The self-coupled Einstein-Cartan-Dirac equations in terms of Dirac bilinears

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Abstract. The Dirac equation for charged fermions interacting with an electromagnetic field can be algebraically inverted, so as to obtain an explicit rational expression of the four-vector potential of the gauge field in terms of the spinors. Substitution of this expression into Maxwell's equations yields the tensor form of the self-interactive Maxwell-Dirac equations. We demonstrate here, how an analogous inversion can be performed on the Dirac equation in curved space-time to obtain *two* four-vector expressions which are gravitational analogues of the electromagnetic vector potential. These potentials appear as irreducible summand components of the spin connection, along with a traceless residual term of mixed symmetry. When taking the torsion field equation into account, the residual term can be written as a function of the object of anholonomity. The vector potentials are comprised from Dirac bilinears, which are interrelated via a rich set of Fierz identities. Furthermore, a local vierbein frame can be chosen in terms of a set of four Dirac bilinear vector fields, normalized by two scalar fields. A corollary of this local frame choice is that the spin connection can be written as an explicit function of the Dirac bilinears. This work constitutes a novel formulation of the self-coupled Einstein-Cartan-Dirac equations in terms of Dirac bilinears, and may provide new insights into the nature of the interaction of Dirac matter with gravity.

1 Introduction

The Dirac equation, the relativistic wave equation for spin-1/2 fermions, can be made to describe particles interacting with a gauge field by replacing the partial derivative with the covariant derivative for the particular field. For a gauge potential of a given form, the Dirac equation may be solved for the spinor field corresponding to the fermion state. One example solution for an electron in an external field is that for the hydrogen atom, where the Dirac equation correctly predicts fine structure as a result of relativistic corrections to the Hamiltonian [20]. However, the external Dirac-Coulomb solution itself does not explain the famous Lamb shift, which requires a consideration of how radiative corrections provided by the Maxwell field affect the energy of the bound electron [3].

An inversion of the Dirac equation can be performed via algebraic rearrangement, such that the gauge potential is written as a rational, explicit function of the spinors [8]. Substituting the inverted Dirac equation into the equations of motion for the gauge field results in a self-coupled system, where the charged fermion field interacts with itself in an internally consistent way. A central aspect of the algebraic inversion procedure is that the spinors do not appear as stand-alone objects, but rather as *bilinear* combinations. An early proponent of using the bilinear description of Dirac states as the objects of primary interest was Takabayasi [24], who promoted the idea of a relativistic hydrodynamical model of Dirac matter. The states of this model were not spinors or wavefunctions, but *tensors* corresponding to quantum observables, such as current and spin densities. This in effect was an early substantial attempt to formulate a *semi-classical* fluid model of relativistic quantum electrodynamics.

There exists a rich set of interrelationships between quadratic combinations of Dirac bilinears, known as *Fierz identities* [9], [21] (alternatively, Fierz-Pauli-Kofink identities [2]); derived via a successive set of *Fierz expansions* over a Dirac Clifford algebra primitive set of sixteen basis elements. Using a similar process, Crawford showed that [6], given a set of sixteen bilinears formed from this set, the spinor field is recoverable up to a constant spinor with arbitrary phase. Additionally, there are two $U(1)$ gauge-dependent bilinears with four-vector Lorentz indices, which appear in bilinear forms of gauge-dependent terms (such as the inverted Dirac equation), and comprise a locally orthonormal tetrad frame along with the standard Lorentz four-vector and axial-vector fields.

In the electromagnetic case, the self-coupled *Maxwell-Dirac* equations were shown to be describable in terms of the gauge *independent* bilinears only, by Inglis and Jarvis [16], manifestly reflecting the physical gauge invariance of the system. Furthermore, these equations were able to be greatly simplified via the applications of infinitesimal invariance under several subgroups of the Poincaré group. These subgroups were chosen from a set of 158 given by Patera, Winternitz & Zassenhaus [22], where a comprehensive list of all the Poincaré Lie subalgebras and their corresponding generators are given. These symmetry reductions aid in the search for solutions to an otherwise intractable set of non-linear equations.

The ability to invert the Dirac equation is not limited to the electromagnetic case either, and we showed in a previous publication [15] that an inversion can be performed for the non-Abelian gauge field $SU(2)$. We found that the algebraic process was very similar to the Abelian case, but with some extra difficulty, and the inverted form was given implicitly. It is currently unclear whether a similar generalisation exists for the strong $SU(3)$ case, although the $U(1) \times SU_L(2)$ electroweak case appears to be promising. Substitution into the Yang-Mills equations yields a fully non-linear self-interactive non-Abelian hydrodynamical theory, relevant to the study of non-perturbative high-energy plasmas. Another, simpler approach to modelling aspects of non-Abelian hydrodynamics, is to generalize the classical fluid mechanical equations to include local internal symmetries. A description of the non-Abelian Lorentz force involving chromoelectric and chromomagnetic field couplings was obtained this way in [4].

In this paper, we demonstrate how the Dirac equation in curved spacetime can be algebraically inverted, in an analogous manner to the $U(1)$ and $SU(2)$ cases. The covariant derivative we use contains the spin connection contracted with the generator for Lorentz transformations. In section 2 we demonstrate how the Dirac equation can be rearranged such that we obtain two gravitational "vector potentials" Ω_a and Ω_{5a} in analogy with the electromagnetic vector potential A_a, and which are functions of the spin connection.

In section 3, we give our definition of the tensor fields resulting from sandwiching elements of the Dirac Clifford algebra basis between Dirac spinors. Using this notation, we then show that by left-multiplying the curved spacetime Dirac equation and its charge conjugate with four different spinors, the resulting set of four equations can be solved explicitly for the *two* gravitational vector potentials. These expressions are rational functions of bilinears and their first derivatives, but are not able to be expressed in terms of our tensor field set without further calculations.

The process by which we can write the inverted expressions in terms of tensor fields is given in 4. Here, we give a brief outline of the process by which Fierz expansions, where an outer product of two Dirac spinors is expanded in the Dirac Clifford algebra basis, are used to derive Fierz identities which are quadratic in the bilinears. These identities are then used to eliminate the explicit appearance of Dirac spinors in the inverted forms of the Dirac equation, replacing them with pure tensor expressions.

Section 5 is given in four parts. In the first part we demonstrate how an irreducible decomposition of the spin connection in $GL(4)$, when taking into account how the gravitational vector potentials are defined, allows it to be written as a sum of three terms. The trace term is a function of Ω_a, and the two traceless terms are a fully antisymmetric function of Ω_{5a} and residual term of mixed symmetry, w_{abc}. In the next two parts, we describe the field equations of the Einstein-Cartan system, for the gravitational dynamics of space-time curvature and spin-torsion respectively. Expressions for the Ricci tensor, scalar, and the torsion are given in terms of the spin connection and the object of anholonomity. In the final part, we show how the algebraic torsion field equation can be used to place constraints on Ω_a and Ω_{5a}, and to derive an explicit expression for w_{abc} in terms of the object of anholonomity.

A summary is given in Section 6. We show that a self-coupled Einstein-Cartan-Dirac system can be obtained by replacing the spin connection terms with their decomposition, including the inverted Dirac equation and torsion constraint equations. A convenient locally orthonormal vierbein frame to use is provided by the set of four four-vector bilinears j^μ, m^μ, n^μ and k^μ, with a scalar and pseudoscalar field-dependent normalising coefficient. As a result, the spin connection can be replaced by Dirac bilinears entirely. A set of constraints provided by the torsion field equation are also given. Further analysis of this system is left for future publications.

2 The curved spacetime Dirac equation and conventions

The Dirac equation in curved spacetime has the form

$$(\mathrm{i}\gamma^a {e_a}^\mu(x)\nabla_\mu - m)\psi = 0, \tag{1}$$

where Greek and Latin indices run from 0 to 3, and correspond to coordinate and locally orthonormal frames respectively. The vierbein field ${e_a}^\mu(x)$ relates these two frames locally at each point x, and is quadratically related to the metric, according to

$$g_{\mu\nu}(x) = {e^a}_\mu(x){e^b}_\nu(x)\eta_{ab}. \tag{2}$$

For the Minkowski flat spacetime metric we use the particle physics sign convention, whereby the signature of the flat spacetime metric is negative in the spatial components:

$$\eta_{ab} \equiv \mathrm{diag}(1, -1, -1, -1). \tag{3}$$

For Dirac spinor fields, the covariant derivative is of the form [7], [25]

$$\nabla_\mu\psi = \partial_\mu\psi + \Gamma_\mu\psi, \tag{4}$$

where the spinor connection coefficients are

$$\Gamma_\mu = \frac{1}{2}{\omega_\mu}^{ab}S_{ab} = -\frac{\mathrm{i}}{2}{\omega_\mu}^{ab}\sigma_{ab}. \tag{5}$$

We shall refer to ${\omega_\mu}^{ab}$ as the *spin connection*. Under the assumption that the tetrad postulate holds

$$\nabla_\mu e_{b\nu} = \partial_\mu e_{b\nu} - \Gamma^\sigma_{\mu\nu} e_{b\sigma} - {\omega_\mu}^c{}_b e_{c\nu} = 0, \tag{6}$$

using the orthonormality of $e_{a\mu}$, we can write the spin connection in terms of the vierbein and affine connection as

$$\omega_{\mu ab} = {e_a}^\nu (\partial_\mu e_{b\nu} - \Gamma^\sigma_{\mu\nu} e_{b\sigma}). \tag{7}$$

Note that because of the intrinsic spin of the Dirac field, $\Gamma^\sigma_{\mu\nu}$ is in general *asymmetric* in μ, ν, resulting in a non-vanishing spacetime torsion [19], [13]. The infinitesimal Lorentz generators in the Dirac spinor representation are

$$S_{ab} = -\frac{\mathrm{i}}{2}\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b], \tag{8}$$

where γ_a are the *Dirac matrices*, and $\sigma_{ab} \equiv \mathrm{i}/2[\gamma_a, \gamma_b]$. Taking account of the Dirac matrix anticommutator

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}, \tag{9}$$

it can be shown that the right-hand side of (8) satisfies the Lie bracket identity for Lorentz generators

$$[S_{ab}, S_{cd}] = \eta_{ad} S_{bc} + \eta_{bc} S_{ad} - \eta_{ac} S_{bd} - \eta_{bd} S_{ac}. \tag{10}$$

Using (5) and (8), we can rewrite the covariant derivative of the Dirac spinor as

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{8}{\omega_\mu}^{ab}[\gamma_a, \gamma_b]\psi. \tag{11}$$

Substituting this into (1), then absorbing the vierbeins and rearranging, the Dirac equation becomes

$$\frac{\mathrm{i}}{8}\omega^{abc}\gamma_a[\gamma_b, \gamma_c]\psi = -\Delta_f \psi, \tag{12}$$

where Δ_f is an abbreviated form of the flat spacetime Dirac operator

$$\Delta_f \psi \equiv (\mathrm{i}\gamma^a \partial_a - m)\psi. \tag{13}$$

Using the Dirac identity

$$\gamma_a \gamma_b \gamma_c = \eta_{ab}\gamma_c + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - \mathrm{i}\epsilon_{abcd}\gamma_5 \gamma^d, \tag{14}$$

we can write the commutator in the last two indices as

$$\gamma_a[\gamma_b, \gamma_c] = 2(\eta_{ab}\gamma_c - \eta_{ac}\gamma_b - \mathrm{i}\epsilon_{abcd}\gamma_5\gamma^d). \tag{15}$$

The conventions we use for γ_5 and the Levi-Civita symbol are those of Itzykson and Zuber [18]:

$$\epsilon^{abcd} = -\epsilon_{abcd} = \begin{cases} +1 & \text{if } \{a, b, c, d\} \text{ even} \\ -1 & \text{odd} \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

$$\gamma^5 = \gamma_5 = -(\mathrm{i}/4!)\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3. \tag{17}$$

The left-hand side operator of (12) therefore becomes

$$\begin{aligned}\frac{\mathrm{i}}{8}\omega^{abc}\gamma_a[\gamma_b,\gamma_c] &= \frac{\mathrm{i}}{4}\omega^{abc}(\eta_{ab}\gamma_c - \eta_{ac}\gamma_b - \mathrm{i}\epsilon_{abcd}\gamma_5\gamma^d) \\ &= \frac{\mathrm{i}}{4}(\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd})\omega^{abc}\gamma^d + \frac{1}{4}\epsilon_{abcd}\omega^{abc}\gamma_5\gamma^d \\ &= \Omega_d\gamma^d + \Omega_{5d}\gamma_5\gamma^d, \end{aligned} \tag{18}$$

where we define the *gravitational vector potentials* as

$$\Omega_d \equiv \frac{1}{4}\delta_{adbc}\omega^{abc}, \tag{19}$$

$$\Omega_{5d} \equiv \frac{1}{4}\epsilon_{abcd}\omega^{abc}, \tag{20}$$

with the mixed symmetry tensor

$$\delta_{abcd} \equiv \mathrm{i}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}). \tag{21}$$

In terms of the Ω-potentials, the Dirac equation now reads

$$(\Omega_a\gamma^a + \Omega_{5a}\gamma_5\gamma^a)\psi = -\Delta_f\psi. \tag{22}$$

This can be compared with the electromagnetically covariant Dirac equation in *flat* space-time

$$-qA_a\gamma^a\psi = -\Delta_f\psi. \tag{23}$$

We can see that there is an analogy between Ω_a and $-qA_a$, in the sense that these terms are coupled to $\gamma^a\psi$. However, in electromagnetism there is no equivalent potential to Ω_{5a}, say $-qA_{5a}$, which couples to $\gamma_5\gamma^a$. Such an analogous term could in principle arise in an Abelian chiral generalization of the electromagnetic gauge group, such as local $U(1)_L \times U(1)_R$ symmetry.

3 The inversion procedure

The inversion of the Dirac equation for the components of the spin connection which couple to spin-1/2 fermions proceeds in a similar fashion to the analogous $U(1)$ electromagnetic [16] and non-Abelian $SU(2)$ [15] cases. In all of these cases, the procedure involves the formation of spinor bilinears, which in the tradition of Takabayasi [24], Zhelnorovich [26], and Halbwachs [11], we can write as a set of 16 tensor fields: scalar, psuedoscalar, four-vector, axial four-vector, and rank-2 tensor

$$\sigma = \overline{\psi}\psi, \tag{24a}$$

$$\omega = \overline{\psi}\gamma_5\psi, \tag{24b}$$

$$j^a = \overline{\psi}\gamma^a\psi, \tag{24c}$$

$$k^a = \overline{\psi}\gamma_5\gamma^a\psi, \tag{24d}$$

$$s^{ab} = \overline{\psi}\sigma^{ab}\psi. \tag{24e}$$

In addition, we also have the dual rank-2 tensor

$$ {}^{*}s^{ab} = \frac{\mathrm{i}}{2}\epsilon^{abcd}s_{cd} = \overline{\psi}\gamma_5\sigma^{ab}\psi, \tag{25} $$

as well as the $U(1)$ gauge dependent four-vectors

$$ m^a + \mathrm{i}n^a = \overline{\psi}^{\mathrm{c}}\gamma^a\psi \tag{26a} $$

$$ m^a = \mathrm{Re}\{\overline{\psi}^{\mathrm{c}}\gamma^a\psi\} = \frac{1}{2}(\overline{\psi}^{\mathrm{c}}\gamma^a\psi + \overline{\psi}\gamma^a\psi^{\mathrm{c}}) \tag{26b} $$

$$ n^a = \mathrm{Im}\{\overline{\psi}^{\mathrm{c}}\gamma^a\psi\} = \frac{\mathrm{i}}{2}(\overline{\psi}\gamma^a\psi^{\mathrm{c}} - \overline{\psi}^{\mathrm{c}}\gamma^a\psi), \tag{26c} $$

where $\overline{\psi}$ and ψ^{c} are the Dirac and charge conjugate spinors respectively. The bilinear set are all *real*, except for ω and ${}^{*}s^{ab}$, which are pure imaginary; this is just a conventional choice, which can be altered by defining the new *real* bilinears $-\mathrm{i}\omega$ and $-\mathrm{i}{}^{*}s^{ab}$. Now, left-multiplying (22) by $\overline{\psi}\gamma^b$ gives

$$ \Omega_a\overline{\psi}\gamma^b\gamma^a\psi + \Omega_{5a}\overline{\psi}\gamma^b\gamma_5\gamma^a\psi = -\mathrm{i}\overline{\psi}\gamma^b\gamma^a(\partial_a\psi) + m\overline{\psi}\gamma^b\psi. \tag{27} $$

Applying the Dirac identities

$$ \gamma^b\gamma^a = \eta^{ba} - \mathrm{i}\sigma^{ba}, \tag{28a} $$

$$ \{\gamma_5, \gamma^a\} = 0, \tag{28b} $$

and writing closed-form bilinears in tensor notation, we get

$$ (\sigma\eta^{ba} - \mathrm{i}s^{ba})\Omega_a + (-\omega\eta^{ba} + \mathrm{i}{}^{*}s^{ba})\Omega_{5a} = -\mathrm{i}\overline{\psi}(\partial^b\psi) - \overline{\psi}\sigma^{ba}(\partial_a\psi) + mj^b. \tag{29} $$

Likewise, left-multiplying (22) by $\overline{\psi}\gamma_5\gamma^b$, and applying the same Dirac identities yields

$$ (\omega\eta^{ba} - \mathrm{i}{}^{*}s^{ba})\Omega_a + (-\sigma\eta^{ba} + \mathrm{i}s^{ba})\Omega_{5a} = -\mathrm{i}\overline{\psi}\gamma_5(\partial^b\psi) - \overline{\psi}\gamma_5\sigma^{ba}(\partial_a\psi) + mk^b. \tag{30} $$

In order to proceed, we require the Dirac equation for the charge conjugated spinor. It can be shown [23] that in the absence of electromagnetic fields, this equation has exactly the same form as (1), such that

$$ (\Omega_a\gamma^a + \Omega_{5a}\gamma_5\gamma^a)\psi^{\mathrm{c}} = -\Delta_f\psi^{\mathrm{c}}. \tag{31} $$

Following the same steps as above, left-multiplying by $\overline{\psi}^{\mathrm{c}}\gamma^b$ and $\overline{\psi}^{\mathrm{c}}\gamma_5\gamma^b$ yields the respective equations

$$ \begin{aligned} (\overline{\psi}^{\mathrm{c}}\psi^{\mathrm{c}}\eta^{ba} - \mathrm{i}\overline{\psi}^{\mathrm{c}}\sigma^{ba}\psi^{\mathrm{c}})\Omega_a &+ (-\overline{\psi}^{\mathrm{c}}\gamma_5\psi^{\mathrm{c}}\eta^{ba} + \mathrm{i}\overline{\psi}^{\mathrm{c}}\gamma_5\sigma^{ba}\psi^{\mathrm{c}})\Omega_{5a} \\ &= -\mathrm{i}\overline{\psi}^{\mathrm{c}}(\partial^b\psi^{\mathrm{c}}) - \overline{\psi}^{\mathrm{c}}\sigma^{ba}(\partial_a\psi^{\mathrm{c}}) + m\overline{\psi}^{\mathrm{c}}\gamma^b\psi^{\mathrm{c}}, \end{aligned} \tag{32} $$

$$ \begin{aligned} (\overline{\psi}^{\mathrm{c}}\gamma_5\psi^{\mathrm{c}}\eta^{ba} - \mathrm{i}\overline{\psi}^{\mathrm{c}}\gamma_5\sigma^{ba}\psi^{\mathrm{c}})\Omega_a &+ (-\overline{\psi}^{\mathrm{c}}\psi^{\mathrm{c}}\eta^{ba} + \mathrm{i}\overline{\psi}^{\mathrm{c}}\sigma^{ba}\psi^{\mathrm{c}})\Omega_{5a} \\ &= -\mathrm{i}\overline{\psi}^{\mathrm{c}}\gamma_5(\partial^b\psi^{\mathrm{c}}) - \overline{\psi}^{\mathrm{c}}\gamma_5\sigma^{ba}(\partial_a\psi^{\mathrm{c}}) + m\overline{\psi}^{\mathrm{c}}\gamma_5\gamma^b\psi^{\mathrm{c}}. \end{aligned} \tag{33} $$

Using the definition for the charge conjugate spinor

$$ \psi^{\mathrm{c}} = C\overline{\psi}^{\mathrm{T}} = \mathrm{i}\gamma^2\gamma^0\overline{\psi}^{\mathrm{T}}, \tag{34} $$

we can derive a relationship between bilinears with *non-Grassmann*[1] charge conjugate spinors and regular spinors

$$\overline{\psi}^{\mathrm{c}}\Gamma\chi^{\mathrm{c}} = -\overline{\chi}C^{-1}\Gamma^{\mathrm{T}}C\psi, \tag{35}$$

where the spinor χ may have tensor indices (ie. $\chi = \partial_a\psi$), and Γ is an element of the same Dirac-Clifford algebra defining the set (24a)-(24e). Applying the Dirac matrix charge conjugation identities [18]

$$C^{-1}\gamma^{a\mathrm{T}}C = -\gamma^a, \tag{36a}$$
$$C^{-1}\gamma_5^{\mathrm{T}}C = \gamma_5, \tag{36b}$$
$$C^{-1}(\gamma_5\gamma^a)^{\mathrm{T}}C = \gamma_5\gamma^a, \tag{36c}$$
$$C^{-1}\sigma^{ab\mathrm{T}}C = -\sigma^{ab}, \tag{36d}$$
$$C^{-1}(\gamma_5\sigma^{ab})^{\mathrm{T}}C = -\gamma_5\sigma^{ab}, \tag{36e}$$

we can rewrite (32) and (33) as

$$(-\sigma\eta^{ba} - \mathrm{i}s^{ba})\Omega_a + (\omega\eta^{ba} + \mathrm{i}^*s^{ba})\Omega_{5a} = \mathrm{i}(\partial^b\overline{\psi})\psi - (\partial_a\overline{\psi})\sigma^{ba}\psi + mj^b, \tag{37}$$
$$(-\omega\eta^{ba} - \mathrm{i}^*s^{ba})\Omega_a + (\sigma\eta^{ba} + \mathrm{i}s^{ba})\Omega_{5a} = \mathrm{i}(\partial^b\overline{\psi})\gamma_5\psi - (\partial_a\overline{\psi})\gamma_5\sigma^{ba}\psi - mk^b \tag{38}$$

respectively. Subtracting (37) from (29), and (38) from (37), yields the respective equations

$$2\sigma\Omega^a - 2\omega\Omega_5^a = -\mathrm{i}\partial^a\sigma - [\overline{\psi}\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\sigma^{ab}\psi], \tag{39}$$
$$2\omega\Omega^a - 2\sigma\Omega_5^a = -\mathrm{i}\partial^a\omega - [\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] + 2mk^a, \tag{40}$$

where we have relabelled the indices. Multiplying (39) and (40) by (σ, ω) and (ω, σ) respectively, then subtracting the second equation from the first gives

$$\begin{aligned}\Omega^a = \frac{1}{2}(\sigma^2 - \omega^2)^{-1}\{&\mathrm{i}[\omega(\partial^a\omega) - \sigma(\partial^a\sigma)] + \omega[\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] \\ &- \sigma[\overline{\psi}\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\sigma^{ab}\psi] - 2m\omega k^a\},\end{aligned} \tag{41}$$
$$\begin{aligned}\Omega_5^a = \frac{1}{2}(\sigma^2 - \omega^2)^{-1}\{&\mathrm{i}[\sigma(\partial^a\omega) - \omega(\partial^a\sigma)] + \sigma[\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] \\ &- \omega[\overline{\psi}\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\sigma^{ab}\psi] - 2m\sigma k^a\},\end{aligned} \tag{42}$$

the inverted form of the Dirac equation in curved spacetime.

4 Bilinear refinement using Fierz identities

It is apparent however, that the bracketed second and third terms in (41) and (42) are not closed-form bilinears, due to the minus sign preventing a simple application of the Leibniz

[1]In the Maxwell-Dirac system, an inconsistency arises if one raises both sides of the inhomogeneous Maxwell equations to a power ≥ 5. If the Dirac spinor components are assumed to be *Grassmann*, then the source term vanishes: $(\partial_\nu F^{\nu\mu})^5 = 0$, due to each term in the spinor expansion of $(j^\mu)^5$ containing $(\psi_\alpha)^2 = 0$ $(\alpha = 1, 2, 3, 4)$. We therefore do not introduce the usual "by hand" sign-reversal of the right-hand side of (35), for Grassmann Dirac spinors under the Fermi statistics rule. The result is an overall sign change compared with the usual transformation of bilinears under a charge conjugation operation [20], however the end result of the inversion process does not depend upon this technicality since both sides of (32) and (33) are equally affected.

rule for derivatives. It is possible to show through a very lengthy algebraic process that Fierz expansions can be used to re-write these terms in closed tensor form. Due to the sheer length and tediousness of these calculations, they are not given here, however their derivation follows a similar process to Appendix C in [16] and Appendix B in [17].

The Fierz expansion can be used to write the outer product of two spinors $\psi\overline{\chi}$, which is a 4×4 matrix in the spinor degrees of freedom, as a sum of terms over the basis of Dirac-Clifford matrices with bilinear coefficients

$$\psi\overline{\chi} = \frac{1}{4}(\overline{\chi}\psi)I + \frac{1}{4}(\overline{\chi}\gamma_a\psi)\gamma^a + \frac{1}{8}(\overline{\chi}\sigma_{ab}\psi)\sigma^{ab} - \frac{1}{4}(\overline{\chi}\gamma_5\gamma_a\psi)\gamma_5\gamma^a + \frac{1}{4}(\overline{\chi}\gamma_5\psi)\gamma_5, \tag{43}$$

which can be derived from the more formal expression

$$\psi\overline{\chi} = \sum_{R=1}^{16} a_R\Gamma_R \tag{44}$$

where $R = 1, ..., 16$ runs over all of the elements of the Dirac-Clifford algebra. Multiplying (44) from the right by Dirac matrix Γ_B [where B runs over the types: *scalar*, ..., *rank-2 tensor* in (24a)-(24e)], and using the trace identities

$$\mathrm{Tr}(\Gamma_R\Gamma_B) = \begin{cases} \mathrm{Tr}(\Gamma_B^2), \text{ if } R = B, \\ 0 \text{ otherwise,} \end{cases} \tag{45}$$

$$\mathrm{Tr}(\psi\overline{\chi}\Gamma_B) = \overline{\chi}\Gamma_B\psi, \tag{46}$$

along with the trace properties of the Dirac matrices, one can easily derive (43).

Following a very tedious process of applying (43) to the terms in (41) and (42) where the spinors are visible, we obtain the purely bilinear expressions

$$\begin{aligned} &\omega[\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] - \sigma[\overline{\psi}\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\sigma^{ab}\psi] \\ &\quad = (\sigma^2 - \omega^2)^{-1}\{s^{ab}[\omega j^c(\partial_b k_c) + \mathrm{i}\sigma m^c(\partial_b n_c)] - {}^*s^{ab}[\sigma j^c(\partial_b k_c) + \mathrm{i}\omega m^c(\partial_b n_c)]\} \\ &\qquad + \delta^{abcd}[k_c(\partial_b k_d) - j_c(\partial_b j_d)], \end{aligned} \tag{47}$$

$$\begin{aligned} &\sigma[\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] - \omega[\overline{\psi}\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\sigma^{ab}\psi] \\ &\quad = (\sigma^2 - \omega^2)^{-1}\{s^{ab}[\sigma j^c(\partial_b k_c) + \mathrm{i}\omega m^c(\partial_b n_c)] - {}^*s^{ab}[\omega j^c(\partial_b k_c) + \mathrm{i}\sigma m^c(\partial_b n_c)]\} \\ &\qquad + \epsilon^{abcd}[k_c(\partial_b k_d) - j_c(\partial_b j_d)]. \end{aligned} \tag{48}$$

Using the Fierz identities [6]

$$s^{ab} = (\sigma^2 - \omega^2)^{-1}(\sigma\epsilon^{abcd} - \omega\delta^{abcd})j_c k_d \tag{49a}$$

$${}^*s^{ab} = (\sigma^2 - \omega^2)^{-1}(\omega\epsilon^{abcd} - \sigma\delta^{abcd})j_c k_d, \tag{49b}$$

$$\mathrm{i}\epsilon^{abcd}j_c k_d = \mathrm{i}(m^a n^b - m^b n^a) = \delta^{abcd}m_c n_d, \tag{49c}$$

$$\mathrm{i}\delta^{abcd}j_c k_d = -j^a k^b + j^b k^a = \epsilon^{abcd}m_c n_d, \tag{49d}$$

the expressions within the curved braces can be recast as

$$\begin{aligned} &s^{ab}[\omega j^c(\partial_b k_c) + \mathrm{i}\sigma m^c(\partial_b n_c)] - {}^*s^{ab}[\sigma j^c(\partial_b k_c) + \mathrm{i}\omega m^c(\partial_b n_c)] \\ &\quad = \delta^{abcd}[j_c j^e k_d(\partial_b k_e) + m_c m^e n_d(\partial_b n_e)], \end{aligned} \tag{50}$$

$$\begin{aligned} &s^{ab}[\sigma j^c(\partial_b k_c) + \mathrm{i}\omega m^c(\partial_b n_c)] - {}^*s^{ab}[\omega j^c(\partial_b k_c) + \mathrm{i}\sigma m^c(\partial_b n_c)] \\ &\quad = \epsilon^{abcd}[j_c j^e k_d(\partial_b k_e) + m_c m^e n_d(\partial_b n_e)]. \end{aligned} \tag{51}$$

To proceed further, we require the tetrad frame of four-vector bilinears, with scalar normalizing factor:

$$\mathrm{t}^a{}_\alpha = (\sigma^2 - \omega^2)^{-1/2}[j^a, m^a, n^a, k^a], \tag{52}$$

where $\alpha = 0, 1, 2, 3$ labels the columns. The tetrad orthonormality implies

$$\mathrm{t}^a{}_\alpha t^\alpha{}_b = \delta^a{}_b = (\sigma^2 - \omega^2)^{-1}(j^a j_b - m^a m_b - n^a n_b - k^a k_b), \tag{53}$$

and taking the derivative yields

$$\mathrm{t}^a{}_\alpha(\partial_b \mathrm{t}_{a\beta}) = -\mathrm{t}^a{}_\beta(\partial_b \mathrm{t}_{a\alpha}), \tag{54}$$

which provides the freedom to switch what bilinear the derivative operator acts on, when the Lorentz index is summed over. In the special case where $\alpha = \beta$, we can replace four-vectors entirely via

$$j^a(\partial_b j_a) = -m^a(\partial_b m_a) = -n^a(\partial_b n_a) = -k^a(\partial_b k_a) = \sigma(\partial_b\sigma) - \omega(\partial_b\omega), \tag{55}$$

which is just the derivative of the invariant length squared Fierz identity [6]. Note that (55) is consistent with (53), when setting $a = b$ and summing. Applying these identities to the square brackets in (50) and (51) gives, after some manipulation

$$\begin{aligned} &j_c j^e k_d(\partial_b k_e) + m_c m^e n_d(\partial_b n_e) \\ &\quad = \frac{1}{2}(\sigma^2 - \omega^2)[j_c(\partial_b j_d) - k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)]. \end{aligned} \tag{56}$$

We now write a much simpler form of (47) and (48):

$$\begin{aligned} &\omega[\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] - \sigma[\overline{\psi}\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\sigma^{ab}\psi] \\ &\quad = \frac{1}{2}\delta^{abcd}[-j_c(\partial_b j_d) + k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)], \end{aligned} \tag{57}$$

$$\begin{aligned} &\sigma[\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] - \omega[\overline{\psi}\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\sigma^{ab}\psi] \\ &\quad = \frac{1}{2}\epsilon^{abcd}[-j_c(\partial_b j_d) + k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)]. \end{aligned} \tag{58}$$

Finally, substituting into (41) and (42), we obtain the gravitational four-vector potentials in terms of *closed-form bilinears* only

$$\begin{aligned} \Omega^a = &\frac{1}{2}(\sigma^2 - \omega^2)^{-1}\{\mathrm{i}[\omega(\partial^a\omega) - \sigma(\partial^a\sigma)] - 2m\omega k^a \\ &+ \frac{1}{2}\delta^{abcd}[-j_c(\partial_b j_d) + k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)]\}, \end{aligned} \tag{59}$$

$$\begin{aligned} \Omega_5^a = &\frac{1}{2}(\sigma^2 - \omega^2)^{-1}\{\mathrm{i}[\sigma(\partial^a\omega) - \omega(\partial^a\sigma)] - 2m\sigma k^a \\ &+ \frac{1}{2}\epsilon^{abcd}[-j_c(\partial_b j_d) + k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)]\}. \end{aligned} \tag{60}$$

Comparing with the inverted Dirac equation in the electromagnetic case [16]

$$\begin{aligned} A^a = &\frac{1}{2q}(\sigma^2 - \omega^2)^{-1}\{\epsilon^{abcd}[j_c(\partial_b k_d) - k_c(\partial_b j_d)] + m^b(\partial^a n_b) - 2m\sigma j^a\} \\ &+ \frac{1}{2q}(\sigma^2 - \omega^2)^{-2}\{\delta^{abcd} j_c k_d[\omega(\partial_b\sigma) - \sigma(\partial_b\omega)] + \epsilon^{abcd} j_c k_d[\omega(\partial_b\omega) - \sigma(\partial_b\sigma)]\}, \end{aligned} \tag{61}$$

where the totality of the $U(1)$ gauge dependence is represented by the $m^b(\partial^a n_b)$ term, we can see that there are some interesting structural similarities.

5 The Einstein-Cartan-Dirac self-coupled system

5.1 Spin Connection Decomposition

From the definitions (19) and (20), we can see that the gravitational vector potentials correspond to components of the spin connection ω_{abc}. Now, since the latter corresponds to a rank-3 tensor representation of the local Lorentz group $SO(1,3)$, we can write it as the sum of a traceless component, plus a component belonging to the orthogonal subspace of trace terms [12]. Taking account of the spin connection antisymmetry in the second and third indices, we have

$$\begin{aligned}\omega_{abc} &= \omega^{(t)}_{abc} + \frac{1}{3}\eta_{ab}\omega_d{}^d{}_c - \frac{1}{3}\eta_{ac}\omega_d{}^d{}_b \\ &= \omega^{(t)}_{abc} - \frac{\mathrm{i}}{3}\delta_{aebc}\omega_d{}^{de}. \end{aligned} \tag{62}$$

Expanding (19), we can make the replacement

$$\omega_d{}^{de} = -2\mathrm{i}\Omega^e, \tag{63}$$

thus obtaining for the decomposition into the traceless, plus irreducible trace part (with Young pattern [1])

$$\omega_{abc} = \omega^{(t)}_{abc} - \frac{2}{3}\delta_{adbc}\Omega^d. \tag{64}$$

Now due to its antisymmetry in the second two indices, the traceless term $\omega^{(t)}_{abc}$ can be split into two further irreducible parts: a fully antisymmetric term $\omega^{(a)}_{abc}$ and a mixed symmetry term w_{abc} with Young patterns $[1^3]$ and [21] respectively, such that

$$\omega_{abc} = (\omega^{(a)}_{abc} + w_{abc}) - \frac{2}{3}\delta_{adbc}\Omega^d. \tag{65}$$

Note that the first two terms on the right-hand side can be written in terms of the spin connection as

$$\omega^{(a)}_{abc} = \frac{1}{3}(\omega_{abc} + \omega_{bca} + \omega_{cab}), \tag{66}$$

$$w_{abc} = \frac{1}{3}(2\omega_{abc} - \omega_{bca} - \omega_{cab}) + \frac{2}{3}\delta_{adbc}\Omega^d. \tag{67}$$

Contracting both sides by $\epsilon^{abce}/4$, and using the definition (20), we find

$$\Omega^e_5 = \frac{1}{4}\epsilon^{abce}\omega^{(a)}_{abc}, \tag{68}$$

since the other terms vanish. Contracting again, but by $\epsilon_{a'b'c'e}$, and using the Levi-Civita contraction identity

$$\begin{aligned}\epsilon_{a'b'c'e}\epsilon^{abce} = & -\delta_{a'}{}^a\delta_{b'}{}^b\delta_{c'}{}^c + \delta_{a'}{}^a\delta_{b'}{}^c\delta_{c'}{}^b - \delta_{a'}{}^b\delta_{b'}{}^c\delta_{c'}{}^a + \delta_{a'}{}^b\delta_{b'}{}^a\delta_{c'}{}^c \\ & - \delta_{a'}{}^c\delta_{b'}{}^a\delta_{c'}{}^b + \delta_{a'}{}^c\delta_{b'}{}^b\delta_{c'}{}^a, \end{aligned} \tag{69}$$

we find on rearrangement

$$\omega^{(a)}_{abc} = -\frac{2}{3}\epsilon_{abcd}\Omega^d_5, \tag{70}$$

which yields upon substitution

$$\omega_{abc} = w_{abc} - \frac{2}{3}\delta_{adbc}\Omega^d - \frac{2}{3}\epsilon_{abcd}\Omega_5^d. \tag{71}$$

We have thus obtained an expression for the spin connection which allows for its replacement in terms of the bilinear Dirac matter states, via the inverted forms of the Dirac equation (59) and (60), with the exception of the residual term w_{abc}. As we shall see in the final subsection, w_{abc} can be replaced by the irreducible traceless mixed-symmetry component of the object of anholonomity (78), which itself can be replaced by Dirac bilinears when the vierbein is chosen to be the bilinear tetrad (52). Thus, we are able to obtain an expression for the spin connection entirely in terms of Dirac bilinears.

5.2 Curvature Field Equations

Now consider Einstein's equations coupled to a source term with generally non-vanishing cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{72}$$

In the present case, where the gravitational field couples to the Dirac field, the asymmetric canonical stress-energy tensor on the right hand side is given by [10]

$$T_{\mu\nu} = \frac{\mathrm{i}}{2}[\overline{\psi}\gamma_\mu(\nabla_\nu\psi) - (\nabla_\nu\overline{\psi})\gamma_\mu\psi]. \tag{73}$$

This can be rewritten in terms of Dirac bilinears with the use of Fierz identities [17], which yields

$$T_{\mu\nu} = \frac{1}{2}(\sigma^2 - \omega^2)^{-1}[\mathrm{i}k_\mu(\omega\partial_\nu\sigma - \sigma\partial_\nu\omega) - g^{-1/2}\epsilon_{\mu\sigma\rho\epsilon}(\nabla_\nu j^\sigma)j^\rho k^\epsilon + j_\mu m^\sigma(\nabla_\nu n_\sigma)]. \tag{74}$$

One the other side of the equation, we have the contractions of the curvature tensor, which in terms of the spin connection is given by [19], [1]

$$R^a{}_{b\mu\nu} = \partial_\nu\omega_\mu{}^a{}_b - \partial_\mu\omega_\nu{}^a{}_b - \omega_\mu{}^a{}_e\omega_\nu{}^e{}_b + \omega_\nu{}^a{}_e\omega_\mu{}^e{}_b. \tag{75}$$

It is important to note that the curvature tensor we use is not the Riemannian one from standard general relativity, due to the presence of a non-vanishing torsion field. The non-Riemannian component of $R^a{}_{b\mu\nu}$ vanishes in the limit where the torsion vanishes. An expression in terms of locally orthonormal components is obtained, as usual, via contraction with the vierbein

$$\begin{aligned} R^{ab}{}_{cd} &\equiv e^\mu{}_c e^\nu{}_d R^{ab}{}_{\mu\nu} \\ &= [e^\mu{}_c(\partial_d e_\mu{}^e) - e^\mu{}_d(\partial_c e_\mu{}^e)]\omega_e{}^{ab} + \partial_d\omega_c{}^{ab} - \partial_c\omega_d{}^{ab} - \omega_c{}^a{}_e\omega_d{}^{eb} + \omega_d{}^a{}_e\omega_c{}^{eb}. \end{aligned} \tag{76}$$

Switching the derivatives on the vierbein terms (which reverses the sign), we can write the curvature tensor as

$$R^{ab}{}_{cd} = \Theta^e{}_{cd}\omega_e{}^{ab} + \partial_d\omega_c{}^{ab} - \partial_c\omega_d{}^{ab} - \omega_c{}^a{}_e\omega_d{}^{eb} + \omega_d{}^a{}_e\omega_c{}^{eb}, \tag{77}$$

where we define the *objects of anholonomity* as

$$\Theta_{abc} \equiv e_{\mu a}(\partial_b e^\mu{}_c - \partial_c e^\mu{}_b), \tag{78}$$

which are representative of the non-commutativity of the tetrad basis [13]. Contracting b and d in the curvature tensor yields the Ricci tensor

$$R^a{}_b = \Theta^c{}_{bd}\omega_c{}^{ad} + \partial_c\omega_b{}^{ac} - \partial_b\omega_c{}^{ac} - \omega_b{}^a{}_c\omega_d{}^{cd} + \omega_d{}^a{}_c\omega_b{}^{cd}, \tag{79}$$

with the final contraction yielding the Ricci scalar

$$R = \Theta_{abc}\omega^{abc} + 2\partial_a\omega_b{}^{ba} - \omega_a{}^a{}_b\omega_c{}^{bc} + \omega_{abc}\omega^{bca}. \tag{80}$$

5.3 Torsion Field Equations

The torsion tensor is defined as the degree to which the affine connection fails to be symmetric:

$$\Upsilon_{\mu\nu}{}^\lambda = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \tag{81}$$

A particle field with intrinsic quantum spin will act as the source of a non-vanishing torsion field, in an analogous manner to stress-energy acting as the source of curvature [13]. The torsion field equation is given by

$$\Upsilon_{\mu\nu}{}^\gamma + \delta^\gamma_\mu\Upsilon_{\nu\sigma}{}^\sigma - \delta^\gamma_\nu\Upsilon_{\mu\sigma}{}^\sigma = 8\pi G\Sigma_{\mu\nu}{}^\gamma. \tag{82}$$

Together, the curvature and torsion gravitational field equations, (72) and (82), comprise the Einstein-Cartan(-Sciama-Kibble) equations.

In terms of the spinor field, the canonical spin momentum tensor in a locally orthonormal frame is

$$\Sigma^{abc} = \frac{\mathrm{i}}{4}\overline{\psi}\gamma^{[a}\gamma^b\gamma^{c]}\psi. \tag{83}$$

Given that

$$\gamma^{[a}\gamma^b\gamma^{c]} = \frac{1}{6}(\gamma^a\gamma^b\gamma^c - \gamma^a\gamma^c\gamma^b + \gamma^b\gamma^c\gamma^a - \gamma^b\gamma^a\gamma^c + \gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a), \tag{84}$$

we can apply the Dirac identities

$$\gamma^a\gamma^b = 2\eta^{ab} - \gamma^b\gamma^a \tag{85a}$$
$$\gamma^a\sigma^{bc} = \mathrm{i}\eta^{ab}\gamma^c - \mathrm{i}\eta^{ac}\gamma^b + \epsilon^{abcd}\gamma_5\gamma_d \tag{85b}$$

to obtain

$$\gamma^{[a}\gamma^b\gamma^{c]} = -\mathrm{i}\epsilon^{abcd}\gamma_5\gamma_d. \tag{86}$$

Substituting into (83), we find

$$\Sigma^{abc} = \frac{1}{4}\epsilon^{abcd}k_d, \tag{87}$$

the spin angular momentum tensor of the Dirac field is proportional to the rank-3 dual of the axial vector bilinear. With regards to the left-hand side of (82), using the tetrad postulate (6), we can write the torsion in terms of the object of anholonomity and spin connection

$$\Upsilon_{abc} \equiv \Upsilon_{\mu\nu}{}^\lambda e^\mu{}_a e^\nu{}_b e_{\lambda c} = \Theta_{cba} - \omega_{abc} + \omega_{bac}. \tag{88}$$

Alternatively, taking an appropriate cyclic combination of the torsion, the spin connection can be written as [5], [1]

$$\omega_{abc} = K_{abc} + \frac{1}{2}(\Theta_{abc} - \Theta_{bca} - \Theta_{cab}), \tag{89}$$

where we define the *contorsion* tensor to be

$$K_{abc} \equiv \frac{1}{2}(-\Upsilon_{abc} + \Upsilon_{bca} - \Upsilon_{cab}). \tag{90}$$

5.4 Constraints Arising From Torsion

We shall now demonstrate how the torsion field equation can be used to obtain a further, very useful set of constraints on the Einstein-Cartan-Dirac system. For convenience, we shall consider the torsion field equation in a local frame

$$\Upsilon_{abc} + \eta_{ac}\Upsilon_b{}^d{}_d - \eta_{bc}\Upsilon_a{}^d{}_d = 8\pi G\Sigma_{abc}. \tag{91}$$

Now, substituting the irreducible decomposition of the spin connection (71) into the torsion (88), we obtain

$$\Upsilon_{abc} = -\Theta_{cab} + w_{cab} - \frac{2}{3}\delta_{cdab}\Omega^d + \frac{4}{3}\epsilon_{cdab}\Omega_5^d, \tag{92}$$

where we have used the cyclic identities

$$\delta_{adbc} - \delta_{bdac} = -\delta_{cdab}, \tag{93}$$
$$-w_{abc} + w_{bac} = w_{cab}. \tag{94}$$

Taking the trace of (92) in the last two indices, the w and Ω_5 terms vanish, and we find

$$\Upsilon_a{}^b{}_b = \Theta_b{}^b{}_a + 2\mathrm{i}\Omega_a, \tag{95}$$

where we have used the antisymmetry of Θ_{abc} in bc. Substituting (92), (95), and (87) into (91), then gathering terms and rearranging, we obtain an explicit expression for the remaining component of the spin connection

$$w_{abc} = \Theta_{abc} + \mathrm{i}\delta_{adbc}\Theta_e{}^{ed} - \frac{4}{3}\delta_{adbc}\Omega^d - \frac{4}{3}\epsilon_{abcd}\Omega_5^d + 2\pi G\epsilon_{abcd}k^d. \tag{96}$$

Taking the trace of (96), the left-hand side and Levi-Civita terms vanish, and we obtain the constraint on the gravitational vector potential

$$\Omega_a = \frac{\mathrm{i}}{2}\Theta_b{}^b{}_a. \tag{97}$$

Similarly, when we fully contract both sides of (96) with the Levi-Civita tensor, the left-hand side and δ-dependent terms vanish. Using the Levi-Civita contraction identity

$$\epsilon_{abcd}\epsilon^{abcf} = -6\delta_d{}^f, \tag{98}$$

we obtain the constraint on the dual gravitational potential

$$\Omega_5^d = -\frac{1}{8}\Theta_{abc}\epsilon^{abcd} + \frac{3\pi}{2}Gk^d. \tag{99}$$

Substituting our constraints (95) and (99) back into (96), and using the Levi-Civita identity (69), we obtain the expression

$$w_{abc} = \frac{1}{3}(2\Theta_{abc} - \Theta_{bca} - \Theta_{cab}) + \frac{\mathrm{i}}{3}\delta_{adbc}\Theta_e{}^{ed}. \quad (100)$$

This expression can be interpreted as that due to the constraints imposed by the torsion equation, the traceless mixed symmetry irreducible component of the spin connection is equal to the traceless mixed symmetry irreducible component of the object of anholonomity. If the local vierbein frame is chosen to be the bilinear tetrad (52), such that

$$e^\mu{}_a = \mathrm{t}^\mu{}_a \equiv (\sigma^2 - \omega^2)^{-1/2}[j^\mu, m^\mu, n^\mu, k^\mu], \quad (101)$$

the *entire spin connection* (71) can be written in terms of Dirac bilinears, when the inverted forms of the Dirac equation (59) and (60) are taken into account. Therefore, in principle Einstein's equation can also be written in terms of Dirac bilinears only.

Now, substituting the constraints (97), (99), and (100) into (92), we obtain the simple form of the torsion

$$\Upsilon_{abc} = 8\pi G\Sigma_{abc}, \quad (102)$$

which is obviously a solution of (91) due to the vanishing trace of the fully antisymmetric spin tensor. Substituting the same constraints into the spin connection (71) we obtain

$$\omega_{abc} = -4\pi G\Sigma_{abc} + \frac{1}{2}(\Theta_{abc} - \Theta_{bca} - \Theta_{cab}), \quad (103)$$

which is consistent with (89), and the contorsion solution

$$K_{abc} = -4\pi G\Sigma_{abc}. \quad (104)$$

6 Summary and conclusions

For the sake of clarity, we shall collate our main results. We have Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (105)$$

where on the right-hand side, we have the Dirac matter stress-energy tensor

$$T_{\mu\nu} = \frac{1}{2}(\sigma^2 - \omega^2)^{-1}[\mathrm{i}k_\mu(\omega\partial_\nu\sigma - \sigma\partial_\nu\omega) - g^{-1/2}\epsilon_{\mu\sigma\rho\epsilon}(\nabla_\nu j^\sigma)j^\rho k^\epsilon + j_\mu m^\sigma(\nabla_\nu n_\sigma)], \quad (106)$$

and on the left, we have the Ricci tensor

$$R^a{}_b = \Theta^c{}_{bd}\omega_c{}^{ad} + \partial_c\omega_b{}^{ac} - \partial_b\omega_c{}^{ac} - \omega_b{}^a{}_c\omega_d{}^{cd} + \omega_d{}^a{}_c\omega_b{}^{cd}, \quad (107)$$

and Ricci scalar

$$R = \Theta_{abc}\omega^{abc} + 2\partial_a\omega_b{}^{ba} - \omega_a{}^a{}_b\omega_c{}^{bc} + \omega_{abc}\omega^{bca}. \quad (108)$$

Note that our curvature terms implicitly contain a non-zero torsion component. The covariant derivatives in the stress-energy tensor contain the affine connection, which due to the tetrad postulate can be written as

$$\Gamma^\lambda{}_{\mu\nu} = \Gamma^\sigma{}_{\mu\nu}e_{b\sigma}e^{b\lambda} = e^{b\lambda}(\partial_\mu e_{b\nu}) - \omega_\mu{}^c{}_b e_{c\nu}e^{b\lambda}. \quad (109)$$

The spin connection can be replaced by its decomposition into irreducible terms

$$\omega_{abc} = w_{abc} - \frac{2}{3}\delta_{adbc}\Omega^d - \frac{2}{3}\epsilon_{abcd}\Omega_5^d, \tag{110}$$

where the first term can be replaced by the torsion equation constrained expression

$$w_{abc} = \frac{1}{3}(2\Theta_{abc} - \Theta_{bca} - \Theta_{cab}) + \frac{\mathrm{i}}{3}\delta_{adbc}\Theta_e{}^{ed}, \tag{111}$$

and the second two terms can be written in terms of Dirac bilinears, via the gravitational vector potentials obtained by inverting the Dirac equation:

$$\begin{aligned}\Omega^a &= \frac{1}{2}(\sigma^2 - \omega^2)^{-1}\{\mathrm{i}[\omega(\partial^a\omega) - \sigma(\partial^a\sigma)] - 2m\omega k^a \\ &\quad + \frac{1}{2}\delta^{abcd}[-j_c(\partial_b j_d) + k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)]\},\end{aligned} \tag{112}$$

$$\begin{aligned}\Omega_5^a &= \frac{1}{2}(\sigma^2 - \omega^2)^{-1}\{\mathrm{i}[\sigma(\partial^a\omega) - \omega(\partial^a\sigma)] - 2m\sigma k^a \\ &\quad + \frac{1}{2}\epsilon^{abcd}[-j_c(\partial_b j_d) + k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)]\},\end{aligned} \tag{113}$$

The object of anholonomity is given in terms of the vierbein as

$$\Theta_{abc} \equiv e_{\mu a}(\partial_b e^\mu{}_c - \partial_c e^\mu{}_b), \tag{114}$$

which can all be replaced by Dirac bilinears by breaking local Lorentz invariance, and as before, choosing the vierbein frame to be the tetrad of Dirac bilinears

$$e^\mu{}_a = \mathrm{t}^\mu{}_a \equiv (\sigma^2 - \omega^2)^{-1/2}[j^\mu, m^\mu, n^\mu, k^\mu]. \tag{115}$$

Furthermore, the torsion field equation provides us with the constraints

$$\Omega_a = \frac{\mathrm{i}}{2}\Theta_b{}^b{}_a, \tag{116}$$

$$\Omega_5^d = -\frac{1}{8}\Theta_{abc}\epsilon^{abcd} + \frac{3\pi}{2}Gk^d, \tag{117}$$

$$\omega_{abc} = -\pi G\epsilon_{abcd}k^d + \frac{1}{2}(\Theta_{abc} - \Theta_{bca} - \Theta_{cab}). \tag{118}$$

Taken together, the equations (105)-(118) describe the gravitationally self-interacting Einstein-Cartan-Dirac equations, in terms of the Lorentz covariant observables of the Dirac field: the Dirac bilinears. We believe the inverted forms of the Dirac equation (112) and (113), the Fierz identities (57) and (58) that lead to their description in terms of Dirac bilinears as opposed to spinors, and their application to the Einstein-Cartan-Dirac system, to be new results.

In the electromagnetic case of the self-coupled Maxwell-Dirac equations, we showed that this system is able to be reduced in the presence of global spacetime symmetries corresponding to subgroups of the Poincaré group, and we gave four specific examples [16]. The approach we used was an infinitesimal method, which involved using the Lie generators of a particular Poincaré subalgebra, provided by Patera, Winternitz & Zassenhaus [22], to calculate joint invariant scalar and vector fields, which were then applied to the physical equations to obtain new exact and numerical solutions [14]. Due to the similar complexity of the Einstein-Cartan-Dirac equations, global symmetry reduction using the same techniques is one way in which solutions to this system can be pursued.

Another avenue of study which the results of this paper highlight is that of the extended Fierz algebra. In order to manipulate expressions involving Dirac bilinears, which emerge naturally in the Dirac equation inversion programme, knowledge of how these bilinears are algebraically related to each other is required. The derivation of Fierz identities is generally straightforward, but it is tedious and time-consuming to do the calculations by hand. Much of the work that has been done to obtain Fierz identities is relevant to the case where the spinors have no tensor indices. However the Dirac equation contains partial derivatives of spinor fields, so a new class of "rank-1" Fierz identities must be obtained; (57) and (58) are two such examples of a much broader set of identities.

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