Mimetic Einstein-Cartan-Kibble-Sciama (ECKS) gravity

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In this paper we formulate Mimetic theory of gravity in first order formalism for differential forms and show that this exercise is equivalent to mimicking Einstein-Cartan-Kibble-Sciama (ECKS) gravity. We consider different possibilities on how torsion is affected by conformal transformations and discuss how this translates into the interpolation between two different conformal transformations of the spin connection, parameterized with a zero-form parameter λ. We prove that regardless of the type of transformation one chooses, in this setting torsion remains as a non propagating field. We also discuss on the conservation of the mimetic energy momentum tensor and show that the trace of the total energy momentum tensor is not null but depends on both, the value of λ and spacetime torsion.

I. INTRODUCTION

General Relativity (GR) is a classical field theory describing the gravitational interaction by means of the Einstein field equations. Remarkably, it has proven success in a wide range phenomena [1] including black-holes as realistic astrophysical objects [2] and the existence of gravitational waves [3–5]. Another important development of GR is in the context of cosmology, in which extending Einstein's field equations by the inclusion of an early inflationary stage as well as a cold dark matter contribution, is in good agreement with observational data [6]. Despite it enormous prosperity, the standard model of cosmology dictates that dark matter is around 85% of the total mass of matter content in the universe. This fact is somehow dramatic since very few is known about the nature of dark matter. This is why the problem of identifying dark matter candidates still attracts so many attention not only form the point of view of modern cosmology, but also from the point of view of particle physics. There have been several attempts in which dark matter candidates have been proposed, namely, as weakly interacting massive particles, sterile neutrinos, axions, cold massive halo objects and primordial black holes [7–9].

In view of difficulties for the standard cosmology models to describe the nature of dark matter (See for instance [10]), there have been a popular trend for considering modified gravity models [11–16]. These models, however, are subjected to too many observational constraints that it becomes really hard to get a consistent model still compatible with the principles GR. Another interesting direction is to consider GR beyond the limits of Riemannian geometry. The canonical model generalizing GR is the Einstein-Cartan-Kibble-Sciama (ECKS) gravity. This modification, or at least its origin, is indeed very old. It came firstly with the works of Elie Cartan in 1922, before the discovery of spin. However, Cartan's model did not bring many attention until the late 1950s, where Sciama and Kibble rediscovered Cartan's results [17, 18]. The main feature of ECSK gravity is that accounts for the presence of spacetime torsion (See [19]). The presence of spacetime torsion could be emanated from spinning properties of matter, in addition to spacetime curvature been triggered by the mere presence of matter [20, 21]. The torsion two-form does not propagate in vacuum and it is thought to be physically relevant only in regions where high spin densities are present such as in the early Universe [22–29].

More recently in [30, 31], Chamseddine and Mukhanov have considered a different approach for addressing the problem of dark matter with the Mimetic gravity theory. In this model it is shown that the conformal degree of freedom of the gravitational field becomes dynamical even in absence of matter. This extra degree of freedom can be identified with the energy density of the mimetic field, which mimics the energy momentum tensor of a presureless dust without needing dark matter particles. Moreover, it has been discussed in [32] that mimetic cosmology derive late-time acceleration as well as inflationary stage of the universe. Nevertheless, during last years, many authors have considered different aspects of mimetic gravity with interesting results. For instance in the context of black holes [33–43], black strings [44] brane-world scenario [45–50], among others. For a more exhaustive survey, see [51–89] and references therein.

In this paper we pursue the goal of constructing a mimetic theory of gravity in first order formalism for differential forms, and show that for a sufficiently general

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family of conformal transformations for the affine connection, the resulting theory is equivalent to "mimicking" the ECSK model. In order to do, we assume that metric and affine properties of spacetime are independently described in terms of the vierbein one-form $e^a(x)$ and the spin connection one-form $\omega^{ab}(x)$. We translate conformal transformation associated for the metric in terms of the vierbein and we extend for the spin connection in a consistent manner, by following a similar approach presented in [90]. Remarkably, applying a conformal transformation to the full spin connection, the contortion tensor or any intermediate case it is not enough to generate propagating torsion field. This implies that in this setting, torsion can only be triggered by the presence of fermion fields which seems to be a generic feature of conformally invariant theories of gravity. This is in contrast with other modified gravity models such as nonminimal couplings of scalars fields with geometry where the torsion two-form propagates in vacuum (See for instance [91, 92]). In addition we discuss about the trace of the energy momentum tensor, which usually vanishes for conformally invariant theories of gravity, and we show that in the Riemann-Cartan setting the trace of the energy momentum tensor depends on the torsion as well as a parameter which characterizes conformal transformations for the spin connection.

This paper is organized as follows. in section II, we summarize the main aspects of mimetic gravity where the connection of mimetic fields and the effective energy momentum tensor of a preasureless dust is identified. In section III, we revisit ECSK gravity and we give a brief description of Cartan's first order formalism for differential forms. In section IV, we introduce some useful mathematical tools to describe conformal structures in the context of Cartan first order formalism. In Section V we derive the equations of mimetic gravity in first order formalism and show how these correspond to the mimetic ECKS model. In addition, the conservation law for the mimetic energy momentum tensor is highlighted. Finally, in section VI we discuss about the trace of the stress energy tensor and its dependency on torsion and conformal parameter λ. The paper concludes in VII with a brief summary and comments regarding some possible physical applications.

II. MIMETIC GRAVITY

Mimetic Gravity was first introduced by A. Chamseddine and V. Mukhanov as a theory of gravity which naturally exhibits conformal symmetry as internal degree of freedom [30]. Let M_4 be a four dimensional spacetime and consider a physical metric $g_{\mu\nu}$, with Lorentz signature $(-, +, +, +)$, depending on an auxiliary metric $\bar{g}_{\mu\nu}$ and a scalar field ϕ, namely

$$g_{\mu\nu} = -\bar{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\,\bar{g}_{\mu\nu}\,. \qquad (1)$$

The metric $g_{\mu\nu}$ is invariant with respect to conformal transformations of the auxiliary metric $\bar{g}_{\mu\nu}$, i.e., it remains unchanged after rescaling

$$\bar{g}_{\mu\nu} \to \Omega^2(x)\,\bar{g}_{\mu\nu}. \qquad (2)$$

Additionally, it follows from (1) that

$$g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = -1\,. \qquad (3)$$

The resultant new degree of freedom associated with the transformation (1) represents the longitudinal mode of gravity which is excited even in the absence of any matter field configurations.

The canonical action of GR is rewritten by considering the physical metric $g_{\mu\nu}$ as function of the scalar field ϕ and the auxiliary metric $\bar{g}_{\mu\nu}$

$$S = \frac{1}{c}\int \mathrm{d}^4x \sqrt{-g(\bar{g}_{\mu\nu}, \phi)}\left[\frac{1}{\kappa_4}\left(\frac{1}{2}R\left(\bar{g}_{\mu\nu}, \phi\right) - \Lambda\right) + L_m\right], \qquad (4)$$

where $\kappa_4 = \frac{8\pi G}{c^4}$ and L_m stands for the matter Lagrangian. The action (4) is invariant under conformal transformation because it only depends on $g_{\mu\nu}$ which is conformally invariant under (2). The resulting dynamics can be directly obtained by starting from the variation of (4) with respect to the physical metric $g_{\mu\nu}$, then expressing $\delta g_{\mu\nu}$ in terms of $\delta\bar{g}_{\mu\nu}$ and $\delta\phi$ and assuming that the last two are independent. Thus,

$$G^{\mu\nu} - \kappa_4 T^{\mu\nu} + (G - \kappa_4 T)\,g^{\lambda\mu}g^{\sigma\nu}\partial_\lambda\phi\partial_\sigma\phi = 0\,, \qquad (5)$$

$$\nabla_\mu\left[(G - \kappa_4 T)\,\partial^\mu\phi\right] = 0\,, \qquad (6)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ the energy momentum tensor, and G, T denote their respective traces. Clearly dynamics given in (5) and (6) departs from pure GR. In Ref. [93], an equivalent formulation of Mimetic Gravity has been proposed where, instead of introducing ϕ through the reparametrization (4), the physical metric $g_{\mu\nu}$ is directly used together with a constrained scalar field, enforcing (3) through a Lagrange multiplier.

Taking the trace in (5), direct calculation shows

$$(G - \kappa_4 T)\left(1 + g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right) = 0\,. \qquad (7)$$

This last equation is automatically satisfied by the constraint (3) even for $G \neq \kappa_4 T$. From this point of view, even in absence of matter, the gravitational field equations have nontrivial solutions for the conformal mode. To understand this extra degree of freedom, rewrite eq.(5)

$$G^{\mu\nu} = \kappa_4\left(T^{\mu\nu} + \bar{T}^{\mu\nu}\right)\,, \qquad (8)$$

where

$$\bar{T}^{\mu\nu} = \left(T - \frac{G}{\kappa_4}\right)g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi\,. \qquad (9)$$

Now compare this expression with the energy momentum tensor for a perfect fluid

$$T^{\mu\nu} = \frac{1}{c^2}\left(\varepsilon + p\right)u^\mu u^\nu - pg^{\mu\nu}\,, \tag{10}$$

where ε is the energy density, p is the pressure and u^μ is the four-velocity which satisfies $\frac{1}{c^2}u^\lambda u_\lambda = -1$. Setting $p = 0$ and making the following identification

$$\varepsilon = \left(T - \frac{G}{\kappa_4}\right)\,, \tag{11}$$

$$u^\mu = c\,g^{\mu\alpha}\partial_\alpha\phi\,, \tag{12}$$

the energy momentum tensor (10) becomes equivalent to $\bar{T}^{\mu\nu}$. Thus, the extra degree of freedom mimics the potential motions of dust with energy density $\left(T - \frac{G}{\kappa_4}\right)$ and the scalar field plays the role of velocity potential. In absence of matter this energy density is proportional to $G = 4\Lambda - R$, which does not vanish for generic solutions. As one can see, normalization condition for the four velocity u^μ and the conservation law for $\bar{T}^{\mu\nu}$, are equivalent to (3) and (6), respectively.

III. ECKS GRAVITY AND FIRST ORDER FORMALISM

So far we have used Greek indices $\mu, \nu, \ldots$ to denote tensor components in the coordinate basis. From now on, we use lower case Latin indices $a, b, \ldots$ for tensors defined in Lorentz (orthonormal) basis. We denote by $\Omega^p(M_4)$ to the set of differential p-forms defined over M_4.

At a particular point $P \in M_4$, the components of the change of base matrix $e^a{}_\mu(x)$ are determined through the relation

$$g_{\mu\nu} = \eta_{ab}e^a{}_\mu e^b{}_\nu\,, \tag{13}$$

where η_{ab} is the Minkowski metric. In terms of $e^a{}_\mu(x)$ we define the vierbein $e^a = e^a{}_\mu(x)\mathrm{d}x^\mu$ as the set of one-forms $\Omega^1(M_4) \in T_x^*(M_4)$. The vierbein contains all the metric properties in such a way that one can shift from $g_{\mu\nu}$ to e^a without any loss of generality. In addition, the affine properties of geometry are described by the one-form spin connection $\omega^{ab} = \omega^{ab}{}_\mu(x)\mathrm{d}x^\mu$. Direct link between ω^{ab} and $\Gamma^\lambda{}_{\mu\nu}$ is established by means of the vierbein postulate

$$\partial_\mu e^a{}_\nu + \omega^a{}_{b\mu}e^b{}_\nu - \Gamma^\lambda{}_{\mu\nu}e^a{}_\lambda = 0\,. \tag{14}$$

The covariant derivative of the vierbein is defined as the two-form torsion $T^a = \mathrm{D}e^a$ where

$$\mathrm{D}e^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b\,. \tag{15}$$

Unlike $\mathrm{d}^2 = 0$, higher order covariant derivatives of the vierbein does not vanish. In fact, direct calculation shows $\mathrm{D}T^a = R^a{}_b \wedge e^b$ where

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}\,, \tag{16}$$

is the Lorentz two-form curvature which transforms covariantly under local Lorentz transformations.

The spin connection can also be decomposed in a torsion free part $\mathring{\omega}^{ab}$ satisfying

$$\mathrm{d}e^a + \mathring{\omega}^a{}_b \wedge e^b = 0\,, \tag{17}$$

and a second rank anti-symmetric one-form κ^{ab} usually called the contorsion. An important observation is that $\mathring{\omega}^{ab}$ is completely determined in terms of the vierbein. This implies that all affine degree of freedom are encoded into the contorsion

$$\kappa^{ab} = \omega^{ab} - \mathring{\omega}^{ab}\,, \tag{18}$$

and consequently $T^a = \kappa^a{}_b \wedge e^b$. With this splitting, the Lorentz curvature can be rewritten as

$$R^{ab} = \mathring{R}^{ab} + \mathring{\mathrm{D}}\kappa^{ab} + \kappa^a{}_c \wedge \kappa^{cb}\,, \tag{19}$$

where $\mathring{R}^{ab} = \mathrm{d}\mathring{\omega}^{ab} + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^{cb}$ is the Riemann curvature two-form and $\mathring{\mathrm{D}}$ stands for the covariant derivative with respect to the torsion free part of the connection $\mathring{\omega}^{ab}$.

A theory of gravity which naturally includes torsional degree of freedom is known as Einstein-Cartan-Kibble-Sciama (ECKS) [94–100]. In this framework, GR extends to the inclusion of spin matter and it has been argued that new effects are produced only for matter densities much larger than the nuclear density. Moreover, it has been shown that torsion appears to prevent cosmological singularities [101–104] and to introduce an effective ultra-violet cutoff in a quantum field theory for fermions [105]. In differential form language, ECKS gravity is described in terms of a four-form Lagrangian

$$\mathcal{L}_{\mathrm{ECKS}} = \mathcal{L}_{\mathrm{G}}\left(e, \omega\right) + \mathcal{L}_{\mathrm{M}}\left(e, \omega, \varphi\right)\,, \tag{20}$$

where

$$\mathcal{L}_{\mathrm{G}} = \frac{1}{4\kappa_4}\epsilon_{abcd}\left(R^{ab} - \frac{\Lambda}{3!}e^a \wedge e^b\right) \wedge e^c \wedge e^d\,, \tag{21}$$

is the four-form Lagrangian for geometry and $\mathcal{L}_{\mathrm{M}}$ is the four-form Lagrangian for any kind of matter fields. Up to boundary terms, variation of the action functional $S = \frac{1}{c}\int_{M_4}\mathcal{L}_{\mathrm{ECKS}}$ is given by

$$\delta S = \frac{1}{c}\int_{M_4}\frac{1}{\kappa_4}\left(\frac{1}{2}\delta\omega^{ab} \wedge \mathcal{W}_{ab} + \mathcal{E}_d \wedge \delta e^d\right)\,, \tag{22}$$

with

$$\mathcal{E}_d = \epsilon_{abcd}\left(\frac{1}{2}R^{ab} - \frac{\Lambda}{3!}e^a \wedge e^b\right) \wedge e^c - \kappa_4 * \mathcal{T}_d\,, \tag{23}$$

$$\mathcal{W}_{ab} = \epsilon_{abcd}T^c \wedge e^d - \kappa_4 * \sigma_{ab}\,, \tag{24}$$

and where the stress-energy one-form $\mathcal{T}^a = \mathcal{T}^a{}_\mu\mathrm{d}x^\mu$ and the spin tensor 1-form $\sigma^{ab} = \sigma^{ab}{}_\mu\mathrm{d}x^\mu$ are defined through

$$\delta_e\mathcal{L}_{\mathrm{M}}^{(4)} = - * \mathcal{T}_d \wedge \delta e^d\,, \tag{25}$$

$$\delta_\omega\mathcal{L}_{\mathrm{M}}^{(4)} = -\frac{1}{2}\delta\omega^{ab} \wedge * \sigma_{ab}\,. \tag{26}$$

Here, $* : \Omega^p(M_d) \to \Omega^{d-p}(M_d)$ denotes the Hodge dual operator. It is important to stress out that in general torsion also contributes to the energy moementum tensor $\mathcal{T}^a$ in (23), trough contorsional terms coming from the Lorentz curvature, according to (19).

IV. CONFORMAL RIEMANN-CARTAN STRUCTURE

In order to characterize conformal structures in differential forms language, let us introduce an operator $\mathrm{I}_{a_1 \ldots a_q} : \Omega^p(M_d) \to \Omega^{p-q}(M_d)$ defined by [91]

$$\mathrm{I}_{a_1 \ldots a_q} = (-1)^{(d-p)(p-q)+\eta_-} * e_{a_1} \wedge \ldots \wedge e_{a_q} \wedge * . \quad (27)$$

Here, η_- stands for the number of minus sign in the metric signature. In particular, we are interested on the special case $q = 1$, $\eta_- = 1$, in four-dimensions

$$\mathrm{I}_a = - * e_a \wedge * . \quad (28)$$

The operator I_a satisfies useful properties such as the Leibniz rule for differential forms and, together with D, the operator I_a defines another important object $\mathcal{D}_a :$ $\Omega^p(M_d) \to \Omega^p(M_d)$ via the anti-commutator

$$\mathcal{D}_a = \{\mathrm{I}_a, \mathrm{D}\} = \mathrm{I}_a\mathrm{D} + \mathrm{D}\mathrm{I}_a , \quad (29)$$

The operators $\mathcal{D}$, I_a and D, form a superalgebra where the two-forms curvature and torsion play the role of structure constants (See [92]).

The conformal transformation

$$g_{\mu\nu} = \exp(2\sigma)\bar{g}_{\mu\nu} \quad (30)$$

that relates the spacetime and the auxiliary manifolds supposes implicitly that a local mapping $\sigma : \mathcal{M} \to \bar{\mathcal{M}}$ has been chosen in such a way that the same coordinates x^μ can be used for $P \in \mathcal{M}$ and $\bar{P} = \sigma(P) \in \bar{\mathcal{M}}$. This means that a coordinate transformation $x'^\mu = x'^\mu(x^\nu)$ in $\mathcal{M}$ induces the same transformation in $\bar{\mathcal{M}}$ and thus, tensors or forms defined on these manifolds transforms with the same Jacobian matrices. This fact allows us to find the relation between the vielbeins associated with these metrics, which by definition satisfy

$$g_{\mu\nu} = e^a_\mu e^a_\nu \eta_{ab} , \quad (31)$$

$$\bar{g}_{\mu\nu} = \bar{e}^a_\mu \bar{e}^b_\nu \eta_{ab} . \quad (32)$$

Indeed, mixing these expressions together with (30) it is direct to see that

$$e^a = \exp(\sigma)\bar{e}^a . \quad (33)$$

Once a vierbein $e^a(x)$ is specified one can always define "structure parameters" $\mathcal{C}_{ab}{}^c(x)$ which satisfy a generalized Maurer-Cartan equation

$$\mathrm{d}e^c = -\frac{1}{2}\mathcal{C}_{ab}{}^c e^a \wedge e^b . \quad (34)$$

This equation allows to solve the torsion-free part of the spin connection

$$\mathring{\omega}_{ab} = \frac{1}{2}(\mathcal{C}_{abc} + \mathcal{C}_{cba} - \mathcal{C}_{cab})e^c . \quad (35)$$

Now, using eq.(33) into (34) and inserting into (35), one finds

$$\mathring{\omega}_{ab} = \bar{\mathring{\omega}}_{ab} + \bar{e}_a \xi_b - \xi_a \bar{e}_b , \quad (36)$$

where

$$\xi^a = \bar{\mathrm{I}}^a \mathrm{d}\sigma . \quad (37)$$

Here,

$$\bar{\mathrm{I}}_a = -\bar{*}(\bar{e}_a \wedge \bar{*} . \quad (38)$$

where the bar in the Hodge dual denotes $\bar{e}^a$-vierbein dependence. In this way, eq.(36) characterizes the conformal transformation associated to the torsion free part of the spin connection.

Notice that we have no information on the conformal transformation of the contorsion κ^{ab} yet. This is due to the fact that in the context of Riemann-Cartan geometry, e^a and κ^{ab} are completely independent degrees of freedom. Therefore, there are multiple possible choices on how κ^{ab} should transform under a Weyl dilatation. An important family of choices can be parameterized as

$$\bar{e}^a \to e^a = \exp(\sigma)\bar{e}^a , \quad (39)$$

$$\bar{\kappa}_{ab} \to \kappa_{ab} = \bar{\kappa}_{ab} + (\lambda - 1)\theta_{ab} , \quad (40)$$

$$\bar{\omega}_{ab} \to \omega_{ab} = \bar{\omega}_{ab} + \lambda\theta_{ab} , \quad (41)$$

where λ is a parameter $0 \leq \lambda \leq 1$ and $\theta^{ab} = -\theta^{ba}$ corresponds to the 1-form

$$\theta^{ab} = \bar{e}^a \xi^b - \xi^a \bar{e}^b . \quad (42)$$

The case $\lambda = 1$ implies,

$$\kappa_{ab} = \bar{\kappa}_{ab} , \quad (43)$$

$$\omega_{ab} = \bar{\omega}_{ab} + \theta_{ab} , \quad (44)$$

which is the "canonical case": the full spin connection changes as the torsionless case, and the contorsion is left untouched by the dilatation. The most "exotic" case corresponds to $\lambda = 0$

$$\kappa_{ab} = \bar{\kappa}_{ab} - \theta_{ab} , \quad (45)$$

$$\omega_{ab} = \bar{\omega}_{ab} , \quad (46)$$

where the spin connection is left untouched by the dilatation and the contorsion absorbs the transformation.

It is clear that the torsionless condition is preserved only for the $\lambda = 1$ case. In fact, the Lorentz curvature and torsion change under the generalized Weyl dilatation (39-41) as

$$\bar{T}_a \to T_a = \exp(\sigma)\bar{T}_a + (\lambda - 1)\bar{e}_a \wedge \mathrm{d}\exp(\sigma) , \quad (47)$$

$$\bar{R}^{ab} \to R^{ab} = \bar{R}^{ab} + \lambda\bar{\mathrm{D}}\theta^{ab} + \lambda^2 \theta^a{}_c \wedge \theta^{cb} . \quad (48)$$

V. MIMETIC ECKS GRAVITY

Mimetic transformations are a particular choice of Weyl dilatations. Let us consider the auxiliary vierbein $\bar{e}^a$ and spin connection $\bar{\omega}^{ab}$ 1-forms, and a scalar field $\phi(x)$. In terms of operators $\bar{\mathrm{I}}_a$ and ϕ let us define a zero-form Lorentz vector

$$\bar{Z}_a = \bar{\mathrm{I}}_a \mathrm{d}\phi\,, \tag{49}$$

and the scalar

$$\bar{Z}^2 = -\eta_{ab}\bar{Z}^a\bar{Z}^b\,. \tag{50}$$

The generalized mimetic vierbein e^a, contorsion κ^{ab}, and spin connection ω^{ab} are defined by

$$\bar{e}^a \to e^a = \bar{Z}\bar{e}^a\,, \tag{51}$$
$$\bar{\kappa}_{ab} \to \kappa_{ab} = \bar{\kappa}_{ab} + (\lambda - 1)\,\theta_{ab}\,, \tag{52}$$
$$\bar{\omega}_{ab} \to \omega_{ab} = \bar{\omega}_{ab} + \lambda\theta_{ab}\,, \tag{53}$$

where the one-form θ^{ab} is given in (42) but now

$$\xi^a = \frac{1}{\bar{Z}}\bar{\mathrm{I}}^a \mathrm{d}\bar{Z}\,. \tag{54}$$

Notice that I_a and $\bar{\mathrm{I}}^a$ operator relate each other by

$$\mathrm{I}_a = \frac{1}{\bar{Z}}\bar{\mathrm{I}}_a\,, \tag{55}$$

and consequently

$$Z_a = \frac{1}{\bar{Z}}\bar{Z}_a\,, \tag{56}$$

so the constraint (3) reads

$$Z^2 = -\eta_{ab}Z^a Z^b = 1\,. \tag{57}$$

A. Mimetic field equations

To construct the mimetic version of ECSK theory, let us consider the Lagrangian (20) and the vierbein e^a and ω^{ab} in terms of the auxiliary variables $\bar{e}^a$, $\bar{\omega}^{ab}$ and ϕ as in eqs. (51,53). A priori, it would seem that different choices of λ would lead us to different dynamics. In particular, the canonical and exotic choices $\lambda = 1$ and $\lambda = 0$ seem to lead to completely different theories. However, nothing is further from truth. The dynamics of the generalized mimetic theory is the same regardless of the choice of λ. Since

$$e^a = \bar{Z}\bar{e}^a, \tag{58}$$
$$\omega_{ab} = \bar{\omega}_{ab} + \lambda\left(\bar{e}_a\xi_b - \xi_a\bar{e}_b\right), \tag{59}$$

we have that the functional variations of the vierbein and the spin connection are given by

$$\delta_{\bar{\omega}}e^d = 0\,, \tag{60}$$
$$\delta_{\bar{e}}e^d = \delta_{\bar{e}}\bar{Z}\bar{e}^d + \bar{Z}\delta\bar{e}^d\,, \tag{61}$$
$$\delta_{\phi}e^d = \delta_{\phi}\bar{Z}\bar{e}^d\,, \tag{62}$$

and

$$\delta_{\bar{\omega}}\omega_{ab} = \delta\bar{\omega}_{ab}\,, \tag{63}$$
$$\delta_{\bar{e}}\omega_{ab} = \lambda\left(\delta\bar{e}_a\xi_b - \xi_a\delta\bar{e}_b\right) + \lambda\left(\bar{e}_a\delta_{\bar{e}}\xi_b - \delta_{\bar{e}}\xi_a\bar{e}_b\right)\,, \tag{64}$$
$$\delta_{\phi}\omega_{ab} = \lambda\left(\bar{e}_a\delta_{\phi}\xi_b - \delta_{\phi}\xi_a\bar{e}_b\right)\,. \tag{65}$$

Notice we need special care when performing functional variation $\delta\bar{Z}^a$. In fact, from definition (49) it is clear that $\bar{Z} = \bar{Z}(\bar{e}, \partial\phi)$. This means we have to consider independent variations of $\bar{Z}^a$ with respect to both, the vierbein $\bar{e}^a(x)$ and the scalar field $\phi(x)$. Since

$$\delta\bar{Z} = -\frac{1}{\bar{Z}}\bar{Z}^a\delta\bar{Z}_a\,, \tag{66}$$

it is possible to prove that

$$\delta_{\bar{e}}\bar{Z} = \bar{Z}^2 Z_a Z_b \mathrm{I}^a\left(\delta\bar{e}^b\right)\,, \tag{67}$$
$$\delta_{\phi}\bar{Z} = -\bar{Z}Z^a\mathrm{I}_a\mathrm{d}\delta\phi\,. \tag{68}$$

Replacing (67)-(68) into (61)-(62) we get the expressions

$$\delta_{\bar{\omega}}e^d = 0\,, \tag{69}$$
$$\delta_{\bar{e}}e^d = \bar{Z}\left[Z_a Z_b \mathrm{I}^a\left(\delta\bar{e}^b\right)e^d + \delta\bar{e}^d\right]\,, \tag{70}$$
$$\delta_{\phi}e^d = -e^d Z^a\mathrm{I}_a\mathrm{d}\delta\phi\,. \tag{71}$$

Up to boundary terms, variation of (20) reads

$$\delta_{\bar{e}}\mathcal{L}_{\mathrm{ECSK}}^{(4)} = \frac{1}{\kappa_4}\left[\frac{1}{2}\delta_{\bar{e}}\omega^{ab}\wedge\mathcal{W}_{ab} + \mathcal{E}_d\wedge\delta_{\bar{e}}e^d\right] = 0\,, \tag{72}$$
$$\delta_{\phi}\mathcal{L}_{\mathrm{ECSK}}^{(4)} = \frac{1}{\kappa_4}\left[\frac{1}{2}\delta_{\phi}\omega^{ab}\wedge\mathcal{W}_{ab} + \mathcal{E}_d\wedge\delta_{\phi}e^d\right] = 0\,, \tag{73}$$
$$\delta_{\bar{\omega}}\mathcal{L}_{\mathrm{ECSK}} = \frac{1}{2}\frac{1}{\kappa_4}\delta_{\bar{\omega}}\omega^{ab}\wedge\mathcal{W}_{ab} = 0\,, \tag{74}$$

where the three-forms $\mathcal{E}_a$ and $\mathcal{W}_{ab}$ are given in (23) and (24) respectively. Since $\delta_{\bar{\omega}}\omega_{ab} = \delta\bar{\omega}_{ab}$, eq.(74) implies $\delta\bar{\omega}^{ab}\wedge\mathcal{W}_{ab} = 0$ and consequently

$$\mathcal{W}_{ab} = 0\,, \tag{75}$$

just as in the standard ECSK model. Inserting $\mathcal{W}_{ab} = 0$ in the equations of motion, we are left with

$$\delta_{\bar{e}}\mathcal{L}_{\mathrm{G}}^{(4)} = \frac{1}{\kappa_4}\mathcal{E}_d\wedge\delta_{\bar{e}}e^d = 0\,, \tag{76}$$
$$\delta_{\phi}\mathcal{L}_{\mathrm{G}}^{(4)} = \frac{1}{\kappa_4}\mathcal{E}_d\wedge\delta_{\phi}e^d = 0\,. \tag{77}$$

From here, using the expressions (70) and (71), and integration by parts in I^a and d, we get the set of mimetic ECSK field equations

$$\mathcal{E}_d - Z_a Z_d \mathrm{I}^a\left(\mathcal{E}_m\wedge e^m\right) = 0\,, \tag{78}$$
$$\mathrm{d}\left[Z^a\mathrm{I}_a\left(\mathcal{E}_d\wedge e^d\right)\right] = 0\,, \tag{79}$$
$$\mathcal{W}_{ab} = 0\,. \tag{80}$$

It is remarkable that they do not depend on the choice of λ. For the mimetic theory, all the choices of conformal transformations for the contorsion lead to the same dynamics.

In order to study the equivalence of these equations written using tensors, it is useful to consider Hodge duality between p-forms and $(d-p)$-forms. For a three-form $\mathcal{E}_d$ in four dimensions, we have

$$\mathcal{E}_d = \mathcal{E}_{md} * e^m. \tag{81}$$

It is straightforward to prove that

$$\mathcal{E}_q \wedge e^q = -\mathcal{E}^p{}_p v_{(4)}, \tag{82}$$
$$\mathrm{I}^m v_{(4)} = *e^m, \tag{83}$$

where $v_{(4)}$ denotes the volume form in four-dimensions and

$$\mathrm{d}\phi = Z_a e^a. \tag{84}$$

Replacing these relations into the field equations (78)-(80) it is possible to write them as

$$\mathcal{E}_d - *(Z_d \mathcal{E}^p{}_p \mathrm{d}\phi) = 0, \tag{85}$$
$$-\mathrm{d} * [\mathcal{E}^p{}_p \mathrm{d}\phi] = 0, \tag{86}$$
$$\mathcal{W}_{ab} = 0. \tag{87}$$

Remarkably, eqs. (85)-(86) have the same form as eqs. (5)-(6) but in terms of the full Lorentz curvature (19) instead of just the Riemannian piece. Note that eq. (87) is the standard field equation for torsion in terms of the spin tensor of matter.

B. Conservation laws

A conservation law

$$\mathrm{d} * J = 0, \tag{88}$$

is always equivalent to

$$\mathring{\mathrm{D}} * J^a = 0 \tag{89}$$

where $\mathring{\mathrm{D}}$ denotes the covariant derivative with respect to $\mathring{\omega}^{ab}$, regardless of the torsion of background geometry. From (85), it is clear one can define a mimetic energy-momentum one-form

$$\bar{\mathcal{T}}_d = \frac{1}{\kappa_4} Z_d \mathcal{E}^p{}_p \mathrm{d}\phi. \tag{90}$$

Conservation law of $\bar{\mathcal{T}}_d$ implies

$$\mathring{\mathrm{D}} * \bar{\mathcal{T}}_d = \frac{1}{\kappa_4} \mathring{\mathrm{D}}\left(Z_d * [\mathcal{E}^p{}_p \mathrm{d}\phi]\right)$$
$$= \frac{1}{\kappa_4}\left(\mathring{\mathrm{D}}Z_d \wedge * [\mathcal{E}^p{}_p \mathrm{d}\phi] + Z_d \wedge \mathring{\mathrm{D}} * [\mathcal{E}^p{}_p \mathrm{d}\phi]\right)$$
$$= \frac{1}{\kappa_4}\left(\mathcal{E}^p{}_p \mathring{\mathrm{D}}Z_d \wedge *\mathrm{d}\phi + Z_d \wedge \mathrm{d} * [\mathcal{E}^p{}_p \mathrm{d}\phi]\right). \tag{91}$$

Using (86), we have

$$\mathring{\mathrm{D}} * \bar{\mathcal{T}}_d = \frac{1}{\kappa_4}\mathcal{E}^p{}_p \mathring{\mathrm{D}}Z_d \wedge *\mathrm{d}\phi. \tag{92}$$

Note that

$$\mathring{\mathrm{D}}Z_d \wedge *\mathrm{d}\phi = \left(e^a \mathring{\mathrm{D}}_a Z_d\right) \wedge *e^b Z_b \tag{93}$$
$$= Z^a \mathring{\mathrm{D}}_a Z_d v_{(4)}, \tag{94}$$

where $v_{(4)}$ denotes the volume form in four-dimensions. Moreover, since $\mathring{\mathrm{D}}_a Z_d = \mathring{\mathrm{D}}_d Z_a$, one obtains

$$\mathring{\mathrm{D}}Z_d \wedge *\mathrm{d}\phi = Z^a \mathring{\mathrm{D}}_d Z_a v_{(4)} = \frac{1}{2}\partial_d(Z^a Z_a)v_{(4)} = 0, \tag{95}$$

where we have used (57). Consequently,

$$\mathring{\mathrm{D}} * \bar{\mathcal{T}}_d = 0, \tag{96}$$

which is the final conservation law for the effective mimetic stress-energy tensor.

VI. THE TRACE OF THE STRESS-ENERGY TENSOR, TORSION AND λ

For the mimetic theory dynamics, the choice of the parameter λ for the conformal transformations (51)-(53) seems to be irrelevant. However, it does not mean that the parameter is meaningless. Actually, it is related with the value of the trace of stress-energy tensor of matter when its Lagrangian has conformal symmetry.

Let us consider a matter Lagrangian $\mathcal{L}_\mathrm{M}$ obeying conformal symmetry by itself

$$\mathcal{L}_\mathrm{M}(e, \omega, \varphi) = \mathcal{L}_\mathrm{M}(\Omega e^a, \omega^{ab} + \frac{\lambda}{\Omega}\left[e^a, \mathrm{I}^b\right]\mathrm{d}\Omega, \frac{1}{\Omega^\alpha}\varphi). \tag{97}$$

where $\left[e^a, \mathrm{I}^b\right] = e^a \mathrm{I}^b - e^b \mathrm{I}^a$. In the standard torsionless case, it would lead to a traceless on-shell stress-energy tensor. This is no longer true in the current context of non-vanishing torsion. In fact, under an infinitesimal dilatation $\Omega = 1 + \varepsilon$, the field content of the matter Lagrangian changes according to

$$\delta_\varepsilon e^a = \varepsilon\, e^a, \tag{98}$$
$$\delta_\varepsilon \omega^{ab} = \lambda\left[e^a, \mathrm{I}^a\right]\mathrm{d}\varepsilon, \tag{99}$$
$$\delta_\varepsilon \varphi = -\alpha\varepsilon\varphi. \tag{100}$$

Moreover, an arbitrary variation of $\mathcal{L}_\mathrm{M}(e, \omega, \varphi)$ is given by

$$\delta\mathcal{L}_\mathrm{M} = -*\mathcal{T}_d \wedge \delta e^d + \frac{1}{2} * \sigma_{ab} \wedge \delta\omega^{ab} + \Phi\delta\varphi \tag{101}$$
$$+ \mathrm{d}\left(\mathcal{B}_a^{(2)} \wedge \delta e^a + \frac{1}{2}\mathcal{B}_{ab}^{(2)} \wedge \delta\omega^{ab} + \mathcal{B}^{(3)}\delta\varphi\right),$$

where Φ denotes the field equation for φ and $\mathcal{B}_a$, $\mathcal{B}_{ab}$, and $\mathcal{B}$ are boundary terms. Therefore, demanding invariance

and consequently

$$G - T = \frac{\mathcal{C}\left(x^i\right)}{\sqrt{\gamma}}, \qquad (120)$$

here $\mathcal{C}$ is an integration constant which only depends on the spatial coordinates x^i. For a flat Friedmann Universe, the metric γ_{ij} is

$$\gamma_{ij} = a^2\left(\tau\right)\delta_{ij}, \qquad (121)$$

so that, (119) leads to

$$G - T = \frac{\mathcal{C}\left(x^i\right)}{a^3\left(\tau\right)}. \qquad (122)$$

Therefore, there is a dark matter source imitated by the scalar field coming from the conformal degree of freedom of the Einstein equations. However, in ECSK theory torsion is present and this implies that there is an additional dark matter source hidden in $G = -R + 4\Lambda$. In fact, splitting the Ricci scalar according to

$$R = \mathring{R} + R\left(\kappa\right), \qquad (123)$$

$$R\left(\kappa\right) = 2\mathring{\nabla}_\mu \kappa^{\mu\nu}{}_\nu + \kappa^\mu{}_{\gamma\mu}\kappa^{\gamma\nu}{}_\nu + \kappa^\mu{}_{\gamma\nu}\kappa^\gamma{}_\mu{}^\nu, \qquad (124)$$

where $\kappa^{\mu\nu}{}_\lambda$ is the contortion, $\mathring{R}$ and $\mathring{\nabla}$ are respectively the Ricci scalar and covariant derivative associated with the Christoffel symbol. Thus, using $\mathring{G} = -\mathring{R} + 4\Lambda$, we get

$$G = \mathring{G} - R\left(\kappa\right). \qquad (125)$$

In order to illustrate the idea let us consider a spin tensor distribution σ_{ab} which may be relevant at cosmological scales. This has been considered, for instance, in [107] where the anstaz for the torsion tensor reads

$$T_{\lambda\mu\nu} = \left[X(\tau)(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\mu}g_{\nu\rho}) - 2\sqrt{g}Y(\tau)\epsilon_{\lambda\mu\nu\rho}\right]u^\rho, \qquad (126)$$

where u^ρ is the co-moving four-velocity which in synchrotonic coordinates is given by $u^0 = 1$ and $u^i = 0$, and X and Y are arbitrary functions of time. It has been argued

that such a configuration may be given by the recently called dark spinors (see for instance [108, 109]). Evaluating (126) for (121), direct calculation shows that the only nonvanishing components are

$$T_{ij0} = X\gamma_{ij}u^0 = a^2 X\delta_{ij}, \qquad (127)$$

$$T_{kij} = -2\sqrt{\gamma}Y\epsilon_{kij}u^0 = -2a^3 Y\epsilon_{kij}, \qquad (128)$$

and since

$$\kappa_{\mu\nu\lambda} = \frac{1}{2}\left(T_{\nu\mu\lambda} - T_{\mu\nu\lambda} + T_{\lambda\mu\nu}\right), \qquad (129)$$

one finds

$$\kappa_{0ij} = -X\gamma_{ij}u^0 = -a^2 X\delta_{ij} \qquad (130)$$

$$\kappa_{ijk} = \sqrt{\gamma}Y\epsilon_{ijk}u^0 = a^3 Y\epsilon_{ijk} \qquad (131)$$

With these components, we can evaluate (124) and therefore, Eqs.(122)-(125) give

$$\mathring{G} - T = \frac{\mathcal{C}}{a^3} + 6\left(\dot{X} + \frac{3}{2}(\dot{a^2})X + \frac{1}{2}X^2 + a^6 Y^2\right) \qquad (132)$$

From here the presence of spin as dark matter has become evident. In mimetic ECKS theory presented here, there are two different dark matter species: one is coming from isolating the conformal mode in a covariant way and the other one comes by considering torsional degrees of freedom.

VIII. ACKNOWLEDGMENTS

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