A Remark on the Coupling of Gravitation and Electron*

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In Einstein's theory of gravitation one may consider either the metric potentials g_{ik}, or both the metric potentials and the components of the affine connection, as independent quantities (metric *vs.* mixed theory) in the coupling of gravitational and electronic fields. This makes a difference for the ensuing equations, but one that can be easily straightened out by a slight change of the Lagrangian; see the Theorem at the end of the paper.

ON the level of classical field theory the welding together of Maxwell's electromagnetic, Dirac's electronic and Einstein's gravitational fields (**F**, **E** and **G**) affords no serious difficulty—whatever interpretation quantum physics may impose upon the resulting field equations, and however unsatisfactory a Lagrangian may be that is made up by addition of four separate parts, one for **F**, two for **E**, and one for **G**. The formulation of Dirac's theory of the electron in the frame of general relativity has to its credit one feature which should be appreciated even by the atomic physicist who feels safe in ignoring the role of gravitation in the building-up of the elementary particles: its explanation of the quantum mechanical principle of "gauge invariance" that connects Dirac's ψ with the electromagnetic potentials.

In contrast to Einstein's original "metric" conception in terms of the g_{pq} there was later developed, by Eddington, by Einstein himself, and recently by Schrödinger, an affine field theory operating with the components $\Gamma_{pq}{}^r$ of an affine connection. But in 1925 Einstein also advocated a "mixed" formulation by means of a Lagrangian in which both the g_{pq} and the $\Gamma_{pq}{}^r$ are taken as basic field quantities and submitted to independent arbitrary infinitesimal variations.[1] In certain respects this seems to be the most natural procedure. But even when the electromagnetic field is taken into account by adding the Lagrangian characteristic for Maxwell's theory, the resulting equations turn out to be the same as in Einstein's purely metric theory. This ceases to be true for the interaction between gravitation and electron; yet, as I shall show here, coincidence can be reestablished by adding to the Lagrangian one simple term, of a structure not dissimilar to that of the Dirac mass term.

A Cartesian frame E in a four-dimensional orthogonal vector space consists of four linearly independent vectors $\mathbf{e}(\alpha)$ ($\alpha=0, 1, 2, 3$), the transition between any two such ("equally admissible") frames being effected by an orthogonal transformation (rotation). Any vector $\mathbf{v}$ is uniquely representable relative to E in the form $\sum_\alpha v(\alpha)\mathbf{e}(\alpha)$ by its "*ortho-components*" $v(\alpha)$. The square of the length of the vector $\mathbf{v}$ is the invariant $\sum_\alpha v^2(\alpha)$. We assume the reality conditions characteristic for the Lorentz-Minkowski vector geometry: $\mathbf{e}(1)$, $\mathbf{e}(2)$, $\mathbf{e}(3)$ are real space-like vectors while $\mathbf{e}(0)/i$ is a time-like vector pointing toward the future. An infinitesimal Lorentz rotation $d\mathbf{e}(\alpha)=\sum_\beta do(\alpha\beta)\cdot\mathbf{e}(\beta)$ is defined by a skew-symmetric matrix $\|do(\alpha\beta)\|$ of which the components $\alpha, \beta=1, 2, 3$ are real and the components $\alpha=0$, $\beta=1, 2, 3$ pure imaginary. With respect to the Lorentz frame E the wave function of the electron has four complex components $\psi_1, \psi_2; \psi_3, \psi_4$ which we arrange in a *column* ψ. Let $\bar{\psi}$ denote the *row* of the conjugate-complex numbers $\bar{\psi}_1, \bar{\psi}_2; \bar{\psi}_3, \bar{\psi}_4$ and form the following Hermitian expressions $s(\alpha)=\bar{\psi}S(\alpha)\psi$, $sp(\alpha)=\bar{\psi}Sp(\alpha)\psi$:

$s(\alpha)$, $sp(\alpha)$	α
$i(\bar{\psi}_1\psi_1+\bar{\psi}_2\psi_2)\pm i(\bar{\psi}_3\psi_3+\bar{\psi}_4\psi_4)$	0
$(\bar{\psi}_1\psi_1-\bar{\psi}_2\psi_2)\mp(\bar{\psi}_3\psi_3-\bar{\psi}_4\psi_4)$	1
$(\bar{\psi}_1\psi_2+\bar{\psi}_2\psi_1)\mp(\bar{\psi}_3\psi_4+\bar{\psi}_4\psi_3)$	2
$i(\bar{\psi}_2\psi_1-\bar{\psi}_1\psi_2)\mp i(\bar{\psi}_4\psi_3-\bar{\psi}_3\psi_4)$	3

where the upper signs refer to $s(\alpha)$, the lower to $sp(\alpha)$. The ψ-components transform under the influence of a Lorentz rotation of the frame so as to make $s(\alpha)$ the ortho-components of an invariant vector (charge-current vector). $sp(\alpha)$ are then the components of a pseudo-vector (the spin, which changes sign if left and right in 3-space are interchanged). We make use also of the bilinear expressions $\psi S(\alpha)\chi$ in which χ is a quantity transforming cogrediently with ψ. The components ψ are determined but for an arbitrary factor $e^{i\lambda}$, the phase λ of which is a real constant.

The metric field at a given point P in the real four-dimensional world is described by assigning to P a Lorentz-frame $E=E(P)$, but does not change if E is submitted to an arbitrary Lorentz rotation. Let the world around P be referred to (real) coordinates x^p ($p=0, 1, 2, 3$). The prototype of a "vector at P" is the line element joining the point P, coordinates x^p, with an infinitely near point P', coordinates x^p+dx^p. The coordinate increments dx^p are the *contravariant components* of this vector. The contravariant components $e^p(\alpha)$ of the four vectors $\mathbf{e}(\alpha)$ that constitute the frame $E(P)$ describe analytically the metric field at P in terms of the coordinates x^p. The laws of nature are

* This is an abbreviated version of a note that appeared under the same title in Actas de la Academía Nacional de Ciencias de Lima **11** (1948), but was made incomprehensible by numerous misprints.

[1] Sitzungsber., Preuss. Akad. der Wissensch. (1925), p. 414.

invariant (1) with respect to arbitrary transformations of the coordinates x^p and (2) with respect to arbitrary Lorentz rotations of the frames $E(P)$ at all points P (the rotation of the frame $E(P)$ at P depending in an arbitrary manner on the point P). Let $\|e_p(\alpha)\|$ denote the matrix reciprocal to $\|e^p(\alpha)\|$,

$$\sum_\alpha e^p(\alpha)\cdot e_q(\alpha)=\delta_q{}^p, \quad \sum_p e^p(\alpha)\cdot e_p(\beta)=\delta(\alpha\beta).$$

Here $\|\delta_q{}^p\|$ and $\|\delta(\alpha\beta)\|$ stand for the unit matrix. Let ϵ^{-1} be the absolute value of the determinant of the $e^p(\alpha)$. $\epsilon\cdot dx^0dx^1dx^2dx^3=\epsilon\cdot dx$ is the invariant volume element of the world. The contravariant, the ortho- and the covariant components v^p, $v(\alpha)$, v_p of a (real) vector $\mathbf{v}$ are connected by the relations

$$\begin{aligned} v^p&=e^p(\alpha)\cdot v(\alpha) \quad \text{or} \quad v(\alpha)=e_p(\alpha)\cdot v^p, \\ v_p&=e_p(\alpha)\cdot v(\alpha) \quad \text{or} \quad v(\alpha)=e^p(\alpha)\cdot v_p \end{aligned} \tag{1}$$

(in the writing-down of which the well-known convention about the omission of summation signs has been adopted). Multiplication by ϵ changes a scalar into a scalar density, a vector into a vector density, etc. We indicate this process by transmuting an italic into the corresponding German letter.

Infinitesimal parallel displacement from P to P' carries the frame $\mathbf{e}(\alpha)=\mathbf{e}(\alpha;P)$ at P into a frame at P' that proceeds from the local frame $\mathbf{e}(\alpha;P')$ at P' by a certain infinitesimal rotation

$$d\Omega=\|do(\alpha\beta)\|, \quad do(\alpha\beta)=o_p(\alpha\beta)\cdot dx^p.$$

We thus come to describe the metric field and the affine connection by the 16+24 quantities $e^p(\alpha)$ and $o_p(\alpha\beta)\{=-o_p(\beta\alpha)\}$.[2] The Riemann curvature tensor is given by

$$R_{pq}(\alpha\beta)=\left\{\frac{\partial o_q(\alpha\beta)}{\partial x^p}-\frac{\partial o_p(\alpha\beta)}{\partial x^q}\right\} + \{o_p(\alpha\gamma)o_q(\gamma\beta)-o_q(\alpha\gamma)o_p(\gamma\beta)\},$$

and hence the scalar curvature equals

$$R=e^p(\alpha)e^q(\beta)R_{pq}(\alpha\beta).$$

The integral $\int\mathfrak{R}\cdot dx$ of $\mathfrak{R}=\epsilon\cdot R$ is an invariant; following Einstein we adopt $\mathfrak{R}$ as the Lagrangian of the gravitational field.

Transcription of Dirac's theory of the electron into general relativity yields for the Lagrangian $\mathfrak{L}+m_0\mathfrak{l}$ of the electronic field the sum of two terms; the principal term $\mathfrak{L}$ is responsible for the most decisive general features of quantum mechanics, the accessory term $m_0\mathfrak{l}$ contains the mass m_0 of the electron as a constant factor:

$$L=\frac{1}{i}\left\{\tilde{\psi}e^p(\alpha)S(\alpha)\frac{\partial\psi}{\partial x^p}-\frac{\partial\tilde{\psi}}{\partial x^p}e^p(\alpha)S(\alpha)\psi\right\} -\frac{1}{i}\sum e^p(\alpha)o_p(\beta\gamma)sp(\delta), \tag{2}$$

with the sum $\sum$ extending over all even permutations $\alpha\beta\gamma\delta$ of 0123;

$$l=(\bar{\psi}_3\psi_1+\bar{\psi}_4\psi_2)+(\bar{\psi}_1\psi_3+\bar{\psi}_2\psi_4). \tag{3}$$

In general relativity the phase λ of the arbitrary factor $e^{i\lambda}$ of the ψ's is an arbitrary function of the position P. The infinitesimal transformation $d\psi=dZ\cdot\psi$ of the ψ's that is induced by an infinitesimal rotation $d\Omega=\|do(\alpha\beta)\|$ of the local frame, therefore, contains an indeterminate additive term $i\cdot df\cdot\psi$ with a pure imaginary scalar factor $i\cdot df$. Hence in a full description of the affine connection there will occur, in addition to the matrix $d\Omega$, also a scalar $df=f_pdx^p$ depending linearly on the dx^p. The derivative $\partial\psi/\partial x^p$ in L has to be replaced by $\partial\psi/\partial x^p+if_p\psi$, where f_p are the covariant components of a vector field, and all laws must be invariant with respect to the "gauge transformation" (or should one rather say "phase transformation"?)

$$\psi\to e^{i\lambda}\cdot\psi, \quad f_p\to f_p-\frac{\partial\lambda}{\partial x^p}.$$

Experience shows that f_p accounts for the effect of the electromagnetic field on the electron and is to be identified with the electromagnetic potential φ_p, measured in a suitable atomic unit, $f_p=(e/\hbar c)\cdot\varphi_p$. Form the field strength $f_{pq}=\partial f_q/\partial x^p-\partial f_p/\partial x^q$. The generation of the electromagnetic field by the electronic charge-current $s(\alpha)$ is accounted for by adding the electromagnetic term

$$\mathfrak{M}=f_{pq}\mathfrak{f}^{pq} \tag{4}$$

to the total Lagrangian.

In the Riemann-Einstein metric geometry the metric quantities $e^p(\alpha)$ determine the affine connection, i.e. the quantities $o_p(\alpha\beta)$, in the following manner. From the Poisson brackets

$$\frac{\partial e^p(\alpha)}{\partial x^q}\cdot e^q(\beta)-\frac{\partial e^p(\beta)}{\partial x^q}\cdot e^q(\alpha)=[e(\alpha),e(\beta)]^p,$$

and in accordance with the rules (1) for the juggling of indices

$$o(\gamma;\alpha\beta)=e^p(\gamma)\cdot o_p(\alpha\beta),$$
$$[e(\alpha),e(\beta)](\gamma)=e_p(\gamma)\cdot[e(\alpha),e(\beta)]^p.$$

Then

$$o(\alpha;\beta\gamma)+o(\beta;\gamma\alpha)=[e(\alpha),e(\beta)](\gamma)$$

or more explicitly

$$2o(\gamma;\alpha\beta)=[e(\gamma),e(\alpha)](\beta) +[e(\beta),e(\gamma)](\alpha)-[e(\alpha),e(\beta)](\gamma). \tag{5}$$

In a quantity like L which involves the $o_p(\alpha\beta)$ one may replace the latter by their expressions $o_p(\alpha\beta)\{e\}$ in terms of the $e^p(\alpha)$ as derived from (5). Let the resulting expression be denoted by L^*, the asterisk indicating that $o_p(\alpha\beta)\{e\}$ has been substituted for $o_p(\alpha\beta)$. In particular R^* is Riemann's scalar curvature

[2] See for this whole analytic treatment: H. Weyl, Zeits. f. Physik **56**, 330 (1929).

written as an algebraic combination of the $e^p(\alpha)$, their first and second derivatives.

For the moment let us ignore the electromagnetic potentials f_p as well as the Dirac mass term, and let κ denote Einstein's gravitational constant multiplied by $\hbar/c$. Then Dirac's and Einstein's equations in the metric theory require the variation of

$$\int (\mathfrak{R}^* + \kappa \mathfrak{L}^*) dx$$

to vanish for such arbitrary infinitesimal variations $\delta\psi$ and $\delta e^p(\alpha)$ as vanish outside a bounded region, while in the mixed theory

$$\delta \int (\mathfrak{R} + \kappa \mathfrak{L}) dx$$

is to vanish for similar variations of ψ, $e^p(\alpha)$ and $o_p(\alpha\beta)$. The latter variations result in a new set of field laws replacing the definitions (5). This set states that the differences

$$\Delta(\gamma; \alpha\beta) = [e(\alpha), e(\beta)](\gamma) - \{o(\alpha; \beta\gamma) + o(\beta; \gamma\alpha)\}$$

instead of being all zero, satisfy the relations

$$\Delta(\gamma; \alpha\beta) = \frac{\kappa}{2i} \, sp(\delta) \tag{6}$$

for any even permutation $\alpha\beta\gamma\delta$ of 0 1 2 3, and the relation

$$\Delta(\gamma; \alpha\beta) = 0 \tag{6'}$$

whenever the three indices $\alpha\beta\gamma$ are not all distinct. *Thus by the influence of matter a slight discordance between affine connection and metric is created.*

Although the list of observables which describe the field in the mixed theory includes $o_p(\alpha\beta)$ beside ψ and $e^p(\alpha)$, it is possible to eliminate the $o_p(\alpha\beta)$ by means of the equations

$$o_p(\alpha\beta) = e_p(\gamma) \cdot o(\gamma; \alpha\beta),$$
$$2o(\gamma; \alpha\beta) = 2o(\gamma; \alpha\beta)\{e\} - \Delta(\gamma; \alpha\beta)$$

and the laws (6), (6′). I have carried out the somewhat laborious calculations. Since the result applies to Lagrangians of a more general type than envisaged above I shall enounce it in this generality. In the expression (2) of L one has now to replace $\partial\psi/\partial x^p$, $\partial\bar{\psi}/\partial x^p$ by $\partial\psi/\partial x^p + if_p\psi$ and $\partial\bar{\psi}/\partial x^p - if_p\bar{\psi}$, respectively. Besides the terms R and L of the Einstein-Dirac theory which involve the $o_p(\alpha\beta)$ there exist four invariants which do not, to wit

$$l = (\bar{\psi}_3\psi_1 + \bar{\psi}_4\psi_2) + (\bar{\psi}_1\psi_3 + \bar{\psi}_2\psi_4),$$
$$l_2 = (\bar{\psi}_3\psi_1 + \bar{\psi}_4\psi_2)(\bar{\psi}_1\psi_3 + \bar{\psi}_2\psi_4) \geq 0,$$
$$M = f_{pq}f^{pq} \quad \text{and} \quad Q = e^p(\alpha)e^q(\beta)F(\alpha\beta)f_{pq}.$$

The skew-symmetric tensor $F(\alpha\beta)$ which enters into Q is constructed out of the 16 products $\bar{\psi}\psi$ as follows:

$$F(01),\ F(23) = (\bar{\psi}_3\psi_1 - \bar{\psi}_4\psi_2) \mp (\bar{\psi}_1\psi_3 - \bar{\psi}_2\psi_4),$$
$$F(02),\ F(31) = (\bar{\psi}_3\psi_2 + \bar{\psi}_4\psi_1) \mp (\bar{\psi}_2\psi_3 + \bar{\psi}_1\psi_4),$$
$$F(03),\ F(12) = i(\bar{\psi}_3\psi_2 - \bar{\psi}_4\psi_1) \pm i(\bar{\psi}_2\psi_3 - \bar{\psi}_1\psi_4)$$

the upper signs to be used for $F(01)$, $F(02)$, $F(03)$, the lower for $F(23)$, $F(31)$, $F(12)$. Let H be of the general form

$$H = R + \kappa\{L + w(l, l_2, M, Q)\},$$

w being any function of four real variables.

THEOREM. *The mixed theory with the Lagrangian ϵH is identical with the metric theory based on the Lagrangian*

$$\epsilon H', \quad H' = R^* + \kappa\{L^* + w'(l, l_2, M, Q)\}$$

where

$$w'(l, l_2, M, Q) = w(l, l_2, M, Q) - \frac{3\kappa}{2} l_2.$$

To this extent then there is complete equivalence between the mixed metric-affine and the purely metric conceptions of gravitation.

Special relativity results if one lets the constant κ tend to zero. Then one obtains $e^p(\alpha) = \text{const.}$, $o_p(\alpha\beta) = 0$, and one may choose

$$e^0(0) = i, \quad e^1(1) = e^2(2) = e^3(3) = 1$$

and all other $e^p(\alpha) = 0$. The Dirac equations and the energy-momentum tensor for the metric and the mixed theory coincide in the limit $\kappa \to 0$.