsions plus a symmetric connection entirely written in terms of the metric; it is essential to remark that if we want all possible connections to give rise to covariant derivatives which, once applied onto the metric, give zero, then we have to restrict the torsion to verify

$$Q_{\alpha\beta\rho}=-Q_{\beta\alpha\rho}$$

spelling its complete antisymmetry [13]. The condition of metric-compatible connection extended to all connections implies the torsion to be completely antisymmetric, once again establishing a link between two structures that are a priori unrelated; that such a link is more profound than we may think can also be seen by considering what is the meaning of the metric tensor in terms of metric concepts and thus the meaning of metric-compatibility in terms of those metric concepts. So let us step back to reconsider the metric-compatibility condition above. It is

$$\partial_\rho g_{\alpha\beta}-g_{\alpha\mu}\Gamma^\mu_{\beta\rho}-g_{\mu\beta}\Gamma^\mu_{\alpha\rho}=0$$

and whenever it holds then the metric tensor becomes constant and that specific combination of connections can be vanished in the same coordinate system; further, if the metric tensor is constant it follows it can be diagonalized and normalized as to be written it in its most trivial form, the Minkowskian matrix: the Minkowskian matrix is the form in which the symmetries of the space-time happen to be manifest. Because this manifestation of space-time symmetries takes place in a system of coordinates that is unique, so the constancy of the metric tensor as well as the vanishing of that specific combination of connections must occur in a system of coordinates that is also unique in its construction: as a consequence of the fact that that specific combination of connections has unicity it follows that torsion has complete antisymmetry. Therefore, that the condition of metric-compatibility for the connection require torsion to be completely antisymmetric looks as a circumstance coming from the existence of a single symmetric part of the connection, itself coming from the fact that this single symmetric part of the connection has to vanish and the metric has to make the symmetries of the space-time manifest in one system of coordinates that is uniquely defined. That completely antisymmetric torsion has a somewhat peculiar role has been discussed in some works such as [14, 15] and [16, 17] and references therein.

So we summarize by saying that the connection

$$\Lambda^\rho_{\alpha\beta}=\tfrac{1}{2}g^{\rho\mu}\left(\partial_\beta g_{\alpha\mu}+\partial_\alpha g_{\mu\beta}-\partial_\mu g_{\alpha\beta}\right)\tag{14}$$

is symmetric and written entirely in terms of the partial derivatives of the metric tensor, and it is called metric connection, while the torsion tensor with all lower indices is taken to be completely antisymmetric and therefore it is possible to write it according to the following form

$$Q_{\alpha\sigma\nu}=\tfrac{1}{6}W^\mu\varepsilon_{\mu\alpha\sigma\nu}\tag{15}$$

in terms of the W^μ pseudo-vector, therefore called torsion pseudo-vector, and with these two quantities we have

$$\Gamma^\rho_{\alpha\beta}=\tfrac{1}{2}g^{\rho\mu}\left[(\partial_\beta g_{\alpha\mu}+\partial_\alpha g_{\mu\beta}-\partial_\mu g_{\alpha\beta})+\tfrac{1}{6}W^\nu\varepsilon_{\nu\mu\alpha\beta}\right]\tag{16}$$

as the most general connection. Such a decomposition is equivalent to the validity of the following conditions

$$(\partial\varepsilon)_{\theta\alpha\beta\mu\nu}\equiv\nabla_\theta\varepsilon_{\alpha\beta\mu\nu}\equiv D_\theta\varepsilon_{\alpha\beta\mu\nu}=0\tag{17}$$
$$\nabla_\mu g_{\alpha\beta}\equiv D_\mu g_{\alpha\beta}=0\tag{18}$$

called metric-compatibility conditions for the connection.

We may proceed on to calculate the commutator of two derivatives, which in the particular case of vectors is

$$[D_\alpha,D_\beta]T^\sigma=(\Gamma^\rho_{\alpha\beta}-\Gamma^\rho_{\beta\alpha})D_\rho T^\sigma+$$
$$+(\partial_\alpha\Gamma^\sigma_{\kappa\beta}-\partial_\beta\Gamma^\sigma_{\kappa\alpha}+\Gamma^\rho_{\kappa\beta}\Gamma^\sigma_{\rho\alpha}-\Gamma^\rho_{\kappa\alpha}\Gamma^\sigma_{\rho\beta})T^\kappa$$

with no second derivatives, the only derivative term left is proportional to the torsion tensor $Q^\rho_{\mu\alpha}$ plus another

$$G^\sigma_{\kappa\alpha\beta}=\partial_\alpha\Gamma^\sigma_{\kappa\beta}-\partial_\beta\Gamma^\sigma_{\kappa\alpha}+\Gamma^\rho_{\kappa\beta}\Gamma^\sigma_{\rho\alpha}-\Gamma^\rho_{\kappa\alpha}\Gamma^\sigma_{\rho\beta}$$

which although written in terms of the connection alone is a tensor; with these expressions we have

$$[D_\alpha, D_\beta]T^\sigma = Q^\rho_{\alpha\beta} D_\rho T^\sigma + G^\sigma_{\kappa\alpha\beta} T^\kappa$$

giving the commutator of vectors in particular: as it has been done for the connection and the most general covariant derivative, the interesting thing is that the definition of tensor $G^\sigma_{\kappa\alpha\beta}$ can be used in the most general case of commutator of covariant derivatives. We also have that

$$\begin{aligned}(\partial\partial T)_{\alpha\beta\rho\ldots\mu} = \partial_{[\alpha}(\partial T)_{\beta\rho\ldots\mu]} = \partial_{[\alpha}\partial_{[\beta}T_{\rho\ldots\mu]]} = \\ = \partial_{[\alpha}\partial_\beta T_{\rho\ldots\mu]} = 0\end{aligned}$$

because partial derivatives always commute and therefore their commutator is always zero; before we have had the opportunity to briefly talk about external calculus, where the external derivatives are used to calculate the border of a manifold, and the above expression refers to the fact that the border has a border that vanishes, or that there is no border of a border. Once again, apart from curiosity, there is no need to deepen these concepts in the following.

To summarize, from the connection we may calculate

$$G^\sigma{}_{\kappa\alpha\beta} = \partial_\alpha \Gamma^\sigma_{\kappa\beta} - \partial_\beta \Gamma^\sigma_{\kappa\alpha} + \Gamma^\sigma_{\rho\alpha}\Gamma^\rho_{\kappa\beta} - \Gamma^\sigma_{\rho\beta}\Gamma^\rho_{\kappa\alpha} \quad (19)$$

which is a tensor antisymmetric in the last two indices and verifying the following cyclic permutation condition

$$\begin{aligned}D_\kappa Q^\rho{}_{\mu\nu} + D_\nu Q^\rho{}_{\kappa\mu} + D_\mu Q^\rho{}_{\nu\kappa} + \\ + Q^\pi{}_{\nu\kappa} Q^\rho{}_{\mu\pi} + Q^\pi{}_{\mu\nu} Q^\rho{}_{\kappa\pi} + Q^\pi{}_{\kappa\mu} Q^\rho{}_{\nu\pi} - \\ - G^\rho{}_{\kappa\nu\mu} - G^\rho{}_{\mu\kappa\nu} - G^\rho{}_{\nu\mu\kappa} \equiv 0\end{aligned} \quad (20)$$

called Riemann curvature tensor, decomposable as

$$\begin{aligned}G^\sigma{}_{\kappa\alpha\beta} = R^\sigma{}_{\kappa\alpha\beta} + \tfrac{1}{2}(\nabla_\alpha Q^\sigma{}_{\kappa\beta} - \nabla_\beta Q^\sigma{}_{\kappa\alpha}) + \\ + \tfrac{1}{4}(Q^\sigma{}_{\rho\alpha} Q^\rho{}_{\kappa\beta} - Q^\sigma{}_{\rho\beta} Q^\rho{}_{\kappa\alpha})\end{aligned} \quad (21)$$

in which it is in terms of the symmetric connection that

$$R^\sigma{}_{\kappa\alpha\beta} = \partial_\alpha \Lambda^\sigma_{\kappa\beta} - \partial_\beta \Lambda^\sigma_{\kappa\alpha} + \Lambda^\sigma_{\rho\alpha}\Lambda^\rho_{\kappa\beta} - \Lambda^\sigma_{\rho\beta}\Lambda^\rho_{\kappa\alpha} \quad (22)$$

is a tensor antisymmetric in the last two indices such that it verifies that cyclic permutation condition

$$R^\rho{}_{\kappa\nu\mu} + R^\rho{}_{\mu\kappa\nu} + R^\rho{}_{\nu\mu\kappa} \equiv 0 \quad (23)$$

called Riemann symmetric curvature tensor and where the torsion tensor is known. By employing torsion and curvature it is possible to demonstrate that we have

$$\begin{aligned}[D_\mu, D_\nu] T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i} = Q^\eta{}_{\mu\nu} D_\eta T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\beta_i} + \\ + \textstyle\sum_{k=1}^{k=j} G^{\alpha_k}{}_{\sigma\mu\nu} T^{\alpha_1\ldots\sigma\ldots\alpha_j}_{\beta_1\ldots\beta_i} - \\ - \textstyle\sum_{k=1}^{k=i} G^\sigma{}_{\beta_k\mu\nu} T^{\alpha_1\ldots\alpha_j}_{\beta_1\ldots\sigma\ldots\beta_i}\end{aligned} \quad (24)$$

as the expression for commutator of covariant derivatives of the tensor field: in particular we have that

$$\partial\partial T = 0 \quad (25)$$

which is valid in the most general circumstance.

Clearly, all these quantities can be written for the symmetric connection only, so that torsionful curvatures can be written as torsionless curvatures plus torsion terms.

We have the validity of the following decomposition

$$\begin{aligned}R_{\kappa\rho\alpha\mu} = \tfrac{1}{2}(\partial_\alpha\partial_\rho g_{\mu\kappa} - \partial_\mu\partial_\rho g_{\kappa\alpha} + \\ + \partial_\mu\partial_\kappa g_{\alpha\rho} - \partial_\kappa\partial_\alpha g_{\mu\rho}) + \\ + \tfrac{1}{4} g^{\sigma\nu}[(\partial_\rho g_{\alpha\nu} + \partial_\alpha g_{\rho\nu} - \partial_\nu g_{\rho\alpha}) \\ (\partial_\kappa g_{\mu\sigma} + \partial_\mu g_{\kappa\sigma} - \partial_\sigma g_{\kappa\mu}) - \\ - (\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\rho\mu}) \\ (\partial_\kappa g_{\alpha\sigma} + \partial_\alpha g_{\kappa\sigma} - \partial_\sigma g_{\kappa\alpha})]\end{aligned} \quad (26)$$

showing the antisymmetry also in the first two indices as well as the symmetry involving all four indices

$$R_{\rho\kappa\mu\nu} = R_{\mu\nu\rho\kappa} \quad (27)$$

called Riemann metric curvature tensor, with one independent contraction as $R_{\mu\sigma} = R^\rho{}_{\mu\rho\sigma}$ which is symmetric and it is called Ricci metric curvature tensor, and whose contraction $R = R_{\mu\sigma} g^{\mu\sigma}$ is called Ricci metric curvature scalar, and such that with torsion we can decompose

$$\begin{aligned}G_{\kappa\rho\alpha\mu} = \tfrac{1}{2}(\partial_\alpha\partial_\rho g_{\mu\kappa} - \partial_\mu\partial_\rho g_{\kappa\alpha} + \\ + \partial_\mu\partial_\kappa g_{\alpha\rho} - \partial_\kappa\partial_\alpha g_{\mu\rho}) + \\ + \tfrac{1}{4} g^{\sigma\nu}[(\partial_\rho g_{\alpha\nu} + \partial_\alpha g_{\rho\nu} - \partial_\nu g_{\rho\alpha}) \\ (\partial_\kappa g_{\mu\sigma} + \partial_\mu g_{\kappa\sigma} - \partial_\sigma g_{\kappa\mu}) - \\ - (\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\rho\mu}) \\ (\partial_\kappa g_{\alpha\sigma} + \partial_\alpha g_{\kappa\sigma} - \partial_\sigma g_{\kappa\alpha})] + \\ + \tfrac{1}{12} \nabla^\eta W^\sigma (g_{\alpha\eta}\varepsilon_{\sigma\kappa\rho\mu} - g_{\mu\eta}\varepsilon_{\sigma\kappa\rho\alpha}) + \\ + \tfrac{1}{144}[W_\sigma W^\sigma (g_{\mu\rho} g_{\alpha\kappa} - g_{\mu\kappa} g_{\alpha\rho}) + \\ + (W_\alpha W_\rho g_{\mu\kappa} - W_\mu W_\rho g_{\alpha\kappa} + \\ + W_\mu W_\kappa g_{\alpha\rho} - W_\alpha W_\kappa g_{\mu\rho})]\end{aligned} \quad (28)$$

showing the antisymmetry in the first two indices, and as a consequence it has one independent contraction chosen according to $G_{\mu\sigma} = G^\rho{}_{\mu\rho\sigma}$ called Ricci curvature tensor, whose contraction $G = G_{\mu\sigma} g^{\mu\sigma}$ is called Ricci curvature scalar, thus extinguishing the curvature contractions.

And finally, we may consider the cyclic permutation of commutator of commutators of covariant derivatives and see that the results are geometric identities.

In general we have that we can write

$$\begin{aligned}D_\mu G^\nu{}_{\iota\kappa\rho} + D_\kappa G^\nu{}_{\iota\rho\mu} + D_\rho G^\nu{}_{\iota\mu\kappa} + \\ + G^\nu{}_{\iota\beta\mu} Q^\beta{}_{\rho\kappa} + G^\nu{}_{\iota\beta\kappa} Q^\beta{}_{\mu\rho} + G^\nu{}_{\iota\beta\rho} Q^\beta{}_{\kappa\mu} \equiv 0\end{aligned} \quad (29)$$

for torsion and curvature valid as a geometric identity.

So far we have introduced the concept of tensor and the way to move its indices, which we recall were coordinate indices; coordinate indices are important since they are the type of indices involved in differentiation, but on the other hand, tensors in coordinate indices always feel the specificity of the coordinate system: tensorial equations do remain formally the same in all coordinate system, but the tensors themselves change in content while changing the coordinate system. The only types of tensors which, also in content, remain the same in all of the coordinate systems are the tensors that are identically equal to zero and the scalars; zero tensors offer little information, but scalars can be used to build a formalism in which tensors can be rendered, both in form and in content, completely invariant. This formalism is known as Lorentz formalism.

In Lorentz formalism, the idea is that of introducing a basis of vectors ξ_a^α having two types of indices: one type of indices (Greek) is the usual coordinate index referring to the component of the vector, whereas the other type of indices (Latin) is a new Lorentz index referring to which vector of the basis we are considering; under the point of view of coordinate transformations the coordinate index ensures the transformation law of a vector, but clearly the other index does not ensure any transformation much as if it were a scalar. To better understand, consider as an example the tensor given by $T_{\alpha\sigma}$ and multiply it by two of the vectors ξ_a^α of the basis contracting the coordinate indices together: the result $T_{\alpha\sigma}\xi_a^\alpha\xi_s^\sigma = T_{as}$ is an object that according to a coordinate transformation law does not transform at all, thus it is completely invariant, which is exactly what we wanted. And the process of making it is the basis of the Lorentz formalism. That tensors with upper indices must be converted into this formalism as well can be taken into account by introducing a dual basis of covectors ξ_α^a as expected: converting a coordinate index to a Lorentz index and then back to a coordinate index requires that $\xi_b^\alpha\xi_\alpha^c = \delta_b^c$ and $\xi_k^\alpha\xi_\sigma^k = \delta_\sigma^\alpha$ as a simple consistency condition. Finally, the operation for moving Lorentz indices is performed in terms of the metric tensor in Lorentz form $g_{\alpha\sigma}\xi_a^\alpha\xi_s^\sigma = g_{as}$ but, because we can always ortho-normalize the basis, the metric tensor in Lorentz form is just the Minkowskian matrix $g_{as} = \eta_{as}$ as it is well known indeed. Once the basis ξ_a^σ is assigned, we may pass to another basis $\xi_a'^\sigma$ linked to the initial according to the transformation $\xi_a'^\sigma = \Lambda_a^b\xi_b^\sigma$ with Λ_a^b chosen as to preserve the structure of the Minkowskian matrix and so such that it has to give $\eta = \Lambda\eta\Lambda^T$ known as Lorentz transformation and justifying the name of this formalism: after that the coordinate tensors are converted into the Lorentz tensors, they are scalars under coordinate transformations, as we have said above, but they are tensors under the Lorentz transformations. In doing so it may seem that we did not gain much, but having coordinate transformations fully converted into Lorentz transformations is an advantage, since Lorentz transformations have a very specific form, which can be made explicit. We know from the theory of Lie groups that any continuous transformation is writable according to a perturbative expansion in products of the infinitesimal parameters times their generators; writing the infinitesimal form $\Lambda = \mathbb{I} + \delta G$ we get that $\delta G\eta$ must be antisymmetric, and we know that 4-dimensional antisymmetric matrices have 6 degrees of freedom: therefore

$$\Lambda = e^{\frac{1}{2}\sigma^{ab}\theta_{ab}}$$

in which $\theta_{ab} = -\theta_{ba}$ and $\sigma_{ab} = -\sigma_{ba}$ amount precisely to 6 parameters and 6 generators, themselves verifying

$$[\sigma_{ab}, \sigma_{cd}] = \eta_{ad}\sigma_{bc} - \eta_{ac}\sigma_{bd} + \eta_{bc}\sigma_{ad} - \eta_{bd}\sigma_{ac}$$

and given according to

$$(\sigma_{ab})^i_j = \delta^i_a\eta_{jb} - \delta^i_b\eta_{ja}$$

as the real representation. This form is the compact way of writing the explicit expressions obtained by considering that $\eta\sigma_{ab}$ are 6 antisymmetric matrices, and thus

$$\sigma^{01} = \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sigma^{02} = \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sigma^{03} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

are the generators of the boosts while

$$\sigma^{23} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\sigma^{31} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$\sigma^{12} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

are the generators of the rotations, verifying the commutation relationships given by the explicit expressions

$$[\sigma_{01}, \sigma_{02}] = -\sigma_{12}$$
$$[\sigma_{02}, \sigma_{03}] = -\sigma_{23}$$
$$[\sigma_{03}, \sigma_{01}] = -\sigma_{31}$$

among boosts and

$$[\sigma_{31}, \sigma_{12}] = \sigma_{23}$$
$$[\sigma_{12}, \sigma_{23}] = \sigma_{31}$$
$$[\sigma_{23}, \sigma_{31}] = \sigma_{12}$$

among rotations and

$$[\sigma_{31},\sigma_{01}]=-\sigma_{03} \qquad [\sigma_{31},\sigma_{03}]=\sigma_{01}$$
$$[\sigma_{23},\sigma_{03}]=-\sigma_{02} \qquad [\sigma_{12},\sigma_{01}]=\sigma_{02}$$
$$[\sigma_{12},\sigma_{02}]=-\sigma_{01} \qquad [\sigma_{23},\sigma_{02}]=\sigma_{03}$$

among different boosts and rotations; as a consequence, by calling $\theta_{01}=\varphi_1$, $\theta_{02}=\varphi_2$, $\theta_{03}=\varphi_3$ for the rapidities and $\theta_{23}=-\theta_1$, $\theta_{31}=-\theta_2$, $\theta_{12}=-\theta_3$ for the angles, it is possible to compute the explicit transformations as

$$\Lambda_{B1}=\left(\begin{array}{c|ccc}\cosh\varphi_1 & -\sinh\varphi_1 & 0 & 0\\ \hline -\sinh\varphi_1 & \cosh\varphi_1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right)$$
$$\Lambda_{B2}=\left(\begin{array}{c|ccc}\cosh\varphi_2 & 0 & -\sinh\varphi_2 & 0\\ \hline 0 & 1 & 0 & 0\\ -\sinh\varphi_2 & 0 & \cosh\varphi_2 & 0\\ 0 & 0 & 0 & 1\end{array}\right)$$
$$\Lambda_{B3}=\left(\begin{array}{c|ccc}\cosh\varphi_3 & 0 & 0 & -\sinh\varphi_3\\ \hline 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\sinh\varphi_3 & 0 & 0 & \cosh\varphi_3\end{array}\right)$$

for the boosts and

$$\Lambda_{R1}=\left(\begin{array}{c|ccc}1 & 0 & 0 & 0\\ \hline 0 & 1 & 0 & 0\\ 0 & 0 & \cos\theta_1 & \sin\theta_1\\ 0 & 0 & -\sin\theta_1 & \cos\theta_1\end{array}\right)$$
$$\Lambda_{R2}=\left(\begin{array}{c|ccc}1 & 0 & 0 & 0\\ \hline 0 & \cos\theta_2 & 0 & -\sin\theta_2\\ 0 & 0 & 1 & 0\\ 0 & \sin\theta_2 & 0 & \cos\theta_2\end{array}\right)$$
$$\Lambda_{R3}=\left(\begin{array}{c|ccc}1 & 0 & 0 & 0\\ \hline 0 & \cos\theta_3 & \sin\theta_3 & 0\\ 0 & -\sin\theta_3 & \cos\theta_3 & 0\\ 0 & 0 & 0 & 1\end{array}\right)$$

for the rotations, such that any product of these specific Lorentz transformations gives the full form of the Lorentz transformation. This Lorentz transformation in full form is the Lorentz transformation that we will employ next.

We may condense everything into the following statements, starting from the fact that given a Lorentz transformation Λ the set of functions $T^{a_1\dots a_i}_{r_1\dots r_j}$ transforming as

$$T'^{a'_1\dots a'_m}_{r'_1\dots r'_n}=(\Lambda^{-1})^{r_1}_{r'_1}\dots(\Lambda^{-1})^{r_n}_{r'_n}(\Lambda)^{a'_1}_{a_1}\dots(\Lambda)^{a'_m}_{a_m}T^{a_1\dots a_m}_{r_1\dots r_n} \tag{30}$$

is called tensor in Lorentz (Latin) indices formalism, and compared to the coordinate (Greek) indices formalism, symmetry properties and contractions are given similarly.

Also algebraic operations hold analogously, but again Lorentz transformations can be local and thus differential operations must be defined by introducing a connection to build Lorentz covariant differentiation: and as we have done before, because this new structure cannot introduce arbitrary concepts, we require $D_\mu\xi^\alpha_a=0$ with the clear consequence that also $D_\mu\eta_{ab}=0$ and these two conditions will ensure that the coordinate formalism and the Lorentz formalism will turn out to be completely equivalent.

Therefore once again we summarize by saying that the set of functions $\Omega^a_{b\mu}$ such that under a general coordinate transformation transforms as a lower Greek index vector and under a Lorentz transformation transforms as

$$\Omega'^{a'}_{b'\nu}=\Lambda^{a'}_a\left[\Omega^a_{b\nu}-(\Lambda^{-1})^a_k(\partial_\nu\Lambda)^k_b\right](\Lambda^{-1})^b_{b'} \tag{31}$$

is called spin connection, and no decomposition nor in particular any torsion can be defined as no transposition of indices of different types is defined. In terms of it

$$D_\mu T^{a_1\dots a_i}_{r_1\dots r_j}=\partial_\mu T^{a_1\dots a_i}_{r_1\dots r_j}+\sum_{k=1}^{k=i}\Omega^{a_k}_{p\mu}T^{a_1\dots p\dots a_i}_{r_1\dots r_j}-$$
$$-\sum_{k=1}^{k=j}\Omega^p_{r_k\mu}T^{a_1\dots a_i}_{r_1\dots p\dots r_j} \tag{32}$$

is covariant derivative of the tensor in Lorentz formalism.

This Lorentz formalism can be made equivalent to the general coordinate formalism by the introduction of the bases of vectors ξ^a_σ and ξ^σ_a dual of one another

$$\xi^a_\mu\xi^\mu_r=\delta^a_r \qquad \xi^a_\mu\xi^\rho_a=\delta^\rho_\mu \tag{33}$$

and such that they verify the ortho-normality conditions

$$g^{\alpha\beta}\xi^a_\alpha\xi^b_\beta=\eta^{ab} \qquad g_{\alpha\beta}\xi^\alpha_a\xi^\beta_b=\eta_{ab} \tag{34}$$

because the η matrices are diagonal and with unitary elements such that the first is positive and the last three are negative, that is they are the Minkowskian matrices, which is the one that is preserved by the Lorentz transformation, in this Lorentz formalism. Accordingly, with this pair of dual bases, ortho-normal with respect to the Minkowskian matrices, it is possible to consider a tensor in general coordinate formalism with at least one Greek index and multiply it by the basis contracting one Greek index with the Greek index of the bases and obtaining the tensor in Lorentz formalism with a Latin index.

Despite in this formalism we cannot define torsion, the torsion defined in coordinate formalism is converted as

$$Q^a{}_{\mu\nu}=-(\partial_\mu\xi^a_\nu-\partial_\nu\xi^a_\mu+\xi^b_\nu\Omega^a_{b\mu}-\xi^b_\mu\Omega^a_{b\nu}) \tag{35}$$

as the spin connection is given by

$$\Omega^a_{b\mu}=\xi^\nu_b\xi^a_\rho\left(\Gamma^\rho_{\nu\mu}-\xi^\rho_k\partial_\mu\xi^k_\nu\right) \tag{36}$$

such that once the two Lorentz indices are brought in the same upper or lower position this is antisymmetric in the two Lorentz indices. The last expression and antisymmetry condition are respectively equivalent to

$$D_\mu\xi^r_\alpha=0 \tag{37}$$

$$D_\mu\eta_{ab}=0 \tag{38}$$

as general coordinate-Lorentz compatibility conditions.

In Lorentz formalism, from the spin connection we get

$$G^{a}{}_{b\alpha\beta} = \partial_{\alpha}\Omega^{a}_{b\beta} - \partial_{\beta}\Omega^{a}_{b\alpha} + \Omega^{a}_{k\alpha}\Omega^{k}_{b\beta} - \Omega^{a}_{k\beta}\Omega^{k}_{b\alpha} \quad (39)$$

as the Riemann curvature tensor. Then we have that

$$[D_{\mu}, D_{\nu}]T^{r_1\ldots r_j} = Q^{\eta}{}_{\mu\nu}D_{\eta}T^{r_1\ldots r_j} + \\ + \sum_{k=1}^{k=j} G^{r_k}{}_{p\mu\nu}T^{r_1\ldots p\ldots r_j} \quad (40)$$

is the general coordinate covariant commutator of covariant derivatives of the tensor field in Lorentz formalism.

As it should be expected by now, we have that

$$G^{a}{}_{b\mu\nu} = \xi^{a}_{\alpha}\xi^{\beta}_{b}G^{\alpha}{}_{\beta\mu\nu} \quad (41)$$

so the Riemann curvature tensor in Lorentz formalism is antisymmetric not only in the two coordinate indices but also in the Lorentz indices, and so as a consequence the Riemann curvature also in this formalism has the same independent contractions, which can therefore be obtained according to $G_{b\sigma} = G^{a}{}_{b\rho\sigma}\xi^{\rho}_{a}$ for the Ricci curvature tensor, and $G = G_{a\sigma}\xi^{\sigma}_{p}\eta^{ap}$ for the Ricci curvature scalar, showing that they can be obtained by contracting either the general coordinate Greek indices or the special Lorentz Latin indices in equivalent manners.

After index renaming we get

$$D_{\mu}G^{a}{}_{j\kappa\rho} + D_{\kappa}G^{a}{}_{j\rho\mu} + D_{\rho}G^{a}{}_{j\mu\kappa} + \\ + G^{a}{}_{j\beta\mu}Q^{\beta}{}_{\rho\kappa} + G^{a}{}_{j\beta\kappa}Q^{\beta}{}_{\mu\rho} + G^{a}{}_{j\beta\rho}Q^{\beta}{}_{\kappa\mu} \equiv 0 \quad (42)$$

with curvature in Lorentz form as a geometric identity.

b. *Gauge fields: the most general abelian case*

In the past section, we have defined general geometric tensors, and now we are going to introduce, by following a similar geometrical spirit, the concept of gauge fields.

Our main goal is going to be focusing on the fact that fields may be complex, and so it makes sense to ask what symmetries can be established for these fields: if a field is complex there arises the issue of phase transformations and correspondingly it is possible to construct a calculus that is in all aspects analogous to the one we just built.

So given a real function α we have that a complex field that transforms according to the transformation

$$\phi' = e^{iq\alpha}\phi \quad (43)$$

is called gauge field of q charge. The sum of gauge fields of equal charges is a gauge field of the same charge and the product of gauge fields of given charges is a gauge field whose charge is the algebraic sum of those charges.

Let it be given a covector field A_{ν} such that for a phase transformation it transforms according to the law

$$A'_{\nu} = A_{\nu} - \partial_{\nu}\alpha \quad (44)$$

then this vector is called gauge potential. With it

$$D_{\mu}\phi = \partial_{\mu}\phi + iqA_{\mu}\phi \quad (45)$$

is said to be the gauge derivative of the gauge field.

For gauge fields we may introduce the operation of complex conjugation without the necessity of introducing any additional structure. For a gauge field of given charge the complex conjugate gauge field has opposite charge.

There is no decomposition of the gauge potential into more fundamental elements. In fact complex conjugation is compatible with gauge derivatives automatically.

From the gauge connection we define

$$F_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha} \quad (46)$$

that is such that $F = \partial A$ and so it is a tensor which is antisymmetric and invariant by a gauge transformation called Maxwell gauge curvature. With it we have that

$$[D_{\mu}, D_{\nu}]\phi = iqF_{\mu\nu}\phi \quad (47)$$

is the commutator of gauge derivatives of gauge fields.

Clearly, the gauge curvature cannot be decomposed in terms of more fundamental underlying structures.

Further we have that

$$\partial_{\nu}F_{\alpha\sigma} + \partial_{\sigma}F_{\nu\alpha} + \partial_{\alpha}F_{\sigma\nu} = 0 \quad (48)$$

or equivalently $\partial F = 0$ as a gauge geometric identity.

2. *Material fields: spinor fields*

In this second subsection, we shall proceed to actually erect the material theory we will employ in the following.

a. *Spinor fields: a geometric combination of Lorentz structure and gauge structure*

In the previous parts we have introduced tensor fields and the way to pass from coordinate into Lorentz indices, specifying that with such a conversion we also had the conversion of the most general coordinate transformation into the specific Lorentz transformation: the advantage of this specific Lorentz transformation is that despite it had been introduced in real representation, nevertheless its explicit form makes it possible to write it in other representations like the complex representation, in which the already-introduced complex fields can find place.

In order to find a Lorentz transformation in complex representation, we specify that these transformations are classified by semi-integer labels known as spin, and here we consider the simplest $\frac{1}{2}$-spin case: so for the complex generators we select those whose irreducible form is given in terms of 2-dimensional matrices. General results from the theory of Lie groups tell us that the Lorentz transformation can be written according to the following form

$$\boldsymbol{\Lambda} = e^{\frac{1}{2}\boldsymbol{\sigma}^{ab}\theta_{ab}}$$

where $\theta^{ab} = -\theta^{ba}$ and $\boldsymbol{\sigma}_{ab} = -\boldsymbol{\sigma}_{ba}$ are 6 parameters and the corresponding 6 generators, which verify the following

$$[\boldsymbol{\sigma}_{ab}, \boldsymbol{\sigma}_{cd}] = \eta_{ad}\boldsymbol{\sigma}_{bc} - \eta_{ac}\boldsymbol{\sigma}_{bd} + \eta_{bc}\boldsymbol{\sigma}_{ad} - \eta_{bd}\boldsymbol{\sigma}_{ac}$$

as the commutation relationships that define the Lorentz algebra and for which the generators $\boldsymbol{\sigma}_{ab}$ have to be given in terms of complex numbers: to actually find these complex generators it is helpful to split these expressions in terms of their time and space projections, respectively labelled by $\boldsymbol{\sigma}_{0A}=\boldsymbol{B}_{A}$ and $\boldsymbol{\sigma}_{AB}=\boldsymbol{R}^{C}\varepsilon_{ABC}$ so that

$$[\boldsymbol{B}_{A},\boldsymbol{B}_{B}]=-\varepsilon_{ABK}\boldsymbol{R}^{K}$$
$$[\boldsymbol{R}_{A},\boldsymbol{R}_{B}]=\varepsilon_{ABK}\boldsymbol{R}^{K}$$
$$[\boldsymbol{R}_{A},\boldsymbol{B}_{B}]=\varepsilon_{ABK}\boldsymbol{B}^{K}$$

showing that $\boldsymbol{B}_{A}$ represents a boost and $\boldsymbol{R}_{A}$ represents a rotation, and as known from the Lie theory we may recombine them as $\frac{1}{2}(\boldsymbol{R}_{A}\pm i\boldsymbol{B}_{A})=A_{A}^{\pm}$ so that

$$[\boldsymbol{A}_{A}^{+},\boldsymbol{A}_{B}^{+}]=\varepsilon_{ABK}\boldsymbol{A}^{+K}\quad [\boldsymbol{A}_{A}^{-},\boldsymbol{A}_{B}^{-}]=\varepsilon_{ABK}\boldsymbol{A}^{-K}$$
$$[\boldsymbol{A}_{A}^{+},\boldsymbol{A}_{B}^{-}]=0$$

as two independent three-dimensional rotations which we have stipulated to write in their irreducible form in terms of some 2-dimensional matrices: these matrices are already known to be the Pauli $\boldsymbol{\sigma}_{A}$ matrices; we recall that the Pauli matrices are given by the following matrices

$$\boldsymbol{\sigma}^{1}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$
$$\boldsymbol{\sigma}^{2}=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$
$$\boldsymbol{\sigma}^{3}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

as it is well known: in their terms the 2-dimensional irreducible forms that we can pick are either the one that is given by $\boldsymbol{A}_{A}^{+}=0$ and $\boldsymbol{A}_{A}^{-}=-\frac{i}{2}\boldsymbol{\sigma}_{A}$ or the complementary one given by $\boldsymbol{A}_{A}^{+}=-\frac{i}{2}\boldsymbol{\sigma}_{A}$ and $\boldsymbol{A}_{A}^{-}=0$ and with which we have $\boldsymbol{B}_{A}^{\pm}=\pm\frac{1}{2}\boldsymbol{\sigma}_{A}$ and $\boldsymbol{R}_{A}=-\frac{i}{2}\boldsymbol{\sigma}_{A}$ so that eventually we obtain $\boldsymbol{\sigma}_{0A}^{\pm}=\pm\frac{1}{2}\boldsymbol{\sigma}_{A}$ and $\boldsymbol{\sigma}_{AB}=-\frac{i}{2}\varepsilon_{ABC}\boldsymbol{\sigma}^{C}$ or explicitly

$$\boldsymbol{\sigma}_{\pm}^{01}=\pm\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$
$$\boldsymbol{\sigma}_{\pm}^{02}=\pm\frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$
$$\boldsymbol{\sigma}_{\pm}^{03}=\pm\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

for boosts and

$$\boldsymbol{\sigma}^{23}=-\frac{i}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$
$$\boldsymbol{\sigma}^{31}=-\frac{i}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$
$$\boldsymbol{\sigma}^{12}=-\frac{i}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

for rotations as the complex generators: in terms of these boosts and rotations as the complex generators, and by still calling $\theta_{01}=\varphi_{1}$, $\theta_{02}=\varphi_{2}$, $\theta_{03}=\varphi_{3}$ for the rapidities and $\theta_{23}=-\theta_{1}$, $\theta_{31}=-\theta_{2}$, $\theta_{12}=-\theta_{3}$ for the angles, it is possible to compute the explicit complex transformations

$$\boldsymbol{\Lambda}_{B1}^{\pm}=\begin{pmatrix}\cosh\frac{\varphi_{1}}{2}&\pm\sinh\frac{\varphi_{1}}{2}\\\pm\sinh\frac{\varphi_{1}}{2}&\cosh\frac{\varphi_{1}}{2}\end{pmatrix}$$
$$\boldsymbol{\Lambda}_{B2}^{\pm}=\begin{pmatrix}\cosh\frac{\varphi_{2}}{2}&\mp i\sinh\frac{\varphi_{2}}{2}\\\pm i\sinh\frac{\varphi_{2}}{2}&\cosh\frac{\varphi_{2}}{2}\end{pmatrix}$$
$$\boldsymbol{\Lambda}_{B3}^{\pm}=\begin{pmatrix}e^{\pm\frac{\varphi_{3}}{2}}&0\\0&e^{\mp\frac{\varphi_{3}}{2}}\end{pmatrix}$$

for boosts and

$$\boldsymbol{\Lambda}_{R1}=\begin{pmatrix}\cos\frac{\theta_{1}}{2}&i\sin\frac{\theta_{1}}{2}\\i\sin\frac{\theta_{1}}{2}&\cos\frac{\theta_{1}}{2}\end{pmatrix}$$
$$\boldsymbol{\Lambda}_{R2}=\begin{pmatrix}\cos\frac{\theta_{2}}{2}&\sin\frac{\theta_{2}}{2}\\-\sin\frac{\theta_{2}}{2}&\cos\frac{\theta_{2}}{2}\end{pmatrix}$$
$$\boldsymbol{\Lambda}_{R3}=\begin{pmatrix}e^{i\frac{\theta_{3}}{2}}&0\\0&e^{-i\frac{\theta_{3}}{2}}\end{pmatrix}$$

for rotations, and such that any product of these specific Lorentz transformations gives the Lorentz transformation in its full form. In the passage from real to complex representation a two-fold multiplicity has arisen since two opposite expressions are possible for the full Lorentz transformation: this is overcome with the 2-dimensional matrices combined into the 4-dimensional matrices

$$\boldsymbol{\Lambda}=\begin{pmatrix}\boldsymbol{\Lambda}^{-}&0\\0&\boldsymbol{\Lambda}^{+}\end{pmatrix}$$

for the Lorentz transformation which can be written according to the usual expansion in terms of the space-time parameters $\varphi_{A}=\theta_{0A}$ and $\theta_{C}=-\frac{1}{2}\varepsilon_{ABC}\theta^{AB}$ and in terms of the 4-dimensional generators that are given by

$$\boldsymbol{\sigma}_{0A}=\frac{1}{2}\begin{pmatrix}-\boldsymbol{\sigma}_{A}&0\\0&\boldsymbol{\sigma}_{A}\end{pmatrix}$$
$$\boldsymbol{\sigma}_{AB}=-\frac{i}{2}\varepsilon_{ABC}\begin{pmatrix}\boldsymbol{\sigma}^{C}&0\\0&\boldsymbol{\sigma}^{C}\end{pmatrix}$$

as boosts and rotations; these can also be written according to the form $\boldsymbol{\sigma}^{ab}=\frac{1}{4}\left[\boldsymbol{\gamma}^{a},\boldsymbol{\gamma}^{b}\right]$ as the commutators of the 4-dimensional matrices that are given by

$$\begin{pmatrix}0&\mathbb{I}\\\mathbb{I}&0\end{pmatrix}=\boldsymbol{\gamma}^{0}$$
$$\begin{pmatrix}0&\boldsymbol{\sigma}^{K}\\-\boldsymbol{\sigma}^{K}&0\end{pmatrix}=\boldsymbol{\gamma}^{K}$$

such that $\{\boldsymbol{\gamma}^a,\boldsymbol{\gamma}^b\}=2\eta^{ab}\mathbb{I}$ in terms of the Minkowskian matrix, and so with a manifest (1+3)-dimensional space-time form, showing that the 4-dimensional matricial form is also the manifestly space-time form. Thus we introduce the set of Clifford matrices implicitly defined by

$$\{\boldsymbol{\gamma}^a,\boldsymbol{\gamma}^b\}=2\eta^{ab}\mathbb{I}$$

which allow the definition of the matrices

$$\boldsymbol{\sigma}^{ab}=\frac{1}{4}\left[\boldsymbol{\gamma}^a,\boldsymbol{\gamma}^b\right]$$

as the generators corresponding to the θ_{ab} parameters in the usual expansion $\boldsymbol{\Lambda}=e^{\frac{1}{2}\boldsymbol{\sigma}^{ab}\theta_{ab}}$ as the Lorentz transformation in full space-time form. This Lorentz transformation considers operations on the space-time and thus it has to be combined with the phase $e^{iq\alpha}$ in order to eventually construct $\boldsymbol{S}=e^{(\frac{1}{2}\boldsymbol{\sigma}^{ab}\theta_{ab}+iq\alpha\mathbb{I})}$ as the Lorentz-phase transformation in complete form. This form is known as spinorial transformation. And it is this spinorial transformation what we employ in the following of the treatment.

The role of the matrices $\boldsymbol{\gamma}^a$ is fundamental and hence we are going to give a few more of their properties, starting from the implicit definition of another matrix as

$$\boldsymbol{\sigma}_{ab}=-\frac{i}{2}\varepsilon_{abcd}\boldsymbol{\pi}\boldsymbol{\sigma}^{cd}$$

and noticing that with it, the set of matrices that is given by the $\mathbb{I}$, $\boldsymbol{\pi}$, $\boldsymbol{\gamma}^a$, $\boldsymbol{\gamma}^a\boldsymbol{\pi}$, $\boldsymbol{\sigma}^{ab}$ is a basis for the 16-dimensional space of complex 4×4 matrices; the normalization condition given by $\boldsymbol{\gamma}_0\boldsymbol{\gamma}_a^\dagger\boldsymbol{\gamma}_0=\boldsymbol{\gamma}_a$ implies that the conditions given by $\boldsymbol{\gamma}_0\boldsymbol{\sigma}_{ab}^\dagger\boldsymbol{\gamma}_0=-\boldsymbol{\sigma}_{ab}$ and $\boldsymbol{\pi}^\dagger=\boldsymbol{\pi}$ hold: because it is known that $\varepsilon_{0123}=1$ then $\boldsymbol{\pi}=i\boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3$ and so we have that it is straightforward to prove also the properties

$$\{\boldsymbol{\pi},\boldsymbol{\gamma}_a\}=0$$

and

$$[\boldsymbol{\pi},\boldsymbol{\sigma}_{ab}]=0$$

in general terms, the last property in particular telling that any representation of the matrices will necessarily be reducible; finally, by direct inspection one can easily demonstrate that we also have the validity of

$$\boldsymbol{\gamma}_a\boldsymbol{\gamma}_b=\eta_{ab}\mathbb{I}+2\boldsymbol{\sigma}_{ab}$$

and perhaps a little longer but still mechanical is

$$\boldsymbol{\gamma}_i\boldsymbol{\gamma}_j\boldsymbol{\gamma}_k=\boldsymbol{\gamma}_i\eta_{jk}-\boldsymbol{\gamma}_j\eta_{ik}+\boldsymbol{\gamma}_k\eta_{ij}+i\varepsilon_{ijkq}\boldsymbol{\pi}\boldsymbol{\gamma}^q$$

from which more identities also follow, although we will not prove them since they are straightforward. To define a procedure of conjugation, we give the general form as

$$\overline{\psi}=\psi^\dagger\boldsymbol{A}$$

and we have to look for $\boldsymbol{A}$ such that the result is in fact the conjugated, that is such that if

$$\psi'=\boldsymbol{S}\psi$$

then

$$\overline{\psi}'=\overline{\psi}\boldsymbol{S}^{-1}$$

in general; this is required so to have that the 16 linearly-independent bi-linear spinorial quantities

$$\begin{gathered}2\overline{\psi}\boldsymbol{\sigma}^{ab}\boldsymbol{\pi}\psi=\Sigma^{ab}\\ 2i\overline{\psi}\boldsymbol{\sigma}^{ab}\psi=S^{ab}\\ \overline{\psi}\boldsymbol{\gamma}^a\boldsymbol{\pi}\psi=V^a\\ \overline{\psi}\boldsymbol{\gamma}^a\psi=U^a\\ i\overline{\psi}\boldsymbol{\pi}\psi=\Theta\\ \overline{\psi}\psi=\Phi\end{gathered}$$

lose all their spinorial transformation laws to remain with the tensorial transformation laws solely: then we have

$$\boldsymbol{S}^\dagger\boldsymbol{A}\boldsymbol{S}=\boldsymbol{A}$$

or equivalently

$$e^{(\frac{1}{2}\boldsymbol{\sigma}^\dagger_{ab}\theta^{ab}-iq\alpha\mathbb{I})}\boldsymbol{A}e^{(\frac{1}{2}\boldsymbol{\sigma}_{ab}\theta^{ab}+iq\alpha\mathbb{I})}=\boldsymbol{A}$$

therefore giving

$$\boldsymbol{\sigma}_{ab}^\dagger\boldsymbol{A}+\boldsymbol{A}\boldsymbol{\sigma}_{ab}=0$$

whose most general solution is given by the expression

$$\boldsymbol{A}=\left(\begin{array}{cc}0 & \lambda\mathbb{I}\\ \omega\mathbb{I} & 0\end{array}\right)$$

with λ and ω being general complex numbers; however, it is also possible to see that requiring the above 16 linearly-independent bi-linear spinorial quantities be real yields that we also have $\lambda=\omega=1$ and eventually

$$\boldsymbol{A}=\boldsymbol{\gamma}^0$$

fixing the relationship

$$\overline{\psi}=\psi^\dagger\boldsymbol{\gamma}^0$$

to be the conjugation of spinors. Furthermore, it is also important to be capable of computing $\psi\overline{\psi}$ and because it is clear that such form is a 4 complex matrix then it can be written as a linear combination of the following form

$$\psi\overline{\psi}\equiv a\mathbb{I}+b_a\boldsymbol{\gamma}^a+c_{ab}\boldsymbol{\sigma}^{ab}+d_{ab}\boldsymbol{\sigma}^{ab}\boldsymbol{\pi}+e_a\boldsymbol{\gamma}^a\boldsymbol{\pi}+f\boldsymbol{\pi}$$

as it is clear: by repeatedly multiplying it by all of the matrices and taking their traces, it is possible to see that one by one the coefficients are determined to be

$$\begin{gathered}\psi\overline{\psi}\equiv\frac{1}{4}\Phi\mathbb{I}+\frac{1}{4}U_a\boldsymbol{\gamma}^a+\frac{i}{8}S_{ab}\boldsymbol{\sigma}^{ab}-\\ -\frac{1}{8}\Sigma_{ab}\boldsymbol{\sigma}^{ab}\boldsymbol{\pi}-\frac{1}{4}V_a\boldsymbol{\gamma}^a\boldsymbol{\pi}-\frac{i}{4}\Theta\boldsymbol{\pi}\end{gathered}$$

as one can also see by direct computation. From such an expression it is also possible to get other identities like

$$\begin{gathered}-V_a\boldsymbol{\gamma}^a\boldsymbol{\pi}\psi=U_a\boldsymbol{\gamma}^a\psi=(\Phi\mathbb{I}+i\Theta\boldsymbol{\pi})\psi\\ -U_a\boldsymbol{\gamma}^a\boldsymbol{\pi}\psi=V_a\boldsymbol{\gamma}^a\psi=(\Phi\boldsymbol{\pi}+i\Theta\mathbb{I})\psi\end{gathered}$$

as well as other relationships such as

$$S_{ab}\Phi-\Sigma_{ab}\Theta=U^{j}V^{k}\varepsilon_{jkab}$$
$$S_{ab}\Theta+\Sigma_{ab}\Phi=U_{[a}V_{b]}$$

with

$$S_{ik}U^{i}=\Theta V_{k}$$
$$\Sigma_{ik}U^{i}=\Phi V_{k}$$
$$S_{ik}V^{i}=\Theta U_{k}$$
$$\Sigma_{ik}V^{i}=\Phi U_{k}$$

and

$$\tfrac{1}{2}S_{ab}S^{ab}=-\tfrac{1}{2}\Sigma_{ab}\Sigma^{ab}=\Phi^{2}-\Theta^{2}$$
$$U_{a}U^{a}=-V_{a}V^{a}=\Theta^{2}+\Phi^{2}$$
$$\tfrac{1}{2}S_{ab}\Sigma^{ab}=-2\Theta\Phi$$
$$U_{a}V^{a}=0$$

all obtained by employing the above re-arrangement and using the properties of the matrices or by simply checking the results with a direct substitution of the general form of spinor and performing calculations straightforwardly.

Finally, it is quite obvious now that if the transformation law $\boldsymbol{S}$ is made local then we have to introduce the spinorial connection transforming according to

$$\boldsymbol{\Omega}'_{\nu}=\boldsymbol{S}\left(\boldsymbol{\Omega}_{\nu}-\boldsymbol{S}^{-1}\partial_{\nu}\boldsymbol{S}\right)\boldsymbol{S}^{-1}$$

so that the spinorial covariant derivative will be

$$\boldsymbol{D}_{\mu}\psi=\partial_{\mu}\psi+\boldsymbol{\Omega}_{\mu}\psi$$
$$\boldsymbol{D}_{\mu}\boldsymbol{\Xi}=\partial_{\mu}\boldsymbol{\Xi}+[\boldsymbol{\Omega}_{\mu},\boldsymbol{\Xi}]$$

according to whether it is applied onto spinors or spinorial matrices, and for the latter we assume the condition given by $\boldsymbol{D}_{\mu}\boldsymbol{\gamma}_{a}=0$ as above: if the spinorial matrix has also a tensorial index the covariant derivative is

$$\boldsymbol{D}_{\mu}\boldsymbol{B}_{a}=\partial_{\mu}\boldsymbol{B}_{a}-\boldsymbol{B}_{b}\Omega^{b}{}_{a\mu}+[\boldsymbol{\Omega}_{\mu},\boldsymbol{B}_{a}]$$

which can be taken for the gamma matrix and hence implementing the above condition, and recalling that these matrices in Lorentz indices are constants, yields

$$-\boldsymbol{\gamma}_{b}\Omega^{b}{}_{a\mu}+[\boldsymbol{\Omega}_{\mu},\boldsymbol{\gamma}_{a}]=0$$

as a relation among connections; by writing a general

$$\boldsymbol{\Omega}_{\mu}=a\Omega^{ij}{}_{\mu}\boldsymbol{\sigma}_{ij}+\boldsymbol{A}_{\mu}$$

and plugging it into the above relation we obtain that

$$-\boldsymbol{\gamma}_{b}\Omega^{b}{}_{k\mu}+a\Omega^{ij}{}_{\mu}[\boldsymbol{\sigma}_{ij},\boldsymbol{\gamma}_{k}]+[\boldsymbol{A}_{\mu},\boldsymbol{\gamma}_{k}]=0$$

and with $[\boldsymbol{\sigma}_{ij},\boldsymbol{\gamma}_{k}]=\eta_{kj}\boldsymbol{\gamma}_{i}-\eta_{ki}\boldsymbol{\gamma}_{j}$ we get $a=1/2$ and

$$[\boldsymbol{A}_{\mu},\boldsymbol{\gamma}_{s}]=0$$

telling that $\boldsymbol{A}_{\mu}$ must commute with all gamma matrices, thus with all possible matrices, and this implies that it is proportional to the identity matrix. By writing it as

$$\boldsymbol{A}_{\mu}=(p+ib)A_{\mu}\mathbb{I}$$

it is possible to see that for $b=q$ it is possible to interpret the vector A_{μ} as the gauge potential; because the other term is related to conformal transformations, which are not symmetries in our case, we set $p=0$ in general. Then we have that all considered we may write the expression

$$\boldsymbol{\Omega}_{\mu}=\tfrac{1}{2}\Omega^{ij}{}_{\mu}\boldsymbol{\sigma}_{ij}+iqA_{\mu}\mathbb{I}$$

as the most general form of the spinorial connection.

We may now summarize by saying that given the most general spinorial transformation $\boldsymbol{S}$ the column and row of complex scalars ψ and $\overline{\psi}$ and the matrix of complex scalars $\boldsymbol{\Xi}$ transforming according to the following law

$$\psi'=\boldsymbol{S}\psi\qquad\overline{\psi}'=\overline{\psi}\boldsymbol{S}^{-1}$$
$$\boldsymbol{\Xi}'=\boldsymbol{S}\boldsymbol{\Xi}\boldsymbol{S}^{-1}\tag{49}$$

are called $\frac{1}{2}$-spin spinorial fields. Operations of sum and product respect spinor transformation laws, as expected.

The coefficients $\boldsymbol{\Omega}_{\nu}$ transforming according to the law

$$\boldsymbol{\Omega}'_{\nu}=\boldsymbol{S}\left(\boldsymbol{\Omega}_{\nu}-\boldsymbol{S}^{-1}\partial_{\nu}\boldsymbol{S}\right)\boldsymbol{S}^{-1}\tag{50}$$

are the most general spinorial connection. Once the connection is assigned, we have the following expressions

$$\boldsymbol{D}_{\mu}\psi=\partial_{\mu}\psi+\boldsymbol{\Omega}_{\mu}\psi\qquad\boldsymbol{D}_{\mu}\overline{\psi}=\partial_{\mu}\overline{\psi}-\overline{\psi}\boldsymbol{\Omega}_{\mu}$$
$$\boldsymbol{D}_{\mu}\boldsymbol{\Xi}=\partial_{\mu}\boldsymbol{\Xi}+[\boldsymbol{\Omega}_{\mu},\boldsymbol{\Xi}]\tag{51}$$

as the covariant derivatives of the spinorial fields.

Clifford matrices are indicated by $\boldsymbol{\gamma}_{a}$ and their implicit definition is given with anticommutation relationships

$$\{\boldsymbol{\gamma}_{a},\boldsymbol{\gamma}_{b}\}=2\eta_{ab}\mathbb{I}\tag{52}$$

so that we define the matrices $\boldsymbol{\sigma}_{ab}$ given by

$$\boldsymbol{\sigma}_{ab}=\tfrac{1}{4}[\boldsymbol{\gamma}_{a},\boldsymbol{\gamma}_{b}]\tag{53}$$

implicitly defining the matrix $\boldsymbol{\pi}$ as

$$\boldsymbol{\sigma}_{ab}=-\tfrac{i}{2}\varepsilon_{abcd}\boldsymbol{\pi}\boldsymbol{\sigma}^{cd}\tag{54}$$

and then we have that because of the normalization

$$\boldsymbol{\gamma}_{0}\boldsymbol{\gamma}_{a}^{\dagger}\boldsymbol{\gamma}_{0}=\boldsymbol{\gamma}_{a}\tag{55}$$

we also have that

$$\boldsymbol{\gamma}_{0}\boldsymbol{\sigma}_{ab}^{\dagger}\boldsymbol{\gamma}_{0}=-\boldsymbol{\sigma}_{ab}\tag{56}$$

with

$$\boldsymbol{\pi}^{\dagger}=\boldsymbol{\pi}\tag{57}$$

coming alongside to the square properties

$$\boldsymbol{\gamma}_a\boldsymbol{\gamma}^a=4\mathbb{I} \tag{58}$$

with

$$\boldsymbol{\sigma}_{ab}\boldsymbol{\sigma}^{ab}=-3\mathbb{I} \tag{59}$$

and

$$\boldsymbol{\pi}^2=\mathbb{I} \tag{60}$$

together with the anticommutation properties

$$\{\boldsymbol{\pi},\boldsymbol{\gamma}_a\}=0 \tag{61}$$
$$\{\boldsymbol{\gamma}_i,\boldsymbol{\sigma}_{jk}\}=i\varepsilon_{ijkq}\boldsymbol{\pi}\boldsymbol{\gamma}^q \tag{62}$$

and the commutation properties

$$[\boldsymbol{\pi},\boldsymbol{\sigma}_{ab}]=0 \tag{63}$$
$$[\boldsymbol{\gamma}_a,\boldsymbol{\sigma}_{bc}]=\eta_{ab}\boldsymbol{\gamma}_c-\eta_{ac}\boldsymbol{\gamma}_b \tag{64}$$
$$[\boldsymbol{\sigma}_{ab},\boldsymbol{\sigma}_{cd}]=\eta_{ad}\boldsymbol{\sigma}_{bc}-\eta_{ac}\boldsymbol{\sigma}_{bd}+\eta_{bc}\boldsymbol{\sigma}_{ad}-\eta_{bd}\boldsymbol{\sigma}_{ac} \tag{65}$$

and in particular we also define $\boldsymbol{\pi}_L$ and $\boldsymbol{\pi}_R$ such that

$$\boldsymbol{\pi}_L=\tfrac{1}{2}(\mathbb{I}-\boldsymbol{\pi}) \tag{66}$$
$$\boldsymbol{\pi}_R=\tfrac{1}{2}(\mathbb{I}+\boldsymbol{\pi}) \tag{67}$$

verifying

$$\boldsymbol{\pi}_L^\dagger=\boldsymbol{\pi}_L \tag{68}$$
$$\boldsymbol{\pi}_R^\dagger=\boldsymbol{\pi}_R \tag{69}$$

alongside to

$$\boldsymbol{\pi}_L^2=\boldsymbol{\pi}_L \tag{70}$$
$$\boldsymbol{\pi}_R^2=\boldsymbol{\pi}_R \tag{71}$$

together with

$$\boldsymbol{\pi}_L\boldsymbol{\pi}_R=\boldsymbol{\pi}_R\boldsymbol{\pi}_L=0 \tag{72}$$
$$\tag{73}$$

and such that

$$\boldsymbol{\pi}_L+\boldsymbol{\pi}_R=\mathbb{I} \tag{74}$$

called left and right projectors, and we have the identities

$$\boldsymbol{\gamma}_a\boldsymbol{\gamma}_b=\eta_{ab}\mathbb{I}+2\boldsymbol{\sigma}_{ab} \tag{75}$$
$$\boldsymbol{\gamma}_i\boldsymbol{\gamma}_j\boldsymbol{\gamma}_k=\boldsymbol{\gamma}_i\eta_{jk}-\boldsymbol{\gamma}_j\eta_{ik}+\boldsymbol{\gamma}_k\eta_{ij}+i\varepsilon_{ijkq}\boldsymbol{\pi}\boldsymbol{\gamma}^q \tag{76}$$

which are useful to bring any product of matrices down to fewer matrices thus showing that the matrices we defined are in fact all we need. Then we have that the temporal gamma matrix $\boldsymbol{\gamma}_0$ is employed to define the conjugation

$$\overline{\psi}=\psi^\dagger\boldsymbol{\gamma}_0 \qquad \boldsymbol{\gamma}_0\overline{\psi}^\dagger=\psi \tag{77}$$

and in particular $\boldsymbol{\pi}_L$and $\boldsymbol{\pi}_R$ define the decomposition of the spinor in its left and right projections

$$\boldsymbol{\pi}_L\psi=\psi_L \qquad \overline{\psi}\boldsymbol{\pi}_R=\overline{\psi}_L \tag{78}$$
$$\boldsymbol{\pi}_R\psi=\psi_R \qquad \overline{\psi}\boldsymbol{\pi}_L=\overline{\psi}_R \tag{79}$$

and such that

$$\overline{\psi}_L+\overline{\psi}_R=\overline{\psi} \qquad \psi_L+\psi_R=\psi \tag{80}$$

is the procedure with which from the chiral projections we re-construct the spinor, and finally we have that with the pair of conjugate spinors we may define 16 linearly-independent bi-linear spinorial quantities according to

$$2\overline{\psi}\boldsymbol{\sigma}^{ab}\boldsymbol{\pi}\psi=\Sigma^{ab} \tag{81}$$
$$2i\overline{\psi}\boldsymbol{\sigma}^{ab}\psi=S^{ab} \tag{82}$$
$$\overline{\psi}\boldsymbol{\gamma}^a\boldsymbol{\pi}\psi=V^a \tag{83}$$
$$\overline{\psi}\boldsymbol{\gamma}^a\psi=U^a \tag{84}$$
$$i\overline{\psi}\boldsymbol{\pi}\psi=\Theta \tag{85}$$
$$\overline{\psi}\psi=\Phi \tag{86}$$

such that

$$\begin{aligned}\psi\overline{\psi}\equiv\tfrac{1}{4}\Phi\mathbb{I}+\tfrac{1}{4}U_a\boldsymbol{\gamma}^a+\tfrac{i}{8}S_{ab}\boldsymbol{\sigma}^{ab}-\\ -\tfrac{1}{8}\Sigma_{ab}\boldsymbol{\sigma}^{ab}\boldsymbol{\pi}-\tfrac{1}{4}V_a\boldsymbol{\gamma}^a\boldsymbol{\pi}-\tfrac{i}{4}\Theta\boldsymbol{\pi}\end{aligned} \tag{87}$$

from which we get the relationships

$$-V_a\boldsymbol{\gamma}^a\boldsymbol{\pi}\psi=U_a\boldsymbol{\gamma}^a\psi=(\Phi\mathbb{I}+i\Theta\boldsymbol{\pi})\psi \tag{88}$$
$$-U_a\boldsymbol{\gamma}^a\boldsymbol{\pi}\psi=V_a\boldsymbol{\gamma}^a\psi=(\Phi\boldsymbol{\pi}+i\Theta\mathbb{I})\psi \tag{89}$$

as well as the relationships

$$S_{ab}\Phi-\Sigma_{ab}\Theta=U^jV^k\varepsilon_{jkab} \tag{90}$$
$$S_{ab}\Theta+\Sigma_{ab}\Phi=U_{[a}V_{b]} \tag{91}$$

together with

$$S_{ik}U^i=\Theta V_k \tag{92}$$
$$\Sigma_{ik}U^i=\Phi V_k \tag{93}$$
$$S_{ik}V^i=\Theta U_k \tag{94}$$
$$\Sigma_{ik}V^i=\Phi U_k \tag{95}$$

and

$$\tfrac{1}{2}S_{ab}S^{ab}=-\tfrac{1}{2}\Sigma_{ab}\Sigma^{ab}=\Phi^2-\Theta^2 \tag{96}$$
$$U_aU^a=-V_aV^a=\Theta^2+\Phi^2 \tag{97}$$
$$\tfrac{1}{2}S_{ab}\Sigma^{ab}=-2\Theta\Phi \tag{98}$$
$$U_aV^a=0 \tag{99}$$

showing how to re-arrange the components of the spinors.

The most general spinorial connection is

$$\boldsymbol{\Omega}_\mu=\tfrac{1}{2}\Omega_{ab\mu}\boldsymbol{\sigma}^{ab}+iqA_\mu\mathbb{I} \tag{100}$$

in terms of the generator-valued spin connection and the gauge potential. This is equivalent to the fact that the spinorial covariant derivatives of the gamma matrices is

$$\boldsymbol{D}_\mu\boldsymbol{\gamma}_a=0 \tag{101}$$

vanishing identically as it is quite straightforward to see.

We have that from the spinorial connection we define

$$\boldsymbol{F}_{\alpha\beta} = \partial_\alpha \boldsymbol{\Omega}_\beta - \partial_\beta \boldsymbol{\Omega}_\alpha + [\boldsymbol{\Omega}_\alpha, \boldsymbol{\Omega}_\beta] \tag{102}$$

as the spinorial curvature. With it

$$[\boldsymbol{D}_\mu, \boldsymbol{D}_\nu]\psi = Q^\alpha{}_{\mu\nu} \boldsymbol{D}_\alpha \psi + \boldsymbol{F}_{\mu\nu}\psi \tag{103}$$

is the commutator of covariant derivatives of spinor fields.

Correspondingly, the curvature is decomposable as

$$\boldsymbol{F}_{\mu\nu} = \tfrac{1}{2} G_{ab\mu\nu} \boldsymbol{\sigma}^{ab} + iqF_{\mu\nu}\mathbb{I} \tag{104}$$

with the Riemann curvature and Maxwell curvature.

For a final step, we have

$$\begin{aligned} D_\mu \boldsymbol{F}_{\kappa\rho} + D_\kappa \boldsymbol{F}_{\rho\mu} + D_\rho \boldsymbol{F}_{\mu\kappa} + \\ + \boldsymbol{F}_{\beta\mu} Q^\beta{}_{\rho\kappa} + \boldsymbol{F}_{\beta\kappa} Q^\beta{}_{\mu\rho} + \boldsymbol{F}_{\beta\rho} Q^\beta{}_{\kappa\mu} \equiv 0 \end{aligned} \tag{105}$$

as spinorial geometrical identities holding in general.

We conclude with some fundamental comments: a first and most important one is about the fact that so far we encountered three types of transformation laws: the first type was the most general coordinate transformation; the second type was the gauge transformation; the third type was the specific Lorentz transformation, which was given in real representation for tensors and complex representation for spinors. The coordinate transformation is known as passive transformation; the Lorentz transformation in real representation is known as active transformation and similarly the Lorentz transformation in complex representation merged with the gauge transformation together into the spinorial transformation is known as active transformation. Because the real Lorentz transformation and the complex Lorentz transformation have the same local parameters then they must be performed simultaneously but it will be the spinorial transformation that will play an important role, as we are going to see in what follows.

Another interesting comment is on the connections and how they are built: the torsion tensor, when the metric tensor is used, gives the connection (16); this connection, when the dual bases of tetrad fields are employed, gives the spin connection (36); this spin connection, when the gamma matrices and their commutators are considered, with the gauge potential, when multiplied by the identity matrix, give the spinorial connection (100). Remarkably, all fields fit within the most general spinorial connection, with no room for anything else; this circumstance can be seen as a sort of geometric unification of all the physical fields that are involved. On the other hand however, in order to see it that way, we have to wait until we interpret these geometric quantities as the actual physical fields.

A final comment regards the structure of the covariant commutator (103), in which by interpreting the covariant derivative as the covariant generators of translations one sees that the completely antisymmetric torsion plays the role that in Lie group theory is played by the completely antisymmetric structure coefficients; we also recall to the reader that in the curvature there appear sigma matrices which are the generators of the Lorentz transformations and therefore of the space-time rotations. An additional interpretation that can be assigned to the covariant commutator is that when a field is moved around it would fail to go back to the starting point and have the initial orientation, the positional mismatch measured by torsion and the directional mismatch measured by curvature, and this is why torsion is also said to describe the dislocations while curvature is also said to describe the disclinations of a round trip. This shows intuitively that both torsion and curvature have to be accounted for the most general description of the properties of the space-time since both the translational and the rotational symmetry is present.

For some introduction to the general theory of spinors and their classifications we refer readers to [18] and [19].

We have now completed the definition in terms of symmetry arguments of the fundamental geometric quantities of the kinematic background, and next we will have them coupled to one another so to assign their dynamics.

B. Geometry and matter in interaction

In this second section, we establish the field dynamics by having all fields linked according to the principle that their coupling would be the most extensive possible.

1. *Geometric-material coupling: covariant-field equations*

In this first subsection, we shall present in what way the geometry and its matter content can be dynamically liked proceeding in the order of increasing exhaustiveness of their coupling, and not surprisingly this presentation coincides with the historical development of the theory.

It is important to specify that it is our main goal that of following Einstein spirit of geometrization, and in order to do so we are going to obtain the field equations for the theory in a genuinely geometric way by finding the most general form of the field equations that is compatible with the constraints given by geometric identities.

a. History: gravity, torsion gravity, independent torsion gravity, propagating torsion gravity

When in 1916 Einstein wanted to construct the theory of gravitation, the idea he wished to follow was inspired geometrically, based on the principle of equivalence.

The principle of equivalence states the equivalence at a local level between inertia and gravitation, in the sense that locally inertial and gravitational forces can simulate one another so well that, when both present, their effects can be made to cancel: it can be stated by saying that one can always find a system of coordinates in which locally the accelerations due to gravitation are negligible.

On the other hand, in absolute differential calculus it is not at all difficult to demonstrate a theorem originally due to Weyl whose statement sounds analogous: it states that one can always find a system of coordinates in which in a point the symmetric part of the connection vanishes.

In the previous sections we have discussed in what way the condition of complete antisymmetry of torsion gives rise to a unique symmetric part of the connection, thus removing any possible ambiguity in the implementation of Weyl theorem: hence for a completely antisymmetric torsion, Weyl theorem is the mathematical implementation of the principle of equivalence insofar as the acceleration due to gravitation is encoded within the symmetric connection; a unique symmetric connection corresponds to a uniquely defined gravitational field as our physical intuition would suggest. Further, that single symmetric connection is entirely written in terms of the derivatives of the metric, and therefore if the gravitational field is encoded within the symmetric connection then the gravitational potential is encoded within the metric tensor.

The metric tensor is a tensor but it cannot vanish nor any of its derived scalar is non-trivial and the connection is not a tensor, so they will always depend on the choice of coordinates: hence, the information about gravity will always be intertwined with inertial information, which is not a surprise since after all we know that they are locally indistinguishable; on the other hand, we wish to have a way to tell gravity apart from inertial information, and to do that it is necessary to take a less local level, then considering the Riemann curvature tensor: if gravity is contained in the metric tensor as well as in the connection, then it is contained in the Riemann curvature tensor too, but the Riemann curvature tensor is a tensor from which non-trivial scalars can be derived or which can be vanished, and this is what makes it able to discriminate gravity from inertial forces. If the metric is Minkowskian and the connection is zero we cannot know whether this is because gravity is absent or compensated by inertial forces, and similarly if the metric is not Minkowskian and the connection is not zero we cannot know whether this is because gravity is present or simulated by inertial forces as above; but if the Riemann curvature tensor is zero we know it is because gravity is absent, and if the Riemann curvature tensor is not zero we know gravity is present in general terms. This has to be so, as there can not be any compensation due to inertial forces since there can be no inertial forces, within the Riemann curvature tensor.

Therefore the principle of equivalence is the manifestation of the interpretative principle telling that gravitation is geometrized, and this is so as a consequence of the fact gravity is contained within the Riemann curvature.

This statement has to be taken into account together with the parallel fact that in Einstein relativity the mass is a form of energy, as it is very well known indeed.

Putting the two things together, it becomes clear that the gravitational field equations that were given in terms of a second-order differential operator of the gravitational potential proportional to the mass density have to be considered as an approximated form of a more general set of gravitational field equations given by a certain (contraction of the) curvature proportional to the energy density.

The energy density is a tensor having two indices and therefore the curvature we are looking for must have two indices as well, which tells that we need the contraction of the Riemann curvature given by the Ricci curvature.

In 1916 all matter forms that were known consisted in macroscopic fluids, scalars and electro-dynamic fields, all of which having an energy density symmetric in the two indices; this might have been a problem as in general the Ricci curvature is not symmetric in its two indices.

And this is where Einstein assumption of the vanishing torsion came about: assuming torsion to be equal to zero meant that the Ricci curvature tensor is symmetric and thus it can be taken proportional to the energy density.

Moreover, the energy density is divergenceless and thus we need to find a specific combination of Ricci curvatures in order to make the geometric side divergenceless too.

To see which, we may consider identity (29) in the case in which torsion vanishes, therefore obtaining identity

$$\nabla_{\mu}R^{\nu}{}_{\iota\kappa\rho}+\nabla_{\kappa}R^{\nu}{}_{\iota\rho\mu}+\nabla_{\rho}R^{\nu}{}_{\iota\mu\kappa}=0$$

whose contraction gives

$$\nabla_{\mu}R^{\mu}{}_{\iota\kappa\rho}-\nabla_{\kappa}R_{\iota\rho}+\nabla_{\rho}R_{\iota\kappa}=0$$

and then

$$\nabla_{\mu}R^{\mu}{}_{\kappa}-\tfrac{1}{2}\nabla_{\kappa}R=0$$

or more in general

$$\nabla_{\mu}(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R-g^{\mu\nu}\Lambda)=0$$

which is symmetric and divergenceless as desired, and so it can be taken to be proportional to the energy density.

Therefore, Einstein geometrical insight is expressed in the form of the gravitational field equations

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R-g^{\mu\nu}\Lambda=\tfrac{1}{2}kT^{\mu\nu}$$

which are then called Einstein field equations.

In summary, Einstein gravitational field equations

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R-g^{\mu\nu}\Lambda=\tfrac{1}{2}kT^{\mu\nu}$$

are the field equations of the theory: from them it follows that the energy density is symmetric and divergenceless

$$T^{[\mu\nu]}=0$$
$$\nabla_{\mu}T^{\mu\nu}=0$$

as the conservation laws that have to be verified by the matter field in the most general of the possible cases.

At least in the most general case without torsion.

Then, one may wonder that what happens if torsion were not neglected in the scheme of Einstein gravity.

Einstein gravitational theory is based on the spirit that geometry gives us a curvature tensor while matter fields have an energy density, and so matter fields filling geometry must have their energy density sourcing the curvature field equations; but then again, one may wonder whether we can keep embracing this spirit also when the torsion tensor is present beside curvature in the geometry.

So the first point to be retained is that a geometry filled with matter fields has to be described by field equations in which geometrical quantities are sourced by material fields in the most general of cases; and the second point to be retained is that in such a most general of the cases geometry gives us curvature and torsion: therefore, if the curvature field equations are sourced by energy density the torsion field equations are sourced by spin density.

In 1928 Dirac has been the first to describe a system of matter fields, named spinors, which possessed an energy density together with a spin density; more important still was the fact that their energy density was described by a tensor that was not symmetric, and to add complications that energy density did not even verify the standard law of conservation as we have accounted just above.

Indeed it is something of the most general validity the result that quantum matter is to be described in terms of irreducible representations of the Poincaré group labelled by quantum numbers of mass and spin [20], and therefore the corresponding matter field has energy density and spin density; general procedures of quantum field theory can be used to assess that spin and energy conservation laws are given according to the following expressions

$$D_\rho S^{\rho\mu\nu}+\frac{1}{2}T^{[\mu\nu]}=0$$

and

$$D_\mu T^{\mu\nu}+T_{\rho\beta}Q^{\rho\beta\nu}-S_{\mu\rho\beta}G^{\mu\rho\beta\nu}=0$$

reducing to the above case of divergenceless and symmetric energy density whenever the spin density vanishes, but which are nevertheless valid in the most general cases.

Correspondingly, in the most general case in which the torsion is allowed to be present beside the curvature we have that identities (29, 20) can be fully contracted as

$$D_\rho Q^{\rho\mu\nu}-G^{[\mu\nu]}=0$$

and

$$D_\mu G^{\mu}{}_{\rho}-\frac{1}{2}D_\rho G-G^{\mu\sigma}Q_{\sigma\mu\rho}-\frac{1}{2}G_{\mu\kappa\sigma\rho}Q^{\sigma\kappa\mu}=0$$

or equivalently

$$D_\rho Q^{\rho\mu\nu}-(G^{[\mu\nu]}-\frac{1}{2}g^{[\mu\nu]}G-g^{[\mu\nu]}\Lambda)=0$$

and

$$D_\mu(G^{\mu\nu}-\frac{1}{2}g^{\mu\nu}G-g^{\mu\nu}\Lambda)-$$
$$-(G_{\mu\sigma}-\frac{1}{2}g_{\mu\sigma}G-g_{\mu\sigma}\Lambda)Q^{\sigma\mu\nu}-\frac{1}{2}G^{\mu\kappa\sigma\nu}Q_{\sigma\kappa\mu}=0$$

being closely resemblant to the conservation laws above.

Then, Einstein field equations can be generalized up to gravitational field equations linking a non-symmetric curvature to a non-symmetric energy density and we have the appearance of a new torsional field equation linking the completely antisymmetric torsion to the completely antisymmetric spin density according to expressions

$$Q^{\rho\mu\nu}=-kS^{\rho\mu\nu}$$

and

$$G^{\mu\nu}-\frac{1}{2}g^{\mu\nu}G-g^{\mu\nu}\Lambda=\frac{1}{2}kT^{\mu\nu}$$

reducing to the above case linking a symmetric curvature to a symmetric energy density whenever the torsion and the spin density vanish, but nevertheless they are valid in the most general case in which torsion is not equal to zero and it is coupled to the spin density, and hence they are called Einstein–Sciama-Kibble field equations [21].

In summary, ESK torsion gravitational field equations

$$Q^{\rho\mu\nu}=-kS^{\rho\mu\nu}$$

and

$$G^{\mu\nu}-\frac{1}{2}g^{\mu\nu}G-g^{\mu\nu}\Lambda=\frac{1}{2}kT^{\mu\nu}$$

are the pair of field equations of the theory: therefore the energy density and spin density verify the relationships

$$D_\rho S^{\rho\mu\nu}+\frac{1}{2}T^{[\mu\nu]}=0$$

and

$$D_\mu T^{\mu\nu}+T_{\sigma\mu}Q^{\sigma\mu\nu}-S_{\mu\kappa\sigma}G^{\mu\kappa\sigma\nu}=0$$

as the conservation laws satisfied by the matter field.

To show that such a non-symmetric energy density and a completely antisymmetric spin density actually exist it is enough to present those pertaining to the spinor field

$$S^{\rho\mu\nu}=\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\rho,\boldsymbol{\sigma}^{\mu\nu}\}\psi$$

and

$$T^{\rho\sigma}=\frac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^\rho\boldsymbol{D}^\sigma\psi-\boldsymbol{D}^\sigma\overline{\psi}\boldsymbol{\gamma}^\rho\psi)$$

as conserved quantities, which are accompanied by

$$i\boldsymbol{\gamma}^\mu\boldsymbol{D}_\mu\psi-m\psi=0$$

as the general spinorial field equations: that these are in fact conserved quantities is clear since when the spinorial field equations are valid then the relationships

$$D_\rho S^{\rho\mu\nu}+\frac{1}{2}T^{[\mu\nu]}=0$$

and

$$D_\mu T^{\mu\nu}+T_{\sigma\mu}Q^{\sigma\mu\nu}-S_{\mu\kappa\sigma}G^{\mu\kappa\sigma\nu}=0$$

are conservation laws that are satisfied in general cases.

As general introduction the interested reader may look at references [22, 23], where all this is presented for the Einstein–Sciama-Kibble torsion gravity with Dirac spinor matter fields according to the usual wisdom; but contrary to what is normally believed in this domain, these field equations are actually not the most general either, and for two reasons, the first being that the new torsional field equations, linking a completely antisymmetric torsion to a complete antisymmetric spin density, can only be valid if the spin is completely antisymmetric, although since this is indeed what happens for the Dirac field, and as this is the only matter field we know, we will not pursue any attempt of generalization in this particular direction.

However, there is another reason to have ESK torsion gravity generalized, and this is related to the fact that while the torsion and gravitational fields are independent, nevertheless their field equations appear to have the same coupling constant, which is a strong restriction.

If we want to make reason of the fact that in general independent fields must have independent coupling to their independent sources, then we must find a way to obtain the ESK field equations generalized as to be given so that the two coupling constants are different [24, 25].

It is tedious but quite straightforward to prove that the generalized system of field equations with two different coupling constants is given by the following expressions

$$Q^{\rho\mu\nu}=-aS^{\rho\mu\nu}$$

and

$$\frac{b}{2a}\left(D_{\rho}Q^{\rho\mu\nu}-\frac{1}{2}Q^{\mu\alpha\sigma}Q^{\nu}{}_{\alpha\sigma}+\frac{1}{4}g^{\mu\nu}Q^{2}\right)+\\ +G^{\mu\nu}-\frac{1}{2}g^{\mu\nu}G-g^{\mu\nu}\Lambda=\frac{1}{2}(a+b)T^{\mu\nu}$$

as the field equations of the theory: then the energy density and spin density verify the relationships

$$D_{\rho}S^{\rho\mu\nu}+\frac{1}{2}T^{[\mu\nu]}=0$$

and

$$D_{\mu}T^{\mu\nu}+T_{\sigma\mu}Q^{\sigma\mu\nu}-S_{\mu\kappa\sigma}G^{\mu\kappa\sigma\nu}=0$$

as the conservation laws satisfied by the matter field.

The spirit of geometrization is thus realized by having both torsion and curvature coupled to the spin density and energy density in terms of independent couplings.

Although this construction clearly extinguishes the use of space-time fields, also gauge fields must be taken into account, and we already know what is the consequence for the system of field equations if we require geometrical quantities to be sourced by the material fields in the most general of cases: if we have torsion-spin field equations as well as curvature-energy field equations, then we must also have gauge-current field equations for completion.

The commutator of covariant derivatives (24) applied to the case of the gauge curvature (46) gives an identity that in its fully contracted form is given by

$$D_{\rho}\left(D_{\sigma}F^{\sigma\rho}+\frac{1}{2}F_{\alpha\mu}Q^{\alpha\mu\rho}\right)=0$$

which looks like the conservation law of the current.

What this suggests is that the previous field equations must be improved with contributions of gauge fields as

$$Q^{\rho\mu\nu}=-aS^{\rho\mu\nu}$$

and

$$\frac{b}{2a}\left(D_{\rho}Q^{\rho\mu\nu}-\frac{1}{2}Q^{\mu\alpha\sigma}Q^{\nu}{}_{\alpha\sigma}+\frac{1}{4}g^{\mu\nu}Q^{2}\right)+\\ +\frac{1}{2}(a+b)\left(F^{\mu\rho}F^{\nu}{}_{\rho}-\frac{1}{4}g^{\mu\nu}F^{2}\right)+\\ +(G^{\mu\nu}-\frac{1}{2}g^{\mu\nu}G-g^{\mu\nu}\Lambda)=\frac{1}{2}(a+b)T^{\mu\nu}$$

and additionally they have to be accompanied by the field equations for the gauge fields given according to

$$\frac{1}{2}F_{\alpha\mu}Q^{\alpha\mu\rho}+D_{\sigma}F^{\sigma\rho}=J^{\rho}$$

yielding the full system of field equations: so the energy density and spin density still verify the relationships

$$D_{\rho}S^{\rho\mu\nu}+\frac{1}{2}T^{[\mu\nu]}=0$$

and

$$D_{\mu}T^{\mu\nu}+T_{\sigma\mu}Q^{\sigma\mu\nu}-S_{\mu\kappa\sigma}G^{\mu\kappa\sigma\nu}+J_{\rho}F^{\rho\nu}=0$$

but now there is also the current density verifying

$$D_{\rho}J^{\rho}=0$$

as the additional conservation law included in the full set of conservation laws satisfied by the matter field.

The non-symmetric energy density and the completely antisymmetric spin density are given according to

$$S^{\rho\mu\nu}=\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\rho},\boldsymbol{\sigma}^{\mu\nu}\}\psi$$

and

$$T^{\rho\sigma}=\frac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{D}^{\sigma}\psi-\boldsymbol{D}^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi)$$

exactly as above and now there is also the current density

$$J^{\rho}=q\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi$$

to complete the conserved quantities, together with the general form of spinorial field equations given by

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{D}_{\mu}\psi-m\psi=0$$

as above: then we have that relationships

$$D_{\rho}S^{\rho\mu\nu}+\frac{1}{2}T^{[\mu\nu]}=0$$

and

$$D_{\mu}T^{\mu\nu}+T_{\sigma\mu}Q^{\sigma\mu\nu}-S_{\mu\kappa\sigma}G^{\mu\kappa\sigma\nu}+J_{\rho}F^{\rho\nu}=0$$

alongside to

$$D_{\rho}J^{\rho}=0$$

are the full conservation laws that are satisfied in general.

All in all, the full system of field equations is given by the torsion-spin and the curvature-energy field equations

$$Q^{\rho\mu\nu}=-\frac{ia}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\rho},\boldsymbol{\sigma}^{\mu\nu}\}\psi$$

and

$$\tfrac{b}{2a}\left(D_\rho Q^{\rho\mu\nu}-\tfrac{1}{2}Q^{\mu\alpha\sigma}Q^{\nu}{}_{\alpha\sigma}+\tfrac{1}{4}g^{\mu\nu}Q^2\right)+$$
$$+\tfrac{(a+b)}{2}\left(F^{\mu\rho}F^{\nu}{}_{\rho}-\tfrac{1}{4}g^{\mu\nu}F^2\right)+$$
$$+(G^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}G-g^{\mu\nu}\Lambda)=\tfrac{i(a+b)}{4}(\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{D}^\nu\psi-\boldsymbol{D}^\nu\overline{\psi}\boldsymbol{\gamma}^\mu\psi)$$

as the pair of field equations describing the space-time structure and with the gauge-current field equations

$$\tfrac{1}{2}F_{\mu\nu}Q^{\mu\nu\rho}+D_\sigma F^{\sigma\rho}=q\overline{\psi}\boldsymbol{\gamma}^\rho\psi$$

as the field equations describing the gauge structure and thus completing the set of field equations describing the geometrical structure, together with field equations

$$i\boldsymbol{\gamma}^\mu\boldsymbol{D}_\mu\psi-m\psi=0$$

as the general form of the spinor field equations and being the field equations that describe the material structure.

These results have been discussed in reference [26] and we invite the reader to have a look there for more details.

But then again, this is not the most general system of field equations either, because the torsion tensor enters algebraically in its coupling to the spin density.

As mentioned, the above system of field equations has the feature that torsion and spin are algebraically related, and this constitutes a conceptual problem because in the case in which the spin density were to vanish then torsion would vanish too, with no possibility of doing otherwise.

That the torsion-spin coupling is algebraic may not be seen as a problem because also the curvature-energy coupling is algebraic, but there are reasons for this situation not be to entirely analogous: the most important is that the torsion that enters in the field equations is the general Cartan torsion, with the consequence that if the spin density were to be equal to zero then the Cartan torsion would be equal to zero, but the curvature that enters the field equations is the Ricci curvature and not the Riemann curvature, with the consequence that even if the energy density were to be equal to zero then the Ricci curvature would be equal to zero, but this would not imply that the Riemann curvature would be equal to zero and some gravitational field may still be present.

Also, the curvature has an internal structure given in terms of first-order derivatives of the connection and thus in terms of second-order derivatives of the metric tensor, so that there exists a dynamics for the gravitational field, but there is no similar dynamics for the torsion field.

If we desire that the torsion dynamics be implemented in the theory, then we have to look for dynamical terms in the torsion-spin field equations, and also for torsional contribution in all of the other field equations as well.

The process is similar to what we have done just above, although very long, so for the moment we will show quite straightforwardly the result, in terms of which we have

$$D_{[\rho}D^\sigma Q_{\mu\nu]\sigma}+Q_{\eta[\mu\nu}G_{\rho]\sigma}g^{\sigma\eta}-G_{\sigma[\rho}Q_{\mu\nu]\eta}g^{\sigma\eta}+$$
$$+M^2Q_{\rho\mu\nu}=\tfrac{1}{12}S_{\rho\mu\nu}$$

and

$$G^{\rho\sigma}-\tfrac{1}{2}Gg^{\rho\sigma}-18k[\tfrac{1}{3}D_\alpha D^{[\alpha}D_\pi Q^{\rho\sigma]\pi}-$$
$$-\tfrac{1}{3}D_\alpha D_\eta Q^{\eta\pi[\alpha}Q^{\rho\sigma]\nu}g_{\nu\pi}-$$
$$-\tfrac{1}{3}Q^{\rho\eta\varphi}D^{[\sigma}D_\pi Q^{\eta\varphi]\pi}-\tfrac{1}{3}Q^{\sigma\eta\varphi}D^{[\rho}D_\pi Q^{\eta\varphi]\pi}+$$
$$+\tfrac{1}{2}D^\pi D_\tau Q^{\tau\mu\nu}Q_{\pi\mu\nu}g^{\rho\sigma}+$$
$$+\tfrac{1}{4}D_\pi Q^{\pi\mu\nu}D^\tau Q_{\tau\mu\nu}g^{\rho\sigma}-D_\pi Q^{\pi\mu\rho}D^\tau Q_{\tau\mu}{}^{\sigma}+$$
$$+\tfrac{1}{3}(Q^{\rho\eta\varphi}D_\tau Q^{\tau\pi\sigma}+Q^{\sigma\eta\varphi}D_\tau Q^{\tau\pi\rho})Q^{\eta\varphi}{}_{\pi}-$$
$$-\tfrac{1}{3}D_\eta Q^{\eta\pi\alpha}D_\alpha Q^{\rho\sigma}{}_{\pi}]-$$
$$-\tfrac{1}{2}k(\tfrac{1}{4}F^2g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha})-$$
$$-(12kM^2+1)(\tfrac{1}{2}D_\alpha Q^{\alpha\rho\sigma}-$$
$$-\tfrac{1}{4}Q^{\rho\alpha\pi}Q^{\sigma}{}_{\alpha\pi}+\tfrac{1}{8}Q^2g^{\rho\sigma})-\Lambda g^{\rho\sigma}=\tfrac{1}{2}kT^{\rho\sigma}$$

alongside to

$$D_\sigma F^{\sigma\mu}+\tfrac{1}{2}F_{\alpha\nu}Q^{\alpha\nu\mu}=J^\mu$$

as the full system of field equations: they imply that

$$D_\rho S^{\rho\mu\nu}+\tfrac{1}{2}T^{[\mu\nu]}=0$$

and

$$D_\mu T^{\mu\nu}+T_{\sigma\mu}Q^{\sigma\mu\nu}-S_{\mu\kappa\sigma}G^{\mu\kappa\sigma\nu}+J_\rho F^{\rho\nu}=0$$

alongside to

$$D_\rho J^\rho=0$$

be the full set of conservation laws that have to be satisfied when the general matter field equations are assigned.

Then we have that the following expressions

$$S^{\rho\mu\nu}=-8X\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\rho,\boldsymbol{\sigma}^{\mu\nu}\}\psi$$

and

$$T^{\rho\sigma}=\tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^\rho\boldsymbol{D}^\sigma\psi-\boldsymbol{D}^\sigma\overline{\psi}\boldsymbol{\gamma}^\rho\psi)+$$
$$+(8X+1)\,D_\alpha(\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\alpha,\boldsymbol{\sigma}^{\rho\sigma}\}\psi)+$$
$$+\tfrac{1}{2}\,(8X+1)\,Q^{\rho\mu\nu}\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\sigma,\boldsymbol{\sigma}_{\mu\nu}\}\psi-$$
$$-(8X+1)\,Q^{\sigma\mu\nu}\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\rho,\boldsymbol{\sigma}_{\mu\nu}\}\psi$$

alongside to

$$J^\mu=q\overline{\psi}\boldsymbol{\gamma}^\mu\psi$$

are the conserved quantities, which are accompanied by

$$i\boldsymbol{\gamma}^\mu\boldsymbol{D}_\mu\psi-i(X+\tfrac{1}{8})Q_{\nu\tau\alpha}\boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\tau\boldsymbol{\gamma}^\alpha\psi-m\psi=0$$

as the most general spinorial field equations: and finally we have that these imply the relationships given by

$$D_\rho S^{\rho\mu\nu}+\tfrac{1}{2}T^{[\mu\nu]}\equiv0$$

and

$$D_\mu T^{\mu\nu}+T_{\rho\beta}Q^{\rho\beta\nu}-S_{\mu\rho\beta}G^{\mu\rho\beta\nu}+J_\rho F^{\rho\nu}\equiv0$$

alongside to

$$D_\rho J^\rho = 0$$

as the full set of conservation laws which are satisfied in the most general of the circumstances that are possible.

And all in all, the system of field equations has form

$$D^{[\rho}D_\sigma Q^{\mu\nu]\sigma} - G^{[\sigma\rho]}Q^{\mu\nu}{}_\sigma - G^{[\sigma\nu]}Q^{\rho\mu}{}_\sigma - G^{[\sigma\mu]}Q^{\nu\rho}{}_\sigma + \\ + M^2 Q^{\rho\mu\nu} = -\tfrac{2}{3}X\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\rho, \boldsymbol{\sigma}^{\mu\nu}\}\psi$$

and

$$G^{\rho\sigma} - \tfrac{1}{2}Gg^{\rho\sigma} - 18k[\tfrac{1}{3}D_\alpha D^{[\alpha}D_\pi Q^{\rho\sigma]\pi} - \\ -\tfrac{1}{3}D_\alpha D_\eta Q^{\eta\pi[\alpha}Q^{\rho\sigma]\nu}g_{\nu\pi} - \\ -\tfrac{1}{3}Q^{\rho\eta\varphi}D^{[\sigma}D_\pi Q^{\eta\varphi]\pi} - \tfrac{1}{3}Q^{\sigma\eta\varphi}D^{[\rho}D_\pi Q^{\eta\varphi]\pi} + \\ +\tfrac{1}{2}D^\pi D_\tau Q^{\tau\mu\nu}Q_{\pi\mu\nu}g^{\rho\sigma} + \\ +\tfrac{1}{4}D_\pi Q^{\pi\mu\nu}D^\tau Q_{\tau\mu\nu}g^{\rho\sigma} - D_\pi Q^{\pi\mu\rho}D^\tau Q_{\tau\mu}{}^\sigma + \\ +\tfrac{1}{3}(Q^{\rho\eta\varphi}D_\tau Q^{\tau\pi\sigma} + Q^{\sigma\eta\varphi}D_\tau Q^{\tau\pi\rho})Q^{\eta\varphi}{}_\pi - \\ -\tfrac{1}{3}D_\eta Q^{\eta\pi\alpha}D_\alpha Q^{\rho\sigma}{}_\pi] - \\ -\tfrac{1}{2}k(\tfrac{1}{4}F^2 g^{\rho\sigma} - F^{\rho\alpha}F^\sigma{}_\alpha) - \\ -(12kM^2+1)(\tfrac{1}{2}D_\alpha Q^{\alpha\rho\sigma} - \\ -\tfrac{1}{4}Q^{\rho\alpha\pi}Q^\sigma{}_{\alpha\pi} + \tfrac{1}{8}Q^2 g^{\rho\sigma}) - \\ -\Lambda g^{\rho\sigma} = \tfrac{1}{2}k[\tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^\rho \boldsymbol{D}^\sigma\psi - \boldsymbol{D}^\sigma\overline{\psi}\boldsymbol{\gamma}^\rho\psi) + \\ +(8X+1)\,D_\alpha(\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\alpha, \boldsymbol{\sigma}^{\rho\sigma}\}\psi) + \\ +\tfrac{1}{2}(8X+1)\,Q^{\rho\mu\nu}\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\sigma, \boldsymbol{\sigma}_{\mu\nu}\}\psi - \\ -(8X+1)\,Q^{\sigma\mu\nu}\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\rho, \boldsymbol{\sigma}_{\mu\nu}\}\psi]$$

alongside to

$$D_\sigma F^{\sigma\mu} + \tfrac{1}{2}F_{\alpha\nu}Q^{\alpha\nu\mu} = q\overline{\psi}\boldsymbol{\gamma}^\mu\psi$$

as the geometric field equations, together with

$$i\boldsymbol{\gamma}^\mu \boldsymbol{D}_\mu\psi - i(X+\tfrac{1}{8})Q_{\nu\tau\alpha}\boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\tau\boldsymbol{\gamma}^\alpha\psi - m\psi = 0$$

as the general form of matter field equations of the theory.

That this is the most general system of field equations is something we told the reader because we wanted to get to the point without being diverted by technicalities, but we have not proved it and so the presentation, despite historically detailed, is mathematically not demonstrated.

In what will be next we are going to re-consider all this the other way around, in order to fully demonstrate that what we presented is the most general case indeed.

b. *Mathematics: the most general theory of propagating torsion gravity*

Before we have told the reader what are the most general field equations without proving it was the case, and now we will demonstrate it by direct construction of the most general system of field equations that is possible.

In order to construct the most general system of field equations, we are going to start by distinguishing them in two different types: the field equations for the geometry-matter coupling, which will be written in the form of second-order derivatives of the metric and torsion and also gauge potentials equal to sources given by the energy and spin and also the current of fields; and the matter field equations, which will be written in the form of a first-order differential operator containing metric and torsion and gauge potentials acting on the spinor field and equalling the spinor field itself. This discrimination comes from the fact that, on the one hand, it is possible to employ spinors to construct sources for the tensor and gauge field equations, but on the other hand, it is not possible to use tensor and gauge fields to build sources of the spinorial field equations; in the spinorial field equations the derivatives of the spinor field must be proportional to the spinor field itself. This discrimination between the form of geometric and matter field equations is therefore intrinsic to the structure of the fields we are employing.

We start by considering the fact that field equations for the metric have to be in the form of some derivative of the metric equal to some source: because the covariant derivative of the metric tensor vanishes identically, then any dynamics of the metric can only be described in terms of the partial derivatives of the metric, or equivalently by the metric connection (14); again the metric connection is not a tensor, and the only way we have from the symmetric connection to form a tensor is to take another partial derivative, therefore forming the metric curvature tensor as given by (22). As equation (26) shows, the metric curvature tensor is one peculiar combination of second-order partial derivatives of the metric, that is arguments of symmetry under the most general coordinate transformations force at least second-order derivatives of the metric in the differential field equations; then arguments of simplicity would require that we do not take any further differential structure. In the following we will see that second-order derivatives in the metric field equations endow them with a character that no other field equation will have, rendering them somewhat peculiar.

For the moment, what we have established is that the metric field equations will have to be given in the form of some combination of the metric curvature tensor, and to see what combination, we start from considering that if the leading term were to be given by the Riemann metric curvature tensor $R^{\alpha\tau\sigma\nu}$ then the vacuum equations would reduce to the condition of vanishing of Riemann metric curvature tensor, so that they would imply that there be only the trivial metric; hence if we want non-trivial metrics to be possible in vacuum, then the Riemann metric curvature tensor must appear contracted as the Ricci metric curvature tensor $R^{\alpha\mu}$ for leading term, and of course we may have contractions such as the Ricci metric curvature scalar $Rg_{\alpha\mu}$ or even $\Lambda g_{\alpha\mu}$ as sub-leading terms in general: so for the moment we will establish the most general form of field equations to be given by

$$R^{\alpha\mu} - ARg^{\alpha\mu} - \Lambda g^{\alpha\mu} = \tfrac{1}{2}kE^{\alpha\mu}$$

where $E^{\alpha\mu}$ will have to be fixed on general grounds, as we are going to do a little later in this treatment.

For now, a first thing to notice is what happens without extra contributions: the vacuum metric field equations are reduced according to the following relationship

$$R^{\mu\nu}-ARg^{\mu\nu}-\Lambda g^{\mu\nu}=0$$

and in absence of torsion identities (29) are given by

$$\nabla_{\mu}R^{\nu}{}_{\iota\kappa\rho}+\nabla_{\kappa}R^{\nu}{}_{\iota\rho\mu}+\nabla_{\rho}R^{\nu}{}_{\iota\mu\kappa}\equiv 0$$

whose contracted form is

$$\nabla_{\mu}R^{\mu}{}_{\iota\kappa\rho}-\nabla_{\kappa}R_{\iota\rho}+\nabla_{\rho}R_{\iota\kappa}\equiv 0$$

and whose fully-contracted form is given by

$$\nabla_{\mu}(R^{\mu\nu}-\tfrac{1}{2}Rg^{\mu\nu})\equiv 0$$

implying that

$$(1-2A)\nabla_{\nu}R=0$$

which is not true unless $2A=1$ holds. So the metric field equations in vacuum are given by the expression

$$R^{\mu\nu}-\tfrac{1}{2}Rg^{\mu\nu}-\Lambda g^{\mu\nu}=0$$

and such a structure must of course be also valid in the case in which extra contributions will be present.

By placing this constraint we obtain field equations

$$R^{\alpha\mu}-\tfrac{1}{2}Rg^{\alpha\mu}-\Lambda g^{\alpha\mu}=\tfrac{1}{2}kE^{\alpha\mu}$$

in which the only constant Λ is still undetermined and it will remain undetermined since there is no way to fix it on geometrical grounds, so that we may think of it as an integration constant, which can always be added and whose value cannot be fixed, unless empirically.

So much for the metric field equations, we next turn our attention to the other field equations, for which the covariant derivatives of the fields will not be identically zero and thus a different path has to be followed.

The field equations for the torsion have to be in the form of covariant derivatives of the torsion axial-vector equal to some source: taking covariant derivatives of the torsion axial-vector implies that we will have to write the field equation in the form of the covariant divergence of the torsion axial-vector equal to a source constituted by a pseudo-scalar field, but the temporal derivative will be specified for the temporal component of the torsion axial-vector solely; therefore we must take two covariant derivatives of the torsion axial-vector as leading term.

To assess what are the most general field equations for the torsion axial-vector we consider that the leading term given in the form of two covariant derivatives of the torsion axial-vector $\nabla_{\sigma}\nabla_{\alpha}W_{\rho}$ is to be such that one of the indices of the derivatives has to be contracted yielding the two forms $\nabla_{\sigma}\nabla^{\sigma}W_{\rho}$ and $\nabla_{\rho}\nabla_{\sigma}W^{\sigma}$ as leading terms: subleading terms may be added eventually and so we may establish the most general form of field equations as

$$\begin{gathered}2\Pi\nabla_{\sigma}\nabla^{\sigma}W^{\eta}-2H\nabla^{\eta}\nabla_{\rho}W^{\rho}-\\-V\nabla_{\alpha}W_{\nu}W_{\rho}\varepsilon^{\alpha\nu\rho\eta}-UW^{\alpha}W_{\alpha}W^{\eta}-\\-2LR^{\eta\rho}W_{\rho}+2NRW^{\eta}+PW^{\eta}=\kappa S^{\eta}\end{gathered}$$

where S^{α} will have to be fixed on general grounds.

This general field equation can be restricted with the Velo-Zwanziger method [27, 28], so taking its divergence

$$\begin{gathered}2(\Pi-H)\nabla_{\eta}\nabla^{\eta}\nabla_{\rho}W^{\rho}+\\+V\nabla_{\eta}\nabla_{\alpha}W_{\nu}W_{\rho}\varepsilon^{\eta\alpha\nu\rho}+\\+V\nabla_{\alpha}W_{\nu}\nabla_{\eta}W_{\rho}\varepsilon^{\eta\alpha\nu\rho}-\\-2[UW^{\rho}W^{\eta}+(L-\Pi)R^{\eta\rho}]\nabla_{\eta}W_{\rho}+\\+(2N-L+\Pi)\nabla_{\eta}RW^{\eta}-\\-(UW^{\alpha}W_{\alpha}-2NR-P)\nabla_{\eta}W^{\eta}=\kappa\nabla_{\eta}S^{\eta}\end{gathered}$$

it becomes possible to see that there appears a third-order time derivative for the temporal component of the torsion axial-vector implying that the constraint obtained from the field equations would actually determine the time evolution of some components of the torsion axial-vector field, and since this would spoil the balance between the number of independent field equations and the amount of degrees of freedom of a given field, then no higher-order derivative terms must be produced in the constraints and so we set $\Pi=H$ identically; once this is done, there is no higher-order derivative nor second-order derivative in time for any components of the field in the constraint, which is thus a true constraint, and when it is substituted back into the field equations, we obtain

$$\begin{gathered}2H\nabla_{\sigma}\nabla^{\sigma}W^{\eta}-2H(UW^{\alpha}W_{\alpha}-2NR-P)^{-1}\cdot\\\cdot\nabla^{\eta}[V\nabla_{\tau}\nabla_{\alpha}W_{\nu}W_{\rho}\varepsilon^{\tau\alpha\nu\rho}+\\+V\nabla_{\alpha}W_{\nu}\nabla_{\tau}W_{\rho}\varepsilon^{\tau\alpha\nu\rho}-\\-2[UW^{\rho}W^{\tau}+(L-H)R^{\tau\rho}]\nabla_{\tau}W_{\rho}+\\+(2N-L+H)\nabla_{\tau}RW^{\tau}-\kappa\nabla_{\tau}S^{\tau}]+\\+2H\nabla^{\eta}(UW^{\alpha}W_{\alpha}-2NR)\cdot\\\cdot(UW^{\alpha}W_{\alpha}-2NR-P)^{-2}\cdot\\\cdot[V\nabla_{\tau}\nabla_{\alpha}W_{\nu}W_{\rho}\varepsilon^{\tau\alpha\nu\rho}+\\+V\nabla_{\alpha}W_{\nu}\nabla_{\tau}W_{\rho}\varepsilon^{\tau\alpha\nu\rho}-\\-2[UW^{\rho}W^{\tau}+(L-H)R^{\tau\rho}]\nabla_{\tau}W_{\rho}+\\+(2N-L+H)\nabla_{\tau}RW^{\tau}-\kappa\nabla_{\tau}S^{\tau}]-\\-V\nabla_{\alpha}W_{\nu}W_{\rho}\varepsilon^{\alpha\nu\rho\eta}-UW^{\alpha}W_{\alpha}W^{\eta}-\\-2LR^{\eta\rho}W_{\rho}+2NRW^{\eta}+PW^{\eta}=\kappa S^{\eta}\end{gathered}$$

which contains second-order time derivatives of all components of the torsion axial-vector, and therefore this is a true field equation. To check the propagation of the field, we have to consider its characteristic equation

$$\begin{gathered}(UW^{\alpha}W_{\alpha}-2NR-P)g^{\nu\eta}n^{2}-\\-V\varepsilon^{\tau\rho\alpha\nu}(\partial W)_{\tau\rho}n_{\alpha}n^{\eta}+\\+2[UW^{\tau}W^{\nu}+(L-H)R^{\tau\nu}]n_{\tau}n^{\eta}=0\end{gathered}$$

giving rise to a characteristic determinant of the form

$$\begin{gathered}(UW^{\alpha}W_{\alpha}-2NR-P)n^{2}+\\+2[UW^{\tau}W^{\nu}+(L-H)R^{\tau\nu}]n_{\tau}n_{\nu}=0\end{gathered}$$

which has to be discussed: because we have no information about $R^{\tau\nu}$ or R then acausality may be possible unless we have that $L=H$ and $N=0$ hold, but even then

$$(UW^2-P)n^2+2U|W\cdot n|^2=0$$

tells that again acausality may occur unless additionally we impose $U=0$ identically, in which case

$$n^2=0$$

spelling that acausality cannot happen. Notice that there is no constraints on V which remains a free parameter.

Placing all constraints together gives field equations

$$4\nabla_{\rho}(\partial W)^{\rho\eta}-VW_{\rho}(\partial W)_{\alpha\nu}\varepsilon^{\rho\alpha\nu\eta}+ \\ +2PW^{\eta}=2\kappa S^{\eta}$$

because without loss of generality constant H can be reabsorbed within a redefinition of all the other constants.

To proceed, we notice that the metric field equations the source contribution from the torsion axial-vector field has to be built with no quartic torsion term, because they would correspond to what in the torsion field equations are cubic torsion term, which are absent, and no second derivatives of torsion, because they would give rise to curvatures, which cannot be present since they are already addressed, and so it is possible to come to the most general form of this contribution as the one given by

$$E^{\mu\nu}=aW^{\mu}W^{\nu}+bW^2g^{\mu\nu}+ \\ +z(W^{\nu}W_{\rho}(\partial W)_{\alpha\sigma}\varepsilon^{\rho\alpha\sigma\mu}+ \\ +W^{\mu}W_{\rho}(\partial W)_{\alpha\sigma}\varepsilon^{\rho\alpha\sigma\nu})+ \\ +y(\nabla_{\sigma}W^{\mu}(\partial W)^{\sigma\nu}+ \\ +\nabla_{\sigma}W^{\nu}(\partial W)^{\sigma\mu})+ \\ +x\nabla^{\mu}W_{\sigma}\nabla^{\nu}W^{\sigma}+ \\ +w\nabla_{\sigma}W^{\mu}\nabla^{\sigma}W^{\nu}+ \\ +v\nabla_{\alpha}W_{\sigma}\nabla^{\alpha}W^{\sigma}g^{\mu\nu}+ \\ +u(\partial W)^{\nu\sigma}(\partial W)^{\mu}{}_{\sigma}+ \\ +t(\partial W)^2g^{\mu\nu}$$

in terms of ten constants: then the metric field equations with such contribution from torsion are given by

$$R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}-\Lambda g^{\mu\nu}=\frac{1}{2}k[aW^{\mu}W^{\nu}+bW^2g^{\mu\nu}+ \\ +z(W^{\nu}W_{\rho}(\partial W)_{\alpha\sigma}\varepsilon^{\rho\alpha\sigma\mu}+W^{\mu}W_{\rho}(\partial W)_{\alpha\sigma}\varepsilon^{\rho\alpha\sigma\nu})+ \\ +y(\nabla_{\sigma}W^{\mu}(\partial W)^{\sigma\nu}+\nabla_{\sigma}W^{\nu}(\partial W)^{\sigma\mu})+ \\ +x\nabla^{\mu}W_{\sigma}\nabla^{\nu}W^{\sigma}+w\nabla_{\sigma}W^{\mu}\nabla^{\sigma}W^{\nu}+ \\ +v\nabla_{\alpha}W_{\sigma}\nabla^{\alpha}W^{\sigma}g^{\mu\nu}+ \\ +u(\partial W)^{\nu\sigma}(\partial W)^{\mu}{}_{\sigma}+ \\ +t(\partial W)^2g^{\mu\nu}]$$

which have to be taken with the torsion field equations

$$4\nabla_{\rho}(\partial W)^{\rho\eta}-VW_{\rho}(\partial W)_{\alpha\nu}\varepsilon^{\rho\alpha\nu\eta}+2PW^{\eta}=0$$

in the vacuum; because for the metric field equations the demonstrated divergencelessness of the left-hand side implies the divergencelessness of the right-hand side, then we must have the divergencelessness of the contribution that comes from the torsion field and which is given by

$$0=(a\nabla\cdot W-\tfrac{z}{2}(\partial W)_{\mu\rho}(\partial W)_{\alpha\sigma}\varepsilon^{\mu\rho\alpha\sigma})W^{\nu}+ \\ +(y\nabla^{\sigma}\nabla_{\mu}W_{\sigma}-u\nabla^{\sigma}(\partial W)_{\sigma\mu}- \\ -2bW_{\mu}-x\nabla^2W_{\mu})(\partial W)^{\mu\nu}+ \\ +(aW_{\mu}-y\nabla^{\sigma}(\partial W)_{\sigma\mu}+w\nabla_{\sigma}\nabla_{\mu}W^{\sigma}+ \\ +zW^{\rho}(\partial W)^{\alpha\sigma}\varepsilon_{\rho\alpha\sigma\mu}+2bW_{\mu}+x\nabla^2W_{\mu})\nabla^{\mu}W^{\nu}- \\ -[(y+u+4t)(\partial W)_{\mu\sigma}+(x-y)\nabla_{\mu}W_{\sigma}]\nabla^{\mu}(\partial W)^{\sigma\nu}+ \\ +[(y+w)(\partial W)_{\sigma\mu}+(x+w)\nabla_{\mu}W_{\sigma}]\nabla^{\mu}\nabla^{\sigma}W^{\nu}+ \\ +2v\nabla_{\mu}W_{\sigma}\nabla^{\nu}\nabla^{\mu}W^{\sigma}+ \\ +zW^{\mu}\nabla_{\mu}W_{\rho}(\partial W)_{\alpha\sigma}\varepsilon^{\rho\alpha\sigma\nu}+ \\ +zW^{\mu}W_{\rho}\nabla_{\mu}(\partial W)_{\alpha\sigma}\varepsilon^{\rho\alpha\sigma\nu}+ \\ +z\nabla\cdot WW_{\rho}(\partial W)_{\alpha\sigma}\varepsilon^{\rho\alpha\sigma\nu}$$

in which the first term has the structure of the divergence of the field equations for the torsion field given by

$$4P\nabla\cdot W+V(\partial W)_{\eta\rho}(\partial W)_{\alpha\nu}\varepsilon^{\eta\rho\alpha\nu}=0$$

while the last term is not related to any other term and therefore these two contributions disappear only if we require that $V=z=0$ hold, so that the above becomes

$$[\tfrac{1}{2}(uP+xP-4b)W_{\mu}+ \\ +(y-x)R_{\sigma\mu}W^{\sigma}](\partial W)^{\mu\nu}+ \\ +[\tfrac{1}{2}(2a+yP+4b-xP)W_{\mu}+ \\ +(w+x)R_{\sigma\mu}W^{\sigma}]\nabla^{\mu}W^{\nu}- \\ -[(y+u+4t)(\partial W)_{\mu\sigma}+ \\ +(x-y)\nabla_{\mu}W_{\sigma}]\nabla^{\mu}(\partial W)^{\sigma\nu}+ \\ +[(y+w)(\partial W)_{\sigma\mu}+ \\ +(x+w)\nabla_{\mu}W_{\sigma}]\nabla^{\mu}\nabla^{\sigma}W^{\nu}+ \\ +2v\nabla_{\mu}W_{\sigma}\nabla^{\nu}\nabla^{\mu}W^{\sigma}=0$$

implying $v=0$ with $x=y=-w$ and $x+u=-4t$ and together with $a=-2b=2tP$ which must hold identically.

As a consequence the metric field equations in presence of the contribution of torsion are writable according to

$$R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}-\Lambda g^{\mu\nu}-\frac{1}{2}k4t[\tfrac{1}{4}(\partial W)^2g^{\mu\nu}- \\ -(\partial W)^{\nu\sigma}(\partial W)^{\mu}{}_{\sigma}+ \\ +\tfrac{1}{2}P(W^{\mu}W^{\nu}-\tfrac{1}{2}W^2g^{\mu\nu})]=\frac{1}{2}kE^{\mu\nu}$$

with torsion field equations that are given by

$$\nabla_{\rho}(\partial W)^{\rho\eta}+\tfrac{1}{2}PW^{\eta}=\tfrac{1}{2}\kappa S^{\eta}$$

which must also be valid in presence of the gauge field.

The field equations for the gauge field are also in the form of covariant derivatives of the gauge potential equal to some source: nevertheless by taking derivatives of the gauge potential means that that we have to consider the gauge strength because this is the only term that is differential in the potential and which is still gauge invariant, but since this is irreducible, any contraction of the gauge strength vanishes and therefore these terms alone cannot be not enough; hence we have to take one more covariant derivative of the gauge strength as leading term.

The most general field equations for the gauge fields have a leading term in the form $\nabla_{\sigma}F_{\alpha\rho}$ and after contraction we get $\nabla_{\sigma}F^{\sigma\rho}$ as the leading term: then we get

$$\nabla_{\sigma}F^{\sigma\eta}-\frac{1}{12}BF_{\alpha\nu}W_{\rho}\varepsilon^{\alpha\nu\rho\eta}=qJ^{\eta}$$

in which the source J^{α} will have to be fixed as well.

The contribution from the gauge field is similarly built in terms of squares of the gauge curvature strength, since any other term would violate gauge symmetry, and thus

$$E^{\mu\nu}=\alpha F^{\mu\rho}F^{\nu}{}_{\rho}+\beta F^{\alpha\pi}F_{\alpha\pi}g^{\mu\nu}$$

in terms of two constants: the metric field equations with contributions of torsion and gauge fields are

$$\begin{gathered}R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}-\Lambda g^{\mu\nu}-\frac{1}{2}k4t[\frac{1}{4}(\partial W)^{2}g^{\mu\nu}-\\-(\partial W)^{\nu\sigma}(\partial W)^{\mu}{}_{\sigma}+\frac{1}{2}P(W^{\mu}W^{\nu}-\\-\frac{1}{2}W^{2}g^{\mu\nu})]=\frac{1}{2}k(\alpha F^{\mu\rho}F^{\nu}{}_{\rho}+\\+\beta F^{\alpha\pi}F_{\alpha\pi}g^{\mu\nu})\end{gathered}$$

with torsion field equations

$$\nabla_{\rho}(\partial W)^{\rho\eta}+\frac{1}{2}PW^{\eta}=0$$

in vacuum and with the gauge field equations

$$\nabla_{\sigma}F^{\sigma\eta}-\frac{1}{12}BF_{\alpha\nu}W_{\rho}\varepsilon^{\alpha\nu\rho\eta}=0$$

in vacuum; again the divergencelessness of the left-hand side implies the divergencelessness right-hand side

$$\begin{gathered}0=\nabla_{\mu}(\alpha F^{\mu\rho}F^{\nu}{}_{\rho}+\beta F^{\alpha\pi}F_{\alpha\pi}g^{\mu\nu})=\\=\alpha\nabla_{\mu}F^{\mu\rho}F^{\nu}{}_{\rho}+\alpha F^{\mu\rho}\nabla_{\mu}F^{\nu}{}_{\rho}+\\+2\beta\nabla^{\nu}F^{\alpha\pi}F_{\alpha\pi}=\\=-\frac{1}{12}\alpha B\varepsilon_{\tau\pi\sigma\rho}F^{\tau\pi}W^{\sigma}F^{\rho\nu}+\\+\alpha F_{\mu\rho}\nabla^{\mu}F^{\nu\rho}+2\beta\nabla^{\nu}F^{\alpha\pi}F_{\alpha\pi}=\\=-\frac{1}{12}\alpha B\varepsilon_{\tau\pi\sigma\rho}F^{\tau\pi}W^{\sigma}F^{\rho\nu}+\\+\frac{1}{2}(\alpha+4\beta)F_{\mu\rho}\nabla^{\nu}F^{\mu\rho}=\\=-\frac{1}{12}\alpha B\varepsilon_{\tau\pi\sigma\rho}F^{\tau\pi}W^{\sigma}F^{\rho\nu}+\\+\frac{1}{4}(\alpha+4\beta)\nabla^{\nu}F^{2}\end{gathered}$$

which implies that $B=0$ and $\alpha=-4\beta$ hold identically.

Placing all constraints together, the metric field equations with contributions of torsion and gauge fields are

$$\begin{gathered}R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}-\Lambda g^{\mu\nu}-\frac{1}{2}k4t[\frac{1}{2}P(W^{\mu}W^{\nu}-\\-\frac{1}{2}W^{2}g^{\mu\nu})+\frac{1}{4}(\partial W)^{2}g^{\mu\nu}-\\-(\partial W)^{\nu\sigma}(\partial W)^{\mu}{}_{\sigma}]-\frac{1}{2}k4\beta(\frac{1}{4}F^{2}g^{\mu\nu}-\\-F^{\mu\rho}F^{\nu}{}_{\rho})=\frac{1}{2}kE^{\mu\nu}\end{gathered}$$

with torsion field equations that are given by

$$\nabla_{\rho}(\partial W)^{\rho\mu}+\frac{1}{2}PW^{\mu}=\frac{1}{2}\kappa S^{\mu}$$

and with the gauge field equations as

$$\nabla_{\sigma}F^{\sigma\mu}=qJ^{\mu}$$

in which we have moved all contributions to the left-hand side in order to stress that these equations will have to be valid in this form even when matter is present.

In the metric field equation the contributions due to torsion and gauge fields are analogous, and torsion and gauge fields are independent, so we may normalize torsion and gauge fields with no loss of generality so to have the two constants t and β with the same value, and it is still without losing generality that they can be reabsorbed in the k constant, and therefore the metric field equations with torsion and gauge fields are given by

$$\begin{gathered}R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}-\Lambda g^{\mu\nu}-\frac{1}{2}k[M^{2}(W^{\mu}W^{\nu}-\frac{1}{2}W^{2}g^{\mu\nu})+\\+\frac{1}{4}(\partial W)^{2}g^{\mu\nu}-(\partial W)^{\nu\sigma}(\partial W)^{\mu}{}_{\sigma}]-\\-\frac{1}{2}k(\frac{1}{4}F^{2}g^{\mu\nu}-F^{\mu\rho}F^{\nu}{}_{\rho})=\frac{1}{2}kE^{\mu\nu}\end{gathered}$$

with torsion field equations in form

$$\nabla_{\rho}(\partial W)^{\rho\mu}+M^{2}W^{\mu}=\frac{1}{2}\kappa S^{\mu}$$

and gauge field equations

$$\nabla_{\sigma}F^{\sigma\mu}=qJ^{\mu}$$

where we set $P=2M^{2}$ because this is just the mass term of the torsion axial-vector field as it is well known.

We notice that in reabsorbing within a renaming of the constant k the values of the constants t and β we did not lose any generality in their absolute value, but in order not to lose any generality also for the sign all constants would have to be positive, and this is general may not be the case: the reason why we did it anyway is that those constants are in front of torsion and gauge fields' energy contributions and energies are positive defined; although we might have assumed those constants to have a generic sign, in the final form of the field equations we would have discovered that the signs were all positive, and thus with no loss of generality we can assume it from the beginning.

We also notice that in torsion and gauge field equations there is no extra contribution: this is because the metric cannot produce extra contributions for torsion and gauge fields, and torsion and gauge fields cannot interact with one another without violating the gauge invariance.

To proceed with the inclusion of matter fields, it is fundamental to notice that spinor fields are defined in terms of gamma matrices that can also be used in building fundamental quantities, whose employment allows to lower the order of derivatives in all such quantities because every time covariance demands for a single covariant index to be present one gamma matrix can be used instead of one spinorial derivatives of the spinor field: to include the matter fields, we have to write the general form of their contribution in the metric field equations, and this can be constructed by employing no more than one spinorial derivative of the spinor field, since gamma matrices can be used to saturate indices, and eventually we have that the most general expression is given according to

$$\begin{aligned}E^{\rho\sigma}=\zeta[\boldsymbol{\nabla}^{\rho}(\overline{\psi}\boldsymbol{\gamma}^{\sigma}\psi)+\boldsymbol{\nabla}^{\sigma}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi)]+\\+i\xi(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\nabla}^{\sigma}\psi-\boldsymbol{\nabla}^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi+\\+\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\nabla}^{\rho}\psi-\boldsymbol{\nabla}^{\rho}\overline{\psi}\boldsymbol{\gamma}^{\sigma}\psi)+\\+\chi\boldsymbol{\nabla}_{\alpha}(\overline{\psi}\boldsymbol{\gamma}^{\alpha}\psi)g^{\rho\sigma}+\\+\lambda i(\overline{\psi}\boldsymbol{\gamma}^{\alpha}\boldsymbol{\nabla}_{\alpha}\psi-\boldsymbol{\nabla}_{\alpha}\overline{\psi}\boldsymbol{\gamma}^{\alpha}\psi)g^{\rho\sigma}+\\+\tau(W^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\pi}\psi+W^{\rho}\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi)+\\+\upsilon W_{\alpha}\overline{\psi}\boldsymbol{\gamma}^{\alpha}\boldsymbol{\pi}\psi g^{\sigma\rho}+\mu\overline{\psi}\psi g^{\rho\sigma}\end{aligned}$$

in general; the contribution as source of the torsion field equations is the spin density of the material field and it can be taken with no spinorial derivative at all when gamma matrices are considered, therefore obtaining that

$$S^{\mu}=\omega\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi$$

also in general; the contribution as source of the gauge field equations is the current density of the material field and similarly it is given according to

$$J^{\rho}=p\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi$$

again in the most general case: the full set of geometry-matter coupling field equations is for the metric field

$$\begin{aligned}R^{\rho\sigma}-\tfrac{1}{2}Rg^{\rho\sigma}-\tfrac{k}{2}[\tfrac{1}{4}(\partial W)^{2}g^{\rho\sigma}-(\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha}]-\\-\tfrac{k}{2}(\tfrac{1}{4}F^{2}g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha})-\\-\tfrac{k}{2}M^{2}(W^{\rho}W^{\sigma}-\tfrac{1}{2}W^{2}g^{\rho\sigma})-\\-\Lambda g^{\rho\sigma}=\tfrac{1}{2}k[\zeta[\boldsymbol{\nabla}^{\rho}(\overline{\psi}\boldsymbol{\gamma}^{\sigma}\psi)+\boldsymbol{\nabla}^{\sigma}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi)]+\\+i\xi(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\nabla}^{\sigma}\psi-\boldsymbol{\nabla}^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi+\\+\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\nabla}^{\rho}\psi-\boldsymbol{\nabla}^{\rho}\overline{\psi}\boldsymbol{\gamma}^{\sigma}\psi)+\\+\chi\boldsymbol{\nabla}_{\alpha}(\overline{\psi}\boldsymbol{\gamma}^{\alpha}\psi)g^{\rho\sigma}+\\+\lambda i(\overline{\psi}\boldsymbol{\gamma}^{\alpha}\boldsymbol{\nabla}_{\alpha}\psi-\boldsymbol{\nabla}_{\alpha}\overline{\psi}\boldsymbol{\gamma}^{\alpha}\psi)g^{\rho\sigma}+\\+\tau(W^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\pi}\psi+W^{\rho}\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi)+\\+\upsilon W_{\alpha}\overline{\psi}\boldsymbol{\gamma}^{\alpha}\boldsymbol{\pi}\psi g^{\sigma\rho}+\mu\overline{\psi}\psi g^{\rho\sigma}]\end{aligned}$$

for the torsion field

$$\nabla_{\rho}(\partial W)^{\rho\mu}+M^{2}W^{\mu}=2\kappa\omega\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi$$

and for the gauge field

$$\nabla_{\sigma}F^{\sigma\mu}=qp\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi$$

with nine constants, some of which to be removed.

The most general field equations for the spinor field have a leading term containing $\boldsymbol{\nabla}_{\mu}\psi$ so that after multiplying by the matrix $\boldsymbol{\gamma}^{\nu}$ it is possible to contract the indices getting $\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi$ as leading term: therefore we may establish the most general form of field equations as

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-XW_{\sigma}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi-m\psi=0$$

where the imaginary unit has been placed because in free cases $i\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-m\psi=0$ so that taking the square of the derivative gives $\boldsymbol{\nabla}^{2}\psi+m^{2}\psi=0$ and m can be interpreted as the mass term, which is what is expected in general.

By considering the fact that the metric field equations are divergenceless on the left-hand side then they must be divergenceless on their right-hand side, so that

$$\begin{aligned}0=M^{2}(\partial W)^{\sigma\rho}W_{\sigma}+M^{2}W^{\rho}\nabla\cdot W-\\-\nabla_{\sigma}(\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha}-F^{\rho\alpha}\nabla_{\sigma}F^{\sigma}{}_{\alpha}+\\+\zeta[-R^{\sigma\rho}\overline{\psi}\boldsymbol{\gamma}_{\sigma}\psi+\nabla^{2}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi)]+\\+2\xi\boldsymbol{\nabla}_{\sigma}[\overline{\psi}\boldsymbol{\sigma}^{\rho\sigma}(i\boldsymbol{\gamma}^{\alpha}\boldsymbol{\nabla}_{\alpha}\psi)+(i\boldsymbol{\nabla}_{\alpha}\overline{\psi}\boldsymbol{\gamma}^{\alpha})\boldsymbol{\sigma}^{\rho\sigma}\psi]-\\-\xi\nabla_{\alpha}\nabla_{\nu}(i\overline{\psi}\{\boldsymbol{\gamma}^{\rho},\boldsymbol{\sigma}^{\alpha\nu}\}\psi)+\\+2\xi[(i\boldsymbol{\nabla}_{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\sigma})\boldsymbol{\nabla}^{\rho}\psi-\boldsymbol{\nabla}^{\rho}\overline{\psi}(i\boldsymbol{\gamma}^{\sigma}\boldsymbol{\nabla}_{\sigma}\psi)]+\\+2\xi[\overline{\psi}\boldsymbol{\nabla}^{\rho}(i\boldsymbol{\gamma}^{\sigma}\boldsymbol{\nabla}_{\sigma}\psi)-\boldsymbol{\nabla}^{\rho}(i\boldsymbol{\nabla}_{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\sigma})\psi]+\\+i\xi R^{ab\sigma\rho}\overline{\psi}\{\boldsymbol{\gamma}_{\sigma},\boldsymbol{\sigma}_{ab}\}\psi-4q\xi F^{\sigma\rho}\overline{\psi}\boldsymbol{\gamma}_{\sigma}\psi+\\+\tau\nabla\cdot W\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\pi}\psi+\tau W^{\rho}\nabla_{\sigma}(\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi)+\\+[\tau\nabla^{\sigma}W^{\rho}+(\upsilon+2\lambda X)\nabla^{\rho}W^{\sigma}]\overline{\psi}\boldsymbol{\gamma}_{\sigma}\boldsymbol{\pi}\psi+\\+W_{\sigma}[(\upsilon+2\lambda X)\nabla^{\rho}(\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi)+\tau\nabla^{\sigma}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\pi}\psi)]+\\+(\mu+2\lambda m)\nabla^{\rho}(\overline{\psi}\psi)=(\tau+2\xi X)\nabla\cdot W\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\pi}\psi+\\+(2\kappa\omega+\tau-2\xi X)\nabla_{\mu}(\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi)W^{\rho}+\\+(\tau+\upsilon+2\lambda X+2\xi X)\overline{\psi}\boldsymbol{\gamma}_{\sigma}\boldsymbol{\pi}\psi\nabla^{\sigma}W^{\rho}+\\+(\tau+\upsilon+2\lambda X+2\xi X)W_{\sigma}\nabla^{\sigma}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\pi}\psi)-\\-(\upsilon+2\lambda X+4\xi X-2\kappa\omega)\overline{\psi}\boldsymbol{\gamma}_{\sigma}\boldsymbol{\pi}\psi(\partial W)^{\sigma\rho}-\\-(\upsilon+2\lambda X)W_{\sigma}\partial^{[\sigma}(\overline{\psi}\boldsymbol{\gamma}^{\rho]}\boldsymbol{\pi}\psi)+\\+q(p-4\xi)\overline{\psi}\boldsymbol{\gamma}_{\sigma}\psi F^{\sigma\rho}-\zeta R^{\sigma\rho}\overline{\psi}\boldsymbol{\gamma}_{\sigma}\psi+\\+\zeta\nabla^{2}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi)+(\mu+2\lambda m)\nabla^{\rho}(\overline{\psi}\psi)\end{aligned}$$

in which the only second-order derivative is the squared derivative and therefore there is no further curvature that can arise, so that those that are already present do not cancel unless $\zeta=0$ hold, and with similar arguments we get that $\mu=-2\lambda m$ and $p=4\xi$ as well as the constraining forms $\tau=-2\xi X$ and $\upsilon=-2\lambda X$ with $\kappa\omega=2\xi X$ which have to hold in order for the fields in interaction to have a total energy that is divergenceless. The full system of field equations is given for the metric field equations as

$$\begin{aligned}R^{\rho\sigma}-\tfrac{1}{2}Rg^{\rho\sigma}-\tfrac{k}{2}[\tfrac{1}{4}(\partial W)^{2}g^{\rho\sigma}-(\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha}]-\\-\tfrac{k}{2}(\tfrac{1}{4}F^{2}g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha})-\tfrac{k}{2}M^{2}(W^{\rho}W^{\sigma}-\tfrac{1}{2}W^{2}g^{\rho\sigma})-\\-\Lambda g^{\rho\sigma}=\tfrac{1}{2}k4\xi[\tfrac{i}{4}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\nabla}^{\sigma}\psi-\boldsymbol{\nabla}^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi+\\+\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\nabla}^{\rho}\psi-\boldsymbol{\nabla}^{\rho}\overline{\psi}\boldsymbol{\gamma}^{\sigma}\psi)-\\-\tfrac{1}{2}X(W^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\pi}\psi+W^{\rho}\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi)]\end{aligned}$$

and torsion field equations

$$\nabla_{\rho}(\partial W)^{\rho\mu}+M^{2}W^{\mu}=4\xi X\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi$$

with gauge field equations as

$$\nabla_{\sigma}F^{\sigma\mu}=4\xi q\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi$$

and matter field equations given by

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-XW_{\sigma}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi-m\psi=0$$

which are valid in the most general case possible.

Finally we notice that without affecting the metric nor the torsion or the gauge fields, the spinor field may be renormalized in such a way that without losing generality we can always set $4\xi=1$ and as a consequence it is possible to see that the full system of field equations has been completely determined, and it is constituted by the metric field equations given according to the expression

$$\begin{gathered}R^{\rho\sigma}-\frac{1}{2}Rg^{\rho\sigma}-\frac{k}{2}[\frac{1}{4}(\partial W)^2g^{\rho\sigma}-(\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha}]-\\-\frac{k}{2}(\frac{1}{4}F^2g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha})-\frac{k}{2}M^2(W^{\rho}W^{\sigma}-\frac{1}{2}W^2g^{\rho\sigma})-\\-\Lambda g^{\rho\sigma}=\frac{1}{2}k[\frac{i}{4}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\nabla}^{\sigma}\psi-\boldsymbol{\nabla}^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi+\\+\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\nabla}^{\rho}\psi-\boldsymbol{\nabla}^{\rho}\overline{\psi}\boldsymbol{\gamma}^{\sigma}\psi)-\\-\frac{1}{2}X(W^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\pi}\psi+W^{\rho}\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi)]\end{gathered}$$

and the torsion field equations given according to

$$\nabla_{\rho}(\partial W)^{\rho\mu}+M^2W^{\mu}=X\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi$$

with gauge field equations given as

$$\nabla_{\sigma}F^{\sigma\mu}=q\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi$$

and matter field equations as

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-XW_{\sigma}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi-m\psi=0$$

with parameters Λ and M and also m describing intrinsic properties of metric and torsion and also spinor fields, while parameters k and X and q are the constants that measure the strength with which metric and torsion and gauge fields couple to the energy and spin and current.

To write the above system of coupled field equations into the system of coupled field equations with respect to which all the torsionless derivatives and curvatures are the torsionful derivatives and curvatures we start by considering precisely the torsion field equations which, after multiplying by the completely antisymmetric pseudotensor, are equivalently written for the completely antisymmetric torsion field according to the expression

$$M^2(\partial Q)_{\alpha\rho\mu\nu}=-\frac{2}{3}X\partial_{[\alpha}(\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}_{\rho},\boldsymbol{\sigma}_{\mu\nu]}\}\psi)$$

as the constraint coming from the covariant gradient of the completely antisymmetric torsion field equations

$$\nabla^{[\rho}\nabla_{\sigma}Q^{\mu\nu]\sigma}+M^2Q^{\rho\mu\nu}=-\frac{2}{3}X\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\rho},\boldsymbol{\sigma}^{\mu\nu}\}\psi$$

which will be useful next: by writing the torsionless covariant derivatives and curvatures in terms of the torsionful covariant derivatives and curvatures we obtain that the completely antisymmetric torsion field equation are given according to the following simple expression

$$\begin{gathered}D^{[\rho}D_{\sigma}Q^{\mu\nu]\sigma}-D_{\alpha}Q^{\alpha\sigma[\rho}Q^{\mu\nu]\pi}g_{\pi\sigma}+\\+M^2Q^{\rho\mu\nu}=-\frac{2}{3}X\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\rho},\boldsymbol{\sigma}^{\mu\nu}\}\psi\end{gathered}$$

or if torsionful curvatures are considered they can equivalently be written in the more complicated form

$$\begin{gathered}D^{[\rho}D_{\sigma}Q^{\mu\nu]\sigma}-G^{[\sigma\rho]}Q^{\mu\nu}{}_{\sigma}-G^{[\sigma\nu]}Q^{\rho\mu}{}_{\sigma}-G^{[\sigma\mu]}Q^{\nu\rho}{}_{\sigma}+\\+M^2Q^{\rho\mu\nu}=-\frac{2}{3}X\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\rho},\boldsymbol{\sigma}^{\mu\nu}\}\psi\end{gathered}$$

in terms of the completely antisymmetric spin density of the matter field; the metric field equations are field equations for the symmetric curvature equivalently written as field equations of the torsionful curvature in the form

$$\begin{gathered}G^{\rho\sigma}-\frac{1}{2}Gg^{\rho\sigma}-18k[\frac{1}{4}D_{\pi}Q^{\pi\mu\nu}D^{\tau}Q_{\tau\mu\nu}g^{\rho\sigma}-\\-D_{\pi}Q^{\pi\mu\rho}D^{\tau}Q_{\tau\mu}{}^{\sigma}+\\+\frac{1}{2}M^2(Q^{\rho\eta\pi}Q^{\sigma}{}_{\eta\pi}-\frac{1}{6}Q^2g^{\rho\sigma})]+\\+4XkD_{\alpha}(\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\alpha},\boldsymbol{\sigma}^{\rho\sigma}\}\psi)-\\-\frac{1}{2}k(\frac{1}{4}F^2g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha})-\\-(\frac{1}{2}D_{\alpha}Q^{\alpha\rho\sigma}-\frac{1}{4}Q^{\rho\alpha\pi}Q^{\sigma}{}_{\alpha\pi}+\frac{1}{8}Q^2g^{\rho\sigma})+\\+4Xk(\frac{1}{2}Q^{\alpha\pi\tau}\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}_{\alpha},\boldsymbol{\sigma}_{\pi\tau}\}\psi g^{\rho\sigma}-\\-Q^{\rho\pi\tau}\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\sigma},\boldsymbol{\sigma}_{\pi\tau}\}\psi-Q^{\sigma\pi\tau}\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\rho},\boldsymbol{\sigma}_{\pi\tau}\}\psi)-\\-\Lambda g^{\rho\sigma}=\frac{1}{2}k[\frac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{D}^{\sigma}\psi-\boldsymbol{D}^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi)+\\+(8X+1)\,D_{\alpha}(\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\alpha},\boldsymbol{\sigma}^{\rho\sigma}\}\psi)+\\+\frac{1}{2}\,(8X+1)\,Q^{\rho\mu\nu}\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\sigma},\boldsymbol{\sigma}_{\mu\nu}\}\psi-\\-\,(8X+1)\,Q^{\sigma\mu\nu}\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\rho},\boldsymbol{\sigma}_{\mu\nu}\}\psi]\end{gathered}$$

or by considering the completely antisymmetric torsion field equations again these curvature field equations can be written as non-symmetric curvature field equations

$$\begin{gathered}G^{\rho\sigma}-\frac{1}{2}Gg^{\rho\sigma}-18k[\frac{1}{3}D_{\alpha}(D^{[\alpha}D_{\pi}Q^{\rho\sigma]\pi}-\\-D_{\eta}Q^{\eta\pi[\alpha}Q^{\rho\sigma]\nu}g_{\nu\pi})+\frac{1}{2}D^{\pi}D_{\tau}Q^{\tau\mu\nu}Q_{\pi\mu\nu}g^{\rho\sigma}-\\-\frac{1}{3}Q^{\rho\eta\varphi}D^{[\sigma}D_{\pi}Q^{\eta\varphi]\pi}-\frac{1}{3}Q^{\sigma\eta\varphi}D^{[\rho}D_{\pi}Q^{\eta\varphi]\pi}+\\+\frac{1}{4}D_{\pi}Q^{\pi\mu\nu}D^{\tau}Q_{\tau\mu\nu}g^{\rho\sigma}-D_{\pi}Q^{\pi\mu\rho}D^{\tau}Q_{\tau\mu}{}^{\sigma}+\\+\frac{1}{3}(Q^{\rho\eta\varphi}D_{\tau}Q^{\tau\pi\sigma}+Q^{\sigma\eta\varphi}D_{\tau}Q^{\tau\pi\rho})Q^{\eta\varphi}{}_{\pi}]-\\-\frac{1}{2}k(\frac{1}{4}F^2g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha})-\\-(12kM^2+1)(\frac{1}{2}D_{\alpha}Q^{\alpha\rho\sigma}-\frac{1}{4}Q^{\rho\alpha\pi}Q^{\sigma}{}_{\alpha\pi}+\frac{1}{8}Q^2g^{\rho\sigma})-\\-\Lambda g^{\rho\sigma}=\frac{1}{2}k[\frac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{D}^{\sigma}\psi-\boldsymbol{D}^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi)+\\+(8X+1)\,D_{\alpha}(\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\alpha},\boldsymbol{\sigma}^{\rho\sigma}\}\psi)+\\+\frac{1}{2}\,(8X+1)\,Q^{\rho\mu\nu}\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\sigma},\boldsymbol{\sigma}_{\mu\nu}\}\psi-\\-\,(8X+1)\,Q^{\sigma\mu\nu}\frac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\rho},\boldsymbol{\sigma}_{\mu\nu}\}\psi]\end{gathered}$$

which are given in terms of the non-symmetric energy density of the matter field. The gauge field equations are

$$D_{\sigma}F^{\sigma\mu}+\frac{1}{2}F_{\alpha\nu}Q^{\alpha\nu\mu}=q\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi$$

given in terms of the current density of the material field.

Finally the material field equations are given by

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{D}_{\mu}\psi-i(X+\frac{1}{8})Q_{\nu\tau\alpha}\boldsymbol{\gamma}^{\nu}\boldsymbol{\gamma}^{\tau}\boldsymbol{\gamma}^{\alpha}\psi-m\psi=0$$

as a condition on the matter field equal to zero.

This condition is a spinor equation and therefore it is possible to multiply it by the $\boldsymbol{\gamma}^{\rho}$ matrix and then by the adjoint spinor $\overline{\psi}$ getting the scalar complex expression

$$i\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\gamma}^{\mu}\boldsymbol{D}_{\mu}\psi-i(X+\frac{1}{8})Q_{\nu\tau\alpha}\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\gamma}^{\nu}\boldsymbol{\gamma}^{\tau}\boldsymbol{\gamma}^{\alpha}\psi-m\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi=0$$

and which can be added to its complex conjugate in order to obtain the scalar real expression as in the following

$$i\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\gamma}^{\mu}\boldsymbol{D}_{\mu}\psi-i\boldsymbol{D}_{\mu}\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\gamma}^{\rho}\psi-\\ -i(X+\tfrac{1}{8})Q_{\nu\tau\alpha}\overline{\psi}(\boldsymbol{\gamma}^{\rho}\boldsymbol{\gamma}^{\nu}\boldsymbol{\gamma}^{\tau}\boldsymbol{\gamma}^{\alpha}+\boldsymbol{\gamma}^{\nu}\boldsymbol{\gamma}^{\tau}\boldsymbol{\gamma}^{\alpha}\boldsymbol{\gamma}^{\rho})\psi-\\ -2m\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi=0$$

which is an expression we will work out in terms of the identity properties of (75, 76) of the gamma matrices: so

$$i\overline{\psi}\boldsymbol{D}^{\rho}\psi-i\boldsymbol{D}^{\rho}\overline{\psi}\psi+2i\overline{\psi}\boldsymbol{\sigma}^{\rho\mu}\boldsymbol{D}_{\mu}\psi+2i\boldsymbol{D}_{\mu}\overline{\psi}\boldsymbol{\sigma}^{\rho\mu}\psi+\\ +(X+\tfrac{1}{8})Q^{\nu\tau\alpha}\varepsilon_{\nu\tau\alpha\sigma}4\overline{\psi}\boldsymbol{\sigma}^{\sigma\rho}\boldsymbol{\pi}\psi-\\ -2m\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi=0$$

or in terms of the bi-linear spinorial quantities

$$i(\overline{\psi}\boldsymbol{D}^{\rho}\psi-\boldsymbol{D}^{\rho}\overline{\psi}\psi)-\boldsymbol{D}_{\mu}S^{\mu\rho}+\\ +(2X+\tfrac{1}{4})\varepsilon_{\nu\tau\alpha\sigma}Q^{\nu\tau\alpha}\Sigma^{\sigma\rho}-2mU^{\rho}=0$$

as it is easy to check; from the scalar complex expression

$$i\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\gamma}^{\mu}\boldsymbol{D}_{\mu}\psi-i(X+\tfrac{1}{8})Q_{\nu\tau\alpha}\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\gamma}^{\nu}\boldsymbol{\gamma}^{\tau}\boldsymbol{\gamma}^{\alpha}\psi-m\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi=0$$

we could have subtracted its complex conjugate but also in this case we would have gotten a scalar real expression which could be worked out with gamma matrices, giving

$$\boldsymbol{D}_{\alpha}\Phi-2(\overline{\psi}\boldsymbol{\sigma}_{\mu\alpha}\boldsymbol{D}^{\mu}\psi-\boldsymbol{D}^{\mu}\overline{\psi}\boldsymbol{\sigma}_{\mu\alpha}\psi)+\\ +(2X+\tfrac{1}{4})\Theta Q^{\pi\tau\eta}\varepsilon_{\pi\tau\eta\alpha}=0$$

and it is left to the reader. If the spinorial field equation

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{D}_{\mu}\psi-i(X+\tfrac{1}{8})Q_{\nu\tau\alpha}\boldsymbol{\gamma}^{\nu}\boldsymbol{\gamma}^{\tau}\boldsymbol{\gamma}^{\alpha}\psi-m\psi=0$$

is multiplied by all of the remaining linearly independent matrices and then by the adjoint and the above procedure of splitting real and imaginary parts is followed, then one would have obtained the validity of all the expressions

$$\tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{D}_{\mu}\psi-\boldsymbol{D}_{\mu}\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi)-\\ -(X+\tfrac{1}{8})Q^{\pi\tau\eta}V^{\sigma}\varepsilon_{\pi\tau\eta\sigma}-m\Phi=0$$

$$D_{\mu}U^{\mu}=0$$

$$\tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\boldsymbol{D}_{\mu}\psi-\boldsymbol{D}_{\mu}\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi)-\\ -(X+\tfrac{1}{8})Q^{\pi\tau\eta}U^{\sigma}\varepsilon_{\pi\tau\eta\sigma}=0$$

$$D_{\mu}V^{\mu}-2m\Theta=0$$

$$D_{\nu}\Theta-2i(\overline{\psi}\boldsymbol{\sigma}_{\mu\nu}\boldsymbol{\pi}\boldsymbol{D}^{\mu}\psi-\boldsymbol{D}^{\mu}\overline{\psi}\boldsymbol{\sigma}_{\mu\nu}\boldsymbol{\pi}\psi)-\\ -(2X+\tfrac{1}{4})\Phi Q^{\pi\tau\eta}\varepsilon_{\pi\tau\eta\nu}+2mV_{\nu}=0$$

$$(\boldsymbol{D}_{\alpha}\overline{\psi}\boldsymbol{\pi}\psi-\overline{\psi}\boldsymbol{\pi}\boldsymbol{D}_{\alpha}\psi)+D^{\mu}\Sigma_{\mu\alpha}+\\ +(2X+\tfrac{1}{4})\varepsilon^{\pi\tau\eta\mu}Q_{\pi\tau\eta}S_{\mu\alpha}=0$$

$$D^{\mu}V^{\rho}\varepsilon_{\mu\rho\alpha\nu}+i(\overline{\psi}\boldsymbol{\gamma}_{[\alpha}\boldsymbol{D}_{\nu]}\psi-\boldsymbol{D}_{[\nu}\overline{\psi}\boldsymbol{\gamma}_{\alpha]}\psi)+\\ +(2X+\tfrac{1}{4})Q^{\pi\tau\eta}\varepsilon_{\pi\tau\eta[\alpha}V_{\nu]}=0$$

$$D^{[\alpha}U^{\nu]}-i\varepsilon^{\alpha\nu\mu\rho}(\boldsymbol{D}_{\mu}\overline{\psi}\boldsymbol{\gamma}_{\rho}\boldsymbol{\pi}\psi-\overline{\psi}\boldsymbol{\gamma}_{\rho}\boldsymbol{\pi}\boldsymbol{D}_{\mu}\psi)-\\ -(12X+\tfrac{3}{2})Q^{\alpha\nu\rho}U_{\rho}-2mS^{\alpha\nu}=0$$

again as it is easy to check and we leave this to the reader.

All in all, the full system of field equations is given by the torsion-spin and the curvature energy-field equations

$$D^{[\rho}D_{\sigma}Q^{\mu\nu]\sigma}-G^{[\sigma\rho]}Q^{\mu\nu}{}_{\sigma}-G^{[\sigma\nu]}Q^{\rho\mu}{}_{\sigma}-G^{[\sigma\mu]}Q^{\nu\rho}{}_{\sigma}+\\ +M^{2}Q^{\rho\mu\nu}=-\tfrac{2}{3}X\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\rho},\boldsymbol{\sigma}^{\mu\nu}\}\psi \tag{106}$$

and

$$G^{\rho\sigma}-\tfrac{1}{2}Gg^{\rho\sigma}-18k[\tfrac{1}{3}D_{\alpha}D^{[\alpha}D_{\pi}Q^{\rho\sigma]\pi}-\\ -\tfrac{1}{3}D_{\alpha}D_{\eta}Q^{\eta\pi[\alpha}Q^{\rho\sigma]\nu}g_{\nu\pi}-\\ -\tfrac{1}{3}Q^{\rho\eta\varphi}D^{[\sigma}D_{\pi}Q^{\eta\varphi]\pi}-\tfrac{1}{3}Q^{\sigma\eta\varphi}D^{[\rho}D_{\pi}Q^{\eta\varphi]\pi}+\\ +\tfrac{1}{2}D^{\pi}D_{\tau}Q^{\tau\mu\nu}Q_{\pi\mu\nu}g^{\rho\sigma}+\\ +\tfrac{1}{4}D_{\pi}Q^{\pi\mu\nu}D^{\tau}Q_{\tau\mu\nu}g^{\rho\sigma}-D_{\pi}Q^{\pi\mu\rho}D^{\tau}Q_{\tau\mu}{}^{\sigma}+\\ +\tfrac{1}{3}(Q^{\rho\eta\varphi}D_{\tau}Q^{\tau\pi\sigma}+Q^{\sigma\eta\varphi}D_{\tau}Q^{\tau\pi\rho})Q^{\eta\varphi}{}_{\pi}-\\ -\tfrac{1}{3}D_{\eta}Q^{\eta\pi\alpha}D_{\alpha}Q^{\rho\sigma}{}_{\pi}]-\\ -\tfrac{1}{2}k(\tfrac{1}{4}F^{2}g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha})-\\ -(12kM^{2}+1)(\tfrac{1}{2}D_{\alpha}Q^{\alpha\rho\sigma}-\\ -\tfrac{1}{4}Q^{\rho\alpha\pi}Q^{\sigma}{}_{\alpha\pi}+\tfrac{1}{8}Q^{2}g^{\rho\sigma})-\\ -\Lambda g^{\rho\sigma}=\tfrac{1}{2}k[\tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{D}^{\sigma}\psi-\boldsymbol{D}^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi)+\\ +(8X+1)\,D_{\alpha}(\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\alpha},\boldsymbol{\sigma}^{\rho\sigma}\}\psi)+\\ +\tfrac{1}{2}\,(8X+1)\,Q^{\rho\mu\nu}\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\sigma},\boldsymbol{\sigma}_{\mu\nu}\}\psi-\\ -(8X+1)\,Q^{\sigma\mu\nu}\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^{\rho},\boldsymbol{\sigma}_{\mu\nu}\}\psi] \tag{107}$$

as the pair of field equations describing the space-time structure, coming with the gauge-current field equations

$$D_{\sigma}F^{\sigma\mu}+\tfrac{1}{2}F_{\alpha\nu}Q^{\alpha\nu\mu}=q\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi \tag{108}$$

as the field equations describing the gauge structure, and thus completing the set of field equations describing the geometrical structure; they come with

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{D}_{\mu}\psi-i(X+\tfrac{1}{8})Q_{\nu\tau\alpha}\boldsymbol{\gamma}^{\nu}\boldsymbol{\gamma}^{\tau}\boldsymbol{\gamma}^{\alpha}\psi-m\psi=0 \tag{109}$$

which is the general form of the spinorial field equations and they can be decomposed according to

$$\tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{D}_{\mu}\psi-\boldsymbol{D}_{\mu}\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi)-\\ -(X+\tfrac{1}{8})Q^{\pi\tau\eta}V^{\sigma}\varepsilon_{\pi\tau\eta\sigma}-m\Phi=0 \tag{110}$$

$$D_{\mu}U^{\mu}=0 \tag{111}$$

$$\tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\boldsymbol{D}_{\mu}\psi-\boldsymbol{D}_{\mu}\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi)-\\ -(X+\tfrac{1}{8})Q^{\pi\tau\eta}U^{\sigma}\varepsilon_{\pi\tau\eta\sigma}=0 \tag{112}$$

$$D_\mu V^\mu - 2m\Theta = 0 \tag{113}$$

$$i(\overline{\psi}\boldsymbol{D}^\alpha\psi - \boldsymbol{D}^\alpha\overline{\psi}\psi) - D_\mu S^{\mu\alpha} + \\ +(2X+\tfrac{1}{4})\varepsilon_{\pi\tau\eta\sigma}Q^{\pi\tau\eta}\Sigma^{\sigma\alpha} - 2mU^\alpha = 0 \tag{114}$$

$$D_\alpha\Phi - 2(\overline{\psi}\boldsymbol{\sigma}_{\mu\alpha}\boldsymbol{D}^\mu\psi - \boldsymbol{D}^\mu\overline{\psi}\boldsymbol{\sigma}_{\mu\alpha}\psi) + \\ +(2X+\tfrac{1}{4})\Theta Q^{\pi\tau\eta}\varepsilon_{\pi\tau\eta\alpha} = 0 \tag{115}$$

$$D_\nu\Theta - 2i(\overline{\psi}\boldsymbol{\sigma}_{\mu\nu}\boldsymbol{\pi}\boldsymbol{D}^\mu\psi - \boldsymbol{D}^\mu\overline{\psi}\boldsymbol{\sigma}_{\mu\nu}\boldsymbol{\pi}\psi) - \\ -(2X+\tfrac{1}{4})\Phi Q^{\pi\tau\eta}\varepsilon_{\pi\tau\eta\nu} + 2mV_\nu = 0 \tag{116}$$

$$(\boldsymbol{D}_\alpha\overline{\psi}\boldsymbol{\pi}\psi - \overline{\psi}\boldsymbol{\pi}\boldsymbol{D}_\alpha\psi) + D^\mu\Sigma_{\mu\alpha} + \\ +(2X+\tfrac{1}{4})\varepsilon^{\pi\tau\eta\mu}Q_{\pi\tau\eta}S_{\mu\alpha} = 0 \tag{117}$$

$$D^\mu V^\rho\varepsilon_{\mu\rho\alpha\nu} + i(\overline{\psi}\boldsymbol{\gamma}_{[\alpha}\boldsymbol{D}_{\nu]}\psi - \boldsymbol{D}_{[\nu}\overline{\psi}\boldsymbol{\gamma}_{\alpha]}\psi) + \\ +(2X+\tfrac{1}{4})Q^{\pi\tau\eta}\varepsilon_{\pi\tau\eta[\alpha}V_{\nu]} = 0 \tag{118}$$

$$D^{[\alpha}U^{\nu]} - i\varepsilon^{\alpha\nu\mu\rho}(\boldsymbol{D}_\mu\overline{\psi}\boldsymbol{\gamma}_\rho\boldsymbol{\pi}\psi - \overline{\psi}\boldsymbol{\gamma}_\rho\boldsymbol{\pi}\boldsymbol{D}_\mu\psi) - \\ -(12X+\tfrac{3}{2})Q^{\alpha\nu\rho}U_\rho - 2mS^{\alpha\nu} = 0 \tag{119}$$

which taken altogether are equivalent to the spinor field equations we have written above, and for this reason they are the field equations describing the material structure.

Now it is possible to write the geometric field equations as field equations given by derivatives of the geometrical fields equal to some source according to expressions

$$D_{[\rho}D^\sigma Q_{\mu\nu]\sigma} + Q_{\eta[\mu\nu}G_{\rho]\sigma}g^{\sigma\eta} - G_{\sigma[\rho}Q_{\mu\nu]\eta}g^{\sigma\eta} + \\ +M^2 Q_{\rho\mu\nu} = \tfrac{1}{12}S_{\rho\mu\nu} \tag{120}$$

and

$$\begin{aligned} &G^{\rho\sigma} - \tfrac{1}{2}Gg^{\rho\sigma} - 18k[\tfrac{1}{3}D_\alpha D^{[\alpha}D_\pi Q^{\rho\sigma]\pi} - \\ &-\tfrac{1}{3}D_\alpha D_\eta Q^{\eta\pi[\alpha}Q^{\rho\sigma]\nu}g_{\nu\pi} - \\ &-\tfrac{1}{3}Q^{\rho\eta\varphi}D^{[\sigma}D_\pi Q^{\eta\varphi]\pi} - \tfrac{1}{3}Q^{\sigma\eta\varphi}D^{[\rho}D_\pi Q^{\eta\varphi]\pi} + \\ &+\tfrac{1}{2}D^\pi D_\tau Q^{\tau\mu\nu}Q_{\pi\mu\nu}g^{\rho\sigma} + \\ &+\tfrac{1}{4}D_\pi Q^{\pi\mu\nu}D^\tau Q_{\tau\mu\nu}g^{\rho\sigma} - D_\pi Q^{\pi\mu\rho}D^\tau Q_{\tau\mu}{}^\sigma + \\ &+\tfrac{1}{3}(Q^{\rho\eta\varphi}D_\tau Q^{\tau\pi\sigma} + Q^{\sigma\eta\varphi}D_\tau Q^{\tau\pi\rho})Q^{\eta\varphi}{}_\pi - \\ &-\tfrac{1}{3}D_\eta Q^{\eta\pi\alpha}D_\alpha Q^{\rho\sigma}{}_\pi] - \\ &-\tfrac{1}{2}k(\tfrac{1}{4}F^2g^{\rho\sigma} - F^{\rho\alpha}F^\sigma{}_\alpha) - \\ &-(12kM^2+1)(\tfrac{1}{2}D_\alpha Q^{\alpha\rho\sigma} - \\ &-\tfrac{1}{4}Q^{\rho\alpha\pi}Q^\sigma{}_{\alpha\pi} + \tfrac{1}{8}Q^2g^{\rho\sigma}) - \Lambda g^{\rho\sigma} = \tfrac{1}{2}kT^{\rho\sigma} \end{aligned} \tag{121}$$

alongside to

$$D_\sigma F^{\sigma\mu} + \tfrac{1}{2}F_{\alpha\nu}Q^{\alpha\nu\mu} = J^\mu \tag{122}$$

in general, where the sources are given by

$$S^{\rho\mu\nu} = -8X\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\rho, \boldsymbol{\sigma}^{\mu\nu}\}\psi \tag{123}$$

and

$$\begin{aligned} &T^{\rho\sigma} = \tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^\rho\boldsymbol{D}^\sigma\psi - \boldsymbol{D}^\sigma\overline{\psi}\boldsymbol{\gamma}^\rho\psi) + \\ &+(8X+1)\,D_\alpha(\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\alpha, \boldsymbol{\sigma}^{\rho\sigma}\}\psi) + \\ &+\tfrac{1}{2}\,(8X+1)\,Q^{\rho\mu\nu}\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\sigma, \boldsymbol{\sigma}_{\mu\nu}\}\psi - \\ &-(8X+1)\,Q^{\sigma\mu\nu}\tfrac{i}{4}\overline{\psi}\{\boldsymbol{\gamma}^\rho, \boldsymbol{\sigma}_{\mu\nu}\}\psi \end{aligned} \tag{124}$$

alongside to

$$J^\mu = q\overline{\psi}\boldsymbol{\gamma}^\mu\psi \tag{125}$$

all given in terms of the matter field; and thus the matter field equations are written according to expressions

$$i\boldsymbol{\gamma}^\mu\boldsymbol{D}_\mu\psi - i(X+\tfrac{1}{8})Q_{\nu\tau\alpha}\boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\tau\boldsymbol{\gamma}^\alpha\psi - m\psi = 0 \tag{126}$$

in the most general case: as a consequence we have that

$$D_\rho S^{\rho\mu\nu} + \tfrac{1}{2}T^{[\mu\nu]} \equiv 0 \tag{127}$$

and

$$D_\mu T^{\mu\nu} + T_{\rho\beta}Q^{\rho\beta\nu} - S_{\mu\rho\beta}G^{\mu\rho\beta\nu} + J_\rho F^{\rho\nu} \equiv 0 \tag{128}$$

alongside to

$$D_\rho J^\rho = 0 \tag{129}$$

are constraints that are satisfied in the most general case.

Intriguingly, we notice that in the spinor field equations the mass appears linearly and thus it may be positive as well as negative, and therefore it is possible to have two different types of spinor field equations; such a possibility is clear because if $m \to -m$ is accompanied by the discrete transformation $\psi \to \boldsymbol{\pi}\psi$ then the system of field equations is invariant, and consequently any solutions of the first is also a solution of the second. Therefore, the fact that we may have two different types of spinor field equations is translated into the fact that we may have two different solutions linked by $\psi \to \boldsymbol{\pi}\psi$ as it is discussed in [29, 30].

The full system of field equations is invariant under the transformation of parity reflection [31] and it is the most general under the restriction of being at the least-order differential form [32]; the requirement of having all fields coupled in terms of field equations having derivative order at their minimum does imply that if we write the field equations in their most general form, then we do end up with the above result: what in fact this means is that we might have ignored all previous steps, and in particular it also means that we could have ignored all considerations on the equivalence principle. The principle of equivalence may be a guide in writing Einstein field equations but in order to constitute the structure of the dynamics of the metric tensor it is not necessary, its role being simply that of allowing us to interpret that metric as what encrypts the gravitational information; moreover, it is instructive to recall that once Einstein field equations are given, then linearized and taken in the static case for small velocities, they reduce to Newton equations, in which the metric is the gravitational potential. The principle of equivalence can not only be reduced to nothing more than a mean to interpret the metric as gravitational potential but in addition it can also be dispensed altogether. The principle of equivalence gives important insights but we can drop it if we only want to interpret the gravitational potential as the metric tensor and write the metric field equations in the most general complete system of field equations.

III. TORSION-SPIN INTERACTIONS

In the first part we have obtained, under the restriction of staying at the least-order derivative, the most general system of differential field equations which determine the dynamical properties of all fields, and in this second part we shall study the system of field equations. Eventually, we will study the effects of the presence of torsion for the chiral dynamics of the single spinorial field in general.

A. Torsion and spinor covariant decomposition

In this first section we will simplify the system of field equations above. This requires two things: one is to have the torsion field separated from all other fields; the other is to have the spinor field split in its two chiral parts.

1. *Torsional decomposition: torsion as an axial-vector massive field*

Among all geometric fields, torsion has a special property indeed: the gauge potential is a gauge field for phase transformations and the metric tensor can be considered a gauge field for coordinate transformations, so both are always depending on the phase or the coordinate system, while torsion is a tensor that does not have any relation with gauge properties. So torsion can be split from gauge and metric connections, with all the covariant derivatives and curvatures being written as covariant derivatives and curvatures with no torsion but with all the torsion terms appearing in the form of separate contributions.

To have the most general connection decomposed into the simplest symmetric connection plus torsion terms we substitute (16) in (36) and this in (100) too.

Thus done the system of field equations reduces to

$$\nabla_{\rho}(\partial W)^{\rho\mu}+M^{2}W^{\mu}=X\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi \tag{130}$$

and

$$\begin{aligned}R^{\rho\sigma}-\tfrac{1}{2}Rg^{\rho\sigma}-\Lambda g^{\rho\sigma}=\tfrac{k}{2}[\tfrac{1}{4}F^{2}g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha}+\\+\tfrac{1}{4}(\partial W)^{2}g^{\rho\sigma}-(\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha}+\\+M^{2}(W^{\rho}W^{\sigma}-\tfrac{1}{2}W^{2}g^{\rho\sigma})+\\+\tfrac{i}{4}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\nabla}^{\sigma}\psi-\boldsymbol{\nabla}^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi+\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\nabla}^{\rho}\psi-\boldsymbol{\nabla}^{\rho}\overline{\psi}\boldsymbol{\gamma}^{\sigma}\psi)-\\-\tfrac{1}{2}X(W^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\pi}\psi+W^{\rho}\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi)]\end{aligned} \tag{131}$$

for the torsion-spin and curvature-energy coupling, and

$$\nabla_{\sigma}F^{\sigma\mu}=q\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi \tag{132}$$

for the gauge-current coupling; and finally

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-XW_{\sigma}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi-m\psi=0 \tag{133}$$

for the spinor field equations which again can be split as

$$\tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-\boldsymbol{\nabla}_{\mu}\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi)-XW_{\sigma}V^{\sigma}-m\Phi=0 \tag{134}$$

$$\nabla_{\mu}U^{\mu}=0 \tag{135}$$

$$\tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\boldsymbol{\nabla}_{\mu}\psi-\boldsymbol{\nabla}_{\mu}\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi)-XW_{\sigma}U^{\sigma}=0 \tag{136}$$

$$\nabla_{\mu}V^{\mu}-2m\Theta=0 \tag{137}$$

$$\begin{aligned}i(\overline{\psi}\boldsymbol{\nabla}^{\alpha}\psi-\boldsymbol{\nabla}^{\alpha}\overline{\psi}\psi)-\nabla_{\mu}S^{\mu\alpha}+\\+2XW_{\sigma}\Sigma^{\sigma\alpha}-2mU^{\alpha}=0\end{aligned} \tag{138}$$

$$\nabla_{\alpha}\Phi-2(\overline{\psi}\boldsymbol{\sigma}_{\mu\alpha}\boldsymbol{\nabla}^{\mu}\psi-\boldsymbol{\nabla}^{\mu}\overline{\psi}\boldsymbol{\sigma}_{\mu\alpha}\psi)+2X\Theta W_{\alpha}=0 \tag{139}$$

$$\begin{aligned}\nabla_{\nu}\Theta-2i(\overline{\psi}\boldsymbol{\sigma}_{\mu\nu}\boldsymbol{\pi}\boldsymbol{\nabla}^{\mu}\psi-\boldsymbol{\nabla}^{\mu}\overline{\psi}\boldsymbol{\sigma}_{\mu\nu}\boldsymbol{\pi}\psi)-\\-2X\Phi W_{\nu}+2mV_{\nu}=0\end{aligned} \tag{140}$$

$$(\boldsymbol{\nabla}_{\alpha}\overline{\psi}\boldsymbol{\pi}\psi-\overline{\psi}\boldsymbol{\pi}\boldsymbol{\nabla}_{\alpha}\psi)+\nabla^{\mu}\Sigma_{\mu\alpha}+2XW^{\mu}S_{\mu\alpha}=0 \tag{141}$$

$$\begin{aligned}\nabla^{\mu}V^{\rho}\varepsilon_{\mu\rho\alpha\nu}+i(\overline{\psi}\boldsymbol{\gamma}_{[\alpha}\boldsymbol{\nabla}_{\nu]}\psi-\boldsymbol{\nabla}_{[\nu}\overline{\psi}\boldsymbol{\gamma}_{\alpha]}\psi)+\\+2XW_{[\alpha}V_{\nu]}=0\end{aligned} \tag{142}$$

$$\begin{aligned}\nabla^{[\alpha}U^{\nu]}+i\varepsilon^{\alpha\nu\mu\rho}(\overline{\psi}\boldsymbol{\gamma}_{\rho}\boldsymbol{\pi}\boldsymbol{\nabla}_{\mu}\psi-\boldsymbol{\nabla}_{\mu}\overline{\psi}\boldsymbol{\gamma}_{\rho}\boldsymbol{\pi}\psi)-\\-2XW_{\sigma}U_{\rho}\varepsilon^{\alpha\nu\sigma\rho}-2mS^{\alpha\nu}=0\end{aligned} \tag{143}$$

together equivalent to the spinor field equations above.

It is now possible to interpret torsion: just a quick look at the torsion-spin and curvature-energy field equations reveals that torsion can be seen as an axial-vector massive field verifying Proca field equations with corresponding energy and torsion-spin coupling in the gravitational field equations [33]. With this insight, one might now wonder if there really was the necessity to go through the trouble of insisting on the presence of torsion if all comes to the presence of an axial-vector massive field, asking why we could not simply impose torsion equal to zero and then allowing an axial-vector massive field to be included into the theory: the answer is that although mathematically it is equivalent to follow both approaches, conceptually the former approach is the most straightforward construction in which all quantities are defined and all relationships are built in the most general manner, while on the other hand the latter approach would be afflicted by a number of arbitrary assumptions. If this latter approach were the one to be followed, we would have to justify why torsion albeit in general present should be removed, why among all fields that could be included we pick precisely a vector field with pseudo-tensorial properties and why it would have to be massive, making a list of assumptions of which none can be justified, and thus resulting into an approach having three unjustified assumptions in alternative to the other approach in which assumptions are either justified or not assumed at all. In order to avoid this high degree of arbitrariness we prefer to follow the approach that we actually followed here: this leads after all to the presence of an axial-vector massive field, and hence it can justify the presence of such a field. And that is, if in the theory there were to appear new physics that could somehow be reconducted to the presence an axial-vector massive field then we would know that these effects would come from the presence of torsion. We shall argue that these effects might be something we have already observed even if we ignored they could come from the torsion tensor itself.

Notice also that there is no torsion modifying the gauge field equations, as expected: in fact, if there were a coupling between torsion and gauge fields then in particular the torsion-spin field equations would have to be sourced by a spin density containing also gauge contributions, but it is not possible to build such an object without violating gauge invariance. Gauge fields do not couple to torsion.

And finally, the torsional contributions in the spinorial field equations will be discussed in the following.

2. *Spinorial reduction: spinors as a combination of chiral parts*

Analogously to the covariant decomposition of torsion, there is also a perfectly covariant split of the spinor field into its two chiral parts according to (78, 79) and therefore in the following we will proceed to perform it.

It is quite easily, although long, to see that the system of field equations reduces to the one for which we have

$$\nabla_{\rho}(\partial W)^{\rho\mu}+M^{2}W^{\mu}=X(\overline{\psi}_{R}\boldsymbol{\gamma}^{\mu}\psi_{R}-\overline{\psi}_{L}\boldsymbol{\gamma}^{\mu}\psi_{L}) \tag{144}$$

and

$$\begin{aligned}
R^{\rho\sigma}-\tfrac{1}{2}Rg^{\rho\sigma}-\Lambda g^{\rho\sigma}=\tfrac{k}{2}[\tfrac{1}{4}F^{2}g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha}+\\
+\tfrac{1}{4}(\partial W)^{2}g^{\rho\sigma}-(\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha}+\\
+M^{2}(W^{\rho}W^{\sigma}-\tfrac{1}{2}W^{2}g^{\rho\sigma})+\\
+\tfrac{i}{4}(\overline{\psi}_{L}\boldsymbol{\gamma}^{\rho}\boldsymbol{\nabla}^{\sigma}\psi_{L}-\boldsymbol{\nabla}^{\sigma}\overline{\psi}_{L}\boldsymbol{\gamma}^{\rho}\psi_{L}+\\
+\overline{\psi}_{L}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\nabla}^{\rho}\psi_{L}-\boldsymbol{\nabla}^{\rho}\overline{\psi}_{L}\boldsymbol{\gamma}^{\sigma}\psi_{L}+\\
+\overline{\psi}_{R}\boldsymbol{\gamma}^{\rho}\boldsymbol{\nabla}^{\sigma}\psi_{R}-\boldsymbol{\nabla}^{\sigma}\overline{\psi}_{R}\boldsymbol{\gamma}^{\rho}\psi_{R}+\\
+\overline{\psi}_{R}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\nabla}^{\rho}\psi_{R}-\boldsymbol{\nabla}^{\rho}\overline{\psi}_{R}\boldsymbol{\gamma}^{\sigma}\psi_{R})-\\
-\tfrac{1}{2}X(W^{\sigma}\overline{\psi}_{R}\boldsymbol{\gamma}^{\rho}\psi_{R}+W^{\rho}\overline{\psi}_{R}\boldsymbol{\gamma}^{\sigma}\psi_{R}-\\
-W^{\sigma}\overline{\psi}_{L}\boldsymbol{\gamma}^{\rho}\psi_{L}-W^{\rho}\overline{\psi}_{L}\boldsymbol{\gamma}^{\sigma}\psi_{L})]
\end{aligned} \tag{145}$$

for the torsion-spin and curvature-energy coupling, and

$$\nabla_{\sigma}F^{\sigma\mu}=q(\overline{\psi}_{R}\boldsymbol{\gamma}^{\mu}\psi_{R}+\overline{\psi}_{L}\boldsymbol{\gamma}^{\mu}\psi_{L}) \tag{146}$$

for the gauge-current coupling; and finally

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi_{L}+XW_{\sigma}\boldsymbol{\gamma}^{\sigma}\psi_{L}-m\psi_{R}=0 \tag{147}$$

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi_{R}-XW_{\sigma}\boldsymbol{\gamma}^{\sigma}\psi_{R}-m\psi_{L}=0 \tag{148}$$

for the spinor field equations which are also split as

$$\begin{aligned}
\tfrac{i}{2}(\overline{\psi}_{R}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi_{R}-\boldsymbol{\nabla}_{\mu}\overline{\psi}_{R}\boldsymbol{\gamma}^{\mu}\psi_{R}+\\
+\overline{\psi}_{L}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi_{L}-\boldsymbol{\nabla}_{\mu}\overline{\psi}_{L}\boldsymbol{\gamma}^{\mu}\psi_{L})-XW_{\sigma}V^{\sigma}-m\Phi=0
\end{aligned} \tag{149}$$

$$\nabla_{\mu}U^{\mu}=0 \tag{150}$$

$$\begin{aligned}
\tfrac{i}{2}(\overline{\psi}_{R}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi_{R}-\boldsymbol{\nabla}_{\mu}\overline{\psi}_{R}\boldsymbol{\gamma}^{\mu}\psi_{R}-\\
-\overline{\psi}_{L}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi_{L}+\boldsymbol{\nabla}_{\mu}\overline{\psi}_{L}\boldsymbol{\gamma}^{\mu}\psi_{L})-XW_{\sigma}U^{\sigma}=0
\end{aligned} \tag{151}$$

$$\nabla_{\mu}V^{\mu}-2m\Theta=0 \tag{152}$$

$$\begin{aligned}
i(\overline{\psi}_{L}\boldsymbol{\nabla}^{\alpha}\psi_{R}-\boldsymbol{\nabla}^{\alpha}\overline{\psi}_{R}\psi_{L}+\overline{\psi}_{R}\boldsymbol{\nabla}^{\alpha}\psi_{L}-\boldsymbol{\nabla}^{\alpha}\overline{\psi}_{L}\psi_{R})-\\
-\nabla_{\mu}S^{\mu\alpha}+2XW_{\sigma}\Sigma^{\sigma\alpha}-2mU^{\alpha}=0
\end{aligned} \tag{153}$$

$$\begin{aligned}
\nabla_{\alpha}\Phi-2(\overline{\psi}_{R}\boldsymbol{\sigma}_{\mu\alpha}\boldsymbol{\nabla}^{\mu}\psi_{L}-\boldsymbol{\nabla}^{\mu}\overline{\psi}_{L}\boldsymbol{\sigma}_{\mu\alpha}\psi_{R}+\\
+\overline{\psi}_{L}\boldsymbol{\sigma}_{\mu\alpha}\boldsymbol{\nabla}^{\mu}\psi_{R}-\boldsymbol{\nabla}^{\mu}\overline{\psi}_{R}\boldsymbol{\sigma}_{\mu\alpha}\psi_{L})+2X\Theta W_{\alpha}=0
\end{aligned} \tag{154}$$

$$\begin{aligned}
\nabla_{\nu}\Theta-2i(\overline{\psi}_{L}\boldsymbol{\sigma}_{\mu\nu}\boldsymbol{\nabla}^{\mu}\psi_{R}+\boldsymbol{\nabla}^{\mu}\overline{\psi}_{R}\boldsymbol{\sigma}_{\mu\nu}\psi_{L}-\\
-\overline{\psi}_{R}\boldsymbol{\sigma}_{\mu\nu}\boldsymbol{\nabla}^{\mu}\psi_{L}-\boldsymbol{\nabla}^{\mu}\overline{\psi}_{L}\boldsymbol{\sigma}_{\mu\nu}\psi_{R})-\\
-2X\Phi W_{\nu}+2mV_{\nu}=0
\end{aligned} \tag{155}$$

$$\begin{aligned}
(\boldsymbol{\nabla}_{\alpha}\overline{\psi}_{L}\psi_{R}+\overline{\psi}_{R}\boldsymbol{\nabla}_{\alpha}\psi_{L}-\boldsymbol{\nabla}_{\alpha}\overline{\psi}_{R}\psi_{L}-\overline{\psi}_{L}\boldsymbol{\nabla}_{\alpha}\psi_{R})+\\
+\nabla^{\mu}\Sigma_{\mu\alpha}+2XW^{\mu}S_{\mu\alpha}=0
\end{aligned} \tag{156}$$

$$\begin{aligned}
\nabla^{\mu}V^{\rho}\varepsilon_{\mu\rho\alpha\nu}+i(\overline{\psi}_{L}\boldsymbol{\gamma}_{[\alpha}\boldsymbol{\nabla}_{\nu]}\psi_{L}-\boldsymbol{\nabla}_{[\nu}\overline{\psi}_{L}\boldsymbol{\gamma}_{\alpha]}\psi_{L}+\\
+\overline{\psi}_{R}\boldsymbol{\gamma}_{[\alpha}\boldsymbol{\nabla}_{\nu]}\psi_{R}-\boldsymbol{\nabla}_{[\nu}\overline{\psi}_{R}\boldsymbol{\gamma}_{\alpha]}\psi_{R})+\\
+2XW_{[\alpha}V_{\nu]}=0
\end{aligned} \tag{157}$$

$$\begin{aligned}
\nabla^{[\alpha}U^{\nu]}+i\varepsilon^{\alpha\nu\mu\rho}(\boldsymbol{\nabla}_{\mu}\overline{\psi}_{L}\boldsymbol{\gamma}_{\rho}\psi_{L}-\overline{\psi}_{L}\boldsymbol{\gamma}_{\rho}\boldsymbol{\nabla}_{\mu}\psi_{L}-\\
-\boldsymbol{\nabla}_{\mu}\overline{\psi}_{R}\boldsymbol{\gamma}_{\rho}\psi_{R}+\overline{\psi}_{R}\boldsymbol{\gamma}_{\rho}\boldsymbol{\nabla}_{\mu}\psi_{R})-\\
-2XW_{\sigma}U_{\rho}\varepsilon^{\alpha\nu\sigma\rho}-2mS^{\alpha\nu}=0
\end{aligned} \tag{158}$$

together equivalent to the spinorial field equations above.

As it is quite clear we may interpret the spinors in this manner: albeit spinors may be considered fundamental, nevertheless they are reducible as they are constituted by two parts which are the left-handed and the right-handed chiral semi-spinor projections; these two chiral parts are independent from one another and when they are taken individually each is irreducible. The two chiral parts are the truly fundamental independent degrees of freedom of the spinor field; and because they are two then the spinor possesses an internal structure. Consequently we have to expect that also an internal dynamics will be displayed.

Each of the chiral semi-spinors can be further split into its two components; however each of these components cannot be taken independently from the other. Hence, if we want to further split each semi-spinor down to its most elementary parts, we must accept that a decomposition will be a completely frame-dependent decomposition.

In this case we will see that there will be much more information that we can extract from the spinor field.

B. Torsion spinor coupling in a special frame

In this second section we will look for a specific frame in which the splitting can reach the single components.

1. Spinorial reduction: spinors as bound states of chiral parts

In the previous subsection in which we had defined the spinor fields, we had also introduced a list of 16 linearly-independent bi-linear spinorial quantities given according to the (81, 82, 83, 84, 85, 86) and these definitions have been given with the intent of having some symmetry between all such bi-linear spinorial quantities: nevertheless, because of the definition (54) we have that it is clearly

$$\Sigma^{ab}=-\tfrac{1}{2}\varepsilon^{abij}S_{ij}$$

showing that only one of the two antisymmetric tensors is really necessary, so that the (87) reduces to the form

$$\psi\overline{\psi}\equiv\tfrac{1}{4}\Phi\mathbb{I}+\tfrac{1}{4}U_a\boldsymbol{\gamma}^a+\tfrac{i}{4}S_{ab}\boldsymbol{\sigma}^{ab}-\tfrac{1}{4}V_a\boldsymbol{\gamma}^a\boldsymbol{\pi}-\tfrac{i}{4}\Theta\boldsymbol{\pi}$$

and making it manifest that indeed there are 16 linearly-independent bi-linear spinor quantities since they have to span the entire space of 4×4 matrices; on the other hand these 16 bi-linear spinor quantities, although independent for linear combinations, are not independent in general as it can be seen from (90, 91) combined as to give

$$S_{ab}(\Phi^2+\Theta^2)=\Phi U^jV^k\varepsilon_{jkab}+\Theta U_{[a}V_{b]}$$

and showing that also the other antisymmetric tensor is not necessary, since it can be written in terms of the two vectors and the two scalars, and so (97, 99) given by

$$U_aU^a=-V_aV^a=|\Theta|^2+|\Phi|^2$$
$$U_aV^a=0$$

remain the only identities. They show that also the norm of the two vectors are not necessary once we have the two scalars: therefore we may define the two directions

$$V^a=(\Theta^2+\Phi^2)^{\frac{1}{2}}v^a$$
$$U^a=(\Theta^2+\Phi^2)^{\frac{1}{2}}u^a$$

together with the re-parametrization

$$\Theta=2\phi^2\sin\beta$$
$$\Phi=2\phi^2\cos\beta$$

in such a way that all relevant information is encoded with the two directions u^a and v^a verifying

$$u_au^a=-v_av^a=1$$
$$u_av^a=0$$

and the two fields ϕ and β in general. The fact that there remains little information may be surprising because the spinor field is defined with 8 real components, but spinors are defined by transformation laws with 6 parameters, so that of the 8 components 6 can be seen as an information related to the reference system, with the consequence for which the true degrees of freedom are 2 and no more.

To properly see this point let us consider the fact that spinors are defined in terms of their transformation law and as such we may employ this law to transfer some of their components away from the spinor field: notice that in the case the spinor has $\Phi=\Theta=0$ then the spinor would lose two degrees of freedom becoming singular, and while being extensively studied [34–40] we will not investigate them here in the following of this work; we therefore turn our full attention to the case in which at least one of the two scalars does not vanish. In this case all of the above relationships among bi-linear spinor quantities happens to be non-degenerate, and so we know that the vector has to be time-like while the axial-vector has to be space-like in general: because U^a is time like we can perform up to three boosts bringing all spatial components to vanish so that by calling $\overline{\psi}=(R^\dagger,L^\dagger)$ these constraints give

$$\begin{pmatrix}R^\dagger & L^\dagger\end{pmatrix}\begin{pmatrix}0 & \boldsymbol{\sigma}^K\\ -\boldsymbol{\sigma}^K & 0\end{pmatrix}\begin{pmatrix}L\\ R\end{pmatrix}=0$$

and therefore

$$R^\dagger\boldsymbol{\sigma}^KR=L^\dagger\boldsymbol{\sigma}^KL$$

showing in particular that the spinor has velocity equal to zero because the two semi-spinor chiral parts have two velocities that are opposite although both different from zero in general; by calling $R^\dagger=(ae^{-i\alpha},be^{-i\beta})$ as well as the complementary $L^\dagger=(ce^{-i\gamma},de^{-i\delta})$ we obtain that

$$\begin{pmatrix}ae^{-i\alpha} & be^{-i\beta}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}ae^{i\alpha}\\ be^{i\beta}\end{pmatrix}-$$
$$-\begin{pmatrix}ce^{-i\gamma} & de^{-i\delta}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}ce^{i\gamma}\\ de^{i\delta}\end{pmatrix}=0$$

$$\begin{pmatrix}ae^{-i\alpha} & be^{-i\beta}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}ae^{i\alpha}\\ be^{i\beta}\end{pmatrix}-$$
$$-\begin{pmatrix}ce^{-i\gamma} & de^{-i\delta}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}ce^{i\gamma}\\ de^{i\delta}\end{pmatrix}=0$$

$$\begin{pmatrix} ae^{-i\alpha} & be^{-i\beta} \end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & -1 \end{pmatrix}\begin{pmatrix} ae^{i\alpha}\\ be^{i\beta} \end{pmatrix}-$$
$$-\begin{pmatrix} ce^{-i\gamma} & de^{-i\delta} \end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & -1 \end{pmatrix}\begin{pmatrix} ce^{i\gamma}\\ de^{i\delta} \end{pmatrix}=0$$

and therefore

$$ab\cos(\beta-\alpha)=cd\cos(\delta-\gamma)$$

$$ab\sin(\beta-\alpha)=cd\sin(\delta-\gamma)$$

$$a^2+d^2=b^2+c^2$$

showing that the two chiral parts have velocities different from zero indeed; by re-parametrizing fields according to the definitions $a=\phi\cos\lambda$ and $d=\phi\sin\lambda$ alongside to the definitions $b=\chi\sin\omega$ and $c=\chi\cos\omega$ we obtain that

$$\cos\lambda\sin\omega\cos(\beta-\alpha)=\cos\omega\sin\lambda\cos(\delta-\gamma)$$

$$\cos\lambda\sin\omega\sin(\beta-\alpha)=\cos\omega\sin\lambda\sin(\delta-\gamma)$$

$$\phi^2=\chi^2$$

from which $\chi=\pm\phi$ and combining the first two gives

$$\lambda=(-1)^n\omega$$
$$\beta-\alpha=\delta-\gamma+n\pi$$

with n integer but having no other constraint. All things together give the final spinor according to the structure

$$\psi=\begin{pmatrix} \pm\cos\omega e^{i\gamma}\\ (-1)^n\sin\omega e^{i\delta}\\ \cos\omega e^{i\alpha}\\ \pm(-1)^n\sin\omega e^{i\alpha}e^{i\delta}e^{-i\gamma} \end{pmatrix}\phi$$

with some sign ambiguities which can be re-collected as

$$\psi=\begin{pmatrix} \pm\cos\omega e^{i\gamma}\\ \pm\sin\omega e^{i\delta}\\ \cos\omega e^{i\alpha}\\ \sin\omega e^{i\alpha}e^{i\delta}e^{-i\gamma} \end{pmatrix}\phi$$

up to the sign of the ω function; equivalently as

$$\psi=\begin{pmatrix} \pm\cos\omega e^{\frac{i}{2}\zeta}e^{-\frac{i}{2}\varsigma}\\ \pm\sin\omega e^{\frac{i}{2}\zeta}e^{\frac{i}{2}\varsigma}\\ \cos\omega e^{-\frac{i}{2}\zeta}e^{-\frac{i}{2}\varsigma}\\ \sin\omega e^{-\frac{i}{2}\zeta}e^{\frac{i}{2}\varsigma} \end{pmatrix}e^{i\varphi}\phi$$

after re-naming the phases α, γ and δ and which is given with five independent fields. We notice that in the ratios of first/second components and third/fourth components the results are the same and this is what makes it possible to employ one single spinorial rotation in order to vanish two components, one for each of the two chiral parts of the spinor. In fact, we still have the freedom to perform the three rotations, and by using only two of them given by the rotation around the first and second axis, we can bring the axial-vector to have its space part aligned with the third axis: then we have the two constraints that can be combined with the previous results giving

$$R^\dagger\boldsymbol{\sigma}^1R=L^\dagger\boldsymbol{\sigma}^1L=0$$
$$R^\dagger\boldsymbol{\sigma}^2R=L^\dagger\boldsymbol{\sigma}^2L=0$$

and consequently

$$\omega=n\tfrac{\pi}{2}$$

in general. Therefore the form of the spinor is either

$$\psi=\begin{pmatrix} \pm e^{\frac{i}{2}\zeta}\\ 0\\ e^{-\frac{i}{2}\zeta}\\ 0 \end{pmatrix}e^{i\varphi}\phi\quad\text{or}\quad\psi=\begin{pmatrix} 0\\ \pm e^{\frac{i}{2}\zeta}\\ 0\\ e^{-\frac{i}{2}\zeta} \end{pmatrix}e^{i\varphi}\phi$$

as according to whether the axial-vector is either aligned or anti-aligned with the third axis respectively and with three independent fields. This form is an eigen-state for the rotation around the third axis. This can be worked out by using the last rotation, since it is precisely the one around the third axis, to have the phase shifted away, so that finally the form of the spinor is either given by

$$\psi=\begin{pmatrix} \pm e^{\frac{i}{2}\beta}\\ 0\\ e^{-\frac{i}{2}\beta}\\ 0 \end{pmatrix}\phi\quad\text{or by}\quad\psi=\begin{pmatrix} 0\\ \pm e^{\frac{i}{2}\beta}\\ 0\\ e^{-\frac{i}{2}\beta} \end{pmatrix}\phi$$

up to the re-naming of the two independent fields made to recover the notation for the fields ϕ and β as we have defined above. These are the truly independent fields.

When any such form is hence plugged into decompositions (134, 135, 136, 137, 138, 139, 140 141, 142, 143) we get ten decompositions of the spinor field equations from which we can extract in particular the following two

$$-XW_\mu-\tfrac{1}{4}g_{\mu\nu}\varepsilon^{\nu\rho\sigma\alpha}\partial_\rho\xi^k_\sigma\xi^j_\alpha\eta_{jk}+qA^\iota u_{[\iota}v_{\mu]}+$$
$$+v_\mu m\cos\beta+\tfrac{1}{2}\nabla_\mu\beta=0$$
$$v_\mu m\sin\beta+qA^\rho u^\nu v^\alpha\varepsilon_{\mu\rho\nu\alpha}+$$
$$+\tfrac{1}{2}|\xi|^{-1}\xi^k_\mu\partial_\alpha(|\xi|\xi^\alpha_k)+\nabla_\mu\ln\phi=0$$

which are very special since we can show that these two expressions imply the spinor field equations (133): in fact by evaluating the spinorial covariant derivative of any of the forms of the above spinors and plugging the two field equations here above, we have that the left-hand side of the spinor field equation (133) can be written as

$$i\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi-XW_\sigma\boldsymbol{\gamma}^\sigma\boldsymbol{\pi}\psi-m\psi=$$
$$=-(iq\boldsymbol{\gamma}^\mu A^\rho u^\nu v^\alpha\varepsilon_{\mu\rho\nu\alpha}+qA^\iota u_{[\iota}v_{\mu]}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}+qA_\mu\boldsymbol{\gamma}^\mu+$$
$$+iv_\mu\boldsymbol{\gamma}^\mu m\sin\beta+v_\mu\boldsymbol{\gamma}^\mu\boldsymbol{\pi}m\cos\beta+m\mathbb{I})\psi$$

in which both torsion and the spin connection straightforwardly simplified away; the gauge field is trickier and to simplify it we have to recall identities (75, 76) together with the re-arrangements (88, 89) from which we have

$$i\boldsymbol{\gamma}^{\mu}\psi U^{\nu}V^{\alpha}\varepsilon_{\mu\rho\nu\alpha}+U_{[\rho}V_{\mu]}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi+U^{2}\boldsymbol{\gamma}_{\rho}\psi=0$$

and therefore giving

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-XW_{\sigma}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi-m\psi=$$
$$-m(iv_{\mu}\boldsymbol{\gamma}^{\mu}\sin\beta+v_{\mu}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\cos\beta+\mathbb{I})\psi$$

which is almost done. By employing (88, 89) we also get

$$iV_{\mu}\boldsymbol{\gamma}^{\mu}\psi\Theta+V_{\mu}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi\Phi+U^{2}\psi=0$$

which gives

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-XW_{\sigma}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi-m\psi=0$$

and therefore (133) is valid. In this way we have proven that the two equations above are together equivalent to the spinor field equations; notice that these 8 spinor field equations are in the form of two 4-dimensional vectorial field equations determining all coordinate derivatives of the two independent fields, and so the balance of degrees of freedom and their equations checks. Finally, we remark that the derivation is general and it can always be done.

To summarize, we define the two directions given by

$$V^{a}=2\phi^{2}v^{a} \tag{159}$$
$$U^{a}=2\phi^{2}u^{a} \tag{160}$$

together with the re-parametrization

$$\Theta=2\phi^{2}\sin\beta \tag{161}$$
$$\Phi=2\phi^{2}\cos\beta \tag{162}$$

so that $u_{a}u^{a}=-v_{a}v^{a}=1$ and $u_{a}v^{a}=0$ and where the two fields ϕ and β are the module and the Takabayashi angle respectively: we can always write the spinor either as

$$\psi=\begin{pmatrix}\pm e^{\frac{i}{2}\beta}\\ 0\\ e^{-\frac{i}{2}\beta}\\ 0\end{pmatrix}\phi\quad\text{or as}\quad\psi=\begin{pmatrix}0\\ \pm e^{\frac{i}{2}\beta}\\ 0\\ e^{-\frac{i}{2}\beta}\end{pmatrix}\phi \tag{163}$$

in which the first is a spin-up eigen-state form while the second is a spin-down eigen-state form whereas the plus and minus signs can be absorbed into the re-definition of the spinors given by $\psi\leftrightarrow\boldsymbol{\pi}\psi$ as a discrete transformation.

When these are plugged into the spinor field equations the latter are equivalent to a pair of field equations that can be written in terms of the vector and axial-vector

$$K_{\mu}=2XW_{\mu}+\frac{1}{2}g_{\mu\nu}\varepsilon^{\nu\rho\sigma\alpha}\partial_{\rho}\xi^{k}_{\sigma}\xi^{j}_{\alpha}\eta_{jk}-2qA^{\iota}u_{[\iota}v_{\mu]} \tag{164}$$
$$G_{\mu}=-|\xi|^{-1}\xi^{k}_{\mu}\partial_{\alpha}(|\xi|\xi^{\alpha}_{k})-2qA^{\rho}u^{\nu}v^{\alpha}\varepsilon_{\mu\rho\nu\alpha} \tag{165}$$

according to the following expressions

$$\nabla_{\mu}\beta-K_{\mu}+v_{\mu}2m\cos\beta=0 \tag{166}$$
$$\nabla_{\mu}\ln\phi^{2}-G_{\mu}+v_{\mu}2m\sin\beta=0 \tag{167}$$

as the most general forms of spinorial field equations.

There is a very important point to be clarified regarding the spinorial active transformations acting on spinorial fields: consider the rotation around the third axis

$$\boldsymbol{\Lambda}=\begin{pmatrix}e^{\frac{i}{2}\theta}&0&0&0\\ 0&e^{-\frac{i}{2}\theta}&0&0\\ 0&0&e^{\frac{i}{2}\theta}&0\\ 0&0&0&e^{-\frac{i}{2}\theta}\end{pmatrix}$$

and the spinors (163) either as

$$\psi=\begin{pmatrix}\pm e^{\frac{i}{2}\beta}\\ 0\\ e^{-\frac{i}{2}\beta}\\ 0\end{pmatrix}\phi\quad\text{or as}\quad\psi=\begin{pmatrix}0\\ \pm e^{\frac{i}{2}\beta}\\ 0\\ e^{-\frac{i}{2}\beta}\end{pmatrix}\phi$$

in general: despite the fact that these spinors are aligned along the axis around which the above rotation operates, that rotation does not leave them unchanged (as we have for vectors); this might already sound problematic, but in addition we also have that when such a rotation is given for an angle $\theta=2\pi$ then $\boldsymbol{\Lambda}=-\mathbb{I}$ implying that the spinor does not go back to the initial configuration (as we have when we perform some passive rotations) but it does that only up to a sign, and it is only when an additional rotation of angle $\theta=2\pi$ is performed that total symmetry is recovered; this too is a situation that sounds peculiar, but we wish now to present an intuitive circumstance in which common objects behave similarly. First of all, we have to take into account the fact that the rotation is an active rotation, and therefore an operation that, keeping fixed the space-time structure, moves the spinor; further, we have to consider the fact that a spinor aligned along a given axis is changed by a rotation around that axis, a situation that forces us to picture the spinor as an object that in addition to a direction also has some structures that feel the rotations around that direction, as if we had a pole to which we had attached a flag or a perch or yet a non-circular pedestal; one of my favourite images for a spinor is that of a book with a pen that is kept orthogonal to the cover and placed on it; to complete the metaphor I can imagine a spinor in the space-time as the book-pen orthogonal system placed in the hand of my wife; then I ask my wife to perform a rotation of the book. As she is going to rotate the book keeping it parallel to the ground with the pen pointing up, she might bring the book under her armpit and then, passing to the exterior, she would manage to complete the full rotation with the book come to the initial position only by having her arm twisted or, if she wants to put her arm back to normal, by flipping the book up-side down so that now the pen would point downwards; and it will only be with yet another full turn that she would be able to have the book-pen orthogonal system back to the initial configuration with no twist in her arm, that is it will only be with a 4π total turn that both space-time and spinor will be back to their original configuration. This example shows that there is nothing really strange that happens when spinors are thought as objects firmly fixed to the space-time where they live [41].

We can now infer the meaning of the various elements of the spinor field: the vector u^a has spatial part that can always be removed by means of a Lorentz boost and so it has to be identified with the velocity vector, and on the other hand the axial-vector v^a has a spatial part that rotates similarly to what a spatial vector would rotate and therefore it has to be identified with the spin vector, while the scalar field ϕ squared is the density, and on the other hand the pseudo-scalar field β is the relative phase between the two non-vanishing components of the spinor fields; the first and the third components and the second and the fourth components are called spin-up and spin-down respectively, whereas the first and the second components and the third and the fourth components are the well known left-handed and right-handed components respectively. We have also shown that there is a further duplicity for the spinorial structure, made clear from the fact that spinors were defined up to the $\psi\rightarrow\boldsymbol{\pi}\psi$ discrete transformation; on the other hand, we have already seen that the $\psi\rightarrow\boldsymbol{\pi}\psi$ and $m\rightarrow -m$ discrete transformation is a symmetry for the field equations. This feature intrinsic to the spinor structure is not related to the specific form of the spinor field equations, as these results are obtained in a pure kinematic way without employing the dynamics.

In the rest frame and for fixed spin alignment, the two chiral parts are complex conjugate of one another, their phase difference is the Takabayashi angle, describing the internal structure [42]; equations (166, 167) manifest that torsion determines the dynamics of the Takabayashi angle and the Takabayashi angle determines the dynamics of the module [43]. Because the two chiral parts are each single-handed, then it follows that for each of them only the vectorial type of bi-linear semi-spinor quantities can be formed, and as a consequence they can only couple to vectorial potentials to give scalar interactions; moreover, the two chiral parts have opposite velocities, so they can only couple to pseudo-tensorial potentials to give parity-conserving interactions. In this work, axial-vector potentials can only be the torsion surrounding spinor fields.

Therefore, spinors are made of two chiral parts interacting through torsion: consequently, we have to expect a spinor to display an internal dynamics with torsion playing the role of the mediator of the binding interaction.

Such an internal dynamics would have to be described in terms of chiral interactions that will have to be taken as mediated by a Proca axial-vector massive field.

On the other hand, it would be desirable to have these general arguments more deeply analyzed, and this is precisely what we will do in the following part.

IV. LIMITING CASES

In the previous parts we have ended by claiming that spinors should display an internal dynamics with torsion being an axial-vector massive field working as mediator of the binding force, and now we will exploit the fact that torsion is a massive axial-vector to find special situations in which the above claims can be better justified.

A. Massive approximations

In the previous part we have been applying a geometric approach leaving out the variational method, and in what follows we will resort to it for a more compact notation.

The above system of field equations, which consists in expressions (130, 131, 132, 133), can entirely be derived by employing the variational formalism from a dynamical action, whose Lagrangian function is given according to

$$\begin{aligned}\mathscr{L}=&-\tfrac{1}{4}(\partial W)^2+\tfrac{1}{2}M^2W^2-\tfrac{1}{k}R-\tfrac{2}{k}\Lambda-\tfrac{1}{4}F^2+\\&+i\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi-X\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi W_\mu-m\overline{\psi}\psi\end{aligned}\tag{168}$$

where torsion is already decomposed, or equivalently

$$\begin{aligned}\mathscr{L}=&-\tfrac{1}{4}(\partial W)^2+\tfrac{1}{2}M^2W^2-\tfrac{1}{k}R-\tfrac{2}{k}\Lambda-\tfrac{1}{4}F^2+\\&+i\overline{\psi}_L\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi_L+i\overline{\psi}_R\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi_R+\\&+X\overline{\psi}_L\boldsymbol{\gamma}^\mu\psi_LW_\mu-X\overline{\psi}_R\boldsymbol{\gamma}^\mu\psi_RW_\mu-\\&-m\overline{\psi}_R\psi_L-m\overline{\psi}_L\psi_R\end{aligned}\tag{169}$$

in which the chiral split is already done: we might write the Lagrangian after the non-covariant split, but as in general varying the Lagrangian does not commute with breaking a symmetry we prefer not to deal with this form.

To begin our investigation, we remark that torsion had a first property that was unlike any other space-time or gauge fields had, and that is it comes as a general feature of the geometry and not from a symmetry principle, with the consequence that it can be massive: so we have that the torsion field equations (130) are such that, in presence of a massive field, they can be taken in the approximation in which the dynamical term is negligible compared to the mass term, with the result that we may approximate

$$M^2W^\mu\approx X\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi\tag{170}$$

yielding an algebraic equation that can be used to have torsion substituted in all other field equations in terms of the spin of the spinor, so that all torsional contributions can effectively be converted into spin-spin interactions.

When this is done in the Lagrangian (168) we can work out with the help of the geometric identity (97) that

$$\begin{aligned}\mathscr{L}_{\mathrm{effective}}=&-\tfrac{1}{k}R-\tfrac{2}{k}\Lambda-\tfrac{1}{4}F^2+\\&+i\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi+\tfrac{1}{2}\tfrac{X^2}{M^2}\overline{\psi}\boldsymbol{\gamma}^\mu\psi\overline{\psi}\boldsymbol{\gamma}_\mu\psi-m\overline{\psi}\psi\end{aligned}\tag{171}$$

or equivalently

$$\begin{aligned}\mathscr{L}_{\mathrm{effective}}=&-\tfrac{1}{k}R-\tfrac{2}{k}\Lambda-\tfrac{1}{4}F^2+\\&+i\overline{\psi}_L\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi_L+i\overline{\psi}_R\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi_R+\\&+\tfrac{X^2}{M^2}\overline{\psi}_L\boldsymbol{\gamma}^\mu\psi_L\overline{\psi}_R\boldsymbol{\gamma}_\mu\psi_R-\\&-m\overline{\psi}_R\psi_L-m\overline{\psi}_L\psi_R\end{aligned}\tag{172}$$

which is exactly the spinor Lagrangian we would have had in the Nambu–Jona-Lasinio model [44, 45]: thus torsion has the effect of creating a field that, within the effective approximation, reduces to the NJL model; consequently, we may use the general results of the NJL model stating that such a contact force is an attractive interaction.

Notice that as (97) shows it is precisely the axial-vector nature of the field what produces the inversion of the sign of the potential turning attractive the interaction.

This confirms that between two chiral parts torsion has the role of mediator of the attractive interaction binding the two chiral parts of the spinor field together.

We recall that the role of the Higgs boson is analogous.

This is not surprising since the torsion-spin coupling is the axial-vector analog of the scalar Yukawa coupling.

In fact, if the effective Lagrangian (171) is further rearranged in terms of (97), it can be put in the form

$$\begin{aligned}\mathscr{L}^{\mathrm{spinor}}_{\mathrm{effective}}=i\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi+\\+\tfrac{1}{2}\tfrac{X^2}{M^2}(|\overline{\psi}\psi|^2+|i\overline{\psi}\boldsymbol{\pi}\psi|^2)-m\overline{\psi}\psi\end{aligned}\tag{173}$$

as the Lagrangian of the spinor field complemented with the torsionally-induced spin-contact interactions.

On the other hand, in the standard model of particle physics [46], we might take into account the Lagrangian for the electron in presence of the Higgs interaction alone together with the Higgs field equations in presence of the electron source solely: if the latter are taken in the case in which the Higgs mass M is very large, then it becomes possible to make the same effective approximation

$$M^2 H\approx -\tfrac{Y}{2}\overline{e}e\tag{174}$$

which is analogous to (170) but it is a scalar relationship.

Plugging it into the standard model Lagrangian gives

$$\mathscr{L}^{\mathrm{electron}}_{\mathrm{effective}}=i\overline{e}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}e+\tfrac{Y^2}{4M^2}|\overline{e}e|^2-m\overline{e}e\tag{175}$$

as the part regarding the electronic field complemented with the Higgs-induced effective interactions.

The comparison between (173) and (175) shows that

$$\mathscr{V}^{\mathrm{spinor}}_{\mathrm{effective}}=-\tfrac{1}{2}\tfrac{X^2}{M^2}(|\overline{\psi}\psi|^2+|i\overline{\psi}\boldsymbol{\pi}\psi|^2)\tag{176}$$

$$\mathscr{V}^{\mathrm{electron}}_{\mathrm{effective}}=-\tfrac{Y^2}{4M^2}|\overline{e}e|^2\tag{177}$$

spelling that torsion gives rise to a chiral self-interaction characterized by both a scalar part and a pseudo-scalar part while the Higgs gives rise to a scalar self-interaction and nothing more: when the Takabayashi angle does not vanish, there is an additional coupling which cannot be codified by the Higgs boson. Thus as a consequence of the presence of the Takabayashi angle, the torsional effective force cannot be reduced to the Higgs effective force.

If in some approximation the Takabayashi angle were to vanish so that the two effective interactions acquired the very same structure, there would still be differences concerning the strengths of those interactions: according to the NJL model, the interaction does not only have to be attractive but also strong enough as to allow formation of bound states, implying that the mediator mass cannot be too large in order not to suppress the interaction.

At the scale at which we perform our experiments, the Higgs mass is too large to ensure bound states, while the torsion mass might still be small enough as to render the effective coupling sufficiently strong for bound states.

So far as we can tell, the two effective forces should in fact have properties quite different from one another.

From the Lagrangian (173) we extract the potential

$$\mathscr{V}=-\tfrac{X^2}{2M^2}\left(|\overline{\psi}\psi|^2+|i\overline{\psi}\boldsymbol{\pi}\psi|^2\right)\tag{178}$$

which is negative, as expected for attractive interactions, and so the energy is the kinetic energy plus the potential energy, given by the general expression according to

$$\mathscr{E}=\mathscr{K}-\tfrac{X^2}{2M^2}\left(|\overline{\psi}\psi|^2+|i\overline{\psi}\boldsymbol{\pi}\psi|^2\right)\tag{179}$$

where we recall all quantities are densities: hence, in the case in which we consider the same approximation above in terms of which the internal structure is neglected, so that the Takabayashi angle may be set to zero, we get

$$E=K-\tfrac{X^2}{2M^2}\tfrac{1}{V}\tag{180}$$

having interpreted the module $\overline{\psi}\psi=V^{-1}$ as inverse volume, which is reasonable at least on dimensional grounds.

On the other hand, it is possible to compute what turns out to be the expression for the internal energy of a van der Waals gas with negative pressure, given by

$$U=T-\tfrac{X^2}{2M^2}\tfrac{1}{V}\tag{181}$$

as it is known from general thermodynamic arguments.

Because thermodynamically the kinetic energy can be interpreted as the temperature, and of course the energy is the internal energy, then the formal similarities of these two apparently unrelated expressions are striking.

In this thermodynamic analogy we have that the single spinor field can be seen as a matter distribution behaving in the same way in which a van der Waals attractive gas with attractive intermolecular forces would [47].

Summarizing, in the effective approximation, torsional interactions give rise to a contact force much in the same way in which the Higgs field would, with these two forces being similarly attractive but with torsion displaying the dependence on the spin that the Higgs lacks; and we have seen that if the Takabayashi angle where to vanish, their common potential would be also analogous to the internal energy of an attractive type of van der Waals gas.

Consequently, insofar as this effective approximation holds there is a clear indication that torsion is a sort of internal binding force, a tension, localizing the spinor.

These considerations are valid for effective approximations obtained when the torsion energy is small compared to its mass and thus in the case of slow torsion.

Similar arguments can also be invoked for spinor fields in the case they are slow since they are massive and therefore it is always possible to boost into the rest frame, but in the case of spinor fields one needs more than rendering small the spatial part of the velocity vector and in order to see what more there is to it we have to employ another form of argument, as the one we will discussed next.

In the space-time the spinorial transformation law has a total of 6 parameters while spinor fields defined in terms of this transformation have a total of 8 real components, and we have seen how to remove 6 components from the spinor field leaving it with just the 2 physical degrees of freedom; if we were to keep out time considering only the space variability, the spinorial transformation law would be the complex representation of the rotation group defined with 3 parameters and spinor fields defined in terms of this transformation would have 4 real components, so that when the above treatment is applied the result would be that we can remove 3 components from the spinor thus leaving it with just 1 physical degree of freedom: because keeping time frozen off merely means that we are taking the non-relativistic approximation, then we have that in the non-relativistic approximation the spinor field would have to lose 1 degree of freedom. The degree of freedom it has to lose can only be the Takabayashi angle: thus in the non-relativistic approximation the spatial part of the vector u^a and the Takabayashi angle β vanish. And since the vector is the velocity then the Takabayashi angle can only be related to the internal motions of the matter.

Taking (135, 137) for (159, 160) and (161, 162) gives

$$\nabla_\mu(\phi^2 u^\mu)=0 \tag{182}$$
$$\nabla_\mu(\phi^2 v^\mu)-2m\phi^2\sin\beta=0 \tag{183}$$

and we see that the density of velocity verifies a continuity equation while the density of spin verifies a continuity equation that is only partially exact because it is sourced by a product of the mass density times the Takabayashi angle and where we do observe that the module describes the density distribution; this last expression tells that for matter distributions that are massive with Takabayashi angle that is non-zero and specifically negative, the axial-vector spin density has a divergence that is also negative, with the consequence that the matter distribution tends to be localized in the coordinates that correspond to the components of the axial-vector spin v^μ that are different from zero. For instance, if matter distributions in spheric coordinates have to be localized in the radial direction, then it is necessary that the axial-vector spin possesses a radial component that is non-zero and that there exists a Takabayashi angle that is non-zero and also negative.

From such a connection between the Takabayashi angle and the axial-vector spin density, and knowing that spin density is a signature of internal structures, we may infer a link between the Takabayashi angle and the presence of internal dynamics, confirming what we have suggested above about some relationship between the Takabayashi angle and *Zitterbewegung* effects of internal motions.

It may be useful to introduce the possibility to switch the representation from chiral to standard, that is changing the representation in such a way that the Takabayashi angle is no longer expressed in terms of imaginary exponentials but real circular functions, a change that is made in terms of a rotation performed by the unitary matrix

$$\boldsymbol{U}=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbb{I} & \mathbb{I}\\ -\mathbb{I} & \mathbb{I}\end{pmatrix}$$

on gamma matrices and spinors: for the gamma matrices we have that in the standard representation they are

$$\boldsymbol{\gamma}^0=\begin{pmatrix}\mathbb{I} & 0\\ 0 & -\mathbb{I}\end{pmatrix}$$
$$\boldsymbol{\gamma}^K=\begin{pmatrix}0 & \boldsymbol{\sigma}^K\\ -\boldsymbol{\sigma}^K & 0\end{pmatrix}$$

so that

$$\boldsymbol{\sigma}_{0A}=\frac{1}{2}\begin{pmatrix}0 & \boldsymbol{\sigma}_A\\ \boldsymbol{\sigma}_A & 0\end{pmatrix}$$
$$\boldsymbol{\sigma}_{AB}=-\frac{i}{2}\varepsilon_{ABC}\begin{pmatrix}\boldsymbol{\sigma}^C & 0\\ 0 & \boldsymbol{\sigma}^C\end{pmatrix}$$

and

$$\boldsymbol{\pi}=\begin{pmatrix}0 & \mathbb{I}\\ \mathbb{I} & 0\end{pmatrix}$$

while the spinors are given according to either the forms

$$\psi=\sqrt{2}\begin{pmatrix}\cos\frac{\beta}{2}\\ 0\\ -i\sin\frac{\beta}{2}\\ 0\end{pmatrix}\phi\ \text{ or }\ \psi=\sqrt{2}\begin{pmatrix}0\\ \cos\frac{\beta}{2}\\ 0\\ -i\sin\frac{\beta}{2}\end{pmatrix}\phi \tag{184}$$

or the forms

$$\psi=\sqrt{2}\begin{pmatrix}-i\sin\frac{\beta}{2}\\ 0\\ \cos\frac{\beta}{2}\\ 0\end{pmatrix}\phi\ \text{ or }\ \psi=\sqrt{2}\begin{pmatrix}0\\ -i\sin\frac{\beta}{2}\\ 0\\ \cos\frac{\beta}{2}\end{pmatrix}\phi \tag{185}$$

the forms on the left being the third axis rotation of the forms on the right and the two upper forms being linked to the two lower forms by $\psi\rightarrow\boldsymbol{\pi}\psi$ with $\boldsymbol{\pi}$ also taken in standard representation as it is obvious; if the dynamics of the Takabayashi angle makes it evolve then the upper and lower components keep rotating into one another.

Above we have established that in the non-relativistic approximation the Takabayashi angle vanishes, and now we can see why for the non-relativistic approximation the standard representation is so useful in the fact that when the Takabayashi angle vanishes then also one of the two components vanishes: for this reason $\sin(\beta/2)$ is said to be the small part whereas $\cos(\beta/2)$ is the large part.

Summarizing, either between left-hand and right-hand components (in chiral representation) or between small and large parts (in standard representation), there arise internal dynamics that can be tied to the presence of the Takabayashi angle, therefore linking this angle to internal motions that could give rise to *Zitterbewegung* effects, in terms of which quantum effects can be described [48, 49].

As a consequence, there may be profound relationships between the effects usually attributed to the quantization and *Zitterbewegung* motion within a particle.

B. Masslessness

In the previous section we have studied what happens in the cases in which some approximation can be done by exploiting the fact that fields can be massive, and next we will be interested in studying the complementary case in which masslessness for all the fields is granted.

In situations of torsion masslessness, the torsional field equations (130) would approximate down to the form

$$\nabla_{\rho}(\partial W)^{\rho\mu} = X\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi \tag{186}$$

which are analogous to the electro-dynamic field equations apart from the fact that these are parity-odd while the electro-dynamic field equations are parity-even.

This aside, both are vector field equations in massless case, and as such we should expect some symmetry to be present: the full Lagrangian in the case of masslessness also for the spinor field is give by the following

$$\begin{aligned}\mathscr{L}_{\text{massless}} = &-\tfrac{1}{4}(\partial W)^{2} - \tfrac{1}{k}R - \tfrac{2}{k}\Lambda - \tfrac{1}{4}F^{2} + \\ &+ i\overline{\psi}_{L}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi_{L} + i\overline{\psi}_{R}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi_{R} + \\ &+ X\overline{\psi}_{L}\boldsymbol{\gamma}^{\mu}\psi_{L}W_{\mu} - X\overline{\psi}_{R}\boldsymbol{\gamma}^{\mu}\psi_{R}W_{\mu}\end{aligned} \tag{187}$$

which is invariant for a transformation of the type

$$W'_{\nu} = W_{\nu} - \partial_{\nu}\omega \tag{188}$$

with

$$\psi'_{L} = e^{-iX\omega}\psi_{L} \qquad \psi'_{R} = e^{iX\omega}\psi_{R} \tag{189}$$

as clear by comparing with (44) and (43) as the transformation laws of the gauge theory of electro-dynamics.

It is possible to see that the above transformation laws for the two chiral components can be written in compact form for the full spinor according to

$$\psi' = e^{iX\omega\boldsymbol{\pi}}\psi \tag{190}$$

known as chiral gauge transformation in full analogy with the gauge symmetry proper to the electro-dynamic field.

Additionally, expressions (163) can also be written as

$$\psi = e^{-\frac{i}{2}\beta\boldsymbol{\pi}}\begin{pmatrix}\pm\phi\\0\\\phi\\0\end{pmatrix} \quad \text{or as} \quad \psi = e^{-\frac{i}{2}\beta\boldsymbol{\pi}}\begin{pmatrix}0\\\pm\phi\\0\\\phi\end{pmatrix} \tag{191}$$

where the Takabayashi angle is written in matrix form.

It is quite clear by combining the last two relationships that regardless the structure of the spinor field it is always possible to perform a chiral gauge transformation taking the local parameter to be $\beta = 2X\omega$ and leaving

$$\psi' = \begin{pmatrix}\pm\phi\\0\\\phi\\0\end{pmatrix} \quad \text{or} \quad \psi' = \begin{pmatrix}0\\\pm\phi\\0\\\phi\end{pmatrix} \tag{192}$$

in terms of the module solely: this has to be expected, as symmetries come with redundant information that can be remove by reducing the fields, and because in this case the chiral symmetry is an additional symmetry with one parameter, we have to expect that one degree of freedom be removed. It is clear that the only degree of freedom to remain is the one that cannot be removed in any way whatsoever, that is the module of the field distribution.

Because in the massless approximation the two chiral Lagrangians become separable, the two chiral parts are independent, and since in such a regime the Takabayashi angle can be vanished, it carries no information, with the consequence that the Takabayashi angle may encode the information on the two chiral parts relative dynamics.

This is yet another fact that supports the evidence for which the Takabayashi angle can be connected to internal dynamics, giving rise to some *Zitterbewegung* effect.

The relationships between quantum properties of fields and Takabayashi angle will be discussed shortly.

V. SUMMARY

In this first part, we started from the most general geometric background in which both torsion and curvature, as well as gauge fields, were present; we have analogously defined spinors: we have written, under the constraint of being at the least-order derivative, what is the most general system of field equations. We proceeded by achieving in the field equations the decomposition of all covariant derivatives and curvatures with torsion into corresponding quantities without torsion plus torsional terms, thus having torsion isolated in a way that showed how torsion is equivalent to an axial-vector massive field; further we accomplished the usual spinor decomposition in its two chiral parts: we argued that the torsion field could be the mediator of an attractive interaction for which a spinor may be a bound-state of its two chiral components where the binding mechanism would be dynamical. We showed that this is indeed the case when effective approximations are taken by indicating how the present theory with these approximations reduces to the NJL theory, therefore we showed that the non-relativistic approximation turns out to be implemented by small spatial part of the velocity as well as small Takabayashi angle and arguing that for this reason the Takabayashi angle encodes information about internal dynamics: we have seen that also in masslessness cases where the two chiral parts no longer have relative dynamics the Takabayashi angle vanishes showing once more that this angle is where information about internal dynamics is stored. With this summary we conclude the first part and in the following we are going to address a topic that at the beginning will look totally unrelated.

TWO: EFFECTS

VI. QUANTA AND SPIN

In the first part, we have presented the general theory of torsion gravity with electro-dynamics for spinor fields, and passing through a discussion on the role of torsion as axial-vector massive mediator of an attractive interaction forming spinor bound-states of the chiral components, we have concluded by speculating on relationships between the Takabayashi angle and internal dynamics; because in literature it is already discussed of a possible link between internal dynamics and quantum effects [49], it may also be instructive to pursue a treatment of quantum physics in this perspective. In this part, we will discuss this.

A. Quantization

The history of the theories of quantization may be separated in three moments: a first involving the gathering of the experimental evidence and the struggle to have it condensed into mathematical concepts; a second involving the endeavour of systematizing these concepts into a rigorous scheme; a third involving such a mathematical scheme valid for mechanics replicated also for fields.

1. *Prelude to quantization*

The first indication that nature might have a quantum character came at the beginning of the twentieth century when Planck was tackling the problem of describing the spectrum of emission for the black-body radiation.

The black-body radiation could be described in terms of the energy density u emitting radiation with frequency given by ν as a function of its temperature T measured in Kelvin: the energy density in the low-frequency regime is described by the Rayleigh-Jeans law given in terms of the light-speed and the Boltzmann constant as

$$u(\nu,T)=8\pi\nu^2 kT/c^3$$

but as frequencies tend to increase this law no longer fits data and additionally it loses sense because when the energy density is integrated over the whole spectrum the total energy becomes infinite; today we would dismiss the problem by assuming that some new physics would enter in the game for those high-frequency regimes at which the infinities would arise, but in those times physicists actually went on to find such new physics, which is expressed by the Wien law given in in terms of two parameters as

$$u(\nu,T)=C\nu^3 e^{-\nu\beta/T}$$

although this formula does no fit data at low frequencies.

Because Wien formula does not work at low frequencies while Rayleigh-Jeans formula is non-sensical at high frequencies, Planck started to think about a way in which the Rayleigh-Jeans and Wien laws could be seen as the low-frequency and high-frequency approximations of a single more universal law: the formula was given by

$$u(\nu,T)=\frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/kT}-1}$$

in terms of a new constant h and additionally, one could integrate this formula over the spectrum of frequencies

$$U(T)=\int_0^\infty u(\nu,T)d\nu=\frac{8\pi h}{c^3}\int_0^\infty\frac{\nu^3}{e^{h\nu/kT}-1}d\nu=$$
$$=\frac{8\pi k^4}{h^3c^3}T^4\int_0^\infty\frac{x^3}{e^x-1}dx=\frac{8\pi k^4}{h^3c^3}T^4\frac{\pi^4}{15}=\frac{8\pi^5k^4}{15h^3c^3}T^4$$

which give the Stefan-Boltzmann expression of the total energy as a quartic function of the absolute temperature.

The success of this formula convinced Planck that it must have been possible to derive it from fundamental principles: the Planck law can be re-written as

$$u(\nu,T)=\frac{8\pi\nu^2kT}{c^3}\frac{h\nu/kT}{e^{h\nu/kT}-1}$$

as the Rayleigh-Jeans formula multiplied by a factor, the former giving the number of modes and the latter giving the average energy per mode; in this form the interest is focused onto the expression of the average energy per mode, which Planck obtained by employing the known Boltzmann probability distribution for the energy modes and a new hypothesis for which these energy modes were multiples of some elementary quantity resulting into

$$\langle u\rangle(\nu,T)=\frac{\sum_n nEe^{-nE/kT}}{\sum_n e^{-nE/kT}}=\frac{E}{e^{E/kT}-1}$$

coinciding with the above so long as $E=h\nu$ is imposed.

Therefore, the spectrum of emission for the black-body radiation can consistently be described whenever the internal modes follow Boltzmann distribution and each of them has an energy given by $E=nh\nu$ and that is, when they come in discrete amounts of an elementary quantity

$$E=h\nu$$

that had to be valid in general circumstances; Planck did not know how the discretization of the spectrum could be obtained nor in what way the energy relationship could be inferred. But its validity had yet another application.

The photoelectric effect is the effect for which a given metal when irradiated emits electrons in such a way that the number of electrons depends on the intensity of the light source while the energy of each electron depends on the frequency of the light source: by assuming that light was made of individual entities behaving like particles called photons each with energy $E = h\nu$ Einstein proved that the photoelectric effect could be explained as the fact that the number of electrons depended on the number of photons while the energy of a single electron depended on the energy of a single photon, quite simply indeed.

If the photoelectric effect consists in the scattering of an electron after total absorption of the incident photon, then one may wonder what would happen if the electron were scattered after having been transferred only a partial amount of the energy of the incident photon, which is an effect first studied by Compton and thus called Compton effect: like Einstein, Compton assumed that photons were responsible for the scattering of individual electrons and that in the process energy and momentum of photons were given by the above $E = h\nu$ and by the relationship given as $P = h\lambda^{-1}$ since for photons $E^2 - P^2c^2 = 0$ due to their masslessness: calling θ the angle between the directions of the incoming and outgoing photon then the shift in its wave-length is given according to the formula

$$\Delta\lambda = \frac{h}{mc}(1 - \cos\theta)$$

where m is the mass of the electron scattered away.

The black-body radiation, photoelectric and Compton effects are all well established facts, and they all have the same underlying idea that light consists of individual bits called photons and being such that

$$E = h\nu$$
$$P = \frac{h}{\lambda}$$

since they are relativistic massless particles.

Some year later, it was de Broglie who brought up the argument that if photons behaved as particle then it might as well be true that electrons behaved as waves with associated wave-length given by

$$\lambda = \frac{h}{P}$$

implying interesting consequences: if this were true then electrons of a given wave-length scattered of comparable spacing had undergo to the phenomenon of diffraction.

And quite astonishingly, diffracted they were.

A system in which corpuscular photons and wave electrons are both present is the description of light emission from atoms, the process that occurs when we have atoms with an external electrons in an excited state falling down to more stable states thus emitting a photon of a given wave-length: in this model of atom, Bohr assumed that electrons in orbits had to have a wave-length as an integer multiple of the orbit length according to

$$2\pi r = \frac{nh}{mv}$$

and where these electrons could go from one orbit to a smaller orbit emitting photons with wave-length

$$\frac{1}{\lambda} = \frac{1}{hc}\left|E_n - E_k\right|$$

where E_n and E_k are the energies of the electron in the orbits that respectively are n and k times the wave-length of the electron: in this model the condition of discretization of the system comes from the requirement of consistency with the periodicity; then the condition of equilibrium between centrifugal and electrostatic force

$$\frac{Ze^2}{r^2} = \frac{mv^2}{r}$$

combined with

$$2\pi r = \frac{nh}{mv}$$

convert into the conditions

$$v = \frac{2\pi e^2 Z}{nh}$$

and

$$r = \frac{n^2h^2}{4\pi^2e^2Zm}$$

eventually yielding the total energy

$$E_n = -\frac{2\pi^2e^4Z^2m}{h^2}\frac{1}{n^2}$$

and thus

$$\frac{1}{\lambda} = \frac{2\pi^2e^4Z^2m}{h^3c}\left|\frac{1}{n^2} - \frac{1}{k^2}\right|$$

in which n and k are the integer numbers associated with the first and second orbits. This is the Rydberg formula.

And it fits all observations for hydrogen atoms.

We notice that the condition of discretization as related to the periodicity of a system is not new because it takes place also in the classical physics of oscillating objects.

The Bohr model is the first instance in which matter is thought as tiny strings and although nowadays nobody would seriously think this is what the electron really is, nevertheless it gave the exact form of the energy levels of electrons in atoms and the frequency of emitted photons precisely, so precisely in fact that despite its intrinsic incorrectness, it cannot have been too far from the truth either: so how can matter behave as a wave and radiation cluster into corpuscular objects similar to particles?

The wave-particle duality of matter and radiation is probably the most intriguing trait of quantum mechanics, possibly even more than quantization itself because as we already mentioned the appearance of discrete spectra is not something new at all. Even more surprisingly, both radiation and matter display such complementary between particles and waves in the very same experiment.

When either matter or radiations is considered in experiments such as the two-slit experiment or in the Mach-Zehnder apparatus, we observe interference patterns that are those proper of waves although we can only detect single individual particles: it is as if particles are waves and behaves as wave but at the moment of interaction only their corpuscular character becomes manifest.

This problem is still an open issue for quantum mechanics, and therefore we are not going to deepen its implications in this pre-historical summary.

2. *Forward the quantization*

In the pre-history of quantum mechanics, we have discussed the photoelectric and Compton effects together with the black-body radiation formula, the diffraction of electrons on a lattice, and the Bohr atom, as indications that light might have a corpuscular character while matter may have an oscillatory character, and that both respect some discretization conditions: we have discussed how quantization may have an origin in some periodicity conditions there were not new, but that the particle-wave dual features of radiation and matter were new indeed.

To continue the history of quantum mechanics, we focus on the fact that matter and radiation seem to be waves appearing as particles when interacting: then is it possible to interpret the particles as localized waves?

Let us assume that the particle is smeared into a localized packet of waves: its mathematical description can be achieved in terms of the superposition of specific waves of specific amplitudes, and so if we decide to describe the profile of the wave-packet as $|\psi(x)|^2$ then

$$\psi(x)=\frac{1}{\sqrt{2\pi}}\int\phi(k)e^{ikx}dk$$

where $\phi(k)$ is what describes the distribution of the amplitudes in terms of what is called wave number; essentially this is the Fourier transform of the initial function, and in fact we can write the complementary relation

$$\phi(k)=\frac{1}{\sqrt{2\pi}}\int\psi(x)e^{-ikx}dx$$

which is valid in general. Also in general is the fact that

$$\int|\psi(x)|^2dx=\int|\phi(k)|^2dk$$

known Parseval relationships: they show that the wave-packet representation and its associated Fourier transform distribution have interesting relationships.

In order to obtain an additional relationship we call

$$\int|\psi(x)|^2dx=\int|\phi(k)|^2dk=N^2$$

and we define the following averages

$$\langle x\rangle=\frac{1}{N^2}\int\psi(x)^\dagger x\psi(x)dx$$

and

$$\langle k\rangle=\frac{1}{N^2}\int\phi(k)^\dagger k\phi(k)dk$$

together with their standard deviations

$$|\Delta x|^2=\langle(x-\langle x\rangle)^2\rangle$$

and

$$|\Delta k|^2=\langle(k-\langle k\rangle)^2\rangle$$

and to try see what happens to their product we split the problem in two sub-problems: one would be to see what specific relationship x and k actually have; the other is to see what relationships hold in general among operators.

Let us start from the latter: considering two Hermitian operators A and B and a generic function f we have

$$\begin{aligned}0\leqslant\int[(A+i\lambda B)f]^\dagger[(A+i\lambda B)f]dx=\\=\int f^\dagger(A^\dagger-i\lambda B^\dagger)(A+i\lambda B)f\,dx=\\=\int f^\dagger(A^2+i\lambda[A,B]+\lambda^2B^2)f\,dx=\\=N^2(\langle A^2\rangle+i\lambda\langle[A,B]\rangle+\lambda^2\langle B^2\rangle)\end{aligned}$$

where the normalization factor can be neglected leaving an expression in terms of a real parameter λ in a relationship that is valid for any admissible value of this real parameter; in particular the relationship

$$\langle A^2\rangle+\lambda i\langle[A,B]\rangle+\lambda^2\langle B^2\rangle\geqslant 0$$

is valid for the value of the parameter that minimizes the expression and given by $i\langle[A,B]\rangle=-2\lambda\langle B^2\rangle$ and so that

$$4\langle A^2\rangle\ \langle B^2\rangle-|i\langle[A,B]\rangle|^2\geqslant 0$$

and thus

$$\langle A^2\rangle\ \langle B^2\rangle\geqslant\left|\tfrac{i}{2}\langle[A,B]\rangle\right|^2$$

in very general circumstances. If now we call the Hermitian operators as $A=a-\langle a\rangle$ and $B=b-\langle b\rangle$ then

$$\Delta a\ \Delta b\geqslant\left|\tfrac{i}{2}\langle[a,b]\rangle\right|$$

since the average is a number and therefore the commutator of two operators shifted by numbers is the commutator of the operators themselves; notice that this result has been found without employing any concept of Fourier analysis. But we still have another problem to address.

The other problem consists in finding the relationship between x and k or alternatively, finding in what way we can express k in the space of x coordinates, and for this problem Fourier analysis is fundamental: to do that, just start from the expression of the $\langle k\rangle$ given in the space of the k coordinates, which we know how to calculate very trivially, and then counter-transform it back to the space of the x coordinates, which is an operation given by the reverse Fourier transformation and we also know it very well: all in all we end up by having that

$$\frac{1}{N^2}\int \phi(k)^\dagger k\phi(k)dk =$$
$$= \frac{1}{2\pi N^2}\iiint \psi(x')^\dagger k\psi(x)e^{-ikx}e^{ikx'}dx'dxdk =$$
$$= \frac{1}{2\pi N^2}\iiint \psi(x')^\dagger \psi(x)i\frac{d}{dx}e^{-ikx}e^{ikx'}dx'dxdk =$$
$$= \frac{1}{2\pi N^2}\iiint i\psi(x')^\dagger \frac{d}{dx}(\psi(x)e^{-ikx})e^{ikx'}dx'dxdk -$$
$$-\frac{1}{2\pi N^2}\iiint i\psi(x')^\dagger \frac{d}{dx}\psi(x)e^{-ikx}e^{ikx'}dx'dxdk$$

where in the last equality the first integral can be easily evaluated and, for functions that vanish on the boundary of integration, such an integral vanishes, leaving

$$\frac{1}{2\pi N^2}\iiint i\psi(x')^\dagger \frac{d}{dx}(\psi(x)e^{-ikx})e^{ikx'}dx'dxdk -$$
$$-\frac{1}{2\pi N^2}\iiint i\psi(x')^\dagger \frac{d}{dx}\psi(x)e^{-ikx}e^{ikx'}dx'dxdk =$$
$$= -\frac{1}{2\pi N^2}\iiint i\psi(x')^\dagger \frac{d}{dx}\psi(x)e^{-ikx}e^{ikx'}dx'dxdk =$$
$$= -\frac{1}{2\pi N^2}\iiint i\psi(x')^\dagger \frac{d}{dx}\psi(x)e^{-ik(x-x')}dx'dxdk =$$
$$= -\frac{1}{N^2}\iint i\psi(x')^\dagger \frac{d}{dx}\psi(x)\delta(x'-x)dx'dx =$$
$$= \frac{1}{N^2}\int \psi(x)^\dagger \left(-i\frac{d}{dx}\right)\psi(x)dx$$

where we only used identities of Fourier analysis; thus

$$\int \phi(k)^\dagger k\phi(k)dk = \int \psi(x)^\dagger \left(-i\frac{d}{dx}\right)\psi(x)dx$$

showing that what in the space of the k coordinates the wave-number becomes in the space of x coordinate space the operator given by minus the imaginary unit times the derivative with respect to the position. What this means is that it becomes possible to calculate the operator form

$$[x,k]\psi(x) = -ix\tfrac{d}{dx}\psi(x) + i\tfrac{d}{dx}(x\psi(x)) = i\psi(x)$$

in which we also remark that the result will have to be independent on the function we have used: consequently

$$[x,k] = i$$

holding as a general relationship between position and wave-number operators. And again we recall that so far we have used only arguments of Fourier analysis.

So far we have demonstrated that for two Hermitian operators we have the validity of the inequality

$$\Delta a\ \Delta b \geqslant \left|\tfrac{i}{2}\langle[a,b]\rangle\right|$$

holding in general; and that

$$[x,k] = i$$

valid in Fourier analysis: to put the two results together we only need to prove that x and k are Hermitian.

The conjugate of the k operator can be calculated and recalling that conjugated operators act back-ward we get

$$\int \psi(x)^\dagger \left(-i\frac{d}{dx}\right)^\dagger \psi(x)dx \equiv$$
$$\equiv \int \left(i\frac{d}{dx}\psi(x)^\dagger\right)\psi(x)dx =$$
$$= \int \psi(x)^\dagger \left(-i\frac{d}{dx}\psi(x)\right)dx \equiv$$
$$\equiv \int \psi(x)^\dagger \left(-i\frac{d}{dx}\right)\psi(x)dx$$

showing that this operator is Hermitian, in the same way in which the position operator is Hermitian since it is the multiplication by the position: therefore we can apply the theorem above in the case of these two operators.

So we have that the theorem above reads

$$\Delta x\ \Delta k \geqslant \left|\tfrac{i}{2}\langle[x,k]\rangle\right|$$

to be combined with the commutation relationship

$$[x,k] = i$$

and therefore

$$\Delta x\ \Delta k \geqslant \left|\tfrac{i}{2}\langle[x,k]\rangle\right| = \left|\tfrac{i}{2}\langle i\rangle\right| = \left|-\tfrac{1}{2}\right| = \tfrac{1}{2}$$

furnishing the final

$$\Delta x\ \Delta k \geqslant \tfrac{1}{2}$$

and thus expressing the impossibility to render both standard deviations as small as one would want.

This result is valid for any wave: in particular, it is valid for de Broglie waves, that is for waves for which we have relationships $\lambda k = 2\pi$ and $P = \hbar k$ giving

$$[x,P] = i\hbar$$

as commutation relationship and so

$$\Delta x\ \Delta P \geqslant \tfrac{\hbar}{2}$$

expressing the impossibility to make both the standard deviations vanish under any circumstance whatsoever.

This result is due to Kennard and it is the mathematical proof of Heisenberg uncertainty relationships.

Heisenberg uncertainty principle tells that when dealing with a quantum object its position and momentum cannot both be know exactly because they come together with associated standard deviations Δx and ΔP which cannot both be vanished since their product is given by

$$\Delta x\Delta P \geqslant \tfrac{\hbar}{2}$$

and similar considerations may be assumed for time and energy standard deviations Δt and ΔE according to

$$\Delta t\Delta E \geqslant \tfrac{\hbar}{2}$$

by exploiting arguments of analogy based on principles of relativity; this principle is the result of the condensation of several observed behaviours of nature finally raised to the status of principle by Heisenberg, and mathematized by Kennard's theorem whenever de Broglie relations

$$P = \hbar k$$
$$E = \hbar\omega$$

happen to hold for a system of waves.

In this sense then, systems of waves always entail uncertainty, and systems of waves verifying the de Broglie relationships $E = \hbar\omega$ and $P = \hbar k$ always entail uncertainty in the form that is give as the Heisenberg principle.

3. *Quantization*

In the previous two sections we have seen that all of the experiments and theoretical constructions fit well into a scheme in which the particles are thought as wave packets verifying $E = \hbar\omega$ and $\vec{P} = \hbar\vec{k}$ in general; wave functions of this sort are mathematically expressed according to

$$\psi(\vec{x}, t) = \frac{1}{(2\pi\hbar)^2}\int\phi(\vec{P}, E)e^{\frac{i}{\hbar}(\vec{P}\cdot\vec{x}-Et)}d^3PdE$$

again valid in general situations: this is the starting point from which to derive everything else from now on.

To begin we can calculate the derivatives

$$i\hbar\frac{\partial}{\partial t}\psi = E\psi$$
$$i\hbar\frac{\partial}{\partial\vec{x}}\psi = -\vec{P}\psi$$

and because in them the form of the wave function is not specified then these relationships have a more general status of generality in the description of quantum matter.

We notice that so far all of conditions of quantization whether they are given in terms of commutators

$$[x, P] = i\hbar$$
$$[t, E] = i\hbar$$

or in terms of the differential conditions

$$i\hbar\frac{\partial}{\partial t}\psi = E\psi$$
$$i\hbar\frac{\partial}{\partial\vec{x}}\psi = -\vec{P}\psi$$

are all relying on the fact that we are working in Galileian coordinates; we notice that the conditions of quantization given in terms of the commutators are themselves given in terms of the position x which is not covariant for coordinate transformations that involve a change to curvilinear coordinates while the conditions of quantization given in terms of differential operators are themselves given in terms of derivatives $\partial/\partial\vec{x}$ which are covariant for coordinate transformations that involve a change to curvilinear coordinates: in view of giving a covariant formulation we drop the condition of quantization given in terms of commutators and we keep only the condition of quantization given in terms of the differential operators, and for these we drop the notation $\partial/\partial\vec{x}$ in favour of $\vec{\nabla}$ valid in general.

Therefore our conditions of quantization are given by

$$i\hbar\vec{\nabla}\psi = -\vec{P}\psi$$
$$i\hbar\frac{\partial}{\partial t}\psi = E\psi$$

leaving the discussion about what happens when time is a variable to the following section. For now, we still keep space and time separated in studying quantum systems.

The dynamics is encoded by the energy condition

$$E = \frac{1}{2m}P^2 + V$$

where V is an external potential taken real.

From this relationship and substituting the quantum operators above we obtain the quantum energy condition

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\vec{\nabla}\cdot\vec{\nabla}\psi + V\psi$$

which is called Schrödinger equation and it describes the dynamics of waves in quantum mechanics under the hypothesis of non-relativistic speed due to the fact that the energy condition is clearly non-relativistic. However, it is manifestly covariant for general 3-dimensional coordinate transformations with time kept as absolute and therefore it is possible to write it also in other coordinate systems.

The most important coordinates are the spherical coordinates (r, θ, φ) where r is the radial coordinate while the θ and φ are the elevation and azimuthal angles: these coordinates give the Laplacian of the wave function as

$$\vec{\nabla}\cdot\vec{\nabla}\psi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\psi\right) + \\ + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\psi\right) + \frac{1}{|\sin\theta|^2}\partial_\varphi\partial_\varphi\psi\right]$$

where the square brackets contain all information about angular dependences; the Schrödinger equation is then given by the above with the potential written in spherical coordinates and the Laplacian we have written here as

$$i\hbar\frac{\partial}{\partial t}\psi = V(r, \theta, \varphi)\psi - \frac{\hbar^2}{2m}\frac{1}{r^2}\partial_r\left(r^2\partial_r\psi\right) - \\ - \frac{\hbar^2}{2m}\frac{1}{r^2}\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\psi\right) + \frac{1}{|\sin\theta|^2}\partial_\varphi\partial_\varphi\psi\right]$$

in which the polar axis is an axis of singularity. This form will be important when treating the Coulomb potential.

Whether in the Galileian coordinates or spherical coordinates, or any other system of coordinates, the explicit structure of the Laplacian is sensitive to the specific coordinates, but formally they are all writable according to the most general covariant expression that is given by

$$i\hbar\frac{\partial}{\partial t}\psi=-\frac{\hbar^2}{2m}\vec{\nabla}\cdot\vec{\nabla}\psi+V\psi$$

which will be the form we will use to deal in general cases.

It is possible to write it in a compact form by recognizing that the energy written in terms of the momentum is the Hamiltonian and so we may indicate it as

$$i\hbar\frac{\partial}{\partial t}\psi=H\psi$$

which is the Schrödinger equation in compact form describing the dynamics of waves in quantum mechanics for non-relativistic speed; the energy condition is hidden within the Hamiltonian but there is still the privileged role of time to tell us that this is a non-relativistic form.

It is also convenient in some circumstances to define the following conserved quantities

$$\rho=\psi^{\dagger}\psi$$

and

$$\vec{j}=-i\frac{\hbar}{2m}\left(\psi^{\dagger}\vec{\nabla}\psi-\vec{\nabla}\psi^{\dagger}\psi\right)$$

in general: with the Schrödinger equation one proves that

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{j}=0$$

is a conservation law that is satisfied in general.

Let us now employ the Schrödinger equation in specific cases, in order to solve some elementary systems.

As an application of the Schrödinger equation we might first consider one of the most important cases given when the potential is that of the harmonic oscillator

$$i\hbar\frac{\partial}{\partial t}\psi+\frac{\hbar^2}{2m}\vec{\nabla}\cdot\vec{\nabla}\psi-\frac{1}{2}\kappa x^2\psi=0$$

with κ constant: stationary states of energy E are obtained for wave functions $\psi(\vec{x},t)=e^{-iEt/\hbar}u(\vec{x})$ and because the potential can be written as the sum of three independent potentials it will be enough to study the unidimensional case for which the above equation reduces to

$$Eu(x)+\frac{\hbar^2}{2m}\frac{d}{dx}\frac{d}{dx}u(x)-\frac{1}{2}\kappa x^2u(x)=0$$

for $u(x)$ that vanishes at infinity; as we already remarked the requirement of the vanishing at infinity allowed us to prove some important formulas and relationships, but in this example such a vanishing at infinity does much more because it provides the condition for the discretization of the energy spectrum, which is found to be given by

$$E_n=\hbar\sqrt{\frac{\kappa}{m}}\left(\frac{1}{2}+n\right)$$

as it can be checked in standard textbooks.

What is important to notice here are two facts about the energy: the first is that if we call the frequency of the harmonic oscillator as $\kappa=m\omega^2$ then

$$E_n=\hbar\omega\left(\tfrac{1}{2}+n\right)$$

which can be split as $E_n=E_0+\Delta E_n$ into

$$E_0=\tfrac{1}{2}\hbar\omega$$

and

$$\Delta E_n=n\hbar\omega$$

from which additional information can be inferred: first, the ground-state does not have zero energy but a minimal

$$E_0=\tfrac{1}{2}\hbar\omega$$

which can be interpreted as the magnitude of the uncertainty with which energies are defined and since also the period is defined up to an uncertainty $\Delta t^{-1}=\omega$ then

$$\Delta E\Delta t=\tfrac{1}{2}\hbar\omega\omega^{-1}=\tfrac{1}{2}\hbar$$

as the special limiting case of Heisenberg uncertainty relationships; second, the energy difference is

$$\Delta E_n=n\hbar\omega$$

which is exactly the Planck assumption we discussed at the beginning when presenting the black-body radiation theory. The second fact about the energy is that its spectrum is discrete and such a quantized character is a direct consequence of the requirement that the wave function tend to vanish at the infinity, as already remarked.

This phenomenon of quantization, now understood as discretization of some quantities, is general and it occurs not only for wave functions that vanish at the infinity, but also when the wave function it is required to be equal to zero at a finite boundary, or when junction conditions are imposed for periodic symmetry: in the previous sections we already remarked that discretization is not something that demands for new principles, and indeed here we have witnessed that the discrete character is connected to specific boundary conditions, more than to the conditions of quantization given by the differential operators above.

Another important application is the one given by the Coulomb potential for which the Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\psi+\frac{\hbar^2}{2m}\vec{\nabla}\cdot\vec{\nabla}\psi+\frac{Ze^2}{r}\psi=0$$

with Z atomic number: stationary states of energy E can be obtained and as the potential has spherical symmetry the wave function factorizes according to the form

$$\psi(t,r,\theta,\varphi)=Ke^{-iEt/\hbar}R(r)\Theta(\theta)\Phi(\varphi)$$

for which the Schrödinger equation splits into

$$ER(r) = -\frac{\hbar^2}{2m}\frac{1}{r^2}\partial_r\left(r^2\partial_r R(r)\right) + \\ + \frac{\hbar^2 l(l+1)}{2mr^2}R(r) - \frac{Ze^2}{r}R(r)$$

together with

$$\sin\theta\partial_\theta\left(\sin\theta\partial_\theta\Theta(\theta)\right) + l(l+1)|\sin\theta|^2\Theta(\theta) = \mu^2\Theta(\theta)$$

and

$$i\partial_\varphi\Phi(\varphi) = \pm\mu\Phi(\varphi)$$

where $R(r)$ must vanish at infinity while $\Theta(\theta)$ and $\Phi(\varphi)$ must be periodic; once again the vanishing at infinity and the periodicity provide the condition for the discretization of the energy spectrum found to be given by

$$E_n = -\frac{1}{2}\frac{e^4 Z^2 m}{\hbar^2}\frac{1}{n^2}$$

in which we have that n is a positive integer, l is a non-negative integer such that $l \leqslant n-1$ and μ is also an integer and such that $-l \leqslant \mu \leqslant l$ hold as constraints among these quantum numbers, as discussed in standard textbooks.

For an assigned n there are n values for l and for each of these l there are $2l+1$ values for μ so that we conclude that there should be a total of n^2 electrons; however, what is observed is that there are $2n^2$ electrons in the outermost layer of the atomic shell: this means that despite all effort of being complete something is still missing, and what is missing is a principle telling us that we must double the occupancy for each level. This is the Pauli principle, but because its implementation would require the concept of spin, we have to accept it for the moment, postponing its mathematical justification to following sections.

Notice the interesting fact that the energy is precisely the one obtained for the Bohr atom by means of entirely different considerations: this does teach us something important about our way of doing physics, because the Bohr atom was built on the assumption that the electrons were strings circling the nucleus, a hypothesis today known to be false, but which nevertheless furnishes the exact formula for the energy levels of the atom, and reminding to us that also from assumptions that are essentially wrong we can obtain results that are numerically correct.

Once again we have found that the energy spectrum is discrete and that such a discrete character is one more time a consequence of asking that the wave function must vanish at infinity or respect some periodicity conditions.

A final important thing to notice can be seen in terms of solutions of the Schrödinger equation in the free case given by the so-called plane waves given according to

$$\psi(\vec{x},t) = K\exp\left(\frac{i}{\hbar}\vec{P}\cdot(\vec{x} - \frac{1}{2m}\vec{P}t)\right)$$

in terms of a generic constant K unspecified; we have

$$\vec{j} = K^2\frac{1}{m}\vec{P}$$

and

$$\rho = K^2$$

and from $m\vec{j} = \rho\vec{P}$ we obtain the interpretation for which if ρ represents a density of matter then $\vec{j}$ represents a density of velocity of the motion of the matter itself.

Notice that the conditions of quantization given by the differential operators are equivalent to the requirement of having plane waves which do not vanish at infinity while the statement that quantum mechanics be determined by the Schrödinger equation is unaffected by such a type of problem: thus we may drop the condition of quantization given in terms of differential operators keeping only the validity of the Schrödinger equation as fundamental.

To conclude, we define the average of an operator

$$\langle Q\rangle = \frac{1}{N^2}\int\psi(x)^\dagger Q\psi(x)dx$$

in terms of which the Schrödinger equation gives

$$\frac{d}{dt}\langle Q\rangle = \left\langle\frac{\partial Q}{\partial t}\right\rangle + \frac{i}{\hbar}\langle[H,Q]\rangle$$

and this is valid in general: in particular for the position

$$\frac{d}{dt}\langle x\rangle = \frac{i}{2m\hbar}\left\langle\left[P^2, x\right]\right\rangle = \frac{i}{2m\hbar}\langle -2i\hbar P\rangle = \frac{1}{m}\langle P\rangle$$

while for the momentum we have

$$\frac{d}{dt}\langle P\rangle = \frac{i}{\hbar}\langle[V(x), P]\rangle$$

and plugging one into the other gives

$$m\frac{d^2}{dt^2}\langle x\rangle = \frac{d}{dt}\langle P\rangle = -\left\langle\frac{dV(x)}{dx}\right\rangle$$

in a form that is similar to Newton's equation of motion.

Expanding the derivative of the potential in a series

$$\frac{dV(x)}{dx} \approx \frac{dV(\langle x\rangle)}{d\langle x\rangle} + (x - \langle x\rangle)\frac{d^2V(\langle x\rangle)}{d^2\langle x\rangle} + \\ + \frac{1}{2}(x - \langle x\rangle)^2\frac{d^3V(\langle x\rangle)}{d^3\langle x\rangle}$$

and therefore

$$\left\langle\frac{dV(x)}{dx}\right\rangle \approx \frac{dV(\langle x\rangle)}{d\langle x\rangle} + \frac{1}{2}\Delta x^2\frac{d^3V(\langle x\rangle)}{d^3\langle x\rangle}$$

showing that if the uncertainty on the position is small

$$\left\langle\frac{dV(x)}{dx}\right\rangle \approx \frac{dV(\langle x\rangle)}{d\langle x\rangle}$$

and the above would coincide with Newton's law for the motion of the peak of the localized matter distribution.

What this means is that whenever the wave function is such that the packet is extremely localized around the average of the position, which can thus be taken as the position of the point-like particle, we have the classical macroscopic limit as it was first discussed by Ehrenfest.

In this macroscopic limit $\hbar$ is no longer present, which is consistent with the fact that the presence of this constant is what gives the presence of quantum effects.

Because we will no longer consider this approximation we will normalize the constant $\hbar=1$ from now on.

4. *Quantization and relativity*

In the previous sections we have seen that conditions of quantization were given by the differential operators

$$i\vec{\nabla}\psi=-\vec{P}\psi$$
$$i\frac{\partial}{\partial t}\psi=E\psi$$

which are covariant for 3-dimensional space transformations and they display some symmetry between space coordinates and the temporal variable; before we have not discussed the analogies between space and time any further but now it is time to discuss it thoroughly since we want to raise the time to the status of the fourth coordinate, to form the (3+1)-dimensional space-time in which to implement the principles of relativity: by calling the time as the zeroth coordinate of the $(3+1)$-dimensional space-time and the energy as the zeroth component of an energy-momentum given by the definition

$$P_\mu=(E,-\vec{P})$$

we may write

$$i\nabla_\alpha\psi=P_\alpha\psi$$

as the (3+1)-dimensional relativistically covariant condition of quantization for the waves of matter.

However, there was another way in terms of which the quantum system could not be relativistic and that was the fact that the dynamics was expresses in terms of

$$E=\frac{1}{2m}P^2$$

which is an energy condition that is clearly not relativistic in itself: the relativistic energy condition is known to be

$$E^2-P^2-m^2=0$$

or by using $P_\mu=(E,-\vec{P})$ as

$$P^\alpha P_\alpha-m^2=0$$

and this the condition that has to be considered.

By employing this relativistic energy condition and the covariant conditions of quantization we obtain

$$\nabla_\alpha\nabla^\alpha\psi+m^2\psi=0 \tag{193}$$

which is the Klein-Gordon equation and it describes the dynamics of waves in quantum mechanics for relativistic cases and therefore in the most general covariant form.

As a consequence of this generality we no longer need specify we are studying waves relativistically in quantum mechanics, and we will talk simply about quantum fields.

It is also convenient to define the conserved quantity

$$mJ_\mu=\tfrac{i}{2}\left(\psi^\dagger\nabla_\mu\psi-\nabla_\mu\psi^\dagger\psi\right)$$

in general: with the Klein-Gordon equation we prove that

$$\nabla_\mu J^\mu=0$$

is a conservation law that is satisfied in general.

Plane waves are given according to the form

$$\psi(x^\mu)=K\exp\left(-iP_\alpha x^\alpha\right)$$

in terms of a generic constant K unspecified; then

$$J^\mu=\frac{1}{m}K^2P^\mu$$

and from $mJ^\mu=\psi^\dagger\psi P^\mu$ we obtain the interpretation for which if $\psi^\dagger\psi$ is the density of matter then J^μ is the density of the velocity of the motion of the matter itself.

Again plane waves do not vanish at the infinity.

Despite the Klein-Gordon equation is a good quantum field equation nevertheless it contains not one but two time derivatives and so one might ask if it is possible to obtain a quantum field equation with one order derivative alone: the problem is purely theoretical and it stems from the fact that the lower is the derivative order of a differential equation, the fewer are the integrations needed to obtain its solutions and the stronger is the solution itself.

It was Dirac who thought about this circumstance, and thus to the occurrence of having relativistic waves with first-order derivative equations, and he noticed that if it were possible to find objects $\boldsymbol{\gamma}^\mu$ verifying the relationships $\{\boldsymbol{\gamma}^\mu,\boldsymbol{\gamma}^\nu\}=2\mathbb{I}\eta^{\mu\nu}$ then it would be possible to write

$$\boldsymbol{\nabla}^2\psi+m^2\psi\equiv(i\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu+m)(-i\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu+m)\psi$$

as a general operatorial identity; it is possible to see that the Klein-Gordon equation would be satisfied if either

$$i\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi-m\psi=0$$

or

$$i\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi+m\psi=0$$

were to be satisfied as well, but these two Dirac equations are first-order derivative equations: each Dirac equation is stronger than the original Klein-Gordon equation since the validity of a single one of them implies the validity of the original equation but the converse is not true.

Given $\boldsymbol{\gamma}^\mu$ matrices with $\{\boldsymbol{\gamma}^\mu,\boldsymbol{\gamma}^\nu\}=2\mathbb{I}\eta^{\mu\nu}$ the equations

$$i\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi\mp m\psi=0 \tag{194}$$

are the Dirac equations describing the dynamics of quantum fields at the least-order derivative possible.

The constraint of being at the least-order of derivatives imposed the introduction of additional objects given by the gamma matrices that were already known to mathematicians under the name of Clifford matrices, and which describe further internal structures for the quantum field.

With the zeroth gamma matrix $\boldsymbol{\gamma}^0$ from the Dirac field we define the adjoint Dirac field $\overline{\psi}=\psi^\dagger\boldsymbol{\gamma}^0$ with which

$$J^\mu=\overline{\psi}\boldsymbol{\gamma}^\mu\psi$$

in general: from the Dirac equations we obtain

$$\nabla_\mu J^\mu=0$$

as a conservation law that is valid in general.

Then plane waves are given according to

$$\psi(x^\mu)=\exp\left(-iP_\alpha x^\alpha\right)u$$

and where $(P_\mu\boldsymbol{\gamma}^\mu\pm m\mathbb{I})u=0$ hold; it is easy to acknowledge that $m\overline{u}\boldsymbol{\gamma}^\mu u=\overline{u}uP^\mu$ and therefore we have that

$$J^\mu=\frac{1}{m}\overline{\psi}\psi P^\mu$$

showing that if $\overline{\psi}\psi$ is the density of matter then J^μ is the density of the velocity of the motion of the matter itself.

But once again plane waves do not vanish at infinity.

An explicit expression of the gamma matrices is what is called standard representation and it is given by

$$\boldsymbol{\gamma}^0=\begin{pmatrix}\mathbb{I} & 0\\ 0 & -\mathbb{I}\end{pmatrix}$$

$$\vec{\boldsymbol{\gamma}}=\begin{pmatrix}0 & \vec{\boldsymbol{\sigma}}\\ -\vec{\boldsymbol{\sigma}} & 0\end{pmatrix}$$

in terms of which writing the Dirac field as

$$\psi=\begin{pmatrix}\phi^-\\ \phi^+\end{pmatrix}$$

allows us to write the Dirac equations as

$$(E\mp m)\phi^-+\vec{P}\cdot\vec{\boldsymbol{\sigma}}\phi^+=0$$
$$(E\pm m)\phi^++\vec{P}\cdot\vec{\boldsymbol{\sigma}}\phi^-=0$$

which can be recombined to give

$$(E+m)(E-m)\phi^+=(\vec{P}\cdot\vec{\boldsymbol{\sigma}})^2\phi^+$$

and

$$(E+m)(E-m)\phi^-=(\vec{P}\cdot\vec{\boldsymbol{\sigma}})^2\phi^-$$

as the general Dirac equations in momentum space; if we consider the limit in which the momentum $\vec{P}\to 0$ then we have that $E\to m$ and in the Dirac equation for negative sign in front of the mass term we get $\phi^+\approx 0$ while for positive sign in front of the mass term we get $\phi^-\approx 0$ and

$$2m(E-m)\phi^-=P^2\phi^-$$
$$2m(E-m)\phi^+=P^2\phi^+$$

both in the free case: for the non-relativistic approximation, according to whether the mass term has a negative or a positive sign there remains the upper or the lower of the components, and in both cases the remaining of the components verifies the equation of the form

$$i\frac{\partial}{\partial t}\phi=-\frac{1}{2m}\vec{\nabla}\cdot\vec{\nabla}\phi+m\phi$$

which can be recognized to be the Schrödinger equation with a potential $V=m$ recovering the expected result.

The Klein-Gordon theory can be obtained by variational methods from the Klein-Gordon Lagrangian

$$\mathscr{L}=\tfrac{1}{2}\left(\nabla_\alpha\psi^\dagger\nabla^\alpha\psi-m^2\psi^\dagger\psi\right)$$

as it is straightforward to check and as very well known.

In this formalism the fact that fields are generally complex and thus displaying a subsequent complex phase symmetry is what through Noether theorem yields the conservation law $\nabla_\mu J^\mu=0$ as it has been given above.

The corresponding Hamiltonian is calculated to be

$$\mathscr{H}=\frac{1}{2}\left(\frac{\partial}{\partial t}\psi^\dagger\frac{\partial}{\partial t}\psi+\vec{\nabla}\psi^\dagger\cdot\vec{\nabla}\psi+m^2\psi^\dagger\psi\right)$$

in which time has become once again privileged, but this expected since the Hamiltonian is an energy density.

And the Dirac theory can be obtained by employing the usual variational methods from the Dirac Lagrangian

$$\mathscr{L}=\tfrac{i}{2}\left(\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi-\boldsymbol{\nabla}_\mu\overline{\psi}\boldsymbol{\gamma}^\mu\psi\right)\pm m\overline{\psi}\psi$$

as it is one more time very straightforward to check.

And once again in this formalism the fields are complex and therefore their subsequent complex phase symmetry is what through Noether theorem yields the conservation law in the form $\nabla_\mu J^\mu=0$ as it is shown above.

The corresponding Hamiltonian has form

$$\mathscr{H}=\frac{i}{2}\left(\psi^\dagger\frac{\partial}{\partial t}\psi-\frac{\partial}{\partial t}\psi^\dagger\psi\right)$$

again with time playing a privileged role.

Having settled the foundations of the Klein-Gordon and the Dirac theories we will now proceed to recall some of the most important concepts of theoretical physics.

5. *Second quantization*

In the previous two sections, we gave the canonical definition of quantum fields whose dynamics is described by field equations, and for them also a variational formalism was discussed: for Klein-Gordon and Dirac cases

$$\mathscr{L}=\frac{1}{2}\left(\nabla_{\alpha}\psi^{\dagger}\nabla^{\alpha}\psi-m^{2}\psi^{\dagger}\psi\right) \tag{195}$$

$$\mathscr{L}=\frac{i}{2}\left(\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-\boldsymbol{\nabla}_{\mu}\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi\right)\pm m\overline{\psi}\psi \tag{196}$$

as it is easy to check by applying the Lagrange equations.

The corresponding Hamiltonians were calculated as

$$\mathscr{H}=\frac{1}{2}\left(\frac{\partial}{\partial t}\psi^{\dagger}\frac{\partial}{\partial t}\psi+\vec{\nabla}\psi^{\dagger}\cdot\vec{\nabla}\psi+m^{2}\psi^{\dagger}\psi\right) \tag{197}$$

$$\mathscr{H}=\frac{i}{2}\left(\psi^{\dagger}\frac{\partial}{\partial t}\psi-\frac{\partial}{\partial t}\psi^{\dagger}\psi\right) \tag{198}$$

in which the role of time was seen to be privileged.

Now we continue the discussion around them by noticing a first important fact, and that is the Hamiltonian for the Klein-Gordon field is always positive defined while the Hamiltonian for the Dirac field is not; Hamiltonians should represent an energy density and as energies should be positive defined this situation may pose problems.

Another theoretical problem we met was the introduction of the Pauli principle, and therefore an appropriate systematization should be in order at this point.

So, there are two issues that seem to have quite a high urgency: there appears to be a problem for the presence of negative energies; and there is the problem of defining the spin and a way to implement the Pauli principle.

The way in which these two issues can be altogether overcome is based on what is usually referred to as second quantization, the argument being due to Dirac, as based on the fact that quantization worked so well in quantizing particles that repeating it a second time may work just as well in quantizing relativistic particles and thus fields.

In order to have quantization repeated a second time we just need to restate the condition of quantization given in terms of commutators in the case of fields, and that is we have to re-interpret the fields and their derivatives as operators verifying specific commutation relationships of some sort: the exact sort of these depends case by case.

As a first guess one may think that by considering the field ψ and its conjugate momentum $\dot{\psi}^{\dagger}$ we could write

$$[\psi_{i}(x),\dot{\psi}_{j}^{\dagger}(y)]=i\delta_{ij}\delta(x-y) \tag{199}$$

where the lower index designates the component of the Dirac field and they are called equal-time commutation relations given in terms of the Dirac delta; although this guess would work for the Klein-Gordon fields nevertheless it does not work for the Dirac field, for which we have that given the field ψ its conjugate momentum is $\psi^{\dagger}$ and

$$\{\psi_{i}(x),\psi_{j}^{\dagger}(y)\}=\delta_{ij}\delta(x-y) \tag{200}$$

where the lower index designates the component of the Dirac field and they are called equal-time anticommutation relations given in terms of the Dirac delta.

To see what second quantization actually is, let us consider solutions of the Dirac equations: in the following we will assume, as it is done in the common treatment, that of the Dirac equations (194) only that with negative sign of the mass term will be considered; for this, and in case of plane waves, solutions are given in terms of the coefficients u verifying $(P_{\mu}\boldsymbol{\gamma}^{\mu}\pm m\mathbb{I})u=0$ and, in the standard representation of the gamma matrices, they are

$$\psi^{+}=e^{-i(Et-Px)}\begin{pmatrix}\sqrt{E+m}\cos\frac{\theta}{2}\\ \sqrt{E+m}\sin\frac{\theta}{2}\\ \sqrt{E-m}\cos\frac{\theta}{2}\\ -\sqrt{E-m}\sin\frac{\theta}{2}\end{pmatrix}$$

and

$$\psi^{-}=e^{+i(Et-Px)}\begin{pmatrix}\sqrt{E-m}\cos\frac{\theta}{2}\\ -\sqrt{E-m}\sin\frac{\theta}{2}\\ \sqrt{E+m}\cos\frac{\theta}{2}\\ \sqrt{E+m}\sin\frac{\theta}{2}\end{pmatrix}$$

with momentum aligned along the third axis. Each form accounts for the two polar projections that are given by the values $\theta=0$ and $\theta=\pi$ and which in the following will be labelled as $s=1$ and $s=2$ respectively; because of the energy condition $E^{2}-P^{2}=m^{2}$ of energy and momentum only one is independent, and thus the expansion

$$\psi(t,x)=\frac{1}{\sqrt{2\pi}}\int\phi(P)\ e^{-i(Et-Px)}dP$$

can be written in terms of explicit coefficients as

$$\psi(t,x)=\frac{1}{\sqrt{2\pi}}\int\sum_{s=1}^{s=2}(e^{-i(Et-Px)}u_{s}^{+}+$$
$$+e^{+i(Et-Px)}u_{s}^{-})dP \tag{201}$$

or after introducing a normalization factor and more general coefficients of expansion it can also be written as

$$\psi(t,x)=\frac{1}{\sqrt{2\pi}}\int\sum_{s=1}^{s=2}(e^{-i(Et-Px)}u_{s}^{+}a_{s}+$$
$$+e^{+i(Et-Px)}u_{s}^{-}b_{s})\frac{dP}{\sqrt{2E}} \tag{202}$$

which gives the Hamiltonian with both positive and negative contributions. Now with the implementation of the conditions of second quantization (200) we may see that

$$\{a_{i}(p),a_{j}^{\dagger}(q)\}=\delta_{ij}\delta(p-q) \tag{203}$$

$$\{b_{i}(p),b_{j}^{\dagger}(q)\}=\delta_{ij}\delta(p-q) \tag{204}$$

and therefore the Hamiltonian becomes

$$\mathscr{H}=\int\sum_{s=1}^{s=2}(a_{s}^{\dagger}a_{s}-b_{s}^{\dagger}b_{s})EdP=$$
$$=\int\sum_{s=1}^{s=2}(a_{s}^{\dagger}a_{s}+b_{s}b_{s}^{\dagger})EdP-\int 2\delta(0)EdP \tag{205}$$

which has only positive energy levels if we forget about the negative infinite contribution of the term on the right.

If we forget about it, renaming $b_s^\dagger = z_s$ yields

$$\mathscr{H} = \int \sum_{s=1}^{s=2} (a_s^\dagger a_s + z_s^\dagger z_s) E dP \tag{206}$$

interpreted by saying that $a_s^\dagger$ annihilates and a_s creates particles while $z_s^\dagger$ annihilates and z_s creates antiparticles and all of them have positive defined energies [50].

The creation/annihilation operators are therefore such that $a|0\rangle = |1\rangle$ and $a^\dagger|1\rangle = |0\rangle$ but if we add the additional requirement that we cannot annihilate anything if nothing is present encoded by $a^\dagger|0\rangle = 0$ then we additionally get $a|1\rangle = aa|0\rangle = -a^\dagger a^\dagger|0\rangle = -a^\dagger 0 = 0$ or $a|1\rangle = 0$ spelling that we cannot create something if something is present entailing the physics of the Pauli exclusion principle [50].

It is quite a general argument that one fields have been quantized according to what we have discussed above the exclusion principle is automatically implemented [51].

In this way the effects of the exclusion principle and ensuring the positivity of the energy have been achieved by means of the principle of second quantization accomplished by lifting the fields up to operators verifying specific commutation or anticommutation relations.

6. *Quantization's boundaries*

In the previous sections we have seen that experimental evidence could be condensed in a theoretical frame built on the principles of quantization encoded by insisting that position and conjugate momentum are operators verifying some commutation relationships, or that there exist fields satisfying certain differential conditions that involved momenta, and that this was also equivalent to having these fields as solutions of a specific field equation, the Schrödinger equation; we have seen that relativistic generalizations yielded the Klein-Gordon and Dirac field equations, and that these fields undergo to a subsequent process of second quantization where the fields and their conjugate momenta are re-interpreted as operators verifying commutation and anticommutation relations: thus we could demonstrate that positive energies were ensured and the Pauli principle was entailed in general [50, 51].

Second quantization gives rise to the consequence that a system constituted by many particles is not to be seen as a system of many fields but as a system of a single field describing many states, each of which being a particle.

Next we will see what more we can actually do in quantum field theory, and specifically about scattering.

In studying interactions the important quantity is the scattering amplitude, that is the probability that a certain process occurs due to the assigned interaction, mathematically given by a specific form which, under certain conditions, can perturbatively be expanded in series, and where each term accounts for the process involving a definite number of those particles that are the excited states of the field describing the interaction; so the interaction of a system of particles can sometimes be written as if all the information about the interaction were actually encoded inside contact vertices while all the rest of the process consisted in particles freely propagating.

In the following we will focus on the interaction of electrons with electro-dynamics, the theory based on the Lagrangian of Dirac fields in interaction with Maxwell fields and for which subsequently conditions of second quantization are imposed, called quantum electro-dynamics.

It is not the place here to dig for the details since the interested reader could simply take advantage of common textbooks, but if we were to write scattering amplitudes perturbatively, in the expansion each term can be interpreted as electrons which absorb/emit photons: external photons are real, but those photons emitted/absorbed by the electron do not verify Maxwell equations and thus they are said to be virtual; the processes for which the electron emits/absorbs virtual photons are electronic self-interactions called photonic loops. All these processes are called radiative corrections, which give rise to a quantum electro-dynamical effect that is entirely new in physics.

More precisely, the electro-dynamic interaction of electrons is described by Maxwell-Dirac Lagrangian

$$\mathscr{L} = \tfrac{1}{4}F^2 + qA_\mu \overline{\psi} \boldsymbol{\gamma}^\mu \psi - \tfrac{i}{2}(\overline{\psi} \boldsymbol{\gamma}^\mu \boldsymbol{\nabla}_\mu \psi - \boldsymbol{\nabla}_\mu \overline{\psi} \boldsymbol{\gamma}^\mu \psi) + m\overline{\psi}\psi$$

where $F_{\alpha\nu} = \partial_\alpha A_\nu - \partial_\nu A_\alpha$ and writable as the Maxwell and Dirac Lagrangians supplemented by the interaction

$$\Delta\mathscr{L} = qA_\mu \overline{\psi} \boldsymbol{\gamma}^\mu \psi$$

containing all information needed to compute scattering amplitudes; if we could calculate all terms of the perturbative expansion we would get the entire analytic expression and despite we do not know it explicitly nevertheless we do know that it has the general structure given by

$$\begin{aligned} qA_\mu \overline{\psi} \boldsymbol{\gamma}^\mu \psi = {} & qF_1(k^2) A_\mu^{\text{ext}} \overline{u} \boldsymbol{\gamma}^\mu u + \\ & + \frac{q}{2m} F_2(k^2) \frac{1}{2} F_{\mu\nu}^{\text{ext}} 2i\overline{u} \boldsymbol{\sigma}^{\mu\nu} u \end{aligned} \tag{207}$$

in which $F_1(k^2)$ and $F_2(k^2)$ are what contains all the information about the scattering in terms of the squared-momentum transfer, called form factors: deviations from the conditions $F_1(k^2) = 1$ and $F_2(k^2) = 0$ tell that some of the loops, or radiative correction, has taken place, and so quantum contributions to electro-dynamical must arise.

It is to be noticed that despite the perturbative series cannot be calculated exactly, we have calculated the first three terms and the results have a remarkable fit with observations in the case of the anomalous magnetic moment of electrons and hyper-fine splittings of spectral lines.

There is however a problem, that is when we said that the scattering amplitudes could be expanded in a perturbative series, each term being a sum of all contributions involving integrals over momentum space, we assumed all this could be done, but the situation is not so straightforward as we have presented: there are general indications that the series may not converge, and this might be a first fatal flaw; additionally, among the various contributions there may be some for which the integral diverges for the large momenta, unless an upper bound to the integration is eventually placed. This upper bound to integration is called cut-off, and it signals the limit beyond which all of our calculations become worthless since we ignore what physical effect might become relevant beyond this point.

Of course it could well happen that the physics beyond the cut-off is such that it renders the dynamic trivial and therefore the integral from the cut-off to infinity is really irrelevant: we did already encounter such an example for the black-body radiation, where the Rayleigh-Jeans law seemed to diverge but it actually did not because of the new physical behaviour given by the Wien law. Therefore beyond the cut-off it may be that physics changes enough to make the integral irrelevant, so that we can get rid of it without too much of a trouble. When we can actually do this, the theory is said to be a renormalizable theory.

We are not going to deepen the systematics of renormalization since it is a task that would bring us too far.

The point that needs to be retained is that these procedures are aimed at restoring the reliability of a theory that is spoiled if infinities cannot be kept under control.

Admittedly, it is preferable to have a theory that does not need to be renormalized, for which infinities appear but they can be removed so to give finite results, and it would be best to have a theory that gives finite results immediately; for the moment such a theory still transcends our means, so having a theory that is renormalizable is the most we can ask: although renormalizability can be discussed only within a heavy formalism, fortunately the final results are easily implementable, as we will see.

A first step is assigning a mass-dimension to quantities, and because we chose $\hbar=1$ and $c=1$ as units, all lengths are dimensionally inverse masses; quantities without an a priori defined dimension of mass are assigned one for which their kinetic term in the Lagrangian has mass-dimension 4 in general: once all quantities have their own mass-dimension, we get the mass-dimension of all terms that can be included in the Lagrangian, and if in the total Lagrangian all the kinetic terms have a mass-dimension of 4 and all of the interactions a have mass-dimension not higher than 4 then the Lagrangian is renormalizable.

To justify this argument intuitively, Wilson came up with the idea that when a mass-dimension, or a length-dimension, is assigned to a field, after a scaling distances, there will be a corresponding scale transformation for the field, and thus a scale transformation for its terms in the Lagrangian: because the action is given in terms of the Lagrangian multiplied by a 4-dimensional volume, then the action is dimensionless, and therefore renormalizability simply means that in the total action all kinetic terms are scale invariant and all interactions are scale invariant or they become negligible at smaller and smaller scales.

For example, taking the electro-dynamic Lagrangian

$$\mathscr{L}=\frac{1}{4}F^2+qA_\mu\overline{\psi}\boldsymbol{\gamma}^\mu\psi-\frac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi-\boldsymbol{\nabla}_\mu\overline{\psi}\boldsymbol{\gamma}^\mu\psi)+m\overline{\psi}\psi$$

we assign to the electrodynamic field A_α mass-dimension 1 and to the Dirac field ψ mass-dimension $\frac{3}{2}$ so that both kinetic terms have mass-dimension 4 while the interaction has mass-dimension 4 and the mass term has mass dimension 3 and thus, the Lagrangian is renormalizable.

As it is clear, the requirement of renormalizability also puts an upper bound to the mass-dimension of the terms that can enter in the Lagrangian, consequently limiting the number of possible terms in the Lagrangian itself.

Therefore, renormalizability is not only useful to justify that we can neglect some integrations in the interactions since they would be irrelevant anyway, but it is also utile in justifying why actions cannot have infinite terms.

7. *Quantization and gravity*

In the previous section, we have discussed that second quantized field in interaction can be studied in terms of scattering amplitudes, we have briefly described how calculations are done in general, and as some prototype interaction we have considered what happens for electrodynamics. We discussed the problem of renormalization, showing the renormalizability of electro-dynamics itself.

The second interaction one might think to add beside electro-dynamics, and possibly even more important than that, is gravity: in fact, despite there exist fields that are neutral and therefore transparent to electro-dynamics, all fields have energy and therefore nothing is transparent to gravity; gravity is described by Einstein field equations

$$R_{\rho\sigma}-\frac{1}{2}Rg_{\rho\sigma}-\Lambda g_{\rho\sigma}=\frac{k}{2}T_{\rho\sigma} \tag{208}$$

where k is the Newton gravitational constant.

These field equations have a kinetic term for gravity that can be derived from the Lagrangian

$$\mathscr{L}=2\Lambda+R \tag{209}$$

varying with respect to the metric field.

In a theory of gravity in which this field is to be quantized, that is in which the metric of the space-time is to be raised to the status of operator of annihilation/creation of virtual particles called gravitons, there is a tension that comes from the fact that gravity couples to anything with an energy and gravitons have energy: therefore, gravitons would be self-interacting, and their dynamics would have non-linear contributions. Gravitons can never be totally free as the propagator of virtual gravitons should be.

Another problem of gravity is that its kinetic term is a curvature, and curvatures have the dimension of an inverse area: so the kinetic term has mass-dimension 2 and this means that Einstein gravity is not renormalizable.

Renormalizability may be obtained for the Lagrangian

$$\mathscr{L}=2\Lambda+R+\frac{A}{2}R^2+\frac{B}{2}R_{\rho\sigma}R^{\rho\sigma} \tag{210}$$

where A and B are constants still undefined [52].

However, despite such a gravitational Lagrangian is renormalizable, nevertheless there remains the fact that gravity is intrinsically non-linear while quantization involves free propagators, and so although there may be a manner to write the gravitational field equations coupling energy to curvature in a renormalizable way, the fact that gravity is encoded as the curvature of the space-time still makes rather problematic its second quantization.

B. Spin

In all previous sections, we have established two consecutive quantizations, a first based on the assumption that dynamical variables were operators undergoing commutation relations, or that there exist fields satisfying differential conditions, or that these fields were solutions of field equations; and a second based on the assumption that these fields are operators undergoing either commutation or anticommutation relations: additionally, second quantization also requires further concepts such as that of renormalizability of interactions. We have discussed how the first quantization stemmed from a certain number of empirical evidences all condensed together into theoretical statements; while second quantization insisted more on the theoretical fact that with it it could be possible to justify at once both the positivity of the energy and the effects of the exclusion principle. A second quantization is very successful in terms of its experimental confirmation despite that theoretically problems are present.

We will discuss how these problems appeared and we will further consider from where the mentioned problems come as a point of departure for possible solutions.

1. *Spin and relativity*

So to begin our critical treatment of quantization, we start by re-summarizing what has been done before filtering out all irrelevant details and focusing on the most fundamental concepts: as a first step, we implemented a condition of quantization given by either

$$[x,P]=i\hbar$$
$$[t,E]=i\hbar$$

or

$$i\hbar\frac{\partial}{\partial t}\psi=E\psi$$
$$i\hbar\frac{\partial}{\partial\vec{x}}\psi=-\vec{P}\psi$$

but we also discussed that only the latter written as

$$i\hbar\vec{\nabla}\psi=-\vec{P}\psi$$
$$i\hbar\frac{\partial}{\partial t}\psi=E\psi$$

are meaningful to eventually give a full covariant formulation and we have seen that by calling the time as zeroth coordinate these could be collected together as

$$i\nabla_{\alpha}\psi=P_{\alpha}\psi \tag{211}$$

in a complete relativistic covariant formulation; we have seen that based on an argument of analogy, it is possible to have this quantization formally repeated a second time by considering that fields could be re-interpreted as operators verifying specific commutation relations

$$[\psi_{i}(x),\dot{\psi_{j}}^{\dagger}(y)]=i\delta_{ij}\delta(x-y) \tag{212}$$

for the Klein-Gordon field and anticommutation relations

$$\{\psi_{i}(x),\psi_{j}^{\dagger}(y)\}=\delta_{ij}\delta(x-y) \tag{213}$$

for the Dirac field: by assuming that of the two possible Dirac equations (194) only that with negative sign of the mass term be allowed, it has been possible to demonstrate that, for plane waves, the Hamiltonian had only positive energy excitations and that the Pauli principle held.

And by computing scattering amplitudes, after having introduced the property of renormalizability, it became a mere technical computation that of calculating the effects of radiative corrections for some elementary processes.

The success of the protocols of second quantization in field theory are so remarkable that it is difficult to believe some problems may arise, but this is what happens as we are going to discuss: a first problem, and clearly the most fundamental one, is that second quantization seems to be genuinely incompatible with gravity; the usual behaviour is that of assuming that gravity is the source of problems, but the fact is that the theory of gravitation as we have presented it in the previous part is a theory that is essentially based on no hypothesis other than those that are also considered in quantum field theory. As we developed it above, we have only required to have the most general system of the least-order derivative field equations, which was enough to let us obtain Einstein equations as those that can be obtained form (209), and from the Einstein equations in weak field approximations it is possible to obtain the known limits by interpreting gravity as what is contained in the metric of the space-time; of course, it is possible to argue that least-order derivative field equations are too restrictive, but even dropping this restriction we would get higher-order derivative field equations as those coming from (210), and from which in the weak field approximations we would obtain the same limits and so we could still interpret gravity as encoded within the metric of the space-time. The problem of incompatibility between gravity and second quantization comes from the fact that we ignore how to second quantize fields that are intrinsically non-linear, like those coupled to themselves due to the fact that they have and are sourced by energy.

The fact that Einstein gravity (or any of its extensions obtained by dropping the constraint of being at the least-order derivative in the field equations) are virtually based on no hypothesis makes it difficult to see how it may be Einstein gravity (or any of its extensions) that has to be modified, and indeed all attempts known at the moment involve a complete re-considerations of all the concepts related to the space-time structure. Because for now all such attempts are unsuccessful, it may not be unwise to try a more humble approach in which second quantization is the key element that has to be changed. After all it is obvious that the procedure of second quantization has a number of arbitrary assumptions that can be modified.

The most important is the fact that albeit renormalization seem to work very well, still it would be desirable to have a theory in which infinities are not removed after showing up but in which they do not show up in the first place: this may mean to get rid of a formalism in which particles are represented as states and fields as operators.

Another assumption that is almost never noticed but which is quite ubiquitous is that solutions are always taken in plane waves, which do not exist: either because every particle has an energy, so it curves the space-time around itself and rectilinear coordinates would simply be non-sensical; or because plane waves as we have already remarked are not square integrable. A way out could be to try a superposition of plane waves that does vanish at infinity; but because matter fields enter non-linearly in the expression of their conserved quantities, we cannot be sure the superposition is still a solution. In any case, these are solutions of the free matter field equation, and once again clearly free matter fields do not really exist.

A hypothesis that also looked quite arbitrary was that of the two Dirac equations (194) only that with negative sign of the mass term was allowed, although we have seen above that both signs of the mass term are allowed, and which correspond to the two types of solutions linked by the discrete transformation $\psi \rightarrow \boldsymbol{\pi}\psi$ in general, so that it is clearly too restrictive to cut off half space of solutions.

As for the commutators, it is to be stressed that as it is clear from (205) the presence of the infinite contribution is quite up-setting, and although in quantum field theory such a contribution is usually neglected because gravity is never considered, nevertheless there is nothing we can do when the coupling to gravitation is taken: getting rid of this term would simply mean that the commutation and anticommutation relationships reduce to

$$[\psi_i(x), \dot{\psi}_j{}^{\dagger}(y)] = 0$$

and

$$\{\psi_i(x), \psi_j^{\dagger}(y)\} = 0$$

implying there are no commutation and anticommutation relationships at all. Additionally, we have seen that such commutation relationships have been assumed based on an analogy with the commutation relationships that give the conditions of quantization for a particle in quantum mechanics, but it is not clear that this could make sense, because we cannot be sure that analogy arguments work and, for that matter, we have already remarked that even in quantum mechanics, among all the conditions of quantization, the one given in terms of commutation relationships was abandoned in favour of the one given with differential operators as the only one to be meaningful in a complete relativistic covariant formulation of the theory.

As commutation relationships make no sense [51] nor we have any definition of field operators [53, 54] it might be wise to dispense with the idea of replicating a second time the condition of quantization: in the following, the condition of quantization will be a condition imposed on mechanics solely. To be fair, we have to insist that even quantization in mechanics is not needed, as we will argue.

Several times we have recalled that in mechanics, the conditions of quantization can be implemented by either re-interpreting position and momentum as operators verifying commutation relationships, or by having fields verifying some differential condition, or yet by assigning the matter field equations: we have already recalled how the commutators should be abandoned in favour of the differential conditions given by the expressions

$$i\nabla_{\alpha}\psi = P_{\alpha}\psi \tag{214}$$

if we want to write everything in a relativistic covariant form, but we may also abandon the differential conditions

$$i\nabla_{\alpha}\psi = P_{\alpha}\psi \tag{215}$$

in favour of the matter field equations

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi \pm m\psi = 0 \tag{216}$$

which are more general and they are the only concept we need after all; if we start from energy conditions defining the dynamics of particles then we need quantization to obtain the matter field equations giving the dynamics of quantum particles, but we may just as well start directly from the matter field equations in the form

$$i\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi \pm m\psi = 0 \tag{217}$$

giving the dynamics of quantum particles immediately.

So, there is no need for quantization if we do not start from particles but directly from quantum particles, given as solutions of matter field equations that can be assigned as a part of the full system of field equations, constructed in terms of general arguments, as it was discussed in the first part, and the field equations are all that is necessary in order to obtain the phenomenology that is so successfully observed; nor could we employ second quantization, since it is not a properly defined concept. Then how can we recover the phenomenology we observed after all?

The starting point is to settle the problems of negative energies and Pauli principle. Others will be treated later.

We begin with the issue of the energies and their being positive as well as negative, and immediately we have to state that we believe this might not be a problem in the first place: there are two reasons for this, and the first is that when dealing with energies it is impossible to neglect what is sourced by energy, and that is the gravitational field; when the gravitational field equations are taken into account, we have a relationship linking the energy tensor to a non-linear expression involving the derivatives of the metric tensor, and therefore two opposite energy tensors give rise to two different structures of the metric and thus to two different gravitational responses implying that in general the negative energy field will not be a solution if the positive energy field is a solution of the entire system of field equations. The second reason is that there is no real problem if energies are negative and in fact this is to be expected for spinors; spinors like any other field have contributions due to overall displacements, which must be positive, but unlike any other field have contributions due to the internal dynamics of the two chiral parts, and if bound-states are to exist, these have to be negative, so the total energy, where both contributions are entangled, is a sum, whose sign cannot be defined in any way at all.

Because either there is no real problem with negative energies or there is nothing to claim until also the gravitational field equations are considered, the problem with positive energy may not be a true problem after all [55].

The other problem that needs to be considered is about the implementation of the Pauli exclusion principle: this principle, as discussed at the beginning, starts from the fact that, in the construction of electronic levels, obtained by solving the non-relativistic matter field equations in a Coulomb potential, the solutions are given in terms of a quantum number n giving the energy level of the external shell, accounting for a total of n^2 electrons; because the number of electrons we observe is $2n^2$ then there must be a two-fold degeneracy, and that is solutions of the matter field equation come in pair of two so that every electronic shell can be filled twice by the same state. The exclusion principle presented thusly is the original one due to Pauli.

Pauli's initial idea to assign a two-fold degeneracy was most straightforwardly that of introducing the concept of spin: the connection is very simple, based on the fact that irreducible representations of particles of spin s have exactly $d=2s+1$ independent components; for particles of spin $s=1/2$ this means $d=2/2+1=2$ components, so that it is possible to think that these two components be precisely the two states that account for the double state of multiplicity. Explicitly, recalling (163) we have that

$$\psi=\begin{pmatrix}\pm\phi^{*}\\0\\\phi\\0\end{pmatrix}\quad\text{or}\quad\psi=\begin{pmatrix}0\\\pm\phi^{*}\\0\\\phi\end{pmatrix}\tag{218}$$

where that on the left is a spin $1/2$ eigen-state while that on the right is a spin $-1/2$ eigen-state: that is for either form and in it, for either of the two chiral parts, we have that upper and lower components have an opposite value of the helicity label. For the superposition of two spinors having two opposite helicities we have for example that

$$\psi_{\mathrm{up}}+\psi_{\mathrm{down}}=\begin{pmatrix}\pm\phi^{*}\\\pm\phi^{*}\\\phi\\\phi\end{pmatrix}\tag{219}$$

which is no longer a spin eigen-state but still in an allowed form, while for the superposition of two spinors that are identical in every respect the sum of two solutions cannot be solution, due to the non-linearity of the sources of the geometrical field equations within the entire system.

This seems a dynamical form of Pauli principle.

Another problem that must be addressed by employing alternative methods is the hyper-fine splitting of spectral lines in atomic emission: this phenomenon, which is also known as Lamb shift, is about the fact that in atoms the electronic levels display an energy shift that is observed but which has otherwise no explanation. Nevertheless it was immediately after Lamb discussed this phenomenon that Welton provided an explanation in terms of the fact that electrons may have a finite size [56]. The idea is that electrons with finite size can never be at the very bottom of the Coulomb potential, and the surplus of energy gives the energy shift of the hyper-fine splittings we observe.

The problem of the anomalous magnetic moments also received immediate and close attention, but all attempts made to explain it without quantum fields were affected by alternative arbitrary assumptions, and it was not until recently that a purely classical field theoretical method of calculation has been used [57]: to recall the main idea we have to consider that electro-dynamics is encoded by exact solutions of the inhomogeneous Maxwell equations

$$A_{\nu}=\frac{q}{4\pi}\int\frac{U'_{\nu}}{|\vec{r}-\vec{r}\,'|}d^{3}r'\tag{220}$$

with $\psi'=\psi'(t-|\vec{r}-\vec{r}\,'|,\vec{r}\,')$ retarded potentials, and assuming that $i\boldsymbol{\nabla}_{\mu}\psi=P_{\mu}\psi$ the decomposition (153) becomes

$$P_{\nu}\Phi-\alpha\int\frac{\Phi U'_{\nu}}{|\vec{r}-\vec{r}\,'|}d^{3}r'-\\-\frac{1}{2}\nabla^{\mu}S_{\mu\nu}+XW^{\sigma}\Sigma_{\sigma\nu}-mU_{\nu}=0\tag{221}$$

with $4\pi\alpha=q^2$ as fine-structure constant; we assume that both gravity and torsion will be neglected, and in case of non-relativistic approximation the spinor has small space part of the velocity and small Takabayashi angle. A small space part of the velocity means that the two chiral parts indicated as $\psi_L=\psi$ and $\psi_R=\psi'$ are such that they verify expressions $\Phi'=\Phi$ and $\vec{U}'=-\vec{U}$ telling that two opposite chiralities have equal module but opposite velocity for a given spin state, and a small Takabayashi angle tells that in standard representation the spinor loses the small part and thus we may set $\overline{\psi}=(\phi^{\dagger},0)$ in general; the two chiral parts will be assumed at a distance $|\vec{r}-\vec{r}\,'|=\lambda$ in average equal to the Compton wave-length. This assumption will imply that the above expression in its space part become

$$\vec{\nabla}\times\left(\phi^{\dagger}\frac{\vec{\boldsymbol{\sigma}}}{2}\phi\right)+\vec{P}\Phi-m\vec{U}\left(1-\frac{\alpha}{2\pi}\right)=0\tag{222}$$

as it can be checked in a rather straightforward manner.

As a consequence of this expression, and for the general definition of the magnetic moment, we obtain that

$$\vec{\mu}=\left(\left\langle\frac{\vec{\boldsymbol{\sigma}}}{2}\right\rangle+\frac{1}{2}\langle\vec{r}\times\vec{P}\rangle\right)\frac{q}{2m}2\left(1-\frac{\alpha}{2\pi}\right)^{-1}\tag{223}$$

in which we recognize the average of the angular momentum of the quantum particle and the average of the spin operator, times the Bohr magneton $\frac{q}{2m}$ times the factor

$$g=2\left(1-\frac{\alpha}{2\pi}\right)^{-1}\approx 2\left(1+\frac{\alpha}{2\pi}\right) \tag{224}$$

to first order in the α parameter: this formula allows for a direct interpretation of the factor 2 as due to the double multiplicity of the chiral structure, while the factor given by the term $(1+\frac{\alpha}{2\pi})$ can be read as the unity contribution coming from the mechanical momentum plus a correction given by $\frac{\alpha}{2\pi}$ at first-order perturbative and due to the fact that the two chiral parts have electro-dynamic interaction with average distance of Compton length. However, there is no reason why such a distance should be the Compton length and we regard this as the last missing element [57].

This picture is merely the application to leptons of the picture that is very successful in the case of hadrons, and that is we exploit chiral internal structure to fit anomalies of the magnetic moment of leptons much in the same way in which the quark internal structure gives the anomalies of the magnetic moment of hadrons as known [58].

To an attentive analysis, it is clear that the theoretical and phenomenological problems of field theory commonly solved using field quantization can also be treated without it in a purely classical field theory: it is essential to notice that whether we discussed about issues related to the energy or the Pauli principle, or computations of the Lamb shift or the anomalous magnetic moment, we have systematically used the fact that the spinor is a particle with finite size having internal structure given in terms of two chiral parts in mutual interaction. Or equivalently that the spinor be allowed to display its spin structure.

In the common paradigm of field quantization the spin structure albeit present does not play a full role, as clear from the fact that in this context the Takabayashi angle is always equal to zero since only solutions in plane waves are considered; no internal structure is considered nor an extension, since particles are always taken point-like.

In the theory of quantum fields, electrons are point-like with quantum effects giving an electronic self-interaction in terms of radiative processes involving loops, while here the self-interaction of the spinor should be regarded as a mutual interaction of its two chiral parts giving internal dynamics for extended fields, and consequently allowing the *Zitterbewegung* to actually influence the particles.

The *Zitterbewegung* of classical fields and quantum effects for structureless particles might coincide [59].

Such a parallel has also been discussed in [60].

VII. SPECIAL SITUATIONS

In the previous section we have seen that, in quantum field theory, the approach is that of considering the particle as a point-like object without internal structure by taking into account only plane-wave solutions since these have no Takabayashi angle and then quantize it, while in the approach followed here, we considered no quantization and we allowed the Takabayashi angle to describe internal dynamics: we have shown or discussed that such an internal dynamics between the two chiral parts of the spinor might give rise to effects that recover those due to field quantization. We recalled that the link between the field quantization and *Zitterbewegung* effects is not new as idea, although admittedly it is very little explored.

Aside from the analogies, there are also distinctive differences, for instance, phenomenologically, quantum field theory is very successful while all of the alternatives still find themselves at the state of the art in which quantum field theory was few years after being conceived; however, theoretically, it is not at all clear if quantum field theory makes any sense while *Zitterbewegung* is the result of spin for spinor fields and therefore it is perfectly defined.

Additionally, although the presence of spin for a spinor field is always considered to have a marginal role, in this work we have the possibility to fully take it into account because such a most general theory of spinors is exactly what we have built in the first part of this work.

A. Macroscopic approximations

In the first part we showed how by exploiting the transformation of the spinor we can boost into the rest frame so to have the most general spinor written in the form

$$\psi=\begin{pmatrix}\pm\cos\omega e^{\frac{i}{2}\beta}e^{-\frac{i}{2}\varsigma}\\ \pm\sin\omega e^{\frac{i}{2}\beta}e^{\frac{i}{2}\varsigma}\\ \cos\omega e^{-\frac{i}{2}\beta}e^{-\frac{i}{2}\varsigma}\\ \sin\omega e^{-\frac{i}{2}\beta}e^{\frac{i}{2}\varsigma}\end{pmatrix}e^{i\alpha}\phi$$

with still three rotations free to achieve; by employing at most two of these we can rotate the spin along the third axis giving the spinor in its most general form as either

$$\psi=\begin{pmatrix}\pm e^{\frac{i}{2}\beta}\\ 0\\ e^{-\frac{i}{2}\beta}\\ 0\end{pmatrix}e^{i\alpha}\phi \quad \text{or} \quad \psi=\begin{pmatrix}0\\ \pm e^{\frac{i}{2}\beta}\\ 0\\ e^{-\frac{i}{2}\beta}\end{pmatrix}e^{i\alpha}\phi$$

according to whether the axial-vector is either aligned or anti-aligned with the third axis respectively, and so either way they are eigen-states of the rotation around the third axis, with still one rotation to accomplish. Before we had proceeded in actually performing the last rotation but as of now we will no longer do it, stopping where we are.

The spinor with the structure that is given by either

$$\psi=\begin{pmatrix}\pm e^{\frac{i}{2}\beta}\\ 0\\ e^{-\frac{i}{2}\beta}\\ 0\end{pmatrix}e^{i\alpha}\phi \quad \text{or} \quad \psi=\begin{pmatrix}0\\ \pm e^{\frac{i}{2}\beta}\\ 0\\ e^{-\frac{i}{2}\beta}\end{pmatrix}e^{i\alpha}\phi$$

can be differentiated with respect to the coordinates and

$$i\partial_\mu\psi=(\tfrac{1}{2}\partial_\mu\beta\boldsymbol{\pi}+i\partial_\mu\ln\phi-\partial_\mu\alpha)\psi$$

and given the spinor connection it is possible to build the most general spinorial covariant derivative; when spinor covariant derivatives are substituted into the spinor differential field equations, we obtain that by defining

$$K_{\mu}=2XW_{\mu}+\frac{1}{2}g_{\mu\nu}\varepsilon^{\nu\rho\sigma\alpha}\partial_{\rho}\xi^{k}_{\sigma}\xi^{j}_{\alpha}\eta_{jk}-$$
$$-2(qA^{\rho}+\nabla^{\rho}\alpha)u_{[\rho}v_{\mu]}$$
$$G_{\mu}=-|\xi|^{-1}\xi^{k}_{\mu}\partial_{\alpha}(|\xi|\xi^{\alpha}_{k})-$$
$$-2(qA^{\rho}+\nabla^{\rho}\alpha)u^{\nu}v^{\alpha}\varepsilon_{\mu\rho\nu\alpha}$$

they are given by the following expressions

$$\nabla_{\mu}\beta-K_{\mu}+v_{\mu}2m\cos\beta=0$$
$$\nabla_{\mu}\ln\phi^{2}-G_{\mu}+v_{\mu}2m\sin\beta=0$$

which are the most general form of spinor field equations as it can be seen by re-following the same passages above.

The structure of the field equations has not changed.

But the structure of the external potentials did change and now there is the extra term $\nabla_{\mu}\alpha$ in both: the reason for writing this form is that now, if the gravitational field can be neglected, we may find a global system of Galileian coordinates in which we may set $\alpha=-P_{k}x^{k}$ so that either

$$\psi=\begin{pmatrix}\pm e^{\frac{i}{2}\beta}\\ 0\\ e^{-\frac{i}{2}\beta}\\ 0\end{pmatrix}e^{-iP_{k}x^{k}}\phi \quad\text{or}\quad \psi=\begin{pmatrix}0\\ \pm e^{\frac{i}{2}\beta}\\ 0\\ e^{-\frac{i}{2}\beta}\end{pmatrix}e^{-iP_{k}x^{k}}\phi \tag{225}$$

with spinorial covariant derivatives that are given by

$$i\boldsymbol{\nabla}_{\mu}\psi=(\tfrac{1}{2}\nabla_{\mu}\beta\boldsymbol{\pi}+i\nabla_{\mu}\ln\phi+P_{\mu}-qA_{\mu})\psi \tag{226}$$

in which we see the condition of quantization with differential operators; plugged into the field equations it gives

$$\nabla_{\mu}\beta-2XW_{\mu}+2(qA^{\rho}-P^{\rho})u_{[\rho}v_{\mu]}+v_{\mu}2m\cos\beta=0 \tag{227}$$
$$\nabla_{\mu}\ln\phi^{2}+2(qA^{\rho}-P^{\rho})u^{\nu}v^{\alpha}\varepsilon_{\mu\rho\nu\alpha}+v_{\mu}2m\sin\beta=0 \tag{228}$$

in terms of the remaining external torsion and gauge potentials and with the dependence on the P_{ρ} constants.

Therefore we see now that, in absence of gravitational fields, we can write the spinor with its spinorial covariant derivatives and spinor differential field equations with the structure of plane waves, with the condition of quantization in terms of differential operators (211) as above.

But here this condition is more general because it does contain also gauge potentials and torsion fields.

From the last two equations we may derive expression

$$P^{\nu}=m\cos\beta u^{\nu}+qA^{\nu}+v^{[\nu}u^{\mu]}(\tfrac{1}{2}\nabla_{\mu}\beta-XW_{\mu})+$$
$$+\tfrac{1}{2}\varepsilon^{\nu\rho\sigma\mu}v_{\rho}u_{\sigma}\nabla_{\mu}\ln\phi^{2} \tag{229}$$

giving the explicit form of the momentum and which can also be derived as a combination of specific decompositions of the spinor field equations; other important ones are expression (137) for the axial-vector spin and known as partially-conserved axial-vector current with (135) for the vector velocity called conserved vector current alongside to (142) for the tensor of spin called conserved tensor current: these last two relationships are the conservation laws for the gauge current and the spin but we also have

$$\nabla_{\rho}[\tfrac{1}{4}F^{2}g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha}+$$
$$+\tfrac{1}{4}(\partial W)^{2}g^{\rho\sigma}-(\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha}+$$
$$+M^{2}(W^{\rho}W^{\sigma}-\tfrac{1}{2}W^{2}g^{\rho\sigma})+$$
$$+\tfrac{i}{4}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\nabla}^{\sigma}\psi-\boldsymbol{\nabla}^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi+\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\nabla}^{\rho}\psi-\boldsymbol{\nabla}^{\rho}\overline{\psi}\boldsymbol{\gamma}^{\sigma}\psi)-$$
$$-\tfrac{1}{2}X(W^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\pi}\psi+W^{\rho}\overline{\psi}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi)]=0$$

for the tensor of energy and which does not appear among the decompositions of the spinor field equations.

With the above expressions of the momentum this gets

$$\nabla_{\rho}[\tfrac{1}{4}F^{2}g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha}+$$
$$+\tfrac{1}{4}(\partial W)^{2}g^{\rho\sigma}-(\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha}+$$
$$+M^{2}(W^{\rho}W^{\sigma}-\tfrac{1}{2}W^{2}g^{\rho\sigma})-$$
$$-XW_{\mu}(v^{[\sigma}u^{\mu]}u^{\rho}+v^{[\rho}u^{\mu]}u^{\sigma}+g^{\mu\sigma}v^{\rho}+g^{\mu\rho}v^{\sigma})\phi^{2}+$$
$$+\tfrac{1}{2}\nabla_{\mu}\beta(v^{[\sigma}u^{\mu]}u^{\rho}+v^{[\rho}u^{\mu]}u^{\sigma}+g^{\mu\sigma}v^{\rho}+g^{\mu\rho}v^{\sigma})\phi^{2}+$$
$$+\tfrac{1}{2}(\varepsilon^{\sigma\eta\alpha\mu}v_{\eta}u_{\alpha}u^{\rho}+\varepsilon^{\rho\eta\alpha\mu}v_{\eta}u_{\alpha}u^{\sigma})\nabla_{\mu}\phi^{2}+$$
$$+2\phi^{2}m\cos\beta u^{\sigma}u^{\rho}]=0$$

quite straightforwardly; it can also be written defining

$$E^{\sigma\rho}=\tfrac{1}{2}\nabla_{\mu}\beta(v^{[\sigma}u^{\mu]}u^{\rho}+v^{[\rho}u^{\mu]}u^{\sigma}+g^{\mu\sigma}v^{\rho}+g^{\mu\rho}v^{\sigma})\phi^{2}+$$
$$+\tfrac{1}{2}(\varepsilon^{\sigma\eta\alpha\mu}v_{\eta}u_{\alpha}u^{\rho}+\varepsilon^{\rho\eta\alpha\mu}v_{\eta}u_{\alpha}u^{\sigma})\nabla_{\mu}\phi^{2}+$$
$$+2\phi^{2}m\cos\beta u^{\sigma}u^{\rho} \tag{230}$$

according to

$$\nabla_{\rho}[\tfrac{1}{4}F^{2}g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha}+$$
$$+\tfrac{1}{4}(\partial W)^{2}g^{\rho\sigma}-(\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha}+$$
$$+M^{2}(W^{\rho}W^{\sigma}-\tfrac{1}{2}W^{2}g^{\rho\sigma})-$$
$$-XW_{\mu}(v^{[\sigma}u^{\mu]}u^{\rho}+v^{[\rho}u^{\mu]}u^{\sigma}+$$
$$+g^{\mu\sigma}v^{\rho}+g^{\mu\rho}v^{\sigma})\phi^{2}+E^{\sigma\rho}]=0$$

where $E^{\sigma\rho}$ is the energy tensor of the pure matter field.

If torsion-matter interactions and torsion dynamics are neglected in the conservation law for the energy while the Takabayashi angle dynamics is neglected in the definition of the energy, we may approximate all to the form

$$E^{\sigma\rho}=\tfrac{1}{2}(\varepsilon^{\sigma\eta\alpha\mu}v_{\eta}u_{\alpha}u^{\rho}+\varepsilon^{\rho\eta\alpha\mu}v_{\eta}u_{\alpha}u^{\sigma})\nabla_{\mu}\phi^{2}+$$
$$+2\phi^{2}mu^{\sigma}u^{\rho}$$

with

$$\nabla_{\rho}(\tfrac{1}{4}F^{2}g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha}+E^{\sigma\rho})=0$$

which hold in the situation where internal dynamics are concealed inside the spinor matter field distribution.

We notice that the time-time component of the energy is given by $E^{tt}=2\phi^{2}m$ which is positive and also

$$\nabla_{\rho}(\tfrac{1}{4}F^{2}g^{\rho\sigma}-F^{\rho\alpha}F^{\sigma}{}_{\alpha}+2\phi^{2}mu^{\sigma}u^{\rho}+$$
$$+\tfrac{1}{2}(\varepsilon^{\sigma\eta\alpha\mu}v_{\eta}u_{\alpha}u^{\rho}+\varepsilon^{\rho\eta\alpha\mu}v_{\eta}u_{\alpha}u^{\sigma})\nabla_{\mu}\phi^{2})=0$$

which can be worked out: by employing the gauge field equations and the identity $\nabla_\rho(2\phi^2 u^\rho)=0$ we obtain

$$2\phi^2 m u^\rho \nabla_\rho u^\sigma + \tfrac{1}{2}\nabla_\rho[\nabla_\mu \phi^2 (\varepsilon^{\sigma\eta\alpha\mu} v_\eta u_\alpha u^\rho + \\ + \varepsilon^{\rho\eta\alpha\mu} v_\eta u_\alpha u^\sigma)] = 2\phi^2 q F^{\sigma\alpha} u_\alpha$$

which is the equation of motion for the matter distribution having a non-trivial contribution of the density field and coupled to an external electro-dynamic field.

The additional term $\nabla_\mu \phi^2$ is because the matter distribution is still a field and it is only when the field vanishes everywhere except in localized regions that

$$m u^\rho \nabla_\rho u^\sigma = q F^{\sigma\alpha} u_\alpha$$

or in the free case

$$u^\rho \nabla_\rho u^\sigma = 0$$

as the equation of motion usually known as Newton law expected to appear in macroscopic approximation.

We started from the most general situation, and first we have neglected gravitation to implement the condition of quantization (211) as above, then we neglected torsion and internal dynamics, finally we assumed the field to be localized and without electro-dynamics, and only in this case we reduced to $u^\rho \nabla_\rho u^\sigma = 0$ as the equation of motion, that is the Newton law: we notice that if we were to write non-relativistic cases of Newton gravitational potential and equation of motion, the term $2\phi^2 m$ is both the energy and the mass density, or equivalently both gravitational and inertial mass density, a fact known as the equivalence of gravitational and inertial mass, established with great experimental accuracy. This equivalence of gravitational and inertial masses is said weak equivalence principle, in parallel to the fact that the equivalence of gravitational and inertial accelerations is the equivalence principle, but despite it is ubiquitously told that the equivalence principle implies the weak equivalence principle, this is true when Newton law is assigned and in this case solely.

As already remarked, the principle of equivalence is a general consequence of the theory, and the accuracy with which the weak equivalence principle is confirmed should not be taken as confirmation of the equivalence principle, for which no proof is required, but as confirmation of the fact that the macroscopic approximation is good.

Finally, we recall that above we have suggested that the problem with energy not being positive may be due to energies that are negative as the contributions of internal dynamics, so only if these are hidden can positive energy be ensured, which is what we proved now [55].

B. Singularity

Having discussed macroscopic approximations resulting from concealing internal structures and disregarding the shape of the matter distribution, we want to address the opposite problem: given that the matter field is very well approximated by a localized distribution, it might be possible that this localization takes the distribution down to point-like objects so much that singularities may form according to the Hawking-Penrose theorem. Henceforth, we want to see what happens if torsion is not neglected.

As a matter of fact, this study has already been started by Kerlick [61], but demonstrating that when in gravity also torsion is considered the singularity formation is not avoided, rather it is enhanced; however, this result is due to the fact that the torsionally-induced spin-spin forces are intrinsically repulsive, with the consequence that they have a positive potential increasing the energy content of the space-time, and thus the possibility of singularities.

This happens to be the case because Kerlick considers the simplest generalization of Einstein gravity, where the torsion does not propagate, and in which the coupling is taken to be the Newton constant, and this results into a spin-spin force having a weak and repulsive character.

On the other hand, however, we have been discussing above that, first of all, in a more general theory of torsion gravity, the torsion-spin coupling is in no way related to the curvature-energy coupling, thus there is no reason for them to be equal, and not only the torsion-spin coupling is allowed to be different from the Newton constant but additionally it is allowed to have opposite sign; moreover, we also remarked that in the most general theory in which torsion propagates, the torsion-spin coupling necessarily has an opposite sign in the effective case. Consequently, the effective forces will be intrinsically attractive, so that their negative potential will decrease the energy content of the space-time, and singularities may be avoided.

To put words in expressions, take (131) contracted as

$$-R - 4\Lambda = \tfrac{k}{2}(-M^2 W^2 + m\Phi) \tag{231}$$

where (134) was used; when this form is plugged into the original equations we obtain that they become

$$R^{\rho\sigma} + \Lambda g^{\rho\sigma} = \tfrac{k}{2}[\tfrac{1}{4}F^2 g^{\rho\sigma} - F^{\rho\alpha}F^\sigma{}_\alpha + \\ + \tfrac{1}{4}(\partial W)^2 g^{\rho\sigma} - (\partial W)^{\sigma\alpha}(\partial W)^\rho{}_\alpha + M^2 W^\rho W^\sigma + \\ + \tfrac{i}{4}(\overline{\psi}\boldsymbol{\gamma}^\rho \boldsymbol{\nabla}^\sigma \psi - \boldsymbol{\nabla}^\sigma \overline{\psi}\boldsymbol{\gamma}^\rho \psi + \overline{\psi}\boldsymbol{\gamma}^\sigma \boldsymbol{\nabla}^\rho \psi - \boldsymbol{\nabla}^\rho \overline{\psi}\boldsymbol{\gamma}^\sigma \psi) - \\ - \tfrac{1}{2}X(W^\sigma V^\rho + W^\rho V^\sigma) - \tfrac{1}{2}m\Phi g^{\rho\sigma}] \tag{232}$$

equivalent to those in the original form. But this form is best suited for application to the singularity theorem.

In fact, for the singularity theorem in Einstein gravity

$$R^{\rho\sigma} u_\rho u_\sigma \geqslant 0 \tag{233}$$

known as the strongest energy condition: neglecting the cosmological constant and electro-dynamics we obtain

$$[\tfrac{1}{4}(\partial W)^2 g^{\rho\sigma} - (\partial W)^{\sigma\alpha}(\partial W)^\rho{}_\alpha + M^2 W^\rho W^\sigma + \\ + \tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^\rho \boldsymbol{\nabla}^\sigma \psi - \boldsymbol{\nabla}^\sigma \overline{\psi}\boldsymbol{\gamma}^\rho \psi) - \tfrac{1}{2}m\Phi g^{\rho\sigma}] u_\rho u_\sigma \geqslant 0 \tag{234}$$

which in the effective approximation becomes

$$\tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^0 \boldsymbol{\nabla}_0 \psi - \boldsymbol{\nabla}_0 \overline{\psi}\boldsymbol{\gamma}^0 \psi) - \tfrac{1}{2}m\Phi \geqslant 0 \tag{235}$$

and because (133) in the effective approximation is

$$i\boldsymbol{\gamma}^{0}\boldsymbol{\nabla}_{0}\psi+i\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{\nabla}}\psi-\frac{X^{2}}{M^{2}}V_{\sigma}\boldsymbol{\gamma}^{\sigma}\boldsymbol{\pi}\psi-m\psi=0 \quad (236)$$

we may use this into the above to get

$$\frac{i}{2}(\vec{\boldsymbol{\nabla}}\overline{\psi}\cdot\vec{\boldsymbol{\gamma}}\psi-\overline{\psi}\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{\nabla}}\psi)+\frac{X^{2}}{M^{2}}V_{\sigma}V^{\sigma}+\frac{1}{2}m\Phi\geqslant 0 \quad (237)$$

whose structure is similar to the condition of Kerlick but with the sign of the non-linear interaction inverted.

We may now follow Kerlick argument by neglecting the derivative term, and by employing (97), we get that

$$-4\frac{X^{2}}{M^{2}}\phi^{4}+m\phi^{2}\cos\beta\geqslant 0 \quad (238)$$

which for specific Takabayashi angles, or in general for large densities, can be violated, and quite easily too.

This is already an improvement compared to Kerlick's model but we would press further and see what happens when no effective approximation is done: in this case

$$[\tfrac{1}{4}(\partial W)^{2}g^{\rho\sigma}-(\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha}+M^{2}W^{\rho}W^{\sigma}+$$
$$+\tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^{\rho}\boldsymbol{\nabla}^{\sigma}\psi-\boldsymbol{\nabla}^{\sigma}\overline{\psi}\boldsymbol{\gamma}^{\rho}\psi)-\tfrac{1}{2}m\Phi g^{\rho\sigma}]u_{\rho}u_{\sigma}\geqslant 0 \quad (239)$$

or equivalently

$$\tfrac{1}{4}(\partial W)^{2}-(\partial W)^{0k}(\partial W)^{0}{}_{k}+M^{2}W^{0}W^{0}+$$
$$+\tfrac{i}{2}(\overline{\psi}\boldsymbol{\gamma}^{0}\boldsymbol{\nabla}_{0}\psi-\boldsymbol{\nabla}_{0}\overline{\psi}\boldsymbol{\gamma}^{0}\psi)-\tfrac{1}{2}m\Phi\geqslant 0 \quad (240)$$

and with (133) we obtain

$$\tfrac{1}{4}(\partial W)^{2}-(\partial W)^{0k}(\partial W)^{0}{}_{k}+M^{2}W^{0}W^{0}+$$
$$+\tfrac{i}{2}(\vec{\boldsymbol{\nabla}}\overline{\psi}\cdot\vec{\boldsymbol{\gamma}}\psi-\overline{\psi}\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{\nabla}}\psi)+XW_{\sigma}V^{\sigma}+\tfrac{1}{2}m\Phi\geqslant 0 \quad (241)$$

which is indeed more general than Kerlick's condition.

With no other approximation to do, and even with all derivative terms positive defined, it is straightforward to see that because of presence of the torsion-spin coupling and the Takabayashi angle, the energy condition need not be verified and the singularity is no longer a necessity.

As a consequence, the widely-spread claims that gravitation would break-down due to singularity formation at high energy, and which appear to be worsened in torsion gravity, are no longer supported if propagating torsion in gravitational backgrounds is properly considered and for a treatment that is not too strong in approximation [62].

When before we talked about the Pauli principle of exclusion we did not consider possible applications such as the stability of neutrons stars due to degeneracy pressure, and it is intriguing that torsion-spin coupling be capable of mimicking degeneracy pressures to such extent.

VIII. COMMENTS

In this second part, we started by re-calling, beginning from the pre-history of quantum theories, the foundations of the quanta and their mathematical implementation in the fiber of our models: we have proceeded to critically discuss the weak points, deconstructing the implementation of quantum protocols, to find that alternatives could be obtained, by exploiting the spin content. We have discussed the approximation in which the internal dynamics is neglected, as a basis for the macroscopic approximation that naturally followed: we have seen that in the theory with propagating torsion in gravitational backgrounds no singularity need form. Some comment is now in order.

The key point of this part is that there are theoretical as well as phenomenological issues that pushes us toward corrections of field theory: the usual paradigm is that of quantum field theory, the theory of fields with implementation of field quantization, where particles are taken to be point-like, mathematically realized in the employment of plane waves, and expanding the interactions in terms of radiative processes involving a definite number of loop diagrams; in what we have presented, no quantization is implemented, and particles are considered to have a finite size, with internal structure given in terms of two chiral parts in interaction through torsion, that is we imposed no quantization, but we insisted on internal dynamics.

It may be that in quantum field theory the process of field quantization is needed as a supplement for the loss of information that is due to the fact that the Takabayashi angle is systematically neglected, so there may be no need for it if the Takabayashi angle is allowed; and in general the Takabayashi angle is different from zero indeed.

The link between *Zitterbewegung* effects of the internal dynamics and quantum aspects may be profound.

THREE: PHYSICS

IX. STANDARD MODELS

In this third part we apply the field equations and the Lagrangians we have obtained above, to investigate six different known open problems in the standard models.

A. Axial-vector interactions

Throughout the entire presentation, we have been considering single spinor fields, but clearly the treatment of two spinor fields, or even more spinor fields, is doable, and it is achieved by replicating the spinor field Lagrangian as many times as the number of independent spinor fields.

For instance, in the case of two spinor fields we have

$$\mathscr{L}=-\tfrac{1}{4}(\partial W)^{2}+\tfrac{1}{2}M^{2}W^{2}-\tfrac{1}{k}R-\tfrac{2}{k}\Lambda-\tfrac{1}{4}F^{2}+$$
$$+i\overline{\psi}_{1}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi_{1}+i\overline{\psi}_{2}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi_{2}+$$
$$-X_{1}\overline{\psi}_{1}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi_{1}W_{\mu}-X_{2}\overline{\psi}_{2}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi_{2}W_{\mu}+$$
$$-m_{1}\overline{\psi}_{1}\psi_{1}-m_{2}\overline{\psi}_{2}\psi_{2} \quad (242)$$

and taking the variation with respect to torsion gives

$$\nabla_{\nu}(\partial W)^{\nu\mu}+M^{2}W^{\mu}=X_{1}\overline{\psi}_{1}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi_{1}+$$
$$+X_{2}\overline{\psi}_{2}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi_{2} \quad (243)$$

as the torsion field equations with two sources: if we consider the effective approximation we obtain expressions

$$M^2 W^\mu \approx X_1 \overline{\psi}_1 \boldsymbol{\gamma}^\mu \boldsymbol{\pi} \psi_1 + X_2 \overline{\psi}_2 \boldsymbol{\gamma}^\mu \boldsymbol{\pi} \psi_2 \tag{244}$$

which can be plugged back into the Lagrangian giving

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{k}R - \tfrac{2}{k}\Lambda - \tfrac{1}{4}F^2 + i\overline{\psi}_1 \boldsymbol{\gamma}^\mu \boldsymbol{\nabla}_\mu \psi_1 + i\overline{\psi}_2 \boldsymbol{\gamma}^\mu \boldsymbol{\nabla}_\mu \psi_2 + \\ &+\tfrac{1}{2}\left|\tfrac{X_1}{M}\right|^2 \overline{\psi}_1 \boldsymbol{\gamma}^\mu \psi_1 \overline{\psi}_1 \boldsymbol{\gamma}_\mu \psi_1 + \tfrac{1}{2}\left|\tfrac{X_2}{M}\right|^2 \overline{\psi}_2 \boldsymbol{\gamma}^\mu \psi_2 \overline{\psi}_2 \boldsymbol{\gamma}_\mu \psi_2 - \\ &-\tfrac{X_1}{M}\tfrac{X_2}{M} \overline{\psi}_1 \boldsymbol{\gamma}^\mu \boldsymbol{\pi} \psi_1 \overline{\psi}_2 \boldsymbol{\gamma}_\mu \boldsymbol{\pi} \psi_2 - m_1 \overline{\psi}_1 \psi_1 - m_2 \overline{\psi}_2 \psi_2\end{aligned} \tag{245}$$

in which each spinor has the self-interaction but between the two spinors there is also a mutual interaction.

The extension to three spinor fields, or n spinor fields, is similar: there are n self-interactions, always attractive, and $\frac{1}{2}n(n-1)$ mutual interactions, being either attractive or repulsive according to $X_i X_j$ being positive or negative.

This extension is interesting for $n = 3$ because such is the situation we have for neutrinos: by neglecting all interactions apart from the effective interactions, and in them neglecting the self-interaction so to have only the mutual interactions, one may calculate the Hamiltonian

$$\mathscr{H} = \sum_{ij} \overline{\nu}_i (U_{ij} - X_i X_j \boldsymbol{\gamma}^\mu \boldsymbol{\pi} \nu_i \overline{\nu}_j \boldsymbol{\pi} \boldsymbol{\gamma}_\mu) \nu_j \tag{246}$$

where the Latin indices run over the three labels associated to the three different flavours of neutrinos; then the matrix $U_{ij} - X_i X_j \boldsymbol{\gamma}^\mu \boldsymbol{\pi} \nu_i \overline{\nu}_j \boldsymbol{\pi} \boldsymbol{\gamma}_\mu$ is the combination of the constant matrix U_{ij} describing kinematic phases that arise from the mass terms as usual plus the field-dependent matrix $X_i X_j \boldsymbol{\gamma}^\mu \boldsymbol{\pi} \nu_i \overline{\nu}_j \boldsymbol{\pi} \boldsymbol{\gamma}_\mu$ describing the dynamical phases that arise from the torsionally-induced non-linear potentials that pertain to this theory.

Dealing with the non-linear potentials is problematic, but in reference [63] this problem is solved by taking neutrinos dense enough to make the torsion field background homogeneous and thus constant: the phase difference is

$$\Delta\Phi \approx \left(\frac{\Delta m^2}{2E} + \frac{1}{4}|W^0 - W^3|\right) L \tag{247}$$

having assumed $W_1 = W_2 = 0$ and where L is the length of the oscillations. In [64] it was seen that (247) in the case in which the neutrino mass difference is small becomes

$$\Delta\Phi \approx \left(\Delta m^2 + m\frac{X_{\text{eff}}^2}{4M^2}|\overline{\nu}\gamma_\mu \nu \overline{\nu}\gamma^\mu \nu|^{\frac{1}{2}}\right) \frac{L}{2E} \tag{248}$$

where m is the value of the nearly-equal masses of neutrinos while X_{eff}^2 is a combination of the coupling constants and with the dependence L/E as the ratio between length and energy of the oscillations as it is well expected.

The phase difference due to the oscillation has the kinematic contribution, as difference of the squared masses, plus a dynamic contribution, proportional to the neutrino mass density distribution: the novelty torsion introduces is that even in the case in which neutrino masses were to be non-zero but with insufficient non-degeneracy in mass spectrum, we might still have oscillations, and therefore ampler margin of freedom before having some tension.

Notice also that both m and X_{eff}^2 depend on the masses and coupling constants of the two neutrinos involved and so that they would be different for another pair of neutrinos, making it clear how the parameters of the oscillation depend on the specific pair of neutrinos, as should be.

If torsion can have effects for neutrino oscillation, then it may also affect the weak interactions among all leptons we know, that is between neutrinos and charged leptons.

The Lagrangian of the standard model [46] is such that after the symmetry breaking it can be written as

$$\begin{aligned}\mathscr{L}^{\text{SM}} = &\tfrac{1}{2}\lambda^2 v^4 - m_H^2\left(\tfrac{H^2+4vH}{4v^2}\right)H^2 - \tfrac{m_e}{v}H\overline{e}e + \\ &+(m_W^2 W^-_\mu W^{\mu+} + \tfrac{1}{2}m_Z^2 Z^2)\left(\tfrac{H^2+2vH}{v^2}\right) + \\ &+\tfrac{1}{4}g^2(W^{\mu-}W^{\nu+} - W^{\nu-}W^{\mu+})\left(W^-_\mu W^+_\nu - W^-_\nu W^+_\mu\right) + \\ &+(g\cos\theta)^2\left(W^+_\mu W^-_\nu Z^\nu Z^\mu - W^-_\mu W^{\mu+} Z^2\right) + \\ &+\tfrac{i}{2}\left(W^+_\nu W^-_\mu - W^+_\mu W^-_\nu\right)\left(g\cos\theta \nabla^{[\mu}Z^{\nu]} - qF^{\mu\nu}\right) + \\ &+ig\cos\theta Z_\nu\left(W^+_\mu \nabla^{[\mu}W^{\nu]-} - W^-_\mu \nabla^{[\mu}W^{\nu]+}\right) + \\ &+\tfrac{g}{\sqrt{2}}\left(W^-_\mu \overline{\nu}\boldsymbol{\gamma}^\mu e_L + W^+_\mu \overline{e}_L \boldsymbol{\gamma}^\mu \nu\right) + \\ &+\tfrac{g}{\cos\theta}Z_\mu\left[\tfrac{1}{2}\left(\overline{\nu}\boldsymbol{\gamma}^\mu\nu - \overline{e}_L\boldsymbol{\gamma}^\mu e_L\right) + |\sin\theta|^2 \overline{e}\boldsymbol{\gamma}^\mu e\right] - \\ &-\tfrac{1}{4}F^2 - \tfrac{1}{4}\left(\nabla_{[\mu}Z_{\nu]}\nabla^{[\mu}Z^{\nu]}\right) + \tfrac{1}{2}m_Z^2 Z^2 - \\ &-\tfrac{1}{2}(\nabla_{[\mu}W^+_{\nu]}\nabla^{[\mu}W^{\nu]-}) + m_W^2 W^-_\mu W^{\mu+} + \\ &+\left(\nabla_\mu H \nabla^\mu H - m_H^2 H^2\right) + \\ &+\left(i\overline{e}\boldsymbol{\gamma}^\mu \boldsymbol{\nabla}_\mu e - m_e \overline{e}e\right) + \\ &+i\overline{\nu}\boldsymbol{\gamma}^\mu \boldsymbol{\nabla}_\mu \nu\end{aligned} \tag{249}$$

where ∇_μ and $\boldsymbol{\nabla}_\mu$ are the general tensorial and spinorial covariant derivatives of the remaining abelian symmetry which, consequently, accounts for the electro-dynamic interaction of all the fields that carry an electric charge.

We already discussed the impact on the Lagrangian of the effective Higgs field, and in the following we focus on the effective weak interactions as also these are mediated by two massive bosons: by taking the field equations for the vector bosons in the effective approximation we get

$$m_Z^2 Z_\mu = -\tfrac{g}{\cos\theta}\left[\tfrac{1}{2}(\overline{\nu}\boldsymbol{\gamma}_\mu\nu - \overline{e}_L\boldsymbol{\gamma}_\mu e_L) + |\sin\theta|^2\overline{e}\boldsymbol{\gamma}_\mu e\right] \tag{250}$$

$$m_W^2 W^+_\mu = -\tfrac{g}{\sqrt{2}}\overline{\nu}\boldsymbol{\gamma}_\mu e_L \tag{251}$$

which can be substituted within the Lagrangian, and by employing (87) as usual, one gets the effective Lagrangian

$$\begin{aligned}\Delta\mathscr{L}^{\text{SM}}_{\text{effective}} = &-\tfrac{g^2}{2m_Z^2}|\sin\theta|^2|\tan\theta|^2 \overline{e}\boldsymbol{\gamma}^\mu e \overline{e}\boldsymbol{\gamma}_\mu e - \\ &-\tfrac{g^2}{2m_Z^2}|\tan\theta|^2\left(\overline{\nu}\boldsymbol{\gamma}^\mu\nu - \overline{e}_L\boldsymbol{\gamma}^\mu e_L\right)\overline{e}\boldsymbol{\gamma}_\mu e + \\ &+\left(\tfrac{g^2}{2m_W^2} + \tfrac{g^2}{4m_Z^2|\cos\theta|^2}\right)\overline{e}_L\boldsymbol{\gamma}^\mu e_L \overline{\nu}\boldsymbol{\gamma}_\mu\nu\end{aligned} \tag{252}$$

in terms of the effective coupling and the mixing angle, and showing that the weak interactions are repulsive.

On the other hand, the Lagrangian for the two spinorial fields, neglecting the gravitational contributions, and calling the spinors e and ν just for clarity, is given by

$$\mathscr{L}^{\mathrm{Q}}=-\tfrac{1}{4}(\partial W)^2+\tfrac{1}{2}M^2W^2-\tfrac{1}{4}F^2-\tfrac{1}{4}G^2+\\ +i\overline{e}\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu e+i\overline{\nu}\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\nu+\\ -X_e\overline{e}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}eW_\mu-X_\nu\overline{\nu}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\nu W_\mu+\\ -m_e\overline{e}e-m_\nu\overline{\nu}\nu \tag{253}$$

in which we have indicated as F^2 and G^2 all gauge terms referring to the $U(1)$ and the $SU(2)$ group respectively, and with effective approximation that is given by

$$\Delta\mathscr{L}^{\mathrm{Q}}_{\mathrm{effective}}=\tfrac{1}{2}\tfrac{X_e^2}{M^2}\overline{e}\boldsymbol{\gamma}^\mu e\overline{e}\boldsymbol{\gamma}_\mu e+\tfrac{1}{2}\tfrac{X_\nu^2}{M^2}\overline{\nu}\boldsymbol{\gamma}^\mu\nu\overline{\nu}\boldsymbol{\gamma}_\mu\nu-\\ -\tfrac{X_eX_\nu}{M^2}\overline{e}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}e\overline{\nu}\boldsymbol{\gamma}_\mu\boldsymbol{\pi}\nu \tag{254}$$

in which we have profited of the identity (97) once again and where $\Delta\mathscr{L}$ means we consider only the spinor sector.

To meaningfully compare this Lagrangian against the Lagrangian of the standard model, it is compulsory that the two Lagrangians have the same field content, and as in the standard model all the neutrinos are massless and left-handed the same should be for this Lagrangian: after re-arranging chiral parts according to (87) we get

$$\Delta\mathscr{L}^{\mathrm{Q-reduced}}_{\mathrm{effective}}=\tfrac{X_e(X_e+X_\nu)}{2M^2}\overline{e}\boldsymbol{\gamma}^\mu e\overline{e}\boldsymbol{\gamma}_\mu e+\\ +\tfrac{X_eX_\nu}{M^2}(\overline{\nu}\boldsymbol{\gamma}_\mu\nu-\overline{e}_L\boldsymbol{\gamma}^\mu e_L)\overline{e}\boldsymbol{\gamma}^\mu e-\\ -2\tfrac{X_eX_\nu}{M^2}\overline{e}_L\boldsymbol{\gamma}^\mu e_L\overline{\nu}\boldsymbol{\gamma}_\mu\nu \tag{255}$$

reduced indeed to the same field content of the standard model Lagrangian and so they can be compared.

To actually do such a comparison, we take the standard model complemented with torsional effects, so that

$$\Delta\mathscr{L}^{\mathrm{total-reduced}}_{\mathrm{effective}}=\Delta\mathscr{L}^{\mathrm{SM}}_{\mathrm{effective}}+\Delta\mathscr{L}^{\mathrm{Q-reduced}}_{\mathrm{effective}}=\\ =-\left(\tfrac{g^2}{2m_Z^2}|\sin\theta|^2|\tan\theta|^2-\tfrac{X_e(X_e+X_\nu)}{2M^2}\right)\overline{e}\boldsymbol{\gamma}^\mu e\overline{e}\boldsymbol{\gamma}_\mu e-\\ -\left(\tfrac{g^2}{2m_Z^2}|\tan\theta|^2-\tfrac{X_eX_\nu}{M^2}\right)(\overline{\nu}\boldsymbol{\gamma}^\mu\nu-\overline{e}_L\boldsymbol{\gamma}^\mu e_L)\,\overline{e}\boldsymbol{\gamma}_\mu e+\\ +\left(\tfrac{g^2}{2m_W^2}+\tfrac{g^2}{4m_Z^2|\cos\theta|^2}-2\tfrac{X_eX_\nu}{M^2}\right)\overline{e}_L\boldsymbol{\gamma}^\mu e_L\overline{\nu}\boldsymbol{\gamma}_\mu\nu \tag{256}$$

showing that in such a case the torsionally-induced spin-contact interaction affects the standard model by correcting its coupling constants as to make them weaker, which is reasonable since torsion is attractive whereas the weak interactions are repulsive. And nevertheless we have that the precision with which these constants are now known imposes the torsional corrections to the standard model to be very small, which is a hint of the fact that either all coupling constants are small or the torsion mass is large.

Or yet again, it might suggest that the approximations with which we worked have been too strong, and that is we should not have considered effective approximations.

Without any effective approximation and allowing sterile right-handed neutrinos, the torsional correction to the standard model in the spinor sector is described by

$$\Delta\mathscr{L}^{\mathrm{total}}=-X_e\overline{e}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}eW_\mu-X_\nu\overline{\nu}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\nu W_\mu+\\ +\tfrac{g}{\sqrt{2}}\left(W^-_\mu\overline{\nu}\boldsymbol{\gamma}^\mu e_L+W^+_\mu\overline{e}_L\boldsymbol{\gamma}^\mu\nu\right)+\\ +\tfrac{g}{\cos\theta}Z_\mu[\tfrac{1}{2}\left(\overline{\nu}\boldsymbol{\gamma}^\mu\nu-\overline{e}_L\boldsymbol{\gamma}^\mu e_L\right)+|\sin\theta|^2\overline{e}\boldsymbol{\gamma}^\mu e] \tag{257}$$

in which care has to be taken in distinguishing the weak boson from torsion (the reason why I used the same letter for both is that when defining torsion I did not intend to study the standard model); clearly if the neutrino is not left-handed (as it should be to grant the mass from which we obtain the kinematic mechanism that is needed to fit oscillations), the two types of neutrino coupling are quite different, according to whether we consider the torsion or the weak interactions: the fact that a sterile neutrino is by construction insensitive to weak interactions but it is sensitive to the universal torsion interaction is the single most important reason for such type of discrepancy.

If we are under the conditions where we can perform an effective approximation, torsionally-induced spin-contact interactions can be re-arranged as to become structurally identical to the weak forces, so that they affect the weak sector only by shifting the coupling constants, but this is already a problem since such a change would be visible in the clean environment of leptonic scattering; on the other hand, if we cannot perform this approximation therefore staying in the most general case, the propagating torsion does not even have the structure of the weak interactions, and its presence is even clearer: as a consequence, it is not likely to find torsional corrections to weak interactions in the case in which only lepton fields are considered.

Hence, if we want to have some hope of finding torsion we should look beyond the standard model.

Before proceeding further to the problem of looking for torsion beyond the standard model, it may be instructive to pause and ask if it is possible that torsion may already be around at the present energy scale, although somehow hidden: we have argued that torsion does not affect weak interactions in the leptonic sector because we would have already seen it in the clear lepton scattering, but torsion may affect the weak interactions in the case of quarks or influence chromo-dynamics and still be hidden in the messy environment involving nucleons. Therefore, it may be that torsion is already present although not manifest.

The question we are asking now is: granted that in the case of leptons all the torsion coupling constants must be small as to comply with very stringent limits, still in the case of quarks the torsion coupling constants are allowed to be larger since experimental limits are less strict, and therefore could it be that in the case of quarks the torsion coupling constants happen to be larger, and in fact large enough to take place beside the weak forces and chromo-dynamics, changing some physical quantity for nucleons?

The answer requires mean beyond my capabilities, but still I am tempted to give a sort of reasonable speculation.

Recently a group of researchers [65] measured the proton radius obtaining a value that was more than five standard deviations off theoretical predictions; this "proton radius problem" may of course be tied to the appearance of new physics, as it is normally expected, but clearly this new physics may actually be some old physics in disguise and what we have in mind is the presence of torsion.

In fact the torsion field, being present, would superpose to chromo-dynamic interactions, and being attractive, it would result into an apparent strengthening of the binding potential, and the subsequent shrinking of the region in which the matter field distribution finds place.

And of course it would not be necessary to invoke this as a new physical effect, torsion being naturally present in the most general geometric background.

After this brief speculation about the way torsion could superpose to chromo-dynamics to change the parton distribution, we go back to our line of thought, and wonder if torsion gives rise to effects beyond the standard model.

The most immediate place in which to look is the standard model of cosmology, and more specifically the dark matter sector; there are a few assumptions we will make throughout this section, quite reasonable nonetheless.

A zeroth point is, of course, that despite we still do not exactly know what dark matter is, nevertheless it has to be a form of matter: albeit many models may fit galactic behaviour in the case of rotations, only dark matter as a real form of matter fits galactic behaviours that are less trivial like those encountered in crossing galaxies [66].

A first point is that, given dark matter as matter, massive and very weakly interacting, we will also assume it to be a $\frac{1}{2}$-spin spinor field: there is no specific reason for this apart from the fact that most attempts to describe dark matter are based on this assumption, and by assuming it we place ourselves in a good position for comparison.

The fact we have assumed dark matter be a spinor also makes it prone to have the torsional interactions in which we are interested: in [67] torsion effects have been studied in a classical context to see how galactic dynamics could be modified by torsion, and in [68] we have applied those results to the case in which torsion was coupled to spinors to see how galactic dynamics could be modified by torsion and how torsion could be sourced by dark matter.

So here as before, torsion is not used as an alternative but as a correction over pre-existing physics. Having this in mind, we recall that in [68] we showed how, if spinors are the source of torsion, the gravitational field in galaxies turns out to be increased: from (131) we see that in the case of the effective approximation (170) we get

$$R^{\rho\sigma}-\frac{1}{2}Rg^{\rho\sigma}-\Lambda g^{\rho\sigma}=\frac{k}{2}(E^{\rho\sigma}-\frac{1}{2}\frac{X^2}{M^2}V^{\mu}V_{\mu}g^{\rho\sigma}) \quad (258)$$

showing that the spinor field with the torsionally-induced non-linear interactions has an effective energy which is written as the usual term plus a non-linear contribution.

For this contribution we have to recall that we are not considering a single spinor field, as we have done when in particle physics, but collective states of spinor fields, as it is natural to assume in cosmology, with the consequence that it is not possible to employ the re-arrangements we used before and thus $V^{\mu}V_{\mu}$ cannot be reduced: generally we do not know how to compute it, but we know it is the square of a density, and it may turn out to be positive.

In reference [68] we have been discussing precisely what would happen if the spin density square happened to be positive, and we have found that the contribution to the energy would change the gravitational field as to allow for a constant behaviour of the rotation curves of galaxies, discussing the value of the torsion-spin coupling constant that is required to fit the galactic observations.

The details of the calculations were based on the fact that in this occurrence and within the approximations of slow rotational velocity and weak gravitational field, the acceleration felt by a point-particle was given by

$$\mathrm{div}\vec{a}\approx -m\rho-K^2\rho^2 \quad (259)$$

in which the Newton gravitational constant has been normalized and where K is the effective value of the torsional constant, with constant tangential velocity obtained for densities scaling down as r^{-2} in general: in the standard approach to dark matter there are only Newtonian source contributions scaling down as r^{-1} and so a modification to the density distribution has to be devised, and it is the well known Navarro-Frenk-White profile; in the presence of torsional corrections, the Newtonian profile suffices because even if the density drops as r^{-1} it is squared in the torsional correction and thus the r^{-2} drop is obtained.

This suggests that the torsion correction may be what gives the Navarro-Frenk-White profile: after all the NFW profile is obtained in n-body dynamics as those assumed here provided that the n spinors interact through torsion.

Nor is it unexpected the idea of modelling dark matter, through the NFW profile, in terms of torsion, since this is precisely what a specific type of effective theories does.

In quite recent years there has been a shift of approach in looking for physics beyond the standard model, and in particular dark matter: the new way of tackling the issue is based on the idea of studying all types of effective interactions that can be put in a Lagrangian, and among all of them there is the axial-vector spin-contact interaction.

However, in even more recent years this approach has been generalized, shifting the attention from the effective interactions to the mediated interactions, known as simplified models [69], but the story does not change, since among all these there is the axial-vector mediated term

$$\Delta\mathscr{L}=-g\overline{\chi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\chi B_{\mu} \quad (260)$$

where χ is the dark matter particle and B_{μ} is the axial-vector mediator, and where the structure of the interaction is that of the torsion-spin coupling, as it should be quite easily recognizable for the reader at this moment.

Since when the standard model has been acknowledged to need a complementation, we have been striving to have it placed within a more general model, which should have contained also some new physics, and in particular dark matter; it has been the constant failure in this project that prompted us to reverse the strategy, pushing us to look for simplified models, namely models that can immediately describe dark matter, or in general new physics, and leaving the task of including them, together with the standard model, into a more general model for later, and better, times: therefore, if we were to see that the dark matter, or generally some new physics, were actually described by one of these simplified models, the following step would be to include it beside the standard model within a more general model, and at this point it should be clear what is our ultimate claim for this entire section.

Our claim is that if such a simplified model is the one described by the axial-vector mediator, then we will need not look very far: the general model would be torsion.

And torsion also has yet another role for cosmology, as we are going to discuss in the following section.

B. Scalar potential

We have just seen some torsional effect for the domain of particle physics in the case of a system of many particles at a galactic level, and we next move to investigate a more direct effect concerning a cosmological situation.

To begin our investigation, the very first thing we want to do is remarking that, as the reader may have noticed, we never treated the scalar field; the reason was merely to keep an already heavy presentation from being heavier still, but it is now time to put some scalar field in.

Lagrangian (168) complemented with a scalar field is

$$\begin{aligned}\mathscr{L}=&-\tfrac{1}{4}(\partial W)^2+\tfrac{1}{2}M^2W^2-\tfrac{1}{k}R-\tfrac{2}{k}\Lambda-\tfrac{1}{4}F^2+\\&+i\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi+\nabla^\mu\phi^\dagger\nabla_\mu\phi-\\&-X\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi W_\mu-\tfrac{1}{2}\Xi\phi^2W^2-Y\overline{\psi}\psi\phi-\\&-m\overline{\psi}\psi+\mu^2\phi^2-\tfrac{1}{2}\lambda^2\phi^4\end{aligned}\tag{261}$$

where the X, Ξ, Y are the coupling constants related to the torsion with spinor and scalar interactions.

It is interesting to notice that in this complementation there is also the term ϕ^2W^2 which couples torsion to the scalar: this may look weird, since torsion is supposed to be sourced by the spin density, which is equal to zero for scalar fields. Therefore we should expect to have torsion without a pure source of scalar fields, although we will have scalar contributions in the torsional field equations.

In fact, upon variation of the Lagrangian, we obtain

$$\nabla_\alpha(\partial W)^{\alpha\nu}+(M^2-\Xi\phi^2)W^\nu=X\overline{\psi}\boldsymbol{\gamma}^\nu\boldsymbol{\pi}\psi\tag{262}$$

in which there is indeed a scalar contribution, although in the form of an interaction giving an effective mass term.

There is, immediately, something rather striking about this expression: in a cosmic scenario, for a universe in a FLRW metric, we would have that the torsion, to respect the same symmetries of isotropy and homogeneity, would have to possess only the temporal component, but in this case the dynamical term would disappear leaving

$$(M^2-\Xi\phi^2)W^\nu=X\overline{\psi}\boldsymbol{\gamma}^\nu\boldsymbol{\pi}\psi\tag{263}$$

as the torsion field equations we would have had in the effective limit, though now the result is exact. The source would have to be the sum of the spin density of all spinors in the universe, and because the spin vector points in all directions, statistically the source vanishes too and

$$(M^2-\Xi\phi^2)W^\nu=0\tag{264}$$

which tells us that, if torsion is present, then

$$M^2=\Xi\phi^2\tag{265}$$

and if Ξ is positive, the scalar acquires the value

$$\phi^2=M^2/\Xi\tag{266}$$

which is of course constant throughout the universe.

A constant scalar all over the universe is the condition needed for slow-roll in inflationary scenarios, and in this case there arises an effective cosmological constant

$$\Lambda_{\text{effective}}=\Lambda+\tfrac{1}{2}\left|\tfrac{\lambda}{2}\left|\tfrac{M}{\Xi}\right|^2\right|^2\tag{267}$$

in the Lagrangian (261), driving the scale factor of the FLRW metric and therefore driving the inflation itself.

Inflation will last, so long as symmetry conditions hold, but as the universe expands and the density of sources decreases, local anisotropies are no longer swamped, and their presence will spoil the symmetries that engaged the above mechanism, bringing inflation to an end [70].

This is a first effect of the presence of the torsion field.

But of course we should not stop here. As the universe expands in a non-inflationary scenario, the torsion field equation would no longer lose the dynamic term due to the symmetries; but it may still lose it because of the possibility to have a massive torsion effective approximation, and in this case we would still have the expression

$$(M^2-\Xi\phi^2)W^\nu\approx X\overline{\psi}\boldsymbol{\gamma}^\nu\boldsymbol{\pi}\psi\tag{268}$$

although only as an approximated form: we may plug it back into the initial Lagrangian (261) obtaining

$$\begin{aligned}\mathscr{L}=&-\tfrac{1}{k}R-\tfrac{2}{k}\Lambda-\tfrac{1}{4}F^2+i\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi+\nabla^\mu\phi^\dagger\nabla_\mu\phi-\\&-\tfrac{1}{2}X^2(M^2-\Xi\phi^2)^{-1}\overline{\psi}\boldsymbol{\gamma}^\nu\boldsymbol{\pi}\psi\overline{\psi}\boldsymbol{\gamma}_\nu\boldsymbol{\pi}\psi-\\&-Y\overline{\psi}\psi\phi-m\overline{\psi}\psi+\mu^2\phi^2-\tfrac{1}{2}\lambda^2\phi^4\end{aligned}\tag{269}$$

as the resulting effective Lagrangian. The presence of an effective interaction involving spinors and scalars, having a structure much richer than that of the Yukawa interaction, is obvious; and we observe that, if for vanishingly small scalar this reduces to the above effective interaction for spinors, in presence of larger values for the scalar it can even become singular. We might speculate that such a value is the maximum allowed for the scalar as the one at which the above mechanism of inflation takes place.

And in addition, we have now at our disposal a potential whose richer structure can be exploited further.

For example, if in the above Lagrangian one considers the starting assumption $m=\mu=0$ then the potential is

$$\begin{aligned}\mathscr{V}=&\tfrac{1}{2}X^2(M^2-\Xi\phi^2)^{-1}\overline{\psi}\boldsymbol{\gamma}^\nu\boldsymbol{\pi}\psi\overline{\psi}\boldsymbol{\gamma}_\nu\boldsymbol{\pi}\psi+\\&+Y\overline{\psi}\psi\phi+\tfrac{1}{2}\lambda^2\phi^4\end{aligned}\tag{270}$$

containing spinor-scalar interactions: for a single spinor field we employ (97) to write this according to the form

$$\begin{aligned}\mathscr{V}=&-\tfrac{1}{2}X^2(M^2-\Xi\phi^2)^{-1}(|\overline{\psi}\psi|^2+|i\overline{\psi}\boldsymbol{\pi}\psi|^2)+\\&+Y\overline{\psi}\psi\phi+\tfrac{1}{2}\lambda^2\phi^4\end{aligned}\tag{271}$$

whose minimum with respect to scalars and spinors is

$$(i\overline{\psi}\boldsymbol{\pi}\psi)_{\mathrm{v}}=0 \tag{272}$$

$$-X^{2}(M^{2}-\Xi\phi_{\mathrm{v}}^{2})^{-1}(\overline{\psi}\psi)_{\mathrm{v}}+Y\phi_{\mathrm{v}}=0 \tag{273}$$

$$\begin{aligned}&-\Xi X^{2}(M^{2}-\Xi\phi_{\mathrm{v}}^{2})^{-2}\phi_{\mathrm{v}}(|\overline{\psi}\psi|_{\mathrm{v}}^{2}+|i\overline{\psi}\boldsymbol{\pi}\psi|_{\mathrm{v}}^{2})+\\&+Y(\overline{\psi}\psi)_{\mathrm{v}}+2\lambda^{2}\phi_{\mathrm{v}}^{3}=0\end{aligned} \tag{274}$$

or plugging $(i\overline{\psi}\boldsymbol{\pi}\psi)_{\mathrm{v}}=0$ into the others

$$X^{2}(\overline{\psi}\psi)_{\mathrm{v}}=Y\phi_{\mathrm{v}}(M^{2}-\Xi\phi_{\mathrm{v}}^{2}) \tag{275}$$

$$\begin{aligned}&-\Xi X^{2}(M^{2}-\Xi\phi_{\mathrm{v}}^{2})^{-2}\phi_{\mathrm{v}}|\overline{\psi}\psi|_{\mathrm{v}}^{2}+\\&+Y(\overline{\psi}\psi)_{\mathrm{v}}+2\lambda^{2}\phi_{\mathrm{v}}^{3}=0\end{aligned} \tag{276}$$

and plugging $X^{2}(\overline{\psi}\psi)_{\mathrm{v}}=Y\phi_{\mathrm{v}}(M^{2}-\Xi\phi_{\mathrm{v}}^{2})$ again

$$\begin{aligned}&-\Xi XY\phi_{\mathrm{v}}^{2}+Y^{\frac{3}{2}}(\phi_{\mathrm{v}})^{\frac{1}{2}}(M^{2}-\Xi\phi_{\mathrm{v}}^{2})^{\frac{3}{2}}+\\&+2\lambda^{2}X\phi_{\mathrm{v}}^{3}(M^{2}-\Xi\phi_{\mathrm{v}}^{2})=0\end{aligned} \tag{277}$$

giving the scalar vacuum; this expression is complicated, but in the case $M^{2}\gg\Xi\phi_{\mathrm{v}}^{2}$ it simplifies to

$$-\Xi XY\phi_{\mathrm{v}}^{2}+Y^{\frac{3}{2}}(\phi_{\mathrm{v}})^{\frac{1}{2}}M^{3}+2\lambda^{2}X\phi_{\mathrm{v}}^{3}M^{2}=0 \tag{278}$$

and if the Yukawa coupling is not large then

$$-\Xi Y\phi_{\mathrm{v}}^{2}+2\lambda^{2}\phi_{\mathrm{v}}^{3}M^{2}=0 \tag{279}$$

admitting the non-trivial solution

$$2\lambda^{2}M^{2}\phi_{\mathrm{v}}=\Xi Y \tag{280}$$

with $2\lambda^{2}X^{2}(\overline{\psi}\psi)_{\mathrm{v}}=\Xi Y^{2}$ and $(i\overline{\psi}\boldsymbol{\pi}\psi)_{\mathrm{v}}=0$ in the above.

Therefore, even in the case we have no mass-like terms for the spinors and scalars, nevertheless it is still possible to have non-trivial minima for spinors and scalars if the torsion field can have interactions with the two of them.

This circumstance is of great importance in view of a possible solution for one of the most unsettling problems that physics is witnessing, lying at the interface between the standard models of cosmology and particle physics, that is the problem of the cosmological constant [71, 72].

The problem is quite simply the fact that the cosmological constant has a measured value that, in natural units, is about one hundred and twenty orders of magnitude off the theoretically predicted one; normally this would have made physicists rejecting the theories in which its value is calculated, but those theories are quantum field theory and the standard model, being very successful otherwise.

Philosophers may argue that in the face of a bad result disproving a theory there can be no good result that can support it: the history of physics is loaded with examples of good agreements between observations and predictions that were based on theories later seen to be false; and in this specific situation, the bad agreement is not only bad, but it is the worst in all of physics since ever. Nowadays, the common behaviour would be to claim that this is not really a bad agreement, since new physics might intervene to make the agreement acceptable: it does not take very experienced philosophers to see that this argument could always be invoked to push the problems under the carpet of an even higher energy frontier, and when this frontier will be unreachable, the predictivity of the theory will be annihilated. In this work we try to embrace a philosophic approach, or merely be reasonable, admitting that such a discrepancy between theory and observation is lethal.

As a consequence of this, it follows that all theories predicting contributions to the cosmological constant must be dramatically re-adjusted: as we said above, these are the general theory of quantum fields, where it is the concept of zero-point energy that comes from vacuum fluctuations what gives rise to a cosmological constant contribution; and the standard model itself, where it is the mechanism of spontaneous symmetry breaking what generates all masses as well as a cosmological constant term.

As for the contribution coming from the general theory of quantum fields in terms of the zero-point energies, we have to recall that the zero-point energies are the result of quantization implemented with commutation relationships; but as we discussed in the second part such commutators are ill-defined: if we abandon them, then there is no zero-point energy, and thus no further contribution to the effective value of the cosmological constant.

Leaving us without zero-point energy, it becomes necessary to find a way to compute the Casimir force without employing vacuum fluctuations: however, the Casimir effects can be obtained without any reference to vacuum fluctuations [75], with radiative processes [76], or by employing more general field theoretical descriptions [77].

The cosmological constant contribution due to vacuum fluctuations of quantized fields may not be there at all.

As for the contribution of the standard model in terms of the mechanism of spontaneous symmetry breaking, we first recall the generalities [46]: the standard model is the local gauge theory of the $U(1)\times SU(2)$ group for which

$$R'=e^{-i\alpha}R \tag{281}$$

$$L'=e^{-\frac{i}{2}(\vec{\boldsymbol{\sigma}}\cdot\vec{\theta}+\mathbb{I}\alpha)}L \tag{282}$$

and

$$\phi'=e^{-\frac{i}{2}(\vec{\boldsymbol{\sigma}}\cdot\vec{\theta}-\mathbb{I}\alpha)}\phi \tag{283}$$

where R is the right-handed spinor and L is the doublet of left-handed spinors while ϕ is the doublet of complex scalars, and that these transformations be local requires the introduction of the gauge fields transforming as

$$\vec{\boldsymbol{\sigma}}\cdot\vec{A}'_{\mu}=e^{-\frac{i}{2}\vec{\boldsymbol{\sigma}}\cdot\vec{\theta}}\left[\vec{\boldsymbol{\sigma}}\cdot\left(\vec{A}_{\mu}-\tfrac{1}{g}\partial_{\mu}\vec{\theta}\right)\right]e^{\frac{i}{2}\vec{\boldsymbol{\sigma}}\cdot\vec{\theta}} \tag{284}$$

$$B'_{\mu}=B_{\mu}-\tfrac{1}{g'}\partial_{\mu}\alpha \tag{285}$$

so that also the derivatives

$$D_{\mu}R=\nabla_{\mu}R-ig'B_{\mu}R \tag{286}$$

$$D_{\mu}L=\nabla_{\mu}L-\tfrac{i}{2}\left(g\vec{\boldsymbol{\sigma}}\cdot\vec{A}_{\mu}+g'\mathbb{I}B_{\mu}\right)L \tag{287}$$

and

$$D_\mu\phi=\nabla_\mu\phi-\frac{i}{2}\left(g\vec{\boldsymbol{\sigma}}\cdot\vec{A}_\mu-g'\mathbb{I}B_\mu\right)\phi \tag{288}$$

are locally symmetric; the gauge curvatures are given by

$$\vec{A}_{\mu\nu}=\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu+g\vec{A}_\mu\times\vec{A}_\nu \tag{289}$$

$$B_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu \tag{290}$$

and they are gauge invariant. With this matter content, the dynamics is assigned by giving the Lagrangian

$$\begin{aligned}\mathscr{L}=i\overline{R}\boldsymbol{\gamma}^\mu D_\mu R+i\overline{L}\boldsymbol{\gamma}^\mu D_\mu L+|D\phi|^2-\tfrac{1}{4}A^2-\tfrac{1}{4}B^2-\\-Y\left(\overline{R}\phi^\dagger L+\overline{L}\phi R\right)+\lambda^2\left(v^2\phi^2-\tfrac{1}{2}\phi^4\right)\end{aligned} \tag{291}$$

written in terms of the Y, λ^2 and v^2 parameters.

For such a Lagrangian, the potential is given by

$$\mathscr{V}=\tfrac{1}{2}\phi^4-v^2\phi^2 \tag{292}$$

whose minimum $\phi_{\mathrm{v}}^2=0$ is invariant but not stable and therefore it will move toward the stable but non-invariant configuration given by $\phi_{\mathrm{v}}^2=v^2$ being the non-trivial minimum; having broken the symmetry, we choose the gauge in which such breakdown of symmetry is manifest as

$$\phi=\begin{pmatrix}0\\ v+H\end{pmatrix} \tag{293}$$

with

$$\vec{A}_\mu=\vec{M}_\mu \tag{294}$$

$$B_\mu=N_\mu \tag{295}$$

and

$$L=\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} \tag{296}$$

$$R=(e_R) \tag{297}$$

and called unitary gauge: the theory can be eventually diagonalized with a field re-configuration given by

$$\cos\theta N_\mu-\sin\theta M^3_\mu=A_\mu \tag{298}$$

$$\sin\theta N_\mu+\cos\theta M^3_\mu=Z_\mu \tag{299}$$

and

$$\frac{1}{\sqrt{2}}\left(M^1_\mu\pm iM^2_\mu\right)=W^\pm_\mu \tag{300}$$

and by recalling that

$$e_L+e_R=e \tag{301}$$

where $g'=g\tan\theta$ gives the mixing angle.

The passage to the unitary gauge was such that from the doublet of complex scalar fields we have a transfer of three degrees of freedom into three gauge fields, with the consequence that they acquire all degrees of freedom they need to be massive: and in fact after the breaking is implemented in the Lagrangian, we find that three mass terms appear for the pair of complex vector fields and for the real vector field $gv=m_W\sqrt{2}=m_Z\sqrt{2}\cos\theta$ together with the mass generation of the spinor $vY=m_e$ and the mass of the scalar $\sqrt{2}\lambda v=m_H$ and additionally there is also a cosmological constant term $v^4\lambda^2=-4\Lambda$ as it can be seen from (249) or with a straightforward calculation.

We notice that the contribution to the Lagrangian

$$\mathscr{L}^{\mathrm{SM}}_{\mathrm{cc}}=\tfrac{1}{2}\lambda^2 v^4 \tag{302}$$

gives a cosmological constant that is negative, as it should have been expected from the fact that it corresponds to the lowest point of the potential, while its numerical value is about 10^{120} in natural units, again as it should have been expected since the lowest point of the potential must be deep enough to grant the vacuum stability; if this term is to disappear, we need vanish either λ or v but as vanishing the former would imply no symmetry breaking, the only possibility is to vanish v so that symmetry breaking can occur, although not spontaneously but dynamically.

Dynamical symmetry breaking is for instance given by the example we have presented right above.

The idea of symmetry breaking is that the scalar has a potential with a trivial minimum, which is symmetric but unstable, and a non-trivial minimum, which is stable but asymmetric, so that the symmetric but unstable configuration will tend to reach stability then breaking the symmetry: spontaneous symmetry breaking takes place when the potential is assigned; dynamical symmetry breaking occurs when the potential is not assigned but induced by other mechanisms. A spontaneous symmetry breaking is that of the standard model; and a dynamical symmetry breaking is induced for instance by having the Higgs be a condensate, or a bound-state, as done in the mentioned Nambu–Jona-Lasinio model, or in reference [73].

More specifically, one starts as in the NJL model from a contact interaction of two spinors ψ and χ as

$$\Delta\mathscr{L}=-g^2\overline{\psi}\chi\overline{\chi}\psi \tag{303}$$

and defining $\overline{\chi}\psi=\phi$ one may have it re-arranged as

$$\Delta\mathscr{L}=-\tfrac{1}{2}(g^2+\mu^2)(\overline{\psi}\phi\chi+\overline{\chi}\phi^\dagger\psi)+\mu^2\phi^\dagger\phi \tag{304}$$

displaying a Yukawa term and the inverse-mass term that provides the symmetry breaking; however, because

$$\overline{\chi}\psi=\phi \tag{305}$$

shows the Higgs to be a bound-state of two spinors, the symmetry breaking is dynamical. Dynamical symmetry breaking may be a consequence of the NJL model, and because here we discussed how the NJL model is the effective approximation of torsion, it follows that a dynamical symmetry breaking may be a consequence of torsion.

Furthermore, even if the Higgs field is not a composite state of spinors, but rather an elementary scalar, it would be possible to have such a scalar coupled to the spinors, and all of them interacting with torsion, and still have a dynamical symmetry breaking, as proven above.

Since the Higgs is not a scalar but a doublet of complex scalars, the relevant Lagrangian (270) is modified to

$$\begin{aligned}\mathscr{V}=\tfrac{1}{2}(M^2-\Xi\phi^2)^{-1}(X_L\overline{L}\boldsymbol{\gamma}^\nu\boldsymbol{\pi}L+X_R\overline{R}\boldsymbol{\gamma}^\nu\boldsymbol{\pi}R)\cdot\\ \cdot(X_L\overline{L}\boldsymbol{\gamma}_\nu\boldsymbol{\pi}L+X_R\overline{R}\boldsymbol{\gamma}_\nu\boldsymbol{\pi}R)+\\ +Y(\overline{R}\phi^\dagger L+\overline{L}\phi R)+\\ +\tfrac{1}{2}\lambda^2\phi^4\end{aligned}\tag{306}$$

but still in the same effective approximation we have

$$\begin{aligned}\Xi(X_R\overline{R}\boldsymbol{\gamma}^\nu R-X_L\overline{L}\boldsymbol{\gamma}^\nu L)\cdot\\ \cdot(X_R\overline{R}\boldsymbol{\gamma}_\nu R-X_L\overline{L}\boldsymbol{\gamma}_\nu L)|_{\mathrm{v}}+2\lambda^2M^4\phi_{\mathrm{v}}^2\approx 0\end{aligned}\tag{307}$$

giving the square of the Higgs vacuum as the square of the density of the spinor vacuum: we do not have a single spinor here, so re-arrangements may not give a negative square density, but anyway if it happens that there is a negative square density of the spinor vacuum, then there is a positive square of the Higgs vacuum, and a dynamical symmetry breaking mechanism occurs eventually.

After dynamical symmetry breaking, the Lagrangian of the standard model reduces to the known Lagrangian of the standard model up to higher-order terms and with a cosmological constant that is given by

$$\mathscr{L}_{\mathrm{cc}}^{\mathrm{SM}}=\lambda^2\phi_{\mathrm{v}}^2M^2\Xi^{-1}\tag{308}$$

which is still negative, but now its value depends on the square of the Higgs vacuum, and hence as the square of the spinor vacuum: within spinors, the scalar vacuum is non-trivial, there is the generation of all the masses and the cosmological constant as in the standard model, and although the cosmological constant is enormous, nevertheless it is also invisible, in particle physics experiments.

But in cosmology, the vacuum spinor density becomes negligibly small, so the scalar vacuum becomes negligible as well, and the generated cosmological constant turns to be negligible too, so much that its value would not even interfere with the value that is actually observed [74].

The picture that emerges is one for which symmetry breaking is no longer a mechanism that happens throughout the universe but only when spinors are present, with the consequence that if spinors are not present the effective cosmological constant is similarly not present.

The cosmological constant due to spontaneous symmetry breaking in the standard model is avoidable.

Getting rid of vacuum fluctuations, as well as any fluctuation, leaves no contribution apart from those due to phase transitions, which can be quenched by a symmetry breaking that is not spontaneous but dynamical, and no effective cosmological constant actually arise.

X. OVERVIEW

In this third part, we have presented and discussed the possible torsional dynamics in the cosmology and particle physics standard models. Now it is time to pull together all the loose ends in order to display the general overview.

We have seen and stated repeatedly that torsion can be thought as an axial-vector massive field coupling to the axial-vector bi-linear spinor field according to the term

$$\Delta\mathscr{L}_{\mathrm{interaction}}^{\mathrm{Q-spinor}}=-X\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi W_\mu\tag{309}$$

of which we have one for every spinor: effective approximations involving two, three or even more spinor fields have been discussed, with a particular care for the case of neutrino oscillations, for which we have detailed in what way the results of [63] can be generalized in order to have

$$\Delta\Phi\approx\tfrac{L}{2E}\left(\Delta m^2+m\tfrac{X_{\mathrm{eff}}^2}{4M^2}|\overline{\nu}\gamma_\mu\nu\overline{\nu}\boldsymbol{\gamma}^\mu\nu|^{\frac{1}{2}}\right)$$

describing the phase difference for almost degenerate neutrino masses, as consisting of the L/E dependence modulating the usual kinetic contribution, plus a new dynamic contribution, so that even if the neutrino mass spectrum were to be degenerate, torsion would still induce an effective mechanism of oscillation; as these considerations have nothing special about neutrinos, and thus they may as well be extended to all leptons, then we proceeded in studying such extension, but once the Lagrangian terms of the weak interaction after symmetry breaking and the torsion for an electron and a left-handed neutrino were taken in the effective approximation, we saw that, due to the cleanliness of the scattering and the precision of the measurements, the standard model correction induced by the torsion had to be very small, and if this occurs because the torsion mass is large then the effective approximation is no longer viable. We have then re-considered the case without effective approximations, allowing also for sterile right-handed neutrinos in order to maintain the feasibility of the dynamical neutrino oscillations discussed above, therefore reaching the general Lagrangian

$$\begin{aligned}\Delta\mathscr{L}_{\mathrm{interaction}}^{\mathrm{Q/weak-spinor}}=-X_e\overline{e}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}eW_\mu-X_\nu\overline{\nu}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\nu W_\mu+\\ +\tfrac{g}{\sqrt{2}}\left(W_\mu^-\overline{\nu}\boldsymbol{\gamma}^\mu e_L+W_\mu^+\overline{e}_L\boldsymbol{\gamma}^\mu\nu\right)+\\ +\tfrac{g}{\cos\theta}Z_\mu[\tfrac{1}{2}\left(\overline{\nu}\boldsymbol{\gamma}^\mu\nu-\overline{e}_L\boldsymbol{\gamma}^\mu e_L\right)+|\sin\theta|^2\overline{e}\boldsymbol{\gamma}^\mu e]\end{aligned}$$

showing that while the sterile right-handed neutrino is by construction insensitive to weak interactions, it is sensitive to the universal torsion interaction, and suggesting that to see torsional interactions on a background of weak interactions we must pass for neutrino physics; we have argued that this situation occurs because weak interactions among leptons provide some very clean scattering, but torsional effects may still be allowed if hidden in less clean processes like the weak or the chromo-dynamical interactions among quarks. We did not dare to deepen the discussion about torsional effects within nucleons, but we have argued that because of the universal attractiveness of torsion, its effect might reasonably be that of shrinking the nucleon radius, providing a possible avenue to tackle the "proton radius problem" that arose in recent years.

After having extensively wandered in the microscopic domain of particle physics, we move to see what type of effect torsion might have for a macroscopic application of a yet unseen particle, dark matter, and we have seen that in the case of effective approximation, the spinor source in the gravitational field equations becomes of the form

$$R^{\rho\sigma}-\frac{1}{2}Rg^{\rho\sigma}-\Lambda g^{\rho\sigma}=\frac{k}{2}(E^{\rho\sigma}-\frac{1}{2}\frac{X^2}{M^2}V^\mu V_\mu g^{\rho\sigma})$$

showing that if the spin density square happens to be positive, the contribution to the energy would change the gravitational field as to allow for a constant behaviour of the rotation curves of galaxies; we have discussed that this behaviour comes from having a matter density scaling according to r^{-2} for large distances; such a behaviour, usually, is granted by the Navarro-Frenk-White profile or, here, is due to the presence of torsion, suggesting that the NFW profile is just the manifestation of torsional interactions, and ultimately that dark matter may be described in terms of the axial-vector simplified model, sorting out one privileged type among all possible simplified models now in fashion in particle physics. Then we proceeded to include into the picture also the scalar fields, getting

$$\begin{aligned}\mathscr{L}=&-\frac{1}{4}(\partial W)^2+\frac{1}{2}M^2W^2-\frac{1}{k}R-\frac{2}{k}\Lambda-\frac{1}{4}F^2+\\&+i\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi+\nabla^\mu\phi^\dagger\nabla_\mu\phi-\\&-X\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi W_\mu-\frac{1}{2}\Xi\phi^2W^2-Y\overline{\psi}\psi\phi-\\&-m\overline{\psi}\psi+\mu^2\phi^2-\frac{1}{2}\lambda^2\phi^4\end{aligned} \tag{310}$$

showing that in general the torsion, beside its coupling to the spinor, may also couple to the scalar, with the scalar behaving as a sort of correction to the mass of torsion and a kind of re-normalization factor in the torsion-spinor effective interactions; we discussed how within a homogeneous isotropic universe, the torsion field equations grant the condition $M^2=\Xi\phi^2$ so that, if Ξ were positive, then the scalar field would acquire a constant value, slow-roll will take place and inflation can engage. And eventually, we have discussed that after inflation has ended, torsional contributions to the scalar sector are such that for single particles we have $W^\alpha W_\alpha<0$ as it is clear in the effective approximation, inducing dynamical symmetry breaking.

This symmetry breaking, being dynamical, may solve the standard model part of the cosmological constant problem; the quantum field theoretical part of the cosmological constant problem may well be a false problem, as we reported above, and as discussed in literature [77].

So for summarizing, we have seen that the torsion field has several potentially interesting effects superposing to the standard models: it might give rise to a dynamical mechanism of oscillations for neutrinos, and although no other effects would be relevant for leptons, it may still have effects for quarks with a possible explanation of the recently emerged "proton radius problem", it may be the fundamental physics behind the NFW profile and as such selecting the axial-vector simplified model as a privileged description of dark matter, it may grant slow-roll and inflation, it may provide the conditions to have a dynamical form of symmetry breaking that would help solve the cosmological constant problem. Presumably, it would be surprising if torsion could do all of those things and we would be the first to be astonished if it actually did, but admittedly there is also no real argument against this.

Moreover, despite having listed six possible scenarios, nevertheless the first four of them were four different applications of the axial-vector coupling while the last two of them were two different applications of the coupling to scalar fields [78]. Only two physical couplings are needed.

And these two couplings are simply all of the possible renormalizable couplings torsion that may have.

FOUR: BEHAVIOUR

XI. DISCUSSION

In this fourth part we are going to provide some general thoughts around torsion in view of the most general, but also difficult, problem of all, finding exact solutions.

We start with some general consideration on the spinorial field equations (166, 167), from which we deduce

$$\begin{aligned}&\phi^{-2}\nabla^2\phi^2+(2m)^2+\\&+2mv^\mu(G_\mu\sin\beta+K_\mu\cos\beta)-\\&-(\nabla_\mu G^\mu+G^2)=0\end{aligned} \tag{311}$$

$$\begin{aligned}&\nabla^2\beta-(2m)^2\sin\beta\cos\beta-\\&-2mv^\mu(G_\mu\cos\beta+K_\mu\sin\beta)-\\&-\nabla_\mu K^\mu=0\end{aligned} \tag{312}$$

which are recognized as a Klein-Gordon equation of real mass $2m$ and a sine–Klein-Gordon equation of imaginary mass $2m$ respectively; both have additional mixing terms that depend on v^μ and which can be made to disappear by working out the products of the G^μ and K^μ potentials to obtain that the final result is given according to

$$\left|\nabla\frac{\beta}{2}\right|^2-m^2-\phi^{-1}\nabla^2\phi+\frac{1}{2}(\nabla G+\frac{1}{2}G^2-\frac{1}{2}K^2)=0 \tag{313}$$

$$\nabla_\mu(\phi^2\nabla^\mu\frac{\beta}{2})-\frac{1}{2}(\nabla K+KG)\phi^2=0 \tag{314}$$

as a Hamilton-Jacobi equation and a continuity equation respectively: the term $\nabla_\mu\beta/2$ is to be interpreted as a momentum density and $\nabla^2\phi/\phi$ is the quantum potential.

In terms of this analysis it becomes clearer that another role which can be attributed to the Takabayashi angle can be that of the action functional of a theory, whenever that theory is written in terms of the Hamiltonian formalism.

We notice that as the action functional is used in path integral quantization [50], then the action functional can be seen as yet another bridge connecting the Takabayashi angle and the peculiar character of quantum fields.

The spinor field equations (166, 167) are the field equations for the Takabayashi angle and for the module, and it is possible to have the former substituted into the latter, taking the limit in which the Takabayashi angle tends to zero, getting an equation for the field $\phi|g|^{\frac{1}{4}}=\varsigma$ as

$$\nabla^2\varsigma-\frac{X^2}{M^2}\frac{4m}{\sqrt{|g|}}\varsigma^3+m^2\varsigma=0 \tag{315}$$

displaying an attractive force; separating space and time coordinates, we may write this equation according to

$$\vec{\nabla}\cdot\vec{\nabla}\varsigma+\frac{X^2}{M^2}\frac{4m}{\sqrt{|g|}}\varsigma^3-(m^2-E^2)\varsigma=0 \tag{316}$$

where we assumed $i\dot{\varsigma}=E\varsigma$ as usually done: then if energy and mass have a small difference it becomes given by

$$\frac{1}{2m}\vec{\nabla}\cdot\vec{\nabla}\varsigma+\frac{X^2}{M^2}\frac{2}{\sqrt{|g|}}\varsigma^3-(m-E)\varsigma=0 \tag{317}$$

looking like a non-relativistic equation with an attractive self-interaction, or a Schrödinger non-linear equation.

We notice that attractive potentials entail the fact that solutions may be trapped in their own potential well, with the consequence that they have a negative potential, and their energy is smaller than the mass: so in (317) we can recognize the structure of a soliton field equation. Since the volume element $\sqrt{|g|}$ is present in most general cases of three dimensions, we know of no analytic solution, but in one dimension the solutions are hyperbolic secants.

Therefore, it is legitimate to believe that also for three dimensional cases we may find localized solutions [79, 80].

Nevertheless, in the large-r regions the non-linear term tends to vanish, and therefore it becomes possible to find solutions because in this approximation (317) reduces to

$$\frac{1}{2m}\vec{\nabla}\cdot\vec{\nabla}\varsigma-(m-E)\varsigma=0 \tag{318}$$

whose solutions are real exponentials

$$\varsigma\approx K\left[\exp\left(r\sqrt{2m|m-E|}\right)\right]^{-1} \tag{319}$$

for any constant K and as an exponential damping with the distance, indeed displaying the drop toward infinity, and in fact such a drop toward infinity is so fast that its volume integral is finite and the distribution is localized.

From the spinor field equations (166, 167) we may also have the latter substituted into the former, and the limit in which the module tends to zero, obtaining an equation for the Takabayashi angle that is given according to

$$\nabla^2\beta-4m^2\beta=0 \tag{320}$$

where the Takabayashi angle was taken small; assuming no time dependence gives the final form

$$\vec{\nabla}\cdot\vec{\nabla}\beta+4m^2\beta=0 \tag{321}$$

with the structure of a purely spatial wave equation.

Therefore solutions are in the form of real circular functions of the spatial coordinates, as for example

$$\beta=K\sin\left(2m\vec{u}\cdot\vec{r}\right) \tag{322}$$

for any given constant K and with $\vec{u}$ unitary vector.

We notice the peculiar circumstance that in free cases, and for small values of their magnitude, the module and the Takabayashi angle have opposite behaviour: along a space coordinate, the module has (exponential) dropping, the Takabayashi angle has (circular) oscillation; a curious fact is that even where the module drops to vanish, there the Takabayashi angle may still be present. The bi-linear spinor quantities are such that in them the Takabayashi angle always appears inside circular functions, so that an oscillating behaviour remains; however, this behaviour is limited, and where the module drops to zero, the bi-linear spinor quantities drop to zero as well. This Takabayashi angle has the intriguing property that it can be non-zero even at infinity without giving divergent quantities.

The limitations of being square-integrable, applicable to observable quantities, are inherited by the module, but the Takabayashi angle is not bounded by this constraint.

In absence of a numerical analysis, we may still obtain some general behaviour of the spinorial field. For this, we take field equations (227, 228) with no electro-dynamics, and in spherical coordinates (t,φ,θ,r) they become

$$\nabla_\mu\beta-2XW_\mu-2P^\rho u_{[\rho}v_{\mu]}+v_\mu 2m\cos\beta=0 \tag{323}$$
$$\nabla_\mu\ln\left(\phi^2r^2\sin\theta\right)-2P^\rho u^\nu v^\alpha\varepsilon_{\mu\rho\nu\alpha}+v_\mu 2m\sin\beta=0 \tag{324}$$

which we may now study by assuming the tetrads to be

$$\xi^0_t=1\quad \xi^t_0=1 \tag{325}$$
$$\xi^2_\varphi=r\sin\theta\quad \xi^\varphi_2=\frac{1}{r\sin\theta} \tag{326}$$
$$\xi^1_\theta=r\quad \xi^\theta_1=\frac{1}{r} \tag{327}$$
$$\xi^3_r=1\quad \xi^r_3=1 \tag{328}$$

and for a momentum of the form

$$P_\mu=(E,0,0,0) \tag{329}$$

where E is the energy of the spinor: this energy is a free parameter, but because of the presence of torsion, which is attractive, the potential is negative, so that the energy is smaller than the mass; then the field equations read

$$\partial_t\beta=2XW_t \tag{330}$$
$$\partial_\varphi\beta=2XW_\varphi \tag{331}$$
$$\partial_\theta\beta=2XW_\theta \tag{332}$$
$$\partial_r\beta+2(E-m\cos\beta)=2XW_r \tag{333}$$

and defining $\phi^2r^2\sin\theta=\varsigma^2$ it turns out this is a function of r alone and such that it has to verify the equation

$$\partial_r\varsigma=m\varsigma\sin\beta \tag{334}$$

as it is possible to check quite straightforwardly.

Notice that solutions with the behaviour of decreasing exponentials are ensured when the Takabayashi angle is negative, with partially-conserved axial-vector current

$$\nabla_\mu(\phi^2v^\mu)=2m\phi^2\sin\beta \tag{335}$$

showing that, when the spinor is coupled to torsion, it is

$$\nabla_\mu(XW^\mu)=4X^2M^{-2}m\phi^2\sin\beta \tag{336}$$

and negative Takabayashi angles are granted, as torsion is attractive; this is intuitive, since attractive torsion acts as some sort of tension over the matter distribution.

If torsion were to be negligible within the spinor field equations, we could find a non-trivial but simple solution as the one given by a constant Takabayashi angle

$$\beta = -\arccos\left(E/m\right) \tag{337}$$

and the exponential

$$\varsigma \approx K\left[\exp\left(r\sqrt{|m^2 - E^2|}\right)\right]^{-1} \tag{338}$$

which is merely the solution (319) before that the non-relativistic approximation $m \approx E$ is assumed.

The above field equations show that an energy smaller that the mass gives an exponential behaviour; a negative Takabayashi angle ensures such an exponential to have a decreasing behaviour. Under these conditions, a material distribution does display stability and localization.

Notice that the localization takes place for those coordinates that correspond to the non-null directions of the spin axial-vector: in the case above, the spin-axial vector has only the third component, and because of the choice of tetrads, it results to have only the radial components, and therefore radial localization takes place [81].

This behaviour is general, and it is appreciated also in the torsionless case, as shown in reference [82].

Before, we have noticed that while the module must be square-integrable, the Takabayashi angle does not suffer any constraint, and it is allowed to be different from zero even at the infinity: could this be taken as a specific form of non-local behaviour? At the beginning of part two, we have very briefly introduced the problems concerning the two-slit experiment and its interpretations: the problems are about the fact that when a single electron hits a two-slit apparatus, it behaves as a wave until it also hits the screen, where it is detected as a particle. That is electrons display wave properties that can stretch up to non-local configurations (since the interference holds up even if the two slits are separated by a very large distance compared to the size of the electron), but nevertheless they appear very localized at the moment of observation (that is when they end up hitting the screen). This situation seems an unsolvable conundrum in the standard view, but as it has been repeated several times along the work, the standard view almost never considers the Takabayashi angle, which might just turn out to be of some help one more time.

Consider an electron sent toward two slits: the electron may be represented by the localized module surrounded by the Takabayashi angle; as (313, 314) showed, the Takabayashi angle can be seen as the action functional, and we have remarked about its oscillatory behaviour. When the surrounding field described by the Takabayashi angle passes through the two slits it behaves as a wave, with a consequent non-local attitude to interfere; on the other hand, the field equation (334) shows that it is where the Takabayashi angle vanishes that the peak of the module is found. On the screen, the Takabayashi angle forms an interference pattern, but it is only in the regions in which it is zero that the module peak could go, and because the module is localized, wherever it is going to hit it will look like a confined matter distribution. So, a possible interpretation for the two-slit phenomenon might just be that there is no entity behaving sometimes as wave and other times as particle, but the Takabayashi angle behaving always as wave and the module localized always as particle.

Then, fields propagates as (complex) waves displaying interference patterns, and it is the (real) localized distributions to be confined like particles; the two parts, being the Takabayashi angle and the module, coexist. However, only particle-like distributions are observable. Therefore, it may be that there is no collapse of waves onto particles, but only that particles are the sole detectable objects.

The wave/particle duality of a field might just as well be the Takabayashi angle/module duality of the spinor.

In reference [82] it is also shown, by presenting an exact solution of the spinor field coupled to its own gravity, that in general it is not possible to have the Takabayashi angle arbitrarily set to zero; admittedly, such a solution is too singular, because of the vanishing of the scalar invariant, to have any chance of representing physical particles, but still it is an exact solution. Because the only element we have left out was torsion, one may be tempted to conclude that torsion could turn out to be essential in its role of forbidding unphysical solutions to actually appear.

To better justify this statement, recall that in the example above, it was torsion which, by ensuring a negative Takabayashi angle and that the total energy was smaller than the mass, ensured that only decreasing exponentials could be solutions of the spinor field equations.

Indications that torsion can in principle be responsible for well-behaved matter distributions are present, but not enough for a strong claim; a boost for the mood can come by finding exact solutions in presence of torsion, and also gravity, but finding exact solutions for such a non-linear system of fully coupled differential field equations is more like a dream than reality, for the present moment.

Help may come from imposing reasonable symmetries of the matter distribution, although it is difficult to see what are the symmetries for such a system.

XII. SYNOPSIS

We have concluded our presentation, which was separated in four parts that are independent on each other.

In the first, we considered the most general geometry, with torsion beside gravity, and gauge potentials, and in it we defined the spinorial matter, then finding the most general system of least-order derivative field equations that was possible: we showed that torsion turns out to be equivalent to an axial-vector massive field, that spinorial fields are composed of two chiral parts, suggesting that torsion could be the mediator of the attraction for which the spinor may form chiral bound states; we showed that this is indeed the case when effective approximations are implemented and we argued that the Takabayashi angle is what encodes information about the internal dynamics and consequently about spin and *Zitterbewegung* effects.

In the second, we recalled the prescriptions of quantum physics, and in front of conceptual problems we returned to the foundations in order to see what could be salvaged or what had to be differently implemented, exhibiting an alternative description that exploited the spin content of spinor fields; we studied spinless approximations, assessing the issue of singularity avoidance for large densities.

In the third, we presented the torsion-spin axial-vector interaction, discussing also the coupling with scalar fields.

In the fourth, we discuss general solution behaviour.

All across these four a priori independent parts is the interconnecting idea that torsion and spin, and their coupling, could play a fundamental role: they can well be the essential reason for the stability of spinors; they give rise to *Zitterbewegung* effects that can describe the anomalies of field theory which are usually ascribed to the quantum correction, with no need of forcing fields to be represented by point particles that are unphysical and without need to assume radiative loops that are troublesome; they have a coupling giving rise to axial-vector models and a special interaction to scalar fields that can solve open problems in the standard models without disrupting consequences like the cosmological constant problem; they can have an effect in sorting out only solutions that are well behaved.

So torsion and spin could be used for addressing open problems in theoretical physics without introducing bad effects as those arising in the common approach.

XIII. OUTLOOK

In a geometry which, in its most general form, is naturally equipped with torsion, and for a physics which, for the most exhaustive form of coupling, has to couple the spin of matter, the fact that torsion couples to the spin of spinor material field distributions is just as well suited as a coupling can possibly be, and its consequences about the stability of such field distributions are certainly worth to receive further attention and to be better understood.

The commonly followed approach to quantum field theory is devised on the prescription that particles be point-like and that their interactions be quantized in terms of radiative corrections, but as it is widely known the point-like character of particles is the reason for the appearance of ultraviolet divergences when radiative corrections are computed, and although renormalization does remove divergences nevertheless it is best to have a theory in which divergences do not appear in the first place; we discussed that renormalization means the existence of a cut-off beyond which we have to stop calculations since we ignore what physical effect might be relevant, and it might just be that this limit is the threshold beyond which particles can no longer be point-like and the novel physical effects may simply be the fact that the now extended field starts to display its internal structure. By providing a model in which extended fields do have an internal dynamics, it is feasible that *Zitterbewegung* effects are what gives rise to the anomalies thought to be due to radiative corrections.

The standard models of cosmology and particle physics are also erected on arbitrary assumptions and we already envisage several ways to promote them to more complete models, such as simplified models or dynamical symmetry breaking; simplified models and dynamical symmetry breaking may be due to axial-vector interactions and a peculiar type of scalar potential, exactly as the ones that are given for the most general coupling of the torsion.

For the general behaviour of matter distributions, the Takabayashi angle tends to behave like a wave, spreading to far away regions, the module tends to be localized like a particle, confined within small regions, and with a peak in those regions in which the Takabayashi angle tends to vanish, therefore opening the possibility of an alternative interpretation of the two-slit experiment that is not based on the wave/particle duality but on the fact that there are two complementary fundamental fields within one spinor.

Where do all this leaves us? For the analogies between quantum point particles and classical spinning fields, and if the analogy actually holds, it may be that these two approaches are fully equivalent. The formalism of quantum field theory, decomposing the scattering into propagation of fields that are free plus interacting vertices encoding all the quantum information, can be visualized as a technologically complicated form of Taylor expansion, where a curve can be decomposed into straight segments and angles at their junctions: as radiative corrections are added, segments are shorter and form more junctions, and if the number of radiative corrections goes to infinity, the segmented line approximates a smooth curve; according to our perspective, the scattering would immediately be described in terms of a smooth line. If this were so, we may expect a formal equivalence; this equivalence would state that if the perturbative expansions were to be calculated exactly then they would yield scattering amplitudes like those we would get if we considered the effects of spin for classical fields. This fact has an implication, and that is even if such equivalence could be demonstrated this can only be done for spinor fields. If instead we consider the scalar field, it does not have spin, and consequently there can be no equivalence with quantum corrections; but conversely, if this equivalence were in fact true, we would be brought to the inescapable conclusion that there can be no quantum corrections for scalar fields. The problem of the vacuum metastability due to quantum corrections of the Higgs potential may not be a problem if this scenario is correct. And this statement has a fairly predictive content, being that no quantum correction can appear in the dynamics of scalar fields so long as they are fundamental.

And what about the standard models? In the standard model of particle physics, there are different facets to be considered: by assuming the existence of right-handed sterile neutrinos, the torsion-spin coupling gives dynamic corrections to the oscillation pattern; by assuming dark matter constituted by spinors, the torsion-spin coupling may give rise to the NFW profile. In the standard model of cosmology, the most urgent of the problems is that of the cosmological constant: in this case a solution has to be sought by generalizing the symmetry breaking from a spontaneous to a dynamical mechanism, and this can be done if the scalar field is allowed to interact with spinors through torsion. In the first two instances, the new contributions are condensed by $\Delta\mathscr{L}=-X\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi W_{\mu}$ as the axial-vector coupling; in the last instance, new physics is represented by $\Delta\mathscr{L}=-X\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi W_{\mu}-\frac{1}{2}\Xi\phi^{2}W^{2}$ as what gives the torsion-spin and scalar interaction, with Ξ being positive as $W^{\alpha}W_{\alpha}<0$ holds for single particles. The last potential for torsion coupling is the most comprehensive, and its good property is that we do not need to postulate it as this is the most general interaction that can be given to torsion, at least within our restriction of allowing only terms for the interactions that are renormalizable.

And finally, what about the general behaviour of spinorial solutions? Here we have discussed how spinor fields are constituted by two fundamental degrees of freedom, the Takabayashi angle β with a wave behaviour, and the module ϕ with a localized character, which could be used to give rise to an interpretation of matter complying with the requirements needed to provide an intuitive image of the two-slit experiment. At the level of this presentation however, there have been generic discussions and reasonable insights, but nothing that was fully mathematically demonstrated, and finding exact solutions would be best, although this task for such a system of field equations is out of the reach of our capabilities, at least for now.

As for the $\Delta\mathscr{L}=-X\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi W_{\mu}-\frac{1}{2}\Xi\phi^{2}W^{2}$ potential, new physics should certainly be hidden within the peculiar properties of such a torsion-spinor/scalar interaction, and in particular one would have to pay attention to the implementation of condition $W^{\alpha}W_{\alpha}<0$ investigating if there are situations where it does not apply; for a single spinor with torsion in effective approximation it is valid, but if the effective approximation cannot be established or we have multiple spinor configurations then it may no longer hold and changes are expected. The role of a scalar interaction may be limited but still quite intriguing.

But what in my opinion is the most intriguing aspect of all is the action of the Takabayashi angle in the dynamics of the module within the spinorial field, an action almost constantly neglected in the usual approaches but that has quite a lot of things to tell us if only we dared to keep it.

And in this case, how can we find exact solutions?

All over the work, there are several assumptions that have been taken into account, such as for instance the fact that, in conditions in which a collective system of spinors was considered, we assumed that they form condensates, which may be reasonable, but not proven; or we assumed that the torsion mass is large enough to allow the effective limits, but despite this may be reasonable since we have not detected torsion yet as an elementary particle, it may still be that torsion has small mass, but that nevertheless it couples to everything weakly. And in discussing what is the general behaviour of solutions, we assumed very little influence of external fields, and this is also an unphysical requirement. Studying the validity of these limits has to be done for the treatment to become more reliable.

Then, there are problems which we have left aside altogether, like the problem of discrete transformations and their violation, giving matter/antimatter asymmetry, for which we suspect that the torsion may have something to tell us, but we never ventured into this type of problems due to the mundane reason of simple lack of time.

Therefore, opportunity for future works might span a large variety of domains, starting from making all results we discussed here more reliable, or complementing them to a greater extent, and reach some yet untackled cases.

An additional problem that torsion gravity has is the fact that there is an astonishingly small number of people working in this domain, and as of this writing, a search in INSPIRE with keywords “torsion” and “spinor” is popping out less than 450 items, a record that is too far from being competitive with any other area of research.

Clearly, progress is faster when there is a critical mass of people working together with a common goal.

In this work we have tried to present what we believe to be the full range of applications that torsion and spin, and their coupling, could have in modern physics, and if, on the one hand, we hope to have clearly made a case for not neglecting the torsional effects on spinorial dynamics, on the other hand, there is still a long way to have these general indications commuted into solid evidence, further completed withing a comprehensive theory, and the effort of a larger number of people would be productive.

I hope this work tickled curiosity in someone.

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[1] E.Cartan, *Annales Sci.Ecole Norm.Sup.* **40**, 325 (1923).
[2] E.Cartan, *Annales Sci.Ecole Norm.Sup.* **41**, 1 (1924).
[3] E.Cartan, *Annales Sci.Ecole Norm.Sup.* **42**, 17 (1925).
[4] E.Cartan, *Compt.Rend.Acad.Sci.* **174**, 593 (1922).
[5] C.Misner, K.Thorne, J.A.Wheeler, *Gravitation* (Freeman, 1973).
[6] D.W.Sciama, in *Recent Developments in General Relativity* (Oxford, 1962).
[7] T.W.B.Kibble, *J.Math.Phys.* **2**, 212 (1961).
[8] F.W.Hehl, P.Von Der Heyde, G.D.Kerlick,

J.M.Nester, *Rev.Mod.Phys.* **48**, 393 (1976).

[9] S.Weinberg, *Gravitation and Cosmology* (Wiley, 1972).

[10] F.W.Hehl, *Found.Phys.***15**, 451 (1985).

[11] F.W.Hehl, Y.N.Obukhov, *Annales Fond. Broglie.***32**, 157 (2007).

[12] L.Fabbri, Ph.D. Thesis: *Higher-Order Theories of Gravitation*, arXiv:0806.2610 [hep-th].

[13] L.Fabbri, *Annales Fond. Broglie.***32**, 215 (2007).

[14] K.Hayashi, *Phys.Lett.B***65**, 437 (1976).

[15] Xin Yu, *Astrophysical and Space Science* **154**, 321 (1989).

[16] J.Audretsch, C.Lämmerzahl, *Class. Quant. Grav.* **5**, 1285 (1988).

[17] A.Macias, C.Lämmerzahl, *J. Math. Phys.* **34**, 4540 (1993).

[18] P.Lounesto, *Clifford Algebras and Spinors* (Cambridge University Press, 2001).

[19] R.T.Cavalcanti, *Int.J.Mod.Phys.D***23**, 1444002 (2014).

[20] E.P.Wigner, *Annals Math.* **40**, 149 (1939).

[21] L.Fabbri, *Annales Fond. Broglie.***33**, 365 (2008).

[22] V.De Sabbata, C.Sivaram, *Spin and Torsion in Gravitation* (World Scientific, Singapore, 1994).

[23] C.Sivaram, K.P.Sinha, *Lett.Nuovo Cim.***13**, 357 (1975).

[24] P.S.Wesson, *Phys.Rev.D***23**, 1730 (1981).

[25] F.A.Kaempffer, *Gen. Rel. Grav.* **7**, 327 (1976).

[26] L.Fabbri, *Gen.Rel.Grav.***45**, 1285 (2013).

[27] G.Velo, D.Zwanziger, *Phys. Rev.* **186**, 1337 (1969).

[28] G.Velo, D.Zwanziger, *Phys. Rev.* **188**, 2218 (1969).

[29] A.O.Barut, G.Ziino, *Mod. Phys. Lett.A***8**, 1011 (1993).

[30] L.Fabbri, *Int.J.Theor.Phys.***53**, 1896 (2014).

[31] L.Fabbri, *Gen.Rel.Grav.***46**, 1663 (2014).

[32] L.Fabbri, *Int.J.Geom.Meth.Mod.Phys.***12**,1550099(2015).

[33] A.S.Belyaev, I.L.Shapiro, *Phys.Lett.B***425**, 246 (1998).

[34] S.Vignolo, L.Fabbri, R.Cianci, *J.Math.Phys.***52**, 112502 (2011).

[35] R.da Rocha,L.Fabbri,J.M.Hoff da Silva,R.T.Cavalcanti, J.A.Silva-Neto,*J.Math.Phys.***54**,102505(2013).

[36] J.M.Hoff da Silva, R.da Rocha, *Phys. Lett. B***718**, 1519 (2013).

[37] R.Abłamowicz, I.Gonçalves, R.da Rocha, *J. Math. Phys.***55**, 103501 (2014).

[38] R.da Rocha, J.M.Hoff da Silva, *Adv. Appl. Clifford Algebras***20**, 847 (2010).

[39] C.H.Coronado Villalobos, J.M.Hoff da Silva, R.da Rocha, *Eur.Phys.J.C***75**, 266 (2015).

[40] R.T.Cavalcanti, J.M.Hoff da Silva, R.da Rocha, *Eur.Phys.J.Plus***129**, 246 (2014).

[41] A.M.Steane, arXiv:1312.3824 [math-ph].

[42] L.Fabbri, *Int.J.Geom.Meth.Mod.Phys.***13**,1650078(2016).

[43] L.Fabbri, *Int.J.Geom.Meth.Mod.Phys.***14**,1750037(2017).

[44] Y.Nambu, G.Jona-Lasinio, *Phys. Rev.* **122**, 345 (1961).

[45] Y.Nambu, G.Jona-Lasinio, *Phys. Rev.* **124**, 246 (1961).

[46] L.Fabbri, *Mod.Phys.Lett.A***26**, 2091 (2011).

[47] L.Fabbri, S.Vignolo, *Int.J.Theor.Phys.***51**,3186(2012).

[48] G.Salesi, E.Recami, *Adv.Appl.Clifford Algebras***7**, S253 (1997).

[49] E.Recami, G.Salesi, *Phys.Rev.A***57**, 98 (1998).

[50] M.E.Peskin, D.V.Schröder, *An Introduction to Quantum Field Theory* (Perseus Books, 1995).

[51] R.F.Streater, A.S.Wightman, *PCT, Spin and Statistics, and All That* (Princeton, 2000).

[52] K.S.Stelle, *Phys.Rev.D***16**, 953 (1977).

[53] J.Schwinger, *Particles, Sources and Fields* (Advanced Book Classics Series, Addison-Wesley, USA, 1989).

[54] J.Schwinger, in *The Physicist's Conception of Nature* (Editor J.Mehra, Springer, 1973).

[55] L.Fabbri, arXiv:1702.04687 [gr-qc].

[56] T.A.Welton, *Phys. Rev.***74**, 1157 (1948).

[57] L.Fabbri, *Int. J. Theor. Phys.***55**, 669 (2016).

[58] B.R.Martin, G.Shaw, *Particle Physics* (Wiley, 2008).

[59] Z.Koba, *Prog. Theor. Phys* **4**, 319 (1949).

[60] D.Hestenes, *Found. Phys.***20**, 1213 (1990).

[61] G.D.Kerlick, *Phys.Rev.D***12**, 3004 (1975).

[62] L.Fabbri, arXiv:1702.03864 [gr-qc].

[63] M.Adak, T.Dereli, L.H.Ryder, *Class.Quant.Grav.* **18**, 1503 (2001).

[64] L.Fabbri, S.Vignolo, *Mod.Phys.Lett.A***31**, 1650014(2016).

[65] R.Pohl *et al.*, *Nature* **466**, 213 (2010).

[66] D.Clowe *et al.*, *Astrophys. J.* **648**, L109 (2006).

[67] A.Tilquin, T.Schucker, *Gen. Rel. Grav.* **43**, 2965 (2011).

[68] L.Fabbri, *Int.J.Mod.Phys.D***22**, 1350071 (2013).

[69] J.Abdallah *et al.*, *Phys. Dark Univ.***9-10**, 8 (2015).

[70] L.Fabbri, arXiv:1609.00276 [gr-qc].

[71] S.E.Rugh, H.Zinkernagel, *Stud. Hist. Philos. Mod. Phys.* **33**, 663 (2002).

[72] J.Martin, *Comptes Rendus Physique***13**, 566 (2012).

[73] W.A.Bardeen, C.T.Hill, M.Lindner, *Phys.Rev.D***41**, 1647 (1990).

[74] L.Fabbri,*Int.J.Geom.Meth.Mod.Phys.***13**,1650068(2016).

[75] H.Nikolic, *Phys. Lett.B***761**, 197 (2016).

[76] R.L.Jaffe, *Phys. Rev. D* **72**, 021301 (2005).

[77] J.Schwinger, *Lett.Math.Phys.***1**, 43 (1975).

[78] A.S.Belyaev,I.L.Shapiro,M.C.Thomas, arXiv:1611.03651 [hep-ph].

[79] K.Takahashi *J.Math.Phys.* **20**, 1232 (1979).

[80] R.B.MacKenzie, M.B.Paranjape, *JHEP***0108**, 003 (2001).

[81] L.Fabbri, arXiv:1611.08474 [gr-qc].

[82] R.Cianci, L.Fabbri, S.Vignolo, *Eur.Phys.J.C***76**,595(2016).