GAUGE GRAVITATION THEORY FROM THE GEOMETRIC VIEWPOINT

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This is the Preface to the special issue of *International Journal of Geometric Methods in Modern Physics* **3**, N.1 (2006) dedicated to the 50th anniversary of gauge gravitation theory. It addresses the geometry underlying gauge gravitation theories, their higher-dimensional, super-gauge, and non-commutative extensions.

At present, Yang–Mills gauge theory provides a universal description of the fundamental electroweak and strong interactions. Gauge gravitation theory from the very beginning [1] aims to extend this description to gravity. It started with gauge models of the Lorentz, Poincaré groups and general covariant transformations [2–4]. At present, gauge gravity is mainly supergravity, higher-dimensional, non-commutative and quantum gravity [5–8]. This Preface can not pretend for any comprehensive analysis of gauge gravitation theories, but addresses only the geometry underlying them. We apologize in advance for that the list of references is far from to be complete.

The first gauge model of gravity was suggested by Utiyama [1] in 1956 just two years after birth of the gauge theory itself. Utiyama was first who generalized the original gauge model of Yang and Mills for $SU(2)$ to an arbitrary symmetry Lie group and, in particular, to the Lorentz group in order to describe gravity. However, he met the problem of treating general covariant transformations and a pseudo-Riemannian metric (a tetrad field) which had no partner in Yang–Mills gauge theory.

In a general setting, fiber bundles provide the adequate geometric formulation of classical field theory where classical fields on a smooth manifold X are represented by sections of fiber bundles over X. In particular, Yang–Mills gauge theory is a gauge theory of principal connections on a principal bundle $P \to X$ and associated bundles with a structure Lie group G. There is the canonical right action of G on P such that $P/G = X$. A typical fiber of P is the group space of G, and G acts on it by left multiplications. Being G-equivariant, principal connections on P (i.e., gauge potentials) are identified to global sections of the quotient bundle $C = J^1P/G$, where J^1P is the first order jet manifold of sections of $P \to X$ [9]. Gauge transformations in Yang–Mills gauge theory are vertical automorphisms of a principal bundle $P \to X$ over $\mathrm{Id}\, X$. Their group $\mathrm{VAut}(P)$ is isomorphic to the group of global sections of the group bundle P_{Ad} associated to P. A typical fiber of P_{Ad} is the group

G acting on itself by the adjoint representation. Under certain conditions, one can define the Sobolev completion of the group $\mathrm{VAut}(P)$ to a Lie group. In classical field theory, it suffices to consider local one-parameter groups of local vertical automorphisms of P whose infinitesimal generators are G-invariant vertical vector fields on P. These vector fields are identified to global sections of the quotient bundle $VP/G \to X$, where VP is the vertical tangent bundle of $P \to X$. They form a Lie algebra $\mathcal{G}$ over the ring $C^\infty(X)$ of smooth real functions on X. It fails to be a Lie algebra of a Lie group, unless its Sobolev completion exists. If $P \to X$ is a trivial bundle, $\mathcal{G}$ is a $C^\infty(X)$-extension (a gauge extension in the physical terminology) of the Lie algebra of G. Let

$$Y = (P \times V)/G \to X \tag{1}$$

be a fiber bundle associated to a principal bundle $P \to X$ whose structure group G acts on the typical fiber V of Y on the left. Any automorphism of P induces a bundle automorphism of Y, and any principal connection on P yields an associated connection on Y. Such a connection is given by the TY-valued form

$$A = dx^\mu \otimes (\partial_\mu + A^i_\mu(x^\nu, y^j)\partial_i) \tag{2}$$

with respect to bundle coordinates (x^μ, y^i) on Y. If Y is a vector bundle, an associated connection (2) takes the form

$$A = dx^\mu \otimes (\partial_\mu - A^r_\mu(x^\nu) I_r{}^i{}_j y^j \partial_i), \tag{3}$$

where I_r are generators of a representation of the Lie algebra of a group G in V. In Yang–Mills gauge theory on a principal bundle P, sections of a P-associated vector bundle (1) describe matter fields. A generalization of the notion of an associated bundle is the category of gauge-natural bundles [10, 11]. In particular, any automorphism of P yields an automorphism of an associated gauge-natural bundle. The above mentioned bundle of principal connections $C = J^1P/G$ exemplifies a gauge-natural bundle.

Studying gauge gravitation theories, it seems reasonable to require that they must incorporate Einstein's General Relativity and, in particular, admit general covariant transformations. Fiber bundles possessing general covariant transformations belong to the category of natural bundles [12]. Given a natural bundle $T \to X$, there exists the canonical lift, called the natural lift, of any diffeomorphism of its base X to a bundle automorphism of T called a general covariant transformation. Accordingly, any vector field τ on X gives rise to a vector field $\overline{\tau}$ on T such that $\tau \mapsto \overline{\tau}$ is a monomorphism of the Lie algebra $\mathcal{T}(X)$ of vector field on X to that on T. One can think of the lift $\overline{\tau}$ as being an infinitesimal generator of a local one-parameter group of local general covariant transformations of T. Note that $\mathcal{T}(X)$ fails to be a Lie algebra of a Lie group. In contrast with the Lie algebra $\mathcal{G}$ of infinitesimal gauge transformations in Yang–Mills gauge theory, $\mathcal{T}(X)$ is not a $C^\infty(X)$-algebra, i.e., it is not a gauge extension of a finite-dimensional Lie algebra.

Gauge gravitation theories thus are gauge theories on natural bundles. The tangent TX, cotangent T^*X and tensor bundles over X exemplify natural bundles. The associated principal bundle with the structure group $GL(n,\mathbb{R})$, $n = \dim X$, is the fiber bundle LX of linear frames in the tangent spaces to X. It is also a natural bundle. Moreover, any gauge-natural bundle associated to LX is a natural bundle. Principal connections on LX are linear connections on the tangent bundle $TX \to X$. If TX is endowed with holonomic bundle coordinates $(x^\mu, \dot x^\mu)$ with respect to the holonomic frames $\{\partial_\mu\}$, such a connection is represent by the TTX-valued form

$$\Gamma = dx^\mu \otimes (\partial_\mu + \Gamma_\mu{}^\alpha{}_\beta \dot x^\beta \dot\partial_\alpha), \tag{4}$$

where $\{\partial_\mu, \dot\partial_\mu = \partial/\partial\dot x^\mu\}$ are holonomic frames in the tangent bundle TTX of TX. The bundle of linear connections

$$C_\Gamma = J^1LX/GL(n,\mathbb{R}) \tag{5}$$

is not associated to LX, but is a natural bundle, i.e., it admits general covariant transformations [9].

General covariant transformations are sufficient in order to restart Einstein's General Relativity and, moreover, the metric-affine gravitation theory [13]. However, one also considers the total group $\mathrm{Aut}(LX)$ of automorphisms of the frame bundle LX over diffeomorphisms of its base X [3]. Such an automorphism is the composition of some general covariant transformation and a vertical automorphism of LX, which is a non-holonomic frame transformation. Subject to vertical automorphisms, the tangent bundle TX is provided with non-holonomic frames $\{\vartheta_a\}$ and the corresponding bundle coordinates (x^μ, y^a). With respect to these coordinates, a linear connection (4) on TX reads

$$\Gamma = dx^\mu \otimes (\partial_\mu + \Gamma_\mu{}^a{}_b(x^\nu) y^b \partial_a), \tag{6}$$

where $\{\partial_\mu, \partial_a\}$ are the holonomic frames in the tangent bundle TTX of TX. A problem is that the Hilbert–Einstein Lagrangian is not invariant under non-holonomic frame transformations. To overcome this difficulty, one can additionally introduce an n-tuple of frame $\vartheta_a = \vartheta^\mu_a \partial_\mu$ (or coframe $\vartheta^a = \vartheta^a_\mu dx^\mu$) fields, which is a section of the frame bundle LX. This section is necessarily local, unless LX is a trivial bundle, i.e., X is a parallelizable manifold.

Furthermore, the vector bundle TX possesses a natural structure of an affine bundle. The associated principal bundle with the structure affine group $A(n,\mathbb{R})$ is the bundle AX of affine frames in TX. There is the canonical bundle monomorphism $LX \to AX$ corresponding to the zero section of the tangent bundle $TX = AX/GL(n,\mathbb{R})$. Principal connections on AX are affine connections on TX. Written with respect to an atlas with linear transition functions, such a connection is a sum $K = \Gamma + \Theta$ of a linear connection Γ on TX

and a VTX-valued (soldering) form Θ, i.e., a section of the tensor bundle $T^*X\otimes VTX$, where VTX is the vertical tangent bundle of $TX\to X$ [9]. Given linear bundle coordinates (x^μ, y^a) on TX, it reads

$$K = dx^\mu \otimes (\partial_\mu + \Gamma_\mu{}^a{}_b(x^\nu)y^b\partial_a + \Theta^a_\mu(x^\nu)\partial_a). \tag{7}$$

Due to the canonical isomorphism $VTX = TX\times TX$, the soldering form $\Theta = \Theta^a_\mu dx^\mu\otimes\partial_a$ yields the TX-valued form $\Theta_X = \Theta^a_\mu dx^\mu\otimes\vartheta_a$, and *vice versa*. For instance, K (7) is a Cartan connection if $\Theta_X = \theta = dx^\mu\otimes\partial_\mu$ is the canonical tangent-valued form on X (i.e., the canonical global section $\theta = \mathbf{1}$ of the group bundle $LX_{Ad}\subset T^*X\otimes TX$). Note that there are different physical interpretations of the translation part Θ of affine connections. In the gauge theory of dislocations, a field Θ describes a distortion [14–16]. At the same time, given a linear frame ϑ_a, the decomposition $\theta = \vartheta^a\otimes\vartheta_a$ motivates many authors to treat a coframe ϑ^a as a translation gauge field (see [3, 17] and references therein). A spinor representation of the Poincaré group is called into play, too [18].

Yang–Mills gauge theory also deals with a Higgs field, besides gauge potentials and matter fields. A Higgs field is responsible for a symmetry breaking. Given a principal bundle $P\to X$ with a structure Lie group G, let H be a closed (consequently, Lie) subgroup H of G. Then we have the composite bundle

$$P \xrightarrow{\pi_{P\Sigma}} P/H \longrightarrow X, \tag{8}$$

where $P\to P/H$ is a principal bundle with the structure group H and $P/H\to X$ is a P-associated bundle with the structure group G acting on its typical fiber G/H on the left. In classical gauge theory on $P\to X$, a symmetry breaking is defined as a reduction of its structure group G to the subgroup H of exact symmetries, i.e., P contains an H-principal subbundle called a G-structure [2,19–23]. There is one-to-one correspondence $P^h = \pi^{-1}_{P\Sigma}(h(X))$ between the reduced H-principal subbundles P^h of P and the global sections h of the quotient bundle $P/H\to X$. These sections are treated as classical Higgs fields. Any principal connection A^h on a reduced subbundle P^h gives rise to a principal connection on P and yields an associated connection on $P/H\to X$ such that the covariant differential $D_{A^h}h$ of h vanishes. Conversely, a principal connection A on P is projected onto P^h iff $D_A h = 0$. At the same time, if the Lie algebra $\mathcal{G}$ of G is the direct sum

$$\mathcal{G} = \mathcal{H}\oplus\mathfrak{m} \tag{9}$$

of the Lie algebra $\mathcal{H}$ of H and a subspace $\mathfrak{m}\subset\mathcal{G}$ such that $ad(g)(\mathfrak{m})\subset\mathfrak{m}$, $g\in H$, then the pull-back of the $\mathcal{H}$-valued component of any principal connection on P onto a reduced subbundle P^h is a principal connection on P^h. This is the case of so-called reductive G-structure [24].

Let $Y \to P/H$ be a vector bundle associated to the H-principal bundle $P \to P/H$. Then, sections of the composite bundle $Y \to P/H \to X$ describe matter fields with the exact symmetry group H in the presence of Higgs fields. A problem is that the typical fiber of a fiber bundle $Y \to X$ fails to carry out a representation of the group G, unless $G \to G/H$ is a trivial bundle. It follows that $Y \to X$ is not associated to P and, it does not admit a principal connection in general. If $G \to G/H$ is a trivial bundle, there exists its global section whose values are representatives of elements of G/H. In this case, the typical fiber of $Y \to X$ is $V \times G/H$, and one can provide it with an induced representation of G [25, 26]. Of course, this representation is not canonical, unless V itself admits a representation of G. If H is a Cartan subgroup of G, the so-called non-linear realization of G in a neighborhood of its unit [27, 28] exemplifies an induced representation.

In order to introduce a covariant differential on $Y \to X$, one can use a principal connection on $Y \to P/H$ [9, 22, 29].

Riemannian and pseudo-Riemannian metrics on a manifold X exemplify classical Higgs fields. Let X be an oriented four-dimensional smooth manifold. The structure group $GL_4 = GL^+(4, \mathbb{R})$ of LX is always reducible to its maximal compact subgroup $SO(4)$. The corresponding global sections of the quotient bundle $LX/SO(4)$ are Riemannian metrics on X. Accordingly, pseudo-Riemannian metrics of signature $(+, - - -)$ on X are global sections of the quotient bundle

$$\Sigma_{\rm PR} = LX/SO(1,3) \to X, \tag{10}$$

corresponding to the reduction of the structure group GL_4 of LX to its Lorentz subgroup $SO(1,3)$ [2, 30, 31]. Such a reduction need not exist, unless X satisfies certain topological conditions. From the physical viewpoint, the existence of a Lorentz reduced structure comes from the geometric equivalence principle and the existence of Dirac's fermion matter [2, 15]. The quotient bundle $\Sigma_{\rm PR}$ (10) is associated to LX. It is a natural bundle. Its global section h, called a tetrad field, defines a principal Lorentz subbundle L^hX of LX. Therefore, h can be represented by a family of local sections $\{h_a\}_\iota$ of LX on trivialization domains U_ι which take values in L^hX and possess Lorentz transition functions. One calls $\{h_a\}$ the tetrad functions, Lorentz frames, or vielbeins. They define an atlas $\Psi^h = \{(\{h_a\}_\iota, U_\iota)\}$ of LX and associated bundles with Lorentz transition functions. There is the canonical imbedding of the bundle $\Sigma_{\rm PR}$ (10) onto an open subbundle of the tensor bundle $\overset{2}{\vee} T^*X$ such that its global section $h = g$ is a pseudo-Riemannian metric $g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab}$ on X, which comes to the the Minkowski metric η with respect to an atlas Ψ^h. Since the Lorentz group is a Cartan subgroup of GL_4, one can locally put $g = \exp\{\sigma^{\alpha\beta}(x^\mu)S_{\alpha\beta}\}(\eta)$, where $S_{\alpha\beta}$ are non-Lorentz generators of GL_4, and treat the parameter functions $\sigma^{\alpha\beta}(x^\mu)$ as Goldstone fields [2, 32, 33].

Any connection on a Lorentz principal bundle L^hX is extended to a connection on LX

and, thus, is a linear connection on TX and other associated bundles. It is called a Lorentz connection. The covariant derivative of g with respect to such a connection vanishes. Thus, gauge theory on the linear frame bundle LX whose structure group is reduced to the Lorentz subgroup restarts the metric-affine gravitation theory [3, 13, 17, 34]. Considering only Lorentz connections, we are in the case of gravity with torsion [4,35–37]. If a connection is flat, its torsion need not vanish. This is the case of a teleparallel gravity [38–40] on a parallelizable manifold. Note that, since orientable three-dimensional manifolds M are parallelizable [41], any product $X = \mathbb{R} \times M$ is parallelizable (see [42] for the case of a compact X).

Note that, if the structure group of LX is reducible to the Lorentz group $SO(1,3)$, it is also reducible to its maximal compact subgroup $SO(3)$. The corresponding global section of the quotient bundle $L^hX/SO(3)$ is a time-like unit vector field h_0 on X which provides a space-time decomposition of TX [15]. Moreover, there is the commutative diagram of structure group reductions

$$\begin{array}{ccc} GL_4 & \longrightarrow & SO(1,3) \\ \downarrow & & \downarrow \\ SO(4) & \longrightarrow & SO(3) \end{array} \tag{11}$$

which leads to the well-known relation $g = 2h_0 \otimes h_0 - g_R$ between pseudo-Riemannian g and Riemannian g_R metrics on X.

Any reduction of the structure group GL_4 of the linear frame bundle LX to the Lorentz one implies the corresponding reduction of the structure group $A(4,\mathbb{R})$ of the affine frame bundle AX to the Poincaré group. This is the case of the so-called Poincaré gauge gravitation theory [3,4,43–47]. Since the Poincaré group comes from the Wigner–Inönii contraction of the de Sitter groups $SO(2,3)$ and $SO(1,4)$ and it is a subgroup of the conformal group, gauge theories on fiber bundles $Y \to X$ with these structure groups, reduced to the Lorentz one, are also considered [45,48–52]. Because these fiber bundles fail to be natural, the lift of the group $Diff(X)$ of diffeomorphisms of X onto Y should be defined [53, 54]. One also meets a problem of physical treating various Higgs fields. In a general setting, one can study a gauge theory on a fiber bundle with the typical fiber $\mathbb{R}^n$ and the topological structure group $Diff(\mathbb{R}^n)$ or its subgroup of analytical diffeomorphisms [55, 56]. Note that any paracompact smooth manifold admits an analytic manifold structure inducing the original smooth one [57]. The Poincaré gauge theory is also generalized to the higher s-spin gauge theory of tensor coframes $\vartheta_\mu^{a_1\ldots a_{s-1}} dx^\mu$ and generalized Lorentz connections $A_\mu^{a_1\ldots a_t, b_1\ldots b_t}$, $t = 1, \ldots, s-1$, which satisfy certain symmetry, skew symmetry and traceless conditions [58, 59].

As was mentioned above, the existence of Dirac's spinor matter implies the existence of a Lorentz reduced structure and, consequently, a (tetrad) gravitational field. Note that, for the purpose of gauge gravitation theory, it is convenient to describe Dirac spinors in the

Clifford algebra terms [60–62]. The Dirac spinor structure on a four-dimensional manifold X is defined as a pair (P^h, z_s) of a principal bundle $P^h \to X$ with the structure spin group $L_s = SL(2,\mathbb{C})$ and its bundle morphism $z_s : P^h \to LX$ to the linear frame bundle LX [63, 64]. Any such morphism factorizes

$$P^h \to L^h X \to LX \tag{12}$$

through some reduced principal subbundle $L^h X \subset LX$ with the structure proper Lorentz group $L = SO^\uparrow(1,3)$, whose universal two-fold covering is L_s. The corresponding quotient bundle $\Sigma_{\rm T} = LX/L$ is a two-fold covering of the bundle $\Sigma_{\rm PR}$ (10). Its global sections are tetrad fields h represented by families of tetrad functions taking values in the proper Lorentz group L. Thus, any Dirac spinor structure is associated to a Lorentz reduced structure, but the converse need not be true. There is the well-known topological obstruction to the existence of a Dirac spinor structure [65, 66]. For instance, a Dirac spinor structure on a non-compact manifold X exists iff X is parallelizable.

Point out that the Dirac spinor structure (12) together with the structure group reduction diagram (11) provides the Ashtekar variables [67].

Given a Dirac spinor structure (12), the associated Dirac spinor bundle S^h can be seen as a subbundle of the bundle of Clifford algebras generated by the Lorentz frames $\{t_a\} \in L^h X$ [64, 68] (see also [69]). This fact enables one to define the Clifford representation

$$\gamma_h(dx^\mu) = h^\mu_a \gamma^a \tag{13}$$

of coframes dx^μ in the cotangent bundle T^*X by Dirac's matrices, and introduce the Dirac operator on S^h with respect to a principal connection on P^h. Then, sections of a spinor bundle S^h describe Dirac spinor fields in the presence of a tetrad field h. Note that there is one-to-one correspondence between the principal connections on P^h and those on the Lorentz frame bundle $L^h X = z_s(P^h)$. Moreover, since the Lie algebras of $G = GL_4$ and $H = L$ obey the decomposition (9), any principal connection Γ on LX yields a spinor connection Γ_s on P^h and S^h [70, 71]. At the same time, the representations (13) for different tetrad fields fail to be equivalent. Therefore, one meets a problem of describing Dirac spinor fields in the presence of different tetrad fields and under general covariant transformations.

Due to the decomposition (9), there is also the canonical lift of any vector field on X onto the bundles P^h and S^h though they are not natural bundles [72-74]. However, this lift, called the Kosmann's Lie derivative, fails to be an infinitesimal generator of general covariant transformations. In order to solve this problem, one can call into play the universal two-fold covering $\widetilde{GL}_4$ of the group GL_4 which obeys the commutative diagram

$$\begin{array}{ccc} \widetilde{GL}_4 & \longrightarrow & GL_4 \\ \uparrow & & \uparrow \\ L_s & \longrightarrow & L \end{array} \qquad (14)$$

Let us consider the $\widetilde{GL}_4$-principal bundle $\widetilde{LX} \to X$ which is the two-fold covering bundle of the frame bundle LX [20, 64, 75, 76]. This covering bundle is unique if X is parallelizable, and it inherits general covariant transformations of the linear frame bundle LX. However, spinor representations of the group $\widetilde{GL}_4$ are infinite-dimensional. Therefore, the $\widetilde{LX}$-associated spinor bundle describes infinite-dimensional "world" spinor fields, but not the Dirac ones [3, 77, 78].

In a different way, we have the commutative diagram

$$\begin{array}{ccc} \widetilde{LX} & \xrightarrow{\zeta} & LX \\ \uparrow & & \uparrow \\ P^h & \longrightarrow & L^hX \end{array} \qquad (15)$$

for any Dirac spinor structure (12) [71, 79]. As a consequence, $\widetilde{LX}/\mathrm{L}_s = LX/L = \Sigma_{\mathrm{T}}$. Let us consider the L_s-principal bundle $\widetilde{LX} \to \Sigma_{\mathrm{T}}$ and the associated spinor bundle $S \to \Sigma_{\mathrm{T}}$. It follows from the diagram (15) that, given a section h of the tetrad fiber bundle $\Sigma_{\mathrm{T}} \to X$, the pull-back of $S \to \Sigma_{\mathrm{T}}$ onto $h(X) \subset \Sigma_{\mathrm{T}}$ is exactly a spinor bundle S^h whose sections describe Dirac spinor fields in the presence of a tetrad field h. Moreover, given the bundle of linear connections C_Γ (5), the pull-back of the spinor bundle $S \to \Sigma_{\mathrm{T}}$ onto $\Sigma_{\mathrm{T}} \times C_\Gamma$ can be provided with a connection and the Dirac operator $\mathcal{D}$ possessing the following property. Given a tetrad field h and a linear connection Γ, the restriction of $\mathcal{D}$ to $S^h = h(X) \times \Gamma(X)$ is the familiar Dirac operator on the spinor bundle $S^h \to X$ with respect to a spinor connection Γ_s [13, 71]. Thus, sections of the composite bundle $S \to \Sigma_{\mathrm{T}} \to X$ describe Dirac spinor fields on X in the presence of different tetrad fields. If X is parallelizable, one can make $S \to X$ associated to the $\widetilde{GL}_4$-principal bundle $\widetilde{LX}$, but not in a canonical way. Accordingly, $S \to X$ admits different lifts of vector fields on X. They differ from each other in vertical fields on $S \to \Sigma_{\mathrm{T}}$ which are infinitesimal generators of Lorentz gauge transformations.

Bearing in mind quantum field theory and unification models, one considers spinor fields on Riemannian and pseudo-Riemannian manifolds of any signature which possess additional symmetries. Let us mention charged spinor fields on a Riemannian manifold X, $\dim X = n$. They are sections of a vector bundle associated to the two-fold covering $P\mathrm{Spin}^c$ of the product $P_{SO} \times P_{U(1)}$ of the $SO(n)$-principal bundle P_{SO} of orthonormal frames in TX and a $U(1)$-principal bundle $P_{U(1)}$ [64, 80, 81]. One says that $P\mathrm{Spin}^c$ defines a spin^c-structure on X. It should be emphasized that a spin^c-structure may prsent even if no spin structure exists. For instance, any oriented four-dimensional Riemannian manifold admits a spin^c-structure. If X is compact, a spin-(spin^c)-structure comes from Connes' commutative geometry characterized by the spectral triple $(\mathcal{A}, E, \mathcal{D})$ of the algebra $\mathcal{A} = C^\infty(X, \mathbb{C})$ of smooth complex functions on X, the Hilbert space $E = L(X, S)$ of square integrable sections of a spinor bundle $S \to X$, and the Dirac operator $\mathcal{D}$ on S [82, 83]. This construction is extended to non-compact manifolds X [84], globally hyperbolic Lorentzian manifolds [85], and pseudo-Riemannian spectral triples [86]. Non-abelian generalizations of a spin^c-structure are also studied [87].

In converse to that General Relativity can be derived from gauge theory, the Kaluza–Klein theory generalized to non-Abelian symmetries shows that higher-dimensional pseudo-Riemannian geometry can lead to Yang–Mills gauge theory on fiber bundles [88–90]. Namely, the following holds [91, 92]. Let E be a smooth manifold provided with a right action of a compact Lie group G such that all isotropy groups are isomorphic to a standard one H. Then $E \to E/G$ is a fiber bundle with the typical fiber G/H and the structure group N/H, where N is the normalize of H in G. Let γ be a G-invariant metric on E. Then it determines a G-invariant metric σ on every fiber of $E \to E/G$, a principal connection A on this fiber bundle and a metric g on E/G, and *vice versa*. Moreover, the scalar curvature of γ falls into the sum of the scalar curvature of g, the Yang–Mills Lagrangian for A and the terms depending on σ. This mathematical result however fails to guarantee a perfect field model on a 'space-time' E/G. A problem is to treat both the extra dimensions differing so markedly from the space-time ones and the metric σ whose components are scalar fields on E/G [93]. It becomes a folklore that the $D = 11$ supergravity provides the most satisfactory solution of this problem [6].

Supergravity theory started with super-extensions of the (anti-) De Sitter and Poincaré Lie algebras [5,94–96]. It is greatly motivated both by the field-unification program and contemporary string and brane theories [7,97–101]. There are various superextensions of pseudo-orthogonal and Poincaré Lie algebras [102–105]. Supergravity is mainly developed as their Yang–Mills theory. The geometric formulation of supergravity as a partner of gravity [106, 107] however meets difficulties.

First of all, it should be noted that the spin spectrum analysis of frame and gauge fields fails to be correct, unless we are in the case of a pseudo-Euclidean space which is not subject to general covariant transformations. Furthermore, odd fields need not be spinor fields. Ghosts and antifields in BRST field theory exemplify odd fields. Among graded commutative algebras, the Arens–Michel algebras of Grassmann type are most suitable for the superanalysis [108], but Grassmann algebras Λ of finite rank are usually called into play. There are several variants of supergeometry over such an algebra [109, 110]. They are graded manifolds, smooth H^∞-, G^∞- GH^∞-supermanifolds, G- and DeWitt supermanifolds. A graded manifold is a pair of a smooth manifold X and a sheaf of Grassmann algebras on X. A smooth GH^∞-supermanifold is a graded local-ringed space $(M, \mathfrak{S})$ which is locally isomorphic to $(B^{n,m}, \mathcal{S})$ where $B^{n,m} = \Lambda_0^n \oplus \Lambda_1^m$ is a supervector space and $\mathcal{S}$ is a sheaf of superfunctions on $B^{n,m}$ taking their values in a Grassmann subalgebra $\Lambda' \subset \Lambda$. One separates the following variants: (i) $\Lambda' = \Lambda$ of G^∞-supermanifolds introduced by A.Rogers [111], (ii) $\mathrm{rank}\,\Lambda - \mathrm{rank}\,\text{Ł}' \geq m$ of GH^∞- supermanifolds, and (iii) $\Lambda' = \mathbb{R}$ of H^∞-supermanifolds. The condition in item (ii) guarantees that odd derivatives can be well defined. For instance, this is not the case of G^∞-supermanifolds, unless $m = 0$. It is essential that the underlying space M of a smooth supermanifold is provided with a structure of a smooth manifold of dimension $2^{\mathrm{rank}\,\Lambda - 1}(n+m)$. At the same time, smooth supermanifolds are effected to serious inconsistencies. For instance, the sheaf of derivations of G^∞-superfunctions is not locally free, and spaces of values of GH^∞-superfunctions fail to be naturally isomorphic. The G-supermanifolds are free of such inconsistences. They are locally isomorphic to $(B^{n,m}, \mathcal{G}_{n,m})$, where the sheaf $\mathcal{G}_{n,m}$ of G-superfunctions is isomorphic to the tensor product $\mathcal{S} \otimes \Lambda$, where $\mathcal{S}$ is a sheaf of H^∞-superfunctions. However, it may happen that non-isomorphic G-manifolds can possess isomorphic underlying smooth manifolds. Smooth supermanifolds and G-supermanifolds become the DeWitt supermanifolds if they are provided with the non-Hausdorff DeWitt topology. This is the coarsest topology such that the body map $B^{n,m} \to \mathbb{R}^n$ is continuous [112]. There is the well-known correspondence between DeWitt supermanifolds and graded manifolds.

Lie supergroups, principal superbundles, associated supervector bundles and principal superconnections are considered in the category of G-supermanifolds in an appropriate way [109]. Thus, they can provide an adequate mathematical language of superextension of Yang–Mills gauge theory. Moreover, on may hope that, since G-supermanifolds are also smooth manifolds, the theorem of reduction of a structure group can be extended to principal superbundles, and then Higgs superfields and supermetrics on supermanifolds can be introduced as true geometric partners of classical Higgs fields and gravity [113]. However, a problem is that even coordinates and variables on supermanifolds are not real (or complex), but contain a nilpotent summand. This is not a standard case of supergauge models [106, 107]. Therefore, one should turn to graded manifolds whose local bases consist of coordinates on a smooth body manifold X and odd generating elements of a Grassmann algebra Λ. The well-known Serre–Swan theorem extended to graded manifolds states that, given a smooth manifold X, a Grassmann exterior algebra generated by elements of a projective $C^\infty(X)$-module of finite rank is isomorphic to the Grassmann algebra of graded functions on a graded manifold with a body X [114]. The theory of graded principal bundles and connections has been developed [116–117], but it involves Hopf algebras and looks rather sophisticated. At the same time, graded manifolds provide the adequate geometric formulation of Lagrangian BRST theory, where supergauge transformations are replaced with a nilpotent BRST operator [118, 119].

There are strong reasons to think that, due to quantum gravity, space-time coordinates become non-commutative [120]. Non-commutative field theory is known to emerge from many mathematical and physical models [121]. There are several approaches to describing non-commutative gravity. One of them lies in the framework of Connes' non-commutative geometry of spectral triples [122–124]. In particular, a gravitational action can be represented by the trace of a suitable function of the Dirac operator [123]. A gauge approach is based on the Seiberg–Witten map replacing the original product of gauge, vielbein and metric fields with the star one [125]. Different variants of the q-deformation (quantum groups) [126, 127] and the Moyal-like (twist) deformation [128–131] of space-time algebras are also considered. Note that the deformation by the twist leads to complex geometry and, in particular, to complexified gravity [132].

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